

A semilinear singular Sturm–Liouville equation involving measure data [☆]

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Abstract

Given $\alpha > 0$ and $p > 1$, let μ be a bounded Radon measure on the interval $(-1, 1)$. We are interested in the equation $-(|x|^{2\alpha}u')' + |u|^{p-1}u = \mu$ on $(-1, 1)$ with boundary condition $u(-1) = u(1) = 0$. We establish some existence and uniqueness results. We examine the limiting behavior of three approximation schemes. The isolated singularity at 0 is also investigated. © 2015 L'Association Publications de l'Institut Henri Poincaré. Published by Elsevier B.V. All rights reserved.

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1. Introduction

In this paper, we consider the following semilinear singular Sturm–Liouville equation

$$\begin{cases} -(|x|^{2\alpha}u')' + |u|^{p-1}u = \mu & \text{on } (-1, 1), \\ u(-1) = u(1) = 0. \end{cases} \quad (1.1)$$

Here we assume that $\alpha > 0$, $p > 1$, and $\mu \in \mathcal{M}(-1, 1)$, where $\mathcal{M}(-1, 1)$ is the space of bounded Radon measures on the interval $(-1, 1)$. We denote

$$C_0[-1, 1] = \{\zeta \in C[-1, 1]; \zeta(-1) = \zeta(1) = 0\}.$$

Then μ can be viewed as a bounded linear functional on $C_0[-1, 1]$. That is,

$$\mathcal{M}(-1, 1) = (C_0[-1, 1])^*.$$

In the previous work [23], we studied the corresponding linear equation (i.e., $p = 1$ in (1.1)). For the linear case, we defined a notion of *solution* for all $\alpha > 0$ and a notion of *good solution* for $0 < \alpha < 1$. We proved the existence and

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uniqueness of the good solution for every measure μ when $0 < \alpha < 1$ and we proved the uniqueness of the solution when $\alpha \geq 1$. We also presented a necessary and sufficient condition on μ for the existence of the solution when $\alpha \geq 1$.

For the semilinear equation (1.1), we can adapt from [23] the notion of solution and the notion of good solution. Rewrite (1.1) as $-(|x|^{2\alpha}u')' + u = u - |u|^{p-1}u + \mu$. Then according to [23], a function u is a solution of (1.1) if

$$u \in L^p(-1, 1) \cap W_{loc}^{1,1}([-1, 1] \setminus \{0\}), \quad |x|^{2\alpha}u' \in BV(-1, 1), \tag{1.2}$$

and u satisfies (1.1) in the usual sense (i.e., in the sense of measures). When $0 < \alpha < 1$, a solution u of (1.1) is called a good solution if it satisfies in addition

$$\begin{cases} \lim_{x \rightarrow 0^+} u(x) = \lim_{x \rightarrow 0^-} u(x), \text{ when } 0 < \alpha < \frac{1}{2}, \\ \lim_{x \rightarrow 0^+} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u(x) = \lim_{x \rightarrow 0^-} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u(x), \text{ when } \alpha = \frac{1}{2}, \\ \lim_{x \rightarrow 0^+} |x|^{2\alpha-1}u(x) = \lim_{x \rightarrow 0^-} |x|^{2\alpha-1}u(x), \text{ when } \frac{1}{2} < \alpha < 1. \end{cases} \tag{1.3}$$

In this work, we are interested in the question of existence and uniqueness, the limiting behavior of three different approximation schemes, and the classification of the isolated singularity at 0.

It turns out that we need to investigate the following four cases separately:

$$0 < \alpha \leq \frac{1}{2}, \quad p > 1, \tag{1.4}$$

$$\frac{1}{2} < \alpha < 1, \quad 1 < p < \frac{1}{2\alpha - 1}, \tag{1.5}$$

$$\frac{1}{2} < \alpha < 1, \quad p \geq \frac{1}{2\alpha - 1}, \tag{1.6}$$

$$\alpha \geq 1, \quad p > 1. \tag{1.7}$$

As we are going to see, the notion of good solution is only necessary for cases (1.4) and (1.5). In fact, for the case (1.6), if the solution exists, it must be the good solution.

Our first result concerns the question of uniqueness.

Theorem 1.1. *If α and p satisfy (1.4) or (1.5), then for every $\mu \in \mathcal{M}(-1, 1)$ there exists at most one good solution of (1.1). If α and p satisfy (1.6) or (1.7), then for every $\mu \in \mathcal{M}(-1, 1)$ there exists at most one solution of (1.1).*

Remark 1.1. In fact, for α and p satisfying (1.4) or (1.5), there exist infinitely many solutions of (1.1); all of them will be identified in Section 7.

The next two theorems answer the question of existence.

Theorem 1.2. *Assume that α and p satisfy (1.4) or (1.5). For every $\mu \in \mathcal{M}(-1, 1)$, there exists a (unique) good solution of (1.1). Moreover, the good solution satisfies*

- (i) $\lim_{x \rightarrow 0} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u(x) = - \lim_{x \rightarrow 0^+} |x|u'(x) = \lim_{x \rightarrow 0^-} |x|u'(x) = \frac{\mu(\{0\})}{2}$ when $\alpha = \frac{1}{2}$ and $p > 1$,
- (ii) $\lim_{x \rightarrow 0} |x|^{2\alpha-1}u(x) = - \lim_{x \rightarrow 0^+} \frac{|x|^{2\alpha}u'(x)}{2\alpha-1} = \lim_{x \rightarrow 0^-} \frac{|x|^{2\alpha}u'(x)}{2\alpha-1} = \frac{\mu(\{0\})}{4\alpha-2}$ when $\frac{1}{2} < \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$,
- (iii) $\| |u|^{p-1}u - |\hat{u}|^{p-1}\hat{u} \|_{L^1} \leq \| \mu - \hat{\mu} \|_{\mathcal{M}}$ and $\| (|u|^{p-1}u - |\hat{u}|^{p-1}\hat{u})^+ \|_{L^1} \leq \| (\mu - \hat{\mu})^+ \|_{\mathcal{M}}$, for $\mu, \hat{\mu} \in \mathcal{M}(-1, 1)$ and their corresponding good solutions u, \hat{u} .

Theorem 1.3. *Assume that α and p satisfy (1.6) or (1.7). For each $\mu \in \mathcal{M}(-1, 1)$, there exists a (unique) solution of (1.1) if and only if $\mu(\{0\}) = 0$. Moreover, if the solution exists, it satisfies*

- (i) $\lim_{x \rightarrow 0} |x|^{2\alpha-1} u(x) = \lim_{x \rightarrow 0} |x|^{2\alpha} u'(x) = 0,$
- (ii) $\| |u|^{p-1} u - |\hat{u}|^{p-1} \hat{u} \|_{L^1} \leq \| \mu - \hat{\mu} \|_{\mathcal{M}}$ and $\| (|u|^{p-1} u - |\hat{u}|^{p-1} \hat{u})^+ \|_{L^1} \leq \| (\mu - \hat{\mu})^+ \|_{\mathcal{M}},$ for $\mu, \hat{\mu} \in \mathcal{M}(-1, 1)$ and their corresponding solutions $u, \hat{u}.$

We now study (1.1) by three different approximation schemes. The first one is the elliptic regularization. Take $0 < \epsilon < 1$ and consider the following regularized equation

$$\begin{cases} -(|x| + \epsilon)^{2\alpha} u'_\epsilon + |u_\epsilon|^{p-1} u_\epsilon = \mu & \text{on } (-1, 1), \\ u_\epsilon(-1) = u_\epsilon(1) = 0. \end{cases} \tag{1.8}$$

Given $\alpha > 0, p > 1$ and $\mu \in \mathcal{M}(-1, 1),$ note that the existence of $u_\epsilon \in H_0^1(-1, 1)$ with $u'_\epsilon \in BV(-1, 1)$ is guaranteed by minimizing the corresponding functional, and the uniqueness of u_ϵ is also standard. Our main results are the following two theorems.

Theorem 1.4. Assume that α and p satisfy (1.4) or (1.5). Then as $\epsilon \rightarrow 0, u_\epsilon \rightarrow u$ uniformly on every compact subset of $[-1, 1] \setminus \{0\},$ where u is the unique good solution of (1.1).

Theorem 1.5. Assume that α and p satisfy (1.6) or (1.7). Denote by δ_0 the Dirac mass at 0. Then as $\epsilon \rightarrow 0, u_\epsilon \rightarrow u$ uniformly on every compact subset of $[-1, 1] \setminus \{0\},$ where u is the unique solution of

$$\begin{cases} -(|x|^{2\alpha} u')' + |u|^{p-1} u = \mu - \mu(\{0\}) \delta_0 & \text{on } (-1, 1), \\ u(-1) = u(1) = 0. \end{cases} \tag{1.9}$$

Remark 1.2. In Section 4 we will present further results about the mode of convergence in Theorems 1.4 and 1.5.

The second approximation scheme consists of truncating the nonlinear term. Fix $p > 1$ and $n \in \mathbb{N}.$ Define $g_{p,n} : \mathbb{R} \rightarrow \mathbb{R}$ as

$$g_{p,n}(t) = (\text{sign } t) \min \left\{ |t|^p, n^{1-\frac{1}{p}} |t| \right\}. \tag{1.10}$$

It is clear that

$$\begin{aligned} 0 \leq g_{p,1}(t) \leq g_{p,2}(t) \leq \dots \leq |t|^{p-1} t, \quad \forall t > 0, \\ |t|^{p-1} t \leq \dots \leq g_{p,2}(t) \leq g_{p,1}(t) \leq 0, \quad \forall t < 0, \\ g_{p,n}(t) \rightarrow |t|^{p-1} t, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Consider the equation

$$\begin{cases} -(|x|^{2\alpha} u'_n)' + g_{p,n}(u_n) = \mu & \text{on } (-1, 1), \\ u_n(-1) = u_n(1) = 0. \end{cases} \tag{1.11}$$

Rewrite (1.11) as $-(|x|^{2\alpha} u'_n)' + u_n = u_n - g_{p,n}(u_n) + \mu.$ Then according to [23], a function u_n is a solution of (1.11) if

$$u_n \in L^1(-1, 1) \cap W_{loc}^{1,1}([-1, 1] \setminus \{0\}), \quad |x|^{2\alpha} u'_n \in BV(-1, 1),$$

and u satisfies (1.11) in the usual sense. When $0 < \alpha < 1,$ a solution u_n of (1.11) is called a good solution if it satisfies in addition (1.3).

We will see in Section 5 that when $0 < \alpha < 1,$ for all $p > 1$ and $n \in \mathbb{N},$ there exists a unique good solution u_n of (1.11). When $\alpha \geq 1,$ for all $p > 1$ and $n \in \mathbb{N},$ there exists a unique solution u_n of (1.11) if and only if $\mu(\{0\}) = 0.$

We have the following results concerning the sequence $\{u_n\}_{n=1}^\infty.$

Theorem 1.6. Assume that α and p satisfy (1.4) or (1.5). Then as $n \rightarrow \infty, u_n \rightarrow u$ uniformly on every compact subset of $[-1, 1] \setminus \{0\},$ where u is the unique good solution of (1.1).

Theorem 1.7. Assume that α and p satisfy (1.6). Then as $n \rightarrow \infty$, $u_n \rightarrow u$ uniformly on every compact subset of $[-1, 1] \setminus \{0\}$, where u is the unique solution of (1.9).

Theorem 1.8. Assume that α and p satisfy (1.7) and $\mu(\{0\}) = 0$. Then as $n \rightarrow \infty$, $u_n \rightarrow u$ uniformly on every compact subset of $[-1, 1] \setminus \{0\}$, where u is the unique solution of (1.9).

Remark 1.3. The more precise mode of convergence in Theorems 1.6, 1.7 and 1.8 will be presented in Section 5.

Remark 1.4. The third approximation scheme consists of approximating the measure μ by a sequence of L^1 -functions under the weak-star topology. This is a delicate subject. For example, for $\frac{1}{2} \leq \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$, let $\mu = \delta_0$ and $f_n = Cn\rho(nx - 1)$, where $\rho(x) = \chi_{\{|x|<1\}} e^{\frac{1}{|x|^{2-1}}}$ and $C^{-1} = \int \rho$, so that $f_n \xrightarrow{*} \delta_0$ in $(C_0[-1, 1])^*$. Let u_n be the good solution corresponding to f_n . Then $u_n \rightarrow u$ but u is not the good solution corresponding to δ_0 . This subject will be discussed in Section 6.

Finally, we study the isolated singularity at 0. The next result asserts that for α and p satisfying (1.6) or (1.7), the isolated singularity at 0 is removable.

Theorem 1.9. Assume that α and p satisfy (1.6) or (1.7). Given $f \in L^1(-1, 1)$, assume that $u \in L^p_{loc}((-1, 1) \setminus \{0\})$ satisfying

$$-\int_{-1}^1 u(|x|^{2\alpha} \zeta')' dx + \int_{-1}^1 |u|^{p-1} u \zeta dx = \int_{-1}^1 f \zeta dx, \quad \forall \zeta \in C_c^\infty((-1, 1) \setminus \{0\}).$$

Then $u \in L^p_{loc}(-1, 1)$ and

$$-\int_{-1}^1 u(|x|^{2\alpha} \zeta')' dx + \int_{-1}^1 |u|^{p-1} u \zeta dx = \int_{-1}^1 f \zeta dx, \quad \forall \zeta \in C_c^\infty(-1, 1). \tag{1.12}$$

Remark 1.5. An easy consequence of Theorem 1.9 is that (1.1) does not have a solution if α and p satisfy (1.6) or (1.7) and $\mu = \delta_0$, which is a special case of Theorem 1.3.

On the other hand, for α and p satisfying (1.4) or (1.5), the isolated singularity at 0 is not removable. In this case, we give a complete classification of the asymptotic behavior of the solutions.

Theorem 1.10. Assume that α and p satisfy (1.4) or (1.5). Let $u \in C^2(0, 1]$ be such that

$$\begin{cases} -(x^{2\alpha} u')' + |u|^{p-1} u = 0 & \text{on } (0, 1), \\ u(1) = 0. \end{cases} \tag{1.13}$$

Then one of the following assertions holds.

- (i) $u \equiv 0$.
- (ii) $u \equiv u_c$ for some constant $c \in (-\infty, 0) \cup (0, +\infty)$, where u_c is the unique solution of (1.13) such that

$$\lim_{x \rightarrow 0^+} \frac{u_c(x)}{E_\alpha(x)} = c, \tag{1.14}$$

and

$$E_\alpha(x) = \begin{cases} 1, & \text{if } 0 < \alpha < \frac{1}{2}, \\ \ln \frac{1}{x}, & \text{if } \alpha = \frac{1}{2}, \\ \frac{1}{x^{2\alpha-1}}, & \text{if } \frac{1}{2} < \alpha < 1. \end{cases} \tag{1.15}$$

(iii) $u \equiv u_{+\infty}$, where $u_{+\infty}$ is the unique solution of (1.13) such that

$$\lim_{x \rightarrow 0^+} x^{\frac{2(1-\alpha)}{p-1}} u_{+\infty}(x) = l_{p,\alpha}, \tag{1.16}$$

and

$$l_{p,\alpha} = \left[(1-\alpha)^2 \left(\frac{2}{p-1} \right) \left(\frac{2p}{p-1} - \frac{1}{1-\alpha} \right) \right]^{\frac{1}{p-1}}. \tag{1.17}$$

(iv) $u \equiv u_{-\infty}$ where $u_{-\infty} = -u_{+\infty}$.

Moreover, $u_{-c} = -u_c$. If $c > 0$ or $c = +\infty$, $u_c \geq 0$. For $c > 0$, $u_c \downarrow 0$ and $u_c \uparrow u_{+\infty}$.

Remark 1.6. The solutions $u_{+\infty}$ and $u_{-\infty}$ are called the *very singular solutions*, which is a terminology introduced by Brezis, Peletier and Terman [8].

Remark 1.7. Given $\mu \in \mathcal{M}(0, 1)$, we can also study the following equation,

$$\begin{cases} -(x^{2\alpha} u')' + |u|^{p-1} u = \mu & \text{on } (0, 1), \\ u(1) = 0. \end{cases} \tag{1.18}$$

In Section 10, we discuss (1.18) under appropriate boundary conditions at 0, and we will also compare the results about (1.18) with some well-known results about the semilinear elliptic equation.

Our study of (1.1) is motivated by various results about the semilinear elliptic equation

$$\begin{cases} -\Delta u + |u|^{p-1} u = \mu & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.19}$$

where $1 < p < \infty$, Ω is a bounded smooth domain in \mathbb{R}^N and μ is a bounded Radon measure on Ω .

The existence and uniqueness of an L^p -solution of (1.19) for all $1 < p < \infty$ and $\mu \in L^1(\Omega)$ is proved by Brezis and Strauss [9]. When μ is just a bounded Radon measure, the following two cases were studied separately:

- (i) $1 < p < \frac{N}{N-2}$ if $N \geq 3$ and no restriction on p if $N = 1, 2$,
- (ii) $p \geq \frac{N}{N-2}$ if $N \geq 3$.

Bénilan and Brezis proved the existence and uniqueness for case (i) and the nonexistence for case (ii) if $\mu = \delta_a$ for some $a \in \Omega$ (see, e.g., [2] and the references therein). For case (ii), a necessary and sufficient condition on μ for the existence of a solution was given by Baras and Pierre [1] (see an equivalent characterization by Gallouët and Morel [17]).

About the isolated (interior) singularity, Brezis and Véron [10] proved that the isolated singularity is removable for case (ii). For case (i), Véron [20] classified the asymptotic behavior of the solutions near the isolated singularity (a different proof was given by Brezis and Oswald [7]).

Brezis [5] observed that, for case (ii) with $\mu = \delta_a$ where $a \in \Omega$, a sequence of approximate solutions may converge to 0, which is obviously not the solution corresponding to $\mu = \delta_a$. This phenomenon was then studied by Brezis, Marcus and Ponce [6] in a more general setting.

We refer to Appendix A of Bénilan and Brezis [2] for a comprehensive review on this subject, and to the monographs of Véron [21,22] for a variety of results about the singularities of solutions for more general classes of PDEs.

The rest of this paper is organized as follows. We present in Section 2 some preliminary results which in particular imply Theorem 1.1. The question of existence is studied in Section 3 where Theorems 1.2 and 1.3 are proved. The three approximation schemes mentioned in the introduction will be investigated respectively in Sections 4, 5 and 6. In Section 7, we describe all the solutions of (1.1) when α and p satisfy (1.4) or (1.5). The removability of the singularity is studied in Section 8 and the classification of the singularity is studied in Section 9. Finally, Section 10 is devoted to (1.18).

2. Preliminary results and the uniqueness

We start with a few results from [23] about the linear operator. The investigation of the linear operator can also be found in [11,12]. We consider the unbounded linear operator $A_\alpha : D(A_\alpha) \subset L^1(-1, 1) \rightarrow L^1(-1, 1)$ where

$$A_\alpha u = -(|x|^{2\alpha} u)', \tag{2.1}$$

$$\tilde{D} = \left\{ u \in L^1(-1, 1) \cap W_{loc}^{2,1}([-1, 1] \setminus \{0\}); u(-1) = u(1) = 0, |x|^{2\alpha} u' \in W^{1,1}(-1, 1) \right\}, \tag{2.2}$$

and

$$D(A_\alpha) = \begin{cases} \tilde{D} \cap C[-1, 1], & \text{when } 0 < \alpha < \frac{1}{2}, \\ \tilde{D} \cap \left\{ u; \left(1 + \ln \frac{1}{|x|}\right)^{-1} u \in C[-1, 1] \right\}, & \text{when } \alpha = \frac{1}{2}, \\ \tilde{D} \cap \{u; |x|^{2\alpha-1} u \in C[-1, 1]\}, & \text{when } \frac{1}{2} < \alpha < 1, \\ \tilde{D}, & \text{when } \alpha \geq 1. \end{cases} \tag{2.3}$$

We have the following properties of the linear operator A_α .

Proposition 2.1. (See Lemma 2.2 in [23].) Assume $0 < \alpha < \frac{1}{2}$. For all $u \in D(A_\alpha)$ we have

$$\lim_{x \rightarrow 0} |x|^{2\alpha} u'(x) = \frac{1}{2} \int_0^1 (A_\alpha u) (1 - s^{1-2\alpha}) ds - \frac{1}{2} \int_{-1}^0 (A_\alpha u) (1 - |s|^{1-2\alpha}) ds, \tag{2.4}$$

$$u(0) = \frac{1}{2(1-2\alpha)} \int_{-1}^1 (A_\alpha u) (1 - |s|^{1-2\alpha}) ds, \tag{2.5}$$

$$\| |x|^{2\alpha} u' \|_{L^\infty} \leq \frac{3}{2} \| A_\alpha u \|_{L^1}, \tag{2.6}$$

$$\| u \|_{W^{1,1}} \leq \frac{6}{1-2\alpha} \| A_\alpha u \|_{L^1}. \tag{2.7}$$

Proposition 2.2. (See Lemma 2.3 in [23].) Assume $\alpha \geq \frac{1}{2}$. Then

$$D(A_\alpha) = \left\{ u \in \tilde{D}; \lim_{x \rightarrow 0} |x|^{2\alpha} u'(x) = 0 \right\}, \tag{2.8}$$

where \tilde{D} is defined by (2.2). For $u \in D(A_\alpha)$ we have

$$\| |x|^{2\alpha} u' \|_{L^\infty} \leq \| A_\alpha u \|_{L^1}, \text{ when } \alpha \geq \frac{1}{2}, \tag{2.9}$$

$$\lim_{x \rightarrow 0} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u(x) = 0, \text{ when } \alpha = \frac{1}{2}, \tag{2.10}$$

$$\left\| \left(1 + \ln \frac{1}{|x|}\right)^{-1} u \right\|_{W^{1,1}} \leq 4 \| A_{\frac{1}{2}} u \|_{L^1}, \text{ when } \alpha = \frac{1}{2}, \tag{2.11}$$

$$\lim_{x \rightarrow 0} |x|^{2\alpha-1} u(x) = 0, \text{ when } \alpha > \frac{1}{2}, \tag{2.12}$$

$$\| |x|^{2\alpha-1} u \|_{W^{1,1}} \leq \frac{4}{2\alpha-1} \| A_\alpha u \|_{L^1}, \text{ when } \alpha > \frac{1}{2}. \tag{2.13}$$

Proposition 2.3. (See Proposition 2.1 in [23].) The operator A_α satisfies the following properties.

- (i) For any $\alpha > 0$, the operator A_α is closed and its domain $D(A_\alpha)$ is dense in $L^1(-1, 1)$.
- (ii) For any $\lambda > 0$ and $\alpha > 0$, $I + \lambda A_\alpha$ maps $D(A_\alpha)$ one-to-one onto $L^1(-1, 1)$ and $(I + \lambda A_\alpha)^{-1}$ is a contraction in $L^1(-1, 1)$.
- (iii) For any $\lambda > 0$, $\alpha > 0$ and $f \in L^1(-1, 1)$, $\text{ess sup}(I + \lambda A_\alpha)^{-1} f \leq \max\{0, \text{ess sup } f\}$.
- (iv) Let γ be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ containing the origin. For any $\alpha > 0$, let $u \in D(A_\alpha)$ and $g \in L^\infty(-1, 1)$ be such that $g(x) \in \gamma(u(x))$ a.e. Then $\int_{-1}^1 A_\alpha u(x)g(x)dx \geq 0$.

We now prove the uniqueness result.

Proof of Theorem 1.1. Fix $\mu \in \mathcal{M}(-1, 1)$. If α and p satisfy (1.4) or (1.5), assume that u and \hat{u} are two good solutions of (1.1) corresponding to μ . Then $u - \hat{u} \in D(A_\alpha)$ and $A_\alpha(u - \hat{u}) = |\hat{u}|^{p-1}\hat{u} - |u|^{p-1}u$.

If α and p satisfy (1.6) or (1.7), assume that u and \hat{u} are two solutions of (1.1) corresponding to μ . Then $-(|x|^{2\alpha}(u - \hat{u})')' = |\hat{u}|^{p-1}\hat{u} - |u|^{p-1}u$. We claim that $u - \hat{u} \in D(A_\alpha)$. For $\alpha \geq 1$, it is clear by the definition of $D(A_\alpha)$. For $\frac{1}{2} < \alpha < 1$ and $p \geq \frac{1}{2\alpha-1}$, by (2.8), it is enough to show that $\lim_{x \rightarrow 0} |x|^{2\alpha}(u - \hat{u})'(x) = 0$. Indeed, since $|x|^{2\alpha}(u - \hat{u})' \in BV(-1, 1)$, the limits $\lim_{x \rightarrow 0^+} |x|^{2\alpha}(u - \hat{u})'(x)$ and $\lim_{x \rightarrow 0^-} |x|^{2\alpha}(u - \hat{u})'(x)$ exist. They have to be zero.

Otherwise, it contradicts the fact that $u - \hat{u} \in L^p(-1, 1)$ with $p \geq \frac{1}{2\alpha-1}$.

Then for all the cases, assertion (iv) of Proposition 2.3 implies that

$$\int_{-1}^1 (|\hat{u}|^{p-1}\hat{u} - |u|^{p-1}u) \text{sign}(u - \hat{u})dx = \int_{-1}^1 A_\alpha(u - \hat{u}) \text{sign}(u - \hat{u})dx \geq 0.$$

On the other hand, $(|\hat{u}|^{p-1}\hat{u} - |u|^{p-1}u) \text{sign}(u - \hat{u}) \leq 0$ a.e. Therefore $u = \hat{u}$ a.e. \square

3. Proof of the existence results

The basic idea in the proof of Theorems 1.2 and 1.3 is to approximate the measures by L^1 -functions. Therefore, we start with the case when $\mu \in L^1(-1, 1)$ in (1.1).

Proposition 3.1. For every $\alpha > 0$, $p > 1$ and $f \in L^1(-1, 1)$, there exists a unique $u \in D(A_\alpha) \cap L^p(-1, 1)$ such that $A_\alpha u + |u|^{p-1}u = f$ a.e. on $(-1, 1)$, where A_α and $D(A_\alpha)$ are given by (2.1) and (2.3) respectively. Moreover, $\| |u|^p \|_{L^1} \leq \|f\|_{L^1}$ and $\|A_\alpha u\|_{L^1} \leq 2\|f\|_{L^1}$.

To prove Proposition 3.1, we need the following result by Brezis and Strauss [9].

Lemma 3.2. (See Theorem 1 in [9].) Let β be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ which contains the origin. Let Ω be any measure space. Let A be an unbounded linear operator on $L^1(\Omega)$ satisfying the following conditions.

- (i) The operator A is closed with dense domain $D(A)$ in $L^1(\Omega)$; for any $\lambda > 0$, $I + \lambda A$ maps $D(A)$ one-to-one onto $L^1(\Omega)$ and $(I + \lambda A)^{-1}$ is a contraction in $L^1(\Omega)$.
- (ii) For any $\lambda > 0$ and $f \in L^1(\Omega)$, $\text{ess sup}_\Omega (I + \lambda A)^{-1} f \leq \max\left\{0, \text{ess sup}_\Omega f\right\}$.
- (iii) There exists $\delta > 0$ such that $\delta \|u\|_{L^1} \leq \|Au\|_{L^1}$, $\forall u \in D(A)$.

Then for every $f \in L^1(\Omega)$, there exists a unique $u \in D(A)$ such that $Au(x) + \beta(u(x)) \ni f(x)$ a.e. Moreover, $\|f - Au\|_{L^1} \leq \|f\|_{L^1}$ and $\|Au\|_{L^1} \leq 2\|f\|_{L^1}$.

We now prove Proposition 3.1. We apply a device by Gallouët and Morel [17].

Proof of Proposition 3.1. We first assume $0 < \alpha < 1$. Applying Proposition 2.3 and the estimates (2.7) and (2.13), we deduce that A_α is an unbounded operator satisfying the conditions (i)–(iii) in Lemma 3.2. Consider $\beta(u) = |u|^{p-1}u$ as a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$. Then Lemma 3.2 implies the desired result.

We then assume $\alpha \geq 1$. For any $n \in \mathbb{N}$, consider the unbounded linear operator

$$A_{\alpha,n}u = -(|x|^{2\alpha}u_n')' + \frac{1}{n}u.$$

Take its domain $D(A_{\alpha,n}) = D(A_\alpha)$. Note that

$$\begin{aligned} A_{\alpha,n} &= A_\alpha + \frac{1}{n}I, \\ \lambda A_{\alpha,n} + I &= \left(\frac{\lambda}{n} + 1\right) \left(\frac{\lambda n}{\lambda + n}A_\alpha + I\right), \\ (\lambda A_{\alpha,n} + I)^{-1} &= \left(\frac{\lambda n}{\lambda + n}A_\alpha + I\right)^{-1} \circ \frac{n}{\lambda + n}I. \end{aligned}$$

It is clear that $A_{\alpha,n}$ satisfies the conditions (i)–(iii) in Lemma 3.2. Therefore, for every $\alpha \geq 1$, $p > 1$, $n \in \mathbb{N}$, and $f \in L^1(-1, 1)$, there exists a unique $u_n \in D(A_\alpha) \cap L^p(-1, 1)$ such that

$$-(|x|^{2\alpha}u_n')' + \frac{1}{n}u_n + |u_n|^{p-1}u_n = f \text{ on } (-1, 1).$$

That is,

$$\int_{-1}^1 |x|^{2\alpha}u_n'\zeta' dx + \int_{-1}^1 \frac{1}{n}u_n\zeta dx + \int_{-1}^1 |u_n|^{p-1}u_n\zeta dx = \int_{-1}^1 f\zeta dx, \quad \forall \zeta \in C_0^1[-1, 1]. \tag{3.1}$$

Moreover, we have

$$\| |u_n|^p \|_{L^1} + \frac{1}{n} \|u_n\|_{L^1} + \| |x|^{2\alpha}u_n' \|_{L^\infty} + \| (|x|^{2\alpha}u_n')' \|_{L^1} \leq C,$$

where C is independent of n . Therefore, passing to a subsequence if necessary, we can assume that there exists $u \in W_{loc}^{1,1}([-1, 1] \setminus \{0\})$ such that $u_n(x) \rightarrow u(x)$, $\forall x \in [-1, 1] \setminus \{0\}$, and $|x|^{2\alpha}u_n' \rightarrow |x|^{2\alpha}u'$ in $L^1(-1, 1)$. It implies that $u(-1) = u(1) = 0$ and $\frac{1}{n}u_n + |u_n|^{p-1}u_n \rightarrow |u|^{p-1}u$ a.e. on $(-1, 1)$.

We now prove that the sequence $\left\{ \frac{1}{n}u_n + |u_n|^{p-1}u_n \right\}_{n=1}^\infty$ is equi-integrable. For this purpose, take a nondecreasing function $\varphi(x) \in C^\infty(\mathbb{R})$ such that $\varphi(x) = 0$ for $x \leq 0$, $\varphi(x) > 0$ for $x > 0$ and $\varphi(x) = 1$ for $x \geq 1$. For fixed $k \in \mathbb{N}$ and $t \in \mathbb{R}^+$, define

$$P_{k,t}(x) = \text{sign } x \varphi(k(|x| - t)).$$

It is clear that $P_{k,t}$ is a maximal monotone graph containing the origin. Moreover,

$$\begin{aligned} \{x : P_{k,t}(x) \neq 0\} &= (-\infty, -t) \cup (t, +\infty), \\ |P_{1,t}(x)| &\leq |P_{2,t}(x)| \leq \dots \leq |P_{k,t}(x)| \leq |P_{k+1,t}(x)| \leq \dots \leq 1, \\ \lim_{k \rightarrow \infty} |P_{k,t}| &= \chi_{\{|x| > t\}}. \end{aligned}$$

Then assertion (iv) in Proposition 2.3 implies that

$$-\int_{-1}^1 (|x|^{2\alpha}u_n')' P_{k,t}(u_n) dx \geq 0.$$

Therefore

$$\int_{-1}^1 |P_{k,t}(u_n)| \left(\frac{1}{n}|u_n| + |u_n|^p\right) dx \leq \int_{-1}^1 |P_{k,t}(u_n)| |f| dx.$$

Passing to the limit as $k \rightarrow \infty$, the Monotone Convergence Theorem implies that

$$\int_{[|u_n|>t]} \left(\frac{1}{n}|u_n| + |u_n|^p \right) dx \leq \int_{[|u_n|>t]} |f| dx, \quad \forall t > 0 \text{ and } \forall n \in \mathbb{N}.$$

Then

$$|[|u_n| > t]| \leq \frac{1}{t^p} \int_{[|u_n|>t]} |u_n|^p dx \leq \frac{C}{t^p}.$$

For any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that

$$\int_{[|u_n|>t_\epsilon]} \left(\frac{1}{n}|u_n| + |u_n|^p \right) dx \leq \int_{[|u_n|>t_\epsilon]} |f| dx \leq \frac{\epsilon}{2}, \quad \forall n \in \mathbb{N}.$$

Take $\delta = \frac{\epsilon}{2(t_\epsilon^p + t_\epsilon)}$. Then for all $K \subset \mathbb{R}$ such that $|K| < \delta$, we have

$$\begin{aligned} \int_K \left(\frac{1}{n}|u_n| + |u_n|^p \right) dx &\leq \int_{K \cap [|u_n| > t_\epsilon]} \left(\frac{1}{n}|u_n| + |u_n|^p \right) dx + \int_{K \cap [|u_n| \leq t_\epsilon]} \left(\frac{1}{n}|u_n| + |u_n|^p \right) dx \\ &\leq \int_{[|u_n| > t_\epsilon]} \left(\frac{1}{n}|u_n| + |u_n|^p \right) dx + (t_\epsilon^p + t_\epsilon)|K| \\ &\leq \epsilon. \end{aligned}$$

Thus, the sequence $\left\{ \frac{1}{n}u_n + |u_n|^{p-1}u_n \right\}_{n=1}^\infty$ is equi-integrable.

A theorem of Vitali implies that $\frac{1}{n}u_n + |u_n|^{p-1}u_n \rightarrow |u|^{p-1}u$ in $L^1(-1, 1)$. Passing to the limit as $n \rightarrow \infty$ in (3.1), we obtain

$$\int_{-1}^1 |x|^{2\alpha} u' \zeta' dx + \int_{-1}^1 |u|^{p-1} u \zeta dx = \int_{-1}^1 f \zeta dx, \quad \forall \zeta \in C_0^1[-1, 1].$$

Therefore, $u \in D(A_\alpha) \cap L^p(-1, 1)$ and $A_\alpha u + |u|^{p-1}u = f$ a.e. on $(-1, 1)$. The uniqueness follows from Theorem 1.1. \square

We now start to prove Theorems 1.2 and 1.3. Given $\mu \in \mathcal{M}(-1, 1)$, there exists a sequence $\{f_n\}_{n=1}^\infty \subset L^1(-1, 1)$ such that $f_n \xrightarrow{*} \mu$ in $(C_0[-1, 1])^*$. For each f_n , by Proposition 3.1, there exists a unique $u_n \in D(A_\alpha) \cap L^p(-1, 1)$ such that

$$\int_{-1}^1 |x|^{2\alpha} u'_n \zeta' dx + \int_{-1}^1 |u_n|^{p-1} u_n \zeta dx = \int_{-1}^1 f_n \zeta dx, \quad \forall \zeta \in C_0^1[-1, 1]. \tag{3.2}$$

Lemma 3.3. Assume that $0 < \alpha < \frac{1}{2}$ and $p > 1$. Let $\{u_n\}_{n=1}^\infty$ be the sequence satisfying (3.2). Then $u_n \rightarrow u$ in $C[-1, 1]$, where u is the (unique) good solution of (1.1).

Proof. Note that $\|f_n\|_{L^1} \leq C$, where C is independent of n . Then Proposition 2.1 implies that $\|u_n\|_{L^\infty} + \| |x|^{2\alpha} u'_n \|_{W^{1,1}} \leq \tilde{C}$, where \tilde{C} is independent of n . Therefore the sequence u_n is bounded in $W^{1,q}(-1, 1)$ for some fixed $q \in (1, \frac{1}{2\alpha})$. By compactness, there exists a subsequence such that $u_{n_k} \rightarrow u$ in $C_0[-1, 1]$ and $|x|^{2\alpha} u'_{n_k} \rightarrow |x|^{2\alpha} u'$ in $L^1(-1, 1)$. Passing to the limit in (3.2) as $n_k \rightarrow \infty$, we obtain that

$$\int_{-1}^1 |x|^{2\alpha} u' \zeta' dx + \int_{-1}^1 |u|^{p-1} u \zeta dx = \int_{-1}^1 \zeta d\mu, \quad \forall \zeta \in C_0^1[-1, 1].$$

We conclude that u is a good solution of (1.1). The uniqueness of the good solution and “the uniqueness of the limit” imply that $u_n \rightarrow u$ in $C[-1, 1]$. \square

Lemma 3.4. Assume that $\alpha = \frac{1}{2}$ and $p > 1$. Let $\{u_n\}_{n=1}^\infty$ be the sequence satisfying (3.2). Then there exists a subsequence $\{n_k\}_{k=1}^\infty$ such that

$$\left(1 + \ln \frac{1}{|x|}\right)^{-1} u_{n_k} \rightarrow \left(1 + \ln \frac{1}{|x|}\right)^{-1} u \text{ in } L^r(-1, 1), \forall r < \infty, \tag{3.3}$$

where u is a solution of (1.1). Moreover, $\left(1 + \ln \frac{1}{|x|}\right)^{-1} u \in BV(-1, 1)$ and

$$\lim_{x \rightarrow 0^+} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u(x) = \lim_{x \rightarrow 0^+} \lim_{k \rightarrow \infty} \left(\int_0^x f_{n_k}(s) ds + \left(\ln \frac{1}{|x|}\right)^{-1} \int_x^1 f_{n_k}(s) \ln \frac{1}{|s|} ds \right), \tag{3.4}$$

$$\lim_{x \rightarrow 0^-} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u(x) = \lim_{x \rightarrow 0^-} \lim_{k \rightarrow \infty} \left(\int_x^0 f_{n_k}(s) ds + \left(\ln \frac{1}{|x|}\right)^{-1} \int_{-1}^x f_{n_k}(s) \ln \frac{1}{|s|} ds \right). \tag{3.5}$$

Proof. Proposition 2.2 implies that

$$\| |x| u'_n \|_{W^{1,1}} + \left\| \left(1 + \ln \frac{1}{|x|}\right)^{-1} u_n \right\|_{W^{1,1}} \leq C,$$

where C is independent of n . As a consequence, we obtain (3.3). Moreover, $\left(1 + \ln \frac{1}{|x|}\right)^{-1} u \in BV(-1, 1)$, $u_{n_k} \rightarrow u$ in $L^p(-1, 1)$ and $|x| u'_{n_k} \rightarrow |x| u'$ in $L^1(-1, 1)$. Passing to the limit in (3.2) as $n_k \rightarrow \infty$, we obtain that u is a solution of (1.1). The proof of (3.4) and (3.5) is the same as the one of Lemma 6.3 in [23]. \square

Lemma 3.5. Assume that $\frac{1}{2} < \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$. Let $\{u_n\}_{n=1}^\infty$ be the sequence satisfying (3.2). Then there exists a subsequence $\{n_k\}_{k=1}^\infty$ such that

$$|x|^{2\alpha-1} u_{n_k} \rightarrow |x|^{2\alpha-1} u \text{ in } L^r(-1, 1), \forall r < \infty, \tag{3.6}$$

where u is a solution of (1.1). Moreover, $|x|^{2\alpha-1} u \in BV(-1, 1)$ and

$$\lim_{x \rightarrow 0^+} |x|^{2\alpha-1} u(x) = \frac{1}{2\alpha-1} \lim_{x \rightarrow 0^+} \lim_{k \rightarrow \infty} \left(\int_0^x f_{n_k}(s) ds + |x|^{2\alpha-1} \int_x^1 f_{n_k}(s) |s|^{1-2\alpha} ds \right), \tag{3.7}$$

$$\lim_{x \rightarrow 0^-} |x|^{2\alpha-1} u(x) = \frac{1}{2\alpha-1} \lim_{x \rightarrow 0^-} \lim_{k \rightarrow \infty} \left(\int_x^0 f_{n_k}(s) ds + |x|^{2\alpha-1} \int_{-1}^x f_{n_k}(s) |s|^{1-2\alpha} ds \right). \tag{3.8}$$

Proof. Proposition 2.2 implies that $\| |x|^{2\alpha} u'_n \|_{W^{1,1}} + \| |x|^{2\alpha-1} u_n \|_{W^{1,1}} \leq C$, where C is independent of n . As a consequence, we obtain (3.6). Moreover, $|x|^{2\alpha-1} u \in BV(-1, 1)$, $u_{n_k} \rightarrow u$ in $L^p(-1, 1)$ and $|x|^{2\alpha} u'_{n_k} \rightarrow |x|^{2\alpha-1} u'$ in $L^1(-1, 1)$. Passing to the limit in (3.2) as $n_k \rightarrow \infty$, we obtain that u is a solution of (1.1). The proof of (3.7) and (3.8) is the same as the one of Lemma 6.4 in [23]. \square

Lemma 3.6. (See Lemma 6.5 in [23].) Fix $\rho \in C(\mathbb{R})$ such that $\text{supp } \rho = [-1, 1]$, $\rho(x) = \rho(-x)$ and $\rho \geq 0$. Let $\rho_n(x) = Cn\rho(nx)$ where $C^{-1} = \int \rho$. For $\mu \in \mathcal{M}(-1, 1)$, let $f_n = \mu * \rho_n$. Then $f_n \in C[-1, 1]$, $\|f_n\|_{L^1} \leq \|\mu\|_{\mathcal{M}}$, and $f_n \xrightarrow{*} \mu$ in $(C_0[-1, 1])^*$. For any $-1 < a < b < 1$ and $y \in [-1, 1]$, we have

$$\lim_{n \rightarrow \infty} \int_{a-y}^{b-y} \rho_n(s) ds = \begin{cases} 0, & \text{for } y \in [-1, a), \\ \frac{1}{2}, & \text{for } y = a, \\ 1, & \text{for } y \in (a, b), \\ \frac{1}{2}, & \text{for } y = b, \\ 0, & \text{for } y \in (b, 1]. \end{cases} \tag{3.9}$$

Moreover,

$$\lim_{x \rightarrow 0^+} \lim_{n \rightarrow \infty} \int_0^x f_n(s) ds = \lim_{x \rightarrow 0^-} \lim_{n \rightarrow \infty} \int_x^0 f_n(s) ds = \frac{1}{2} \mu(\{0\}). \tag{3.10}$$

Proof of Theorem 1.2. The existence of good solution for $0 < \alpha < \frac{1}{2}$ and $p > 1$ has been proved by Lemma 3.3.

Assume now that f_n is the sequence identified in Lemma 3.6. For $\alpha = \frac{1}{2}$ and $p > 1$, we claim that

$$\begin{aligned} & \lim_{x \rightarrow 0^+} \lim_{n \rightarrow \infty} \left(\int_0^x f_n(s) ds + \left(\ln \frac{1}{|x|} \right)^{-1} \int_x^1 f_n(s) \ln \frac{1}{|s|} ds \right) \\ &= \lim_{x \rightarrow 0^-} \lim_{n \rightarrow \infty} \left(\int_x^0 f_n(s) ds + \left(\ln \frac{1}{|x|} \right)^{-1} \int_{-1}^x f_n(s) \ln \frac{1}{|s|} ds \right) \\ &= \frac{1}{2} \mu(\{0\}). \end{aligned}$$

For $\frac{1}{2} < \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$, we claim that

$$\begin{aligned} & \frac{1}{2\alpha-1} \lim_{x \rightarrow 0^+} \lim_{n \rightarrow \infty} \left(\int_0^x f_n(s) ds + |x|^{2\alpha-1} \int_x^1 f_n(s) |s|^{1-2\alpha} ds \right) \\ &= \frac{1}{2\alpha-1} \lim_{x \rightarrow 0^-} \lim_{n \rightarrow \infty} \left(\int_x^0 f_n(s) ds + |x|^{2\alpha-1} \int_{-1}^x f_n(s) |s|^{1-2\alpha} ds \right) \\ &= \frac{1}{2(2\alpha-1)} \mu(\{0\}). \end{aligned}$$

The proof of these two claims is the same as their counterparts in the proof of (i) of Theorem 6.1 in [23]. Therefore, in view of Lemmas 3.4 and 3.5, we proved the existence of good solution for $\frac{1}{2} \leq \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$, as well as assertions (i) and (ii). Assertion (iii) will be proved in Section 4. \square

Lemma 3.7. Assume that α and p satisfy (1.6) or (1.7). Let $\{u_n\}_{n=1}^\infty$ be the sequence satisfying (3.2). Then $|x|^{2\alpha-1} u_n \rightarrow |x|^{2\alpha-1} u$ in $L^r(-1, 1)$, $\forall r < \infty$, where u is the solution of (1.9).

Proof. Proposition 2.2 implies that $\| |x|^{2\alpha} u'_n \|_{W^{1,1}} + \| |x|^{2\alpha-1} u_n \|_{W^{1,1}} \leq C$, where C is independent of n . It follows that $|x|^{2\alpha} u'_{n_k} \rightarrow |x|^{2\alpha} u'$ and $|x|^{2\alpha-1} u_n \rightarrow |x|^{2\alpha-1} u$ in $L^r(-1, 1)$, $\forall r < \infty$. Note that $\|u_n\|_{L^p} \leq C$. Then Fatou’s Lemma implies that $u \in L^p(-1, 1)$. Passing to the limit in (3.2) as $n_k \rightarrow \infty$, we obtain

$$\int_{-1}^1 |x|^{2\alpha} u' \zeta' dx + \int_{-1}^1 |u|^{p-1} u \zeta dx = \int_{-1}^1 \zeta d\mu, \quad \forall \zeta \in C_c^1((-1, 1) \setminus \{0\}). \tag{3.11}$$

Here we use the same device as in Brezis and Véron [10]. Let $\varphi(x) \in C^\infty(\mathbb{R})$ be such that $0 \leq \varphi \leq 1$, $\varphi \equiv 0$ on $(-\frac{1}{2}, \frac{1}{2})$ and $\varphi \equiv 1$ on $\mathbb{R} \setminus (-1, 1)$. Let $\varphi_n(x) = \varphi(nx)$. In (3.11), perform integration by parts and replace ζ by $\varphi_n\phi$ where $\phi \in C_c^2(-1, 1)$. It follows that

$$-\int_{-1}^1 u(|x|^{2\alpha}(\varphi_n\phi)')'dx + \int_{-1}^1 |u|^{p-1}u\varphi_n\phi dx = \int_{-1}^1 \varphi_n\phi d\mu, \quad \forall \phi \in C_c^2(-1, 1). \tag{3.12}$$

For each term on the left-hand side of (3.12), we obtain

$$\begin{aligned} \int_{-1}^1 |x|^{2\alpha}u'(x)\varphi(nx)\phi''(x)dx &\rightarrow \int_{-1}^1 |x|^{2\alpha}u'(x)\phi''(x)dx, \\ 2\alpha \int_{-1}^1 u(x) \operatorname{sign} x |x|^{2\alpha-1}\varphi(nx)\phi'(x)dx &\rightarrow 2\alpha \int_{-1}^1 u(x) \operatorname{sign} x |x|^{2\alpha-1}\phi'(x)dx, \\ \int_{-1}^1 |u(x)|^{p-1}u(x)\varphi(nx)\phi(x)dx &\rightarrow \int_{-1}^1 |u(x)|^{p-1}u(x)\phi(x)dx, \\ \left| 2n \int_{-\frac{1}{n}}^{\frac{1}{n}} |x|^{2\alpha}u(x)\varphi'(nx)\phi'(x)dx \right| &\leq \frac{2}{n^{2\alpha-1}} \|\varphi'\phi'\|_{L^\infty} \|u\|_{L^1(-\frac{1}{n}, \frac{1}{n})} \rightarrow 0, \\ \left| 2\alpha n \int_{-\frac{1}{n}}^{\frac{1}{n}} u(x) \operatorname{sign} x |x|^{2\alpha-1}\varphi'(nx)\phi(x)dx \right| &\leq \frac{2\alpha}{n^{2\alpha-2}} \left(\frac{2}{n}\right)^{\frac{1}{p'}} \|\varphi'\phi\|_{L^\infty} \|u\|_{L^p(-\frac{1}{n}, \frac{1}{n})} \rightarrow 0, \\ \left| n^2 \int_{-\frac{1}{n}}^{\frac{1}{n}} u(x)|x|^{2\alpha}\varphi''(nx)\phi(x)dx \right| &\leq \frac{1}{n^{2\alpha-2}} \left(\frac{2}{n}\right)^{\frac{1}{p'}} \|\varphi''\phi\|_{L^\infty} \|u\|_{L^p(-\frac{1}{n}, \frac{1}{n})} \rightarrow 0, \end{aligned}$$

where p' is the Hölder conjugate of p , which satisfies $\frac{1}{p'} + 2\alpha - 2 \geq 0$. For the right-hand side of (3.12), the Dominated Convergence Theorem implies that

$$\lim_{n \rightarrow \infty} \int_{-1}^1 \varphi(nx)\phi(x)d\mu = \int_{-1}^1 \phi(x)d(\mu - \mu(\{0\})\delta_0).$$

Thus

$$\int_{-1}^1 |x|^{2\alpha}u'\phi'dx + \int_{-1}^1 |u|^{p-1}u\phi dx = \int_{-1}^1 \phi d(\mu - \mu(\{0\})\delta_0), \quad \forall \phi \in C_c^1(-1, 1).$$

Therefore u is the solution of (1.9). \square

Proof of Theorem 1.3. Suppose $\mu(\{0\}) = 0$. Then Lemma 3.7 implies that (1.1) has a solution. Conversely, assume that u is a solution of (1.1). We claim that $\mu(\{0\}) = 0$. Indeed, we have

$$-\int_{-1}^1 u(|x|^{2\alpha}\zeta')'dx + \int_{-1}^1 |u|^{p-1}u\zeta dx = \int_{-1}^1 \zeta d\mu, \quad \forall \zeta \in C_c^\infty(-1, 1). \tag{3.13}$$

Take $\varphi \in C_c^\infty(\mathbb{R})$ such that $\varphi \equiv 1$ on $(-1, 1)$, $\text{supp } \varphi \subset (-2, 2)$ and $0 \leq \varphi \leq 1$. Replace $\zeta(x)$ by $\varphi(nx)$ in (3.13). Then for each term on the left-hand side of (3.13), we have

$$\begin{aligned} \left| n^2 \int_{-\frac{2}{n}}^{\frac{2}{n}} u(x) |x|^{2\alpha} \varphi''(nx) dx \right| &\leq \frac{2^{2\alpha + \frac{2}{p'}}}{n^{2\alpha - 2 + \frac{1}{p'}}} \|\varphi''\|_{L^\infty} \|u\|_{L^p(-\frac{2}{n}, \frac{2}{n})} \rightarrow 0, \\ \left| 2\alpha n \int_{-\frac{2}{n}}^{\frac{2}{n}} u(x) |x|^{2\alpha - 1} \varphi'(nx) \text{sign } x dx \right| &\leq \frac{2^{2\alpha + \frac{2}{p'}} \alpha}{n^{2\alpha - 2 + \frac{1}{p'}}} \|\varphi'\|_{L^\infty} \|u\|_{L^p(-\frac{2}{n}, \frac{2}{n})} \rightarrow 0, \\ \int_{-1}^1 |u(x)|^{p-1} u(x) \varphi(nx) dx &\rightarrow 0. \end{aligned}$$

For the right-hand side of (3.13), we have

$$\int_{-1}^1 \varphi(nx) d\mu = \mu(\{0\}) + \int_{(0, \frac{2}{n}] } \varphi(nx) d\mu + \int_{[-\frac{2}{n}, 0)} \varphi(nx) d\mu.$$

Note that

$$\lim_{n \rightarrow \infty} \int_{(0, \frac{2}{n}] } \varphi(nx) d\mu = \lim_{n \rightarrow \infty} \int_{[-\frac{2}{n}, 0)} \varphi(nx) d\mu = 0.$$

Therefore, $\mu(\{0\}) = 0$.

Assume now that the solution exists. We prove assertion (i). Indeed, since $|x|^{2\alpha - 1} u \in BV(-1, 1)$, the one-side limits $\lim_{x \rightarrow 0^+} |x|^{2\alpha - 1} u(x)$ and $\lim_{x \rightarrow 0^-} |x|^{2\alpha - 1} u(x)$ exist. They must be zero. Otherwise, it contradicts $u \in L^p(-1, 1)$.

The same reason guarantees that $\lim_{x \rightarrow 0} |x|^{2\alpha} u'(x) = 0$. Assertion (ii) will be proved in Section 4. \square

4. The elliptic regularization

For any $0 < \epsilon < 1$, we consider the regularized equation (1.8). Since $\mathcal{M}(-1, 1) \subset H^{-1}(-1, 1)$, the solution u_ϵ of (1.8) is actually the minimizer of the following functional

$$I(u) = \frac{1}{2} \int_{-1}^1 (|x| + \epsilon)^{2\alpha} |u'|^2 dx + \frac{1}{p+1} \int_{-1}^1 |u|^{p+1} dx - \int_{-1}^1 u d\mu, \quad \forall u \in H_0^1(-1, 1).$$

It implies that u_ϵ satisfies the following weak formulation

$$\int_{-1}^1 (|x| + \epsilon)^{2\alpha} u'_\epsilon v' dx + \int_{-1}^1 |u_\epsilon|^{p-1} u_\epsilon v dx = \int_{-1}^1 v d\mu, \quad \forall v \in H_0^1(-1, 1). \tag{4.1}$$

Take $v_n = \varphi(nu_\epsilon)$ where $\varphi \in C^\infty(\mathbb{R})$ and $\varphi' \geq 0$ such that $\varphi \equiv 1$ on $[1, \infty)$, $\varphi \equiv -1$ on $(-\infty, -1]$ and $\varphi(0) = 0$. Notice that

$$\int_{-1}^1 (|x| + \epsilon)^{2\alpha} u'_\epsilon v'_n dx = n \int_{-1}^1 (|x| + \epsilon)^{2\alpha} |u'_\epsilon|^2 \varphi'(nu_\epsilon) dx \geq 0.$$

Then

$$\|u_\epsilon\|_{L^p(-1,1)}^p = \lim_{n \rightarrow \infty} \int_{-1}^1 |u_\epsilon|^{p-1} u_\epsilon v_n dx \leq \lim_{n \rightarrow \infty} \int_{-1}^1 v_n d\mu \leq \|\mu\|_{\mathcal{M}(-1,1)}. \tag{4.2}$$

We now examine the limiting behavior of the family $\{u_\epsilon\}_{\epsilon>0}$ and we are going to establish the following sharper form of [Theorems 1.4 and 1.5](#).

Theorem 4.1. *Given $\alpha > 0$, as $\epsilon \rightarrow 0$, we have*

$$(|x| + \epsilon)^{2\alpha} u'_\epsilon \rightarrow |x|^{2\alpha} u' \text{ in } L^r(-1, 1), \forall r < \infty. \tag{4.3}$$

Moreover,

$$u_\epsilon \rightarrow u \text{ in } C_0[-1, 1], \text{ if } 0 < \alpha < \frac{1}{2}, \tag{4.4}$$

$$\left(1 + \ln \frac{1}{|x| + \epsilon}\right)^{-1} u_\epsilon \rightarrow \left(1 + \ln \frac{1}{|x|}\right)^{-1} u \text{ in } L^r(-1, 1), \forall r < \infty, \text{ if } \alpha = \frac{1}{2}, \tag{4.5}$$

$$(|x| + \epsilon)^{2\alpha-1} u_\epsilon \rightarrow |x|^{2\alpha-1} u \text{ in } L^r(-1, 1), \forall r < \infty, \text{ if } \alpha > \frac{1}{2}. \tag{4.6}$$

Here u is the unique good solution of (1.1) if α and p satisfy (1.4) or (1.5); u is the unique solution of (1.9) if α and p satisfy (1.6) or (1.7).

The proof for the case $0 < \alpha < \frac{1}{2}$ of [Theorem 4.1](#) is the same as the proof for the case $0 < \alpha < \frac{1}{2}$ of [Theorem 5.1](#) in [\[23\]](#), except some obvious modifications due to the nonlinear term. We omit the detail.

Proof of Theorem 4.1 for $\alpha = \frac{1}{2}$. Write $K_\epsilon^+ = \lim_{x \rightarrow 0^+} u'_\epsilon(x)$ and $K_\epsilon^- = \lim_{x \rightarrow 0^-} u'_\epsilon(x)$. One can perform integration by parts (the same as the proof of [Theorem 5.1](#) of [\[23\]](#)) and obtain, for $x \in (0, 1)$,

$$\begin{aligned} u_\epsilon(x) = \ln\left(\frac{1+\epsilon}{x+\epsilon}\right) & \left(-\epsilon K_\epsilon^+ + \int_{(0,x)} d\mu - \int_0^x |u_\epsilon(s)|^{p-1} u_\epsilon(s) ds \right) \\ & - \int_x^1 |u_\epsilon(s)|^{p-1} u_\epsilon(s) \ln\left(\frac{1+\epsilon}{s+\epsilon}\right) ds + \int_{[x,1)} \ln\left(\frac{1+\epsilon}{s+\epsilon}\right) d\mu(s), \end{aligned}$$

and for $x \in (-1, 0)$,

$$\begin{aligned} u_\epsilon(x) = \ln\left(\frac{1+\epsilon}{|x|+\epsilon}\right) & \left(\epsilon K_\epsilon^- + \int_{(x,0)} d\mu - \int_x^0 |u_\epsilon(s)|^{p-1} u_\epsilon(s) ds \right) \\ & - \int_{-1}^x |u_\epsilon(s)|^{p-1} u_\epsilon(s) \ln\left(\frac{1+\epsilon}{|s|+\epsilon}\right) ds + \int_{(-1,x]} \ln\left(\frac{1+\epsilon}{|s|+\epsilon}\right) d\mu(s). \end{aligned}$$

Taking into account the relations $u_\epsilon(0^+) = u_\epsilon(0^-)$ and $\epsilon K_\epsilon^+ - \epsilon K_\epsilon^- = -\mu(\{0\})$, we deduce that

$$\begin{aligned} \epsilon K_\epsilon^+ = -\frac{1}{2} \mu(\{0\}) + \frac{1}{2 \ln\left(\frac{1+\epsilon}{\epsilon}\right)} & \int_{(-1,0) \cup (0,1)} (\text{sign } s) \ln\left(\frac{1+\epsilon}{|s|+\epsilon}\right) d\mu(s) \\ & - \frac{1}{2 \ln\left(\frac{1+\epsilon}{\epsilon}\right)} \int_{-1}^1 (\text{sign } s) |u_\epsilon(s)|^{p-1} u_\epsilon(s) \ln\left(\frac{1+\epsilon}{|s|+\epsilon}\right) ds, \end{aligned}$$

and

$$\begin{aligned} \epsilon K_\epsilon^- &= \frac{1}{2} \mu(\{0\}) + \frac{1}{2 \ln\left(\frac{1+\epsilon}{\epsilon}\right)} \int_{(-1,0) \cup (0,1)} (\text{sign } s) \ln\left(\frac{1+\epsilon}{|s|+\epsilon}\right) d\mu(s) \\ &\quad - \frac{1}{2 \ln\left(\frac{1+\epsilon}{\epsilon}\right)} \int_{-1}^1 (\text{sign } s) |u_\epsilon(s)|^{p-1} u_\epsilon(s) \ln\left(\frac{1+\epsilon}{|s|+\epsilon}\right) ds. \end{aligned}$$

It is easy to check that $|\epsilon K_\epsilon^+| \leq \frac{3}{2} \|\mu\|_{\mathcal{M}}$ and $|\epsilon K_\epsilon^-| \leq \frac{3}{2} \|\mu\|_{\mathcal{M}}$ since $\|u_\epsilon\|_{L^p}^p \leq \|\mu\|_{\mathcal{M}}$. Therefore, we obtain that

$$\left\| \left(1 + \ln \frac{1}{|x|+\epsilon}\right)^{-1} u_\epsilon \right\|_{W^{1,1}(-1,1)} + \|(|x| + \epsilon) u'_\epsilon \|_{BV(-1,1)} \leq C,$$

where C is independent of ϵ . It follows that (4.3) and (4.5) hold for a subsequence $\{u_{\epsilon_n}\}_{n=1}^\infty$. Moreover, the sequence $\{|u_{\epsilon_n}|^{p-1} u_{\epsilon_n}\}_{n=1}^\infty$ is equi-integrable and $|u_{\epsilon_n}|^{p-1} u_{\epsilon_n} \rightarrow |u|^{p-1} u$ in $L^1(-1, 1)$. Passing to the limit as $n \rightarrow \infty$ in (4.1), we obtain

$$\int_{-1}^1 |x| u' v' dx + \int_{-1}^1 |u|^{p-1} u v dx = \int_{-1}^1 v d\mu, \quad \forall v \in C_0^1[-1, 1].$$

Notice that $\|u_\epsilon\|_{L^{p+1}(-1,1)} \leq C$. The same argument as in the proof of Theorem 5.1 in [23] implies that

$$-\lim_{\epsilon \rightarrow 0} \epsilon K_\epsilon^+ = \lim_{\epsilon \rightarrow 0} \epsilon K_\epsilon^- = \frac{1}{2} \mu(\{0\}),$$

and

$$\lim_{x \rightarrow 0^+} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u(x) = \lim_{x \rightarrow 0^-} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u(x) = \frac{1}{2} \mu(\{0\}).$$

Therefore, u is the good solution. The uniqueness of the good solution and the uniqueness of the limit imply that (4.3) and (4.5) hold for the family $\{u_\epsilon\}_{\epsilon>0}$. \square

Proof of Theorem 4.1 for $\frac{1}{2} < \alpha < 1$. We denote $K_\epsilon^+ = \lim_{x \rightarrow 0^+} u'_\epsilon(x)$ and $K_\epsilon^- = \lim_{x \rightarrow 0^-} u'_\epsilon(x)$. Integration by parts yields, for $x \in (0, 1)$,

$$\begin{aligned} u_\epsilon(x) &= \left(\frac{(x+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}}{2\alpha-1} \right) \left(-\epsilon^{2\alpha} K_\epsilon^+ + \int_{(0,x)} d\mu - \int_0^x |u_\epsilon(s)|^{p-1} u_\epsilon(s) ds \right) \\ &\quad - \int_x^1 |u_\epsilon(s)|^{p-1} u_\epsilon(s) \left(\frac{(s+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}}{2\alpha-1} \right) ds \\ &\quad + \int_{[x,1)} \frac{(s+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}}{2\alpha-1} d\mu(s), \end{aligned}$$

and for $x \in (-1, 0)$,

$$\begin{aligned}
 u_\epsilon(x) = & \left(\frac{(|x| + \epsilon)^{1-2\alpha} - (1 + \epsilon)^{1-2\alpha}}{2\alpha - 1} \right) \left(\epsilon^{2\alpha} K_\epsilon^- + \int_{(x,0)} d\mu - \int_x^0 |u_\epsilon(s)|^{p-1} u_\epsilon(s) ds \right) \\
 & - \int_{-1}^x |u_\epsilon(s)|^{p-1} u_\epsilon(s) \left(\frac{(|s| + \epsilon)^{1-2\alpha} - (1 + \epsilon)^{1-2\alpha}}{2\alpha - 1} \right) ds \\
 & + \int_{(-1,x]} \frac{(|s| + \epsilon)^{1-2\alpha} - (1 + \epsilon)^{1-2\alpha}}{2\alpha - 1} d\mu(s).
 \end{aligned}$$

By the relations $u_\epsilon(0^+) = u_\epsilon(0^-)$ and $\epsilon^{2\alpha} K_\epsilon^+ - \epsilon^{2\alpha} K_\epsilon^- = -\mu(\{0\})$, we have

$$\begin{aligned}
 \epsilon^{2\alpha} K_\epsilon^+ = & -\frac{1}{2} \mu(\{0\}) - \frac{\int_{-1}^1 (\text{sign } s) |u_\epsilon(s)|^{p-1} u_\epsilon(s) [(|s| + \epsilon)^{1-2\alpha} - (1 + \epsilon)^{1-2\alpha}] ds}{2[\epsilon^{1-2\alpha} - (1 + \epsilon)^{1-2\alpha}]} \\
 & + \frac{\int_{(-1,0) \cup (0,1)} (\text{sign } s) [(|s| + \epsilon)^{1-2\alpha} - (1 + \epsilon)^{1-2\alpha}] d\mu(s)}{2[\epsilon^{1-2\alpha} - (1 + \epsilon)^{1-2\alpha}]},
 \end{aligned}$$

and

$$\begin{aligned}
 \epsilon^{2\alpha} K_\epsilon^- = & \frac{1}{2} \mu(\{0\}) - \frac{\int_{-1}^1 (\text{sign } s) |u_\epsilon(s)|^{p-1} u_\epsilon(s) [(|s| + \epsilon)^{1-2\alpha} - (1 + \epsilon)^{1-2\alpha}] ds}{2[\epsilon^{1-2\alpha} - (1 + \epsilon)^{1-2\alpha}]} \\
 & + \frac{\int_{(-1,0) \cup (0,1)} (\text{sign } s) [(|s| + \epsilon)^{1-2\alpha} - (1 + \epsilon)^{1-2\alpha}] d\mu(s)}{2[\epsilon^{1-2\alpha} - (1 + \epsilon)^{1-2\alpha}]}.
 \end{aligned}$$

It is easy to check that $|\epsilon^{2\alpha} K_\epsilon^+| \leq \frac{3}{2} \|\mu\|_{\mathcal{M}}$ and $|\epsilon^{2\alpha} K_\epsilon^-| \leq \frac{3}{2} \|\mu\|_{\mathcal{M}}$ since $\|u_\epsilon\|_{L^p}^p \leq \|\mu\|_{\mathcal{M}}$. Therefore, we obtain that

$$\left\| (|x| + \epsilon)^{2\alpha-1} u_\epsilon \right\|_{W^{1,1}(-1,1)} + \left\| (|x| + \epsilon)^{2\alpha} u'_\epsilon \right\|_{BV(-1,1)} \leq C, \tag{4.7}$$

where C is independent of ϵ . It follows that (4.3) and (4.6) hold for a subsequence $\{u_{\epsilon_n}\}_{n=1}^\infty$.

If $1 < p < \frac{1}{2\alpha-1}$, there exists $\theta \in \left(p, \frac{1}{2\alpha-1}\right)$ such that $\|u_\epsilon\|_{L^\theta(-1,1)} \leq C$. Thus the sequence $\{|u_{\epsilon_n}|^{p-1} u_{\epsilon_n}\}_{n=1}^\infty$ is equi-integrable and $|u_{\epsilon_n}|^{p-1} u_{\epsilon_n} \rightarrow |u|^{p-1} u$ in $L^1(-1, 1)$. Passing to the limit as $n \rightarrow \infty$ in (4.1), we obtain

$$\int_{-1}^1 |x|^{2\alpha} u'v'dx + \int_{-1}^1 |u|^{p-1} uvdx = \int_{-1}^1 v d\mu, \quad \forall v \in C_0^1[-1, 1].$$

The same argument as in the proof of Theorem 5.1 in [23] implies that

$$-\lim_{\epsilon \rightarrow 0} \epsilon^{2\alpha} K_\epsilon^+ = \lim_{\epsilon \rightarrow 0} \epsilon^{2\alpha} K_\epsilon^- = \frac{1}{2} \mu(\{0\})$$

and

$$\lim_{x \rightarrow 0^+} |x|^{2\alpha-1} u(x) = \lim_{x \rightarrow 0^-} |x|^{2\alpha-1} u(x) = \frac{1}{2(2\alpha - 1)} \mu(\{0\}).$$

Therefore, u is the good solution.

If $p \geq \frac{1}{2\alpha-1}$, a consequence of (4.7) is that $u_{\epsilon_n} \rightarrow u$ uniformly on any closed interval $I \subset [-1, 1] \setminus \{0\}$. Passing to the limit as $n \rightarrow \infty$ in (4.1), we obtain

$$\int_{-1}^1 |x|^{2\alpha} u'v'dx + \int_{-1}^1 |u|^{p-1} uvdx = \int_{-1}^1 v d\mu, \quad \forall v \in C_c^1((-1, 1) \setminus \{0\}).$$

Since $\|u_\epsilon\|_{L^p}^p \leq \|\mu\|_{\mathcal{M}}$, Fatou’s lemma yields $u \in L^p(-1, 1)$. The same argument as in the proof of Lemma 3.7 implies that u is the solution of (1.9). The uniqueness of the solution and the uniqueness of the limit imply that (4.3) and (4.6) hold for the family $\{u_\epsilon\}_{\epsilon>0}$. \square

We omit the proof for the case $\alpha \geq 1$ of Theorem 4.1 since it is the same as the proof for the case $\frac{1}{2} < \alpha < 1$ and $p \geq \frac{1}{2\alpha-1}$.

If we assume the data to be L^1 , we have a further result about the mode of convergence.

Theorem 4.2. For $\alpha \geq \frac{1}{2}$ and $\mu \in L^1(-1, 1)$, the mode of convergence in (4.5) and (4.6) can be improved as

$$\left(1 + \ln \frac{1}{|x| + \epsilon}\right)^{-1} u_\epsilon \rightarrow \left(1 + \ln \frac{1}{|x|}\right)^{-1} u \text{ in } C_0[-1, 1], \text{ if } \alpha = \frac{1}{2}, \tag{4.8}$$

and

$$(|x| + \epsilon)^{2\alpha-1} u_\epsilon \rightarrow |x|^{2\alpha-1} u \text{ in } C_0[-1, 1], \text{ if } \alpha > \frac{1}{2}. \tag{4.9}$$

To prove Theorem 4.2, one can just perform the same argument as the proof of Theorem 5.2 in [23]. We omit the detail.

As we indicated in the previous section, the following is the

Proof of (iii) of Theorem 1.2 and proof of (ii) of Theorem 1.3. For $\mu, \hat{\mu} \in \mathcal{M}(-1, 1)$, denote by u_ϵ and \hat{u}_ϵ their corresponding solution of (1.8). From (4.1) we have

$$\begin{aligned} & \int_{-1}^1 (|x| + \epsilon)^{2\alpha} (u_\epsilon - \hat{u}_\epsilon)' v' dx + \int_{-1}^1 (|u_\epsilon|^{p-1} u_\epsilon - |\hat{u}_\epsilon|^{p-1} \hat{u}_\epsilon) v dx \\ &= \int_{-1}^1 v d(\mu - \hat{\mu}), \forall v \in H_0^1(-1, 1). \end{aligned}$$

Take $v = \varphi_n (u_\epsilon - \hat{u}_\epsilon)$, where φ_n is the smooth approximation of either $\text{sign } x$ or $(\text{sign } x)^+$. We obtain

$$\left\| |u_\epsilon|^{p-1} u_\epsilon - |\hat{u}_\epsilon|^{p-1} \hat{u}_\epsilon \right\|_{L^1} \leq \|\mu - \hat{\mu}\|_{\mathcal{M}},$$

and

$$\left\| \left(|u_\epsilon|^{p-1} u_\epsilon - |\hat{u}_\epsilon|^{p-1} \hat{u}_\epsilon \right)^+ \right\|_{L^1} \leq \left\| (\mu - \hat{\mu})^+ \right\|_{\mathcal{M}}.$$

Then Fatou’s lemma yields the desired result. \square

5. The approximation via truncation

In this section, we consider the approximation scheme via the truncated problem (1.11). As we mentioned in the introduction, the following lemma ensures the sequence $\{u_n\}_{n=1}^\infty$ is well-defined.

Lemma 5.1. Fix $p > 1$ and $n \in \mathbb{N}$. When $0 < \alpha < 1$, for each $\mu \in \mathcal{M}(-1, 1)$, Eq. (1.11) has a unique good solution u_n . When $\alpha \geq 1$, for each $\mu \in \mathcal{M}(-1, 1)$, Eq. (1.11) has a unique solution u_n if and only if $\mu(\{0\}) = 0$. Moreover, for both cases, $\|g_{p,n}(u_n)\|_{L^1} \leq \|\mu\|_{\mathcal{M}}$ and $\|(x|^{2\alpha} u_n')\|_{\mathcal{M}} \leq 2 \|\mu\|_{\mathcal{M}}$.

Proof. For $\mu \in \mathcal{M}(-1, 1)$, take $f_m = \rho_m * \mu$, where ρ_m is specified in Lemma 3.6. Then $f_m \xrightarrow{*} \mu$ in $(C_0[-1, 1])^*$ as $m \rightarrow \infty$. For fixed $m \in \mathbb{N}$, the same argument as in the proof of Proposition 3.1 implies that there exists $u_{n,m} \in D(A_\alpha)$ such that

$$\int_{-1}^1 |x|^{2\alpha} u'_{n,m} \zeta' dx + \int_{-1}^1 g_{p,n}(u_{n,m}) \zeta dx = \int_{-1}^1 f_m \zeta dx, \quad \forall \zeta \in C_0^1[-1, 1]. \tag{5.1}$$

Moreover,

$$\begin{aligned} \|g_{p,n}(u_{n,m})\|_{L^1} &\leq \|f_m\|_{L^1} \leq \|\mu\|_{\mathcal{M}}, \\ \|(|x|^{2\alpha} u'_{n,m})'\|_{L^1} &\leq 2 \|f_m\|_{L^1} \leq 2 \|\mu\|_{\mathcal{M}}. \end{aligned}$$

If $0 < \alpha < \frac{1}{2}$, then $\{u_{n,m}\}_{m=1}^\infty$ is a bounded sequence in $W^{1,q}(-1, 1)$ for $1 < q < \frac{1}{2\alpha}$. Thus, passing to the limit as $m \rightarrow \infty$ in (5.1), we obtain

$$\int_{-1}^1 |x|^{2\alpha} u'_n \zeta' dx + \int_{-1}^1 g_{p,n}(u_n) \zeta dx = \int_{-1}^1 \zeta d\mu, \quad \forall \zeta \in C_0^1[-1, 1], \tag{5.2}$$

where $u_n \in W^{1,1}(-1, 1)$, $\|g_{p,n}(u_n)\|_{L^1} \leq \|\mu\|_{\mathcal{M}}$ and $\|(|x|^{2\alpha} u'_n)'\|_{\mathcal{M}} \leq 2 \|\mu\|_{\mathcal{M}}$.

If $\frac{1}{2} \leq \alpha < 1$, as $m \rightarrow \infty$, we obtain $|x|^{2\alpha} u'_{n,m} \rightarrow |x|^{2\alpha} u'_n$ and $|x|^{2\alpha-1} u_{n,m} \rightarrow |x|^{2\alpha-1} u_n$ in $L^r(-1, 1)$, $\forall r < \infty$. Then the Dominated Convergence Theorem implies that $g_{p,n}(u_{n,m}) \rightarrow g_{p,n}(u_n)$ in $L^1(-1, 1)$. We again obtain (5.2). The same as the proof of Theorem 1.2, we can check that

$$\begin{aligned} \lim_{x \rightarrow 0^+} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u_n(x) &= \lim_{x \rightarrow 0^-} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u_n(x) = \frac{1}{2} \mu(\{0\}), \quad \text{if } \alpha = \frac{1}{2}, \\ \lim_{x \rightarrow 0^+} |x|^{2\alpha-1} u_n(x) &= \lim_{x \rightarrow 0^-} |x|^{2\alpha-1} u_n(x) = \frac{1}{2(2\alpha-1)} \mu(\{0\}), \quad \text{if } \frac{1}{2} < \alpha < 1. \end{aligned}$$

Therefore, u_n is a good solution of (1.11) such that $\|g_{p,n}(u_n)\|_{L^1} \leq \|\mu\|_{\mathcal{M}}$ and $\|(|x|^{2\alpha} u'_n)'\|_{\mathcal{M}} \leq 2 \|\mu\|_{\mathcal{M}}$.

If $\alpha \geq 1$, as $m \rightarrow \infty$, we obtain $|x|^{2\alpha} u'_{n,m} \rightarrow |x|^{2\alpha} u'_n$ in $L^r(-1, 1)$, $\forall r < \infty$, and $u_{n,m} \rightarrow u_n$ uniformly on any closed interval $I \subset [-1, 1] \setminus \{0\}$. Passing to the limit as $m \rightarrow \infty$, we have $\|g_{p,n}(u_n)\|_{L^1} \leq \|\mu\|_{\mathcal{M}}$ and

$$\int_{-1}^1 |x|^{2\alpha} u'_n \zeta' dx + \int_{-1}^1 g_{p,n}(u_n) \zeta dx = \int_{-1}^1 \zeta d\mu, \quad \forall \zeta \in C_c^1((-1, 1) \setminus \{0\}).$$

The same as the proof of Theorem 1.4 in [23], we have that u_n is a solution of (1.11) if and only if $\mu(\{0\}) = 0$. If u_n is a solution, it clearly satisfies $\|g_{p,n}(u_n)\|_{L^1} \leq \|\mu\|_{\mathcal{M}}$ and $\|(|x|^{2\alpha} u'_n)'\|_{\mathcal{M}} \leq 2 \|\mu\|_{\mathcal{M}}$.

We now proof the uniqueness. Assume that $u_n^{(1)}$ and $u_n^{(2)}$ are two solutions of (1.11) corresponding to μ . Then $u_n^{(1)} - u_n^{(2)} \in D(A_\alpha)$ and

$$-(|x|^{2\alpha} (u_n^{(1)} - u_n^{(2)})')' + g_{p,n}(u_n^{(1)}) - g_{p,n}(u_n^{(2)}) = 0.$$

Assertion (iv) of Proposition 2.3 implies that

$$-\int_{-1}^1 (|x|^{2\alpha} (u_n^{(1)} - u_n^{(2)})')' \text{sign}(u_n^{(1)} - u_n^{(2)}) dx \geq 0.$$

Therefore, $g_{p,n}(u_n^{(1)}) = g_{p,n}(u_n^{(2)})$ and $u_n^{(1)} = u_n^{(2)}$ a.e. \square

We now prove Theorems 1.6, 1.7 and 1.8. Actually, we will prove the following result with a more accurate mode of convergence.

Theorem 5.2. *As $n \rightarrow \infty$, we have*

$$|x|^{2\alpha} u'_n \rightarrow |x|^{2\alpha} u' \text{ in } L^r(-1, 1), \quad \forall r < \infty. \tag{5.3}$$

Moreover,

$$u_n \rightarrow u \text{ in } C_0[-1, 1], \text{ if } 0 < \alpha < \frac{1}{2}, \tag{5.4}$$

$$\left(1 + \ln \frac{1}{|x|}\right)^{-1} u_n \rightarrow \left(1 + \ln \frac{1}{|x|}\right)^{-1} u \text{ in } L^r(-1, 1), \forall r < \infty, \text{ if } \alpha = \frac{1}{2}, \tag{5.5}$$

$$|x|^{2\alpha-1} u_n \rightarrow |x|^{2\alpha-1} u \text{ in } L^r(-1, 1), \forall r < \infty, \text{ if } \alpha > \frac{1}{2}. \tag{5.6}$$

Here u is the unique good solution of (1.1) if α and p satisfy (1.4) or (1.5); u is the unique solution of (1.9) if α and p satisfy (1.6) or (1.7).

Proof. Assume $0 < \alpha < \frac{1}{2}$. We obtain that the sequence $\{u_n\}_{n=1}^\infty$ is bounded in $W^{1,q}(-1, 1)$ for $1 < q < \frac{1}{2\alpha}$. Hence, there exists a subsequence such that

- (i) $u_{n_k} \rightarrow u$ in $C[-1, 1]$,
- (ii) $g_{p,n_k}(u_{n_k}) \rightarrow |u|^{p-1}u$ in $L^1(-1, 1)$,
- (iii) $|x|^{2\alpha}u'_{n_k} \rightarrow |x|^{2\alpha}u'$ in $L^r(-1, 1), \forall r < \infty$.

Passing to the limit as $n_k \rightarrow \infty$, we obtain that

$$\int_{-1}^1 |x|^{2\alpha}u'\zeta' dx + \int_{-1}^1 |u|^{p-1}u\zeta dx = \int_{-1}^1 \zeta d\mu, \forall \zeta \in C_0^1[-1, 1].$$

Thus, u is the good solution of (1.1).

Assume $\alpha = \frac{1}{2}$. Denote $K^+ = \lim_{x \rightarrow 0^+} |x|u'_n(x)$ and $K^- = \lim_{x \rightarrow 0^-} |x|u'_n(x)$. Integration by parts yields, for $x \in (0, 1)$,

$$u_n(x) = \left(\ln \frac{1}{x}\right) \left(-K^+ + \int_{(0,x)} d\mu - \int_0^x g_{p,n}(u_n(s))ds\right) - \int_x^1 g_{p,n}(u_n(s)) \ln \frac{1}{s} ds + \int_{[x,1)} \ln \frac{1}{s} d\mu(s),$$

and for $x \in (0, 1)$,

$$u_n(x) = \left(\ln \frac{1}{|x|}\right) \left(K^- + \int_{(x,0)} d\mu - \int_x^0 g_{p,n}(u_n(s))ds\right) - \int_{-1}^x g_{p,n}(u_n(s)) \ln \frac{1}{|s|} ds + \int_{(-1,x]} \ln \frac{1}{|s|} d\mu(s).$$

One can check that

$$\lim_{x \rightarrow 0^+} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u_n(x) = -K^+,$$

$$\lim_{x \rightarrow 0^-} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u_n(x) = K^-.$$

Since u_n is a good solution, then $K^+ + K^- = 0$. On the other hand, $K^- - K^+ = \mu(\{0\})$. Therefore, $K^+ = -\frac{1}{2}\mu(\{0\})$ and $K^- = \frac{1}{2}\mu(\{0\})$. Furthermore, a direct computation yields that

$$\left\| \left(1 + \ln \frac{1}{|x|} \right)^{-1} u_n \right\|_{W^{1,1}} + \| |x| u'_n \|_{BV} \leq C,$$

where C is independent of n . It implies that (5.3) and (5.5) hold for a subsequence $\{u_{n_k}\}_{k=1}^\infty$. As a result, the sequence $\{g_{p,n_k}(u_{n_k})\}_{k=1}^\infty$ is equi-integrable and $g_{p,n_k}(u_{n_k}) \rightarrow |u|^{p-1}u$ in $L^1(-1, 1)$. Passing to the limit as $n_k \rightarrow \infty$, we obtain that

$$\int_{-1}^1 |x| u' \zeta' dx + \int_{-1}^1 |u|^{p-1} u \zeta dx = \int_{-1}^1 \zeta d\mu, \quad \forall \zeta \in C_0^1[-1, 1].$$

Moreover, we can check that

$$\begin{aligned} \lim_{x \rightarrow 0^+} \left(1 + \ln \frac{1}{|x|} \right)^{-1} u(x) &= \lim_{x \rightarrow 0^+} \lim_{k \rightarrow \infty} \left(1 + \ln \frac{1}{|x|} \right)^{-1} u_{n_k}(x) = -K^+ = \frac{1}{2} \mu(\{0\}), \\ \lim_{x \rightarrow 0^-} \left(1 + \ln \frac{1}{|x|} \right)^{-1} u(x) &= \lim_{x \rightarrow 0^-} \lim_{k \rightarrow \infty} \left(1 + \ln \frac{1}{|x|} \right)^{-1} u_{n_k}(x) = K^- = \frac{1}{2} \mu(\{0\}). \end{aligned}$$

Thus, u is the good solution of (1.1).

Assume $\alpha > \frac{1}{2}$. Denote $K^+ = \lim_{x \rightarrow 0^+} |x|^{2\alpha} u'_n(x)$ and $K^- = \lim_{x \rightarrow 0^-} |x|^{2\alpha} u'_n(x)$. Integration by parts yields, for $x \in (0, 1)$,

$$\begin{aligned} u_n(x) &= \frac{x^{1-2\alpha} - 1}{2\alpha - 1} \left(-K^+ + \int_{(0,x)} d\mu - \int_0^x g_{p,n}(u_n(s)) ds \right) \\ &\quad - \int_x^1 \frac{s^{1-2\alpha} - 1}{2\alpha - 1} g_{p,n}(u_n(s)) ds + \int_{[x,1)} \frac{s^{1-2\alpha} - 1}{2\alpha - 1} d\mu(s), \end{aligned}$$

and for $x \in (-1, 0)$,

$$\begin{aligned} u_n(x) &= \frac{|x|^{1-2\alpha} - 1}{2\alpha - 1} \left(K^- + \int_{(x,0)} d\mu - \int_x^0 g_{p,n}(u_n(s)) ds \right) \\ &\quad - \int_{-1}^x \frac{|s|^{1-2\alpha} - 1}{2\alpha - 1} g_{p,n}(u_n(s)) ds + \int_{(-1,x]} \frac{|s|^{1-2\alpha} - 1}{2\alpha - 1} d\mu(s). \end{aligned}$$

One can check that

$$\begin{aligned} \lim_{x \rightarrow 0^+} |x|^{2\alpha-1} u_n(x) &= -\frac{K^+}{2\alpha - 1}, \\ \lim_{x \rightarrow 0^-} |x|^{2\alpha-1} u_n(x) &= \frac{K^-}{2\alpha - 1}. \end{aligned}$$

When $\frac{1}{2} < \alpha < 1$, since u_n is the good solution, we have $K^+ + K^- = 0$. On the other hand, $K^- - K^+ = \mu(\{0\})$. Thus $K^+ = -\frac{1}{2} \mu(\{0\})$ and $K^- = \frac{1}{2} \mu(\{0\})$. When $\alpha \geq 1$, the fact that $u_n \in L^1(-1, 1)$ implies that $K^+ = K^- = 0$. For either case, we have

$$\| |x|^{2\alpha-1} u_n \|_{W^{1,1}} + \| |x|^{2\alpha} u'_n \|_{BV} \leq C,$$

where C is independent of n . It implies that (5.3) and (5.6) hold for a subsequence $\{u_{n_k}\}_{k=1}^\infty$.

If α and p satisfy (1.5), it implies that $\{g_{p,n_k}(u_{n_k})\}_{n=1}^\infty$ is equi-integrable. Therefore $g_{p,n_k}(u_{n_k}) \rightarrow |u|^{p-1}u$ in $L^1(-1, 1)$. Passing to the limit as $n_k \rightarrow \infty$, we obtain that

$$\int_{-1}^1 |x|^{2\alpha} u' \zeta' dx + \int_{-1}^1 |u|^{p-1} u \zeta dx = \int_{-1}^1 \zeta d\mu, \forall \zeta \in C_0^1[-1, 1].$$

Moreover, we can check that

$$\begin{aligned} \lim_{x \rightarrow 0^+} |x|^{2\alpha-1} u(x) &= \lim_{x \rightarrow 0^+} \lim_{k \rightarrow \infty} |x|^{2\alpha-1} u_{n_k}(x) = -\frac{1}{2\alpha-1} K^+ = \frac{1}{2(2\alpha-1)} \mu(\{0\}), \\ \lim_{x \rightarrow 0^-} |x|^{2\alpha-1} u(x) &= \lim_{x \rightarrow 0^-} \lim_{k \rightarrow \infty} |x|^{2\alpha-1} u_{n_k}(x) = \frac{1}{2\alpha-1} K^- = \frac{1}{2(2\alpha-1)} \mu(\{0\}). \end{aligned}$$

Thus, u is the good solution of (1.1).

If α and p satisfy (1.6) or (1.7), we obtain that $u_{n_k} \rightarrow u$ uniformly on any closed interval $I \subset [-1, 1] \setminus \{0\}$. Therefore,

$$\int_{-1}^1 |x|^{2\alpha} u' \zeta' dx + \int_{-1}^1 |u|^{p-1} u \zeta dx = \int_{-1}^1 \zeta d\mu, \forall \zeta \in C_c^1((-1, 1) \setminus \{0\}).$$

The same argument as in the proof of Lemma 3.7 implies that u is the solution of (1.9).

For all the above cases, the uniqueness of the limit implies that (5.3)–(5.6) hold for the whole sequence $\{u_n\}_{n=1}^\infty$. \square

If we assume the data to be L^1 , we have a further result about the mode of convergence.

Theorem 5.3. For $\alpha \geq \frac{1}{2}$ and $\mu \in L^1(-1, 1)$, the mode of convergence in (5.5) and (5.6) can be improved as

$$\begin{aligned} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u_n &\rightarrow \left(1 + \ln \frac{1}{|x|}\right)^{-1} u \text{ in } C_0[-1, 1], \text{ if } \alpha = \frac{1}{2}, \\ |x|^{2\alpha-1} u_n &\rightarrow |x|^{2\alpha-1} u \text{ in } C_0[-1, 1], \text{ if } \alpha > \frac{1}{2}. \end{aligned}$$

The proof of Theorem 5.3 is just the same as the one of Theorem 5.2 in [23], except some obvious modifications due to the nonlinear term. We omit the detail.

Remark 5.1. The choice of $g_{p,n}$ can be more general than the one given by (1.10). In fact, assume that $g_{p,n}$ satisfies

- (i) $g_{p,n} \in C(\mathbb{R})$, nondecreasing,
- (ii) $0 \leq g_{p,1}(t) \leq g_{p,2}(t) \leq \dots \leq |t|^{p-1}t$, for $t \in (0, \infty)$,
- (iii) $|t|^{p-1}t \leq \dots \leq g_{p,2}(t) \leq g_{p,1}(t) \leq 0$, for $t \in (-\infty, 0)$,
- (iv) $g_{p,n}(t) \rightarrow |t|^{p-1}t$, as $n \rightarrow \infty$,
- (v) for each $p > 1$ and $n \in \mathbb{N}$, there exist constants $C = C(p, n) > 0$ and $M = M(p, n) > 0$ such that

$$\begin{cases} |g_{p,n}(t)| \leq C|t|, & \text{for } |t| \in (M, \infty), \text{ if } 0 < \alpha < 1, \\ |g_{p,n}(t)| = C|t|, & \text{for } |t| \in (M, \infty), \text{ if } \alpha \geq 1. \end{cases}$$

Then all the results in this section still hold and the proof remains the same.

6. The lack of stability of the good solution for $\frac{1}{2} \leq \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$

This section is devoted to the question of stability of the solution with respect to the perturbation of the measure μ under the weak-star topology. Recall that Lemma 3.3 implies that when $0 < \alpha < \frac{1}{2}$ and $p > 1$ the unique good solution is stable. Lemma 3.7 implies that when α and p satisfy (1.6) or (1.7) and $\mu(\{0\}) = 0$, the unique solution is stable. Therefore, we only investigate the stability of the good solution when $\frac{1}{2} \leq \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$. In this case, as we pointed out in Remark 1.4, the stability of the good solution fails.

Assume $\frac{1}{2} \leq \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$. Given $\mu \in \mathcal{M}(-1, 1)$, there exists a sequence $\{f_n\}_{n=1}^\infty \subset L^1(-1, 1)$ such that $f_n \xrightarrow{*} \mu$ in $(C_0[-1, 1])^*$. Let u_n be the unique good solution of the following equation

$$\begin{cases} -(|x|^{2\alpha}u_n')' + |u_n|^{p-1}u_n = f_n & \text{on } (-1, 1), \\ u_n(-1) = u_n(1) = 0. \end{cases} \tag{6.1}$$

By Proposition 3.1, we know that $u_n \in D(A_\alpha) \cap L^p(-1, 1)$ and

$$\int_{-1}^1 |x|^{2\alpha}u_n'\zeta' dx + \int_{-1}^1 |u_n|^{p-1}u_n\zeta dx = \int_{-1}^1 f_n\zeta dx, \quad \forall \zeta \in C_0^1[-1, 1]. \tag{6.2}$$

The limiting behavior of the sequence $\{u_n\}_{n=1}^\infty$ is sensitive to the choice for the sequence $\{f_n\}_{n=1}^\infty$.

Theorem 6.1. Assume that $\frac{1}{2} \leq \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$. Take $\rho \in C(\mathbb{R})$ such that $\text{supp } \rho = [-1, 1]$, $\rho(x) = \rho(-x)$ and $\rho \geq 0$. Let $C^{-1} = \int \rho$ and $\rho_n(x) = Cn\rho(nx)$. For fixed $\tau \in \mathbb{R}$, take

$$f_n = \mu * \rho_n + \tau (Cn\rho(nx - 1) - Cn\rho(nx + 1)). \tag{6.3}$$

Then $f_n \xrightarrow{*} \mu$ in $(C_0[-1, 1])^*$. Let u_n be the unique good solution of (6.1). Then as $n \rightarrow \infty$, we have

$$\left(1 + \ln \frac{1}{|x|}\right)^{-1} u_n \rightarrow \left(1 + \ln \frac{1}{|x|}\right)^{-1} u \text{ in } L^r(-1, 1), \quad \forall r < \infty, \text{ if } \alpha = \frac{1}{2}, \tag{6.4}$$

$$|x|^{2\alpha-1}u_n \rightarrow |x|^{2\alpha-1}u \text{ in } L^r(-1, 1), \quad \forall r < \infty, \text{ if } \frac{1}{2} < \alpha < 1, \tag{6.5}$$

where u is a solution of (1.1) such that, if $\alpha = \frac{1}{2}$,

$$\begin{cases} \lim_{x \rightarrow 0^+} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u(x) = - \lim_{x \rightarrow 0^+} |x|u'(x) = \frac{1}{2}\mu(\{0\}) + \tau, \\ \lim_{x \rightarrow 0^-} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u(x) = \lim_{x \rightarrow 0^-} |x|u'(x) = \frac{1}{2}\mu(\{0\}) - \tau, \end{cases} \tag{6.6}$$

and if $\frac{1}{2} < \alpha < 1$,

$$\begin{cases} \lim_{x \rightarrow 0^+} |x|^{2\alpha-1}u(x) = -\frac{1}{2\alpha-1} \lim_{x \rightarrow 0^+} |x|^{2\alpha}u'(x) = \frac{\mu(\{0\})}{2(2\alpha-1)} + \frac{\tau}{2\alpha-1}, \\ \lim_{x \rightarrow 0^-} |x|^{2\alpha-1}u(x) = \frac{1}{2\alpha-1} \lim_{x \rightarrow 0^-} |x|^{2\alpha}u'(x) = \frac{\mu(\{0\})}{2(2\alpha-1)} - \frac{\tau}{2\alpha-1}. \end{cases} \tag{6.7}$$

Remark 6.1. A straightforward consequence of Theorem 6.1 is that the limiting function u is the good solution if and only if $\tau = 0$. This means that, in general, the stability of the good solution fails.

Proof of Theorem 6.1. Note that we already have (3.3)–(3.8) by Lemmas 3.4 and 3.5. Also note that since u_{n_k} is the good solution of (6.1), we have

$$|x|^{2\alpha}u_{n_k}'(x) = \int_0^x \left(|u_{n_k}(s)|^{p-1}u_{n_k}(s) - f_{n_k}(s)\right) ds, \quad \forall x \in (-1, 1).$$

Therefore,

$$\lim_{x \rightarrow 0^+} |x|^{2\alpha}u'(x) = \lim_{x \rightarrow 0^+} \lim_{k \rightarrow \infty} |x|^{2\alpha}u_{n_k}'(x) = - \lim_{x \rightarrow 0^+} \lim_{k \rightarrow \infty} \int_0^x f_{n_k}(s) ds.$$

Similarly,

$$\lim_{x \rightarrow 0^-} |x|^{2\alpha} u'(x) = \lim_{x \rightarrow 0^-} \lim_{k \rightarrow \infty} \int_x^0 f_{n_k}(s) ds.$$

Then taking into account (6.3), one can obtain (6.6) and (6.7). Finally, the uniqueness of the limit implies (6.4) and (6.5). □

If $\mu \in L^1(-1, 1)$ and the convergence is under the weak topology $\sigma(L^1, L^\infty)$, we can recover the stability of the good solution.

Theorem 6.2. Assume that $\frac{1}{2} \leq \alpha < 1$, $1 < p < \frac{1}{2\alpha-1}$ and $\mu \in L^1(-1, 1)$. Let the sequence $\{f_n\}_{n=1}^\infty \subset L^1(-1, 1)$ be such that $f_n \rightharpoonup \mu$ weakly in $\sigma(L^1, L^\infty)$. Let u_n be the unique good solution of (6.1). Then as $n \rightarrow \infty$, we have

$$\left(1 + \ln \frac{1}{|x|}\right)^{-1} u_n \rightarrow \left(1 + \ln \frac{1}{|x|}\right)^{-1} u \text{ in } C_0[-1, 1], \text{ if } \alpha = \frac{1}{2}, \tag{6.8}$$

$$|x|^{2\alpha-1} u_n \rightarrow |x|^{2\alpha-1} u \text{ in } C_0[-1, 1], \text{ if } \frac{1}{2} < \alpha < 1, \tag{6.9}$$

where u is the good solution of (1.1).

The proof of Theorem 6.2 is the same as the one of Theorem 6.2 in [23], except some obvious modifications due to the nonlinear term. We omit the detail.

7. The non-uniqueness for the cases (1.4) and (1.5)

Throughout this section, we assume that α and p satisfy (1.4) or (1.5). We present a complete description of all the solutions of (1.1). Note that if u is a solution of (1.1), then we have

$$\lim_{x \rightarrow 0^+} |x|^{2\alpha} u'(x) - \lim_{x \rightarrow 0^-} |x|^{2\alpha} u'(x) = -\mu(\{0\}).$$

On the other hand, we have

Theorem 7.1. Assume that α and p satisfy (1.4) or (1.5). For any $\tau \in \mathbb{R}$ and any $\mu \in \mathcal{M}(-1, 1)$, there exists a unique solution u of (1.1) such that

$$\begin{cases} \lim_{x \rightarrow 0^+} |x|^{2\alpha} u'(x) = \tau, \\ \lim_{x \rightarrow 0^-} |x|^{2\alpha} u'(x) = \tau + \mu(\{0\}). \end{cases} \tag{7.1}$$

Proof. We first prove uniqueness. For any $\tau \in \mathbb{R}$ and any $\mu \in \mathcal{M}(-1, 1)$, assume that both u_1 and u_2 are solutions of (1.1) satisfying (7.1). Then

$$-(|x|^{2\alpha}(u_1 - u_2)')' + |u_1|^{p-1}u_1 - |u_2|^{p-1}u_2 = 0,$$

and $\lim_{x \rightarrow 0} |x|^{2\alpha}(u_1 - u_2)'(x) = 0$. When $0 < \alpha < \frac{1}{2}$, take $\phi \in C^\infty(\mathbb{R})$ such that $\phi(0) = 0$, $\phi' \geq 0$, $\phi > 0$ on $(0, +\infty)$, $\phi < 0$ on $(-\infty, 0)$, and $\phi = \text{sign}$ on $\mathbb{R} \setminus (-1, 1)$. Since $u_1 - u_2 \in W^{1,1}(0, 1)$, we have

$$\int_0^1 (|x|^{2\alpha}(u_1 - u_2)')' \phi(u_1 - u_2) dx = - \int_0^1 |x|^{2\alpha} ((u_1 - u_2)')^2 \phi'(u_1 - u_2) dx \leq 0.$$

Therefore,

$$\int_0^1 (|u_1|^{p-1}u_1 - |u_2|^{p-1}u_2) \phi(u_1 - u_2) dx = 0.$$

It implies that $u_1 = u_2$ a.e. on $(0, 1)$. The same argument implies that $u_1 = u_2$ a.e. on $(-1, 0)$. When $\frac{1}{2} \leq \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$, by Proposition 2.2, we have $u_1 - u_2 \in D(A_\alpha)$. Assertion (iv) of Proposition 2.3 implies that

$$\int_{-1}^1 (|x|^{2\alpha}(u_1 - u_2)')' \operatorname{sign}(u_1 - u_2) dx \leq 0.$$

Therefore, $u_1 = u_2$ a.e. on $(-1, 1)$.

Next we prove the existence when $0 < \alpha < \frac{1}{2}$ and $p > 1$. We first claim that for every $v \in \mathcal{M}(0, 1)$ and $\tau \in \mathbb{R}$, there exists $v \in W^{1,1}(0, 1)$ such that $x^{2\alpha}v' \in BV(0, 1)$ and

$$\begin{cases} -(x^{2\alpha}v')' + |v|^{p-1}v = v & \text{on } (0, 1), \\ v(1) = 0, \lim_{x \rightarrow 0^+} x^{2\alpha}v'(x) = \tau. \end{cases} \tag{7.2}$$

Indeed, define a nonlinear operator $A : C[0, 1] \rightarrow C[0, 1]$ as

$$\begin{aligned} Av(x) &= \frac{1 - x^{1-2\alpha}}{1 - 2\alpha} \int_0^x |v(s)|^{p-1}v(s)ds + \int_x^1 |v(s)|^{p-1}v(s) \frac{1 - s^{1-2\alpha}}{1 - 2\alpha} ds \\ &\quad - \int_x^1 \frac{1}{t^{2\alpha}} \int_{(0,t)} dv dt + \tau \frac{1 - x^{1-2\alpha}}{1 - 2\alpha}. \end{aligned}$$

It is clear that A is continuous. Take

$$X^\alpha = \left\{ v \in H_{loc}^1(0, 1); v \in L^2(0, 1), x^\alpha v' \in L^2(0, 1), v(1) = 0 \right\}, \tag{7.3}$$

with the norm

$$\|v\|_{X^\alpha} = \|v\|_{L^2} + \|x^\alpha v'\|_{L^2}.$$

It is easy to check that X^α is compact in $C[0, 1]$ and $A(X^\alpha) \subset X^\alpha$ (see, e.g., [11]). Therefore, the Schauder Fixed Point Theorem implies that there exists a fixed point $v \in X^\alpha$ such that $v = Av$. This fixed point v is precisely a solution of (7.2).

For any $\mu \in \mathcal{M}(-1, 1)$, take $\mu_1 = \mu|_{(0,1)}$ and $\mu_2 = \mu|_{(-1,0)}$. For any $\tau \in \mathbb{R}$, we deduce from the above claim that there exist $u_1 \in W^{1,1}(0, 1)$ and $u_2 \in W^{1,1}(-1, 0)$ such that $x^{2\alpha}u_1' \in BV(0, 1)$ and $|x|^{2\alpha}u_2' \in BV(-1, 0)$, which satisfy

$$\begin{cases} -(x^{2\alpha}u_1')' + |u_1|^{p-1}u_1 = \mu_1 & \text{on } (0, 1), \\ u_1(1) = 0, \lim_{x \rightarrow 0^+} x^{2\alpha}u_1'(x) = \tau, \end{cases}$$

and

$$\begin{cases} -(|x|^{2\alpha}u_2')' + |u_2|^{p-1}u_2 = \mu_2 & \text{on } (-1, 0), \\ u_2(-1) = 0, \lim_{x \rightarrow 0^-} |x|^{2\alpha}u_2'(x) = \tau + \mu(\{0\}). \end{cases}$$

Take

$$u = \begin{cases} u_1 & \text{on } (0, 1), \\ u_2 & \text{on } (-1, 0). \end{cases}$$

Then u is a solution of (1.1) satisfying (7.1).

When $\frac{1}{2} \leq \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$, the existence of the solution of (1.1) with property (7.1) is a direct consequence of Theorem 6.1. \square

8. Removable singularity

In this section, we prove [Theorem 1.9](#). The idea of the proof is the same as Brezis–Véron [\[10\]](#) and Brezis [\[5\]](#).

Lemma 8.1. *Assume that $\alpha > 0$, $p > 1$ and $f \in L^1(-1, 1)$. Let $u \in L^p_{loc}((-1, 1) \setminus \{0\})$ be such that*

$$-\int_{-1}^1 u(|x|^{2\alpha} \zeta')' dx + \int_{-1}^1 |u|^{p-1} u \zeta dx = \int_{-1}^1 f \zeta dx, \quad \forall \zeta \in C_c^\infty((-1, 1) \setminus \{0\}).$$

Then $u \in W^{2,1}_{loc}((-1, 1) \setminus \{0\})$ and

$$-(|x|^{2\alpha} u')' + |u|^{p-1} u = f \quad \text{on } (a, b), \quad \forall (a, b) \subset\subset (-1, 1) \setminus \{0\}.$$

The proof of [Lemma 8.1](#) is standard.

Lemma 8.2. *Assume that $\alpha > 0$, $p > 1$ and $f \in L^1(-1, 1)$. Assume that $u \in W^{2,1}_{loc}((-1, 1) \setminus \{0\})$ and*

$$-(|x|^{2\alpha} u')' + |u|^{p-1} u = f \quad \text{on } (a, b), \quad \forall (a, b) \subset\subset (-1, 1) \setminus \{0\}.$$

Then

$$-\int_{-1}^1 u^+ (|x|^{2\alpha} \zeta')' dx + \int_{-1}^1 (u^+)^p \zeta dx \leq \int_{-1}^1 f^+ \zeta dx, \quad \forall \zeta \in C_c^\infty((-1, 1) \setminus \{0\}) \text{ and } \zeta \geq 0. \tag{8.1}$$

Proof. Denote $\mathcal{L}u = (|x|^{2\alpha} u')'$. Fix an interval $(a, b) \subset\subset (-1, 1) \setminus \{0\}$. We recall the following Kato’s inequality (Lemma A in [\[19\]](#)),

$$\mathcal{L}|u| \geq (\mathcal{L}u) \operatorname{sign} u \quad \text{in } \mathcal{D}'(a, b).$$

By the same argument as Lemma 1 of [\[10\]](#), we obtain

$$\mathcal{L}(u^+) \geq (\mathcal{L}u) \operatorname{sign}^+ u \quad \text{in } \mathcal{D}'(a, b), \tag{8.2}$$

where

$$\operatorname{sign}^+ x = \begin{cases} 1 & \text{when } x > 0, \\ \frac{1}{2} & \text{when } x = 0, \\ 0 & \text{when } x < 0. \end{cases}$$

Since $\mathcal{L}u = |u|^{p-1} u - f$ on (a, b) , it implies that

$$\mathcal{L}(u^+) \geq |u|^{p-1} u \operatorname{sign}^+ u - f^+ = (u^+)^p - f^+ \quad \text{in } \mathcal{D}'(a, b).$$

Therefore

$$-\int_a^b u^+ (|x|^{2\alpha} \zeta')' dx + \int_{-1}^1 (u^+)^p \zeta dx \leq \int_{-1}^1 f^+ \zeta dx, \quad \forall \zeta \in C_c^\infty(a, b) \text{ and } \zeta \geq 0.$$

Since (a, b) is arbitrary in $(-1, 1) \setminus \{0\}$, we derived [\(8.1\)](#). \square

Lemma 8.3 (Maximum principle). *Let $\alpha > 0$. Assume that $(a, b) \subset\subset (-1, 1) \setminus \{0\}$ and $u \in L^1(a, b)$ satisfying $u \geq 0$ a.e., $\operatorname{supp} u \subset\subset (a, b)$ and*

$$(|x|^{2\alpha} u')' \geq 0 \quad \text{in } \mathcal{D}'(a, b).$$

Then $u = 0$ a.e. on (a, b) .

Proof. Assume $\text{supp } u \subset (\bar{a}, \bar{b}) \subset\subset (\bar{a}, \bar{b}) \subset\subset (a, b)$. Take the positive smooth mollifiers $\rho_n(x) = Cn\rho(nx)$ where $\rho(x) = \chi_{[|x|<1]}e^{-\frac{1}{|x|^2-1}}$ and $C^{-1} = \int \rho$. Consider $u_n = u * \rho_n$ with n large enough such that $(\bar{a} - \frac{1}{n}, \bar{b} + \frac{1}{n}) \subset (a, b)$. Notice that $u_n \geq 0$ and $u_n \in C_c^\infty(a, b)$. We claim that

$$\int_a^b (|x|^{2\alpha} u_n')' \zeta dx \geq 0, \forall \zeta \in C_c^\infty(a, b) \text{ with } \zeta \geq 0. \tag{8.3}$$

Indeed, we have

$$\begin{aligned} \int_a^b (|x|^{2\alpha} u_n')' \zeta dx &= \int_a^b u_n (|x|^{2\alpha} \zeta')' dx \\ &= \int_{-\frac{1}{n}}^{\frac{1}{n}} \rho_n(y) \left(\int_{\bar{a}}^{\bar{b}} u(z) (|z+y|^{2\alpha} \zeta'(z+y))' dz \right) dy. \end{aligned}$$

It is enough to show

$$\int_{\bar{a}}^{\bar{b}} u(z) (|z+y|^{2\alpha} \zeta'(z+y))' dz \geq 0, \forall y \in (-\frac{1}{n}, \frac{1}{n}), \forall \zeta \in C_c^\infty(a, b) \text{ with } \zeta \geq 0.$$

We already know

$$\int_a^b u(z) (|z|^{2\alpha} \varphi'(z))' dz \geq 0, \forall \varphi \in C_c^\infty(a, b) \text{ with } \varphi \geq 0.$$

Given $y \in (-\frac{1}{n}, \frac{1}{n})$ and $\zeta \in C_c^\infty(a, b)$ with $\zeta \geq 0$, define

$$\bar{\varphi}(z) = \int_{\bar{a}}^z \frac{|t+y|^{2\alpha}}{|t|^{2\alpha}} \zeta'(t+y) dt + \int_{\bar{a}}^{\bar{b}} \frac{|t+y|^{2\alpha}}{|t|^{2\alpha}} |\zeta'(t+y)| dt \quad \text{on } [\bar{a}, \bar{b}].$$

Take $\varphi = \bar{\varphi}h$ where h is the cut-off function such that $h \in C_c^\infty(a, b)$, $h \geq 0$, $h \equiv 1$ on (\bar{a}, \bar{b}) and $\text{supp } h \subset (\bar{a}, \bar{b})$. Then $\varphi \in C_c^\infty(a, b)$ with $\varphi \geq 0$. Therefore

$$\int_{\bar{a}}^{\bar{b}} u(z) (|z+y|^{2\alpha} \zeta'(z+y))' dz = \int_a^b u(z) (|z|^{2\alpha} \varphi'(z))' dz \geq 0.$$

Thus we proved (8.3). It implies that $(|x|^{2\alpha} u_n')' \geq 0$ on (a, b) . The standard Maximum Principle yields that $u_n = 0$. Since $u_n \rightarrow u$ in $L^1(a, b)$, we have $u = 0$ a.e. on (a, b) . \square

Lemma 8.4 (Keller–Osserman estimate). Assume that $\alpha > 0$, $p > 1$ and $f \in L^1(-1, 1)$. Let $u \in W_{loc}^{2,1}((-1, 1) \setminus \{0\})$ be such that

$$-(|x|^{2\alpha} u')' + |u|^{p-1} u = f \quad \text{on } (a, b), \forall (a, b) \subset\subset (-1, 1) \setminus \{0\}.$$

Then

$$u(x) \leq C(\alpha, p) |x|^{\frac{2\alpha-2}{p-1}} + u_0(x), \forall 0 < |x| \leq \frac{1}{2}, \tag{8.4}$$

where $C(\alpha, p)$ is a positive constant depending only on α and p , and $u_0 \in D(A_\alpha) \cap L^p(-1, 1)$ is the unique solution of

$$\begin{cases} -(|x|^{2\alpha}u_0')' + u_0^p = |f| & \text{on } (-1, 1), \\ u_0(-1) = u_0(1) = 0. \end{cases}$$

Proof. We fix x_0 such that $0 < |x_0| \leq \frac{1}{2}$. Consider the interval

$$I_{x_0} = \left(x_0 - \frac{|x_0|}{2}, x_0 + \frac{|x_0|}{2}\right) \subset\subset (-1, 1) \setminus \{0\}.$$

Define

$$v(x) = \lambda \left(\frac{|x_0|^2}{4} - (x - x_0)^2\right)^{-\frac{2}{p-1}} \quad \text{on } I_{x_0},$$

where $\lambda > 0$ is a constant to be determined so that

$$-(|x|^{2\alpha}v')' + v^p \geq 0 \quad \text{on } I_{x_0}. \tag{8.5}$$

Indeed, we have

$$(|x|^{2\alpha}v')' = \frac{4\lambda}{p-1} \left(\frac{|x_0|^2}{4} - (x - x_0)^2\right)^{-\frac{2}{p-1}-2} \times J$$

where

$$J = 2 \left(\frac{2}{p-1} + 1\right) (x - x_0)^2 |x|^{2\alpha} + \left(\frac{|x_0|^2}{4} - (x - x_0)^2\right) (|x|^{2\alpha} + 2\alpha(x - x_0)|x|^{2\alpha-1} \text{sign } x).$$

Since $x \in I_{x_0}$, we have $|J| \leq A(\alpha)|x_0|^{2\alpha+2}$ where $A(\alpha)$ is a constant only depending on α . Notice that $-\frac{2}{p-1} - 2 = -\frac{2p}{p-1}$. Therefore,

$$-(|x|^{2\alpha}v')' + v^p \geq \left(-A(\alpha)\frac{4\lambda}{p-1}|x_0|^{2\alpha+2} + \lambda^p\right) \left(\frac{|x_0|^2}{4} - (x - x_0)^2\right)^{-\frac{2p}{p-1}}.$$

Take λ such that

$$-A(\alpha)\frac{4\lambda}{p-1}|x_0|^{2\alpha+2} + \lambda^p = 0,$$

i.e.

$$\lambda = \left(\frac{4A(\alpha)}{p-1}|x_0|^{2\alpha+2}\right)^{\frac{1}{p-1}}.$$

Then the inequality (8.5) holds. Now take $\bar{v} = v + u_0$ which satisfies

$$-(|x|^{2\alpha}\bar{v}')' + \bar{v}^p \geq |f| \quad \text{on } I_{x_0}.$$

Denote $\mathcal{L}u = (|x|^{2\alpha}u')'$. We have

$$\mathcal{L}(u - \bar{v}) \geq |u|^{p-1}u - \bar{v}^p \quad \text{on } I_{x_0}.$$

Applying the revised Kato's inequality (8.2), we obtain

$$\mathcal{L}((u - \bar{v})^+) \geq (|u|^{p-1}u - \bar{v}^p) \text{sign}^+(u - \bar{v}) \geq 0 \quad \text{in } \mathcal{D}'(I_{x_0}).$$

Notice that $\lim_{x \rightarrow \partial I_{x_0}} \bar{v}(x) = +\infty$ and $u \in L^\infty(I_{x_0})$. It follows that $(u - \bar{v})^+ = 0$ near ∂I_{x_0} . Then Lemma 8.3 implies that $(u - \bar{v})^+ = 0$ on I_{x_0} . In particular,

$$u(x_0) \leq \bar{v}(x_0) = \left(\frac{1}{4}\right)^{-\frac{2}{p-1}} \left(\frac{4A(\alpha)}{p-1}\right)^{\frac{1}{p-1}} |x_0|^{\frac{2\alpha-2}{p-1}} + u_0(x_0).$$

Let $C(\alpha, p) = \left(\frac{1}{4}\right)^{-\frac{2}{p-1}} \left(\frac{4A(\alpha)}{p-1}\right)^{\frac{1}{p-1}}$. Note that x_0 is arbitrary in $(0, \frac{1}{2}]$, so we obtain (8.4). \square

Lemma 8.5. *Under the assumption of Theorem 1.9, we have $u \in L^p_{loc}(-1, 1)$.*

Proof. We first prove that $u^+ \in L^p_{loc}(-1, 1)$. Applying Lemmas 8.1 and 8.2, we find

$$-\int_{-1}^1 u^+ (|x|^{2\alpha} \zeta')' dx + \int_{-1}^1 (u^+)^p \zeta dx \leq \int_{-1}^1 f^+ \zeta dx, \quad \forall \zeta \in C_c^\infty((-1, 1) \setminus \{0\}) \text{ with } \zeta \geq 0.$$

Take $\varphi(x) \in C^\infty(\mathbb{R})$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 0$ on $(-\frac{1}{2}, \frac{1}{2})$ and $\varphi \equiv 1$ on $\mathbb{R} \setminus (-1, 1)$. Define $\varphi_n(x) = \varphi(nx) \in C^\infty[-1, 1]$. For any $\zeta \in C_c^\infty(-1, 1)$ with $\zeta \geq 0$, we have

$$\int_{-1}^1 (u^+)^p \varphi_n \zeta dx \leq \int_{-1}^1 u^+ (|x|^{2\alpha} (\varphi_n \zeta)')' dx + \int_{-1}^1 f^+ \varphi_n \zeta dx.$$

Notice that

$$\begin{aligned} \int_{-1}^1 u^+ (|x|^{2\alpha} (\varphi_n \zeta)')' dx &= 2\alpha n \int_{-\frac{1}{n}}^{\frac{1}{n}} u^+ \operatorname{sign} x |x|^{2\alpha-1} \varphi'(nx) \zeta dx + 2\alpha \int_{-1}^1 u^+ \operatorname{sign} x |x|^{2\alpha-1} \varphi(nx) \zeta' dx \\ &\quad + \int_{-1}^1 u^+ |x|^{2\alpha} \varphi_n \zeta'' dx + 2n \int_{-\frac{1}{n}}^{\frac{1}{n}} u^+ |x|^{2\alpha} \varphi'(nx) \zeta' dx + n^2 \int_{-\frac{1}{n}}^{\frac{1}{n}} u^+ |x|^{2\alpha} \varphi''(nx) \zeta dx. \end{aligned}$$

In view of Lemma 8.4 and Proposition 3.1, we know

$$\left\| u^+ |x|^{2\alpha-1} \right\|_{L^\infty(-\frac{1}{2}, \frac{1}{2})} + \left\| nu^+ |x|^{2\alpha} \right\|_{L^\infty(-\frac{1}{n}, \frac{1}{n})} \leq C,$$

where C is independent of n . Also notice that

$$\int_{-\frac{1}{n}}^{\frac{1}{n}} n |\varphi'(nx)| dx = \int_{-1}^1 |\varphi'(x)| dx$$

and

$$\int_{-\frac{1}{n}}^{\frac{1}{n}} n |\varphi''(nx)| dx = \int_{-1}^1 |\varphi''(x)| dx.$$

Therefore,

$$\int_{-1}^1 u^+ (|x|^{2\alpha} (\varphi_n \zeta)')' dx \leq C,$$

where C is independent of n . It implies that

$$\int_{-1}^1 (u^+)^p \varphi_n \zeta \, dx \leq C.$$

Passing to the limit as $n \rightarrow \infty$, we have $(u^+)^p \zeta \in L^1(-1, 1)$. Hence, $u^+ \in L^p_{loc}(-1, 1)$. Similarly, $u^- \in L^p_{loc}(-1, 1)$. \square

Proof of Theorem 1.9. Take $\varphi(x) \in C^\infty(\mathbb{R})$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 0$ on $(-\frac{1}{2}, \frac{1}{2})$ and $\varphi \equiv 1$ on $\mathbb{R} \setminus (-1, 1)$. Define $\varphi_n(x) = \varphi(nx) \in C^\infty[-1, 1]$. Then we have

$$-\int_{-1}^1 u(|x|^{2\alpha}(\varphi_n \zeta)')' \, dx + \int_{-1}^1 |u|^{p-1} u \varphi_n \zeta \, dx = \int_{-1}^1 f \varphi_n \zeta \, dx, \quad \forall \zeta \in C_c^\infty(-1, 1). \tag{8.6}$$

Note that $u \in L^p_{loc}(-1, 1)$ by Lemma 8.5. Passing to the limit as $n \rightarrow \infty$ in (8.6), the same argument as in the proof of Lemma 3.7 implies (1.12). \square

9. Classification of singularity

In this section, we prove Theorem 1.10. The proof combines ideas by Véron [20,21] and Brezis and Oswald [7].

Lemma 9.1. Assume that $\alpha > 0$ and $p \geq 1$. Let $u \in C^2(0, 1]$ satisfying (1.13). Then u cannot change signs, i.e., either $u \geq 0$, or $u \leq 0$ on $(0, 1]$.

Proof. For a fixed $t \in (0, 1)$, multiply (1.13) on both sides by $u(x)$ and integrate by parts on the interval $(t, 1)$. We obtain that

$$-\frac{1}{2} t^{2\alpha} \frac{d}{dt} (u^2(t)) = \int_t^1 x^{2\alpha} u'(x) u'(x) \, dx + \int_t^1 |u(x)|^{p+1} \, dx \geq 0.$$

It implies that $|u|$ is decreasing on $(0, 1]$ and therefore changing sign is not permitted for u . \square

Lemma 9.2. Assume that $\alpha > 0$ and $p > 1$. Let $u \in C^2(0, 1]$ be such that $u \geq 0$ and u satisfies (1.13). Let

$$v(r) = \left(\frac{1}{1-\alpha} \right)^{\frac{2}{p-1}} u \left(r^{\frac{1}{1-\alpha}} \right) \in C^2(0, 1]. \tag{9.1}$$

Then v solves

$$\begin{cases} -v''(r) - \left(\frac{\alpha}{1-\alpha} \right) \frac{1}{r} v'(r) + v^p(r) = 0 & \text{on } (0, 1), \\ v(1) = 0. \end{cases} \tag{9.2}$$

Moreover $r^{\frac{2}{p-1}} v(r) \in L^\infty(0, 1)$.

Proof. One can directly check that v solves (9.2). From Lemma 8.4, we have $x^{\frac{2-2\alpha}{p-1}} u(x) \in L^\infty(0, 1)$. Therefore $r^{\frac{2}{p-1}} v(r) \in L^\infty(0, 1)$. \square

Lemma 9.3. Assume that α and p satisfy (1.4) or (1.5). Assume that $v \in C^2(0, 1]$, $v \geq 0$ and v solves (9.2). Denote

$$\bar{l}_{p,\alpha} = \left[\left(\frac{2}{p-1} \right) \left(\frac{2p}{p-1} - \frac{1}{1-\alpha} \right) \right]^{\frac{1}{p-1}}. \tag{9.3}$$

Then one of the following assertions holds.

- (i) $\lim_{r \rightarrow 0^+} r^{\frac{2}{p-1}} v(r) = \bar{l}_{p,\alpha}$.
- (ii) $\lim_{r \rightarrow 0^+} r^{\frac{2}{p-1}} v(r) = 0$.

Moreover, if v satisfies (i), then

$$\left| v(r) - \bar{l}_{p,\alpha} r^{-\frac{2}{p-1}} \right| \leq \bar{l}_{p,\alpha} r^{\frac{2p}{p-1} - \frac{1}{1-\alpha}}, \quad \forall r \in (0, 1]. \tag{9.4}$$

Proof. Write $\bar{l}_{p,\alpha}^{-1} r^{\frac{2}{p-1}} v(r) = \phi(x)$ where $x = r^{\frac{2(p+1)}{p-1} - \frac{1}{1-\alpha}}$. It is easy to obtain that $\phi(x) \in L^\infty(0, 1)$ and it solves

$$\begin{cases} x^2 \phi''(x) = \frac{\bar{l}_{p,\alpha}^{p-1}}{\left(\frac{2(p+1)}{p-1} - \frac{1}{1-\alpha}\right)^2} (\phi^p(x) - \phi(x)) & \text{on } (0, 1), \\ \phi(1) = 0. \end{cases}$$

We claim that $0 \leq \phi(x) \leq 1$. Indeed, if $\phi(x_0) > 1$ for some $x_0 \in (0, 1)$, then ϕ is convex and increasing on $(0, x_0)$. Therefore $\phi''(x) \geq \frac{c}{x^2}$ on $(0, x_0)$, and thus $\phi(x) \geq \tilde{c} - c \ln x$, which contradicts $\phi \in L^\infty(0, 1)$. Hence $0 \leq \phi(x) \leq 1$.

As a result, ϕ is concave and $\lim_{x \rightarrow 0^+} \phi(x)$ exists. If $0 < \lim_{x \rightarrow 0^+} \phi(x) < 1$, then $\phi''(x) \leq -\frac{c}{x^2}$ for x near 0, and thus $\phi(x) \leq -\tilde{c} + c \ln x$, which again contradicts $\phi \in L^\infty(0, 1)$. Therefore either $\lim_{x \rightarrow 0^+} \phi(x) = 1$ or $\lim_{x \rightarrow 0^+} \phi(x) = 0$. If $\lim_{x \rightarrow 0^+} \phi(x) = 1$, since ϕ is concave, it implies that $1 \geq \phi(x) \geq 1 - x, \forall x \in (0, 1]$, which is precisely (9.4). \square

Lemma 9.4. Assume that $\frac{1}{2} < \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$. Assume that $v \in C^2(0, 1], v \geq 0$ and v solves (9.2). If $\lim_{r \rightarrow 0^+} r^{\frac{2}{p-1}} v(r) = 0$, then there exists $\epsilon_0 > 0$ such that $r^{\frac{2}{p-1} - \epsilon_0} v(r) \in L^\infty(0, 1)$.

In order to prove Lemma 9.4, we need the following lemma from [21], which is originally due to Chen, Matano and Véron [13].

Lemma 9.5. (See Lemma 2.1 on p. 67 of [21].) Let $y(t) \in C[0, \infty)$ be such that $y \geq 0$ and

- (i) $\lim_{t \rightarrow \infty} y(t) = 0$,
- (ii) $\limsup_{t \rightarrow \infty} e^{\epsilon t} y(t) = +\infty, \forall \epsilon > 0$.

Then there exists $\eta \in C^\infty[0, \infty)$ such that

- (i) $\eta > 0, \eta' < 0, \lim_{t \rightarrow \infty} \eta(t) = 0$,
- (ii) $\lim_{t \rightarrow \infty} e^{\epsilon t} \eta(t) = +\infty, \forall \epsilon > 0$,
- (iii) $0 < \limsup_{t \rightarrow \infty} \frac{y(t)}{\eta(t)} < \infty$,
- (iv) $\left(\frac{\eta'}{\eta}\right)', \left(\frac{y'}{\eta}\right)'' \in L^1(0, \infty)$,
- (v) $\lim_{t \rightarrow \infty} \frac{\eta'(t)}{\eta(t)} = \lim_{t \rightarrow \infty} \frac{\eta''(t)}{\eta(t)} = 0$.

Proof of Lemma 9.4. Write $v(r) = r^{-\frac{2}{p-1}} y(t)$ where $t = \ln \frac{1}{r}$ and $t \in [0, \infty)$. Denote $\beta = \frac{2(p+1)}{p-1} - \frac{1}{1-\alpha}$. Then $y(t) \in C^2[0, \infty), \lim_{t \rightarrow \infty} y(t) = 0$ and $y(t)$ solves

$$\begin{cases} y''(t) + \beta y'(t) + \bar{l}_{p,\alpha}^{-1} y(t) - y^p(t) = 0 & \text{on } (0, \infty), \\ y(0) = 0. \end{cases}$$

Assume $\limsup_{t \rightarrow \infty} e^{\epsilon t} y(t) = +\infty, \forall \epsilon > 0$. Denote $w(t) = \frac{y(t)}{\eta(t)}$ where η is given by Lemma 9.5. Then $w \in L^\infty(0, \infty) \cap C^2[0, \infty)$ and w satisfies

$$w''(t) + \left(\beta + 2 \frac{\eta'(t)}{\eta(t)} \right) w'(t) = f(t) \quad \text{on } (0, \infty), \tag{9.5}$$

where

$$f(t) = \eta^{p-1}(t)w^p(t) - \left(\bar{l}_{p,\alpha}^{p-1} + \frac{\eta''(t)}{\eta(t)} + \beta \frac{\eta'(t)}{\eta(t)} \right) w(t) \in L^\infty(0, \infty).$$

We claim that

$$\lim_{t \rightarrow \infty} w'(t) = \lim_{t \rightarrow \infty} w''(t) = 0. \tag{9.6}$$

We only show $\lim_{t \rightarrow \infty} w'(t) = 0$ since one can show the other part of (9.6) by the same idea. To show $\lim_{t \rightarrow \infty} w'(t) = 0$, it is enough to obtain that w' is uniformly continuous and $w' \in L^2(0, \infty)$. To do so, we first need $w' \in L^\infty(0, \infty)$. Indeed, from (9.5) we obtain

$$(\eta^2(t)e^{\beta t} w'(t))' = \eta^2(t)e^{\beta t} f(t).$$

That is,

$$w'(t) = \frac{\int_0^t \eta^2(s)e^{\beta s} f(s) ds}{e^{\beta t} \eta^2(t)} + \frac{w'(0)\eta^2(0)}{e^{\beta t} \eta^2(t)}.$$

Note that the Mean Value Theorem yields

$$\frac{\int_0^t \eta^2(s)e^{\beta s} f(s) ds}{e^{\beta t} \eta^2(t) - \eta^2(0)} = \frac{\eta^2(\xi)e^{\beta \xi} f(\xi)}{\beta e^{\beta \xi} \eta^2(\xi) + 2e^{\beta \xi} \eta'(\xi)\eta(\xi)}, \tag{9.7}$$

where $\xi \in (0, t)$ and ξ depends on t . One can check that the right hand side of (9.7) is in $L^\infty(0, \infty)$. Therefore $w' \in L^\infty(0, \infty)$. As a consequence, w is uniformly continuous. To show the uniform continuity of w' , note that (9.5) implies

$$\left(w'(t) + \left(\beta + 2 \frac{\eta'(t)}{\eta(t)} \right) w(t) \right)' = f(t) + 2 \left(\frac{\eta'(t)}{\eta(t)} \right)' w(t). \tag{9.8}$$

One can check that the right hand side of (9.8) is in $L^\infty(0, \infty)$. Therefore $w'(t) + \left(\beta + 2 \frac{\eta'(t)}{\eta(t)} \right) w(t)$ is uniformly continuous and so is w' . Now, multiplying (9.5) by $w'(t)$, we obtain

$$\begin{aligned} & \left(\beta + 2 \frac{\eta'(t)}{\eta(t)} \right) (w'(t))^2 \\ &= -\frac{1}{2} \frac{d}{dt} (w'(t))^2 - \frac{1}{2} \frac{d}{dt} \left[\left(\bar{l}_{p,\alpha}^{p-1} + \frac{\eta''(t)}{\eta(t)} + \beta \frac{\eta'(t)}{\eta(t)} \right) w^2(t) \right] \\ & \quad + \frac{1}{2} \left(\frac{\eta''(t)}{\eta(t)} + \beta \frac{\eta'(t)}{\eta(t)} \right)' w^2(t) + \frac{d}{dt} \left(\frac{\eta^{p-1}(t)w^{p+1}(t)}{p+1} \right) - \frac{p-1}{p+1} \eta^{p-2}(t)\eta'(t)w^{p+1}(t). \end{aligned}$$

Notice that $\eta^{p-2}\eta'w^{p+1} \in L^1(0, \infty)$ since

$$\int_0^n \left| \eta^{p-2}(s)\eta'(s)w^{p+1}(s) \right| ds \leq \left| w^{p+1}(\xi) \right| \left| \eta^{p-1}(0) - \eta^{p-1}(n) \right| \leq 2 \|w\|_{L^\infty}^{p+1} \|\eta\|_{L^\infty}^{p-1},$$

where n is any integer, $\xi \in (0, n)$ and the choice of ξ depends on n . By Lemma 9.5, there exists $t_n \rightarrow \infty$ such that $\lim_{n \rightarrow \infty} w(t_n) = \theta > 0$. Since $w' \in L^\infty(0, \infty)$, without loss of generality, one can assume that $\lim_{n \rightarrow \infty} w'(t_n)$ exists. As a result, we obtain that $\lim_{n \rightarrow \infty} \int_0^{t_n} (w'(t))^2 dt$ exists. Therefore $w' \in L^2(0, \infty)$.

Note that (9.5) and (9.6) imply $\lim_{t \rightarrow \infty} w(t) = 0$, which is a contradiction with $\lim_{n \rightarrow \infty} w(t_n) = \theta > 0$. Hence, there exists $\epsilon_0 > 0$ such that $e^{\epsilon_0 t} y(t) \in L^\infty(0, \infty)$, i.e., $r^{\frac{2}{p-1}-\epsilon_0} v(r) \in L^\infty(0, 1)$. \square

Lemma 9.6. Assume that $\frac{1}{2} < \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$. Assume that $v \in C^2(0, 1]$, $v \geq 0$ and v solves (9.2). If $r^{\frac{2\alpha-1}{1-\alpha}} v(r) \notin L^\infty(0, 1)$, then $r^\theta v(r) \notin L^\infty(0, 1)$, $\forall \theta < \frac{2}{p-1}$.

Proof. Fix $k \in \left[\frac{2\alpha-1}{1-\alpha}, \frac{2}{p-1} \right)$. Write $v(r) = Mr^{-k}h(s)$ where $s = \frac{r^j}{j}$ with $j = 2k - \frac{2\alpha-1}{1-\alpha} > 0$ and M is a positive constant such that $M^{p-1} j^{\frac{2-k(p-1)}{j}-2} = 1$. Then $h(s) \in C^2(0, 1/j]$, $h \geq 0$ and h solves

$$\begin{cases} h''(s) = s^{\frac{2-k(p-1)}{j}-2} h^p(s) - k \left(k - \frac{2\alpha-1}{1-\alpha} \right) j^{-2} s^{-2} h(s) & \text{on } (0, 1/j), \\ h(1/j) = 0. \end{cases}$$

Integrating the above equation, we obtain, for $s \in (0, 1/j)$,

$$h(s) + k \left(k - \frac{2\alpha-1}{1-\alpha} \right) j^{-2} \int_s^{1/j} t^{-2} h(t)(t-s) dt = -h'(1/j)(1/j-s) + \int_s^{1/j} t^{\frac{2-k(p-1)}{j}-2} h^p(t)(t-s) dt.$$

Therefore,

$$|h(s) + h'(1/j)(1/j-s)| \leq \int_s^{1/j} t^{\frac{2-k(p-1)}{2j}} h^p(t) t^{\frac{2-k(p-1)}{2j}-1} dt.$$

Assume $r^k v(r) \notin L^\infty(0, 1)$. Then $h(s) \notin L^\infty(0, 1/j)$. The above inequality then implies that

$$s^{\frac{2-k(p-1)}{2j}} h^p(s) \notin L^\infty(0, 1/j).$$

The definition of h implies that $r^{k+\frac{2-k(p-1)}{2p}} v(r) \notin L^\infty(0, 1)$. By induction, we obtain a sequence $k_n \in \left[\frac{2\alpha-1}{1-\alpha}, \frac{2}{p-1} \right)$ such that $r^{k_n} v(r) \notin L^\infty(0, 1)$, $\forall n \in \mathbb{N}$, $k_0 = \frac{2\alpha-1}{1-\alpha}$ and

$$k_n = k_{n-1} + \frac{2 - k_{n-1}(p-1)}{2p}.$$

That is,

$$k_n = \frac{2}{p-1} - \left(\frac{p+1}{2p} \right)^n \left(\frac{2}{p-1} - \frac{2\alpha-1}{1-\alpha} \right).$$

Therefore, $r^\theta v(r) \notin L^\infty(0, 1)$, $\forall \theta < \frac{2}{p-1}$. \square

Lemma 9.7. Assume that $\frac{1}{2} \leq \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$. Let $u \in C^2(0, 1]$ be such that $u \geq 0$, $\frac{u}{E_\alpha} \notin L^\infty(0, 1)$ and u solves (1.13), where E_α is defined by (1.15). Then $\lim_{x \rightarrow 0^+} x^{\frac{2(1-\alpha)}{p-1}} u(x) = l_{p,\alpha}$.

Proof. Since $\frac{u}{E_\alpha} \notin L^\infty$, it implies

$$\limsup_{x \rightarrow 0^+} \frac{u(x)}{E_\alpha(x)} = +\infty.$$

Consider v defined by (9.1). We have that

$$\limsup_{r \rightarrow 0^+} \frac{v(r)}{I_\alpha(r)} = +\infty,$$

where

$$I_\alpha(r) = \begin{cases} \ln \frac{1}{r}, & \text{if } \alpha = \frac{1}{2}, \\ r^{-\frac{2\alpha-1}{1-\alpha}}, & \text{if } \frac{1}{2} < \alpha < 1. \end{cases}$$

It is then equivalent to show that

$$\lim_{r \rightarrow 0^+} r^{\frac{2}{p-1}} v(r) = \bar{l}_{p,\alpha}, \tag{9.9}$$

where $\bar{l}_{p,\alpha}$ is given by (9.3). If $\alpha = \frac{1}{2}$, one can check that v is the radially symmetric and positive solution of the following equation

$$\begin{cases} -\Delta v + v^p = 0 & \text{on } B_1 \setminus \{0\}, \\ v = 0 & \text{on } \partial B_1, \end{cases}$$

where $B_1 \subset \mathbb{R}^2$ is the unit ball centered at the origin. Then Theorem 4.1 by Véron [20] implies (9.9). If $\frac{1}{2} < \alpha < 1$, Lemmas 9.3, 9.4 and 9.6 imply (9.9). \square

Lemma 9.8. Assume that $\frac{1}{2} \leq \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$. Let $u \in C^2(0, 1]$ be such that $u \geq 0$, $\frac{u}{E_\alpha} \in L^\infty(0, 1)$ and u solves (1.13), where E_α is defined by (1.15). Then its even extension $\bar{u}(x) := u(|x|)$ is the good solution of the following equation

$$\begin{cases} -(|x|^{2\alpha} \bar{u}')' + \bar{u}^p = c_0 \delta_0 & \text{on } (-1, 1), \\ \bar{u}(-1) = \bar{u}(1) = 0, \end{cases} \tag{9.10}$$

where c_0 is some nonnegative constant.

Proof. We first claim that there is a sequence $\{a_n\}_{n=1}^\infty \subset (0, 1)$ such that $\lim_{n \rightarrow \infty} a_n = 0$ and that the sequence $\{a_n^{2\alpha} u'(a_n)\}_{n=1}^\infty$ is bounded. Otherwise, it means that $\lim_{x \rightarrow 0^+} x^{2\alpha} u'(x) = -\infty$ since u is non-increasing. Then for all $M > 0$, there exists $a_M \in (0, 1)$ such that $\lim_{M \rightarrow +\infty} a_M = 0$ and

$$u'(x) \leq -\frac{M}{x^{2\alpha}}, \quad \forall x \in (0, a_M).$$

It follows that

$$\frac{u(a_M^2)}{E_\alpha(a_M^2)} \geq \frac{M}{2}, \quad \text{if } \alpha = \frac{1}{2},$$

and

$$\frac{u(a_M/2)}{E_\alpha(a_M/2)} \geq \frac{M}{2\alpha-1} \left[1 - \left(\frac{1}{2}\right)^{2\alpha-1} \right], \quad \text{if } \frac{1}{2} < \alpha < 1,$$

which contradicts $\frac{u}{E_\alpha} \in L^\infty(0, 1)$. Therefore, such a sequence $\{a_n\}_{n=1}^\infty$ exists. Without loss of generality, assume $\lim_{n \rightarrow \infty} a_n^{2\alpha} u'(a_n) = -\frac{c_0}{2}$.

The assumptions $\frac{u}{E_\alpha} \in L^\infty(0, 1)$ and $1 < p < \frac{1}{2\alpha-1}$ imply that $u \in L^p(0, 1)$. For any $\zeta \in C_0^1[-1, 1]$, from (1.13) one obtains

$$\int_{a_n}^1 |x|^{2\alpha} u' \zeta' dx + \int_{a_n}^1 u^p \zeta dx = -a_n^{2\alpha} u'(a_n) \zeta(a_n).$$

Passing to the limit as $n \rightarrow \infty$, it yields that $x^{2\alpha} u' \in L^1(0, 1)$ and

$$\int_0^1 |x|^{2\alpha} u' \zeta' dx + \int_0^1 u^p \zeta dx = \frac{c_0}{2} \zeta(0).$$

A similar computation for \bar{u} yields that $|x|^{2\alpha}\bar{u}' \in L^1(-1, 1)$ and

$$\int_{-1}^1 |x|^{2\alpha}\bar{u}'\zeta' dx + \int_{-1}^1 \bar{u}^p \zeta dx = c_0\zeta(0), \quad \forall \zeta \in C_0^1[-1, 1].$$

Thus $|x|^{2\alpha}\bar{u}' \in BV(-1, 1)$. Denote $\lim_{x \rightarrow 0^+} |x|^{2\alpha}\bar{u}'(x) = K^+$. We can check that

$$\lim_{x \rightarrow 0^+} \left(1 + \ln \frac{1}{|x|}\right)^{-1} \bar{u}(x) = K^+, \quad \text{if } \alpha = \frac{1}{2},$$

$$\lim_{x \rightarrow 0^+} |x|^{2\alpha-1}\bar{u}(x) = \frac{K^+}{2\alpha-1}, \quad \text{if } \frac{1}{2} < \alpha < 1.$$

Since \bar{u} is an even function, we have

$$\lim_{x \rightarrow 0^+} \left(1 + \ln \frac{1}{|x|}\right)^{-1} \bar{u}(x) = \lim_{x \rightarrow 0^-} \left(1 + \ln \frac{1}{|x|}\right)^{-1} \bar{u}(x), \quad \text{if } \alpha = \frac{1}{2},$$

$$\lim_{x \rightarrow 0^+} |x|^{2\alpha-1}\bar{u}(x) = \lim_{x \rightarrow 0^-} |x|^{2\alpha-1}\bar{u}(x), \quad \text{if } \frac{1}{2} < \alpha < 1.$$

Then we can conclude that \bar{u} is the good solution of (9.10). \square

Proof of Theorem 1.10 for $0 < \alpha < \frac{1}{2}$. Lemma 9.1 implies that u does not change its sign. Therefore we only need to consider $u \geq 0$ in (1.13).

We first prove the uniqueness. For solutions of type (ii), if there are two solutions u_1 and u_2 solving (1.13) with $\lim_{x \rightarrow 0^+} u_i(x) = c, i = 1, 2$, then

$$\int_0^1 x^{2\alpha}((u_1 - u_2)')^2 \phi'(u_1 - u_2) dx + \int_0^1 (u_1^p - u_2^p) \phi(u_1 - u_2) dx = 0,$$

where $\phi \in C^\infty(\mathbb{R})$ such that $\phi(0) = 0, \phi' \geq 0, \phi > 0$ on $(0, \infty), \phi < 0$ on $(-\infty, 0)$, and $\phi = \text{sign}$ on $\mathbb{R} \setminus (-1, 1)$. It follows that $u_1 = u_2$ on $[0, 1]$. For solutions of type (iii), if there are two solutions u_1 and u_2 solving (1.13) with

$\lim_{x \rightarrow 0^+} x^{\frac{2(1-\alpha)}{p-1}} u_i(x) = l_{p,\alpha}, i = 1, 2$, then estimate (9.4) implies

$$|u_1(x) - u_2(x)| \leq 2l_{p,\alpha}x^{\sigma_0}, \quad \forall x \in (0, 1],$$

for some $\sigma_0 > 0$. Also notice that

$$-(x^{2\alpha}(u_1(x) - u_2(x))')' + c(x)(u_1(x) - u_2(x)) = 0 \quad \text{on } (0, 1),$$

where

$$c(x) = \begin{cases} \frac{u_1^p(x) - u_2^p(x)}{u_1(x) - u_2(x)}, & \text{if } u_1(x) \neq u_2(x), \\ pu_1^{p-1}(x), & \text{if } u_1(x) = u_2(x). \end{cases}$$

It is easy to check that $c \in C(0, 1]$ and $c \geq 0$. A maximum principle on $(\epsilon, 1)$ implies

$$\max_{x \in (\epsilon, 1)} |u_1(x) - u_2(x)| \leq |u_1(\epsilon) - u_2(\epsilon)| \leq 2l_{p,\alpha}\epsilon^{\sigma_0}.$$

Let $\epsilon \rightarrow 0^+$ and then $u_1 = u_2$ on $(0, 1)$.

We now claim that, for $u \geq 0$ satisfying (1.13), one of the following assertions holds.

- (i) $\lim_{x \rightarrow 0^+} x^{\frac{2(1-\alpha)}{p-1}} u(x) = l_{p,\alpha}$.
- (ii) $\lim_{x \rightarrow 0^+} u(x) = c$, for some $c \geq 0$.

Indeed, denote

$$v(r) = \left(\frac{1-2\alpha}{1-\alpha} \right)^{\frac{p}{p-1} + \frac{3-4\alpha}{(p-1)(1-2\alpha)}} r^{\frac{1-2\alpha}{1-\alpha}} h \left(\frac{1-\alpha}{1-2\alpha} r^{-\frac{1-2\alpha}{1-\alpha}} \right), \tag{9.11}$$

where v is defined in (9.1). Then $h(s) \in C^2 \left[\frac{1-\alpha}{1-2\alpha}, \infty \right)$ and h satisfies

$$h''(s) = s^{-p-2-\frac{1}{1-2\alpha}} h^p(s) \quad \text{on} \quad \left(\frac{1-\alpha}{1-2\alpha}, \infty \right).$$

A result of Fowler (p. 288 in [16]) implies that, as $s \rightarrow \infty$, either

$$h(s) = \left[\frac{(p(1-2\alpha) + 1)(2-2\alpha)}{(p-1)^2(1-2\alpha)} \right]^{\frac{1}{p-1}} s^{\frac{p(1-2\alpha)+1}{(p-1)(1-2\alpha)}} (1 + o(1)),$$

or

$$h(s) = As + B + \frac{A^p(1-2\alpha)^2}{2-2\alpha} s^{-\frac{1}{1-2\alpha}} (1 + o(1)),$$

for some constants A and B . Therefore, the relation (9.11) implies our claim.

We then show the existence of the u_c and the $u_{+\infty}$. Consider the Hilbert space X^α given by (7.3). Note that $X^\alpha \subset C[0, 1]$. It is straightforward to check that there is a minimizer of the following constraint minimization problem,

$$\min_{u \in X^\alpha, u(0)=c} \left\{ \frac{1}{2} \int_0^1 x^{2\alpha} (u'(x))^2 dx + \frac{1}{p+1} \int_0^1 |u(x)|^{p+1} dx \right\},$$

and the minimizer is indeed the u_c . Moreover, a comparison principle implies that $u_{c_1} \geq u_{c_2}$ if $c_1 \geq c_2$. On the other hand, Lemma 8.4 implies that $u_c(x) \leq C(\alpha, p)x^{-\frac{2(1-\alpha)}{p-1}}$ for $0 < x \leq \frac{1}{2}$. Since u_c is decreasing, $u_c(x) \leq C(\alpha, p)2^{\frac{2(1-\alpha)}{p-1}}$ for $\frac{1}{2} < x \leq 1$. Therefore $\lim_{c \rightarrow \infty} u_c(x) < \infty$ for all $x \in (0, 1]$. We claim that $u_{+\infty}(x) = \lim_{c \rightarrow \infty} u_c(x)$. Indeed, since

$$\limsup_{x \rightarrow 0^+} u_{+\infty}(x) \geq \lim_{x \rightarrow 0^+} u_c(x) = c,$$

we have

$$\limsup_{x \rightarrow 0^+} u_{+\infty}(x) = +\infty.$$

Note that $u_{+\infty}$ is still a solution of (1.13). The previous claim implies that $u_{+\infty}$ satisfies (1.16).

Finally, denote $u_0(x) = \lim_{c \rightarrow 0^+} u_c(x)$. Then $\lim_{x \rightarrow 0^+} u_0(x) = 0$. Therefore $u_0 = 0$. \square

Proof of Theorem 1.10 for $\frac{1}{2} \leq \alpha < 1$. The same as the case $0 < \alpha < \frac{1}{2}$, we only need to consider $u \geq 0$ in (1.13).

We first prove the uniqueness. Note that the even extension of u_c is the good solution of (9.10) with $c_0 = 2c$. The uniqueness of the good solution of (9.10) implies the uniqueness of u_c . The proof for the uniqueness of $u_{+\infty}$ is the same as the case $0 < \alpha < \frac{1}{2}$.

We now prove that, for $u \geq 0$ satisfying (1.13), one of the following three assertions holds.

- (i) $u \equiv 0$.
- (ii) $\lim_{x \rightarrow 0^+} \frac{u(x)}{E_\alpha(x)} = c$, for some $c > 0$.
- (iii) $\lim_{x \rightarrow 0^+} x^{\frac{2(1-\alpha)}{p-1}} u(x) = l_{p,\alpha}$.

We consider $\limsup_{x \rightarrow 0^+} \frac{u(x)}{E_\alpha(x)}$. If $\limsup_{x \rightarrow 0^+} \frac{u(x)}{E_\alpha(x)} = 0$, Lemma 9.8 implies that $\bar{u}(x) := u(|x|)$ is the good solution of (9.10) with $c_0 = 0$. Therefore the uniqueness of the good solution of (9.10) forces $u \equiv 0$. If $0 < \limsup_{x \rightarrow 0^+} \frac{u(x)}{E_\alpha(x)} < \infty$, then \bar{u} satisfies (9.10) with $c_0 > 0$. Therefore by Theorem 1.2, we have $\lim_{x \rightarrow 0^+} \frac{u(x)}{E_\alpha(x)} = c_0/2$. If $\limsup_{x \rightarrow 0^+} \frac{u(x)}{E_\alpha(x)} = \infty$, Lemma 9.7 implies $\lim_{x \rightarrow 0^+} x^{\frac{2(1-\alpha)}{p-1}} u(x) = l_{p,\alpha}$.

The existence of u_c is already given by Theorem 1.2. Note that the limits $\lim_{c \rightarrow \infty} u_c(x)$ and $\lim_{c \rightarrow 0^+} u_c(x)$ are well-defined for $x \in (0, 1]$. The same as the case $0 < \alpha < \frac{1}{2}$, we can check that $u_{+\infty}(x) = \lim_{c \rightarrow \infty} u_c(x)$ and $0 = \lim_{c \rightarrow 0^+} u_c(x)$. \square

After Theorem 1.10 was proved, the author was informed the recent work by Brandolini, Chiacchio, Cîrstea and Trombetti [4]. The authors in [4] studied the positive solutions of the following equation

$$-\operatorname{div}(\mathcal{A}(|x|)\nabla u) + u^p = 0 \quad \text{on } B_1^* := B_1 \setminus \{0\},$$

where $B_1 \subset \mathbb{R}^N$ is the unit ball centered at the origin, $N \geq 3$, and \mathcal{A} is a positive $C^1(0, 1]$ -function such that

$$\lim_{t \rightarrow 0^+} \frac{t\mathcal{A}'(t)}{\mathcal{A}(t)} = \vartheta, \text{ for some } \vartheta \in (2 - N, 2).$$

For the special case when $\mathcal{A}(r) = r^\vartheta$ with $\vartheta \in (2 - N, 2)$, a consequence of the main result in [4] is

Theorem 9.9. Assume $1 < p < \frac{N}{N-2+\vartheta}$. For a positive solution $u \in C^2(0, 1]$ satisfying

$$\begin{cases} u''(r) + (N - 1 + \vartheta)\frac{u'(r)}{r} = \frac{u^p(r)}{r^\vartheta} & \text{on } (0, 1), \\ u(1) = 0, \end{cases} \tag{9.12}$$

one of the following cases occurs.

- (i) $u \equiv 0$,
- (ii) $\lim_{r \rightarrow 0^+} r^{N-2+\vartheta} u(r) = \lambda$, for some $\lambda \in (0, \infty)$,
- (iii) $\lim_{r \rightarrow 0^+} r^{\frac{2-\vartheta}{p-1}} u(r) = \left[\frac{(N-(N-2+\vartheta)p)(2-\vartheta)}{(p-1)^2} \right]^{\frac{1}{p-1}}$.

Remark 9.1. Let $\tilde{u}(x) = N^{-\frac{2}{p-1}} u(x^{1/N})$, where u satisfies (9.12). Then \tilde{u} satisfies

$$\begin{cases} -(x^{2\alpha}\tilde{u}')' + \tilde{u}^p = 0 & \text{on } (0, 1), \\ \tilde{u}(1) = 0, \end{cases}$$

where $\alpha = 1 - \frac{\vartheta-2}{N} \in (\frac{1}{2}, 1)$. It is now easy to check that Theorem 9.9 coincides with the case $\frac{1}{2} < \alpha < 1$ of Theorem 1.10. However, the proofs of these two theorems are different.

10. The equation on the interval (0, 1)

In this section, we first consider the following equation,

$$\begin{cases} -(x^{2\alpha}u')' + |u|^{p-1}u = \mu & \text{on } (0, 1), \\ \lim_{x \rightarrow 0^+} x^{2\alpha}u'(x) = \beta, \\ u(1) = 0, \end{cases} \tag{10.1}$$

where $\mu \in \mathcal{M}(0, 1)$, $\alpha > 0$, $p > 1$ and $\beta \in \mathbb{R}$.

A function u is a solution of (10.1) if

$$u \in L^p(0, 1) \cap W_{loc}^{1,1}(0, 1], \quad x^{2\alpha}u' \in BV(0, 1), \tag{10.2}$$

and u satisfies (10.1) in the usual sense.

The following result concerns the existence and uniqueness of the solution of (10.1).

Theorem 10.1. *Let $\mu \in \mathcal{M}(0, 1)$.*

(i) *If α and p satisfy (1.4) or (1.5), then there exists a unique solution of (10.1) for all $\beta \in \mathbb{R}$. Moreover, this unique solution satisfies*

$$\begin{aligned} \lim_{x \rightarrow 0^+} \left(1 + \ln \frac{1}{x}\right)^{-1} u(x) &= - \lim_{x \rightarrow 0^+} x u'(x) = -\beta \text{ when } \alpha = \frac{1}{2} \text{ and } p > 1, \\ \lim_{x \rightarrow 0^+} x^{2\alpha-1} u(x) &= - \lim_{x \rightarrow 0^+} \frac{x^{2\alpha} u'(x)}{2\alpha - 1} = -\frac{\beta}{2\alpha - 1} \text{ when } \frac{1}{2} < \alpha < 1 \text{ and } 1 < p < \frac{1}{2\alpha - 1}. \end{aligned}$$

(ii) *If α and p satisfy (1.6) or (1.7), then there exists a solution of (10.1) if and only if $\beta = 0$. Moreover, if the solution exists, then it is unique and it satisfies*

$$\lim_{x \rightarrow 0^+} x^{2\alpha-1} u(x) = \lim_{x \rightarrow 0^+} x^{2\alpha} u'(x) = 0.$$

Proof. We first prove the existence in assertion (i). Take $\bar{\mu} \in \mathcal{M}(-1, 1)$ as the zero extension of μ , i.e., $\bar{\mu}(A) = \mu(A \cap (0, 1))$, where $A \subset (-1, 1)$ is a Borel set. Then Theorem 7.1 implies that there exists a solution \bar{u} satisfying

$$\begin{cases} -(|x|^{2\alpha} \bar{u}')' + |\bar{u}|^{p-1} \bar{u} = \bar{\mu} & \text{on } (-1, 1), \\ \lim_{x \rightarrow 0} |x|^{2\alpha} \bar{u}'(x) = \beta, \\ \bar{u}(-1) = \bar{u}(1) = 0. \end{cases}$$

Therefore, $u = \bar{u}|_{(0,1)}$ is a solution of (10.1).

We then prove the existence in assertion (ii). We still take $\bar{\mu}$ as the zero extension of μ . Notice that $\bar{\mu}(\{0\}) = 0$. Then Theorem 1.3 implies that there exists a solution \bar{u} satisfying

$$\begin{cases} -(|x|^{2\alpha} \bar{u}')' + |\bar{u}|^{p-1} \bar{u} = \bar{\mu} & \text{on } (-1, 1), \\ \lim_{x \rightarrow 0} |x|^{2\alpha} \bar{u}'(x) = 0, \\ \bar{u}(-1) = \bar{u}(1) = 0. \end{cases}$$

Therefore, $u = \bar{u}|_{(0,1)}$ is a solution of (10.1) with $\beta = 0$. On the other hand, if (10.1) has a solution with $\beta \neq 0$, it implies that $u \sim \frac{1}{x^{2\alpha-1}}$ near $x = 0$. It is a contradiction with the fact that $u \in L^p(0, 1)$.

We now prove the uniqueness for both cases. Assume that there are two solutions u_1 and u_2 . Then we have

$$\begin{cases} -(x^{2\alpha}(u_1 - u_2))' + |u_1|^{p-1}u_1 - |u_2|^{p-1}u_2 = 0 & \text{on } (0, 1), \\ \lim_{x \rightarrow 0^+} x^{2\alpha}(u_1 - u_2)'(x) = 0, \\ u_1(1) = u_2(1) = 0. \end{cases}$$

Define $\bar{u}_i \in W_{loc}^{1,1}([-1, 1] \setminus \{0\})$, $i = 1, 2$, such that $\bar{u}_i = u_i$ on $(0, 1)$ and $\bar{u}_i = 0$ on $(-1, 0)$. Then the same argument for the uniqueness of Theorem 7.1 implies that $\bar{u}_1 = \bar{u}_2$. Thus, $u_1 = u_2$. \square

Remark 10.1. When $0 < \alpha < \frac{1}{2}$, we can also consider the following equation,

$$\begin{cases} -(x^{2\alpha}u')' + |u|^{p-1}u = \mu & \text{on } (0, 1), \\ \lim_{x \rightarrow 0^+} u(x) = \beta, \\ u(1) = 0, \end{cases} \tag{10.3}$$

where $\mu \in \mathcal{M}(0, 1)$, $p > 1$ and $\beta \in \mathbb{R}$. Indeed, the uniqueness of the solution of (10.3) has been proved in Theorem 1.10. The existence of the solution of (10.3) follows from the existence of the minimizer of the following minimization problem,

$$\min_{u \in X^\alpha, u(0)=\beta} \left\{ \frac{1}{2} \int_0^1 x^{2\alpha} (u'(x))^2 dx + \frac{1}{p+1} \int_0^1 |u(x)|^{p+1} dx - \int_0^1 u(x) d\mu(x) \right\},$$

where X^α is given by (7.3). Moreover, a direct computation shows that this unique solution u satisfies

$$\lim_{x \rightarrow 0^+} x^{2\alpha} u'(x) = - \int_0^1 |u(s)|^{p-1} u(s) (1 - s^{1-2\alpha}) ds + \int_0^1 (1 - s^{1-2\alpha}) d\mu(s) - (1 - 2\alpha)\beta.$$

We now discuss the connections between Theorem 10.1 and the well-known existence results about the semilinear elliptic equation. Let $B_1 \subset \mathbb{R}^N$ be the unit ball centered at the origin and $\mu \in \mathcal{M}(B_1)$. For $p > 1$, consider the following equation,

$$\begin{cases} -\Delta u + |u|^{p-1}u = \mu & \text{on } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases} \tag{10.4}$$

Recall that a function u is a weak solution of (10.4) if $u \in L^p(B_1) \cap W_0^{1,1}(B_1)$ and

$$\int_{B_1} \nabla u \nabla \zeta dx + \int_{B_1} |u|^{p-1} u \zeta dx = \int_{B_1} \zeta d\mu, \quad \forall \zeta \in C_0^\infty(B_1).$$

Although the general existence theory about (10.4) is well-known, the following corollary provides a more precise information when μ is rotationally invariant, i.e., $\mu(A) = \mu(OA)$, where A is any Borel set in B_1 and O is any $N \times N$ orthogonal matrix.

Corollary 10.2. Assume that $\mu \in \mathcal{M}(B_1)$ is rotationally invariant. Let $|\mathbb{S}^{N-1}|$ be the surface area of \mathbb{S}^{N-1} . Define $\tilde{\mu} \in \mathcal{M}(0, 1)$ as

$$\tilde{\mu}(A) = \mu \left(\left\{ r\theta; r \in A, \theta \in \mathbb{S}^{N-1} \right\} \right), \quad \forall A \subset (0, 1) \text{ such that } A \text{ is a Borel set.} \tag{10.5}$$

Let $f_*\tilde{\mu}$ be the push-forward measure of $\tilde{\mu}$ under the map $f : [0, 1] \rightarrow [0, 1]$ with $f(r) = r^N$, i.e., $f_*\tilde{\mu}(A) = \tilde{\mu}(f^{-1}(A))$, $\forall A \subset (0, 1)$, Borel set.

(i) Assume that $1 < p < \frac{N}{N-2}$ for $N \geq 3$ or $p > 1$ for $N = 2$. Then $u(x) = N^{\frac{2}{p-1}} \tilde{u}(|x|^N)$ is a weak solution of (10.4), where \tilde{u} satisfies

$$\begin{cases} -(t^{2(1-\frac{1}{N})} \tilde{u}'(t))' + |\tilde{u}(t)|^{p-1} \tilde{u}(t) = N^{-\frac{2p}{p-1}} |\mathbb{S}^{N-1}|^{-1} f_*\tilde{\mu} & \text{on } (0, 1), \\ \lim_{t \rightarrow 0^+} t^{2(1-\frac{1}{N})} \tilde{u}'(t) = N^{-\frac{2p}{p-1}} |\mathbb{S}^{N-1}|^{-1} \mu(\{0\}), \\ \tilde{u}(1) = 0. \end{cases} \tag{10.6}$$

(ii) Assume that $p \geq \frac{N}{N-2}$ for $N \geq 3$. Eq. (10.4) has a weak solution if and only if $\mu(\{0\}) = 0$. Moreover, if $\mu(\{0\}) = 0$, then $u(x) = N^{\frac{2}{p-1}} \tilde{u}(|x|^N)$ is a weak solution of (10.4), where \tilde{u} satisfies

$$\begin{cases} -(t^{2(1-\frac{1}{N})} \tilde{u}'(t))' + |\tilde{u}(t)|^{p-1} \tilde{u}(t) = N^{-\frac{2p}{p-1}} |\mathbb{S}^{N-1}|^{-1} f_*\tilde{\mu} & \text{on } (0, 1), \\ \lim_{t \rightarrow 0^+} t^{2(1-\frac{1}{N})} \tilde{u}'(t) = \tilde{u}(1) = 0. \end{cases} \tag{10.7}$$

To prove Corollary 10.2, we need the following lemma.

Lemma 10.3. Assume that $\mu \in \mathcal{M}(B_1)$ is rotationally invariant. Assume that $u \in L^p(B_1) \cap W_0^{1,1}(B_1)$, u is radially symmetric, and

$$\int_{B_1} \nabla u \nabla \zeta \, dx + \int_{B_1} |u|^{p-1} u \zeta \, dx = \int_{B_1} \zeta \, d\mu, \quad \forall \zeta \in C_0^\infty(B_1) \text{ such that } \zeta \text{ is radially symmetric.}$$

Then u is a weak solution of (10.4).

Proof. We use the same idea as the proof of Proposition 5.1 in [18]. We first take $w \in L^p(B_1) \cap W_0^{1,1}(B_1)$ as a weak solution of

$$\Delta w = |u|^{p-1} u - \mu \quad \text{on } B_1.$$

Then w is radially symmetric and

$$\int_{B_1} \nabla w \nabla \zeta \, dx + \int_{B_1} |u|^{p-1} u \zeta \, dx = \int_{B_1} \zeta \, d\mu, \quad \forall \zeta \in C_0^\infty(B_1).$$

For any $\zeta \in C_0^\infty(B_1)$ such that ζ is radially symmetric, we have

$$\int_{B_1} w(\Delta \zeta) = \int_{B_1} u(\Delta \zeta).$$

Moreover, for any $\phi \in C_c^\infty(B_1)$ such that ϕ is radially symmetric, there exists $\zeta \in C_0^\infty(B_1)$ such that ζ is radially symmetric and $\Delta \zeta = \phi$ on B_1 . It implies that

$$\int_{\Omega} (w - u)\phi \, dx = 0, \quad \forall \phi \in C_c^\infty(B_1) \text{ such that } \phi \text{ is radially symmetric.}$$

Then

$$\int_0^1 (w(t) - u(t))\varphi(t)t^{N-1} \, dt = 0, \quad \forall \varphi \in C_c^\infty(0, 1).$$

Therefore $w = u$ a.e. \square

Proof of Corollary 10.2. Note that Theorem 10.1 ensures the existence of \tilde{u} in (10.6) and (10.7).

We first prove assertion (i). For any $\zeta \in C_0^\infty(B_1)$ such that ζ is radially symmetric, we denote $g(|x|^N) = \zeta(x)$. Then $g(t) \in C[0, 1]$, $g(1) = 0$ and $g'(t) \in L^1(0, 1)$. Therefore,

$$\begin{aligned} & \int_0^1 t^{2(1-\frac{1}{N})} \tilde{u}'(t)g'(t) \, dt + \int_0^1 |\tilde{u}(t)|^{p-1} \tilde{u}(t)g(t) \, dt \\ &= N^{-\frac{2p}{p-1}} \left| \mathbb{S}^{N-1} \right|^{-1} \int_0^1 g(t) \, d(f_* \tilde{\mu})(t) + N^{-\frac{2p}{p-1}} \left| \mathbb{S}^{N-1} \right|^{-1} g(0)\mu(\{0\}). \end{aligned} \tag{10.8}$$

Note that $\int_0^1 g(t) \, d(f_* \tilde{\mu})(t) = \int_0^1 g(r^N) \, d\tilde{\mu}(r)$ by Theorem 3.6.1 on p. 190 of [3]. Let $t = r^N$ in (10.8). We have

$$\begin{aligned} & \int_0^1 g(r^N) \, d\tilde{\mu}(r) + g(0)\mu(\{0\}) = N^{\frac{2p}{p-1}} \left| \mathbb{S}^{N-1} \right| \int_0^1 r^{2N-2} \tilde{u}'(r^N)g'(r^N)Nr^{N-1} \, dr \\ & \quad + N^{\frac{2p}{p-1}} \left| \mathbb{S}^{N-1} \right| \int_0^1 |\tilde{u}(r^N)|^{p-1} \tilde{u}(r^N)g(r^N)Nr^{N-1} \, dr. \end{aligned}$$

Let $u(x) = N^{\frac{2}{p-1}} \tilde{u}(|x|^N)$ with $x \in B_1$. Then $u \in L^p(B_1) \cap W_0^{1,1}(B_1)$. Moreover,

$$\begin{aligned} \int_{B_1} \nabla u \nabla \zeta \, dx &= N^{\frac{2}{p-1}+2} \left| \mathbb{S}^{N-1} \right| \int_0^1 r^{2N-2} \tilde{u}'(r^N) g'(r^N) N r^{N-1} \, dr, \\ \int_{B_1} |u|^{p-1} u \zeta \, dx &= N^{\frac{2p}{p-1}} \left| \mathbb{S}^{N-1} \right| \int_0^1 |\tilde{u}(r^N)|^{p-1} \tilde{u}(r^N) g(r^N) N r^{N-1} \, dr, \\ \int_{B_1} \zeta \, d\mu &= \int g(r^N) d\tilde{\mu}(r) + g(0) \mu(\{0\}). \end{aligned}$$

Therefore,

$$\int_{B_1} \nabla u \nabla \zeta \, dx + \int_{B_1} |u|^{p-1} u \zeta \, dx = \int_{B_1} \zeta \, d\mu, \quad \forall \zeta \in C_0^\infty(B_1) \text{ such that } \zeta \text{ is radially symmetric.}$$

By Lemma 10.3, u is a weak solution of (10.4).

We now prove assertion (ii). If $\mu(\{0\}) = 0$, then the same proof as the above shows that u is a weak solution of (10.4). On the other hand, if μ is rotationally invariant and (10.4) has a weak solution, then

$$\int_{B_1} \nabla u \nabla \zeta \, dx + \int_{B_1} |u|^{p-1} u \zeta \, dx = \int_{B_1} \zeta \, d\mu, \quad \forall \zeta \in C_0^\infty(B_1) \text{ such that } \zeta \text{ is radially symmetric.}$$

Write $g(r) = \zeta(x)$ where $r = |x|$. Then $g \in W^{1,\infty}(0, 1)$ and $g(1) = 0$. Write $u(r) = u(x)$ where $r = |x|$. Then $|u|^p r^{N-1} \in L^1(0, 1)$ and, by Theorem 2.3 in [18], $u \in W_{loc}^{1,1}(0, 1]$ such that $r^{N-1} u' \in L^1(0, 1)$. Therefore

$$\begin{aligned} &\left| \mathbb{S}^{N-1} \right| \int_0^1 r^{N-1} u'(r) g'(r) \, dr + \left| \mathbb{S}^{N-1} \right| \int_0^1 N r^{N-1} |u(r)|^{p-1} u(r) g(r) \, dr \\ &= \int_0^1 g(r) d\tilde{\mu}(r) + g(0) \mu(\{0\}). \end{aligned}$$

That is

$$\lim_{r \rightarrow 0^+} r^{N-1} u'(r) = \left| \mathbb{S}^{N-1} \right|^{-1} \mu(\{0\}).$$

It forces $\mu(\{0\}) = 0$. Otherwise, $u \sim r^{-N+2}$ near $r = 0$. Therefore $|u|^p r^{N-1} \sim r^{-(N-2)p+N-1}$ near $r = 0$. Since $p \geq \frac{N}{N-2}$, it implies that $|u|^p r^{N-1} \notin L^1(0, 1)$, which is a contradiction. \square

The well-known result by Baras and Pierre [1] states that for $\mu \in \mathcal{M}(B_1)$, $p \geq \frac{N}{N-2}$ and $N \geq 3$, Eq. (10.4) has a weak solution if and only if

$$\mu(E) = 0, \quad \forall E \subset B_1 \text{ such that } Cap_{2,p'}(E) = 0, \tag{10.9}$$

where $Cap_{2,p'}$ is the capacity associated with the $W^{2,p'}(\mathbb{R}^N)$ -norm and p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Remark 10.2. In the case when μ is rotationally invariant, the criterion (10.9) is equivalent to $\mu(\{0\}) = 0$. Therefore, the necessary and sufficient condition in assertion (ii) of Corollary 10.2 is consistent with (10.9).

The proof of this remark relies on the following lemma.

Lemma 10.4. Let $\mu \in \mathcal{M}(B_1)$ be rotationally invariant, $\tilde{\mu}$ be defined by (10.5), and \mathcal{H}^{N-1} be the $(n - 1)$ -dimensional Hausdorff measure on \mathbb{S}^{N-1} . Then for any μ -integrable function f , we have

$$\int_{B_1} f(x) d\mu(x) = \frac{1}{|\mathbb{S}^{N-1}|} \int_{(0,1)} \left(\int_{\mathbb{S}^{N-1}} f(r\theta) d\mathcal{H}^{N-1}(\theta) \right) d\tilde{\mu}(r) + f(0)\mu(\{0\}), \tag{10.10}$$

where $r = |x|$ and $\theta = \frac{x}{|x|}$, $\forall x \in B_1 \setminus \{0\}$.

Proof. By a standard linearity and approximation argument, we only need to prove (10.10) for characteristic functions. Moreover, by a standard argument involving the properties of Borel algebra and Radon measure (see, e.g., the proof of Theorem 2.49 in [15]), we only need to show that

$$\mu((0, a] \times U) = \frac{1}{|\mathbb{S}^{N-1}|} \tilde{\mu}((0, a]) \times \mathcal{H}^{N-1}(U), \quad \forall a \in (0, 1), \quad \forall U \subset \mathbb{S}^{N-1} \text{ such that } U \text{ is open.}$$

Apply once again the standard approximation argument. It is further reduced to show that

$$\int_{(0,a] \times \mathbb{S}^{N-1}} \phi\left(\frac{x}{|x|}\right) d\mu(x) = \frac{\tilde{\mu}((0, a])}{|\mathbb{S}^{N-1}|} \int_{\mathbb{S}^{N-1}} \phi(\theta) d\mathcal{H}^{N-1}(\theta), \quad \forall \phi \in C(\mathbb{S}^{N-1}). \tag{10.11}$$

We use some ideas by Christensen [14] to show (10.11). For fixed $x \in \mathbb{S}^{N-1}$ and $\epsilon > 0$, denote

$$C(x; \epsilon) = \left\{ y \in \mathbb{S}^{N-1}; d(x, y) < \epsilon \right\},$$

the so-called spherical cap, where $d(\cdot, \cdot)$ is the standard Riemannian distance on \mathbb{S}^{N-1} . Define

$$C(\epsilon) = \mu((0, a] \times C(x; \epsilon)).$$

Note that $C(\epsilon)$ is well-defined since μ is rotationally invariant and $\mu((0, a] \times C(x; \epsilon))$ is independent of $x \in \mathbb{S}^{N-1}$. Denote $B_a = (0, a] \times \mathbb{S}^{N-1}$. Define

$$K_\epsilon(x, y) : B_a \times B_a \rightarrow \mathbb{R},$$

as

$$K_\epsilon(x, y) = \begin{cases} \frac{1}{C(\epsilon)}, & \text{if } d\left(\frac{x}{|x|}, \frac{y}{|y|}\right) < \epsilon, \\ 0, & \text{otherwise.} \end{cases}$$

For any $x \in B_a$, write $\varphi(x) = \phi\left(\frac{x}{|x|}\right)$. Define

$$K_\epsilon \varphi(x) = \int_{B_a} K_\epsilon(x, y) \varphi(y) d\mu(y), \quad \forall x \in B_a.$$

It is clear that $K_\epsilon \varphi(x) \rightarrow \varphi(x)$ as $\epsilon \rightarrow 0$ for all $x \in B_a$. Therefore, the Dominated Convergence Theorem implies that

$$\lim_{\epsilon \rightarrow 0} \int_{B_a} K_\epsilon \varphi(x) d(\mathcal{H}^{N-1} \times \tilde{\mu})(x) = \int_{B_a} \varphi(x) d(\mathcal{H}^{N-1} \times \tilde{\mu})(x).$$

Note that

$$\begin{aligned} \int_{B_a} K_\epsilon \varphi(x) d(\mathcal{H}^{N-1} \times \tilde{\mu})(x) &= \int_{B_a} \varphi(y) \left(\int_{B_a} K_\epsilon(x, y) d(\mathcal{H}^{N-1} \times \tilde{\mu})(x) \right) d\mu(y) \\ &= \frac{\tilde{\mu}((0, a]) \mathcal{H}^{N-1}(C(x; \epsilon))}{C(\epsilon)} \int_{B_a} \varphi(y) d\mu(y). \end{aligned}$$

Therefore, there exists $\lambda \in \mathbb{R}$ such that

$$\lim_{\epsilon \rightarrow 0} \frac{\bar{\mu}((0, a]) \mathcal{H}^{N-1}(C(x; \epsilon))}{C(\epsilon)} = \lambda.$$

Take $\varphi \equiv 1$. It implies that $\lambda = |\mathbb{S}^{N-1}|$. Hence, identity (10.11) holds and the proof is complete. \square

Proof of Remark 10.2. Assume that μ satisfies (10.9). Since $\text{Cap}_{2,p'}(\{0\}) = 0$, it is clear that $\mu(\{0\}) = 0$. On the other hand, assume that μ is rotationally invariant and $\mu(\{0\}) = 0$. For any $E \subset B_1$ such that $\text{Cap}_{2,p'}(E) = 0$, it holds that $\dim_{\mathcal{H}}(E) \leq N - 2$, where $\dim_{\mathcal{H}}$ is the Hausdorff dimension. Therefore,

$$\int_{\mathbb{S}^{N-1}} \chi_E(r\theta) d\mathcal{H}^{N-1}(\theta) = 0, \quad \forall r \in (0, 1).$$

Hence Lemma 10.4 implies that

$$\mu(E) = \frac{1}{|\mathbb{S}^{N-1}|} \int_{(0,1)} \left(\int_{\mathbb{S}^{N-1}} \chi_E(r\theta) d\mathcal{H}^{N-1}(\theta) \right) d\tilde{\mu}(r) + \mu(\{0\}) = 0. \quad \square$$

Conflict of interest statement

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