Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 33 (2016) 1081–1101

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Global bifurcation theory for periodic traveling interfacial gravity–capillary waves

David M. Ambrose [a,*,1], Walter A. Strauss [b,2], J. Douglas Wright [a,3]

[a] Department of Mathematics, Drexel University, Philadelphia, PA 19104, United States
[b] Department of Mathematics, Brown University, Providence, RI 02912, United States

Received 20 November 2014; received in revised form 18 March 2015; accepted 24 March 2015

Available online 1 April 2015

Abstract

We consider the global bifurcation problem for spatially periodic traveling waves for two-dimensional gravity–capillary vortex sheets. The two fluids have arbitrary constant, non-negative densities (not both zero), the gravity parameter can be positive, negative, or zero, and the surface tension parameter is positive. Thus, included in the parameter set are the cases of pure capillary water waves and gravity–capillary water waves. Our choice of coordinates allows for the possibility that the fluid interface is not a graph over the horizontal. We use a technical reformulation which converts the traveling wave equations into a system of the form "identity plus compact." Rabinowitz' global bifurcation theorem is applied and the final conclusion is the existence of either a closed loop of solutions, or an unbounded set of nontrivial traveling wave solutions which contains waves which may move arbitrarily fast, become arbitrarily long, form singularities in the vorticity or curvature, or whose interfaces self-intersect.

Keywords: Global bifurcation; Surface tension; Traveling wave; Interfacial flow; Water wave

1. Introduction

We consider the case of two two-dimensional fluids, of infinite vertical extent and periodic in the horizontal direction (of period $M > 0$) and separated by an interface which is free to move. Each fluid has a constant, non-negative density: $\rho_2 \geq 0$ in the upper fluid and $\rho_1 \geq 0$ in the lower. Of course, we do not allow both densities to be zero, but if one of the densities is zero, then it is known as the water wave case. The velocity of each fluid satisfies the incompressible, irrotational Euler equations. The restoring forces in the problem include non-zero surface tension (with surface tension constant $\tau > 0$) on the interface and a gravitational body force (with acceleration $g \in \mathbf{R}$, possibly

* Corresponding author.
E-mail address: ambrose@math.drexel.edu (D.M. Ambrose).
1 DMA gratefully acknowledges support from the National Science Foundation through grant DMS-1016267.
2 WAS gratefully acknowledges support from the National Science Foundation through grant DMS-1007960.
3 JDW acknowledges gratefully support from the National Science Foundation through grant DMS-1105635.

http://dx.doi.org/10.1016/j.anihpc.2015.03.005

zero) which acts in the vertical direction. Since the fluids are irrotational, the interface is a vortex sheet, meaning that the vorticity in the problem is an amplitude times a Dirac mass supported on the interface. We call this problem "the two-dimensional gravity–capillary vortex sheet problem." The average vortex strength on the interface is denoted by $\bar{\gamma}$.

In [2], two of the authors and Akers established a new formulation for the traveling wave problem for parameterized curves, and applied it to the vortex sheet with surface tension (in case the two fluids have the same density). The curves in [2] may have multi-valued height. This is significant since it is known that there exist traveling waves in the presence of surface tension which do indeed have multi-valued height; the most famous such waves are the Crapper waves [14], and there are other, related waves known [24,4,15]. The results of [2] were both analytical and computational; the analytical conclusion was a local bifurcation theorem, demonstrating that there exist traveling vortex sheets with surface tension nearby to equilibrium. In the present work, we establish a *global* bifurcation theorem for the problem with general densities. We now state a somewhat informal version of this theorem:

Theorem 1 *(Main theorem). For all choices of the constants $\tau > 0$, $M > 0$, $\bar{\gamma} \in \mathbf{R}$, $\rho_1, \rho_2 \geq 0$ (not both zero) and $g \in \mathbf{R}$, there exist a countable number of connected sets of smooth*[4] *non-trivial symmetric periodic traveling wave solutions, bifurcating from a quiescent equilibrium, for the two-dimensional gravity–capillary vortex sheet problem. If $\bar{\gamma} \neq 0$ or $\rho_1 \neq \rho_2$, then each of these connected sets has at least one of the following properties:*

(a) *it contains waves whose interfaces have lengths per period which are arbitrarily long;*
(b) *it contains waves whose interfaces have arbitrarily large curvature;*
(c) *it contains waves where the jump of the tangential component of the fluid velocity across the interface or its derivative is arbitrarily large;*
(d) *its closure contains a wave whose interface has a point of self-intersection;*
(e) *it contains a sequence of waves whose interfaces converge to a flat configuration but whose speeds contain at least two convergent subsequences whose limits differ.*

In the case that $\bar{\gamma} = 0$ and $\rho_1 = \rho_2$, each connected set has at least one of the properties (a)–(f), where (f) is the following:

(f) *it contains waves which have speeds which are arbitrarily large.*

We mention that in the case of pure gravity waves, it has sometimes been possible to rule out the possibility of an outcome like (e) above; one such paper, for example, is [11]. The argument to eliminate such an outcome is typically a maximum principle argument, and this type of argument appears to be unavailable in the present setting because of the larger number of derivatives stemming from the presence of surface tension. In a forthcoming numerical work, computations will be presented which indicate that in some cases, outcome (e) can in fact occur for gravity–capillary waves [3].

Following [2], we start from the formulation of the problem introduced by Hou, Lowengrub, and Shelley, which uses geometric dependent variables and a normalized arclength parameterization of the free surface [19,20]. This formulation follows from the observation that the tangential velocity can be chosen arbitrarily, while only the normal velocity needs to be chosen in accordance with the physics of the problem. The tangential velocity can then be selected in a convenient fashion which allows us to specialize the equations of motion to the periodic traveling wave case in a way that does not require the interface to be a graph over the horizontal coordinate. The resulting equations are non-local, nonlinear and involve the singular Birkhoff–Rott integral. Despite their complicated appearance, using several well-known properties of the Birkhoff–Rott integral we are able to recast the traveling wave equations in the form of "identity plus compact." Consequently, we are able to use an abstract version of the Rabinowitz global-bifurcation theory [29] to prove our main result. An interesting feature of our formulation is that, unlike similar formulations that allow for overturning waves by using a conformal mapping, an extension of the present method to the case of 3D waves, using for instance ideas like those in [6], seems entirely possible.

[4] Here and below, when we say a function is "smooth" we mean that its derivatives of all orders exist.

The main theorem allows for both positive and negative gravity; equivalently, we could say we allow a heavier fluid above or below a lighter fluid. As remarked in [4], this is an effect that relies strongly on the presence of surface tension. In the case of pure gravity waves, there are some theorems in the literature demonstrating the nonexistence of traveling waves in the case of negative gravity [21,31].

A similar problem was treated by Amick and Turner [7]. As with the present paper they treat the global bifurcation of interfacial waves between two fluids. However, they require the non-stagnation condition that the horizontal velocity of the fluid is less than the wave speed ($u < c$). Thus their global connected set stops once $u = c$ and there cannot be any overturning waves. Their paper has some other less important differences as well, namely it treats solitary waves and the top and bottom are fixed ($0 < y < 1$). Their methodology is very different from ours as well, since they handle the case of a smooth density first without using the Birkhoff–Rott formulation, and only later let the density approach a step function. Another paper [8] by the same authors only treats small solutions. Small-amplitude interfacial periodic traveling gravity and capillary–gravity waves on finite depth, now allowing vorticity in the fluid region, were proved to exist in [26]; this work does allow stagnation points.

Global bifurcation with $\rho_2 \equiv 0$, that is, in the water wave case, has been studied by a variety of authors. In particular, global bifurcation that permits overturning waves in the case of constant vorticity is treated in [12]. Another recent paper is [16], in which a global bifurcation theorem is proved in the case $\rho_2 \equiv 0$ for capillary–gravity waves on finite depth, also with constant vorticity. Both of these works allow for multi-valued waves by means of a conformal mapping. Walsh treats global bifurcation for capillary water waves with general non-constant vorticity in [32], with the requirement that the interface be a graph with respect to the horizontal coordinate. The methodologies of all of these papers are completely different from the present work.

Our reformulation of the traveling wave problem into the form "identity plus compact" uses the presence of surface tension in a fundamental way. In particular, the surface tension enters the problem through the curvature of the interface, and the curvature involves derivatives of the free surface. By inverting these derivatives, we gain the requisite compactness. The paper [27] uses a similar idea to gain compactness in order to prove a global bifurcation theorem for capillary–gravity water waves with constant vorticity and single-valued height.

We mention that the current work finds examples of solutions for interfacial irrotational flow which exist for all time. The relevant initial value problems are known to be well-posed at short times [5], but behavior at large times is in general still an open question. Some works on existence or nonexistence of singularities for these problems are [10,17,13]. For small-amplitude, pure capillary water waves, global solutions are known to exist in general [18,22].

The plan of the paper is as follows: in Section 2, we describe the equations of motion for the relevant interfacial fluid flows. In Section 3, we detail our traveling wave formulation which uses the arclength formulation and which allows for waves with multi-valued height. In Section 4, we explore the consequences of the assumption of spatial periodicity for our traveling wave formulation. In Section 5, we continue to work with the traveling wave formulation, now reformulating into an equation of the form "identity plus compact." This sets the stage for Section 6, in which we state a more detailed version of our main theorem and provide the proof.

2. The equations of motion

We consider two two-dimensional fluids separated by a one-dimensional sharp interface, $(x(\alpha, t), y(\alpha, t))$, with t being time and α being the spatial parameter along the curve. We consider (x, y) to be horizontally periodic, and we consider both of the fluid regions to be of infinite vertical extent. In the interior of each fluid region, the fluid velocities satisfy the irrotational, incompressible Euler equations:

$$
\begin{aligned}
&u_t + u \cdot \nabla u = -\nabla p, \\
&\mathrm{div}(u) = 0, \\
&u = \nabla \phi. \qquad (1)
\end{aligned}
$$

Because of condition (1), which we reiterate holds in the interior of either fluid region, the vorticity of each fluid is zero in the interior of either fluid region. The vorticity is not identically zero, however, because the velocity can jump across the free surface; therefore, there is measure-valued vorticity present, supported on the free surface (x, y). In particular, the vorticity is equal to an amplitude $\gamma(\alpha, t)$ multiplied by the Dirac mass of the interface. The specific jump conditions at the interface for the velocity and pressure are given below in (7) and (9). For a detailed discussion of the equations of motion, we refer the interested reader to [25] or [30].

If we make the canonical identification[5] of $\mathbf{R}^2$ with the complex plane $\mathbf{C}$, we may represent the free surface at time t, denoted by $S(t)$, as the graph (with respect to the parameter α) of

$$z(\alpha,t)=x(\alpha,t)+iy(\alpha,t).$$

The unit tangent and upward normal vectors to S are, respectively:

$$T=\frac{z_\alpha}{|z_\alpha|} \text{ and } N=i\frac{z_\alpha}{|z_\alpha|}. \tag{2}$$

(A derivative with respect to α is denoted either as a subscript or as ∂_α.) Thus we have uniquely defined real valued functions $U(\alpha,t)$ and $V(\alpha,t)$ such that

$$z_t=UN+VT \tag{3}$$

for all α and t. We call U the normal velocity of the interface and V the tangential velocity. The normal velocity U is determined from fluid mechanical considerations and is given by:

$$U=\Re(W^*N) \tag{4}$$

where

$$W^*(\alpha,t):=\frac{1}{2\pi i}\text{PV}\int_{\mathbf{R}}\frac{\gamma(\alpha',t)}{z(\alpha,t)-z(\alpha',t)}d\alpha' \tag{5}$$

is commonly referred to as the Birkhoff–Rott integral. (We use "$*$" to denote complex conjugation.) Furthermore, the operator $\Re:\mathbf{C}\to\mathbf{R}$ is the so-called real-part operator, which for any $(a,b)\in\mathbf{R}^2$ satisfies $\Re(a+ib)=a$.

The real-valued quantity γ is called in [19] "the unnormalized vortex sheet-strength," though in this document we will primarily refer to it as simply the "vortex sheet-strength." It can be used to recover the Eulerian fluid velocity (denoted by u) in the bulk at time t and position $w\notin S(t)$ via

$$u(w,t):=\left[\frac{1}{2\pi i}\int_{\mathbf{R}}\frac{\gamma(\alpha',t)}{w-z(\alpha',t)}d\alpha'\right]^*. \tag{6}$$

The quantity γ is also related to the jump in the tangential velocity of the fluid. Specifically, using the Plemelj formulas [28], one finds that:

$$[[u]]:=\lim_{w\to z(\alpha,t)^+}u(w,t)-\lim_{w\to z(\alpha,t)^-}u(w,t)=\frac{\gamma(\alpha,t)}{z^*_\alpha(\alpha,t)}. \tag{7}$$

In the above, the "+" and "−" modifying $z(\alpha,t)$ mean that the limit is taken from "above" or "below" $S(t)$, respectively. If we let $j(\alpha,t):=\Re([[u]]^*T)$ be the component of $[[u]]$ which is tangent to $S(t)$ at $z(\alpha,t)$, then the preceding formula shows:

$$\gamma(\alpha,t)=j(\alpha,t)\,|z_\alpha(\alpha,t)|, \tag{8}$$

which is to say that $\gamma(\alpha,t)$ is a scaled version of the *jump in the tangential velocity* of the fluid across the interface. For completeness, we mention that the jump in pressure across the boundary is given by the Laplace–Young condition,

$$[[p]]=\tau\kappa, \tag{9}$$

where τ is the positive, constant coefficient of surface tension, and κ is the curvature of the interface. We mention that the boundary conditions for the velocity at vertical infinity can be computed from (6), namely, $u\to\pm\bar{\gamma}/2$, where $\bar{\gamma}$ is the average value of γ over one period, as y goes to vertical infinity in either direction.

[5] Throughout this paper we make this identification for any vector in $\mathbf{R}^2$.

As shown in [5], γ evolves according to the equation

$$\gamma_t = \tau\frac{\theta_{\alpha\alpha}}{|z_\alpha|} + \frac{((V-\Re(W^*T))\gamma)_\alpha}{|z_\alpha|} \\ -2A\left(\frac{\Re(W_t^*T)}{|z_\alpha|} + \frac{1}{8}\frac{(\gamma^2)_\alpha}{|z_\alpha|^2} + gy_\alpha - (V-\Re(W^*T))\Re(W_\alpha^*T)\right). \tag{10}$$

Here A is the *Atwood number*,

$$A := \frac{\rho_1-\rho_2}{\rho_1+\rho_2}.$$

Note that A can be taken as any value in the interval $[-1,1]$. Lastly, $\theta(\alpha,t)$ is the *tangent angle* to $S(t)$ at the point $z(\alpha,t)$. Specifically it is defined by the relation

$$z_\alpha = |z_\alpha|e^{i\theta}.$$

Observe that we have the following nice representations of the tangent and normal vectors in terms of θ:

$$T = e^{i\theta} \text{ and } N = ie^{i\theta}. \tag{11}$$

We further mention that part of the great utility of using θ as one of our dependent variables is its relationship with the curvature (this is relevant because of (9) above), $\kappa = \theta_\alpha/|z_\alpha|$.

As observed above, the tangential velocity V has no impact on the geometry of $S(t)$. As such, we are free to make V anything we wish. In this way, one sees that Eqs. (3) and (10) form a closed dynamical system. In [19], the authors make use of the flexibility in the choice of V to design an efficient and non-stiff numerical method for the solution of the dynamical system. In the article [5], V is selected in a way which is helpful in making *a priori* energy estimates, and in completing a proof of local-in-time well-posedness of the initial value problem. We leave V arbitrary for now.

3. Traveling waves

We are interested in finding traveling wave solutions, which is to say solutions where both the interface and Eulerian fluid velocity propagate horizontally with no change in form and at constant speed. To be precise:

Definition 1. We say $(z(\alpha,t),\gamma(\alpha,t))$ is a traveling wave solution of (3) and (10) if there exists $c \in \mathbf{R}$ such that for all $t \in \mathbf{R}$ we have

$$S(t) = S(0) + ct \tag{12}$$

and, for all $w \notin S(t)$,

$$u(w,t) = u(w-ct,0) \tag{13}$$

where u is determined from $(z(\alpha,t),\gamma(\alpha,t))$ by way of (6).

Later on the speed c will serve as our bifurcation parameter. We have the following results concerning traveling wave solutions of (3) and (10).

Proposition 1 *(Traveling wave ansatz). (i) Suppose that $(z(\alpha,t),\gamma(\alpha,t))$ solves (3) and (10) and, moreover, there exists $c \in \mathbf{R}$ such that*

$$z_t = c \quad \textit{and} \quad \gamma_t = 0 \tag{14}$$

hold for all α and t. Then $(z(\alpha,t),\gamma(\alpha,t))$ is a traveling wave solution with speed c.

(ii) If $(\check{z},\check{\gamma})$ is a traveling wave solution with speed c of (3) and (10) then there exists a reparameterization of $S(t)$ which maps $(\check{z},\check{\gamma}) \mapsto (z,\gamma)$ where (z,γ) satisfies (14).

Proof. First we prove (i). Since $z_t = c$, we have $z(\alpha,t) = z(\alpha,0) + ct$ which immediately gives (12). Then, since $\gamma_t = 0$ we have $\gamma(\alpha,t) = \gamma(\alpha,0)$ and thus

$$u^*(w,t) = \frac{1}{2\pi i}\int_{\mathbf{R}} \frac{\gamma(\alpha',t)}{w - z(\alpha',t)}d\alpha' = \frac{1}{2\pi i}\int_{\mathbf{R}} \frac{\gamma(\alpha',0)}{w - (z(\alpha',0)+ct)}d\alpha' = u^*(w-ct,0). \tag{15}$$

And so we have (13).

Now we prove (ii). Suppose $(\check{z}(\beta,t), \check{g}(\beta,t))$ gives a traveling wave solution. The reparameterization which yields (14) can be written explicitly. Specifically, condition (12) implies that $z(\alpha,t) := \check{z}(\alpha,0) + ct$ is a parameterization of $S(t)$. Clearly $z_t = c$, and we have the first equation in (14).

Now let $\gamma(\alpha,t)$ be the corresponding vortex sheet-strength for the parameterization of $S(t)$ given by $z(\alpha,t)$. (For concreteness, $\gamma(\alpha,t)$ can be computed directly from (8).) Since we have a traveling wave, we have (13). Define

$$m(w,t) =: \frac{1}{2\pi i}\int_{\mathbf{R}} \frac{\gamma(\alpha',t)-\gamma(\alpha',0)}{w - ct - \check{z}(\alpha',0)}d\alpha'.$$

Then for $w \notin S(t)$ we have

$$\begin{aligned} m(w,t) &= \frac{1}{2\pi i}\int_{\mathbf{R}} \frac{\gamma(\alpha',t)}{w - (\check{z}(\alpha',0)+ct)}d\alpha - \frac{1}{2\pi i}\int_{\mathbf{R}} \frac{\gamma(\alpha',0)}{(w-ct) - \check{z}(\alpha',0)}d\alpha' \\ &= u(w,t) - u(w-ct,0) = 0. \end{aligned} \tag{16}$$

However, for a point $w_0 = \check{z}(\alpha,0) + ct \in S(t)$, the Plemelj formulas state that

$$\lim_{w\to w_0^{\pm}} m(w) = \mathrm{PV}\frac{1}{2\pi i}\int_{\mathbf{R}} \frac{\gamma(\alpha',t)-\gamma(\alpha',0)}{\check{z}(\alpha,0) - \check{z}(\alpha',0)}d\alpha \pm \frac{1}{2}\frac{\gamma(\alpha,t)-\gamma(\alpha,0)}{\check{z}_\alpha(\alpha,0)}$$

where the "+" and "−" signs modifying w_0 in the limit indicate that the limit is taken from "above" or "below" $S(t)$, respectively. But, of course, m is identically zero so that

$$\frac{1}{2}\left(\gamma(\alpha,t) - \gamma(\alpha,0)\right) = \pm\check{z}_\alpha(\alpha,0)\mathrm{PV}\frac{1}{2\pi i}\int_{\mathbf{R}} \frac{\gamma(\alpha',t)-\gamma(\alpha',0)}{\check{z}(\alpha,0) - \check{z}(\alpha',0)}d\alpha,$$

which in turn implies $\gamma(\alpha,t) = \gamma(\alpha,0)$. Since this is true for any t and any α, we see that $\gamma_t = 0$, the second equation in (14). □

Remark 1. We additionally assume that $S(t)$ is parameterized to be proportional to arclength, *i.e.*

$$|z_\alpha| = \sigma = \text{constant} > 0 \tag{17}$$

for all (α,t). One may worry that the enforcement of the parameterization such that $z_t = c$ in (14) is at odds with this sort of arclength parameterization. However, notice that $z_t = c$ implies that $z_{\alpha t} = 0$ which in turn implies that z_α (and thus $|z_\alpha|$) does not depend on time. Then the parameterization of $S(t)$ given by $\tilde{z}(\beta(\alpha),t) = z(\alpha,t)$ where $d\beta/d\alpha = |z_\alpha|/\sigma$ has $|\tilde{z}_\beta| = \sigma$. Thus it is merely a convenience to assume (17). We will select a convenient choice for σ later. Arguments parallel to the above show that $z_t = c$ implies that $\theta_t = 0$ and thus we will view θ as being a function of α only.

Now we insert the ansatz (14) and the arclength parameterization (17) into the equations of motion (3) and (10). First, as observed in [2], we see that elementary trigonometry shows that $z_t = c$ and (3) are equivalent to

$$U = -c\sin\theta \tag{18}$$

and

$$V = c\cos\theta. \tag{19}$$

Notice this last equation selects V in terms of the tangent angle θ. That is to say (19) should be viewed as the definition of V. On the other hand (18) should be viewed as one of the equations we wish to solve. Using (4), we rewrite it as

$$\Re(W^*N) + c\sin\theta = 0. \tag{20}$$

The above considerations transform (10) to:

$$0 = \tau\frac{\theta_{\alpha\alpha}}{\sigma} + \frac{\{(c\cos\theta - \Re(W^*T))\gamma\}_\alpha}{\sigma} - 2A\left(\frac{1}{8}\frac{(\gamma^2)_\alpha}{\sigma^2} + g\sin\theta - (c\cos\theta - \Re(W^*T))\Re(W^*_\alpha T)\right). \tag{21}$$

The last part of this expression may be rewritten as follows. Observe that

$$-\frac{1}{2}\partial_\alpha\{(c\cos\theta - \Re(W^*T))^2\} = (c\cos\theta - \Re(W^*T))\left(c\sin\theta\theta_\alpha + \Re(W^*T_\alpha) + \Re(W^*_\alpha T)\right). \tag{22}$$

Using (11), we see that $T_\alpha = N\theta_\alpha$. Thus since θ is real valued and by virtue of (20), we have

$$c\sin\theta\theta_\alpha + \Re(W^*T_\alpha) = (c\sin\theta + \Re(W^*N))\theta_\alpha = 0.$$

So (22) simplifies to

$$-\frac{1}{2}\partial_\alpha(c\cos\theta - \Re(W^*T))^2 = (c\cos\theta - \Re(W^*T))\Re(W^*_\alpha T).$$

Hence

$$0 = \tau\frac{\theta_{\alpha\alpha}}{\sigma} + \frac{\{(c\cos\theta - \Re(W^*T))\gamma\}_\alpha}{\sigma} - 2A\left(\frac{1}{8}\frac{(\gamma^2)_\alpha}{\sigma^2} + g\sin\theta + \frac{1}{2}\partial_\alpha(c\cos\theta - \Re(W^*T))^2\right), \tag{23}$$

which we rewrite as

$$-\theta_{\alpha\alpha} = \Phi(\theta,\gamma;c,\sigma) := \frac{1}{\tau}(\partial_\alpha\{(c\cos\theta - \Re(W^*T))\gamma\}) - \frac{A}{\tau}\left(\frac{1}{4\sigma}\partial_\alpha(\gamma^2) + 2g\sigma\sin\theta + \sigma\partial_\alpha\{c\cos\theta - \Re(W^*T))^2\}\right). \tag{24}$$

Note that we have not specified z as one of the dependencies of Φ. This may seem unusual, given the prominent role of z in computing the Birkhoff–Rott integral W^*. However, given σ in (17) one can determine $z(\alpha,t)$ solely from the tangent angle $\theta(\alpha)$, at least up to a rigid translation. Specifically, and without loss of generality, we have

$$z(\alpha,0) = z(\alpha,t) - ct = \sigma\int_0^\alpha e^{i\theta(\alpha')}d\alpha'. \tag{25}$$

In this way, we view W^* as being a function of θ, γ and σ.

In short, we have shown the following:

Lemma 2 *(Traveling wave equations, general version). Given time independent functions θ and γ and constants $c \in \mathbf{R}$ and $\sigma > 0$, compute $z(\alpha,t)$ from (25), W^* from (5) and N and T from (11). If*

$$\Re(W^*N) + c\sin\theta = 0 \quad \textit{and} \quad \theta_{\alpha\alpha} + \Phi(\theta,\gamma;c,\sigma) = 0 \tag{26}$$

hold then (z,γ) is a traveling wave solution with speed c for (3) and (10).

It happens that under the assumption that the traveling waves are spatially periodic, (26) can be reformulated as "identity plus compact" which, in turn will allow us to employ powerful abstract global bifurcation results. The next section deals with how to deal with spatial periodicity.

4. Spatial periodicity

To be precise, by spatial periodicity we mean the following:

Definition 2. Suppose that $(z(\alpha, t), \gamma(\alpha, t))$ is a solution of (3) and (10) such that

$$S(t) = S(t) + M$$

and

$$u(w + M, t) = u(w, t)$$

for all t and $w \notin S(t)$, then the solution is said to be (horizontally) spatially periodic with period M.

It is clear if one has a spatially periodic curve $S(t)$ then it can be parameterized in such a way that the parameterization is 2π-periodic in its dependence on the parameter. That is to say, the curve can be parameterized such that

$$z(\alpha + 2\pi, t) = z(\alpha, t) + M. \tag{27}$$

It is here that we encounter a sticky issue. As described in Lemma 2, our goal is to find θ and γ such that (26) holds and additionally (27) holds. The issue is that, given a function $\theta(\alpha)$ which is 2π-periodic with respect to α, it may not be the case that the curve z reconstructed from it via (25) satisfies (27). In fact, due to (17), the periodicity (27) is valid if and only if

$$2\pi\overline{\cos\theta} := \int_0^{2\pi} \cos(\theta(\alpha'))d\alpha' = \frac{M}{\sigma} \quad \text{and} \quad 2\pi\overline{\sin\theta} := \int_0^{2\pi} \sin(\theta(\alpha'))d\alpha' = 0. \tag{28}$$

We could impose (28) on θ. However, we follow another strategy which leaves θ free by modifying (26) so that (28) holds.

Indeed, we first fix the spatial period $M > 0$. Suppose we are given a real 2π-periodic function $\theta(\alpha)$ for which

$$\overline{\cos\theta} \neq 0 \tag{29}$$

so that the period M of the curve will not vanish. Then we define the "renormalized curve" as

$$\widetilde{Z}[\theta](\alpha) = \frac{M}{2\pi\overline{\cos\theta}} \left\{ \int_0^{\alpha} e^{i\theta(\beta)} d\beta - i\alpha\, \overline{\sin\theta} \right\}. \tag{30}$$

Of course, this function is one derivative smoother than θ. A direct calculation shows that

$$\widetilde{Z}[\theta](\alpha + 2\pi) = \widetilde{Z}[\theta](\alpha) + M. \tag{31}$$

Thus $w = \widetilde{Z}[\theta]$ is the parameterization of a curve which satisfies

$$w(\alpha + 2\pi) = w(\alpha) + M \text{ for all } \alpha \text{ in } \mathbf{R}. \tag{32}$$

Now $\partial_\alpha \widetilde{Z}[\theta] = \dfrac{M}{2\pi\overline{\cos\theta}}(\exp(i\theta[\alpha]) - i\overline{\sin\theta})$, and the tangent and normal vectors for $\widetilde{Z}[\theta]$ are given by:

$$\widetilde{T}[\theta] := \partial_\alpha \widetilde{Z}[\theta]/|\partial_\alpha \widetilde{Z}[\theta]| \quad \text{and} \quad \widetilde{N}[\theta] := i\partial_\alpha \widetilde{Z}[\theta]/|\partial_\alpha \widetilde{Z}[\theta]|. \tag{33}$$

These expressions are not equal to $e^{i\theta}$ and $ie^{i\theta}$, as was the case for T and N in (11).

For a given real function $\gamma(\alpha)$ and parametrized curve $w(\alpha)$, define the *Birkhoff–Rott integral*

$$B[w]\gamma(\alpha) := \frac{1}{2\pi i} \mathrm{PV} \int_{\mathbf{R}} \frac{\gamma(\alpha')}{w(\alpha) - w(\alpha')} d\alpha'.$$

Thus $W^* = B[z]\gamma$. If w satisfies (32), we can rewrite this integral as

$$B[w]\gamma(\alpha) = \frac{1}{2iM}\mathrm{PV}\int_0^{2\pi} \gamma(\alpha')\cot\left(\frac{\pi}{M}(w(\alpha)-w(\alpha'))\right)d\alpha' \tag{34}$$

by means of Mittag–Leffler's famous series expansion for the cotangent (see, e.g., Chapter 3 of [1]). Finally, for any real 2π-periodic functions θ and γ and any constant $c \in \mathbf{R}$, define

$$\begin{aligned}\widetilde{\Phi}(\theta,\gamma;c) &:= \frac{1}{\tau}\partial_\alpha\{c\cos\theta - \Re(B[\widetilde{Z}[\theta]]\gamma\ \widetilde{T}[\theta]))\gamma\} \\ &\quad - \frac{A}{\tau}\left(\frac{\pi\overline{\cos\theta}}{2M}\partial_\alpha(\gamma^2) + \frac{gM}{\pi\overline{\cos\theta}}\left(\sin\theta - \overline{\sin\theta}\right) + \frac{M}{2\pi\overline{\cos\theta}}\partial_\alpha\{(c\cos\theta - \Re(B[\widetilde{Z}[\theta]]\gamma\ \widetilde{T}[\theta]))^2\}\right).\end{aligned} \tag{35}$$

In terms of these definitions the basic equations are rewritten as follows:

Proposition 2 *(Traveling wave equations, spatially periodic version). If the 2π-periodic functions $\theta(\alpha)$, $\gamma(\alpha)$ and the constant $c \neq 0$ satisfy (29) and*

$$\Re(B[\widetilde{Z}[\theta]]\gamma\ \widetilde{N}[\theta]) + c\sin\theta = 0 \quad \textit{and} \quad \theta_{\alpha\alpha} + \widetilde{\Phi}(\theta,\gamma;c) = 0, \tag{36}$$

then $(\widetilde{Z}[\theta](\alpha)+ct, \gamma(\alpha,t))$ is a spatially periodic traveling wave solution with speed c and period M for (3) and (10).

Proof. Putting $w = \widetilde{Z}[\theta]$, from the definitions above we have $\widetilde{N}[\theta] = iw_\alpha/|w_\alpha|$. Thus by Lemma 3 below,

$$\int_0^{2\pi} \Re(B[\widetilde{Z}[\theta]]\gamma\ \widetilde{N}[\theta])\,d\alpha = 0.$$

Together with the first equation in (36) and the fact $c \neq 0$, this gives

$$\overline{\sin\theta} = 0. \tag{37}$$

Now we let $\sigma = M/(2\pi\overline{\cos\theta})$ and compute $z(\alpha,t)$ from (25), W^* from (5), and N and T from (2). By (25) and (30) we see that $z(\alpha,t) = \widetilde{Z}[\theta](\alpha) + ct$. This in turn gives $W^* = B[\widetilde{Z}[\theta]]\gamma$, $N = \widetilde{N}[\theta]$ and $T = \widetilde{T}[\theta]$. Together with the fact that $\overline{\sin\theta} = 0$, this shows that

$$\widetilde{\Phi}(\theta,\gamma;c) = \Phi(\theta,\gamma;c,\sigma).$$

Thus both equations in (36) coincide exactly their counterparts in (26). Proposition 2 then shows that $z(\alpha,t)$ is a traveling wave with speed $c \in \mathbf{R}$. We know that $z(\alpha,t)$ is M-periodic since it was constructed from θ with $\widetilde{Z}$. □

Lemma 3. *If $w(\cdot)$ satisfies (32) and $\gamma(\cdot)$ is a 2π-periodic function, then*

$$\int_0^{2\pi} \Re\left(B[w]\gamma(\alpha)\frac{iw_\alpha(\alpha)}{|w_\alpha(\alpha)|}\right)d\alpha = 0.$$

Proof. This lemma says that the mean value of the normal component of $B[w](\gamma)$ is equal to zero. This follows from the fact that $B[w](\gamma)$ extends to a divergence-free field in the interior of the fluid region, and from the Divergence Theorem. □

5. Reformulation as "identity plus compact"

5.1. Mapping properties

Let $\mathcal{H}^s_{\text{per}} := \mathcal{H}^s_{\text{per}}[0, 2\pi]$ be the usual Sobolev space of 2π-periodic functions from $\mathbf{R}$ to $\mathbf{C}$ whose first $s \in \mathbf{N}$ weak derivatives are square integrable. Likewise for intervals $I \subset \mathbf{R}$, let $\mathcal{H}^s(I)$ be the usual Sobolev space of functions from I to $\mathbf{C}$ whose first $s \in \mathbf{N}$ weak derivatives are square integrable. Finally, $\mathcal{H}^s_{\text{loc}}$ is the set of all functions from $\mathbf{R}$ to $\mathbf{C}$ which are in $\mathcal{H}^s(I)$ for all bounded intervals $I \subset \mathbf{R}$.

By (31), $\widetilde{Z}[\theta](\alpha) - M\alpha/2\pi$ is periodic. Let

$$\mathcal{H}^s_M := \left\{ w \in \mathcal{H}^s_{\text{loc}} : w(\alpha) - M\alpha/2\pi \in \mathcal{H}^s_{\text{per}} \right\}.$$

Clearly $\mathcal{H}^s_M$ is a complete metric space with the metric of $\mathcal{H}^s_{\text{per}}$ We have the following lemma concerning the renormalized curve $\widetilde{Z}[\theta]$:

Lemma 4. *For $s \geq 1$ and $h \geq 0$ let*

$$\mathcal{U}^s_h := \left\{ \theta \in \mathcal{H}^s_{\text{per}} : \int_0^{2\pi} \cos(\theta(\alpha)) d\alpha > h \right\}.$$

Then the map $\widetilde{Z}[\theta]$ defined in (30) is smooth from $\mathcal{U}^s_h$ into $\mathcal{H}^{s+1}_M$ and the maps $\widetilde{N}[\theta]$ and $\widetilde{T}[\theta]$ given in (33) are smooth from $\mathcal{U}^s_h$ into $\mathcal{H}^s_{\text{per}}$. Moreover, for any $h > 0$, there exists $C > 0$ such that

$$\|\widetilde{T}[\theta]\|_{\mathcal{H}^s_{\text{per}}} + \|\widetilde{N}[\theta]\|_{\mathcal{H}^s_{\text{per}}} + \|\widetilde{Z}[\theta]\|_{\mathcal{H}^{s+1}(0,2\pi)} + \left\| \frac{1}{\partial_\alpha \widetilde{Z}[\theta]} \right\|_{\mathcal{H}^s_{\text{per}}} \leq C(1 + \|\theta\|_{\mathcal{H}^s_{\text{per}}}) \tag{38}$$

for all $\theta \in \mathcal{U}^s_h$.

Proof. As already mentioned, $\widetilde{Z}[\theta]$ is one derivative smoother than θ. A series of naive estimates leads to the bound on $\widetilde{Z}[\theta]$ in (38). Next, since θ belongs to $\mathcal{U}^s_h$,

$$\overline{\sin\theta} + \frac{1}{100\pi^2} h^2 \leq \frac{1}{2\pi} \left[\int_0^{2\pi} \sin(\theta(a)) da + \frac{1}{50\pi} \left(\int_0^{2\pi} \cos(\theta(a)) da \right)^2 \right].$$

The Cauchy–Schwarz inequality on the cosine term leads to

$$\overline{\sin\theta} + \frac{1}{100\pi} h^2 \leq \frac{1}{2\pi} \int_0^{2\pi} \left(\sin(\theta(a)) + \frac{1}{25} \cos^2(\theta(a)) \right) da \leq 1$$

since $\sin(x) + (1/25)\cos^2(x) \leq 1$. Thus

$$\overline{\sin\theta} \leq 1 - \frac{1}{100\pi^2} h^2.$$

For $h > 0$, this implies that $e^{i\theta} - i\,\overline{\sin\theta}$ cannot vanish. Hence $1/\partial_\alpha \widetilde{Z}[\theta] \in \mathcal{H}^s_{\text{per}}$ and the remaining bounds in (38) follow by routine estimates. The smooth dependence of $\widetilde{Z}$, $\widetilde{N}$ and $\widetilde{T}$ on θ is a consequence of standard results on compositions. □

The most singular part of the Birkhoff–Rott operator B is essentially the periodic Hilbert transform H, which is defined as

$$H\gamma(\alpha) := \frac{1}{2\pi} \text{PV} \int_0^{2\pi} \gamma(\alpha') \cot\left(\frac{1}{2}(\alpha - \alpha') \right) d\alpha'.$$

It is well-known that for any $s \geq 0$, H is a bounded linear map from $\mathcal{H}^s_{\text{per}}$ to $\mathcal{H}^s_{\text{per},0}$ (the subscript 0 here indicates that the average over a period vanishes). Moreover, H annihilates the constant functions and $H^2\gamma = -\gamma + \overline{\gamma}$, where $\overline{\gamma} =: \frac{1}{2\pi}\int_0^{2\pi} \gamma(a)da$. In order to conclude that the leading singularity of the function $B[w]\gamma$ is given in terms of $H\gamma$, we require a "chord-arc" condition, as stated in the following lemma.

Lemma 5. *For $b \geq 0$ and $s \geq 2$, let the "chord-arc space" be*

$$\mathcal{C}^s_b := \left\{ w(\alpha) \in \mathcal{H}^s_M : \inf_{\alpha,\alpha' \in [0,2\pi]} \left| \frac{w(\alpha') - w(\alpha)}{\alpha' - \alpha} \right| > b \right\}$$

and the remainder operator K be

$$K[w]\gamma(\alpha) := B[w]\gamma(\alpha) - \frac{1}{2i w_\alpha(\alpha)} H\gamma(\alpha).$$

Then $(w, \gamma) \mapsto K[w]\gamma$ is a smooth map from $\mathcal{C}^s_b \times \mathcal{H}^1_{\text{per}} \to \mathcal{H}^{s-1}_{\text{per}}$. If $b > 0$, then there exists a constant $C > 0$ such that for all $w \in \mathcal{C}^s_b$ and for all $\gamma \in \mathcal{H}^1_{\text{per}}$,

$$\|K[w]\gamma\|_{\mathcal{H}^{s-1}_{\text{per}}} \leq C\|\gamma\|_{\mathcal{H}^1_{\text{per}}} \exp\left\{C\|w\|_{\mathcal{H}^s(0,2\pi)}\right\}.$$

Proof. See Lemma 3.5 of [5]. We mention that related lemmas can be found elsewhere in the literature, such as in [9]. □

The set $\mathcal{C}^s_b$ is the open subset of $\mathcal{H}^s_M$ of functions whose graphs satisfy the "chord-arc" condition. *This condition precludes self-intersection of the graph.* Note that this is true even in the case where $b = 0$ since we have selected the strict inequality in the definition. Of course if $b > 0$, membership of w in this set $\mathcal{C}^s_b$ implies that $|w_\alpha(\alpha)| \geq b$ for all α.

Note that $H\gamma$ is real because γ is real-valued. Also note that the definition of $\widetilde{T}[\theta]$ implies that $\widetilde{T}[\theta]/\partial_\alpha \widetilde{Z}[\theta] = 1/|\partial_\alpha \widetilde{Z}[\theta]|$ is also real. Thus

$$\begin{aligned} \Re\left((B[\widetilde{Z}[\theta]]\gamma\ \widetilde{T}[\theta]\right) &= \Re\left(\left(K[\widetilde{Z}[\theta]]\gamma\right)\widetilde{T}[\theta]\right) + \Re\left(\left(\frac{1}{2i\partial_\alpha \widetilde{Z}[\theta]} H\gamma\right)\widetilde{T}[\theta]\right) \\ &= \Re\left(\left(K[\widetilde{Z}[\theta]]\gamma\right)\widetilde{T}[\theta]\right) \end{aligned} \tag{39}$$

and similarly

$$\Re((B[\widetilde{Z}[\theta]]\gamma\ \widetilde{N}[\theta]) = \frac{1}{2|\partial_\alpha \widetilde{Z}[\theta]|} H\gamma + \Re(K[\widetilde{Z}[\theta]]\gamma\ \widetilde{N}[\theta]). \tag{40}$$

Therefore, counting derivatives and applying Lemmas 4 and 5, we directly obtain the following regularity.

Corollary 1. *Let $s, s_1 \geq 1$, $b > 0$, $h > 0$ and*

$$\mathcal{U}^s_{b,h} := \left\{\theta \in \mathcal{U}^s_h : \widetilde{Z}[\theta] \in \mathcal{C}^{s+1}_b\right\}.$$

Then the mappings $(\theta, \gamma) \to B[\widetilde{Z}[\theta]]\gamma$ and $(\theta, \gamma) \to \Re(B[\widetilde{Z}[\theta]]\gamma\ \widetilde{N}[\theta])$ are smooth from $\mathcal{U}^s_{b,h} \times \mathcal{H}^{s_1}_{\text{per}}$ into $\mathcal{H}^{\min\{s,s_1\}}_{\text{per}}$. Furthermore, $\Re(B[\widetilde{Z}[\theta]]\gamma\ \widetilde{T}[\theta])$ is a smooth map from $\mathcal{U}^s_{b,h} \times \mathcal{H}^1_{\text{per}}$ into $\mathcal{H}^s_{\text{per}}$.

Corollary 2. *$\widetilde{\Phi}(\theta, \gamma; c)$ is a smooth map from $\mathcal{U}^1_{b,h} \times \mathcal{H}^1_{\text{per}} \times \mathbf{R}$ into $L^2_{\text{per},0} := \mathcal{H}^0_{\text{per},0}$.*

Proof. The fact that $\widetilde{\Phi}(\theta, \gamma; c)$ is a smooth map from $\mathcal{U}^1_{b,h} \times \mathcal{H}^1_{\text{per}} \times \mathbf{R}$ into L^2_{per} follows from the previous corollary and the definition of $\widetilde{\Phi}$. Examination of the terms in $\widetilde{\Phi}$ shows that all but one is a perfect derivative, and thus will have mean value zero on $[0, 2\pi]$. The remaining term is a constant times $\sin\theta - \overline{\sin\theta}$, which also has mean zero. Thus $\widetilde{\Phi} \in L^2_{\text{per},0}$. □

We introduce the "inverse" operator

$$\partial_\alpha^{-2} f(\alpha) := \int_0^\alpha \int_0^a f(s)dsda - \frac{\alpha}{2\pi}\int_0^{2\pi}\int_0^a f(s)dsda,$$

which is bounded from $H^s_{\mathrm{per},0}$ to H^{s+2}_{per}. Indeed, it is obvious that $\partial_\alpha^2(\partial_\alpha^{-2} f) = f$, so we only need to demonstrate the periodicity of $\partial_\alpha^{-2} f$ for any $f \in H^s_{\mathrm{per},0}$. To this end, we compute

$$\begin{aligned}\partial_\alpha^{-2} f(\alpha+2\pi) &= \int_0^{\alpha+2\pi}\int_0^a f(s)dsda - \frac{\alpha+2\pi}{2\pi}\int_0^{2\pi}\int_0^a f(s)dsda \\ &= \int_{2\pi}^{\alpha+2\pi}\int_0^a f(s)dsda - \frac{\alpha}{2\pi}\int_0^{2\pi}\int_0^a f(s)dsda \\ &= \int_0^\alpha\int_0^b f(s)dsdb - \frac{\alpha}{2\pi}\int_0^{2\pi}\int_0^a f(s)dsda = \partial_\alpha^{-2} f(\alpha).\end{aligned}$$

5.2. *Final reformulation*

Using (40) in the first equation of (36) yields the equation

$$H\gamma + 2\left|\partial_\alpha \widetilde{Z}[\theta]\right| \Re\left((K[\widetilde{Z}[\theta]]\gamma)\widetilde{N}[\theta]\right) + 2c\left|\partial_\alpha \widetilde{Z}[\theta]\right| \sin\theta = 0.$$

It will be helpful to break γ up into the sum of its average value and a mean zero piece, so we let

$$\gamma_1 := \gamma - \overline{\gamma}.$$

Applying H to both sides and using $H^2\gamma = -\gamma + \overline{\gamma} = -\gamma_1$, we obtain

$$\gamma_1 - H\left\{2\left|\partial_\alpha \widetilde{Z}[\theta]\right| \Re\left((K[\widetilde{Z}[\theta]](\overline{\gamma}+\gamma_1))\widetilde{N}[\theta]\right) + 2c\left|\partial_\alpha \widetilde{Z}[\theta]\right| \sin\theta\right\} = 0. \tag{41}$$

It will turn out that we are free to specify $\overline{\gamma}$ in advance, and so henceforth we will view $\overline{\gamma}$ as a constant in the equations, akin to g, M, A or τ.

Now one of the equations we wish to solve is $\theta_{\alpha\alpha} + \widetilde{\Phi}(\theta, \gamma; c) = 0$. We use ∂_α^{-2} to "solve" this equation for θ. Keeping in mind that $\gamma = \overline{\gamma} + \gamma_1$, we define

$$\Theta(\theta, \gamma_1; c) := -\partial_\alpha^{-2}\widetilde{\Phi}(\theta, \overline{\gamma}+\gamma_1; c). \tag{42}$$

Then the second equation in (36) is equivalent to $\theta - \Theta(\theta, \gamma_1; c) = 0$. Knowing that $\theta = \Theta$, we are also free to rewrite (41) as $\gamma_1 - \Gamma(\theta, \gamma_1; c) = 0$, where

$$\begin{aligned}\Gamma(\theta, \gamma_1; c) := 2H\Big\{&\left|\partial_\alpha \widetilde{Z}[\Theta(\theta,\gamma_1;c)]\right| \Re\left(\left(K\left[\widetilde{Z}[\Theta(\theta,\gamma_1;c)]\right](\overline{\gamma}+\gamma_1)\right)\widetilde{N}[\Theta(\theta,\gamma_1;c)]\right) \\ &+ c\left|\partial_\alpha \widetilde{Z}[\Theta(\theta,\gamma_1;c)]\right| \sin(\Theta(\theta,\gamma_1;c))\Big\}.\end{aligned} \tag{43}$$

Summarizing, our equations now have the form

$$\theta - \Theta(\theta, \gamma_1; c) = 0, \quad \gamma_1 - \Gamma(\theta, \gamma_1; c) = 0. \tag{44}$$

The set where the solutions will be situated is $\mathcal{U} = \mathcal{U}_{0,0}$ where

$$\begin{aligned}\mathcal{U}_{b,h} := \Big\{(\theta, \gamma_1; c) \in \mathcal{H}^1_{\mathrm{per}} \times \mathcal{H}^1_{\mathrm{per},0} \times \mathbf{R} : \theta \text{ is odd}, \gamma_1 \text{ is even}, \overline{\cos\theta} > h, \\ \widetilde{Z}[\theta] \in \mathcal{C}^2_b \text{ and } \widetilde{Z}[\Theta(\theta,\gamma_1;c)] \in \mathcal{C}^3_b\Big\}.\end{aligned} \tag{45}$$

These sets are given the topology of $\mathcal{H}^1_{\mathrm{per}} \times \mathcal{H}^1_{\mathrm{per},0} \times \mathbf{R}$. Note that they are defined so that the functions have one derivative. The following theorem states that Θ and Γ gain an extra derivative.

Theorem 6 *("Identity plus compact" formulation). For all $b, h > 0$, the pair (Θ, Γ) is a compact map from $\mathcal{U}_{b,h}$ into $\mathcal{H}^2_{\text{per,odd}} \times \mathcal{H}^2_{\text{per,0,even}}$ and is smooth from $\mathcal{U}$ into $\mathcal{H}^2_{\text{per,odd}} \times \mathcal{H}^2_{\text{per,0,even}}$. If $(\theta, \gamma_1; c) \in \mathcal{U}$ solves (44), then the pair $(\widetilde{Z}[\theta](\alpha) + ct, \overline{\gamma} + \gamma_1(\alpha))$ is a spatially periodic, symmetric traveling wave solution of (3) and (10) with speed c and period M.*

Proof. Observe that from the results of the previous section, $\Theta(\theta, \gamma_1; c)$ is a smooth map from $\mathcal{U}^1_{b,h} \times \mathcal{H}^1_{\text{per},0} \times \mathbf{R}$ into $\mathcal{H}^2_{\text{per}}$ for any $b, h > 0$. Careful unraveling of the definitions shows that $\Gamma(\theta, \gamma_1; c)$ is a smooth map from the set $\{(\theta, \gamma_1; c) \in \mathcal{U}^1_{b,h} \times \mathcal{H}^1_{\text{per},0} \times \mathbf{R} : \widetilde{Z}[\Theta(\theta, \gamma_1; c)] \in \mathcal{C}^3_b\}$ into $\mathcal{H}^2_{\text{per},0}$. These facts, together with the uniform bound for fixed $b > 0$ on the remainder operator K in Lemma 5, imply that the mapping (Θ, Γ) is compact from $\mathcal{U}_{b,h}$ into $\mathcal{H}^2_{\text{per,odd}} \times \mathcal{H}^2_{\text{per,0,even}}$, since $\mathcal{H}^2_{\text{per}}$ is compactly embedded in $\mathcal{H}^1_{\text{per}}$. We conclude that Θ and Γ are also smooth maps, but not necessarily compact, on the union of the previous sets over all $b, h > 0$, which is to say that (Θ, Γ) is smooth on $\mathcal{U}$. The second statement in the theorem is obvious from the previous discussion. Finally, the subscripts "odd" and "even" in the target space for (Θ, Γ) above simply denote the subspaces which consist of odd and even functions. That (Θ, Γ) preserves this symmetry can be directly checked from its definition; the computation is not short, but neither is it difficult. So we omit it. □

6. Global bifurcation

6.1. General considerations

Our basic tool is the following global bifurcation theorem, which is based on the use of Leray–Schauder degree. It is fundamentally due to Rabinowitz [29] and was later generalized by Kielhöfer [23].

Theorem 7 *(General bifurcation theorem). Let X be a Banach space and U be an open subset of $X \times \mathbf{R}$. Let F map U continuously into X. Assume that*

(a) *the Frechet derivative $D_\xi F(0, \cdot)$ belongs to $C(\mathbf{R}, L(X, X))$,*
(b) *the mapping $(\xi, c) \to F(\xi, c) - \xi$ is compact from $X \times \mathbf{R}$ into X, and*
(c) *$F(0, c_0) = 0$ and $D_x F(0, c)$ has an odd crossing number at $c = c_0$.*

Let $\mathcal{S}$ denote the closure of the set of nontrivial solutions of $F(\xi, c) = 0$ in $X \times \mathbf{R}$. Let $\mathcal{C}$ denote the connected component of $\mathcal{S}$ to which $(0, c_0)$ belongs. Then one of the following alternatives is valid:

(i) *$\mathcal{C}$ is unbounded; or*
(ii) *$\mathcal{C}$ contains a point $(0, c_1)$ where $c_0 \neq c_1$; or*
(iii) *$\mathcal{C}$ contains a point on the boundary of U.*

The *crossing number* is the number of eigenvalues of $D_x F(0, c)$ that pass through 0 as c passes through c_0. In his original paper [29] Rabinowitz assumed that F has the form $F(\xi, c) = \xi - cG(\xi)$. Kielhöffer's book [23] permits the general form as above. Theorem II.3.3 of [23] states Theorem 7 in the case that $U = X \times \mathbf{R}$. The proof of Theorem 7 with an open set U is practically identical to that in [23].

We apply this theorem to our problem by fixing $b, h > 0$ and setting $U = \mathcal{U}_{b,h}$, $X = \mathcal{H}^1_{\text{per,odd}} \times \mathcal{H}^1_{\text{per,0,even}}$, $\xi = (\theta, \gamma_1)$, and $F(\xi, c) = \xi - (\Theta(\xi, c), \Gamma(\xi, c))$. Then the problem laid out in Theorem 6 fits into the framework of this theorem. All we need to do is to choose c_0 so that the linearization has an odd crossing number when $c = c_0$. In fact, the simplest case with crossing number one will suffice.

6.2. Computation of the crossing number

This calculation is difficult primarily due to the large number of terms we must differentiate. Thus we introduce some notation which will help to compress the calculations. For any map $\mu(\theta, \gamma_1; c)$, we use $(\breve{\theta}, \breve{\gamma})$ to denote the direction of differentiation. To wit, we define:

$$\mu_0 := \mu(0,0;c) \quad \text{and}$$
$$D\mu := D_{\theta,\gamma_1}\mu(\theta,\gamma_1;c)\big|_{(0,0;c)}(\breve{\theta},\breve{\gamma}) := \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\mu(\epsilon\breve{\theta},\epsilon\breve{\gamma};c)-\mu(0,0;c)\Big). \tag{46}$$

We let $Q(\theta,\gamma_1) := (\overline{\gamma}+\gamma_1)^2$, $Y(\theta) = \overline{\sin\theta}$, $\Sigma(\theta) = M/(2\pi\overline{\cos\theta})$, and $\widetilde{\mathsf{W}}^*[\theta,\gamma_1] := B[\widetilde{Z}[\theta]](\overline{\gamma}+\gamma_1)$. It is to be understood that by sin and cos we mean the maps $\theta \to \sin\theta$ and $\theta \to \cos\theta$, respectively. We will first compute the linearizations of Θ and Γ while ignoring the restrictions to symmetric (even/odd) functions; of course, computation of the full linearization will restrict in a natural way to the linearization of the symmetric problem.

The following quantities and derivatives thereof are elementary:

$$\sin_0 = 0,\quad \cos_0 = 1,\quad Q_0 = \overline{\gamma}^2,\quad Y_0 = 0,\quad \Sigma_0 = 1,\quad \widetilde{Z}_0 = \frac{M}{2\pi}\alpha,$$
$$(\partial_\alpha\widetilde{Z})_0 = \big|\partial_\alpha\widetilde{Z}\big|_0 = \frac{M}{2\pi},\quad \widetilde{T}_0 = 1,\quad \widetilde{N}_0 = i \quad\text{and}\quad \widetilde{\mathsf{W}}_0^* = 0.$$
$$D\sin = \breve{\theta},\quad D\cos = 0,\quad DQ = 2\overline{\gamma}\breve{\gamma},\quad DY = \frac{1}{2\pi}\int_0^{2\pi}\breve{\theta}(a)da,\quad D\Sigma = 0,$$
$$D\widetilde{Z} = \frac{iM}{2\pi}\left(\int_0^{\alpha}\breve{\theta}(a)da - \frac{\alpha}{2\pi}\int_0^{2\pi}\breve{\theta}(a)da\right),\quad D\partial_\alpha\widetilde{Z} = \frac{iM}{2\pi}\left(\breve{\theta} - \frac{1}{2\pi}\int_0^{2\pi}\breve{\theta}(a)da\right),$$
$$D\left|\partial_\alpha\widetilde{Z}\right| = 0,\quad D\widetilde{T} = i\left(\breve{\theta} - \frac{1}{2\pi}\int_0^{2\pi}\breve{\theta}(a)da\right)\quad\text{and}\quad D\widetilde{N} = -\breve{\theta} + \frac{1}{2\pi}\int_0^{2\pi}\breve{\theta}(a)da.$$

The computation of $D\widetilde{\mathsf{W}}^*$ is somewhat more complicated. By the product and chain rules,

$$\begin{aligned}
D\widetilde{\mathsf{W}}^* &= D\left[B[\widetilde{Z}[\theta]](\overline{\gamma}+\gamma_1)\right](\breve{\theta},\breve{\gamma})\\
&= \frac{1}{2iM}\mathrm{PV}\int_0^{2\pi} D\left[(\overline{\gamma}+\gamma_1(\alpha'))\cot\left(\frac{\pi}{M}\left(\widetilde{Z}[\theta](\alpha)-\widetilde{Z}[\theta](\alpha')\right)\right)\right]d\alpha'\\
&= \frac{1}{2iM}\mathrm{PV}\int_0^{2\pi}\breve{\gamma}(\alpha')\cot\left(\frac{\pi}{M}\left(\widetilde{Z}_0(\alpha)-\widetilde{Z}_0(\alpha')\right)\right)d\alpha'\\
&\quad -\frac{\pi}{2iM^2}\mathrm{PV}\int_0^{2\pi}\overline{\gamma}\csc^2\left(\frac{\pi}{M}\left(\widetilde{Z}_0(\alpha)-\widetilde{Z}_0(\alpha')\right)\right)\left(D\widetilde{Z}(\alpha)-D\widetilde{Z}(\alpha')\right)(\breve{\theta})\,d\alpha'.
\end{aligned}\tag{47}$$

Now we use the fact that $\widetilde{Z}_0(\alpha) = M/2\pi\alpha$ and the definition of H to see that the first of the two terms above is exactly $(\pi/iM)H\breve{\gamma}$. The second term T_2 is

$$\begin{aligned}
&-\frac{\pi\overline{\gamma}}{2iM^2}\mathrm{PV}\int_0^{2\pi}\csc^2\left(\frac{1}{2}\left(\alpha-\alpha'\right)\right)\left(D\widetilde{Z}(\alpha)-D\widetilde{Z}(\alpha')\right)d\alpha'\\
&\quad = -\frac{\pi\overline{\gamma}}{iM^2}\mathrm{PV}\int_0^{2\pi}\cot\left(\frac{1}{2}\left(\alpha-\alpha'\right)\right)\frac{\partial}{\partial\alpha'}D\widetilde{Z}(\alpha')d\alpha' = -\frac{2\pi^2\overline{\gamma}}{iM^2}H\left(\partial_\alpha D\widetilde{Z}\right).
\end{aligned}\tag{48}$$

But,

$$\partial_\alpha D\tilde{Z} = \frac{iM}{2\pi}\left(\breve{\theta}(\alpha') - \frac{1}{2\pi}\int_0^{2\pi}\breve{\theta}(a)da\right).$$

Since H annihilates constants, the second term T_2 is equal to $-(\pi\bar{\gamma}/M)H\breve{\theta}$. Thus

$$D\widetilde{W}^* = \frac{\pi}{iM}H\breve{\gamma} - \frac{\pi\bar{\gamma}}{M}H\breve{\theta}. \tag{49}$$

In order to evaluate $D\Theta$, we have $\Theta_0 = 0$ and

$$\begin{aligned} D\Theta = &-\frac{1}{\tau}\partial_\alpha^{-2}\partial_\alpha D\left[(c\cos - \Re(\widetilde{W}^*\widetilde{T}))(\bar{\gamma}+\gamma_1)\right] \\ &+ \frac{A}{\tau}\partial_\alpha^{-2} D\left[\frac{\partial_\alpha Q}{4\Sigma} + 2g\Sigma(\sin - Y) + \Sigma\partial_\alpha(c\cos - \Re(\widetilde{W}^*\widetilde{T}))^2\right]. \end{aligned} \tag{50}$$

Carrying out D, we have

$$\begin{aligned} D\Theta = &-\frac{1}{\tau}\partial_\alpha^{-2}\partial_\alpha\left[(cD\cos - \Re((D\widetilde{W}^*)\widetilde{T}_0) - \Re(\widetilde{W}_0^*(D\widetilde{T})))\bar{\gamma} + (c\cos_0 - \Re(\widetilde{W}_0^*\widetilde{T}_0))\breve{\gamma}\right] \\ &+ \frac{A}{\tau}\partial_\alpha^{-2}\left[\frac{\Sigma_0\partial_\alpha DQ - \partial_\alpha(Q_0)D\Sigma}{4\Sigma_0^2} + 2g\Sigma_0(D\sin - DY) + 2gD\Sigma(\sin_0 - Y_0)\right] \\ &+ \frac{A}{\tau}\partial_\alpha^{-2}\left[(D\Sigma)\partial_\alpha(c\cos_0 - \Re(\widetilde{W}_0^*\widetilde{T}_0))^2\right] \\ &+ \frac{A}{\tau}\partial_\alpha^{-2}\left[2\Sigma_0\partial_\alpha[(c\cos_0 - \Re(\widetilde{W}_0^*\widetilde{T}_0))(cD\cos - \Re((D\widetilde{W}^*)\widetilde{T}_0) - \Re(\widetilde{W}_0^*(D\widetilde{T})))]\right]. \end{aligned}$$

When we use the expressions at the start of this section, this quantity reduces to

$$\begin{aligned} D\Theta = &-\frac{1}{\tau}\partial_\alpha^{-2}\partial_\alpha\left[-\bar{\gamma}\Re(D\widetilde{W}^*) + c\breve{\gamma}\right] \\ &+ \frac{A}{\tau}\partial_\alpha^{-2}\left[\frac{\bar{\gamma}\partial_\alpha\breve{\gamma}}{2(M/2\pi)} + \frac{gM}{\pi}P\breve{\theta}\right] + \frac{A}{\tau}\partial_\alpha^{-2}\left[\frac{M}{\pi}\partial_\alpha[-c\Re(D\widetilde{W}^*)]\right], \end{aligned} \tag{51}$$

where

$$P\breve{\theta} := \breve{\theta} - \frac{1}{2\pi}\int_0^{2\pi}\breve{\theta}(a)da.$$

Using (49) in this expression, we get

$$D\Theta = -\frac{\pi\bar{\gamma}}{M}\left(\frac{\bar{\gamma}}{\tau} - \frac{cAM}{\pi\tau}\right)\partial_\alpha^{-2}\partial_\alpha H\breve{\theta} + \frac{AgM}{\pi\tau}\partial_\alpha^{-2}P\breve{\theta} + \left(\frac{A\bar{\gamma}\pi}{\tau M} - \frac{c}{\tau}\right)\partial_\alpha^{-2}\partial_\alpha\breve{\gamma}. \tag{52}$$

Lastly, to compute $D\Gamma$, we have $\Gamma_0 = 0$ and by (40) and (43),

$$\Gamma = 2H\left(\left|\partial_\alpha\widetilde{Z}[\Theta]\right|\Re\left(\widetilde{W}^*\left[\Theta,\bar{\gamma}+\gamma_1\right]\widetilde{N}[\Theta]\right) - \frac{1}{2}H\gamma_1\right) + 2cH\left(\left|\partial_\alpha\widetilde{Z}[\Theta]\right|\sin(\Theta)\right).$$

Differentiating, we get

$$\begin{aligned} D\Gamma &= 2H\Big\{D\left|\partial_\alpha\widetilde{Z}\right|\circ D\Theta\,\Re\left(\widetilde{W}_0^*\widetilde{N}_0\right) + \left|\partial_\alpha\widetilde{Z}_0\right|\Re\left(D\left(\widetilde{W}[\Theta,\bar{\gamma}+\gamma_1]\right)\widetilde{N}_0\right) \\ &\qquad + \left|\partial_\alpha\widetilde{Z}_0\right|\Re\left(\widetilde{W}_0^*\,D\widetilde{N}\circ D\Theta\right) - \frac{1}{2}H\breve{\gamma}\Big\} + 2cH\left\{D\left|\partial_\alpha\widetilde{Z}\right|\circ D\Theta\ \sin_0 + \left|\partial_\alpha\widetilde{Z}_0\right|D\sin\circ D\Theta\right\} \\ &= \breve{\gamma} + \frac{M}{\pi}H\Re\left(iD\left(\widetilde{W}[\Theta,\bar{\gamma}+\gamma_1]\right)\right) + \frac{cM}{\pi}HD\Theta \end{aligned} \tag{53}$$

because $D|\partial_\alpha\widetilde{Z}| = 0$ and $\widetilde{W}_0^* = 0$. By (49), we have $D\left(\widetilde{W}[\Theta,\bar{\gamma}+\gamma_1]\right) = \frac{\pi}{iM}H\breve{\gamma} - \frac{\pi\bar{\gamma}}{M}HD\Theta$, so that

$$D\Gamma = \breve{\gamma} - \Re(\breve{\gamma} - i\bar{\gamma}D\Theta) + \frac{cM}{\pi}HD\Theta = \frac{cM}{\pi}HD\Theta. \tag{54}$$

Combining (49), (52) and (54), we see that the linearization of the mapping $(\theta, \gamma_1) \to (\theta - \Theta, \gamma_1 - \Gamma)$ at $(0, 0; c)$ is

$$L_c \begin{bmatrix} \breve{\theta} \\ \breve{\gamma} \end{bmatrix} := \begin{bmatrix} \breve{\theta} - D\Theta \\ \breve{\gamma} - D\Gamma \end{bmatrix}$$
$$= \begin{bmatrix} 1 + \frac{\pi\overline{\gamma}}{M}\left(\frac{\overline{\gamma}}{\tau} - \frac{cAM}{\pi\tau}\right)\partial_\alpha^{-2}\partial_\alpha H - \frac{AgM}{\pi\tau}\partial_\alpha^{-2}P & -\left(\frac{A\overline{\gamma}\pi}{\tau M} - \frac{c}{\tau}\right)\partial_\alpha^{-2}\partial_\alpha \\ \overline{\gamma}c\left(\frac{\overline{\gamma}}{\tau} - \frac{cAM}{\pi\tau}\right)H\partial_\alpha^{-2}\partial_\alpha H - \frac{cM^2Ag}{\pi^2\tau}H\partial_\alpha^{-2}P & 1 - c\left(\frac{A\overline{\gamma}}{\tau} - \frac{cM}{\pi\tau}\right)H\partial_\alpha^{-2}\partial_\alpha \end{bmatrix} \begin{bmatrix} \breve{\theta} \\ \breve{\gamma} \end{bmatrix}. \tag{55}$$

Our goal is to find those values of c such that L_c has a one-dimensional nullspace. Because we are working with 2π-periodic functions, we may expand them as

$$\breve{\theta}(\alpha) = \sum_{k\in\mathbf{Z}} \widehat{\breve{\theta}}(k)e^{ik\alpha} \quad \text{and} \quad \breve{\gamma}(\alpha) = \sum_{k\in\mathbf{Z}'} \widehat{\breve{\gamma}}(k)e^{ik\alpha}.$$

We have denoted $\mathbf{Z}' := \mathbf{Z}/\{0\}$. We can eliminate the $k = 0$ coefficient for $\breve{\gamma}$ since it has zero mean. The operators ∂_α, H, P and ∂_α^{-2} can be represented on the Fourier side in the usual way:

$$\widehat{\partial_\alpha \mu}(k) = ik\widehat{\mu}(k), \quad \widehat{H\mu}(k) = -i\,\mathrm{sgn}(k)\widehat{\mu}(k),$$
$$\widehat{P\mu}(k) = (1 - \delta_0(k))\widehat{\mu}(k) \quad \text{and} \quad \widehat{\partial_\alpha^{-2}\mu}(k) = -\frac{1}{k^2}\widehat{\mu}(k),$$

where $\delta_0(k) = 1$ for $k = 0$ and is otherwise zero. Thus L_c is represented on the frequency side as the Fourier multiplier

$$\widehat{L_c \begin{bmatrix} \breve{\theta} \\ \breve{\gamma} \end{bmatrix}}(k) = \widehat{L_c}(k) \begin{bmatrix} \widehat{\breve{\theta}}(k) \\ \widehat{\breve{\gamma}}(k) \end{bmatrix} \tag{56}$$

where, for $k \neq 0$

$$\widehat{L_c}(k) = \begin{bmatrix} 1 - \frac{\pi\overline{\gamma}}{M}\left(\frac{\overline{\gamma}}{\tau} - \frac{cAM}{\pi\tau}\right)|k|^{-1} + \frac{AgM}{\pi\tau}k^{-2} & i\left(\frac{A\overline{\gamma}\pi}{\tau M} - \frac{c}{\tau}\right)k^{-1} \\ i\overline{\gamma}c\left(\frac{\overline{\gamma}}{\tau} - \frac{cAM}{\pi\tau}\right)k^{-1} - i\frac{cM^2Ag}{\pi^2\tau}\,\mathrm{sgn}(k)k^{-2} & 1 + c\left(\frac{A\overline{\gamma}}{\tau} - \frac{cM}{\pi\tau}\right)|k|^{-1} \end{bmatrix} \tag{57}$$

and $\widehat{L_c}(0)$ is the identity.

Now we can easily compute the point spectrum of L_c. In particular, $\lambda \in \mathbf{C}$ is an eigenvalue of L_c if and only if λ is an eigenvalue of $\widehat{L_c}(k)$ for some integer k. For any nonzero integer k, an elementary computation shows that the two eigenvalues of $\widehat{L_c}(k)$ are 1 and

$$\lambda_k(c) := 1 + \frac{2\overline{\gamma}cAM\pi - M^2c^2 - \overline{\gamma}^2\pi^2}{M\pi\tau}|k|^{-1} + \frac{gAM}{\pi\tau}|k|^{-2}.$$

Since this expression is even in k, every eigenvalue of L_c has even multiplicity. So any crossing number for L_c will necessarily be even. However, the eigenvector of $\widehat{L_c}(k)$ associated to this eigenvalue is $\begin{bmatrix} \mathrm{sgn}(k)i\pi/cM \\ 1 \end{bmatrix}$ which in turn implies that

$$\begin{bmatrix} i\pi/cM \\ 1 \end{bmatrix} e^{ik\alpha} \quad \text{and} \quad \begin{bmatrix} -i\pi/cM \\ 1 \end{bmatrix} e^{-ik\alpha}$$

are the corresponding eigenfunctions for L_c with eigenvalue $\lambda_k(c)$. Of course, we can break these up into real an imaginary parts to get an equivalent basis for the eigenspace, namely,

$$\begin{bmatrix} -(\pi/cM)\sin(k\alpha) \\ \cos(k\alpha) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} (\pi/cM)\cos(k\alpha) \\ \sin(k\alpha) \end{bmatrix}.$$

Only the first of these satisfies the symmetry properties (θ odd, γ_1 even) required by our function space (45). Thus when we take account of the symmetry, the dimension of the eigenspace equals one. We summarize the spectral analysis as follows.

Proposition 3 *(Spectrum of L_c). Let L_c be the linearization of the mapping $(\theta, \gamma, c) \in \mathcal{U} \to (\theta - \Theta, \gamma - \Gamma) \in \mathcal{H}^2_{\text{per}} \times \mathcal{H}^2_{\text{per},0}$ at $(0, 0; c)$. The spectrum of L_c consists of 1 and the point spectrum*

$$\sigma_{\text{pt}} := \left\{ \lambda_k(c) := 1 + \frac{2\overline{\gamma} c A M \pi - M^2 c^2 - \overline{\gamma}^2 \pi^2}{M \pi \tau} k^{-1} + \frac{g A M}{\pi \tau} k^{-2} : k \in \mathbf{N} \right\}.$$

Moreover, each eigenvalue λ has geometric and algebraic multiplicity $N_\lambda(c)$ where

$$N_\lambda(c) := |\{k \in \mathbf{N} \text{ such that } \lambda_k(c) = \lambda\}|.$$

The eigenspace for λ is

$$E_\lambda(c) := \text{span} \left\{ \begin{bmatrix} -(\pi/cM) \sin(k\alpha) \\ \cos(k\alpha) \end{bmatrix} : k \in \mathbf{N} \text{ such that } \lambda_k(c) = \lambda \right\}.$$

Corollary 3. *Fix A, g, $\overline{\gamma} \in \mathbf{R}$ and $\tau, M > 0$. Let*

$$\mathcal{K} := \left\{ k \in \mathbf{Z} : \pi^2 \overline{\gamma}^2 A^2 k^2 + \pi \tau k^3 M - \pi^2 k^2 \overline{\gamma}^2 + k A g M^2 > 0 \text{ and } AgM/\pi \tau k \notin \mathbf{N} \setminus \{k\} \right\}.$$

For $k \in \mathbf{N}$, let

$$c_\pm(k) := \frac{\pi \overline{\gamma} A}{M} \pm \frac{1}{kM} \sqrt{\pi^2 \overline{\gamma}^2 A^2 k^2 + \pi \tau k^3 M - \pi^2 k^2 \overline{\gamma}^2 + k A g M^2}.$$

Then $|\mathcal{K}| = \infty$ and L_c has crossing number equal to one at a real value $c = c_\pm(k)$ if and only if $k \in \mathcal{K}$. If $A = 0$, then $\mathcal{K} = \mathbf{Z}$.

Proof. Fix $k \in \mathbf{N}$. We are looking for the real values of $c \in \mathbf{R}$ for which $\lambda_k = 0$. There are at most two roots. A routine calculation shows that $\lambda_k(c) = 0$ if and only if $c = c_\pm(k)$. The first condition in the definition of $\mathcal{K}$ shows that $c_\pm(k)$ is a real number. Thus $\lambda = 0$ is an eigenvalue. We must compute its crossing number and the first step is to calculate its multiplicity, denoted $N_0(c_\pm(k))$. Thus, given $c = c_\pm(k)$, we must find all $l \in \mathbf{N}$ such that $\lambda_l(c_\pm(k)) = 0$. Clearly $l = k$ works. A small amount of algebra shows that the only other possible solution of $\lambda_l(c_\pm(k)) = 0$ is

$$l = l_k := AgM/\pi \tau k.$$

Another calculation shows that

$$\lambda_k(c_\pm(k) + \epsilon) = \mp \frac{2\sqrt{\pi^2 \overline{\gamma}^2 A^2 k^2 + \pi \tau k^3 M - \pi^2 k^2 \overline{\gamma}^2 + k A g M^2}}{k^2 \pi \tau} \epsilon - \frac{M}{\tau k \pi} \epsilon^2$$

and

$$\lambda_{l_k}(c_\pm(k) + \epsilon) = \mp \frac{2\sqrt{\pi^2 \overline{\gamma}^2 A^2 k^2 + \pi \tau k^3 M - \pi^2 k^2 \overline{\gamma}^2 + k A g M^2}}{AgM} \epsilon - \frac{k}{Ag} \epsilon^2.$$

Now assume that $k \in \mathcal{K}$. The second condition in the definition of $\mathcal{K}$ shows that $l_k \notin \mathbf{N} \setminus \{k\}$ and thus $N_0(c_\pm(k)) = 1$. The first condition guarantees that coefficient of ϵ in the expansion of $\lambda_k(c_\pm(k) + \epsilon)$ is non-zero. Thus we see $\lambda_\pm(c)$ changes sign as c passes through $c_\pm(k)$: the crossing number is equal to one and thus is odd. Thus we have shown the "if" direction in the corollary. The "only if" direction follows by showing that the crossing number is either two or zero when one of the conditions is not met. The details are simple so we omit them.

To show that $|\mathcal{K}| = \infty$, observe that, since $\tau, M > 0$, we have

$$\lim_{k \to \infty} \left(\pi^2 \overline{\gamma}^2 A^2 k^2 + \pi \tau k^3 M - \pi^2 k^2 \overline{\gamma}^2 + k A g M^2 \right) = \infty.$$

Therefore the first condition in the definition of $\mathcal{K}$ is met for all k sufficiently large. Likewise, no matter the choices of the parameters A, g, M and τ, $\lim_{k \to \infty} AgM/\pi \tau k = 0$ and the second condition holds for k sufficiently large. Thus $|\mathcal{K}|$ contains all $k > k_0$ for some finite $k_0 \in \mathbf{N}$. □

6.3. Application of the abstract theorem

An appeal to Theorem 7 has the following consequence.

Theorem 8 *(Global bifurcation). Let the surface tension $\tau > 0$, period $M > 0$, Atwood number $A \in \mathbf{R}$, and average vortex strength $\bar{\gamma} \in \mathbf{R}$ be given. Let $\mathcal{U}$, $\mathcal{K}$ and $c_{\pm}(k)$ be defined as above. Let $\mathcal{S} \subset \mathcal{U}$ be the closure (in $\mathcal{H}^1_{\mathrm{per}} \times \mathcal{H}^1_{\mathrm{per},0} \times \mathbf{R}$) of the set of all solutions of (44) for which either $\theta \neq 0$ or $\gamma_1 \neq 0$. Given $k \in \mathcal{K}$, let $\mathcal{C}_{\pm}(k)$ be the connected component of $\mathcal{S}$ which contains $(0, 0; c_{\pm}(k))$.*

Then

(I) *either $\mathcal{C}_{\pm}(k)$ is unbounded;*
(II) *or $\mathcal{C}_{\pm}(k) = \mathcal{C}_{+}(l)$ or $\mathcal{C}_{\pm}(k) = \mathcal{C}_{-}(l)$ for some $l \in \mathcal{K}$ with $l \neq k$;*
(III) *or $\mathcal{C}_{\pm}(k)$ contains a point on the boundary of $\mathcal{U}$.*

Proof. We saw in Theorem 6 that the mapping (Θ, Γ) was compact on $\mathcal{U}_{b,h}$ for all $b, h > 0$ and we saw that there are always choices of c which result in an odd crossing number in Corollary 3. Thus Theorem 7 can be applied with outcomes which coincide with the outcomes *(I)–(III)* in Theorem 8 except with the replacement of $\mathcal{U}$ with the $\mathcal{U}_{b,h}$ in *(III)*. Since $\mathcal{U} = \cup_{b,h>0}\mathcal{U}_{b,h}$ an easy topological argument gives *(III)* as above. □

This general statement leads in turn to our main conclusion.

Proof of Theorem 1. By Proposition 6, a solution $(\theta, \gamma_1; c)$ of (44) gives rise to symmetric periodic traveling wave solutions of the two dimensional gravity–capillary vortex sheet problem by taking $z(\alpha, t) = ct + \widetilde{Z}[\theta](\alpha)$ and $\gamma = \bar{\gamma} + \gamma_1$. (Note that, as in the proof of Proposition 2, this implies that $N = \widetilde{N}[\theta]$ and $T = \widetilde{T}[\theta]$. We will use the two equivalent notations interchangeably in what follows.) Note that in the statement of Theorem 1 it is stated that the traveling wave solutions are smooth, whereas the solutions given in Theorem 8 are stated to merely belong to $\mathcal{H}^1_{\mathrm{per}} \times \mathcal{H}^1_{\mathrm{per},0}$. However, the maps Θ and Γ in (44) are smoothing. Therefore a simple bootstrap argument shows that θ and γ_1 are in $\mathcal{H}^s_{\mathrm{per}}$ for any s and thus in C^∞. The corresponding traveling waves are likewise smooth; the details are routine and omitted.

Each of the outcomes (a)–(f) in Theorem 1 corresponds to one of the alternatives (I)–(III) of Theorem 8. Fix $k \in \mathcal{K}$. It is straightforward to see that alternative (II) in Theorem 8 is interpreted as outcome (e) in Theorem 1.

Now consider alternative (III). If $(\theta, \gamma_1; c) \in \mathcal{C}_{\pm}(k)$ is on the boundary of $\mathcal{U}$, then inspection of the definition of $\mathcal{U}$ shows that

$$\text{either } \overline{\cos\theta} = 0 \quad \text{or} \quad \widetilde{Z}[\theta] = \widetilde{Z}[\Theta] \notin \mathcal{C}^3_0. \tag{58}$$

The reconstruction of the curve $S(t)$ from θ via $\widetilde{Z}$ (recalling that $\overline{\sin\theta} = 0$ for solutions) shows that the length of $S(t)$ per period is given by

$$L[\theta] := \int_0^{2\pi} \left|\partial_\alpha \widetilde{Z}[\theta](a)\right| da = \frac{M}{\overline{\cos\theta}}.$$

In case $\overline{\cos\theta} = 0$, the length of the curve reconstructed from $(\theta, \gamma_1; c)$ is formally infinite. Since $(\theta, \gamma_1; c)$ is in the closure of the set of nontrivial traveling wave solutions, the more precise statement is that there is sequence of solutions whose lengths diverge, which is outcome (a).

Now suppose we have the other alternative in (58), namely, that $\widetilde{Z}[\theta] \notin \mathcal{C}^3_0$. Since $h = 0$ in this space, it means that

$$\inf_{\alpha,\alpha' \in [0,2\pi]} \zeta = 0, \qquad \text{where } \zeta(\alpha, \alpha') := \left|\frac{\widetilde{Z}[\theta](\alpha') - \widetilde{Z}[\theta](\alpha)}{\alpha' - \alpha}\right|.$$

Moreover,

$$\lim_{\alpha' \to \alpha} \zeta(\alpha, \alpha') = \left|\partial_\alpha \widetilde{Z}[\theta](\alpha)\right| = \frac{M}{2\pi\,\overline{\cos\theta}} = \frac{L[\theta]}{2\pi} \geq 1. \tag{59}$$

But clearly $(\widetilde{Z}[\theta](\alpha') - \widetilde{Z}[\theta](\alpha))/(\alpha' - \alpha)$ is a continuous function of (α, α') for $\alpha \neq \alpha'$. Thus its infimum, which vanishes, is attained at some pair of values $(\alpha_\star, \alpha'_\star)$ where $\alpha_\star \neq \alpha'_\star$. Hence $\zeta(\alpha_*, \alpha'_*) = 0$, which in turn implies that

$$\left|\widetilde{Z}[\theta](\alpha'_\star) - \widetilde{Z}[\theta](\alpha_\star)\right| = 0.$$

This means that the curve reconstructed from θ intersects itself, outcome (d).

Now consider alternative (I). Then $\mathcal{C}_\pm(k)$ contains a sequence of solutions $\{(\theta_n, \gamma_{1,n}; c_n)\}$ for which

$$\lim_{n\to\infty}\left(|c_n| + \|\theta_n\|_{\mathcal{H}^1_{\text{per}}} + \|\gamma_{1,n}\|_{\mathcal{H}^1_{\text{per}}}\right) = \infty,$$

so that at least one of the three terms on the left diverges.

By combining (28), (30), and (37), we see that $|\partial_\alpha \widetilde{Z}[\theta_n]| = \sigma_n$, and the length of one period of the interface is thus proportional to σ_n. If $\sigma_n \to \infty$, then outcome (a) has occurred; thus, we may assume that σ_n remains bounded above independently of n. Since the length of one period of the curve may not vanish (by periodicity), we also see that σ_n is bounded below (away from zero) independently of n. Considering again (28), we see that σ_n being bounded above implies that $\overline{\cos\theta_n}$ is bounded away from zero.

Suppose first that $|c_n| \to \infty$ but $\|\theta_n\|_{\mathcal{H}^1_{\text{per}}} + \|\gamma_{1,n}\|_{\mathcal{H}^1_{\text{per}}}$ is bounded. We see from (20) that

$$\|c_n \sin\theta_n\|_{H^1} = \|\Re(W_n^* N_n)\|_{H^1}.$$

Since $W_n^* = B[\widetilde{Z}[\theta_n]]\gamma_n$, this implies

$$\|c_n \sin\theta_n\|_{H^1} = \|\Re(B[\widetilde{Z}[\theta_n]]\gamma_n\, N_n)\|_{H^1}.$$

We then use (40) to write this as

$$\|c_n \sin\theta_n\|_{H^1} = \left\|\frac{1}{2\sigma_n} H\gamma_n + \Re(K[\widetilde{Z}[\theta_n]]\gamma_n\, N_n)\right\|_{H^1}.$$

We have remarked above that $\frac{1}{\sigma_n}$ is bounded independently of n, and we see that $H\gamma_n$ is uniformly bounded in H^1 since γ_n is. Applying Lemma 5 gives an estimate for the operator K, and we find the following:

$$\|c_n \sin\theta_n\|_{H^1} \le C\|\gamma_n\|_{H^1} + C\exp\{C\|\widetilde{Z}[\theta_n]\|_{H^1}\}\|\gamma_n\|_{H^1}\|N_n\|_{H^1}.$$

Since θ_n is, by assumption, bounded in H^1, we see from (30) and (33) that $\widetilde{Z}[\theta_n]$ and N_n are as well. We conclude that $c_n \sin\theta_n$ is bounded in $\mathcal{H}^1_{\text{per}}$, independently of n.

Therefore $\|\sin\theta_n\|_{\mathcal{H}^1_{\text{per}}} \to 0$, and by Sobolev embedding, $\sin\theta_n \to 0$ uniformly. This implies that θ_n converges to a multiple of π; the uniform convergence and the continuity and oddness of θ_n make it is straightforward to see that this multiple must be zero. Note also, then, that $|\cos\theta_n| \to 1$, uniformly. Continuing, we integrate (23) once, finding that the quantity

$$(c_n\cos\theta_n - \Re(W_n^* T_n))\gamma_n - 2A\left(\frac{1}{8}\frac{\gamma_n^2}{\sigma_n} + g\sigma_n \int^\alpha \sin\theta_n\, d\alpha + \frac{\sigma_n}{2}\{c_n\cos\theta_n - \Re(W_n^* T_n)\}^2\right) \tag{60}$$

is bounded in L^2_{per}.

Recalling again that $W_n^* = B[\widetilde{Z}(\theta_n)]\gamma_n$, we see that (39) gives a formula for $\Re(W_n^* T_n)$. Similarly to our previous use of Lemma 5, we see that Lemma 5 then implies that $\Re(W_n^* T_n)$ is bounded in $\mathcal{H}^1_{\text{per}}$. If $A = 0$, we then deduce that $c_n\gamma_n\cos\theta_n$ is bounded in L^2_{per} and therefore $\gamma_n \to 0$ in L^2_{per}. Thus the average, which is a constant, must satisfy $\bar{\gamma} = 0$. If we have $\bar{\gamma} \neq 0$, then this is a contradiction.

Now assume that $A \neq 0$. Dividing (60) by c_n^2, we see that all the terms then go to zero as $n \to \infty$ except $\frac{\sigma_n}{2}(\cos\theta_n)^2$. This then implies that $\frac{\sigma_n}{2}(\cos\theta_n)^2$ goes to zero, which is a contradiction since $\cos\theta_n \to 1$ uniformly.

If $\bar{\gamma} = 0$ and $A = 0$, then we do not rule out $|c_n| \to \infty$; this is possibility (f) of the theorem.

If $\|\theta_n\|_{\mathcal{H}^1_{\text{per}}}$ diverges, then either θ_n or $\partial_\alpha\theta_n$ diverges in L^2_{per}. Since θ is the tangent angle to $S(t)$, the curvature is exactly $\kappa(\alpha) = \partial_\alpha\theta(\alpha)/\sigma$. Inspection of (28) indicates that if $\sigma_n \to 0$, then $\overline{\cos\theta_n} \to \infty$. This clearly cannot be the case, however, and thus σ_n cannot go to zero. Recall that we assume that σ_n is bounded above. If it is $\partial_\alpha\theta_n$ that diverges, then we see that the curvature diverges, which is to say that we have outcome (b). On the other hand, suppose that it is θ_n that diverges in L^2_{per}. Since θ_n is odd and periodic, $\theta_n(0) = \theta_n(2\pi) = 0$. If the L^2_{per}-norm of θ_n diverges then of course its L^∞-norm also diverges and so does $\partial_\alpha\theta_n$. This means that the curvature for the reconstructed interface is diverging. Thus $\|\theta_n\|_{\mathcal{H}^1_{\text{per}}}$ diverging implies outcome (b).

If $\|\gamma_{1,n}\|_{\mathcal{H}^1_{\text{per}}}$ diverges, then either $\gamma_{1,n}$ or $\partial_\alpha\gamma_{1,n}$ diverges in L^2_{per}. Suppose that it is the former. From Section 2, the jump in the tangential velocity across the interface is related to γ by (8). By the reconstruction method and the length L above, it implies that $j(\alpha,t) = 2\pi(\overline{\gamma} + \gamma_1(\alpha,t))/L[\theta]$. If $L[\theta_n]$ remains bounded, then clearly the jump j given above diverges in L^2_{per}, which is outcome (c). If $L[\theta_n]$ diverges, we have outcome (a). Likewise, if $\partial_\alpha\gamma_{1,n}$ diverges in L^2_{per}, then either the derivative of the jump diverges, outcome (c), or else the length diverges, outcome (a). □

Conflict of interest statement

The authors have no conflicts of interest.

References

[1] M.J. Ablowitz, A.S. Fokas, Complex Variables: Introduction and Applications, Cambridge Texts Appl. Math., Cambridge University Press, Cambridge, 1997.

[2] B. Akers, D.M. Ambrose, J.D. Wright, Traveling waves from the arclength parameterization: vortex sheets with surface tension, Interfaces Free Bound. 15 (3) (2013) 359–380.

[3] B.F. Akers, D.M. Ambrose, K. Pond, J.D. Wright, Internal capillary gravity waves, 2015, in preparation.

[4] B.F. Akers, D.M. Ambrose, J.D. Wright, Gravity perturbed Crapper waves, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 470 (2161) (2014), 20130526, 14 pp.

[5] D.M. Ambrose, Well-posedness of vortex sheets with surface tension, SIAM J. Math. Anal. 35 (1) (2003) 211–244 (electronic).

[6] D.M. Ambrose, N. Masmoudi, Well-posedness of 3D vortex sheets with surface tension, Commun. Math. Sci. 5 (2) (2007) 391–430.

[7] C.J. Amick, R.E.L. Turner, A global theory of internal solitary waves in two-fluid systems, Trans. Am. Math. Soc. 298 (2) (1986) 431–484.

[8] C.J. Amick, R.E.L. Turner, Small internal waves in two-fluid systems, Arch. Ration. Mech. Anal. 108 (2) (1989) 111–139.

[9] J.T. Beale, T.Y. Hou, J.S. Lowengrub, Growth rates for the linearized motion of fluid interfaces away from equilibrium, Commun. Pure Appl. Math. 46 (9) (1993) 1269–1301.

[10] A. Castro, D. Cordoba, C. Fefferman, F. Gancedo, J. Gomez-Serrano, Finite time singularities for water waves with surface tension, J. Math. Phys. 53 (11) (2012).

[11] A. Constantin, W. Strauss, Exact steady periodic water waves with vorticity, Commun. Pure Appl. Math. 57 (4) (2004) 481–527.

[12] A. Constantin, W. Strauss, E. Varvaruca, Global bifurcation of steady gravity water waves with critical layers, preprint.

[13] D. Coutand, S. Shkoller, On the impossibility of finite-time splash singularities for vortex sheets, preprint, arXiv:1407.1479, 2014.

[14] G.D. Crapper, An exact solution for progressive capillary waves of arbitrary amplitude, J. Fluid Mech. 2 (1957) 532–540.

[15] P. de Boeck, Existence of capillary–gravity waves that are perturbations of Crapper's waves, preprint.

[16] P. de Boeck, Global bifurcation for steady finite-depth capillary–gravity waves with constant vorticity, preprint.

[17] C. Fefferman, A.D. Ionescu, V. Lie, On the absence of "splash" singularities in the case of two-fluid interfaces, preprint, arXiv:1312.2917, 2013.

[18] P. Germain, N. Masmoudi, J. Shatah, Global existence for capillary water waves, Commun. Pure Appl. Math. 68 (2014) 625–687.

[19] T.Y. Hou, J.S. Lowengrub, M.J. Shelley, Removing the stiffness from interfacial flows with surface tension, J. Comput. Phys. 114 (2) (1994) 312–338.

[20] T.Y. Hou, J.S. Lowengrub, M.J. Shelley, The long-time motion of vortex sheets with surface tension, Phys. Fluids 9 (7) (1997) 1933–1954.

[21] V.M. Hur, No solitary waves exist on 2D deep water, Nonlinearity 25 (12) (2012) 3301–3312.

[22] A.D. Ionescu, F. Pusateri, Global regularity for 2d water waves with surface tension, preprint, arXiv:1408.4428, 2014.

[23] H. Kielhöfer, Bifurcation Theory: An Introduction with Applications to Partial Differential Equations, second edition, Appl. Math. Sci., vol. 156, Springer, New York, 2012.

[24] W. Kinnersley, Exact large amplitude capillary waves on sheets of fluid, J. Fluid Mech. 77 (1976) 229.

[25] Andrew J. Majda, Andrea L. Bertozzi, Vorticity and Incompressible Flow, Cambridge Texts Appl. Math., vol. 27, Cambridge University Press, Cambridge, 2002.

[26] A.-V. Matioc, Steady internal water waves with a critical layer bounded by the wave surface, J. Nonlinear Math. Phys. 19 (1) (2012), 1250008, 21 pp.

[27] B.-V. Matioc, Global bifurcation for water waves with capillary effects and constant vorticity, Monatshefte Math. 174 (3) (2014) 459–475.

[28] N.I. Muskhelishvili, Singular Integral Equations: Boundary Problems of Function Theory and Their Application to Mathematical Physics, Dover Publications, Inc., New York, 1992, translated from the second (1946) Russian edition and with a preface by J.R.M. Radok, corrected reprint of the 1953 English translation.

[29] P.H. Rabinowitz, Some global results for nonlinear eigenvalue problems, J. Funct. Anal. 7 (1971) 487–513.

[30] P.G. Saffman, Vortex Dynamics, Camb. Monogr. Mech. Appl. Math., Cambridge University Press, New York, 1992.
[31] J.F. Toland, On a pseudo-differential equation for Stokes waves, Arch. Ration. Mech. Anal. 162 (2) (2002) 179–189.
[32] S. Walsh, Steady stratified periodic gravity waves with surface tension II: global bifurcation, Discrete Contin. Dyn. Syst. 34 (8) (2014) 3287–3315.