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High magnetic field equilibria for the Fokker–Planck–Landau equation

Mihai Bostan

Aix Marseille Université, CNRS, Centrale Marseille, LATP, UMR 7353, 13453 Marseille, France

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Abstract

The subject matter of this paper concerns the equilibria of the Fokker–Planck–Landau equation under the action of strong magnetic fields. Averaging with respect to the fast cyclotronic motion when the Larmor radius is supposed to be finite leads to an integro-differential version of the Fokker–Planck–Landau collision kernel, combining perpendicular space coordinates (with respect to the magnetic lines) and velocity. We determine the equilibria of this gyroaveraged Fokker–Planck–Landau kernel and derive the macroscopic equations describing the evolution around these equilibria, in the parallel direction.

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1. Introduction

We investigate the transport of charged particles under the action of strong magnetic fields, which is motivated by the magnetic confinement for tokamak plasmas. We neglect the self-consistent electro-magnetic field, but we take into account the interactions between particles. The external electric field $E = -\nabla_x \Phi$ is fixed, and the external magnetic field writes

$$B^\varepsilon = \frac{B(x)}{\varepsilon} d(x), \quad |d| = 1$$

where $\varepsilon > 0$ is a small parameter, destinated to converge to 0, in order to describe strong magnetic fields. The scalar function ϕ stands for the electric potential, $B(x) > 0$ is the rescaled magnitude of the magnetic field and $d(x)$ denotes its direction.

E-mail addresses: bostan@cmi.univ-mrs.fr, mihai.bostan@univ-amu.fr.

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The presence density $f^\varepsilon = f^\varepsilon(t,x,v) \geq 0$ of a population of charged particles with mass m and charge q satisfies

$$\partial_t f^\varepsilon + v\cdot\nabla_x f^\varepsilon + \frac{q}{m}\big(E + v\wedge B^\varepsilon\big)\cdot\nabla_v f^\varepsilon = Q\big(f^\varepsilon, f^\varepsilon\big), \quad (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3, \tag{1}$$

$$f^\varepsilon(0,x,v) = f^{\text{in}}(x,v), \quad (x,v)\in\mathbb{R}^3\times\mathbb{R}^3. \tag{2}$$

Here Q denotes the Fokker–Planck–Landau collision kernel cf. [21,13,14]

$$Q(f,f)(v) = \operatorname{div}_v\left\{\int_{\mathbb{R}^3}\sigma\big(|v-v'|\big)S\big(v-v'\big)\big[f\big(v'\big)\nabla_v f(v) - f(v)\nabla_{v'} f\big(v'\big)\big]\,\mathrm{d}v'\right\}$$

where $\sigma > 0$ stands for the scattering cross section and $S(w) = I - \frac{w\otimes w}{|w|^2}$ is the orthogonal projection on the plane of normal $w \neq 0$. The interpretation of the density f^ε is straightforward: the number of charged particles contained at time t inside the infinitesimal volume $\mathrm{d}x\mathrm{d}v$ around the point (x,v) of the position-velocity phase space is given by $f^\varepsilon(t,x,v)\mathrm{d}x\mathrm{d}v$. Eq. (1) describes the evolution of the density f^ε due to the transport and to the particle interactions.

The behavior of (1), (2) without collisions, when $\varepsilon \searrow 0$, is now well understood [20,24,15,3–6]. It reduces to homogenization analysis and can be solved using the concept of two-scale convergence [17,18,16].

Gyroaveraged collision operators have been proposed in [25,11,12,19]. The main difficulty lies on the relaxation of the distribution function towards an equilibrium. Many of these gyroaveraged collision operators fail to relax to equilibria, in particular those obtained by linearization around Maxwellians (which are not gyrokinetic equilibria, at least in the finite Larmor radius regime). Very recently, the averaging techniques developed in [3–5] have been extended to the collisional framework. Gyroaveraged collision kernels have been proposed for the relaxation Boltzmann operator, the Fokker–Planck and Fokker–Planck–Landau operators [7–10].

There are mainly two asymptotic regimes describing the transport of charged particles under strong magnetic fields: the guiding center and the finite Larmor radius approximations. In the guiding center approximation, the ratio between the perpendicular and parallel spatial lengths is much smaller (and thus neglected) with respect to the ratio between the cyclotronic period and the observation time unit. In this case, any Larmor circle reduces to its center. Therefore, the particle positions are left invariant at the cyclotronic time scale, the magnetic field becomes locally uniform, and the gyroaverage plays only in the perpendicular velocity space. For these reasons, the derivation of the guiding center approximation is relatively simple, and explicit models are available for general tridimensional magnetic geometry [5,6,10]. The situation is quite different for the finite Larmor radius approximation. In this case, we assume that the ratio between the perpendicular and parallel spatial lengths is small, remaining of the same order as the ratio between the cyclotronic period and the observation time unit

$$\frac{L_\perp}{L_\parallel} = \frac{T_c}{T_{\text{obs}}} = \varepsilon \ll 1.$$

The particles move on small Larmor circles, the position is not anymore left invariant at the cyclotronic scale, the magnetic field is no more locally uniform, and the gyroaverage combines now position and velocity. Think that the average of a particle position, which is the Larmor center, depends not only on the initial position, but also on the initial perpendicular velocity. This fact will impact a lot the structure of the Fokker–Planck–Landau kernel. Indeed, after average, the collision kernel will be not anymore local in space and the equilibria will be given by profile in velocity and perpendicular position. The computations require much effort, and most of the times, the limit models are not completely explicit. Generally we start analyzing the case of uniform magnetic fields, eventually we generalize these results by linearization around the Larmor center (since the magnetic field does not change a lot along a Larmor radius). The finite Larmor radius regime provides a more realistic description for the tokamak plasmas.

In this paper we concentrate on the finite Larmor radius approximation. Assuming that the magnetic field is homogeneous and stationary

$$B^\varepsilon = \left(0,0,\frac{B}{\varepsilon}\right)$$

for some constant $B > 0$, Eq. (1) becomes

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}\big(v_1\partial_{x_1} f^\varepsilon + v_2\partial_{x_2} f^\varepsilon\big) + v_3\partial_{x_3} f^\varepsilon + \frac{q}{m}E\cdot\nabla_v f^\varepsilon + \frac{\omega_c}{\varepsilon}\big(v_2\partial_{v_1} f^\varepsilon - v_1\partial_{v_2} f^\varepsilon\big) = Q\big(f^\varepsilon, f^\varepsilon\big) \tag{3}$$

where $\omega_c = qB/m$ stands for the rescaled cyclotronic frequency. When ε is small, the density f^ε writes as a combination between a dominant density f and corrections of orders $\varepsilon, \varepsilon^2, \dots$

$$f^\varepsilon = f + \varepsilon f^1 + \varepsilon^2 f^2 + \cdots. \tag{4}$$

Plugging (4) into (3) and using the notations $\overline{x} = (x_1, x_2)$, $\overline{v} = (v_1, v_2)$, ${}^\perp\overline{v} = (v_2, -v_1)$ yield

$$\mathcal{T} f := \overline{v} \cdot \nabla_{\overline{x}} f + \omega_c {}^\perp\overline{v} \cdot \nabla_{\overline{v}} f = 0, \tag{5}$$

$$\partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f + \mathcal{T} f^1 = Q(f, f), \tag{6}$$

$$\vdots$$

where $\mathcal{T}$ is the linear operator defined in $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ by

$$\mathcal{T} u = \operatorname{div}_{x,v}(ub), \quad b = (\overline{v}, 0, \omega_c {}^\perp\overline{v}, 0), \ \omega_c = \frac{qB}{m}$$

for any function u in the domain

$$\mathrm{D}(\mathcal{T}) = \{u(x, v) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3): \ \operatorname{div}_{x,v}(ub) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)\}.$$

At any time t the density $f(t, \cdot, \cdot)$ remains constant along the flow $(X, V)(s; x, v)$ associated to the transport operator $\overline{v} \cdot \nabla_{\overline{x}} + \omega_c {}^\perp\overline{v} \cdot \nabla_{\overline{v}}$

$$\frac{\mathrm{d}\overline{X}}{\mathrm{d}s} = \overline{V}(s), \qquad \frac{\mathrm{d}X_3}{\mathrm{d}s} = 0, \qquad \frac{\mathrm{d}\overline{V}}{\mathrm{d}s} = \omega_c {}^\perp\overline{V}(s), \qquad \frac{\mathrm{d}V_3}{\mathrm{d}s} = 0, \qquad (X, V)(0; x, v) = (x, v) \tag{7}$$

and therefore, at any time t, the density $f(t, \cdot, \cdot)$ depends only on the invariants of (7)

$$f(t, x, v) = g\left(t, x_1 + \frac{v_2}{\omega_c}, x_2 - \frac{v_1}{\omega_c}, x_3, r = |\overline{v}|, v_3\right).$$

The time evolution for f comes by (6), after eliminating f^1. The antisymmetry of $\mathcal{T}$ ensures that the range of $\mathcal{T}$ is orthogonal to its kernel, which allows us to get rid of f^1 in (6) by taking the orthogonal projection onto $\ker \mathcal{T}$

$$\mathrm{Proj}_{\ker \mathcal{T}}\left\{\partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f\right\} = \mathrm{Proj}_{\ker \mathcal{T}}\{Q(f, f)\}, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3. \tag{8}$$

Actually taking the orthogonal projection on $\ker \mathcal{T}$ reduces to averaging along the characteristic flow of $\mathcal{T}$ in (7) cf. [3–5]. This flow is $T_c = \frac{2\pi}{\omega_c}$ periodic and writes

$$\overline{V}(s) = R(-\omega_c s)\overline{v}, \qquad \overline{X}(s) = \overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} - \frac{{}^\perp\overline{V}(s)}{\omega_c}, \qquad X_3(s) = x_3, \qquad V_3(s) = v_3$$

where $R(\alpha)$ stands for the rotation of angle α

$$R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

For any function $u \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, the average operator is defined by

$$\langle u \rangle(x, v) = \frac{1}{T_c} \int_0^{T_c} u\big(X(s; x, v), V(s; x, v)\big) \, \mathrm{d}s$$

$$= \frac{1}{2\pi} \int_0^{2\pi} u\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} - \frac{{}^\perp\{R(\alpha)\overline{v}\}}{\omega_c}, x_3, R(\alpha)\overline{v}, v_3\right) \mathrm{d}\alpha. \tag{9}$$

We introduce the notation $e^{i\varphi}$ for the $\mathbb{R}^2$ vector $(\cos\varphi, \sin\varphi)$. If the vector $\overline{v}$ writes $\overline{v} = |\overline{v}|e^{i\varphi}$, then $R(\alpha)\overline{v} = |\overline{v}|e^{i(\alpha+\varphi)}$ and the expression for $\langle u \rangle$ becomes

$$\begin{aligned}\langle u\rangle(x,v) &= \frac{1}{2\pi}\int_0^{2\pi} u\left(\overline{x}+\frac{{}^\perp\overline{v}}{\omega_c}-\frac{{}^\perp\{|\overline{v}|e^{i(\alpha+\varphi)}\}}{\omega_c}, x_3, |\overline{v}|e^{i(\alpha+\varphi)}, v_3\right)\mathrm{d}\alpha\\ &= \frac{1}{2\pi}\int_0^{2\pi} u\left(\overline{x}+\frac{{}^\perp\overline{v}}{\omega_c}-\frac{{}^\perp\{|\overline{v}|e^{i\alpha}\}}{\omega_c}, x_3, |\overline{v}|e^{i\alpha}, v_3\right)\mathrm{d}\alpha.\end{aligned}$$

The properties of the average operator (9) are summarized below (see Propositions 2.1, 2.2 in [5] for proof details). We denote by $\|\cdot\|$ the standard norm of $L^2(\mathbb{R}^3\times\mathbb{R}^3)$.

Proposition 1.1. *The average operator is linear and continuous. Moreover it coincides with the orthogonal projection on the kernel of $\mathcal{T}$* i.e.*,*

$$\langle u\rangle \in \ker\mathcal{T} \quad \textit{and} \quad \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\big(u-\langle u\rangle\big)\varphi\,\mathrm{d}v\mathrm{d}x = 0, \quad \forall\varphi\in\ker\mathcal{T}. \tag{10}$$

Remark 1.1. Notice that $(\overline{X},\overline{V})$ depends only on s and $(\overline{x},\overline{v})$ and thus the variational characterization in (10) holds true at any fixed $(x_3,v_3)\in\mathbb{R}^2$. Indeed, for any $\varphi\in\ker\mathcal{T}$, $(x_3,v_3)\in\mathbb{R}^2$ we have

$$\begin{aligned}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(u\varphi)(x,v)\,\mathrm{d}\overline{v}\mathrm{d}\overline{x} &= \frac{1}{T_c}\int_0^{T_c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u(x,v)\varphi\big(\overline{X}(-s;x,v),x_3,\overline{V}(-s;x,v),v_3\big)\,\mathrm{d}\overline{v}\mathrm{d}\overline{x}\mathrm{d}s\\ &= \frac{1}{T_c}\int_0^{T_c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u\big(\overline{X}(s;x,v),x_3,\overline{V}(s;x,v),v_3\big)\varphi(x,v)\,\mathrm{d}\overline{v}\mathrm{d}\overline{x}\mathrm{d}s\\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\langle u\rangle(x,v)\varphi(x,v)\,\mathrm{d}\overline{v}\mathrm{d}\overline{x}.\end{aligned}$$

We have the orthogonal decomposition of $L^2(\mathbb{R}^3\times\mathbb{R}^3)$ into invariant functions along the characteristics (7) and zero average functions

$$u = \langle u\rangle + \big(u-\langle u\rangle\big), \qquad \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\big(u-\langle u\rangle\big)\langle u\rangle\,\mathrm{d}v\mathrm{d}x = 0.$$

Notice that $\mathcal{T}^\star = -\mathcal{T}$ and thus the equality $\langle\cdot\rangle = \mathrm{Proj}_{\ker\mathcal{T}}$ implies

$$\ker\langle\cdot\rangle = (\ker\mathcal{T})^\perp = \big(\ker\mathcal{T}^\star\big)^\perp = \overline{\mathrm{Range}\,\mathcal{T}}.$$

In particular $\mathrm{Range}\,\mathcal{T}\subset\ker\langle\cdot\rangle$. Actually we show that $\mathrm{Range}\,\mathcal{T}$ is closed, which will give a solvability condition for $\mathcal{T}u = w$ (cf. [5, Propositions 2.2]).

Proposition 1.2. *The restriction of $\mathcal{T}$ to* $\ker\langle\cdot\rangle$ *is one-to-one map onto* $\ker\langle\cdot\rangle$*. Its inverse belongs to* $\mathcal{L}(\ker\langle\cdot\rangle,\ker\langle\cdot\rangle)$ *and we have the Poincaré inequality*

$$\|u\| \le \frac{2\pi}{|\omega_c|}\|\mathcal{T}u\|, \quad \omega_c = \frac{qB}{m}\neq 0$$

for any $u\in\mathrm{D}(\mathcal{T})\cap\ker\langle\cdot\rangle$.

A very useful result when averaging transport operators is given by the following commutation formula between divergence and average (cf. Proposition 3.3 in [8]).

Proposition 1.3. *For any smooth field $\xi = (\xi_x, \xi_v) \in \mathbb{R}^6$ we have the equality*

$$\langle \operatorname{div}_{x,v} \xi \rangle = \operatorname{div}_{\overline{x}} \left\{ \left\langle \xi_{\overline{x}} + \frac{^\perp \xi_{\overline{v}}}{\omega_c} \right\rangle + \left\langle \xi_{\overline{v}} \cdot \frac{^\perp \overline{v}}{|\overline{v}|} \right\rangle \frac{\overline{v}}{\omega_c |\overline{v}|} - \left\langle \xi_{\overline{v}} \cdot \frac{\overline{v}}{|\overline{v}|} \right\rangle \frac{^\perp \overline{v}}{\omega_c |\overline{v}|} \right\} + \partial_{x_3} \langle \xi_{x_3} \rangle$$
$$+ \operatorname{div}_{\overline{v}} \left\{ \left\langle \xi_{\overline{v}} \cdot \frac{^\perp \overline{v}}{|\overline{v}|} \right\rangle \frac{^\perp \overline{v}}{|\overline{v}|} + \left\langle \xi_{\overline{v}} \cdot \frac{\overline{v}}{|\overline{v}|} \right\rangle \frac{\overline{v}}{|\overline{v}|} \right\} + \partial_{v_3} \langle \xi_{v_3} \rangle.$$

In particular we have for any smooth field $\xi_x \in \mathbb{R}^3$

$$\langle \operatorname{div}_x \xi_x \rangle = \operatorname{div}_x \langle \xi_x \rangle$$

and for any smooth field $\xi_v \in \mathbb{R}^3$

$$\langle \operatorname{div}_v \xi_v \rangle = \operatorname{div}_{\overline{x}} \left\{ \left\langle \frac{^\perp \xi_{\overline{v}}}{\omega_c} \right\rangle + \left\langle \xi_{\overline{v}} \cdot \frac{^\perp \overline{v}}{|\overline{v}|} \right\rangle \frac{\overline{v}}{\omega_c |\overline{v}|} - \left\langle \xi_{\overline{v}} \cdot \frac{\overline{v}}{|\overline{v}|} \right\rangle \frac{^\perp \overline{v}}{\omega_c |\overline{v}|} \right\}$$
$$+ \operatorname{div}_{\overline{v}} \left\{ \left\langle \xi_{\overline{v}} \cdot \frac{^\perp \overline{v}}{|\overline{v}|} \right\rangle \frac{^\perp \overline{v}}{|\overline{v}|} + \left\langle \xi_{\overline{v}} \cdot \frac{\overline{v}}{|\overline{v}|} \right\rangle \frac{\overline{v}}{|\overline{v}|} \right\} + \partial_{v_3} \langle \xi_{v_3} \rangle.$$

Coming back to (8), on the one hand, averaging $\partial_t + v_3 \partial_{x_3} + \frac{q}{m} E \cdot \nabla_v$ leads to another transport operator. This is a straightforward consequence of the commutation formula between the divergence and average in Proposition 1.3. For the presentation clarity, the proof of this result is sketched in Appendix A.

Proposition 1.4. *Assume that the electric field derives from a smooth potential* i.e., *$E = -\nabla_x \phi$. Then for any $f \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3) \cap \ker \mathcal{T}$ we have*

$$\left\langle \partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f \right\rangle = \partial_t f + \frac{\langle ^\perp \overline{E} \rangle}{B} \cdot \nabla_{\overline{x}} f + v_3 \partial_{x_3} f + \frac{q}{m} \langle E_3 \rangle \partial_{v_3} f. \tag{11}$$

On the other hand, the average of the Fokker–Planck–Landau kernel *i.e.*, $\langle Q \rangle (f, f) := \langle Q(f, f) \rangle$ writes cf. Proposition 4.10 in [9]

$$\omega_c^{-2} \langle Q \rangle (f, f)(x, v)$$
$$= \operatorname{div}_{\omega_c x, v} \left\{ \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sum_{i=1}^{4} f\left(\overline{x'}, x_3, v'\right) \xi^i \left(\overline{x}, v, \overline{x'}, v'\right) \otimes \xi^i \left(\overline{x}, v, \overline{x'}, v'\right) \nabla_{\omega_c x, v} f(x, v) \, \mathrm{d}v' \mathrm{d}\overline{x'} \right\}$$
$$- \operatorname{div}_{\omega_c x, v} \left\{ \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \sum_{i=1}^{4} f(x, v) \xi^i \left(\overline{x}, v, \overline{x'}, v'\right) \otimes \varepsilon_i \xi^i \left(\overline{x'}, v', \overline{x}, v\right) \nabla_{\omega_c x', v'} f\left(\overline{x'}, x_3, v'\right) \mathrm{d}v' \mathrm{d}\overline{x'} \right\}. \tag{12}$$

Up to our knowledge, the above averaged Fokker–Planck–Landau kernel has never been reported in the plasma physics literature, before [9]. Its calculation relies on gyroaveraging differential operators and velocity convolutions. Some results regarding the behavior of the gyroaverage with respect to velocity convolutions have been obtained in [10] (in the framework of the guiding center approximation).

The operator in (12) is completely explicit. We indicate below the expressions for the vector fields entering it. Notice that their derivation is not of all trivial. The reader may refer to [9] for details. Nevertheless, we are using these expressions in order to determine the equilibria of the averaged Fokker–Planck–Landau kernel.

The notation $\operatorname{div}_{\omega_c x, v}$ stands for the divergence with respect to the variables $\omega_c x$ and v (like that all variables entering the divergence are homogeneous). Here $\varepsilon_1 = \varepsilon_2 = -1$, $\varepsilon_3 = \varepsilon_4 = 1$ and the explicit formulae of the fields $(\xi^i)_{1 \le i \le 4}$ are given by

$$\xi^1 \left(\overline{x}, v, \overline{x'}, v'\right) = \{\sigma \chi\}^{1/2} \frac{r' \sin \varphi (v_3 - v_3')}{|z| \sqrt{|z|^2 + (v_3 - v_3')^2}} \left(\frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{(^\perp \overline{v}, 0)}{|\overline{v}|} \right),$$

$$\xi^2(\overline{x}, v, \overline{x'}, v') = \{\sigma\chi\}^{1/2}\left[\frac{r - r'\cos\varphi}{|z|}\left(\frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{({}^\perp\overline{v}, 0)}{|\overline{v}|}\right) + \left(\frac{({}^\perp z, 0)}{|z|}, 0\right)\right],$$

$$\xi^3(\overline{x}, v, \overline{x'}, v') = \{\sigma\chi\}^{1/2}\frac{r'\sin\varphi}{|z|}\left(\frac{({}^\perp\overline{v}, 0)}{|\overline{v}|}, -\frac{(\overline{v}, 0)}{|\overline{v}|}\right),$$

$$\frac{\xi^4(\overline{x}, v, \overline{x'}, v')}{\{\sigma\chi\}^{1/2}} = \frac{(r'\cos\varphi - r)(v_3 - v_3')}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}}\left(\frac{({}^\perp\overline{v}, 0)}{|\overline{v}|}, -\frac{(\overline{v}, 0)}{|\overline{v}|}\right) + \frac{((v_3 - v_3')\frac{(z,0)}{|z|}, -|z|e_3)}{\sqrt{|z|^2 + (v_3 - v_3')^2}}$$

where $v_3, v_3' \in \mathbb{R}$, $r = |\overline{v}|$, $r' = |\overline{v'}|$, $z = (\omega_c\overline{x} + {}^\perp\overline{v}) - (\omega_c\overline{x'} + {}^\perp\overline{v'})$, $\sigma = \sigma\sqrt{|z|^2 + (v_3 - v_3')^2}$, the angle $\varphi \in (0, \pi)$ satisfies

$$|z|^2 = r^2 + (r')^2 - 2rr'\cos\varphi, \qquad |r - r'| < |z| < r + r'$$

and

$$\chi(r, r', z) = \frac{\mathbf{1}_{\{|r-r'|<|z|<r+r'\}}}{\pi^2\sqrt{|z|^2 - (r - r')^2}\sqrt{(r + r')^2 - |z|^2}}, \quad r, r' \in \mathbb{R}_+,\ z \in \mathbb{R}^2.$$

For every $r, r' \in \mathbb{R}_+$, $\chi(r, r', z)\mathrm{d}z$ is a probability measure on $\mathbb{R}^2$

$$\int_{\mathbb{R}^2} \chi(r, r', z)\,\mathrm{d}z = 1, \quad r, r' \in \mathbb{R}_+.$$

This measure characterizes the interaction between the Larmor circles of centers $\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}$, $\overline{x'} + \frac{{}^\perp\overline{v'}}{\omega_c}$ and radii $\frac{|\overline{v}|}{|\omega_c|}$, $\frac{|\overline{v'}|}{|\omega_c|}$, and charges only the circle pairs having non-empty intersection *i.e.*,

$$\frac{||\overline{v}| - |\overline{v'}||}{|\omega_c|} < \left|\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} - \left(\overline{x'} + \frac{{}^\perp\overline{v'}}{\omega_c}\right)\right| < \frac{|\overline{v}| + |\overline{v'}|}{|\omega_c|}.$$

More exactly, the measure χ appears when averaging integrals with respect to v (see Proposition 4.2 in [8] for details)

$$\left\langle \int_{\mathbb{R}^3} f(x, v')\,\mathrm{d}v' \right\rangle(x, v) = \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \chi(r, r', z) f(\overline{x'}, x_3, v')\,\mathrm{d}v'\mathrm{d}\overline{x'}$$

for any $f = f(x, v) \in \ker\mathcal{T}$.

Clearly, the kernel $\langle Q\rangle$ in (12) is an integro-differential operator in $(\overline{x}, v)$ (observe that there is no derivative with respect to x_3 since $\xi^i_{x_3} = 0$, $1 \le i \le 4$) and therefore will satisfy the mass, momentum and kinetic energy balances only globally in $(\overline{x}, v)$. Indeed, the averaged kernel writes as a divergence with respect to $(\overline{x}, v)$ and therefore there is no reason why its integral with respect to v vanishes. Only the integral with respect to $(\overline{x}, v)$ balances, assuming that the integrand has nice decay at infinity. Similarly, the averaged Fokker–Planck–Landau kernel will decrease the entropy $f\ln f$ globally in $(\overline{x}, v)$. Finally, combining (8), (11), (12) leads to the following model for the dominant density $f = \lim_{\varepsilon\searrow 0} f^\varepsilon$ in (4)

$$\partial_t f + \frac{\langle{}^\perp\overline{E}\rangle}{B}\cdot\nabla_{\overline{x}} f + v_3\partial_{x_3} f + \frac{q}{m}\langle E_3\rangle\partial_{v_3} f = \langle Q\rangle(f, f) \tag{13}$$

with

$$\langle Q\rangle(f, f) = \omega_c^2\,\mathrm{div}_{\omega_c x, v}\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\sum_{i=1}^4 f(\overline{x'}, x_3, v')\xi^i(\overline{x}, v, \overline{x'}, v') \otimes \xi^i(\overline{x}, v, \overline{x'}, v')\nabla_{\omega_c x, v} f(x, v)\,\mathrm{d}v'\mathrm{d}\overline{x'}\right\}$$
$$- \omega_c^2\,\mathrm{div}_{\omega_c x, v}\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\sum_{i=1}^4 f(x, v)\xi^i(\overline{x}, v, \overline{x'}, v') \otimes \varepsilon_i\xi^i(\overline{x'}, v', \overline{x}, v)\nabla_{\omega_c x', v'} f(\overline{x'}, x_3, v')\,\mathrm{d}v'\mathrm{d}\overline{x'}\right\}.$$

We concentrate on the equilibria of $\langle Q\rangle$, which are local in x_3, but global in $(\bar{x}, v)$. For doing that we establish an H-theorem. Thanks to the H theorem satisfied by $\langle Q\rangle$ (see Theorem 2.1 for precise statements and notations), the positive equilibria of $\langle Q\rangle$ are determined by the constraints

$$\xi^i \cdot \nabla \ln f - \varepsilon_i (\xi^i)' \cdot \nabla' \ln f' = 0, \quad 1 \le i \le 4.$$

It happens that the densities above are parametrized by six quantities $\rho > 0$, $u = (u_1, u_2, u_3) \in \mathbb{R}^3$, $K > 0$, $K + G > 0$

$$\rho = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} f(\bar{x}, v)\,\mathrm{d}v\mathrm{d}\bar{x}, \qquad \rho\bar{u} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (\omega_c \bar{x} + {}^\perp\bar{v}) f(\bar{x}, v)\,\mathrm{d}v\mathrm{d}\bar{x}, \qquad \rho u_3 = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 f(\bar{x}, v)\,\mathrm{d}v\mathrm{d}\bar{x},$$

$$\rho K = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\bar{v}|^2 + (v_3 - u_3)^2}{2} f(\bar{x}, v)\,\mathrm{d}v\mathrm{d}\bar{x}, \qquad \rho G = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\omega_c \bar{x} + {}^\perp\bar{v} - \bar{u}|^2 - |\bar{v}|^2}{2} f(\bar{x}, v)\,\mathrm{d}v\mathrm{d}\bar{x}$$

which are linear combinations of the moments of f with respect to the average collision invariants (cf. Proposition 2.1)

$$1,\ \omega_c \bar{x} + {}^\perp\bar{v},\ v_3,\ \frac{|v|^2}{2},\ \frac{|\omega_c \bar{x} + {}^\perp\bar{v}|^2 - |\bar{v}|^2}{2}.$$

Clearly ρ represents the total number of particles in the phase space $(\bar{x}, v)$ and u_3 is the mean parallel velocity in $(\bar{x}, v)$. The mean perpendicular velocity do not enter the numbers parametrizing these equilibria. Indeed, any density f satisfying the constraint $\mathcal{T} f = 0$ has zero mean perpendicular velocity

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \bar{v} f(\bar{x}, v)\,\mathrm{d}v\mathrm{d}\bar{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \langle\bar{v}\rangle f(\bar{x}, v)\,\mathrm{d}v\mathrm{d}\bar{x} = (0, 0).$$

The role of the mean perpendicular velocity is played by the displacement of the mean Larmor center over one cyclotronic period

$$\bar{u} = \frac{2\pi}{T_c} \frac{\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (\bar{x} + \frac{{}^\perp\bar{v}}{\omega_c}) f(\bar{x}, v)\,\mathrm{d}v\mathrm{d}\bar{x}}{\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} f(\bar{x}, v)\,\mathrm{d}v\mathrm{d}\bar{x}}.$$

The moment in the definition of $\rho\bar{u}$ is associated to the Larmor center $\bar{x} + \frac{{}^\perp\bar{v}}{\omega_c}$ which is balanced by the kernel $\langle Q\rangle$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \left(\bar{x} + \frac{{}^\perp\bar{v}}{\omega_c}\right) \langle Q\rangle(f, f)\,\mathrm{d}v\mathrm{d}\bar{x} = 0.$$

The parameter K is related to the kinetic energy $|v|^2/2$ which remains balanced by $\langle Q\rangle$. The parameter G corresponds to a new collision invariant $(|\omega_c \bar{x} + {}^\perp\bar{v}|^2 - |\bar{v}|^2)/2$ *i.e.*,

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\omega_c \bar{x} + {}^\perp\bar{v}|^2 - |\bar{v}|^2}{2} \langle Q\rangle(f, f)\,\mathrm{d}v\mathrm{d}\bar{x} = 0$$

and characterizes the gyrokinetic framework. Indeed, in the absence of the magnetic field, that is if $\omega_c = 0$, then $\bar{u} = (0, 0)$ and G vanishes.

The equilibria appear as Maxwellians of the form

$$f = \frac{\rho\omega_c^2}{(2\pi)^{5/2} \frac{\mu^2\theta^{3/2}}{\mu - \theta}} \exp\left(-\frac{|\bar{v}|^2 + (v_3 - u_3)^2}{2\theta}\right) \exp\left(-\frac{|\omega_c \bar{x} + {}^\perp\bar{v} - \bar{u}|^2 - |\bar{v}|^2}{2\mu}\right) \tag{14}$$

where θ and μ are uniquely determined by imposing the moment equalities defining K and G

$$\frac{\mu\theta}{\mu - \theta} + \frac{\theta}{2} = K, \qquad \mu - \frac{\mu\theta}{\mu - \theta} = G, \quad \mu > \theta > 0.$$

At a first glance, these equilibria may appear very complicated. The point is that the average operator combines position and velocity in such a way that, at equilibrium, the particle density satisfies given profiles in velocity and perpendicular position.

Determining the equilibria of $\langle Q\rangle$ is a crucial issue for understanding the behavior of the tokamak plasmas, in the gyrokinetic approximation. The complete characterization of these equilibria is far to be obvious since they are no more local in space and depend on a larger set of parameters, including several new moments associated to new collision invariants. In particular we focus on the dissipation mechanisms, the main goal being the derivation of fluid models, much easier to understand and to simulate numerically. Once we have determined the equilibria of $\langle Q\rangle$, we can search for the dynamics in (13) near local (in (t, x_3)) equilibria. In other words we concentrate on strongly collisional regimes of (13) and we obtain an Euler type system of six equations and six unknowns in the parallel direction. Up to our knowledge, this result has not been reported yet and represents a first research work in this direction. This Euler system represents a new hyperbolic model, enjoying new features, coming from the averaging process with respect to the fast cyclotronic motion. Its study could be very important for a better comprehension of classical fluid mechanics, combined with fast rotations or, more generally, when fast oscillations play an important role. For simplicity we discard here all technical difficulties related to the smoothness of the solution of (13), the validity of the Hilbert expansion we are using, etc. We restrict ourselves to formal computations and write down the expected macroscopic limit model in the parallel direction.

Theorem 1.1. *Assume that the electric field is parallel and depends only on the time and the parallel space coordinate* $E = (0, 0, E_3(t, x_3))$ *and let* $f^{\text{in}} \in \ker\mathcal{T}$ *be a positive smooth density with rapid decay at infinity. For any* $\tau > 0$ *the density* f^τ *stands for the solution (assumed smooth and having nice decay at infinity) of the problem*

$$\partial_t f^\tau + v_3\partial_{x_3} f^\tau + \frac{q}{m}E_3(t, x_3)\partial_{v_3} f^\tau = \frac{1}{\tau}\langle Q\rangle\big(f^\tau, f^\tau\big), \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3,$$
$$f^\tau(t = 0, x, v) = f^{\text{in}}(x, v) \geq 0, \quad (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3. \tag{15}$$

Therefore the leading order term in the expansion $f^\tau = f + \tau f^1 + \cdots$ (i.e., $f = \lim_{\tau\searrow 0} f^\tau$) *is a local equilibrium (see (14)) parametrized by the functions* $\rho = \rho(t, x_3) > 0$, $u = u(t, x_3)$, $\theta = \theta(t, x_3) > 0$, $\mu = \mu(t, x_3) > \theta(t, x_3) > 0$, *which satisfy the system of conservation laws*

$$\partial_t\rho + \partial_{x_3}(\rho u_3) = 0, \qquad \partial_t(\rho u) + \partial_{x_3}\big(\rho\big(u_3 u + (0, 0, \theta)\big)\big) - \rho\frac{q}{m}(0, 0, E_3) = 0, \quad (t, x_3) \in \mathbb{R}_+ \times \mathbb{R},$$
$$\partial_t\left[\rho\left(\frac{\mu\theta}{\mu-\theta} + \frac{\theta}{2} + \frac{(u_3)^2}{2}\right)\right] + \partial_{x_3}\left[u_3\rho\left(\frac{\mu\theta}{\mu-\theta} + \frac{3\theta}{2} + \frac{(u_3)^2}{2}\right)\right] - \frac{q}{m}E_3\rho u_3$$
$$= \partial_t\left[\rho\left(\frac{\mu\theta}{\mu-\theta} + \frac{\theta}{2}\right)\right] + \partial_{x_3}\left[\rho u_3\left(\frac{\mu\theta}{\mu-\theta} + \frac{\theta}{2}\right)\right] + \rho\theta\partial_{x_3}u_3 = 0, \quad (t, x_3) \in \mathbb{R}_+ \times \mathbb{R},$$
$$\partial_t\left[\rho\left(\mu - \frac{\mu\theta}{\mu-\theta}\right)\right] + \partial_{x_3}\left[\rho u_3\left(\mu - \frac{\mu\theta}{\mu-\theta}\right)\right] = 0, \quad (t, x_3) \in \mathbb{R}_+ \times \mathbb{R}$$

and the initial conditions

$$\rho(0, x_3) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} f^{\text{in}}(x, v)\,\mathrm{d}v\mathrm{d}\bar{x}, \qquad \rho(0, x_3)u(0, x_3) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (\omega_c\bar{x} + {}^\perp\bar{v}, v_3)f^{\text{in}}(x, v)\,\mathrm{d}v\mathrm{d}\bar{x},$$
$$\rho(0, x_3)\left(\frac{\mu(0, x_3)\theta(0, x_3)}{\mu(0, x_3) - \theta(0, x_3)} + \frac{\theta(0, x_3)}{2}\right) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\bar{v}|^2 + (v_3 - u_3(0, x_3))^2}{2} f^{\text{in}}(x, v)\,\mathrm{d}v\mathrm{d}\bar{x},$$
$$\rho(0, x_3)\left(\mu(0, x_3) - \frac{\mu(0, x_3)\theta(0, x_3)}{\mu(0, x_3) - \theta(0, x_3)}\right) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\omega_c\bar{x} + {}^\perp\bar{v} - \bar{u}(0, x_3)|^2 - |\bar{v}|^2}{2} f^{\text{in}}\,\mathrm{d}v\mathrm{d}\bar{x}.$$

The solution (ρ, u, θ, μ) *also verifies*

$$\partial_t\left(\rho\ln\frac{\rho(\mu-\theta)}{\mu^2\theta^{3/2}}\right) + \partial_{x_3}\left(\rho u_3\ln\frac{\rho(\mu-\theta)}{\mu^2\theta^{3/2}}\right) = 0, \quad (t, x_3) \in \mathbb{R}_+ \times \mathbb{R}.$$

For numerical simulations it is useful to write simplified versions of the averaged Fokker–Planck–Landau kernel which preserve the equilibria and the relaxation property towards these equilibria. The key point is to consider first order approximation near the equilibria, by neglecting all second order fluctuation terms around these equilibria. The averaged collision kernel $\langle Q\rangle$ being quadratic, the computation of the first order approximation L follows in a natural way, leading to a complete explicit formula. In particular we check that L has exactly the same equilibria as $\langle Q\rangle$.

Theorem 1.2. *For any positive density $f = f(\bar{x}, v)$ we denote by $\mathcal{E}_f$ the equilibrium of $\langle Q\rangle$ having the same moments as f*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (f - \mathcal{E}_f)\varphi(\bar{x}, v)\,\mathrm{d}v\mathrm{d}\bar{x} = 0, \quad \varphi \in \left\{1, \omega_c\bar{x} + {}^\perp\bar{v}, v_3, |v|^2/2, \left(|\omega_c\bar{x} + {}^\perp\bar{v}|^2 - |\bar{v}|^2\right)/2\right\}.$$

The linearized of $\langle Q\rangle(f, f)$ around the equilibrium $\mathcal{E}_f$ writes

$$\omega_c^{-2}L(f) = \sum_{i=1}^{4} \mathrm{div}_{\omega_c x, v} \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \mathcal{E}_f\mathcal{E}'_f \left\{\xi^i \cdot \nabla\left(\frac{f}{\mathcal{E}_f}\right) - \varepsilon_i\left(\xi^i\right)' \cdot \nabla'\left(\frac{f'}{\mathcal{E}'_f}\right)\right\}\xi^i\,\mathrm{d}v'\mathrm{d}\overline{x'}.$$

Moreover, the following statements hold:

1. *For any two functions $f = f(\bar{x}, v)$, $\varphi = \varphi(\bar{x}, v)$ we have*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} L(f)\varphi\,\mathrm{d}v\mathrm{d}\bar{x} = -\frac{\omega_c^2}{2}\sum_{i=1}^{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \mathcal{E}_f\mathcal{E}'_f\left\{\xi^i \cdot \nabla\left(\frac{f}{\mathcal{E}_f}\right) - \varepsilon_i\left(\xi^i\right)' \cdot \nabla'\left(\frac{f'}{\mathcal{E}'_f}\right)\right\}$$
$$\times\left\{\xi^i \cdot \nabla\varphi - \varepsilon_i\left(\xi^i\right)' \cdot \nabla'\varphi'\right\}\mathrm{d}v'\mathrm{d}\overline{x'}\,\mathrm{d}v\mathrm{d}\bar{x}. \tag{16}$$

2. *For any positive density f we have the inequality*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{f}{\mathcal{E}_f}L(f)\,\mathrm{d}v\mathrm{d}\bar{x} \le 0 \tag{17}$$

with equality iff

$$\xi^i \cdot \nabla\left(\frac{f}{\mathcal{E}_f}\right) - \varepsilon_i\left(\xi^i\right)' \cdot \nabla'\left(\frac{f'}{\mathcal{E}'_f}\right) = 0, \quad 1 \le i \le 4.$$

3. *The positive equilibria of L are the positive equilibria of $\langle Q\rangle$*

$$f > 0, \qquad L(f) = 0 \quad \Leftrightarrow \quad f = \mathcal{E}_f.$$

As usual, it is possible to further simplify the average Fokker–Planck–Landau operator, using its BGK approximation $L_{\mathrm{BGK}} = -(f - \mathcal{E}_f)$, whose behavior regarding the equilibria is very similar to that of $\langle Q\rangle$ (see Theorem 5.1).

Our paper is organized as follows. In Section 2 we investigate the main properties of the average Fokker–Planck–Landau collision operator. In particular we characterize its equilibria, thanks to an H type theorem. These equilibria are computed in Section 3. They are special Maxwellians depending on six parameters, which correspond to six moments. Section 4 is devoted to the fluid model near gyrokinetic equilibria, when the collisions dominate the transport. Simplified versions of the averaged Fokker–Planck–Landau collision operator are studied in the last section (the linearized around equilibria and the BGK approximation). Some technical proofs and computations have been postponed to Appendix A.

2. The averaged Fokker–Planck–Landau collision operator

In this section we present the main properties of the operator $\langle Q\rangle(f, f) := \langle Q(f, f)\rangle$, whose expression (12) has been obtained in [9] for any density $f = f(x, v)$ satisfying the constraint $\mathcal{T}f = 0$. The main goal is how to determine the equilibria of $\langle Q\rangle$. These equilibria are local in x_3 (since $\langle Q\rangle$ is local in x_3) and we expect that they are special Maxwellians depending on the velocity v, but also on the perpendicular spatial coordinates x_1, x_2. We will see that the set of these equilibria is parametrized by six numbers

$$\rho(x_3) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} f(x,v)\,\mathrm{d}v\mathrm{d}\overline{x}, \tag{18}$$

$$\rho(x_3)\overline{u}(x_3) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (\omega_c\overline{x} + {}^\perp\overline{v}) f(x,v)\,\mathrm{d}v\mathrm{d}\overline{x}, \tag{19}$$

$$\rho(x_3)u_3(x_3) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 f(x,v)\,\mathrm{d}v\mathrm{d}\overline{x}, \tag{20}$$

$$\rho(x_3)K(x_3) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\overline{v}|^2 + (v_3-u_3)^2}{2} f(x,v)\,\mathrm{d}v\mathrm{d}\overline{x}, \tag{21}$$

$$\rho(x_3)G(x_3) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\omega_c\overline{x} + {}^\perp\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2} f(x,v)\,\mathrm{d}v\mathrm{d}\overline{x}. \tag{22}$$

Clearly u_3 represents the mean parallel velocity, $\overline{u}/\omega_c$ is the mean Larmor circle center and K represents the temperature. Notice that the mean perpendicular velocity vanishes for any density satisfying the constraint $\mathcal{T}f$ since

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \overline{v} f(x,v)\,\mathrm{d}v\mathrm{d}\overline{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \langle\overline{v}\rangle f(x,v)\,\mathrm{d}v\mathrm{d}\overline{x} = (0,0).$$

Therefore the mean perpendicular velocity will not enter the parameter family characterizing the equilibria. The interpretation of the quantity in (22) comes by observing that the Larmor circle power with respect to the mean Larmor center $\overline{u}/\omega_c$ is

$$\left|\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} - \frac{\overline{u}}{\omega_c}\right|^2 - \frac{|\overline{v}|^2}{|\omega_c|^2}$$

and thus $2G/\omega_c^2$ is the mean Larmor circle power with respect to the mean Larmor center. The quantities in (18), (19), (20), (21), (22) are the moments of f with respect to the functions in the set

$$\mathcal{C} = \left\{1, \omega_c\overline{x} + {}^\perp\overline{v}, v_3, \frac{|\overline{v}|^2 + (v_3-u_3)^2}{2}, \frac{|\omega_c\overline{x} + {}^\perp\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2}\right\}.$$

All the functions in $\mathcal{C}$ are balanced by $\langle Q\rangle$. This is a consequence of the balances satisfied by Q and the definition of $\langle Q\rangle$, as the average of Q.

Proposition 2.1. *For any function $f = f(x,v) \in \ker\mathcal{T}$ we have*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \langle Q\rangle(f,f)\,\mathrm{d}v\mathrm{d}\overline{x} = 0, \qquad \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (\omega_c\overline{x} + {}^\perp\overline{v}, v_3)\langle Q\rangle(f,f)\,\mathrm{d}v\mathrm{d}\overline{x} = (0,0,0),$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\overline{v}|^2 + (v_3-u_3)^2}{2}\langle Q\rangle(f,f)\,\mathrm{d}v\mathrm{d}\overline{x} = 0,$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\omega_c\overline{x} + {}^\perp\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2}\langle Q\rangle(f,f)\,\mathrm{d}v\mathrm{d}\overline{x} = 0.$$

Proof. Observe that any function $\varphi \in \mathcal{C}$ belongs to $\ker \mathcal{T}$, since it depends only on the invariants of $\mathcal{T}$, that is only on $\omega_c \overline{x} + {}^\perp\overline{v}, x_3, |\overline{v}|, v_3$. Therefore, for any such function we can write, thanks to Remark 1.1

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \varphi \langle Q\rangle(f,f)\,\mathrm{d}v\mathrm{d}\overline{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \varphi \big\langle Q(f,f)\big\rangle \mathrm{d}v\mathrm{d}\overline{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \varphi Q(f,f)\,\mathrm{d}v\mathrm{d}\overline{x}. \tag{23}$$

Notice also that any function $\varphi \in \mathcal{C}$ writes as a linear combination of $1, v, |v|^2/2$, with coefficients depending only on x. Therefore the mass, momentum and kinetic energy balances of the Fokker–Planck–Landau kernel guarantee that

$$\int_{\mathbb{R}^3} \varphi(x,v) Q(f,f)\,\mathrm{d}v = 0, \quad x \in \mathbb{R}^3. \tag{24}$$

Our conclusion follows from (23) and (24). □

We are looking now for the equilibria of $\langle Q\rangle$. The crucial point is to establish an H type theorem for the kernel $\langle Q\rangle$. Most of the results in the sequel are valid for all densities f, not necessarily in the kernel of $\mathcal{T}$, but with respect to some particular extension of $\langle Q\rangle$ to the space of all densities f. It happens that the good choice is to define $\langle Q\rangle(f,f)$ by the same formula as in (12). The particular structure of the fields $(\xi^i)_{1\le i\le 4}$ allows us to obtain the following characterization of the kernel $\langle Q\rangle$ in the distribution sense cf. Proposition 4.11 in [9].

Theorem 2.1. *Consider two functions $f = f(x,v) > 0$, $\varphi = \varphi(x,v)$ (not necessarily in the kernel of $\mathcal{T}$).*

1. *For any $x_3 \in \mathbb{R}$ we have*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \langle Q\rangle(f,f)\varphi\,\mathrm{d}v\mathrm{d}\overline{x}$$
$$= -\frac{\omega_c^2}{2}\sum_{i=1}^4 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} ff'\big(\xi^i\cdot\nabla\ln f - \varepsilon_i\big(\xi^i\big)'\nabla'\ln f'\big)\big(\xi^i\cdot\nabla\varphi - \varepsilon_i\big(\xi^i\big)'\nabla'\varphi'\big)\,\mathrm{d}v'\mathrm{d}\overline{x'}\,\mathrm{d}v\mathrm{d}\overline{x} \tag{25}$$

where

$$f = f(x,v), \qquad f' = f'\big(x_1', x_2', x_3, v'\big),$$
$$\nabla\varphi = \nabla_{\omega_c x, v}\varphi(x,v), \qquad \nabla'\varphi' = \nabla_{\omega_c x', v'}\varphi\big(x_1', x_2', x_3, v'\big),$$
$$\xi^i = \xi^i\big(x_1, x_2, v, x_1', x_2', v'\big), \qquad \big(\xi^i\big)' = \xi^i\big(x_1', x_2', v', x_1, x_2, v\big).$$

2. *For any positive density f we have the inequality*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \ln f\langle Q\rangle(f,f)\,\mathrm{d}v\mathrm{d}\overline{x} \le 0$$

with equality iff

$$\xi^i\cdot\nabla\ln f - \varepsilon_i\big(\xi^i\big)'\cdot\nabla'\ln f' = 0, \quad 1\le i\le 4. \tag{26}$$

3. *The positive equilibria of the averaged Fokker–Planck–Landau kernel* i.e., $f > 0$, $\langle Q\rangle(f,f) = 0$ *are the positive functions verifying* (26).

Proof. 1. Notice that for any $1 \le i \le 4$ we have $\xi^i\cdot(e_3, 0) = 0$ and therefore the operator $\mathrm{div}_{\omega_c x, v}$ acts only in (x_1, x_2, v). Thus, for any fixed $x_3 \in \mathbb{R}$ we can perform integration by parts with respect to (x_1, x_2, v).

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \langle Q\rangle(f,f)\varphi\,\mathrm{d}v\mathrm{d}\overline{x} = -\sum_{i=1}^4 \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} ff'$$
$$\times\big\{\big(\xi^i\cdot\nabla\varphi\big)\big(\xi^i\cdot\nabla\ln f\big) - \varepsilon_i\big(\xi^i\cdot\nabla\varphi\big)\big(\big(\xi^i\big)'\cdot\nabla'\ln f'\big)\big\}\,\mathrm{d}v'\mathrm{d}\overline{x'}\,\mathrm{d}v\mathrm{d}\overline{x}. \tag{27}$$

Performing the change of variables $(x_1', x_2', v') \leftrightarrow (x_1, x_2, v)$ yields

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \langle Q\rangle(f,f)\varphi \,\mathrm{d}v\mathrm{d}\overline{x} = -\sum_{i=1}^{4}\omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} ff' \times \left\{\left(\left(\xi^i\right)'\cdot\nabla'\varphi'\right)\left(\left(\xi^i\right)'\cdot\nabla'\ln f'\right) - \varepsilon_i\left(\left(\xi^i\right)'\cdot\nabla'\varphi'\right)\left(\xi^i\cdot\nabla\ln f\right)\right\}\mathrm{d}v\mathrm{d}\overline{x}\,\mathrm{d}v'\mathrm{d}\overline{x'}. \tag{28}$$

Combining (27), (28) one gets by Fubini's theorem

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \langle Q\rangle(f,f)\varphi \,\mathrm{d}v\mathrm{d}\overline{x} = -\frac{\omega_c^2}{2}\sum_{i=1}^{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} ff'T^i\,\mathrm{d}v'\mathrm{d}\overline{x'}\,\mathrm{d}v\mathrm{d}\overline{x}$$

where

$$T^i = \left(\xi^i\cdot\nabla\varphi - \varepsilon_i\left(\xi^i\right)'\cdot\nabla'\varphi'\right)\left(\xi^i\cdot\nabla\ln f - \varepsilon_i\left(\xi^i\right)'\cdot\nabla'\ln f'\right),\quad 1\le i\le 4.$$

2. Applying (25) with $\varphi = \ln f$ yields

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \ln f\,\langle Q\rangle(f,f)\,\mathrm{d}v\mathrm{d}\overline{x} = -\frac{\omega_c^2}{2}\sum_{i=1}^{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} ff' \times\left(\xi^i\cdot\nabla\ln f - \varepsilon_i\left(\xi^i\right)'\cdot\nabla'\ln f'\right)^2\mathrm{d}v'\mathrm{d}\overline{x'}\,\mathrm{d}v\mathrm{d}\overline{x}\le 0,\quad x_3\in\mathbb{R}$$

with equality iff $\xi^i\cdot\nabla\ln f - \varepsilon_i(\xi^i)'\cdot\nabla'\ln f' = 0$, $1\le i\le 4$.

3. Consider f a positive equilibrium of $\langle Q\rangle$. Therefore we have the equality

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \ln f\,\langle Q\rangle(f,f)\,\mathrm{d}v\mathrm{d}\overline{x} = 0$$

and by the previous assertion we deduce (26). Conversely, let f be a positive density satisfying (26). Then, for any function φ we have, thanks to (25)

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \varphi\langle Q\rangle(f,f)\,\mathrm{d}v\mathrm{d}\overline{x} = 0$$

implying that $\langle Q\rangle(f,f) = 0$. □

Remark 2.1. It is remarkable that the extension we have considered for $\langle Q\rangle$ (to the space of all positive densities) still satisfies the balances stated in Proposition 2.1. This can be checked directly, thanks to (25), verifying that for any $\varphi\in\mathcal{C}$

$$\xi^i\cdot\nabla\varphi - \varepsilon_i\left(\xi^i\right)'\cdot\nabla'\varphi' = 0,\quad 1\le i\le 4.$$

Actually, as $\xi^i\cdot\nabla x_3 = \xi^i\cdot(e_3, 0)/\omega_c = 0$, $1\le i\le 4$, it is enough to do it for the functions

$$1,\ \omega_c\overline{x} + {}^\perp\overline{v},\ v_3,\ \frac{|v|^2}{2},\ \frac{|\omega_c\overline{x} + {}^\perp\overline{v}|^2 - |\overline{v}|^2}{2}.$$

For example, let us verify that

$$\xi^i\cdot\nabla\frac{|v|^2}{2} - \varepsilon_i\left(\xi^i\right)'\cdot\nabla'\frac{|v'|^2}{2} = 0,\quad 1\le i\le 4.$$

The above condition is trivially satisfied for $i\in\{1,2\}$. For $i=3$ we have

$$\xi^3\cdot\nabla\frac{|v|^2}{2} - \varepsilon_3\left(\xi^3\right)'\cdot\nabla'\frac{|v'|^2}{2} = -\{\sigma\chi\}^{1/2}\frac{r'\sin\varphi}{|z|}r + \{\sigma\chi\}^{1/2}\frac{r\sin\varphi}{|z|}r' = 0.$$

Finally, when $i = 4$ we obtain

$$\begin{aligned}
&\xi^4 \cdot \nabla \frac{|v|^2}{2} - \varepsilon_4 \big(\xi^4\big)' \cdot \nabla' \frac{|v'|^2}{2} \\
&= \{\sigma\chi\}^{1/2} \left\{ - \frac{(r'\cos\varphi - r)(v_3 - v_3')r + |z|^2 v_3}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} + \frac{(r\cos\varphi - r')(v_3' - v_3)r' + |z|^2 v_3'}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} \right\} \\
&= \{\sigma\chi\}^{1/2} \frac{v_3 - v_3'}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} \big[r^2 + \big(r'\big)^2 - 2rr'\cos\varphi - |z|^2\big] = 0.
\end{aligned}$$

Remark 2.2. The previous balances follow also by the argument below. Any local (in x) Maxwellian $f(x, v) = \exp(\alpha(x)|v|^2 + \beta(x) \cdot v + \gamma(x))$ which belongs to $\ker \mathcal{T}$ is an equilibrium for $\langle Q \rangle$, since

$$\langle Q \rangle(f, f) = \big\langle Q(f, f)\big\rangle = \langle 0 \rangle = 0.$$

We deduce by the third statement of Theorem 2.1 that

$$\xi^i \cdot \nabla\varphi - \varepsilon_i \big(\xi^i\big)' \cdot \nabla'\varphi' = 0, \quad 1 \le i \le 4$$

for any function $\varphi(x, v) = \alpha(x)|v|^2 + \beta(x) \cdot v + \gamma(x)$ in the kernel of $\mathcal{T}$, and in particular for the functions

$$1,\ \omega_c \bar{x} + {}^\perp\bar{v},\ v_3,\ \frac{|v|^2}{2},\ \frac{|\omega_c \bar{x} + {}^\perp\bar{v}|^2 - |\bar{v}|^2}{2} = \frac{\omega_c^2\,|\bar{x}|^2 - 2\omega_c({}^\perp\bar{x} \cdot \bar{v})}{2}.$$

We conclude by the first statement in Theorem 2.1.

3. The equilibria of the averaged Fokker–Planck–Landau collision operator

We determine now the positive equilibria of $\langle Q \rangle$ by solving (26) for any $1 \le i \le 4$. We recall that

$$\psi_1 = x_1 + \frac{v_2}{\omega_c}, \qquad \psi_2 = x_2 - \frac{v_1}{\omega_c}, \qquad \psi_3 = x_3, \qquad \psi_4 = |\bar{v}|, \qquad \psi_5 = v_3$$

is a family of independent invariants for $\mathcal{T} = \bar{v} \cdot \nabla_{\bar{x}} + \omega_c {}^\perp\bar{v} \cdot \nabla_{\bar{v}}$. We start solving Eq. (26) which corresponds to $i = 1$. Then we restrict this set of solutions by imposing successively Eq. (26) with $i = 2$, $i = 3$ and $i = 4$. It is the only place where we use the explicit form of the vector fields $(\xi^i)_{1 \le i \le 4}$, entering the expression of $\langle Q \rangle$. These computations are a little bit tedious, but finally they will provide the product of Maxwellians realizing the equilibria of $\langle Q \rangle$, parametrized by the moments ρ, u, K, G. Moreover, we should pay attention to the fact that the probability measure χ enters as a factor any vector field $(\xi^i)_{1 \le i \le 4}$ and therefore each equality in (26) is non-trivial only on the support of χ, that is, only for pairs of Larmor circles having non-empty intersection. All these proofs are postponed to Appendix A. For another proof, which avoids the explicit computation of the vector fields $(\xi^i)_{1 \le i \le 4}$, we refer to Proposition 3.5. For simplicity we do not care about the regularity of the solutions. All the derivatives are understood in the classical sense and we are looking for smooth solutions.

Proposition 3.1. *The positive densities satisfying*

$$\xi^1 \cdot \nabla \ln f + \big(\xi^1\big)' \cdot \nabla' \ln f' = 0 \tag{29}$$

are those in the kernel of $\mathcal{T}$.

Proposition 3.2. *The positive densities satisfying (29) and*

$$\xi^2 \cdot \nabla \ln f + \big(\xi^2\big)' \cdot \nabla' \ln f' = 0 \tag{30}$$

are those of the form

$$f(x, v) = \exp\left(\frac{\alpha(x_3)}{2} \left| \bar{x} + \frac{{}^\perp\bar{v}}{\omega_c} \right|^2 + \beta(x_3) \cdot \left(\bar{x} + \frac{{}^\perp\bar{v}}{\omega_c} \right) + \lambda(x_3, |\bar{v}|, v_3) \right)$$

for some functions $\alpha : \mathbb{R} \to \mathbb{R}$, $\beta = (\beta_1, \beta_2) : \mathbb{R} \to \mathbb{R}^2$, $\lambda : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$.

Solving for $i = 3$ in (26), we will determine the particular form of the function $\lambda(x_3, |\overline{v}|, v_3)$.

Proposition 3.3. *The positive densities satisfying (29), (30) and*

$$\xi^3 \cdot \nabla \ln f - (\xi^3)' \cdot \nabla' \ln f' = 0 \tag{31}$$

are of the form

$$f(x,v) = \exp\left(\frac{\alpha(x_3)}{2}\left|\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}\right|^2 + \beta(x_3)\cdot\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}\right) + \gamma(x_3)\frac{|\overline{v}|^2}{2} + \mu(x_3, v_3)\right)$$

for some functions $\alpha, \gamma : \mathbb{R} \to \mathbb{R}$, $\beta : \mathbb{R} \to \mathbb{R}^2$, $\mu : \mathbb{R}^2 \to \mathbb{R}$.

It remains to determine the function $\mu(x_3, v_3)$. This will be done by solving (26) with $i = 4$, and we deduce that μ is a quadratic function of v_3, with coefficients depending on x_3.

Proposition 3.4. *The positive densities satisfying (29), (30), (31) and*

$$\xi^4 \cdot \nabla \ln f - (\xi^4)' \cdot \nabla' \ln f' = 0 \tag{32}$$

are of the form

$$\begin{aligned} f(x,v) = \exp\Bigg\{&\frac{\alpha(x_3)}{2}\left|\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}\right|^2 + \beta(x_3)\cdot\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}\right) + \gamma(x_3)\frac{|\overline{v}|^2}{2} \\ &+ \left(\gamma(x_3) + \frac{\alpha(x_3)}{\omega_c^2}\right)\frac{(v_3)^2}{2} + \delta(x_3)v_3 + \eta(x_3)\Bigg\}\end{aligned}$$

for some functions $\alpha, \gamma, \delta, \eta : \mathbb{R} \to \mathbb{R}$, $\beta : \mathbb{R} \to \mathbb{R}^2$.

We present now an alternative proof of the results stated in Propositions 3.1, 3.2, 3.3, 3.4. This approach does not require neither the exact computation of the averaged Fokker–Planck–Landau collision kernel, nor the resolution of (26).

Proposition 3.5. *The positive densities f in the kernel of $\mathcal{T}$ satisfying $\langle Q\rangle(f, f) = 0$ are of the form*

$$\begin{aligned} \ln f(x,v) = &\frac{\alpha(x_3)}{2}\left|\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}\right|^2 + \overline{\beta}(x_3)\cdot\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}\right) + \gamma(x_3)\frac{|\overline{v}|^2}{2} \\ &+ \left(\gamma(x_3) + \frac{\alpha(x_3)}{\omega_c^2}\right)\frac{(v_3)^2}{2} + \delta(x_3)v_3 + \eta(x_3)\end{aligned} \tag{33}$$

for some functions $\alpha, \gamma, \delta, \eta : \mathbb{R} \to \mathbb{R}$, $\overline{\beta} : \mathbb{R} \to \mathbb{R}^2$.

Proof. Clearly any positive density f in (33) is a Maxwellian satisfying the constraint $\mathcal{T} f = 0$ and

$$\langle Q\rangle(f, f) = \big\langle Q(f, f)\big\rangle = \langle 0\rangle = 0.$$

Conversely, let us consider a positive density f satisfying $\mathcal{T} f = 0$, $\langle Q\rangle(f, f) = 0$ and observe that for any $x_3 \in \mathbb{R}$ we can write

$$\begin{aligned} 0 &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \ln f \langle Q\rangle(f,f)\,\mathrm{d}v\mathrm{d}\overline{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \ln f \big\langle Q(f,f)\big\rangle \mathrm{d}v\mathrm{d}\overline{x} \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \ln f(x,v) Q\big(f(x,\cdot), f(x,\cdot)\big)(v)\,\mathrm{d}v\mathrm{d}\overline{x} \le 0\end{aligned}$$

since for any $x = (\overline{x}, x_3)$ we have the inequality

$$\int_{\mathbb{R}^3} \ln f(x,v) Q\big(f(x,\cdot), f(x,\cdot)\big)(v)\,\mathrm{d}v \le 0. \tag{34}$$

We deduce that for any $x = (\overline{x}, x_3)$ we have equality in (34), which implies that $f(x,\cdot)$ is a local Maxwellian *i.e.*,

$$\ln f(x,v) = \frac{A(x)}{\omega_c^2}\frac{|v|^2}{2} + \overline{B}(x)\cdot\frac{{}^\perp\overline{v}}{\omega_c} + \delta(x)v_3 + C(x)$$

for some functions $A, B_1, B_2, \delta, C : \mathbb{R}^3 \to \mathbb{R}$. We have to determine the structure of the previous functions, such that the constraint $\mathcal{T} f = 0$ holds true. Observe that

$$0 = \mathcal{T}\ln f = \frac{\overline{v}\cdot\nabla_{\overline{x}}A}{\omega_c^2}\frac{|v|^2}{2} - \frac{\partial_{\overline{x}}{}^\perp\overline{B} : \overline{v}\otimes\overline{v}}{\omega_c} - \overline{B}\cdot\overline{v} + \overline{v}\cdot\nabla_{\overline{x}}\delta v_3 + \overline{v}\cdot\nabla_{\overline{x}}C.$$

Clearly, the third (higher) order term in velocity vanishes, saying that $\nabla_{\overline{x}}A = 0$, or equivalently $A = A(x_3)$ and

$$-\frac{\partial_{\overline{x}}{}^\perp\overline{B} : \overline{v}\otimes\overline{v}}{\omega_c} - \overline{B}\cdot\overline{v} + \overline{v}\cdot\nabla_{\overline{x}}\delta v_3 + \overline{v}\cdot\nabla_{\overline{x}}C = 0.$$

Similarly $\delta = \delta(x_3)$ and the second order term in $\overline{v}$ vanishes

$$\partial_{\overline{x}}{}^\perp\overline{B} : \overline{v}\otimes\overline{v} = 0$$

implying that $\partial_{\overline{x}}{}^\perp\overline{B}$ is antisymmetric

$$\partial_{x_1}B_2 = \partial_{x_2}B_1 = 0, \qquad \partial_{x_1}B_1 = \partial_{x_2}B_2, \qquad \nabla_{\overline{x}}C = \overline{B}.$$

We obtain immediately that there is a function $\alpha = \alpha(x_3)$ such that

$$\partial_{x_1}B_1(x_1,x_3) = \alpha(x_3) = \partial_{x_2}B_2(x_2,x_3)$$

and thus $\overline{B} = \overline{\beta}(x_3) + \alpha(x_3)\overline{x}$ for some functions $\overline{\beta} = (\beta_1(x_3), \beta_2(x_3))$. The function C writes

$$C(x) = \overline{\beta}(x_3)\cdot\overline{x} + \alpha(x_3)\frac{|\overline{x}|^2}{2} + \eta(x_3)$$

and finally

$$\begin{aligned}\ln f(x,v) &= \frac{A(x_3)}{\omega_c^2}\frac{|v|^2}{2} + \overline{\beta}(x_3)\cdot\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}\right) + \alpha(x_3)\overline{x}\cdot\frac{{}^\perp\overline{v}}{\omega_c} + \delta(x_3)v_3 + \alpha(x_3)\frac{|\overline{x}|^2}{2} + \eta(x_3)\\ &= \frac{\alpha(x_3)}{2}\left|\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}\right|^2 + \overline{\beta}(x_3)\cdot\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}\right) + \frac{A(x_3) - \alpha(x_3)}{\omega_c^2}\frac{|\overline{v}|^2}{2} + \frac{A(x_3)}{\omega_c^2}\frac{(v_3)^2}{2} + \delta(x_3)v_3 + \eta(x_3).\end{aligned}$$

We have obtained for $\ln f$ the form in (33), taking $\gamma(x_3) = (A(x_3) - \alpha(x_3))/\omega_c^2$. □

It is easily seen that any equilibrium of the averaged Fokker–Planck–Landau kernel can be written as

$$\ln f(x,v) = \frac{\alpha(x_3)}{\omega_c^2}\frac{|\omega_c\overline{x} + {}^\perp\overline{v}|^2 - |\overline{v}|^2}{2} + \frac{\beta(x_3)}{\omega_c}\cdot(\omega_c\overline{x} + {}^\perp\overline{v}) + \left(\gamma(x_3) + \frac{\alpha(x_3)}{\omega_c^2}\right)\frac{|v|^2}{2} + \delta(x_3)v_3 + \eta(x_3)$$

and appears as a linear combination (with coefficients depending on x_3) of functions which are balanced by $\langle Q\rangle$, globally in $(\overline{x}, v)$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\langle Q\rangle\,\mathrm{d}v\mathrm{d}\overline{x} = 0, \qquad \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}(\omega_c\overline{x} + {}^\perp\overline{v}, v_3)\langle Q\rangle\,\mathrm{d}v\mathrm{d}\overline{x} = (0,0,0),$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\frac{|v|^2}{2}\langle Q\rangle\,\mathrm{d}v\mathrm{d}\overline{x} = 0, \qquad \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\frac{|\omega_c\overline{x} + {}^\perp\overline{v}|^2 - |\overline{v}|^2}{2}\langle Q\rangle\,\mathrm{d}v\mathrm{d}\overline{x} = 0.$$

Clearly, up to a factor depending on x_3, the equilibrium f writes

$$f \sim \exp\Big(-\frac{|\overline{v}|^2 + (v_3 - u_3(x_3))^2}{2\theta(x_3)}\Big)\exp\Big(-\frac{|\omega_c\overline{x} + {}^\perp\overline{v} - \overline{u}(x_3)|^2 - |\overline{v}|^2}{2\mu(x_3)}\Big)$$

for some functions $u(x_3) = (u_1, u_2, u_3)(x_3)$, $\theta(x_3)$, $\mu(x_3)$, or equivalently as a product of three Maxwellians

$$f \sim \frac{1}{2\pi\frac{\mu\theta}{\mu-\theta}}\exp\Big(-\frac{|\overline{v}|^2}{2\frac{\mu\theta}{\mu-\theta}}\Big)\frac{1}{(2\pi\theta)^{1/2}}\exp\Big(-\frac{(v_3-u_3)^2}{2\theta}\Big)\frac{1}{2\pi\mu}\exp\Big(-\frac{|\omega_c\overline{x} + {}^\perp\overline{v} - \overline{u}|^2}{2\mu}\Big).$$

Motivated by the above considerations, we parametrize the equilibria of $\langle Q\rangle$ by six functions $\rho, u = (u_1, u_2, u_3), \theta, \mu$, as announced by (14). It will be very useful, for the moment computations, to introduce the following representation for such equilibria. These decomposition will be the starting point for many development involving the moments, the entropy, ...

$$\begin{aligned} f(x, v) &= \frac{\rho(x_3)\omega_c^2}{(2\pi)^{5/2}\frac{\mu^2\theta^{3/2}}{\mu-\theta}}\exp\Big(-\frac{|\overline{v}|^2 + (v_3 - u_3(x_3))^2}{2\theta(x_3)}\Big)\exp\Big(-\frac{|\omega_c\overline{x} + {}^\perp\overline{v} - \overline{u}(x_3)|^2 - |\overline{v}|^2}{2\mu(x_3)}\Big) \\ &= \frac{\rho(x_3)}{2\pi\frac{\mu\theta}{\mu-\theta}}\exp\Big(-\frac{|\overline{v}|^2}{2\frac{\mu\theta}{\mu-\theta}}\Big)\frac{1}{(2\pi\theta)^{1/2}}\exp\Big(-\frac{(v_3-u_3(x_3))^2}{2\theta}\Big)\frac{\omega_c^2}{2\pi\mu}\exp\Big(-\frac{|\omega_c\overline{x} + {}^\perp\overline{v} - \overline{u}(x_3)|^2}{2\mu}\Big). \end{aligned} \tag{35}$$

For integrability reasons we assume that $\mu > \theta > 0$. The functions ρ, u, θ, μ are uniquely determined by the moments of f with respect to

$$1,\ \omega_c\overline{x} + {}^\perp\overline{v},\ v_3,\ \frac{|v|^2}{2},\ \frac{|\omega_c\overline{x} + {}^\perp\overline{v}|^2 - |\overline{v}|^2}{2}.$$

Proposition 3.6. *For any $(\rho, u_1, u_2, u_3, K, G) \in \mathbb{R}^6$, $\rho > 0$, $K > 0$, $K + G > 0$ there is a unique local (in x_3) equilibrium $f = f(\overline{x}, v)$ for $\langle Q\rangle$ satisfying*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} f \,\mathrm{d}v\mathrm{d}\overline{x} = \rho, \qquad \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (\omega_c\overline{x} + {}^\perp\overline{v}, v_3) f \,\mathrm{d}v\mathrm{d}\overline{x} = \rho u,$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|v|^2}{2} f \,\mathrm{d}v\mathrm{d}\overline{x} = \rho\frac{(u_3)^2}{2} + \rho K, \qquad \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\omega_c\overline{x} + {}^\perp\overline{v}|^2 - |\overline{v}|^2}{2} f \,\mathrm{d}v\mathrm{d}\overline{x} = \rho\frac{|\overline{u}|^2}{2} + \rho G.$$

Proof. We are searching for a positive local equilibrium $f = f(\overline{x}, v)$ parametrized by $\tilde{\rho}, \tilde{u}, \theta, \mu$. For any dimension d and real number $T > 0$, the notation $\mathcal{M}_T^d(w)$ stands for the Maxwellian of temperature T in $\mathbb{R}^d$

$$\mathcal{M}_T^d(w) = \frac{1}{(2\pi T)^{d/2}}\exp\Big(-\frac{|w|^2}{2T}\Big), \quad w \in \mathbb{R}^d.$$

For simplicity we drop the index d, but the reader should keep in mind that the Maxwellian dimension is that of the variable taken as argument. The equilibrium f writes, cf. (35)

$$f(\overline{x}, v) = \tilde{\rho}\mathcal{M}_{\frac{\mu\theta}{\mu-\theta}}(\overline{v})\mathcal{M}_\theta(v_3 - \tilde{u}_3)\omega_c^2\mathcal{M}_\mu(\omega_c\overline{x} + {}^\perp\overline{v} - \overline{\tilde{u}}).$$

Clearly, integrating first with respect to $\overline{x}$ for any fixed v and performing the change of variable $\omega_c^2\mathrm{d}\overline{x} = \mathrm{d}(\omega_c\overline{x} + {}^\perp\overline{v} - \overline{\tilde{u}})$ yield

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} f(\overline{x}, v)\,\mathrm{d}v\mathrm{d}\overline{x} = \tilde{\rho}$$

and thus $\tilde{\rho} = \rho$. Similarly

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (\omega_c \overline{x} + {}^\perp\overline{v}) f \,\mathrm{d}v\mathrm{d}\overline{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (\omega_c \overline{x} + {}^\perp\overline{v} - \overline{\tilde{u}} + \overline{\tilde{u}}) f \,\mathrm{d}v\mathrm{d}\overline{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \overline{\tilde{u}} f \,\mathrm{d}v\mathrm{d}\overline{x} = \tilde{\rho}\overline{\tilde{u}},$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 f \,\mathrm{d}v\mathrm{d}\overline{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (v_3 - \tilde{u}_3 + \tilde{u}_3) f \,\mathrm{d}v\mathrm{d}\overline{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \tilde{u}_3 f \,\mathrm{d}v\mathrm{d}\overline{x} = \tilde{\rho}\tilde{u}_3.$$

Therefore $\tilde{u} = u$ and the parameters $\frac{(\tilde{u}_1, \tilde{u}_2)}{\omega_c}, \tilde{u}_3$ appear as the mean Larmor center and the mean parallel velocity of the local equilibrium $f(\overline{x}, v)$. It remains to determine θ and μ. On the one hand notice that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\overline{v}|^2 + (v_3 - u_3)^2}{2} f(\overline{x}, v) \,\mathrm{d}v\mathrm{d}\overline{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|v|^2}{2} f \,\mathrm{d}v\mathrm{d}\overline{x} - \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{(u_3)^2}{2} f \,\mathrm{d}v\mathrm{d}\overline{x} = \rho K$$

and

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\omega_c \overline{x} + {}^\perp\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2} f(\overline{x}, v) \,\mathrm{d}v\mathrm{d}\overline{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\omega_c \overline{x} + {}^\perp\overline{v}|^2 - |\overline{v}|^2}{2} f \,\mathrm{d}v\mathrm{d}\overline{x} - \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\overline{u}|^2}{2} f \,\mathrm{d}v\mathrm{d}\overline{x} = \rho G.$$

On the other hand, using several times the formula

$$\int_{\mathbb{R}^d} |w|^2 \mathcal{M}_T(w) \,\mathrm{d}w = T \int_{\mathbb{R}^d} |w|^2 \mathcal{M}_1(w) \,\mathrm{d}w = -T \int_{\mathbb{R}^d} w \cdot \nabla_w \mathcal{M}_1(w) \,\mathrm{d}w = Td \tag{36}$$

yields

$$\begin{aligned}
&\frac{1}{\rho}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\overline{v}|^2 + (v_3 - u_3)^2}{2} f \,\mathrm{d}v\mathrm{d}\overline{x} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\overline{v}|^2}{2} \mathcal{M}_{\frac{\mu\theta}{\mu-\theta}}(\overline{v}) \mathcal{M}_\theta(v_3 - u_3) \omega_c^2 \mathcal{M}_\mu(\omega_c \overline{x} + {}^\perp\overline{v} - \overline{u}) \,\mathrm{d}v\mathrm{d}\overline{x} \\
&\quad + \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \mathcal{M}_{\frac{\mu\theta}{\mu-\theta}}(\overline{v}) \frac{(v_3 - u_3)^2}{2} \mathcal{M}_\theta(v_3 - u_3) \omega_c^2 \mathcal{M}_\mu(\omega_c \overline{x} + {}^\perp\overline{v} - \overline{u}) \,\mathrm{d}v\mathrm{d}\overline{x} \\
&= \frac{\mu\theta}{\mu - \theta} + \frac{\theta}{2}
\end{aligned} \tag{37}$$

and

$$\begin{aligned}
&\frac{1}{\rho}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\omega_c \overline{x} + {}^\perp\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2} f \,\mathrm{d}v\mathrm{d}\overline{x} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \mathcal{M}_{\frac{\mu\theta}{\mu-\theta}}(\overline{v}) \mathcal{M}_\theta(v_3 - u_3) \frac{|\omega_c \overline{x} + {}^\perp\overline{v} - \overline{u}|^2}{2} \omega_c^2 \mathcal{M}_\mu(\omega_c \overline{x} + {}^\perp\overline{v} - \overline{u}) \,\mathrm{d}v\mathrm{d}\overline{x} \\
&\quad - \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\overline{v}|^2}{2} \mathcal{M}_{\frac{\mu\theta}{\mu-\theta}}(\overline{v}) \mathcal{M}_\theta(v_3 - u_3) \omega_c^2 \mathcal{M}_\mu(\omega_c \overline{x} + {}^\perp\overline{v} - \overline{u}) \,\mathrm{d}v\mathrm{d}\overline{x} \\
&= \mu - \frac{\mu\theta}{\mu - \theta}.
\end{aligned} \tag{38}$$

We are done if we prove that there is a unique solution θ, μ satisfying $\mu > \theta > 0$, for the system

$$\frac{\mu\theta}{\mu - \theta} + \frac{\theta}{2} = K, \qquad \mu - \frac{\mu\theta}{\mu - \theta} = G.$$

We solve with respect to $\nu := \frac{\mu}{\theta} > 1$ which can be expressed in terms of $S := \frac{G}{K}$. Indeed, ν satisfies

$$2\nu\frac{\nu-2}{3\nu-1} = \frac{\mu - \frac{\mu\theta}{\mu-\theta}}{\frac{\mu\theta}{\mu-\theta} + \frac{\theta}{2}} = \frac{G}{K} = S > -1$$

or equivalently

$$2(\nu-1)^2 - 3S(\nu-1) - 2(S+1) = 0.$$

The above equation of the unknown $(\nu - 1)$ has one positive and one negative roots, since their product is $-(S+1) = -\frac{G+K}{K} < 0$. Then the ratio $\nu = \frac{\mu}{\theta} > 1$ is given by

$$\nu = \frac{4 + 3S + \sqrt{9S^2 + 16(S+1)}}{4}.$$

Combining with the equation $\frac{\theta}{2} + \mu = K + G$ we obtain

$$\theta = \frac{K+G}{1/2+\nu} > 0, \qquad \mu = \nu\theta = \nu\frac{K+G}{1/2+\nu} > \theta. \qquad \square$$

Remark 3.1. Any positive density $f(\overline{x}, v)$ satisfies

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\overline{v}|^2 + (v_3 - u_3)^2}{2} f\,\mathrm{d}v\mathrm{d}\overline{x} + \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\omega_c\overline{x} + {}^\perp\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2} f\,\mathrm{d}v\mathrm{d}\overline{x}$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\omega_c\overline{x} + {}^\perp\overline{v} - \overline{u}|^2 + (v_3 - u_3)^2}{2} f\,\mathrm{d}v\mathrm{d}\overline{x} > 0$$

which justifies the hypothesis $K + G > 0$.

4. The fluid model near gyrokinetic equilibria

In this section we investigate the fluid approximation of the model (13) when the collision mechanism dominates the transport. Clearly we are interested on regimes close to gyrokinetic equilibria. For simplicity we neglect the perpendicular electric field and we assume that the parallel electric field depends only on (t, x_3) and thus $\langle E_3\rangle = E_3$. Eq. (13) becomes

$$\partial_t f^\tau + v_3\partial_{x_3} f^\tau + \frac{q}{m}E_3(t, x_3)\partial_{v_3} f^\tau = \frac{1}{\tau}\langle Q\rangle\big(f^\tau, f^\tau\big), \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \tag{39}$$

and we intend to analyze the asymptotic behavior for small τ. Formally we have

$$f^\tau = f + \tau f^1 + \tau^2 f^2 + \cdots. \tag{40}$$

Following the standard arguments which allow us to derive the Euler equations starting from the kinetic description when the collisions dominate the transport [1,2,22,23], we determine the leading order term in the expansion (40) by the conditions

$$\langle Q\rangle(f, f) = 0, \qquad \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \left\{\partial_t f + v_3\partial_{x_3} f + \frac{q}{m}E_3\partial_{v_3} f\right\}\varphi(\overline{x}, v)\,\mathrm{d}v\mathrm{d}\overline{x} = 0, \quad (t, x_3) \in \mathbb{R}_+ \times \mathbb{R}$$

for any average collision invariant φ of the family

$$1,\ \omega_c\overline{x} + {}^\perp\overline{v},\ v_3,\ \frac{|v|^2}{2},\ \frac{|\omega_c\overline{x} + {}^\perp\overline{v}|^2 - |\overline{v}|^2}{2}.$$

For any $(t, x_3) \in \mathbb{R}_+ \times \mathbb{R}$, the density $(\overline{x}, v) \to f(t, \overline{x}, x_3, v)$ is a local gyrokinetic equilibrium and writes, cf. (35)

$$f(t, x, v) = \rho(t, x_3)\mathcal{M}_{\frac{\mu\theta}{\mu-\theta}}(\overline{v})\mathcal{M}_\theta\big(v_3 - u_3(t, x_3)\big)\omega_c^2\mathcal{M}_\mu\big(\omega_c\overline{x} + {}^\perp\overline{v} - \overline{u}(t, x_3)\big) \tag{41}$$

for some functions $\rho, u = (u_1, u_2, u_3), \theta, \mu$ depending on (t, x_3). The microscopic density f is determined by its moments whose evolution comes by imposing the balances corresponding to each collision invariant. Using the collision invariant $\varphi = 1$ leads to the continuity equation

$$\partial_t \rho + \partial_{x_3}(\rho u_3) = 0, \quad (t, x_3) \in \mathbb{R}_+ \times \mathbb{R}. \tag{42}$$

In order to obtain the other conservation laws in Theorem 1.1 we need essentially to compute the first and second order moments, together with their fluxes (see Appendix A for details).

Lemma 4.1. *For any local gyrokinetic equilibria cf.* (35)

$$f(x, v) = \rho(x_3)\mathcal{M}_{\frac{\mu\theta}{\mu-\theta}}(\overline{v})\mathcal{M}_\theta\big(v_3 - u_3(x_3)\big)\omega_c^2\mathcal{M}_\mu\big(\omega_c\overline{x} + {}^\perp\overline{v} - \overline{u}(x_3)\big)$$

we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3(\omega_c\overline{x} + {}^\perp\overline{v}, v_3) f(x, v)\,\mathrm{d}v\mathrm{d}\overline{x} = \rho\big(u_3\overline{u}, (u_3)^2 + \theta\big)$$

and

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (\omega_c\overline{x} + {}^\perp\overline{v}, v_3)\partial_{v_3} f\,\mathrm{d}v\mathrm{d}\overline{x} = (0, 0, -\rho).$$

Lemma 4.2. *For any local gyrokinetic equilibria cf.* (35)

$$f(x, v) = \rho(x_3)\mathcal{M}_{\frac{\mu\theta}{\mu-\theta}}(\overline{v})\mathcal{M}_\theta\big(v_3 - u_3(x_3)\big)\omega_c^2\mathcal{M}_\mu\big(\omega_c\overline{x} + {}^\perp\overline{v} - \overline{u}(x_3)\big)$$

we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 \frac{|\overline{v}|^2 + (v_3 - u_3)^2}{2} f(x, v)\,\mathrm{d}v\mathrm{d}\overline{x} = \rho u_3\left(\frac{\mu\theta}{\mu - \theta} + \frac{\theta}{2}\right),$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 \frac{|\omega_c\overline{x} + {}^\perp\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2} f(x, v)\,\mathrm{d}v\mathrm{d}\overline{x} = \rho u_3\left(\mu - \frac{\mu\theta}{\mu - \theta}\right),$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 \frac{|\overline{v}|^2 + (v_3 - u_3)^2}{2} \partial_{x_3} f(x, v)\,\mathrm{d}v\mathrm{d}\overline{x} = \partial_{x_3}\left[\rho u_3\left(\frac{\mu\theta}{\mu - \theta} + \frac{\theta}{2}\right)\right] + \rho\theta\partial_{x_3}u_3,$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 \frac{|\omega_c\overline{x} + {}^\perp\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2} \partial_{x_3} f\,\mathrm{d}v\mathrm{d}\overline{x} = \partial_{x_3}\left[\rho u_3\left(\mu - \frac{\mu\theta}{\mu - \theta}\right)\right],$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\overline{v}|^2 + (v_3 - u_3)^2}{2} \partial_{v_3} f(x, v)\,\mathrm{d}v\mathrm{d}\overline{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\omega_c\overline{x} + {}^\perp\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2} \partial_{v_3} f\,\mathrm{d}v\mathrm{d}\overline{x} = 0.$$

We will also need to compute the macroscopic entropy $\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} f \ln f\,\mathrm{d}v\mathrm{d}\overline{x}$ and its parallel flux $\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 \times f \ln f\,\mathrm{d}v\mathrm{d}\overline{x}$ associated to any local gyrokinetic equilibrium f (see Appendix A for details).

Lemma 4.3. *For any local gyrokinetic equilibrium cf.* (35)

$$f(x, v) = \rho(x_3)\mathcal{M}_{\frac{\mu\theta}{\mu-\theta}}(\overline{v})\mathcal{M}_\theta\big(v_3 - u_3(x_3)\big)\omega_c^2\mathcal{M}_\mu\big(\omega_c\overline{x} + {}^\perp\overline{v} - \overline{u}(x_3)\big)$$

we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} f \ln f\,\mathrm{d}v\mathrm{d}\overline{x} = \rho\ln\left(\frac{\rho\omega_c^2}{(2\pi)^{5/2}\frac{\mu^2\theta^{3/2}}{\mu-\theta}}\right) - \frac{5}{2}\rho$$

and

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 f \ln f \,\mathrm{d}v\mathrm{d}\overline{x} = \rho u_3 \ln\left(\frac{\rho\omega_c^2}{(2\pi)^{5/2}\frac{\mu^2\theta^{3/2}}{\mu-\theta}}\right) - \frac{5}{2}\rho u_3.$$

We are ready to derive the macroscopic limit model stated in Theorem 1.1 for strong collisional regimes in the gyrokinetic framework.

Proof of Theorem 1.1. We have already deduced the continuity equation (42), appealing to the collision invariant $\varphi = 1$. Using the collision invariants $\omega_c\overline{x} + {}^\perp\overline{v}$, v_3 yields

$$\partial_t \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}(\omega_c\overline{x} + {}^\perp\overline{v}) f \,\mathrm{d}v\mathrm{d}\overline{x} + \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3(\omega_c\overline{x} + {}^\perp\overline{v})\partial_{x_3} f \,\mathrm{d}v\mathrm{d}\overline{x} + \frac{q}{m}E_3 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}(\omega_c\overline{x} + {}^\perp\overline{v})\partial_{v_3} f \,\mathrm{d}v\mathrm{d}\overline{x} = 0,$$

$$\partial_t \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 f \,\mathrm{d}v\mathrm{d}\overline{x} + \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}(v_3)^2\partial_{x_3} f \,\mathrm{d}v\mathrm{d}\overline{x} + \frac{q}{m}E_3 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3\partial_{v_3} f \,\mathrm{d}v\mathrm{d}\overline{x} = 0.$$

Thanks to Lemma 4.1 one gets

$$\partial_t(\rho\overline{u}) + \partial_{x_3}(\rho u_3\overline{u}) = 0, \tag{43}$$

$$\partial_t(\rho u_3) + \partial_{x_3}\left[\rho\left((u_3)^2 + \theta\right)\right] - \frac{q}{m}E_3\rho = 0. \tag{44}$$

Appealing now to the collision invariant $\frac{|\overline{v}|^2+(v_3-u_3)^2}{2}$ yields

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\frac{|\overline{v}|^2+(v_3-u_3)^2}{2}\partial_t f \,\mathrm{d}v\mathrm{d}\overline{x} + \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3\frac{|\overline{v}|^2+(v_3-u_3)^2}{2}\partial_{x_3} f \,\mathrm{d}v\mathrm{d}\overline{x}$$
$$+ \frac{q}{m}E_3\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\frac{|\overline{v}|^2+(v_3-u_3)^2}{2}\partial_{v_3} f \,\mathrm{d}v\mathrm{d}\overline{x} = 0. \tag{45}$$

Notice that (37) allows us to write

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\frac{|\overline{v}|^2+(v_3-u_3)^2}{2}\partial_t f \,\mathrm{d}v\mathrm{d}\overline{x} = \partial_t\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\frac{|\overline{v}|^2+(v_3-u_3)^2}{2} f \,\mathrm{d}v\mathrm{d}\overline{x} - \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}(u_3 - v_3)\partial_t u_3 f \,\mathrm{d}v\mathrm{d}\overline{x}$$
$$= \partial_t\left[\rho\left(\frac{\mu\theta}{\mu-\theta} + \frac{\theta}{2}\right)\right]$$

and therefore, thanks to Lemma 4.2, (45) reduces to

$$\partial_t\left[\rho\left(\frac{\mu\theta}{\mu-\theta} + \frac{\theta}{2}\right)\right] + \partial_{x_3}\left[\rho u_3\left(\frac{\mu\theta}{\mu-\theta} + \frac{\theta}{2}\right)\right] + \rho\theta\partial_{x_3}u_3 = 0. \tag{46}$$

The previous equation can be written in conservative form, replacing the collision invariant $\frac{|\overline{v}|^2+(v_3-u_3)^2}{2}$ by $\frac{|v|^2}{2}$. In this case we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\frac{|v|^2}{2} f \,\mathrm{d}v\mathrm{d}\overline{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\frac{|\overline{v}|^2+(v_3-u_3)^2}{2} f \,\mathrm{d}v\mathrm{d}\overline{x} + \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\frac{(u_3)^2}{2} f \,\mathrm{d}v\mathrm{d}\overline{x} = \rho\left(\frac{\mu\theta}{\mu-\theta} + \frac{\theta}{2} + \frac{(u_3)^2}{2}\right),$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3\frac{|v|^2}{2} f \,\mathrm{d}v\mathrm{d}\overline{x} = u_3\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\frac{|v|^2}{2} f \,\mathrm{d}v\mathrm{d}\overline{x} + \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} u_3(v_3-u_3)^2 f \,\mathrm{d}v\mathrm{d}\overline{x} = u_3\rho\left(\frac{\mu\theta}{\mu-\theta} + \frac{3\theta}{2} + \frac{(u_3)^2}{2}\right),$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\frac{|v|^2}{2}\partial_{v_3} f \,\mathrm{d}v\mathrm{d}\overline{x} = -\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 f \,\mathrm{d}v\mathrm{d}\overline{x} = -\rho u_3.$$

We obtain

$$\partial_t \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|v|^2}{2} f \,\mathrm{d}v\mathrm{d}\overline{x} + \partial_{x_3} \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 \frac{|v|^2}{2} f \,\mathrm{d}v\mathrm{d}\overline{x} + \frac{q}{m} E_3 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|v|^2}{2} \partial_{v_3} f \,\mathrm{d}v\mathrm{d}\overline{x} = 0$$

or equivalently

$$\partial_t \left[\rho\left(\frac{\mu\theta}{\mu-\theta} + \frac{\theta}{2} + \frac{(u_3)^2}{2}\right)\right] + \partial_{x_3}\left[u_3\rho\left(\frac{\mu\theta}{\mu-\theta} + \frac{3\theta}{2} + \frac{(u_3)^2}{2}\right)\right] - \frac{q}{m} E_3 \rho u_3 = 0.$$

Finally, the last collision invariant $\frac{|\omega_c \overline{x} + {}^\perp\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2}$ gives

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\omega_c \overline{x} + {}^\perp\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2} \partial_t f \,\mathrm{d}v\mathrm{d}\overline{x} + \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 \frac{|\omega_c \overline{x} + {}^\perp\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2} \partial_{x_3} f \,\mathrm{d}v\mathrm{d}\overline{x} \\
&\quad + \frac{q}{m} E_3 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\omega_c \overline{x} + {}^\perp\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2} \partial_{v_3} f \,\mathrm{d}v\mathrm{d}\overline{x} = 0.
\end{aligned} \tag{47}$$

Using (38) we deduce that

$$\begin{aligned}
\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\omega_c \overline{x} + {}^\perp\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2} \partial_t f \,\mathrm{d}v\mathrm{d}\overline{x} &= \partial_t \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\omega_c \overline{x} + {}^\perp\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2} f \,\mathrm{d}v\mathrm{d}\overline{x} \\
&\quad - \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (\overline{u} - \omega_c \overline{x} - {}^\perp\overline{v}) \cdot \partial_t \overline{u} f \,\mathrm{d}v\mathrm{d}\overline{x} = \partial_t \left[\rho\left(\mu - \frac{\mu\theta}{\mu-\theta}\right)\right]
\end{aligned}$$

and Lemma 4.2 applied to the other terms in (47) implies

$$\partial_t \left[\rho\left(\mu - \frac{\mu\theta}{\mu-\theta}\right)\right] + \partial_{x_3}\left[\rho u_3\left(\mu - \frac{\mu\theta}{\mu-\theta}\right)\right] = 0. \tag{48}$$

We write the balance of the microscopic entropy $f \ln f$ and we deduce a new conservation law (in other words we construct a macroscopic entropy). Indeed, multiplying (39) by $1 + \ln f^\tau$ yields after integration with respect to $(\overline{x}, v)$

$$\begin{aligned}
\partial_t \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} f^\tau \ln f^\tau \,\mathrm{d}v\mathrm{d}\overline{x} + \partial_{x_3} \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 f^\tau \ln f^\tau \,\mathrm{d}v\mathrm{d}\overline{x} &= \frac{1}{\tau} \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \left(1 + \ln f^\tau\right) \langle Q\rangle\left(f^\tau, f^\tau\right) \mathrm{d}v\mathrm{d}\overline{x} \\
&= \frac{1}{\tau} \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \ln f^\tau \langle Q\rangle\left(f^\tau, f^\tau\right) \mathrm{d}v\mathrm{d}\overline{x}.
\end{aligned} \tag{49}$$

But thanks to Theorem 2.1 we know that for any $(t, x_3) \in \mathbb{R}_+ \times \mathbb{R}$ and $\tau > 0$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \ln f^\tau \langle Q\rangle\left(f^\tau, f^\tau\right) \mathrm{d}v\mathrm{d}\overline{x} \le 0$$

and therefore, passing formally to the limit when $\tau \searrow 0$ in (49) implies

$$\partial_t \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} f \ln f \,\mathrm{d}v\mathrm{d}\overline{x} + \partial_{x_3} \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 f \ln f \,\mathrm{d}v\mathrm{d}\overline{x} \le 0. \tag{50}$$

By Lemma 4.3 we know that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} f \ln f \,\mathrm{d}v\mathrm{d}\overline{x} = \rho \ln\left(\frac{\rho\omega_c^2}{(2\pi)^{5/2} \frac{\mu^2\theta^{3/2}}{\mu-\theta}}\right) - \frac{5}{2}\rho,$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 f \ln f \,\mathrm{d}v\mathrm{d}\overline{x} = \rho u_3 \ln\left(\frac{\rho\omega_c^2}{(2\pi)^{5/2} \frac{\mu^2\theta^{3/2}}{\mu-\theta}}\right) - \frac{5}{2}\rho u_3$$

and (50) reduces to

$$\partial_t\left[\rho\ln\left(\frac{\rho\omega_c^2}{(2\pi)^{5/2}\frac{\mu^2\theta^{3/2}}{\mu-\theta}}\right)-\frac{5}{2}\rho\right]+\partial_{x_3}\left[\rho u_3\ln\left(\frac{\rho\omega_c^2}{(2\pi)^{5/2}\frac{\mu^2\theta^{3/2}}{\mu-\theta}}\right)-\frac{5}{2}\rho u_3\right]\le 0.$$

Combining with the continuity equation (42), we obtain the entropy inequality

$$\partial_t\left[\rho\ln\left(\frac{\rho}{\frac{\mu^2\theta^{3/2}}{\mu-\theta}}\right)\right]+\partial_{x_3}\left[\rho u_3\ln\left(\frac{\rho}{\frac{\mu^2\theta^{3/2}}{\mu-\theta}}\right)\right]\le 0. \tag{51}$$

When the solution (ρ,u,θ,μ) is smooth, the reader can check by standard computations, similar to those used when dealing with the Euler equations, that the inequality in (51) becomes equality, being a consequence of the previous conservation laws (42), (43), (44), (46), (48). □

5. Linearization of the averaged Fokker–Planck–Landau operator

Another important issue is the derivation of a simplified averaged Fokker–Planck–Landau operator, when the density is close to the equilibrium. The natural way to do it is to neglect the second order fluctuations around the equilibrium, which makes sense for example in the strongly collisional regime. The key point is that the resulting simplified kernel still keeps the main features of the original averaged Fokker–Planck–Landau kernel. For any positive density $f=f(\overline{x},v)$ we denote by $\mathcal{E}_f$ the equilibrium of $\langle Q\rangle$ having the same moments as f

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}(\mathcal{E}_f-f)\,\mathrm{d}v\mathrm{d}\overline{x}=0,\qquad \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}(\omega_c\overline{x}+{}^\perp\overline{v},v_3)(\mathcal{E}_f-f)\,\mathrm{d}v\mathrm{d}\overline{x}=0,$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\frac{|v|^2}{2}(\mathcal{E}_f-f)\,\mathrm{d}v\mathrm{d}\overline{x}=0,\qquad \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\frac{|\omega_c\overline{x}+{}^\perp\overline{v}|^2-|\overline{v}|^2}{2}(\mathcal{E}_f-f)\,\mathrm{d}v\mathrm{d}\overline{x}=0.$$

Proof of Theorem 1.2. We assume that f is close to $\mathcal{E}_f$ and by neglecting the terms of order $(f-\mathcal{E}_f)^2$ one gets the first order approximation, denoted by $L(f)$

$$\begin{aligned}
\omega_c^{-2}\langle Q\rangle(f,f) &= \omega_c^{-2}\langle Q\rangle(f,f)-\omega_c^{-2}\langle Q\rangle(\mathcal{E}_f,\mathcal{E}_f)\\
&=\sum_{i=1}^{4}\mathrm{div}_{\omega_c x,v}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\big\{f(\overline{x'},v')\xi^i(\overline{x},v,\overline{x'},v')\otimes\xi^i(\overline{x},v,\overline{x'},v')\nabla_{\omega_c x,v}f(\overline{x},v)\\
&\quad-\mathcal{E}_f(\overline{x'},v')\xi^i(\overline{x},v,\overline{x'},v')\otimes\xi^i(\overline{x},v,\overline{x'},v')\nabla_{\omega_c x,v}\mathcal{E}_f(\overline{x},v)\big\}\,\mathrm{d}v'\mathrm{d}\overline{x'}\\
&\quad-\sum_{i=1}^{4}\mathrm{div}_{\omega_c x,v}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\big\{f(\overline{x},v)\xi^i(\overline{x},v,\overline{x'},v')\otimes\varepsilon_i\xi^i(\overline{x'},v',\overline{x},v)\nabla_{\omega_c x',v'}f(\overline{x'},v')\\
&\quad-\mathcal{E}_f(\overline{x},v)\xi^i(\overline{x},v,\overline{x'},v')\otimes\varepsilon_i\xi^i(\overline{x'},v',\overline{x},v)\nabla_{\omega_c x',v'}\mathcal{E}_f(\overline{x'},v')\big\}\,\mathrm{d}v'\mathrm{d}\overline{x'}\\
&\approx\sum_{i=1}^{4}\mathrm{div}_{\omega_c x,v}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\big\{\mathcal{E}_f'\xi^i\otimes\xi^i\nabla(f-\mathcal{E}_f)+(f'-\mathcal{E}_f')\xi^i\otimes\xi^i\nabla\mathcal{E}_f\big\}\,\mathrm{d}v'\mathrm{d}\overline{x'}\\
&\quad-\sum_{i=1}^{4}\mathrm{div}_{\omega_c x,v}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\big\{\mathcal{E}_f\xi^i\otimes\varepsilon_i(\xi^i)'\nabla'(f'-\mathcal{E}_f')+(f-\mathcal{E}_f)\xi^i\otimes\varepsilon_i(\xi^i)'\nabla'\mathcal{E}_f'\big\}\,\mathrm{d}v'\mathrm{d}\overline{x'}\\
&=:\omega_c^{-2}L(f).
\end{aligned}\tag{52}$$

We have used the notations

$$f=f(\overline{x},v),\qquad f'=f(\overline{x'},v'),\qquad \mathcal{E}_f=\mathcal{E}_f(\overline{x},v),\qquad \mathcal{E}_f'=\mathcal{E}_f(\overline{x'},v'),$$

$$\xi^i=\xi^i(\overline{x},v,\overline{x'},v'),\qquad (\xi^i)'=\xi^i(\overline{x'},v',\overline{x},v),\qquad \nabla=\nabla_{\omega_c x,v},\qquad \nabla'=\nabla_{\omega_c x',v'}.$$

Since $\mathcal{E}_f$ is an equilibrium, we know by Theorem 2.1 that

$$\xi^i \cdot \nabla \ln \mathcal{E}_f - \varepsilon_i (\xi^i)' \cdot \nabla' \ln \mathcal{E}'_f = 0, \quad 1 \le i \le 4$$

and therefore

$$\begin{aligned}
&\mathcal{E}'_f \xi^i \otimes \xi^i \nabla (f - \mathcal{E}_f) - (f - \mathcal{E}_f) \xi^i \otimes \varepsilon_i (\xi^i)' \nabla' \mathcal{E}'_f \\
&= \mathcal{E}_f \mathcal{E}'_f \left\{ \xi^i \otimes \xi^i \frac{\nabla (f - \mathcal{E}_f)}{\mathcal{E}_f} - \frac{f - \mathcal{E}_f}{\mathcal{E}_f} \xi^i \otimes \varepsilon_i (\xi^i)' \nabla' \ln \mathcal{E}'_f \right\} \\
&= \mathcal{E}_f \mathcal{E}'_f \left\{ \xi^i \otimes \xi^i \frac{\nabla (f - \mathcal{E}_f)}{\mathcal{E}_f} - \frac{f - \mathcal{E}_f}{\mathcal{E}_f} \xi^i \otimes \xi^i \nabla \ln \mathcal{E}_f \right\} \\
&= \mathcal{E}_f \mathcal{E}'_f \xi^i \otimes \xi^i \nabla \left(\frac{f - \mathcal{E}_f}{\mathcal{E}_f} \right) = \mathcal{E}_f \mathcal{E}'_f \xi^i \otimes \xi^i \nabla \left(\frac{f}{\mathcal{E}_f} \right).
\end{aligned} \tag{53}$$

Similarly one gets

$$\begin{aligned}
&(f' - \mathcal{E}'_f) \xi^i \otimes \xi^i \nabla \mathcal{E}_f - \mathcal{E}_f \xi^i \otimes \varepsilon_i (\xi^i)' \nabla' (f' - \mathcal{E}'_f) \\
&= \mathcal{E}_f \mathcal{E}'_f \left\{ \frac{f' - \mathcal{E}'_f}{\mathcal{E}'_f} \xi^i \otimes \xi^i \nabla \ln \mathcal{E}_f - \xi^i \otimes \varepsilon_i (\xi^i)' \frac{\nabla' (f' - \mathcal{E}'_f)}{\mathcal{E}'_f} \right\} \\
&= \mathcal{E}_f \mathcal{E}'_f \left\{ \frac{f' - \mathcal{E}'_f}{\mathcal{E}'_f} \xi^i \otimes \varepsilon_i (\xi^i)' \nabla' \ln \mathcal{E}'_f - \xi^i \otimes \varepsilon_i (\xi^i)' \frac{\nabla' (f' - \mathcal{E}'_f)}{\mathcal{E}'_f} \right\} \\
&= -\mathcal{E}_f \mathcal{E}'_f \xi^i \otimes \varepsilon_i (\xi^i)' \nabla' \left(\frac{f' - \mathcal{E}'_f}{\mathcal{E}'_f} \right) = -\mathcal{E}_f \mathcal{E}'_f \xi^i \otimes \varepsilon_i (\xi^i)' \nabla' \left(\frac{f'}{\mathcal{E}'_f} \right).
\end{aligned} \tag{54}$$

Combining (52), (53), (54) leads to the following expression for the first order approximation of $\langle Q \rangle$ near equilibrium

$$\begin{aligned}
\omega_c^{-2} L(f) &= \sum_{i=1}^{4} \operatorname{div}_{\omega_c x, v} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{E}_f \mathcal{E}'_f \left\{ \xi^i \otimes \xi^i \nabla \left(\frac{f}{\mathcal{E}_f} \right) - \xi^i \otimes \varepsilon_i (\xi^i)' \nabla' \left(\frac{f'}{\mathcal{E}'_f} \right) \right\} \mathrm{d}v' \mathrm{d}\overline{x'} \\
&= \sum_{i=1}^{4} \operatorname{div}_{\omega_c x, v} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{E}_f \mathcal{E}'_f \left\{ \xi^i \cdot \nabla \left(\frac{f}{\mathcal{E}_f} \right) - \varepsilon_i (\xi^i)' \cdot \nabla' \left(\frac{f'}{\mathcal{E}'_f} \right) \right\} \xi^i \, \mathrm{d}v' \mathrm{d}\overline{x'}.
\end{aligned} \tag{55}$$

We justify now the properties of L.

1. Integrating by parts with respect to $(\overline{x}, v)$ we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} L(f) \varphi \, \mathrm{d}v \mathrm{d}\overline{x} = -\sum_{i=1}^{4} \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{E}_f \mathcal{E}'_f \\
\times \left\{ (\xi^i \cdot \nabla \varphi) \left[\xi^i \cdot \nabla \left(\frac{f}{\mathcal{E}_f} \right) \right] - \varepsilon_i (\xi^i \cdot \nabla \varphi) \left[(\xi^i)' \cdot \nabla' \left(\frac{f'}{\mathcal{E}'_f} \right) \right] \right\} \mathrm{d}v' \mathrm{d}\overline{x'} \, \mathrm{d}v \mathrm{d}\overline{x}.
\end{aligned}$$

Performing the change of variables $(\overline{x'}, v') \leftrightarrow (\overline{x}, v)$ yields

$$\begin{aligned}
\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} L(f) \varphi \, \mathrm{d}v \mathrm{d}\overline{x} = -\sum_{i=1}^{4} \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{E}_f \mathcal{E}'_f \\
\times \left\{ ((\xi^i)' \cdot \nabla' \varphi') \left[(\xi^i)' \cdot \nabla' \left(\frac{f'}{\mathcal{E}'_f} \right) \right] - \varepsilon_i ((\xi^i)' \cdot \nabla' \varphi') \left[\xi^i \cdot \nabla \left(\frac{f}{\mathcal{E}_f} \right) \right] \right\} \mathrm{d}v \mathrm{d}\overline{x} \, \mathrm{d}v' \mathrm{d}\overline{x'}.
\end{aligned}$$

Combining the above equalities gives

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} L(f)\varphi \,\mathrm{d}v\mathrm{d}\overline{x} = -\frac{\omega_c^2}{2}\sum_{i=1}^{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \mathcal{E}_f\mathcal{E}_f' S^i \,\mathrm{d}v'\mathrm{d}\overline{x'}\,\mathrm{d}v\mathrm{d}\overline{x}$$

where

$$S^i = \big(\xi^i\cdot\nabla\varphi - \varepsilon_i\big(\xi^i\big)'\cdot\nabla'\varphi'\big)\left[\xi^i\cdot\nabla\left(\frac{f}{\mathcal{E}_f}\right) - \varepsilon_i\big(\xi^i\big)'\cdot\nabla'\left(\frac{f'}{\mathcal{E}_f'}\right)\right], \quad 1\le i\le 4.$$

2. It comes immediately by taking $\varphi = f/\mathcal{E}_f$ in (16).

3. If f is a positive equilibrium of $\langle Q\rangle$, we have $f = \mathcal{E}_f$ and therefore $L(f) = 0$. Conversely, assume that f is a positive equilibrium of L. Then we have equality in (17), saying that

$$\xi^i\cdot\nabla\left(\frac{f}{\mathcal{E}_f}\right) - \varepsilon_i\big(\xi^i\big)'\cdot\nabla'\left(\frac{f'}{\mathcal{E}_f'}\right) = 0, \quad 1\le i\le 4. \tag{56}$$

We consider the Hilbert space $L^2_{\mathcal{E}_f} = \{g(\overline{x}, v): \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} g^2/\mathcal{E}_f \,\mathrm{d}v\mathrm{d}\overline{x} < +\infty\}$ endowed with the scalar product

$$(g,h)_{L^2_{\mathcal{E}_f}} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{gh}{\mathcal{E}_f}\,\mathrm{d}v\mathrm{d}\overline{x}, \quad g,h\in L^2_{\mathcal{E}_f}$$

and the linear operator l_f given by

$$\omega_c^{-2} l_f(g) = \sum_{i=1}^{4}\mathrm{div}_{\omega_c x, v}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\mathcal{E}_f\mathcal{E}_f'\left\{\xi^i\cdot\nabla\left(\frac{g}{\mathcal{E}_f}\right) - \varepsilon_i\big(\xi^i\big)'\cdot\nabla'\left(\frac{g'}{\mathcal{E}_f'}\right)\right\}\xi^i\,\mathrm{d}v'\mathrm{d}\overline{x'}.$$

Obviously $f - \mathcal{E}_f$ belongs to the kernel of l_f. By the first statement we deduce that

$$\begin{aligned}\big(l_f(g), h\big)_{L^2_{\mathcal{E}_f}} = &-\frac{\omega_c^2}{2}\sum_{i=1}^{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\mathcal{E}_f\mathcal{E}_f'\left(\xi^i\cdot\nabla\left(\frac{g}{\mathcal{E}_f}\right) - \varepsilon_i\big(\xi^i\big)'\cdot\nabla'\left(\frac{g'}{\mathcal{E}_f'}\right)\right)\\ &\times\left(\xi^i\cdot\nabla\left(\frac{h}{\mathcal{E}_f}\right) - \varepsilon_i\big(\xi^i\big)'\cdot\nabla'\left(\frac{h'}{\mathcal{E}_f'}\right)\right)\mathrm{d}v'\mathrm{d}\overline{x'}\,\mathrm{d}v\mathrm{d}\overline{x}\end{aligned}$$

saying that l_f is symmetric with respect to the scalar product $(\cdot,\cdot)_{L^2_{\mathcal{E}_f}}$. Moreover, it is easily seen that $g\in\ker l_f$ iff

$$\xi^i\cdot\nabla\ln\left(\exp\left(\frac{g}{\mathcal{E}_f}\right)\right) - \varepsilon_i\big(\xi^i\big)'\cdot\nabla'\ln\left(\exp\left(\frac{g'}{\mathcal{E}_f'}\right)\right) = 0, \quad 1\le i\le 4. \tag{57}$$

Thanks to Proposition 3.4, (57) implies that $g\in\ker l_f$ iff $g/\mathcal{E}_f = \ln\exp(g/\mathcal{E}_f)$ is a linear combination of the collision invariants

$$1,\ \omega_c\overline{x} + {}^{\perp}\overline{v},\ v_3,\ \frac{|v|^2}{2},\ \frac{|\omega_c\overline{x} + {}^{\perp}\overline{v}|^2 - |\overline{v}|^2}{2}.$$

In particular, since f and $\mathcal{E}_f$ have the same moments with respect to the above collision invariants, for any $g\in\ker l_f$ one gets

$$(f-\mathcal{E}_f, g)_{L^2_{\mathcal{E}_f}} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}(f-\mathcal{E}_f)\frac{g}{\mathcal{E}_f}\,\mathrm{d}v\mathrm{d}\overline{x} = 0$$

saying that $f-\mathcal{E}_f\in(\ker l_f)^{\perp}$. Finally $f-\mathcal{E}_f\in\ker l_f\cap(\ker l_f)^{\perp} = \{0\}$ and thus $f = \mathcal{E}_f$. $\square$

The first order approximation of $\langle Q\rangle$ near equilibria (see (55)) inherits all the properties of $\langle Q\rangle$, nevertheless its structure remains complex. Using L instead of $\langle Q\rangle$ requires almost the same computational effort. A classical way to circumvent these efforts relies on the BGK approximation of $\langle Q\rangle$, which writes

$$L_{\mathrm{BGK}} = -(f - \mathcal{E}_f).$$

The properties of the BGK operator associated to $\langle Q\rangle$ are summarized below.

Theorem 5.1.

1. *For any* $f = f(\overline{x}, v)$ *and* $\varphi = \varphi(\overline{x}, v) > 0$ *we have*
$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} L_{\mathrm{BGK}}(f)\ln\varphi \,\mathrm{d}v\mathrm{d}\overline{x} = -\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (f-\mathcal{E}_f)(\ln\varphi - \ln\mathcal{E}_\varphi)\,\mathrm{d}v\mathrm{d}\overline{x}.$$
2. *For any positive density* f *we have the inequality*
$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} L_{\mathrm{BGK}}(f)\ln f \,\mathrm{d}v\mathrm{d}\overline{x} \le 0$$
with equality iff $f = \mathcal{E}_f$.
3. *The positive equilibria of* L_{BGK} *are the positive equilibria of* $\langle Q\rangle$
$$f > 0, \qquad L_{\mathrm{BGK}}(f) = 0 \quad \Leftrightarrow \quad f = \mathcal{E}_f.$$

Proof. 1. For any $\varphi > 0$, $\ln\mathcal{E}_\varphi$ is a linear combination of the collision invariants

$$1,\ \omega_c\overline{x} + {}^\perp\overline{v},\ v_3,\ \frac{|v|^2}{2},\ \frac{|\omega_c\overline{x} + {}^\perp\overline{v}|^2 - |\overline{v}|^2}{2}.$$

By the definition of $\mathcal{E}_f$ we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (f-\mathcal{E}_f)\ln\mathcal{E}_\varphi \,\mathrm{d}v\mathrm{d}\overline{x} = 0$$

implying that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} L_{\mathrm{BGK}}(f)\ln\varphi \,\mathrm{d}v\mathrm{d}\overline{x} = -\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (f-\mathcal{E}_f)\ln\varphi\,\mathrm{d}v\mathrm{d}\overline{x} = -\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (f-\mathcal{E}_f)(\ln\varphi - \ln\mathcal{E}_\varphi)\,\mathrm{d}v\mathrm{d}\overline{x}.$$

2. Taking $\varphi = f > 0$ in the previous statement, we obtain

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} L_{\mathrm{BGK}}(f)\ln f \,\mathrm{d}v\mathrm{d}\overline{x} = -\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (f-\mathcal{E}_f)(\ln f - \ln\mathcal{E}_f)\,\mathrm{d}v\mathrm{d}\overline{x} \le 0$$

with equality iff $f = \mathcal{E}_f$.

3. Clearly $L_{\mathrm{BGK}}(f) = 0$ iff $f - \mathcal{E}_f = 0$. □

Conflict of interest statement

The author declares there is no conflict of interest.

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Appendix A. Proofs of Propositions 1.4, 3.1, 3.2, 3.3, 3.4 and Lemmas A.1, 4.1, 4.2, 4.3

Proof of Proposition 1.4. By the linearity of the average operator we obtain

$$\left\langle \partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f \right\rangle = \langle \partial_t f \rangle + \langle v_3 \partial_{x_3} f \rangle + \frac{q}{m} \langle E \cdot \nabla_v f \rangle.$$

It is easily seen that ∂_t and ∂_{x_3} commute with the average operator and thus, taking into account that $f \in \ker \mathcal{T}$ one gets

$$\langle \partial_t f \rangle = \partial_t \langle f \rangle = \partial_t f, \qquad \langle v_3 \partial_{x_3} f \rangle = v_3 \langle \partial_{x_3} f \rangle = v_3 \partial_{x_3} \langle f \rangle = v_3 \partial_{x_3} f.$$

Observe that $\mathcal{T}(f\phi) = f \overline{v} \cdot \nabla_{\overline{x}} \phi = -f \overline{v} \cdot \overline{E}$ and thus $\langle f \overline{v} \cdot \overline{E} \rangle = 0$. Thanks to Proposition 1.3 one gets

$$\langle E \cdot \nabla_v f \rangle = \langle \mathrm{div}_v \{ f E \} \rangle = \mathrm{div}_{\overline{x}} \left\langle f \frac{{}^\perp \overline{E}}{\omega_c} \right\rangle + \mathcal{T} \left\langle f \frac{{}^\perp \overline{v} \cdot \overline{E}}{\omega_c |\overline{v}|^2} \right\rangle + \partial_{v_3} \langle f E_3 \rangle = \mathrm{div}_{\overline{x}} \left\{ f \left\langle \frac{{}^\perp \overline{E}}{\omega_c} \right\rangle \right\} + \partial_{v_3} \{ f \langle E_3 \rangle \}$$

implying that

$$\frac{q}{m} \langle E \cdot \nabla_v f \rangle = \mathrm{div}_{\overline{x}} \left\{ f \frac{\langle {}^\perp \overline{E} \rangle}{B} \right\} + \frac{q}{m} \partial_{v_3} \{ f \langle E_3 \rangle \}.$$

Using again Proposition 1.3 we deduce that ∂_{v_3} and $\mathrm{div}_{\overline{x}}$ commute with the average operator, implying that

$$\partial_{v_3} \langle E_3 \rangle = \langle \partial_{v_3} E_3 \rangle = 0, \qquad \mathrm{div}_{\overline{x}} \langle {}^\perp \overline{E} \rangle = \langle \mathrm{div}_{\overline{x}} {}^\perp \overline{E} \rangle = 0$$

and our statement follows. □

Proof of Proposition 3.1. Observe that

$$\left(\frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{({}^\perp \overline{v}, 0)}{|\overline{v}|} \right) \cdot \nabla_{\omega_c x, v} = \frac{\mathcal{T}}{\omega_c |\overline{v}|}$$

and

$$\left(\frac{(\overline{v'}, 0)}{|\overline{v'}|}, \frac{({}^\perp \overline{v'}, 0)}{|\overline{v'}|} \right) \cdot \nabla_{\omega_c x', v'} = \frac{\mathcal{T}'}{\omega_c |\overline{v'}|}$$

where $\mathcal{T}' = \overline{v'} \cdot \nabla_{\overline{x'}} + \omega_c {}^\perp \overline{v'} \cdot \nabla_{\overline{v'}}$. Therefore (29) writes

$$\{\sigma \chi\}^{1/2} \frac{r' \sin\varphi (v_3 - v_3')}{|z| \sqrt{|z|^2 + (v_3 - v_3')^2}} \frac{\mathcal{T} \ln f}{\omega_c |\overline{v}|} + \{\sigma \chi\}^{1/2} \frac{r \sin\varphi (v_3' - v_3)}{|z| \sqrt{|z|^2 + (v_3 - v_3')^2}} \frac{\mathcal{T}' \ln f'}{\omega_c |\overline{v'}|} = 0$$

which reduces to

$$\frac{\mathcal{T} \ln f}{r^2} = \frac{\mathcal{T}' \ln f'}{(r')^2}, \quad \text{if } |r - r'| < |z| < r + r', \ v_3 \neq v_3'. \tag{58}$$

We claim that $\mathcal{T} \ln f$ depends only on the invariants of $\mathcal{T}$ *i.e.*,

$$\mathcal{T} \ln f(x, v) = \mathcal{T} \ln f(y, w)$$

for any $(x, v), (y, w) \in \mathbb{R}^6$ such that

$$\omega_c \overline{x} + {}^\perp \overline{v} = \omega_c \overline{y} + {}^\perp \overline{w}, \qquad x_3 = y_3, \qquad |\overline{v}| = |\overline{w}|, \qquad v_3 = w_3. \tag{59}$$

Take $(x, v), (y, w)$ verifying (59) and $(x', v') \in \mathbb{R}^6$ such that

$$v_3 \neq v_3', \qquad \left| \frac{|\overline{v}|}{|\omega_c|} - \frac{|\overline{v'}|}{|\omega_c|} \right| < \left| \overline{x} + \frac{{}^\perp \overline{v}}{\omega_c} - \left(\overline{x'} + \frac{{}^\perp \overline{v'}}{\omega_c} \right) \right| < \frac{|\overline{v}|}{|\omega_c|} + \frac{|\overline{v'}|}{|\omega_c|}$$

meaning that the Larmor circles of centers $\overline{x'} + {}^\perp\overline{v'}/\omega_c, \overline{x} + {}^\perp\overline{v}/\omega_c$ and radii $|\overline{v'}|/|\omega_c|, |\overline{v}|/|\omega_c|$ have non-empty intersection. We also have

$$w_3 \neq v_3', \qquad \left| \frac{|\overline{w}|}{|\omega_c|} - \frac{|\overline{v'}|}{|\omega_c|} \right| < \left| \overline{y} + \frac{{}^\perp\overline{w}}{\omega_c} - \left(\overline{x'} + \frac{{}^\perp\overline{v'}}{\omega_c} \right) \right| < \frac{|\overline{w}|}{|\omega_c|} + \frac{|\overline{v'}|}{|\omega_c|}$$

and (58) implies

$$\frac{\mathcal{T} \ln f(x,v)}{|\overline{v}|^2} = \frac{\mathcal{T}' \ln f'}{|\overline{v'}|^2} = \frac{\mathcal{T} \ln f(y,w)}{|\overline{w}|^2}.$$

As $|\overline{v}| = |\overline{w}|$, we deduce that $\mathcal{T} \ln f(x,v) = \mathcal{T} \ln f(y,w)$ for any $(x,v), (y,w)$ verifying (59), and therefore $\mathcal{T} \ln f$ remains constant along the characteristic flow of $\mathcal{T}$. Thus

$$\mathcal{T} \ln f = \langle \mathcal{T} \ln f \rangle = \mathrm{Proj}_{\ker \mathcal{T}} \mathcal{T} \ln f = 0$$

and finally $\ln f$ and f belong to the kernel of $\mathcal{T}$. □

In the sequel we will need the following easy lemma.

Lemma A.1. *Let $F = F(y,p) : \mathbb{R}^2 \times \mathbb{R}^m \to \mathbb{R}^2$ be a smooth field satisfying*

$$\left[F\left(y', p'\right) - F(y,p)\right] \cdot {}^\perp\left(y' - y\right) = 0, \quad y, y' \in \mathbb{R}^2,\ p, p' \in \mathbb{R}^m. \tag{60}$$

Then there is $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^2$ such that $F(y,p) = \alpha y + \beta$, $(y,p) \in \mathbb{R}^2 \times \mathbb{R}^m$.

Proof. Observe that F does not depend on p. Indeed, taking $y' = y + hz$, $p' = p + hq$ we have

$$\frac{[F(y+hz, p+hq) - F(y,p)]}{h} \cdot {}^\perp z = 0.$$

Letting $h \to 0$ we obtain

$$\left[\partial_y F(y,p) z + \partial_p F(y,p) q\right] \cdot {}^\perp z = 0, \quad y, z \in \mathbb{R}^2,\ p, q \in \mathbb{R}^m.$$

Replacing z by tu with $t \in \mathbb{R}^\star$, $u \in \mathbb{R}^2$, one gets

$$\left(t \partial_y F(y,p) u + \partial_p F(y,p) q\right) \cdot {}^\perp u = 0.$$

Passing to the limit when $t \to 0$, we deduce that

$$\partial_p F(y,p) q \cdot {}^\perp u = 0, \quad u \in \mathbb{R}^2,\ q \in \mathbb{R}^m$$

and thus $\partial_p F = 0$, saying that $F(y,p) = F^0(y)$, with $F^0(y) = F(y,0)$.

Taking $y' = y + hz$, $h \in \mathbb{R}^\star$, $z \in \mathbb{R}^2$ in (60) we obtain

$$\frac{[F^0(y+hz) - F^0(y)]}{h} \cdot {}^\perp z = 0.$$

Passing to the limit when $h \to 0$ yields

$$\left(\partial_y F^0(y) z\right) \cdot {}^\perp z = 0, \quad y, z \in \mathbb{R}^2$$

which is equivalent to

$$R(\pi/2) \partial_y F^0(y) : z \otimes z = 0, \quad y, z \in \mathbb{R}^2.$$

Therefore $R(\pi/2) \partial_y F^0(y)$ is antisymmetric, saying that

$$\partial_{y_1} F_1^0(y) = \partial_{y_2} F_2^0(y) = \alpha, \qquad \partial_{y_2} F_1^0(y) = \partial_{y_1} F_2^0(y) = 0, \quad y \in \mathbb{R}^2.$$

Notice that

$$\partial_{y_1} \alpha = \partial_{y_1} \partial_{y_2} F_2^0 = \partial_{y_2} \partial_{y_1} F_2^0 = 0, \qquad \partial_{y_2} \alpha = \partial_{y_2} \partial_{y_1} F_1^0 = \partial_{y_1} \partial_{y_2} F_1^0 = 0$$

saying that α is constant. Finally we have

$$\nabla_y\{F_1^0 - \alpha y_1\} = (0,0) = \nabla_y\{F_2^0 - \alpha y_2\}$$

and thus there is $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$ such that

$$F^0(y) = \alpha y + \beta, \quad y \in \mathbb{R}^2. \qquad \square$$

Proof of Proposition 3.2. We have

$$\xi^2 \cdot \nabla \ln f = \{\sigma\chi\}^{1/2} \frac{r - r'\cos\varphi}{\omega_c |z||\overline{v}|} \mathcal{T} \ln f + \{\sigma\chi\}^{1/2} \frac{{}^\perp z}{\omega_c |z|} \cdot \nabla_{\overline{x}} \ln f = \{\sigma\chi\}^{1/2} \frac{{}^\perp z}{\omega_c |z|} \cdot \nabla_{\overline{x}} \ln f$$

and

$$(\xi^2)' \cdot \nabla' \ln f' = \{\sigma\chi\}^{1/2} \frac{r' - r\cos\varphi}{\omega_c |z||\overline{v'}|} \mathcal{T}' \ln f' - \{\sigma\chi\}^{1/2} \frac{{}^\perp z}{\omega_c |z|} \cdot \nabla_{\overline{x'}} \ln f' = -\{\sigma\chi\}^{1/2} \frac{{}^\perp z}{\omega_c |z|} \cdot \nabla_{\overline{x'}} \ln f'.$$

Thus (30) becomes

$$ {}^\perp z \cdot \left(\nabla_{\overline{x}} \ln f - \nabla_{\overline{x'}} \ln f'\right) = 0, \qquad \left||\overline{v}| - |\overline{v'}|\right| < \left|\omega_c \overline{x} + {}^\perp\overline{v} - \left(\omega_c \overline{x'} + {}^\perp\overline{v'}\right)\right| < |\overline{v}| + |\overline{v'}|. \tag{61}$$

Since the positive density f satisfies (29), $\ln f$ belongs to $\ker \mathcal{T}$ and thus there is a function g such that

$$\ln f(x,v) = g\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}, x_3, |\overline{v}|, v_3\right), \quad (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

Observe that

$$\nabla_{\overline{x}} \ln f(x,v) = \nabla_{\overline{\psi}} g\left(\psi_1(x,v), \psi_2(x,v), x_3, |\overline{v}|, v_3\right), \quad (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$$

and therefore (61) reduces

$$ {}^\perp\left(\overline{\psi} - \overline{\psi'}\right) \cdot \left(\nabla_{\overline{\psi}} g(\overline{\psi}, x_3, r, v_3) - \nabla_{\overline{\psi'}} g\left(\overline{\psi'}, x_3, r', v_3'\right)\right) = 0 \tag{62}$$

for any $(\overline{\psi}, x_3, r, v_3)$, $(\overline{\psi'}, x_3, r', v_3')$ satisfying $|r - r'|/|\omega_c| < |\overline{\psi} - \overline{\psi'}| < (r + r')/|\omega_c|$. We cannot apply directly Lemma A.1, since (62) holds only for pairs of Larmor circles with non-empty intersection. Nevertheless we can proceed as in the proof of Lemma A.1, taking $h \in \mathbb{R}^\star$ small enough, $s, u_3 \in \mathbb{R}$, $\overline{u} \in \mathbb{R}^2 \setminus (0,0)$

$$r' = r + hs, \qquad v_3' = v_3 + hu_3, \qquad \overline{\psi'} = \overline{\psi} + h\overline{u}$$

such that

$$|h| \frac{|s|}{|\omega_c|} < |h||\overline{u}| < \frac{2r + hs}{|\omega_c|}.$$

Therefore (62) holds true, implying that

$$ {}^\perp\overline{u} \cdot \frac{G(\overline{\psi} + h\overline{u}, x_3, r + hs, v_3 + hu_3) - G(\overline{\psi}, x_3, r, v_3)}{h} = 0 \tag{63}$$

where $G(\overline{\psi}, x_3, r, v_3) = \nabla_{\overline{\psi}} g(\overline{\psi}, x_3, r, v_3)$. Letting $h \to 0$ we deduce that G depends only on $\overline{\psi}$ and x_3

$$\nabla_{\overline{\psi}} g(\overline{\psi}, x_3, r, v_3) = G(\overline{\psi}, x_3, r, v_3) = G^0(\overline{\psi}, x_3).$$

Coming back to (63) we obtain

$$ {}^\perp\overline{u} \cdot \frac{G^0(\overline{\psi} + h\overline{u}, x_3) - G^0(\overline{\psi}, x_3)}{h} = 0$$

and we deduce by Lemma A.1 that

$$\nabla_{\overline{\psi}} g(\overline{\psi}, x_3, r, v_3) = G^0(\overline{\psi}, x_3) = \alpha(x_3)\overline{\psi} + \beta(x_3) = \nabla_{\overline{\psi}} \left\{\alpha(x_3) \frac{|\overline{\psi}|^2}{2} + \beta(x_3) \cdot \overline{\psi}\right\}.$$

Finally one gets

$$f(x,v) = \exp\big(g\big(\overline{\psi}(x,v), x_3, |\overline{v}|, v_3\big)\big)$$
$$= \exp\left(\frac{\alpha(x_3)}{2}\left|\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}\right|^2 + \beta(x_3)\cdot\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}\right) + \lambda(x_3, |\overline{v}|, v_3)\right). \qquad \square$$

Proof of Proposition 3.3. We introduce the field $b^4 \cdot \nabla_{x,v} = -\frac{{}^\perp\overline{v}}{\omega_c|\overline{v}|}\cdot\nabla_{\overline{x}} + \frac{\overline{v}}{|\overline{v}|}\cdot\nabla_{\overline{v}}$. We have

$$\xi^3\cdot\nabla\ln f = -\{\sigma\chi\}^{1/2}\frac{r'\sin\varphi}{|z|}b^4\cdot\nabla_{x,v}\ln f$$

and

$$\big(\xi^3\big)'\cdot\nabla'\ln f' = -\{\sigma\chi\}^{1/2}\frac{r\sin\varphi}{|z|}\big(b^4\big)'\cdot\nabla_{x',v'}\ln f'.$$

Thanks to Proposition 3.1 we have

$$\ln f(x,v) = g\left(\psi_1 = x_1 + \frac{v_2}{\omega_c}, \psi_2 = x_2 - \frac{v_1}{\omega_c}, x_3, r = |\overline{v}|, v_3\right)$$

and by direct computations one gets

$$\nabla_{\overline{x}}\ln f(x,v) = \nabla_{\overline{\psi}}g(\overline{\psi}, x_3, |\overline{v}|, v_3), \qquad \nabla_{\overline{v}}\ln f = -\frac{{}^\perp\nabla_{\overline{\psi}}g}{\omega_c} + \frac{\overline{v}}{|\overline{v}|}\partial_r g.$$

Therefore $b^4\cdot\nabla_{x,v}$ is the derivative with respect to $r = |\overline{v}|$

$$b^4\cdot\nabla_{x,v}\ln f = -\frac{{}^\perp\overline{v}}{\omega_c|\overline{v}|}\cdot\nabla_{\overline{\psi}}g + \frac{\overline{v}}{|\overline{v}|}\cdot\left(-\frac{{}^\perp\nabla_{\overline{\psi}}g}{\omega_c} + \frac{\overline{v}}{|\overline{v}|}\partial_r g\right) = \partial_r g$$

and (31) reduces to

$$\frac{\partial_r g(\overline{\psi}, x_3, r, v_3)}{r} = \frac{\partial_{r'}g'(\overline{\psi}', x_3, r', v_3')}{r'}, \qquad \frac{|r-r'|}{|\omega_c|} < \big|\overline{\psi} - \overline{\psi}'\big| < \frac{(r+r')}{|\omega_c|}.$$

Replacing $(\overline{\psi}', r', v_3')$ by small perturbations of $(\overline{\psi}, r, v_3)$ such that $|r-r'|/|\omega_c| < |\overline{\psi} - \overline{\psi}'| < (r+r')/|\omega_c|$ hold true, we deduce immediately that $\frac{\partial_r g}{r}$ depends only on x_3 and thus

$$\partial_r g(\overline{\psi}, x_3, r, v_3) = r\gamma(x_3).$$

By Proposition 3.2 we know that

$$g = \ln f = \alpha(x_3)\frac{|\overline{\psi}|^2}{2} + \beta(x_3)\cdot\overline{\psi} + \lambda(x_3, r, v_3)$$

implying that $\partial_r\lambda = r\gamma(x_3)$. Finally $\lambda(x_3, r, v_3) = \gamma(x_3)\frac{r^2}{2} + \mu(x_3, v_3)$ saying that

$$f(x,v) = \exp\left(\frac{\alpha(x_3)}{2}\left|\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}\right|^2 + \beta(x_3)\cdot\left(\overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}\right) + \gamma(x_3)\frac{|\overline{v}|^2}{2} + \mu(x_3, v_3)\right). \qquad \square$$

Proof of Proposition 3.4. The formula of the vector field ξ^4 allows us to write

$$\xi^4\cdot\nabla\ln f = -\{\sigma\chi\}^{1/2}\frac{(r'\cos\varphi - r)(v_3 - v_3')}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}}b^4\cdot\nabla_{x,v}\ln f$$
$$+ \{\sigma\chi\}^{1/2}\frac{v_3 - v_3'}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}}\frac{z}{\omega_c}\cdot\nabla_{\overline{x}}\ln f - \frac{\{\sigma\chi\}^{1/2}|z|}{\sqrt{|z|^2 + (v_3 - v_3')^2}}\partial_{v_3}\ln f$$

and

$$\left(\xi^4\right)' \cdot \nabla' \ln f' = -\{\sigma\chi\}^{1/2} \frac{(r\cos\varphi - r')(v_3' - v_3)}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} \left(b^4\right)' \cdot \nabla_{x',v'} \ln f' + \{\sigma\chi\}^{1/2} \frac{v_3 - v_3'}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} \frac{z}{\omega_c} \cdot \nabla_{\overline{x'}} \ln f' - \frac{\{\sigma\chi\}^{1/2}|z|}{\sqrt{|z|^2 + (v_3 - v_3')^2}} \partial_{v_3'} \ln f'.$$

By Proposition 3.1 we have

$$\ln f(x,v) = g\left(\overline{\psi}(x,v), x_3, |\overline{v}|, v_3\right)$$

and by Proposition 3.3 we know that

$$\frac{b^4 \cdot \nabla_{x,v} \ln f}{|\overline{v}|} = \frac{\partial_r g}{r} = \gamma(x_3).$$

Therefore (32) reduces to

$$\gamma(x_3)\left(v_3 - v_3'\right)\left[r\left(r - r'\cos\varphi\right) + r'\left(r' - r\cos\varphi\right)\right] + \frac{v_3 - v_3'}{\omega_c} z \cdot \left(\nabla_{\overline{x}} \ln f - \nabla_{\overline{x'}} \ln f'\right) - |z|^2\left(\partial_{v_3} \ln f - \partial_{v_3'} \ln f'\right) = 0$$

when $|r - r'| < |z| < r + r'$. Taking into account that

$$r\left(r - r'\cos\varphi\right) + r'\left(r' - r\cos\varphi\right) = r^2 + \left(r'\right)^2 - 2rr'\cos\varphi = |z|^2$$

we obtain for any $|r - r'| < |z| < r + r'$

$$\gamma(x_3)\left(v_3 - v_3'\right) + \frac{v_3 - v_3'}{\omega_c|z|} \frac{z}{|z|} \cdot \left(\nabla_{\overline{x}} \ln f - \nabla_{\overline{x'}} \ln f'\right) = \partial_{v_3} \ln f - \partial_{v_3'} \ln f'. \tag{64}$$

But $\nabla_{\overline{x}} \ln f = \nabla_{\overline{\psi}} g = \alpha(x_3)\overline{\psi}(x,v) + \beta(x_3)$, implying that

$$\frac{z}{\omega_c|z|^2} \cdot \left(\nabla_{\overline{x}} \ln f - \nabla_{\overline{x'}} \ln f'\right) = \frac{\alpha(x_3)}{\omega_c^2}$$

and therefore (64) is equivalent to

$$\partial_{v_3} g(\overline{\psi}, x_3, r, v_3) - \partial_{v_3'} g\left(\overline{\psi'}, x_3, r', v_3'\right) = \left(v_3 - v_3'\right)\left[\gamma(x_3) + \frac{\alpha(x_3)}{\omega_c^2}\right], \qquad \frac{|r - r'|}{|\omega_c|} < \left|\overline{\psi} - \overline{\psi'}\right| < \frac{r + r'}{|\omega_c|}.$$

We introduce the function $G(\overline{\psi}, x_3, r, v_3) = \partial_{v_3} g(\overline{\psi}, x_3, r, v_3)$ and let us consider $h, s \in \mathbb{R}^\star$, $\overline{u} \in \mathbb{R}^2 \setminus \{(0,0)\}$, $u_3 \in \mathbb{R}$

$$\overline{\psi'} = \overline{\psi} + h\overline{u}, \qquad r' = r + hs, \qquad v_3' = v_3 + hu_3$$

such that

$$\frac{|h||s|}{|\omega_c|} < |h||\overline{u}| < \frac{2r + hs}{|\omega_c|}.$$

We deduce that

$$\frac{G(\overline{\psi} + h\overline{u}, x_3, r + hs, v_3 + hu_3) - G(\overline{\psi}, x_3, r, v_3)}{h} = u_3\left(\gamma(x_3) + \frac{\alpha(x_3)}{\omega_c^2}\right)$$

which implies

$$\nabla_{\overline{\psi}} G \cdot \overline{u} + \partial_r G s + \left(\partial_{v_3} G - \gamma(x_3) - \frac{\alpha(x_3)}{\omega_c^2}\right) u_3 = 0, \qquad \frac{|s|}{|\omega_c|} < |\overline{u}|$$

saying that

$$\nabla_{\overline{\psi}} G = (0,0), \qquad \partial_r G = 0, \qquad \partial_{v_3} G = \gamma(x_3) + \frac{\alpha(x_3)}{\omega_c^2}$$

and

$$\partial_{v_3} g(\overline{\psi}, x_3, r, v_3) = G(\overline{\psi}, x_3, r, v_3) = \left(\gamma(x_3) + \frac{\alpha(x_3)}{\omega_c^2}\right) v_3 + \delta(x_3).$$

The previous equality allows us to determine the function $\mu = \mu(x_3, v_3)$ in the expression of $g = \ln f$

$$g(\overline{\psi}, x_3, r, v_3) = \alpha(x_3)\frac{|\overline{\psi}|^2}{2} + \beta(x_3)\cdot\overline{\psi} + \gamma(x_3)\frac{|\overline{v}|^2}{2} + \mu(x_3, v_3).$$

Taking the derivative with respect to v_3 yields

$$\left(\gamma(x_3) + \frac{\alpha(x_3)}{\omega_c^2}\right) v_3 + \delta(x_3) = \partial_{v_3} g = \partial_{v_3}\mu$$

and therefore

$$\mu(x_3, v_3) = \left(\gamma(x_3) + \frac{\alpha(x_3)}{\omega_c^2}\right)\frac{(v_3)^2}{2} + \delta(x_3) v_3 + \eta(x_3). \qquad \square$$

Proof of Lemma 4.1. We have

$$v_3(\omega_c\overline{x} + {}^\perp\overline{v}) f = u_3(\omega_c\overline{x} + {}^\perp\overline{v}) f + \rho\mathcal{M}_{\frac{\mu\theta}{\mu-\theta}}(\overline{v})(v_3 - u_3)\mathcal{M}_\theta(v_3 - u_3)\omega_c^2\mathcal{M}_\mu(\omega_c\overline{x} + {}^\perp\overline{v} - \overline{u})$$

and thus

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3(\omega_c\overline{x} + {}^\perp\overline{v}) f(x,v)\,\mathrm{d}v\mathrm{d}\overline{x} = \rho u_3\overline{u}.$$

It is easily seen, thanks to (36), that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (v_3)^2 f\,\mathrm{d}v\mathrm{d}\overline{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (v_3 - u_3 + u_3)^2 f\,\mathrm{d}v\mathrm{d}\overline{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (v_3 - u_3)^2 f\,\mathrm{d}v\mathrm{d}\overline{x} + \rho(u_3)^2 = \rho\big((u_3)^2 + \theta\big).$$

Clearly we have, integrating by parts

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (\omega_c\overline{x} + {}^\perp\overline{v})\partial_{v_3} f\,\mathrm{d}v\mathrm{d}\overline{x} = (0,0), \qquad \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3\partial_{v_3} f\,\mathrm{d}v\mathrm{d}\overline{x} = -\rho. \qquad \square$$

Proof of Lemma 4.2. Clearly

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (v_3 - u_3)\frac{|\overline{v}|^2 + (v_3 - u_3)^2}{2} f(x,v)\,\mathrm{d}v\mathrm{d}\overline{x} = 0$$

and thus (37) yields

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3\frac{|\overline{v}|^2 + (v_3 - u_3)^2}{2} f(x,v)\,\mathrm{d}v\mathrm{d}\overline{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} u_3\frac{|\overline{v}|^2 + (v_3 - u_3)^2}{2} f(x,v)\,\mathrm{d}v\mathrm{d}\overline{x} = \rho u_3\left(\frac{\mu\theta}{\mu-\theta} + \frac{\theta}{2}\right).$$

Similarly, thanks to (38) we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3\frac{|\omega_c\overline{x} + {}^\perp\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2} f(x,v)\,\mathrm{d}v\mathrm{d}\overline{x} &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} u_3\frac{|\omega_c\overline{x} + {}^\perp\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2} f(x,v)\,\mathrm{d}v\mathrm{d}\overline{x} \\ &= \rho u_3\left(\mu - \frac{\mu\theta}{\mu-\theta}\right).\end{aligned}$$

It is easily seen that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 \frac{|\overline{v}|^2 + (v_3 - u_3)^2}{2} \partial_{x_3} f(x,v)\,\mathrm{d}v\mathrm{d}\overline{x} - \partial_{x_3} \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 \frac{|\overline{v}|^2 + (v_3 - u_3)^2}{2} f\,\mathrm{d}v\mathrm{d}\overline{x}$$

$$= -\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3(u_3 - v_3)\partial_{x_3} u_3 f\,\mathrm{d}v\mathrm{d}\overline{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (v_3 - u_3)^2 f(x,v)\,\mathrm{d}v\mathrm{d}\overline{x}\partial_{x_3} u_3 = \rho\theta\partial_{x_3} u_3$$

and

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 \frac{|\omega_c \overline{x} + {}^\perp\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2} \partial_{x_3} f\,\mathrm{d}v\mathrm{d}\overline{x} - \partial_{x_3} \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\omega_c \overline{x} + {}^\perp\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2} f\,\mathrm{d}v\mathrm{d}\overline{x}$$

$$= -\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3(\overline{u} - \omega_c\overline{x} - {}^\perp\overline{v}) \cdot \partial_{x_3}\overline{u} f\,\mathrm{d}v\mathrm{d}\overline{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (v_3 - u_3)(\omega_c\overline{x} + {}^\perp\overline{v} - \overline{u}) \cdot \partial_{x_3}\overline{u} f\,\mathrm{d}v\mathrm{d}\overline{x} = 0.$$

Therefore we obtain

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 \frac{|\overline{v}|^2 + (v_3 - u_3)^2}{2} \partial_{x_3} f\,\mathrm{d}v\mathrm{d}\overline{x} = \partial_{x_3}\left[\rho u_3\left(\frac{\mu\theta}{\mu - \theta} + \frac{\theta}{2}\right)\right] + \rho\theta\partial_{x_3} u_3$$

and

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 \frac{|\omega_c \overline{x} + {}^\perp\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2} \partial_{x_3} f\,\mathrm{d}v\mathrm{d}\overline{x} = \partial_{x_3}\left[\rho u_3\left(\mu - \frac{\mu\theta}{\mu - \theta}\right)\right]. \qquad \square$$

Proof of Lemma 4.3. By direct computation one gets

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} f \ln f\,\mathrm{d}v\mathrm{d}\overline{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \left[\ln\left(\frac{\rho\omega_c^2}{(2\pi)^{5/2}\frac{\mu^2\theta^{3/2}}{\mu-\theta}}\right) - \frac{|\overline{v}|^2 + (v_3 - u_3)^2}{2\theta} - \frac{|\omega_c \overline{x} + {}^\perp\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2\mu}\right] f\,\mathrm{d}v\mathrm{d}\overline{x}$$

$$= \rho\ln\left(\frac{\rho\omega_c^2}{(2\pi)^{5/2}\frac{\mu^2\theta^{3/2}}{\mu-\theta}}\right) - \frac{\rho}{\theta}\left(\frac{\mu\theta}{\mu - \theta} + \frac{\theta}{2}\right) - \frac{\rho}{\mu}\left(\mu - \frac{\mu\theta}{\mu - \theta}\right)$$

$$= \rho\ln\left(\frac{\rho\omega_c^2}{(2\pi)^{5/2}\frac{\mu^2\theta^{3/2}}{\mu-\theta}}\right) - \frac{5}{2}\rho$$

and

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} v_3 f \ln f\,\mathrm{d}v\mathrm{d}\overline{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (v_3 - u_3) f \ln f\,\mathrm{d}v\mathrm{d}\overline{x} + \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} u_3 f \ln f\,\mathrm{d}v\mathrm{d}\overline{x} = u_3 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} f \ln f\,\mathrm{d}v\mathrm{d}\overline{x}$$

$$= \rho u_3 \ln\left(\frac{\rho\omega_c^2}{(2\pi)^{5/2}\frac{\mu^2\theta^{3/2}}{\mu-\theta}}\right) - \frac{5}{2}\rho u_3. \qquad \square$$

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