Pseudoholomorphic strips in symplectisations I: Asymptotic behavior

Bandes pseudo-holomorphes en symplectisations I: comportement asymptotique

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Abstract

This paper is part of a larger program, the investigation of the Chord Problem in three dimensional contact geometry. The main tool will be pseudoholomorphic strips in the symplectisation of a three dimensional contact manifold with two totally real submanifolds L_0, L_1 as boundary conditions. The submanifolds L_0 and L_1 do not intersect transversally. The subject of this paper is to study the asymptotic behavior of such pseudoholomorphic strips.
Résumé

Cet article fait partie d'un programme de travail plus grand : la recherche sur le problème de Chord en géométrie de contact en dimension trois. L'outil essentiel sont des bandes pseudo-holomorphes dans la symplectisation d'une variété contact à dimension trois avec la condition au bord suivante : les deux composantes de la frontière sont contenues dans deux sous-variétés totalement réelles L_0, L_1. Les variétés L_0 et L_1 ont des intersections non-transverses. Le sujet de cet article est l'étude du comportement asymptotique des solutions.

1. Introduction

This paper is the first part of a larger program, the investigation of the chord problem in three dimensional contact geometry [4–6]. Let (M, λ) be a $(2n + 1)$-dimensional contact manifold, i.e. λ is a 1-form on M such that $\lambda \wedge (d\lambda)^n$ is a volume form on M. The contact structure associated to λ is the $2n$-dimensional vector bundle

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$\xi = \ker \lambda \to M$, which is a symplectic vector bundle with symplectic structure $d\lambda|_{\xi \oplus \xi}$. There is a distinguished vector field associated to a contact form, the Reeb vector field X_λ, which is defined by the equations

$$i_{X_\lambda} d\lambda \equiv 0, \qquad i_{X_\lambda} \lambda \equiv 1.$$

We denote by $\pi_\lambda : TM \to \xi$ the projection along the Reeb vector field. The Chord Problem is about the global dynamics of the Reeb vector field. More precisely, the issue is the existence of so-called 'characteristic chords'. These are trajectories x of the Reeb vector field which hit a given Legendrian submanifold $\mathcal{L}^n \subset (M, \lambda)$ at two different times $t = 0$, $T > 0$. We also ask for $x(0) \neq x(T)$, otherwise the chord would actually be a periodic orbit. Recall that a submanifold L in a $(2n + 1)$-dimensional contact manifold (M, λ) is called Legendrian if it is everywhere tangent to the hyperplane field ξ and if it has dimension n. We are mostly interested in the three-dimensional situation, the question is then whether a given Legendrian knot has a characteristic chord. The Chord problem should be viewed as the relative version of the Weinstein conjecture which deals with the existence of periodic orbits of the Reeb vector field.

Characteristic chords occur naturally in classical mechanics. In this context they are referred to as 'brake–orbits', and were investigated by Seifert in 1948 [20] and others since the 1970's [7,9,21,22].

In 1986, V.I. Arnold conjectured the existence of characteristic chords on the three sphere for any contact form inducing the standard contact structure and for any Legendrian knot [8]. After a partial result by the author in [3] this conjecture was finally confirmed by K. Mohnke in [17]. It is natural to ask the existence question for characteristic chords not only for $M = S^3$, but also for general contact manifolds. A new invariant for Legendrian knots and contact manifolds proposed by Y. Eliashberg, A. Givental and H. Hofer in [11] ('Relative Contact Homology') is actually based on counting characteristic chords and periodic orbits of the Reeb vector field.

The subject of the paper [6] is an existence result for characteristic chords which goes beyond the special classes of contact three manifolds investigated so far. The purpose of this paper and [4,5] is to establish a filling method by pseudoholomorphic curves where we use a surface $F \subset M = M^3$ with boundary, and where we start filling from a tangency at the boundary. Pseudoholomorphic curves are maps from a Riemann surface into an almost complex manifold W satisfying a nonlinear Cauchy Riemann type equation. In our case, the manifold W is the symplectisation $(\mathbf{R} \times M, d(e^t \lambda))$ of the contact manifold (M, λ). We are going to consider a special type of almost complex structures $\tilde{J}$ on $\mathbf{R} \times M$. We pick a complex structure $J : \xi \to \xi$ such that $d\lambda \circ (\mathrm{Id} \times J)$ is a bundle metric on ξ. We then define an almost complex structure on $\mathbf{R} \times M$ by demanding $\tilde{J} \equiv J$ on ξ and sending $\partial/\partial t$ (the generator of the $\mathbf{R}$-component) onto the Reeb vector field. Then $\tilde{J}(p)$ has to map $X_\lambda(p)$ onto $-\partial/\partial t$.

If S is a Riemann surface with complex structure j then we define a map

$$\tilde{u} = (a, u) : S \to \mathbf{R} \times M$$

to be a pseudoholomorphic curve if

$$D\tilde{u}(z) \circ j(z) = \tilde{J}(\tilde{u}(z)) \circ D\tilde{u}(z) \quad \text{for all } z \in S.$$

If (s, t) are conformal coordinates on S then this becomes:

$$\partial_s \tilde{u} + \tilde{J}(\tilde{u}) \partial_t \tilde{u} = 0.$$

We are interested only in pseudoholomorphic curves which have finite energy in the sense that

$$E(\tilde{u}) := \sup_{\phi \in \Sigma} \int_S \tilde{u}^* d(\phi \lambda) < +\infty,$$

where $\Sigma := \{\phi \in C^\infty(\mathbf{R}, [0, 1]) \mid \phi' \geqslant 0\}$. The Riemann surface S in this paper is an infinite strip $S = \mathbf{R} \times [0, 1]$, and we will impose a mixed boundary condition as follows: Let $\mathcal{L} \subset M$ be a homologically trivial Legendrian knot bounding an embedded surface $\mathcal{D}$. A point $p \in \mathcal{D}$ is called singular if $T_p \mathcal{D} = \ker \lambda(p)$. If the surface is oriented (by a volume form σ) and if $j : \mathcal{D} \hookrightarrow M$ is the inclusion, then we define a vector field Z on $\mathcal{D}$ by $i_Z \sigma = j^* \lambda$.

This vector field vanishes precisely in the singular points. The flow lines of Z determine a singular foliation of the surface $\mathcal{D}$ which does not depend on the particular choice of the volume form or the contact form. This singular foliation is also called the characteristic foliation of $\mathcal{D}$ (induced by $\ker\lambda$). Let $p \in \mathcal{D}$ be a singular point and denote by $Z'(p) : T_p\mathcal{D} \to T_p\mathcal{D}$ the linearization of the vector field Z in p. Let λ_1, λ_2 be the eigenvalues of $Z'(p)$. We say that p is non-degenerate if none of the eigenvalues lie on the imaginary axis. A non-degenerate singular point p is called elliptic if $\lambda_1\lambda_2 > 0$ and hyperbolic if $\lambda_1\lambda_2 < 0$. In the elliptic case the critical point $Z(p) = 0$ is either a source or a sink, and in the hyperbolic case it is a saddle point.

Choosing $\mathcal{D}$ appropriately we may assume that there are only non-degenerate singular points, in particular there are only finitely many. We denote the surface without the singular points by $\mathcal{D}^*$. We consider the boundary value problem

$$\tilde{u} = (a, u) : S \to \mathbf{R} \times M,$$
$$\partial_s\tilde{u} + \tilde{J}(\tilde{u})\partial_t\tilde{u} = 0,$$
$$\tilde{u}(s, 0) \subset \mathbf{R} \times \mathcal{L}, \tag{1}$$
$$\tilde{u}(s, 1) \subset \{0\} \times \mathcal{D}^*,$$
$$0 < E(\tilde{u}) < +\infty.$$

The subject of this paper is to investigate the behavior of solutions $\tilde{u}$ for large $|s|$. The finiteness condition on the energy actually forces the solutions to converge to points $\tilde{p}_\pm \in \{0\} \times \mathcal{L}$ at an exponential rate. We first introduce suitable coordinates near the Legendrian knot, and we deform $\mathcal{D}$ near its boundary, keeping $\mathcal{L} = \partial\mathcal{D}$ fixed, in order to achieve a certain normal form for $\mathcal{D}$ near its boundary. We then derive exponential decay estimates for $\tilde{u} - \tilde{p}_\pm$ and all its derivatives in coordinates. In local coordinates near $\tilde{p}_\pm$ the almost complex structure $\tilde{J}$ corresponds to some real (4×4)-matrix valued function which we denote by M. The main result of this paper is the following asymptotic formula

Theorem 1.1. *For sufficiently large s_0 and $s \geqslant s_0$ we have the following asymptotic formula for nonconstant solutions v of* (1) *having finite energy*:

$$v(s, t) = e^{\int_{s_0}^{s} \alpha(\tau)\,d\tau}\big(e(t) + r(s, t)\big), \tag{2}$$

where $\alpha : [s_0, \infty) \to \mathbf{R}$ is a smooth function satisfying $\alpha(s) \to \lambda < 0$ as $s \to \infty$ with λ being an eigenvalue of the selfadjoint operator

$$A_\infty : L^2\big([0, 1], \mathbf{R}^4\big) \supset H_L^{1,2}\big([0, 1], \mathbf{R}^4\big) \to L^2\big([0, 1], \mathbf{R}^4\big)$$
$$\gamma \mapsto -M_\infty\dot{\gamma}, \quad M_\infty := \lim_{s \to \infty} M\big(v(s, t)\big).$$

Moreover, $e(t)$ is an eigenvector of A_∞ belonging to the eigenvalue λ with $e(t) \neq 0$ for all $t \in [0, 1]$, and r is a smooth function so that r and all its derivatives converge to zero uniformly in t as $s \to \infty$.

We will prove more about the decay of $|\lambda - \alpha(s)|$, r and their derivatives:

Theorem 1.2. *Let r and $\alpha(s)$ be as in Theorem* 1.1. *Then there is a constant $\delta > 0$ such that for each integer $l \geqslant 0$ and each multi-index $\beta \in \mathbf{N}^2$*

$$\sup_{0 \leqslant t \leqslant 1} \big|D^\beta r(s, t)\big|, \left|\frac{d^l}{ds^l}\big(\alpha(s) - \lambda\big)\right| \leqslant c_{\beta, l}\, e^{-\delta|s|}$$

with suitable constants $c_{\beta, l} > 0$.

The subscript 'L' in $H_L^{1,2}([0, 1], \mathbf{R}^4)$ indicates the boundary condition (see (21) for a precise definition). This formula is an essential ingredient for the rest of the program [4–6].

The asymptotic behavior of holomorphic strips with mixed boundary conditions similar to ours was investigated in [19], but only for non-degenerate ends. We are dealing with a degenerate situation, i.e. the manifolds $L_0 = \mathbf{R} \times \mathcal{L}$ and $L_1 = \{0\} \times \mathcal{D}$ do not intersect transversally. The degenerate situation is much more delicate: In the non-degenerate case the intersection $L_0 \cap L_1$ would consist of isolated points. Having shown that a pseudoholomorphic strip $\tilde{u}(s, t)$ with finite energy approaches $L_0 \cap L_1$ as $|s| \to \infty$ one can fairly easily see that oscillations between two points in $L_0 \cap L_1$ would cost too much energy, i.e. it would contradict $E(\tilde{u}) < \infty$. In our case we have to show that the end of the solution cannot move along the 1–dimensional set $L_0 \cap L_1$ while $|s|$ grows. Analytically, degeneracy means that the operator A_∞ above has a nontrivial kernel. The strategy is to derive estimates for the 'components' of $\tilde{u}$ orthogonal to the kernel of A_∞ (in a suitable sense). We will then show that they decay fast enough to force the component along the kernel of A_∞ to zero as well.

Degenerate ends were investigated in the paper [14], but only for pseudoholomorphic cylinders $S = \mathbf{R} \times S^1$ (periodic boundary condition in t). Our problem requires a different approach. The paper [19] contains the decay estimate of Theorem 1.2 for the case $\beta = 0$. Eduardo Mora proved Theorem 1.2 for pseudoholomorphic cylinders independently of the author in his Ph.D. thesis [18]. Because we are choosing special $\tilde{J}$ and $\mathcal{D}$ near $\{0\} \times \mathcal{L}$ solutions to the boundary value problem (1) can be constructed explicitly near elliptic singular points on the boundary of $\mathcal{D}$ (see [5]).

2. Simplifying the spanning surface $\mathcal{D}$ near the boundary

In this section we will simplify the surface $\mathcal{D}$ near its boundary to obtain a normal form in coordinates near the knot $\mathcal{L} = \partial \mathcal{D}$. This is useful for the analysis later. In particular, we will be able to produce explicitly a family of finite energy strips coming out from elliptic singular points on the boundary.

If (M, λ) is a three dimensional contact manifold and $\mathcal{L}$ a Legendrian knot in M then, by a well-known theorem of A. Weinstein (see [23,24,1]), there are open neighborhoods $U \subset M$ of the knot $\mathcal{L}$, $V \subset S^1 \times \mathbf{R}^2$ of $S^1 \times \{(0, 0)\}$ and a diffeomorphism $\Psi : U \to V$, so that $\Psi^*(dy + x \, d\theta) = \lambda|_U$, where θ denotes the coordinate on $S^1 \approx \mathbf{R}/\mathbf{Z}$ and x, y are coordinates on $\mathbf{R}^2$. We will refer to this result as the 'Legendrian neighborhood theorem'. If we are working near the knot $\mathcal{L}$ we may assume that our contact manifold is $(S^1 \times \mathbf{R}^2, \lambda = dy + x \, d\theta)$ and the knot is given by $S^1 \times \{(0, 0)\}$. We will denote the piece of the spanning surface $\mathcal{D} \cap U$ again by $\mathcal{D}$. Choosing U sufficiently small we may assume that all the singular points on the piece $\mathcal{D} \cap U$ lie on the boundary, i.e.

$$\left\{ p \in \mathcal{D} \cap U \mid T_p \mathcal{D} = \ker \lambda(p) \right\} = \left\{ (\theta_k, 0, 0) \in S^1 \times \mathbf{R}^2 \right\}_{1 \leqslant k \leqslant N}, \quad N \in \mathbf{N}.$$

We parameterize $\mathcal{D}$ as follows:

$$\mathcal{D} = \left\{ \left(\theta, x(\theta, r), y(\theta, r) \right) \in S^1 \times \mathbf{R}^2 \mid r, \theta \in [0, 1] \right\},$$

where x, y are suitable smooth functions which are 1-periodic in θ and satisfy

$$x(\theta, 0) \equiv y(\theta, 0) \equiv 0.$$

Moreover we orient $\mathcal{D}$ in such a way that the above parameterization $([0, 1] \times [0, 1], d\theta \wedge dr) \to \mathcal{D}$ is orientation preserving. We orient $\mathcal{L}$ by $v = d/d\theta$, so that (v, ν) is positively oriented, where ν denotes the inward normal vector. A point $(\theta_0, 0, 0)$ is a singular point if and only if $\partial_r y(\theta_0, 0) = 0$. Since also $\partial_\theta y(\theta_0, 0) = \partial_\theta x(\theta_0, 0) = 0$ and $\mathcal{D}$ is embedded, we conclude that $\partial_r x(\theta_0, 0) \neq 0$. The tangent space $T_{(\theta_0, 0, 0)} \mathcal{D}$ is oriented by the basis $(\partial/\partial\theta, \partial_r x(\theta_0, 0)\partial/\partial x)$. On the other hand, the contact structure $\ker \lambda(\theta_0, 0, 0)$ is oriented by $(\partial/\partial\theta, -\partial/\partial x)$. The singular point $(\theta_0, 0, 0)$ is called positive if these two orientations coincide, which is the case for $\partial_r x(\theta_0, 0) < 0$, otherwise $(\theta_0, 0, 0)$ is called negative. Hence in the case of a positive (negative) singularity, the surface $\mathcal{D}$ lies on the side of the negative (positive) x-axis. We would like to perturb $\mathcal{D}$ near its boundary, leaving the boundary fixed, so that the number and type of the singularities does not change and the new surface has some kind of normal form

near its boundary. the following is the main result of this section. It is an immediate consequence of Propositions 2.2 and 2.3 below.

Proposition 2.1. *Let (M, λ) be a three-dimensional contact manifold. Further, let $\mathcal{L}$ be a Legendrian knot and $\mathcal{D}$ an embedded surface with $\partial \mathcal{D} = \mathcal{L}$ so that all the singular points are non-degenerate. We denote the finitely many singular points on the boundary by e_k, $1 \leqslant k \leqslant N$ (ordered by moving in the direction of the orientation of $\mathcal{L}$).*
   Then there is an embedded surface $\mathcal{D}'$ having the same boundary as $\mathcal{D}$ which differs from $\mathcal{D}$ only by a C^0-small perturbation supported near $\mathcal{L}$ having the same singular points as $\mathcal{D}$ so that the following holds:
   There is a neighborhood U of $\mathcal{L}$ and a diffeomorphism $\Phi : U \to S^1 \times \mathbf{R}^2$ so that

- $\Phi^*(dy + x\,d\theta) = \lambda|_U$, $(\theta, x, y) \in S^1 \times \mathbf{R}^2$,
- $\Phi(\mathcal{L}) = S^1 \times \{(0, 0)\}$,
- $\Phi(e_k) = (\theta_k, 0, 0)$, $0 \leqslant \theta_1 < \cdots < \theta_N < 1$,
- $\Phi(U \cap \mathcal{D}') = \{(\theta, a(\theta)r, b(\theta)r) \in S^1 \times \mathbf{R}^2 \mid \theta, r \in [0, 1]\}$,

where a, b are smooth 1-periodic functions with:

- $b(\theta_k) = 0$ *and* $b(\theta)$ *is nonzero if* $\theta \neq \theta_k$,
- $a(\theta_k) < 0$ *if* e_k *is a positive singular point,* $a(\theta_k) > 0$ *if* e_k *is a negative singular point,*
- *if* e_k *is elliptic then* $-1 < b'(\theta_k)/a(\theta_k) < 0$,
- *if* e_k *is hyperbolic then the quotient* $\frac{b'(\theta_k)}{a(\theta_k)}$ *is either strictly smaller than -1 or positive,*
- a *has exactly one zero in each of the intervals* $[\theta_k, \theta_{k+1}]$, $k = 1, \ldots, N-1$ *and* $[\theta_N, 1] \cup [0, \theta_1]$,
- *If* e_k *is an elliptic singular point and if* $|\theta - \theta_k|$ *is sufficiently small then we have* $b(\theta) = -\frac{1}{2}a(\theta)(\theta - \theta_k)$.

   We consider first the situation near boundary singular points.

2.1. A normal form for the spanning surface near boundary singularities

   We first simplify the surface $\mathcal{D}$ near singular points on the boundary:

Proposition 2.2. *Let $\mathcal{L}$ be a Legendrian knot in a three dimensional contact manifold (M, λ) and let $\mathcal{D} \subset M$ be an embedded surface with $\partial \mathcal{D} = \mathcal{L}$. Assume that the singular points of the characteristic foliation on $\mathcal{D}$ are nondegenerate. Denote by $(\theta, x, y) \in S^1 \times \mathbf{R}$ the coordinates near $\mathcal{L}$ provided by the Legendrian neighborhood theorem. If $\mathcal{D}$ is parameterized by*

$$\{(\theta, x(\theta, r), y(\theta, r)) \in S^1 \times \mathbf{R} \mid r \in [0, 1]\}$$

near $\mathcal{L}$ then there is an embedded surface $\mathcal{D}'$ with the following properties:

- $\mathcal{D}'$ *is obtained from $\mathcal{D}$ by a C^0-small perturbation supported near the boundary singular points leaving the boundary fixed, i.e. $\partial \mathcal{D}' = \partial \mathcal{D} = \mathcal{L}$.*
- $\mathcal{D}'$ *has the same singular points as $\mathcal{D}$.*
- *If $(\theta_0, 0, 0)$ is a boundary singularity and*

$$\mathcal{D}' = \{(\theta, x'(\theta, r), y'(\theta, r)) \in S^1 \times \mathbf{R} \mid r \in [0, 1]\}$$

then

$$y'(\theta, r) = cx'(\theta, r)(\theta - \theta_0) + \frac{b}{2}x'(\theta, r)^2,$$

where
1. $c = -\frac{1}{2}$ *and* $b = 0$ *if* $(\theta_0, 0, 0)$ *is elliptic,*
2. $c \in (-\infty, -1) \cup (0, +\infty)$ *if* $(\theta_0, 0, 0)$ *is hyperbolic.*

Proof. Let us first point out how to recognize the type of the singularity $(\theta_0, 0, 0)$ in the above parameterization. Since the Jacobian of the map $\Psi(\theta, r) = (\theta, x(\theta, r))$ at the point $(\theta_0, 0)$ has rank 2, there is a local inverse and we parameterize $\mathcal{D}$ by

$$\mathcal{D} = \left\{ \left(\theta, x, (y \circ \Psi^{-1})(\theta, r) \right) \right\},$$

where (θ, x) is sufficiently near to $(\theta_0, 0)$. Note that in the case of a positive (negative) singular point $(\theta_0, 0, 0)$ the map Ψ^{-1} is only defined for non-positive (non-negative) x. We write

$$f(\theta, x) := (y \circ \Psi^{-1})(\theta, x)$$

and note that

- $f(\theta, 0) \equiv 0$,
- $\partial_x f(\theta_0, 0) = 0$ since $(\theta_0, 0, 0)$ is a singular point.

We may extend f smoothly so that it is defined for small $|x|$ regardless of the sign of x. Write

$$f(\theta, x) = \frac{a}{2}(\theta - \theta_0)^2 + \frac{b}{2}x^2 + c(\theta - \theta_0)x + h(\theta, x)$$

with $a = \partial_{\theta\theta} f(\theta_0, 0)$, $b = \partial_{xx} f(\theta_0, 0)$, $c = \partial_{x\theta} f(\theta_0, 0)$, and h of order at least 3 in $(\theta - \theta_0, x)$. Note that $a = 0$ and also $h(\theta, 0) = 0$, hence

$$f(\theta, x) = \frac{b}{2}x^2 + c(\theta - \theta_0)x + h(\theta, x).$$

Investigate now the admissible values for the constants b and c. The surface $\mathcal{D}$ is given by $H^{-1}(0)$, where

$$H(\theta, x, y) := y - f(\theta, x).$$

Then the vector field $\widehat{V}_H$, which is defined by $i_{\widehat{V}_H} d\lambda = (i_{X_\lambda} dH) d\lambda - dH$ and $i_{\widehat{V}_H} \lambda = 0$, is given by

$$\widehat{V}_H(\theta, x, y) = -\partial_x f(\theta, x) \frac{\partial}{\partial \theta} + \left(x + \partial_\theta f(\theta, x) \right) \frac{\partial}{\partial x} + x \partial_x f(\theta, x) \frac{\partial}{\partial y},$$

and it induces the characteristic foliation on $\mathcal{D}$. Its linearization $\widehat{V}_H'(\theta, x, y)$ is given by

$$\begin{pmatrix} -\partial_{x\theta} f(\theta, x) & -\partial_{xx} f(\theta, x) & 0 \\ \partial_{\theta\theta} f(\theta, x) & 1 + \partial_{x\theta} f(\theta, x) & 0 \\ x \partial_{x\theta} f(\theta, x) & \partial_x f(\theta, x) + x \partial_{xx} f(\theta, x) & 0 \end{pmatrix}.$$

The contact structure $\ker \lambda(\theta_0, 0, 0)$ is generated by the vectors $(1, 0, 0)$ and $(0, 1, 0)$. We represent $\widehat{V}_H'(\theta_0, 0, 0)$ by the matrix

$$\begin{pmatrix} -\partial_{x\theta} f(\theta_0, 0) & -\partial_{xx} f(\theta_0, 0) \\ 0 & 1 + \partial_{x\theta} f(\theta_0, 0) \end{pmatrix} = \begin{pmatrix} -c & -b \\ 0 & 1 + c \end{pmatrix}.$$

The singular point $(\theta_0, 0, 0)$ is then hyperbolic if $c(c + 1) > 0$ and elliptic if $c(c + 1) < 0$. Let us translate this into our original parameterization $(\theta, x(\theta, r), y(\theta, r))$ of $\mathcal{D}$. Using $f \circ \Psi = y$, we compute

$$\partial_{\theta r} y(\theta_0, 0) = c \, \partial_r x(\theta_0, 0),$$

so that we are in the following situation: A singular point $(\theta_0, 0, 0)$ is

1. positive if $\partial_r x(\theta_0, 0) < 0$,
2. negative if $\partial_r x(\theta_0, 0) > 0$,
3. elliptic if $(\partial_{\theta r} y / \partial_r x)(\theta_0, 0) \in (-1, 0)$, and
4. hyperbolic if $(\partial_{\theta r} y / \partial_r x)(\theta_0, 0) \in (-\infty, -1) \cup (0, +\infty)$.

We will remove now the higher order term h by a perturbation. Take a smooth function $\beta : [0, \infty) \to [0, 1]$ with $\beta \equiv 0$ on $[0, 1]$, $\beta \equiv 1$ on $[2, \infty)$ and $\beta' \geqslant 0$. For small $\delta > 0$ we define $\beta_\delta := \beta((\theta^2 + x^2)/\delta^2)$ and

$$f_\delta(\theta, x) := \frac{b}{2}x^2 + cx(\theta - \theta_0) + \beta_\delta(\theta - \theta_0, x)h(\theta, x).$$

This perturbation takes place in a small neighborhood of the singular point $(\theta_0, 0, 0)$. We have to show that the new surface given by the graph of f_δ has the same singularities as $\mathcal{D}$ provided $\delta > 0$ was chosen sufficiently small. We proceed indirectly. Assume that for any sequence $\delta_n \searrow 0$ there is a singular point $(\theta_n, x_n, f_{\delta_n}(\theta_n, x_n))$ on the surface $\mathcal{D}_{\delta_n}$ given by the graph of f_{δ_n} which satisfies $(\theta_n - \theta_0)^2 + x_n^2 \leqslant 2\delta_n^2$ and is different from $(\theta_0, 0, 0)$. If $(\theta_n, x_n, f_{\delta_n}(\theta_n, x_n))$ is singular then

$$0 = \partial_\theta f_{\delta_n}(\theta_n, x_n) + x_n$$
$$= (c + 1)x_n + \beta_{\delta_n}(\theta_n - \theta_0, x_n)\partial_\theta h(\theta_n, x_n) + \frac{2(\theta_n - \theta_0)}{\delta_n^2}\beta'\left(\frac{(\theta_n - \theta_0)^2 + x_n^2}{\delta_n^2}\right)h(\theta_n, x_n)$$

and

$$0 = \partial_x f_{\delta_n}(\theta_n, x_n)$$
$$= bx_n + c(\theta_n - \theta_0) + \beta_{\delta_n}(\theta_n - \theta_0, x_n)\partial_x h(\theta_n, x_n) + \frac{2x_n}{\delta_n^2}\beta'\left(\frac{(\theta_n - \theta_0)^2 + x_n^2}{\delta_n^2}\right)h(\theta_n, x_n).$$

We write shortly

$$0 = (c + 1)x_n + \beta_{\delta_n}\partial_\theta h + \frac{2(\theta_n - \theta_0)}{\delta_n^2}\beta'_{\delta_n}h, \qquad 0 = bx_n + c(\theta_n - \theta_0) + \beta_{\delta_n}\partial_x h + \frac{2x_n}{\delta_n^2}\beta'_{\delta_n}h. \tag{3}$$

Remark. The reader should be aware that β'_{δ_n} is not the derivative of β_{δ_n}, but the rescaled derivative of β

$$\beta'_{\delta_n} := \beta'\left(\frac{(\theta_n - \theta_0)^2 + x_n^2}{\delta_n^2}\right).$$

Eq. (3) is the same as

$$\begin{pmatrix} x_n \\ \theta_n - \theta_0 \end{pmatrix} = \frac{1}{c(c + 1)}\begin{pmatrix} c & 0 \\ -b & c + 1 \end{pmatrix} H(\theta_n, x_n) \tag{4}$$

with

$$H(\theta_n, x_n) = -\begin{pmatrix} \beta_{\delta_n}\partial_\theta h + \frac{2(\theta_n - \theta_0)}{\delta_n^2}\beta'_{\delta_n}h \\ \beta_{\delta_n}\partial_x h + \frac{2x_n}{\delta_n^2}\beta'_{\delta_n}h \end{pmatrix},$$

which satisfies

$$\frac{H(\theta_n, x_n)}{\sqrt{(\theta_n - \theta_0)^2 + x_n^2}} \to 0$$

as $n \to \infty$ since H is of order at least 2 in $(\theta_n - \theta_0, x_n)$. Dividing Eq. (4) by $\sqrt{(\theta_n - \theta_0)^2 + x_n^2}$ and passing to the limit $n \to \infty$ we obtain a contradiction.

Hence we may assume that $\mathcal{D}$ is given by the graph of

$$f(\theta, x) = \frac{b}{2}x^2 + cx(\theta - \theta_0)$$

for $x^2 + (\theta - \theta_0)^2$ sufficiently small. Consider now the case where $(\theta_0, 0, 0)$ is an elliptic singularity. Take the same smooth function β as before and define for $\delta > 0$

$$f_\delta(\theta, x) := \beta\left(\frac{(\theta - \theta_0)^2 + x^2}{\delta^2}\right)\frac{b}{2}x^2 + cx(\theta - \theta_0),$$

so that $f_\delta \equiv f$ if $(\theta - \theta_0)^2 + x^2 \geqslant 2\delta^2$ and $f_\delta(\theta, x) = cx(\theta - \theta_0)$ if $(\theta - \theta_0)^2 + x^2 \leqslant \delta^2$. Writing $\beta_\delta := \beta(((\theta - \theta_0)^2 + x^2)/\delta^2)$, $\beta'_\delta := \beta'(((\theta - \theta_0)^2 + x^2)/\delta^2)$ as before, the condition of $(\theta, x, f_\delta(\theta, x))$ being a singular point is

$$0 = \begin{pmatrix} c & b\beta_\delta + \frac{bx^2}{\delta^2}\beta'_\delta \\ \frac{bx^2}{\delta^2}\beta'_\delta & c + 1 \end{pmatrix}\begin{pmatrix} \theta - \theta_0 \\ x \end{pmatrix}$$

which implies

$$0 = c(c + 1) - b^2\left(\frac{x^2}{\delta^2}\beta_\delta\beta'_\delta + \frac{x^4}{\delta^4}(\beta'_\delta)^2\right) \leqslant c(c + 1)$$

in contradiction to the fact that $(\theta_0, 0, 0)$ is an elliptic singularity. Hence we may assume that

$$f(\theta, x) = cx(\theta - \theta_0)$$

near an elliptic singularity $(\theta_0, 0, 0)$, where $c \in (-1, 0)$. Now we will carry out a last modification to achieve $c = -\frac{1}{2}$. We take a smooth function

$$\beta : \mathbf{R} \to \left[\min\{c, -1/2\}, \max\{c, -1/2\}\right] \subset (-1, 0)$$

with $\beta(s) = -\frac{1}{2}$ for $|s| \leqslant 1$ and $\beta(s) = c$ for $|s| \geqslant 2$. Define for small $\delta > 0$

$$f_\delta(\theta, x) := \beta\left(\frac{(\theta - \theta_0)^2 + x^2}{\delta^2}\right)x(\theta - \theta_0).$$

Again, we did not create any new singular points. This completes the proof of Proposition 2.2. $\quad\square$

2.2. Perturbing the spanning surface near the Legendrian knot

We will now show the following:

Proposition 2.3. *Let $\mathcal{L}$ be a Legendrian knot in a three dimensional contact manifold (M, λ) and let $\mathcal{D}$ be an embedded surface with $\partial\mathcal{D} = \mathcal{L}$. Assume that the singular points on $\mathcal{D}$ are non-degenerate.*
   Then there is an embedded surface $\mathcal{D}'$ with $\partial\mathcal{D}' = \mathcal{L}$ which differs from $\mathcal{D}$ by a C^0-small perturbation supported near $\mathcal{L}$ and leaving $\mathcal{L}$ fixed so that $\mathcal{D}$ and $\mathcal{D}'$ have the same singular points and the following holds:
   There is a neighborhood U of $\mathcal{L}$ and a diffeomorphism $\Phi : U \to S^1 \times \mathbf{R}^2$ so that

1. $\Phi^*(dy + x\,d\theta) = \lambda|_U$, $(\theta, x, y) \in S^1 \times \mathbf{R}$,
2. $\Phi(\mathcal{L}) = S^1 \times \{(0, 0)\}$,
3. $\Phi(U \cap \mathcal{D}') = \{(\theta, a(\theta)r, b(\theta)r) \in S^1 \times \mathbf{R} \mid \theta \in S^1 \approx \mathbf{R}/\mathbf{Z}, \ r \in [0, 1]\}$, *where*

$$\theta \mapsto \begin{pmatrix} a(\theta) \\ b(\theta) \end{pmatrix}$$

is a smooth closed curve in $\mathbf{R}^2\backslash\{0\}$ *with the following properties*:

(a) $\qquad$ $\mathrm{tb}(\mathcal{L}) = \deg\left[\theta \mapsto \begin{pmatrix} a(\theta) \\ b(\theta) \end{pmatrix}\right],$

$\qquad$ *where* tb *denotes denotes the Thurston–Bennequin invariant of the Legendrian knot (see* [10]).

(b) $b(\theta_0) = 0$ *if and only if* $\Phi^{-1}(\theta_0, 0, 0)$ *is a singular point on* $\mathcal{L}$,

(c) *a singular point* $\Phi^{-1}(\theta_0, 0, 0)$ *is*

$\qquad$ (i) *positive (negative) if* $a(\theta_0) < 0$ $(a(\theta_0) > 0)$,

$\qquad$ (ii) *elliptic if* $c = b'(\theta_0)/a(\theta_0) \in (-1, 0)$,

$\qquad$ (iii) *hyperbolic if* $c = b'(\theta_0)/a(\theta_0) \in (-\infty, -1) \cup (0, +\infty)$,

(d) *for* θ *near* θ_0, *where* $\Phi^{-1}(\theta_0, 0, 0)$ *is a singular point, we have* $b(\theta) = c(\theta - \theta_0)a(\theta)$,

(e) *if* $\Phi^{-1}(\theta_0, 0, 0)$ *and* $\Phi^{-1}(\theta_1, 0, 0)$ *are singular points of opposite sign with* $\theta_0 < \theta_1$, *so that all the points* $(\theta, 0, 0)$ *are non-singular for* $\theta \in (\theta_0, \theta_1)$, *then* a *has exactly one zero in the interval* (θ_0, θ_1).

Proof. We parameterize $\mathcal{D}$ again by

$$\mathcal{D} = \left\{\left(\theta, x(\theta, r), y(\theta, r)\right) \in S^1 \times \mathbf{R}^2 \mid \theta \in S^1, \ r \in [0, 1]\right\}$$

and we expand x, y as follows:

$$x(\theta, r) = \partial_r x(\theta, 0)r + h(\theta, r),$$
$$y(\theta, r) = \partial_r y(\theta, 0)r + k(\theta, r),$$

where h, k are of order at least 2 in r and 1-periodic in θ. For small r and $|\theta - \theta_0|$, where $(\theta_0, 0, 0)$ is a boundary singular point, we have

$$y(\theta, r) = cx(\theta, r)(\theta - \theta_0) + \frac{b}{2}x^2(\theta, r) \tag{5}$$

by Proposition 2.2. In the case of an elliptic singularity we may assume that $c = -\frac{1}{2}$ and $b = 0$. We want to perturb $\mathcal{D}$ near its boundary leaving $\partial\mathcal{D}$ fixed, so that the higher order terms h and k disappear. We will only indicate the necessary steps and leave the details to the reader. The verification that no new singularities are created is completely straight forward using the normal form (5) near the singular points. Pick a smooth function $\beta:[0, \infty) \to [0, 1]$ with $\beta \equiv 0$ on $[0, 1]$, $\beta \equiv 1$ on $[2, \infty)$ and $0 \leqslant \beta'(s) \leqslant 2$ for all $s \geqslant 0$. We define

$$x_\delta(\theta, r) := \partial_r x(\theta, 0)r + \beta\left(\frac{r}{\delta}\right)h(\theta, r),$$
$$y_\delta(\theta, r) := \partial_r y(\theta, 0)r + \beta\left(\frac{r}{\delta}\right)k(\theta, r)$$

and

$$\mathcal{D}_\delta = \left\{\left(\theta, x_\delta(\theta, r), y_\delta(\theta, r)\right) \in S^1 \times \mathbf{R}^2 \mid \theta \in S^1, \ r \in [0, 1]\right\}.$$

For $r \geqslant 2\delta$ the perturbed surface $\mathcal{D}_\delta$ coincides with $\mathcal{D}$ and we have $\partial\mathcal{D}_\delta = \partial\mathcal{D} = \mathcal{L}$. The surface $\mathcal{D}_\delta$ has the same singularities on the boundary as $\mathcal{D}$ and that $\mathcal{D}_\delta$ has no singularities in the range $0 < r < 2\delta$ provided $\delta > 0$ was chosen sufficiently small. It remains to verify that the surface $\mathcal{D}_\delta$ is embedded for sufficiently small δ. If it were not then we could find sequences $\delta_k \searrow 0$, $0 \leqslant r_k, r'_k \leqslant 2\delta_k$ and θ_k such that

$$\left(\partial_r x_{\delta_k}(\theta_k, r_k), \partial_r y_{\delta_k}(\theta_k, r_k)\right) = (0, 0)$$

for all k (surface not immersed) or

$$x_{\delta_k}(\theta_k, r_k) = x_{\delta_k}(\theta_k, r_k'),$$
$$y_{\delta_k}(\theta_k, r_k) = y_{\delta_k}(\theta_k, r_k')$$

for all k (surface has self-intersections). Both assertions contradict the fact that $(\partial_r x(\theta, 0), \partial_r y(\theta, 0)) \neq (0, 0)$ for all θ, and can therefore not occur.

Hence we may assume that near $\mathcal{L}$ we have

$$\mathcal{D} = \{(\theta, a(\theta)r, b(\theta)r) \in S^1 \times \mathbf{R}^2 \mid r \in [0, 1]\},$$

where the map

$$\theta \mapsto \begin{pmatrix} a(\theta) \\ b(\theta) \end{pmatrix}$$

is a closed curve in $\mathbf{R}^2 \backslash \{0\}$. A point $(\theta_0, 0, 0)$ is a singular point if and only if $b(\theta_0) = 0$ and it is

1. positive if $a(\theta_0) < 0$,
2. negative if $a(\theta_0) > 0$,
3. elliptic if $b'(\theta_0)/a(\theta_0) \in (-1, 0)$ and
4. hyperbolic if $b'(\theta_0)/a(\theta_0) \in (-\infty, -1) \cup (0, +\infty)$.

If r and $\theta - \theta_0$ are sufficiently small, where $(\theta_0, 0, 0)$ is a singular point, then we compute with the normal form (5):

$$b(\theta) = \partial_r y(\theta, 0) = c\partial_r x(\theta, 0)(\theta - \theta_0) = ca(\theta)(\theta - \theta_0), \tag{6}$$

so if we use the parameters $(\theta, \rho = a(\theta)r)$ instead of (θ, r) then $\mathcal{D}$ is given by

$$\mathcal{D} = \{(\theta, \rho, c\rho(\theta - \theta_0))\} \quad \text{near } (\theta_0, 0, 0).$$

Hence the modification that we carried out on $\mathcal{D}$ in this section did not affect the normal form near boundary singularities that we have constructed in the previous section.

In this picture it is easy to understand the Thurston–Bennequin invariant of the knot $\mathcal{L}$. Let us shift $\mathcal{L}$ along the Reeb vector field to get a knot

$$\mathcal{L}' := \{(\theta, 0, \delta) \in S^1 \times \mathbf{R}^2 \mid \theta \in [0, 1]\}$$

with some small constant δ. Then $\mathcal{L}'$ and $\mathcal{D}$ intersect if and only if

$$a(\theta) = 0 \quad \text{and} \quad r = \frac{\delta}{b(\theta)}.$$

The condition $a(\theta) = 0$ means that the Reeb vector field X_λ is tangent to $\mathcal{D}$ at the point $(\theta, 0, \delta)$. Without affecting the value of the intersection number $\text{int}(\mathcal{L}', \mathcal{D})$ we may perturb the loop $(a(\theta), b(\theta))$ slightly so that $a'(\theta) \neq 0$ whenever $a(\theta) = 0$. Then we compute with $\lambda = dy + x \, d\theta$ and

$$\mathcal{S} := \{\theta \in [0, 1] \mid a(\theta) = 0 \text{ and } \operatorname{sign} b(\theta) = \operatorname{sign} \delta\} :$$

$$\begin{aligned}
\text{tb}(\mathcal{L}) &= \sum_{\theta \in \mathcal{S}} \operatorname{sign}\left[(\lambda \wedge d\lambda)_{(\theta,0,\delta)}\left(\begin{pmatrix} 1 \\ a'(\theta)\delta/b(\theta) \\ b'(\theta)\delta/b(\theta) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ b(\theta) \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)\right] \\
&= \sum_{\theta \in \mathcal{S}} \operatorname{sign}[-\delta a'(\theta)] \\
&= \deg\left[\theta \mapsto \begin{pmatrix} a(\theta) \\ b(\theta) \end{pmatrix}\right].
\end{aligned} \tag{7}$$

Assume now that $(\theta_0, 0, 0)$ and $(\theta_1, 0, 0)$ are singularities of opposite sign with $\theta_0 < \theta_1$, so that all the points $(\theta, 0, 0)$ with $\theta \in (\theta_0, \theta_1)$ are not singular. Let us assume that $(\theta_0, 0, 0)$ is the negative singularity. Then

- $b(\theta_0) = b(\theta_1) = 0$ and b is nonzero on (θ, θ_1).
- $a(\theta_0) > 0$, $a(\theta_1) < 0$.

We would like to perturb $\mathcal{D}$ near $\mathcal{L}$, leaving the boundary fixed, so that a has only one zero in the interval (θ_0, θ_1). Let $\delta > 0$ and pick a smooth function β so that $\beta \equiv 0$ on $[0, \delta]$ and $\beta \equiv 1$ on $[2\delta, \infty)$. Let $\hat{a}$ be a 1-periodic function which coincides with a except on some interval $[\theta_0 + \varepsilon, \theta_1 - \varepsilon]$, and which has exactly one zero between θ_0 and θ_1. We define

$$\tilde{a}(\theta, r) := \bigl(1 - \beta(r)\bigr)\hat{a}(\theta) + \beta(r)a(\theta)$$

and denote the new surface by

$$\mathcal{D}_\delta := \bigl\{(\theta, \tilde{a}(\theta, r)r, b(\theta)r)\bigr\},$$

which has the same number and type of singularities as $\mathcal{D}$ because we did not change the function b and because $\tilde{a}$ coincides with a near θ_0 and θ_1. Moreover, $\mathcal{D}_\delta$ is embedded since it is immersed and the map $(\theta, r) \mapsto (\theta, \tilde{a}(\theta, r), b(\theta))$ is also injective. This completes the proof of Proposition 2.3. $\square$

Remark. The negative singularities correspond to the points where the curve $\theta \overset{\gamma}{\longmapsto} a(\theta) + ib(\theta) \in \mathbf{C} \backslash \{0\}$ hits the positive real axis. Similarly, positive singularities corresponds to the intersection of γ with the negative real axis.

2.3. The non-Lagrangian part of the boundary condition

The submanifold $\mathbf{R} \times \mathcal{L}$ is a Lagrangian submanifold of the symplectisation $(\mathbf{R} \times M, d(e^t\lambda))$. However, the submanifold $\{0\} \times \mathcal{D}$ is only totally real with respect to any $\tilde{J}$ away from the singular points. These two submanifold serve as boundary conditions for our boundary value problem, and we have to find a way to deal with $\{0\} \times \mathcal{D}$ in order to derive apriori estimates. The problem is the following: The fibers of the vector bundles $T(\mathbf{R} \times \mathcal{L})$ and $\tilde{J}T(\mathbf{R} \times \mathcal{L})$ are orthogonal with respect to the $\tilde{J}$-invariant metric $g = d(e^t\lambda) \circ (\tilde{J} \times \mathrm{Id})$ while $T(\{0\} \times \mathcal{D})$ and $\tilde{J}T(\{0\} \times \mathcal{D})$ are only transverse, but not orthogonal. On the other hand, we will need this orthogonality to prove asymptotic decay estimates later (without it certain operators would fail to be self-adjoint). The way out is the following: Instead of using the metric g above, we use a different one where we have orthogonality. We will be able to control this metric if we do estimates later on. There is a 2-form ω near the intersection set $\{0\} \times \mathcal{L}$ of $\mathbf{R} \times \mathcal{L}$ and $\{0\} \times \mathcal{D}$ which is nondegenerate away from the singular points so that both submanifolds become Lagrangian with respect to ω, and ω is compatible with the almost complex structure $\tilde{J}$. In general, we cannot expect ω to be closed, unless we weaken our requirements and replace compatibility by tameness (i.e. $\omega(v, \tilde{J}v) > 0$ for all $v \neq 0$). It will turn out that we need the compatibility condition, but we do not need ω to be closed. We construct such a 2-form explicitly in local coordinates. We will confine ourselves to a special almost complex structure $\tilde{J}$ near $\{0\} \times \mathcal{L}$ which will also be used in the subsequent papers [4–6].

From now on we pick an almost complex structure $\tilde{J}$ on $\mathbf{R} \times M$, where the corresponding $J : \xi \to \xi$ has the following form in local coordinates near $\{0\} \times \mathcal{L}$:

$$J(\theta, x, y) \cdot (1, 0, -x) := (0, -1, 0), \qquad J(\theta, x, y) \cdot (0, 1, 0) := (1, 0, -x). \tag{8}$$

Lemma 2.4. *If $U_k \subset M$ are disjoint open neighborhoods of the singular points e_k, $k = 1, \ldots, N$, on the boundary $\mathcal{L} = \partial \mathcal{D}$ then there exist an open neighborhood $V \subset M$ of $\mathcal{L}$ and a nondegenerate 2-form ω defined on $W = \mathbf{R} \times (V \setminus \cup_k U_k) \subset \mathbf{R} \times M$, so that $\omega|_{T(\{0\} \times \mathcal{D})} \equiv 0$, $\omega|_{T(\mathbf{R} \times \mathcal{L})} \equiv 0$ and the form ω is compatible with $\tilde{J}$, i.e. $\omega \circ (\mathrm{Id} \times \tilde{J})$ is a Riemannian metric.*

Proof. Use the coordinates $(\theta, x, y) \in \mathbf{R}^3$ near $\mathcal{L}$ which we derived in Section 2, where the contact form equals $dy + x\, d\theta$ and $\{0\} \times \mathcal{D}$ is represented by

$$\{(0, \theta, a(\theta)r, b(\theta)r) \in \{0\} \times \mathbf{R}^3 \mid r, \theta \in [0, 1]\}.$$

Denoting the standard Euclidean product on $\mathbf{R}^4$ by $\langle \cdot, \cdot \rangle$, we have to find a function with values in the set of skew-symmetric 4×4-matrices $\Omega(\tau, \theta, x, y)$ such that

1. $\langle \cdot, \Omega \tilde{J} . \rangle$ is a metric,
2. $\langle v, \Omega w \rangle|_{(\tau, \theta, 0, 0)} = 0$ for all $v, w \in T_{(\tau, \theta, 0, 0)}(\mathbf{R} \times \mathcal{L})$,
3. $\langle v, \Omega w \rangle|_{(0, \theta, q(\theta)y, y)} = 0$ for all $v, w \in T_{(0, \theta, q(\theta)y, y)}(\{0\} \times \mathcal{D})$, where $q(\theta) := a(\theta)/b(\theta)$.

The matrix of $\tilde{J}$ is given by

$$\tilde{J}(\tau, \theta, x, y) = \begin{pmatrix} 0 & -x & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -x & 0 \end{pmatrix}.$$

We write $\Omega = (\omega_{kl})_{1 \leqslant k, l \leqslant 4}$ with $\omega_{kl} = -\omega_{lk}$. If we choose

$$\omega(\tau, \theta, x, y) = -xC\, d\tau \wedge d\theta - q(\theta)d\tau \wedge dx + C\, d\tau \wedge dy - d\theta \wedge dx + q(\theta)\, d\theta \wedge dy$$

and $\omega = \langle \cdot, \Omega . \rangle$, where

$$C > \max\{0, \sup q^2(\theta)\},$$

then the matrix $\Omega \tilde{J}$ is positive definite if x, y are sufficiently small. $\quad \square$

3. Asymptotic behavior at infinity

Assume we have a solution of:

$$\begin{cases} \tilde{u} = (a, u) : S \to \mathbf{R} \times M, \\ \partial_s \tilde{u} + \tilde{J}(\tilde{u})\partial_t \tilde{u} = 0, \\ \tilde{u}(s, 0) \subset \mathbf{R} \times \mathcal{L}, \\ \tilde{u}(s, 1) \subset \{0\} \times \mathcal{D}^{**}, \\ E(\tilde{u}) < +\infty, \end{cases} \tag{9}$$

where $S := \mathbf{R} \times [0, 1]$ and $\mathcal{D}^{**}$ is the spanning surface $\mathcal{D}$ without some open neighborhood U of the set of singular points Γ. We will show that the condition of finite energy forces the solution to converge to points on the knot $\mathcal{L}$ for $|s| \to \infty$, more precisely

$$\tilde{u}(s, t) \to \tilde{p}_\pm \in \{0\} \times \mathcal{L}$$

as $s \to \pm\infty$ uniformly in t. We will also show that this convergence is of exponential nature. This fact will be crucial for the nonlinear Fredholm theory in [4].

3.1. The solutions approach the Legendrian asymptotically

As a first step, we will show that the ends of a finite energy strip $\tilde{u}$ have to approach the knot $\{0\} \times \mathcal{L} \subset \mathbf{R} \times M$ asymptotically. This actually works under the weaker assumption $\tilde{u}(s, 1) \in \{0\} \times \mathcal{D}$. The main result of this section is Proposition 3.4 below.

Lemma 3.1. *Assume $\tilde{u} : S \to \mathbf{R} \times M$ satisfies Eq.* (9) *above. If in addition*

$$\int_S u^* \, d\lambda = 0$$

then $\tilde{u}$ must be constant.

Proof. The map $\tilde{u} = (a, u)$ satisfies the following system of equations:

$$\pi_\lambda \partial_s u + J(u)\pi_\lambda \partial_t u = 0,$$
$$\partial_s a - \lambda(u)\partial_t u = 0,$$
$$\partial_t a + \lambda(u)\partial_s u = 0.$$

Since

$$\int_S u^* \, d\lambda = \int_S d\lambda(\pi_\lambda \partial_s u, \pi_\lambda \partial_t u) \, ds \wedge dt$$
$$= \frac{1}{2} \int_S \left[|\pi_\lambda \partial_s u|_J^2 + |\pi_\lambda \partial_t u|_J^2 \right] ds \wedge dt$$
$$= 0,$$

where $| \cdot |_J^2 = d\lambda(\cdot, J.)$, we conclude that $\pi_\lambda \partial_s u = \pi_\lambda \partial_t u \equiv 0$ and therefore

$$\triangle a \, ds \wedge dt = -d(da \circ i) = u^* \, d\lambda = 0,$$

hence $a : S \to \mathbf{R}$ is harmonic and satisfies $a(s, 1) \equiv 0$. Because of $u(s, 0) \in \mathcal{L}$ we also have

$$\partial_t a(s, 0) = -\lambda\big(u(s, 0)\big)\partial_s u(s, 0) \equiv 0.$$

Define now $f : S \to \mathbf{R}$ by

$$f(s, t) := \int_0^t \partial_s a(s, \tau) \, d\tau$$

so that $\partial_t f = \partial_s a$ and

$$\partial_s f(s, t) = -\int_0^t \partial_{tt} a(s, \tau) \, d\tau = -\partial_t a(s, t) + \partial_t a(s, 0) = -\partial_t a(s, t).$$

Then $\Phi := a + if : S \to \mathbf{C}$ is holomorphic and satisfies

$$\Phi(s, 0) \in \mathbf{R}, \qquad \Phi(s, 1) \in i\mathbf{R}.$$

Case 1. $|\nabla\Phi|$ *is bounded.* We define

$$\widehat{\Phi} : \widehat{S} := \mathbf{R} \times [-1, +1] \to \mathbf{C}$$

by

$$\widehat{\Phi}(s,t) := \begin{cases} \Phi(s,t) & \text{if } t \geqslant 0, \\ \overline{\Phi(s,-t)} & \text{if } t < 0. \end{cases}$$

Note that $\widehat{\Phi}$ is holomorphic. Let

$$\hat{b} := \partial_s(\operatorname{Re}\widehat{\Phi}) : \widehat{S} \to \mathbf{R}.$$

Then $\hat{b}$ is harmonic, $C := \sup_{\widehat{S}} |\hat{b}| < +\infty$ by assumption and $\hat{b}(s,\pm 1) \equiv 0$. Defining

$$\hat{c}(s,t) := \int_0^t \partial_s \hat{b}(s,\tau)\,d\tau - \int_0^s \partial_t \hat{b}(\sigma,0)\,d\sigma,$$

we compute $\partial_t \hat{c} = \partial_s \hat{b}$ and $\partial_s \hat{c} = -\partial_t \hat{b}$, hence $\delta := \hat{b} + i\hat{c} : \widehat{S} \to \mathbf{C}$ is holomorphic with bounded real part. The function $g := e^\delta$ is also holomorphic and satisfies

$$|g| \leqslant e^C, \quad |g(s,\pm 1)| = 1.$$

Let $\varepsilon > 0$ and define a holomorphic function on $\widehat{S}$ by

$$h_\varepsilon(z) := \frac{1}{1 - i\varepsilon(z+i)}.$$

We compute with $z = s + it$

$$|h_\varepsilon(z)|^2 = \frac{1}{(1 + \varepsilon(1+t))^2 + \varepsilon^2 s^2} \leqslant 1.$$

For $s \neq 0$ we have $|h_\varepsilon(z)|^2 \leqslant 1/(\varepsilon^2 s^2)$, hence the holomorphic function gh_ε satisfies

$$|g(z)h_\varepsilon(z)| \leqslant 1$$

whenever $z \in \partial\Omega$, where

$$\Omega := \left[-\varepsilon^{-1}e^C, \varepsilon^{-1}e^C \right] \times [-1, +1].$$

Using the maximum principle, we conclude that $|gh_\varepsilon| \leqslant 1$ on all of Ω, but outside Ω we also have

$$|g(z)h_\varepsilon(z)| \leqslant \frac{e^C}{\varepsilon|s|} \leqslant 1.$$

Keeping $z \in \widehat{S}$ fixed and passing to the limit $\varepsilon \searrow 0$ we conclude that $|g(z)| = e^{\hat{b}(z)} \leqslant 1$, hence $\hat{b}(z) \leqslant 0$. Repeating the same argument with $-\delta$ instead of δ, we also obtain $-\hat{b}(z) \leqslant 0$, hence $\partial_s(\operatorname{Re}\widehat{\Phi}) = \hat{b}(z) \equiv 0$. We know now that $\operatorname{Re}\widehat{\Phi}$ is harmonic, does not depend on s and satisfies $\operatorname{Re}\widehat{\Phi}(s,\pm 1) \equiv 0$. This implies that $\operatorname{Re}\widehat{\Phi}$ is identically zero and therefore also $a \equiv 0$. In view of

$$\partial_s u = \pi_\lambda \partial_s u + (\lambda(u)\partial_s u) X_\lambda(u)$$

and

$$\partial_t u = \pi_\lambda \partial_t u + (\lambda(u)\partial_t u) X_\lambda(u)$$

we conclude that u must be constant.

Case 2. $|\nabla\Phi|$ is unbounded. Pick sequences $z_k' \in S$, $\varepsilon_k' \searrow 0$ so that

$$\varepsilon_k' |\nabla\Phi(z_k')| \to +\infty.$$

By a lemma of Hofer (see [15], Chapter 6.4, Lemma 5 and [1]) we find sequences $z_k = s_k + it_k \in S$, $\varepsilon_k \searrow 0$ so that

- $\varepsilon_k R_k := \varepsilon_k |\nabla\Phi(z_k)| \to +\infty$,

- $|z_k - z_k'| \leqslant \varepsilon_k'$,
- $|\nabla \Phi(z)| \leqslant 2R_k$ whenever $|z - z_k| \leqslant \varepsilon_k$.

We may assume without loss of generality that $t_k \to t_0 \in [0, 1]$. We consider the following cases after choosing a suitable subsequence:

1. $-t_k R_k \to -\infty$,
   - (a) $R_k(1 - t_k) \to l \in [0, +\infty)$,
   - (b) $R_k(1 - t_k) \to +\infty$,
2. $-t_k R_k \to -l \in (-\infty, 0]$, then $R_k(1 - t_k) \to +\infty$.

Let us begin with the case 1(b). We define

$$\Omega_k := \mathbf{R} \times \left[-t_k R_k, R_k(1 - t_k) \right]$$

and the holomorphic maps $\Phi_k : \Omega_k \to \mathbf{C}$ by

$$\Phi_k(z) := \Phi \left(z_k + z R_k^{-1} \right) - \Phi(z_k)$$

so that

$$\left| \nabla \Phi_k(0) \right| = 1, \quad \Phi_k(0) = 0$$

and

$$\left| \nabla \Phi_k(z) \right| \leqslant 2$$

if $z \in B_{\varepsilon_k R_k}(0) \cap \Omega_k$. Using the Cauchy integral formula for higher derivatives we find for each compact subset K of $\mathbf{C}$ a number k_0 so that $K \subset B_{\varepsilon_k R_k}(0) \cap \Omega_k$ for all $k \geqslant k_0$ and all the maps Φ_k are bounded in $C^\infty(K)$ uniformly in $k \geqslant k_0$. By the Ascoli–Arzela theorem, some subsequence of (Φ_k) converges in C^∞_{loc} to an entire holomorphic function Ψ satisfying

$$\left| \nabla \Psi(z) \right| \leqslant 2, \quad \left| \nabla \Psi(0) \right| = 1 \quad \text{and} \quad \Psi(0) = 0.$$

By Liouville's theorem Ψ must be an affine function. Let $\phi \in \Sigma$ and define $\phi_k \in \Sigma$ by

$$\phi_k(s) := \phi \left(s - \operatorname{Re} \Phi(z_k) \right)$$

and

$$\tau_\phi(s, t) := \phi'(s) \, ds \wedge dt.$$

We estimate using $u^* d\lambda = 0$:

$$\int_{\Omega_k} \Phi_k^* \tau_\phi = \int_S \Phi^* \tau_{\phi_k} = \int_S \phi_k'(a) \Phi^*(ds \wedge dt)$$

$$= \int_S \phi_k'(a) \, da \wedge u^* \lambda = \int_S \tilde{u}^* d(\phi_k \lambda) \leqslant E(\tilde{u}).$$

For every compact $K \subset \mathbf{C}$ we have

$$\int_K \Phi_k^* \tau_\phi \xrightarrow{k \to \infty} \int_K \Psi^* \tau_\phi.$$

It follows for non constant $\phi \in \Sigma$:

$$+\infty = \int\limits_{\mathbf{C}} \phi'(s)\, ds \wedge dt = \int\limits_{\mathbf{C}} \tau_\phi = \int\limits_{\mathbf{C}} \Psi^* \tau_\phi \leqslant E(\tilde{u}).$$

This contradiction to $E(\tilde{u}) < +\infty$ shows that case 1(b). cannot occur. We will proceed similarly with the remaining cases 1(a). and 2. Let us continue with case 2. We define

$$\Omega_k := \mathbf{R} \times [0, R_k]$$

and for $z = s + it \in \Omega_k$, $z_k = s_k + it_k$

$$\Phi_k(z) := \Phi\left(s_k + z R_k^{-1}\right) - \Phi_k(s_k),$$

so that

$$\left|\nabla \Phi_k(i R_k t_k)\right| = 1, \quad \Phi_k(0) = 0$$

and

$$\left|\nabla \Phi_k(z)\right| \leqslant 2$$

whenever $z \in B_{\varepsilon_k R_k}(i R_k t_k) \cap \Omega_k$. Reasoning as before we obtain C^∞_{loc}-convergence of some subsequence of (Φ_k) to a holomorphic map $\Psi : H^+ \to \mathbf{C}$, where H^+ denotes the upper half plane in $\mathbf{C}$. Since we have $\Phi_k(\mathbf{R}) \subset \mathbf{R}$ for all k, we also obtain

$$\Psi(\partial H^+) \subset \mathbf{R}.$$

Moreover $|\nabla \psi(z)| \leqslant 2$, $\Psi(0) = 0$ and Ψ is not constant. Using the Schwarz reflection principle we can extend Ψ to an entire holomorphic function with bounded derivative, so that Ψ must be an affine function by Liouville's theorem. In view of $\Psi(0) = 0$ and the real boundary values we have actually $\Psi(z) = \alpha z$ with some nonzero real number α. We compute as before with nonconstant $\phi \in \Sigma$:

$$\int\limits_{\Omega_k} \Phi_k^* \tau_\phi = \int\limits_{S} \Phi^* \tau_{\phi_k} \leqslant E(\tilde{u}),$$

where $\phi_k(s) := \phi(s - \operatorname{Re}\Phi(s_k))$, which implies

$$\int\limits_{H^+} \Psi^* \tau_\phi \leqslant E(\tilde{u}) < +\infty.$$

But on the other hand

$$\int\limits_{H^+} \Psi^* \tau_\phi = |\alpha| \int\limits_{H^+} \phi'(s)\, ds \wedge dt = +\infty,$$

so that case 2. is impossible. We are left with case 1a. We define $\Omega_k := \mathbf{R} \times [0, R_k]$ and for $z \in \Omega_k$

$$\Phi_k(z) := \Phi\left(s_k + i - R_k^{-1} z\right) - \Phi(s_k + i).$$

We have

$$\left|\nabla \Phi_k\left(i R_k(1 - t_k)\right)\right| = 1, \quad \Phi_k(0) = 0$$

and

$$\left|\nabla \Phi_k(z)\right| \leqslant 2$$

whenever $z \in B_{\varepsilon_k R_k}(i R_k(1 - t_k)) \cap \Omega_k$. Moreover

$$\Phi_k(\mathbf{R}) \subset i\mathbf{R}.$$

Again, a subsequence of (Φ_k) converges in C^∞_{loc} to a holomorphic map

$$\Psi : H^+ \to \mathbf{C}$$

with $|\nabla \Psi(z)| \leqslant 2$, $\Psi(0) = 0$ and Ψ is not constant. Defining

$$\tilde{\Psi}(z) := \begin{cases} \Psi(z) & \text{if } \operatorname{Im}(z) \geqslant 0, \\ -\overline{\Psi(\bar{z})} & \text{if } \operatorname{Im}(z) < 0, \end{cases}$$

we obtain an entire holomorphic function with bounded gradient which has to be affine. Since $\Psi(\partial H^+) \subset i\mathbf{R}$ we have $\Psi(z) = i\alpha z$ with some nonzero real number α. Then

$$E(\tilde{u}) \geqslant \int_{H^+} \tilde{u}^* d(\phi\lambda) = \int_{H^+} \phi'(a) \, da \wedge df$$

$$= \int_{H^+} \alpha^2 \phi'(-\alpha t) \, ds \wedge dt = |\alpha| \cdot \int_{H^+} \phi'(t) \, ds \wedge dt,$$

but if we take a $\phi \in \Sigma$ which is not constant on $[0, +\infty)$, then $\int_{H^+} \phi'(t) \, ds \wedge dt = +\infty$. This is a contradiction to the finite energy condition. Hence we have shown that $|\nabla \Phi|$ must be bounded, and therefore $\tilde{u}$ is constant. $\quad\square$

Remark. There are similar results for $\tilde{u}$ defined on the whole plane $\mathbf{C}$ [12,1] and for $\tilde{u}$ defined on H^+ with boundary condition $\mathbf{R} \times \mathcal{L}$ [2]. In the case of a finite energy strip $\tilde{u} : S \to \mathbf{R} \times M$ with boundary condition $\tilde{u}(\partial S) \subset \mathbf{R} \times \mathcal{L}$ we cannot conclude from $\int_S u^* d\lambda = 0$ that $\tilde{u}$ is constant (see [2]).

We will omit the proof of the following lemma since it is similar to the proof of Lemma 3.1:

Lemma 3.2. *Let $\tilde{u} = (a, u) : H^+ \to \mathbf{R} \times M$ be a solution of $\partial_s \tilde{u} + \tilde{J}(\tilde{u})\partial_t \tilde{u} = 0$ satisfying the boundary condition $\tilde{u}(\partial H^+) \subset \{0\} \times \mathcal{D}^*$ and the finite energy condition $E(\tilde{u}) < +\infty$. If also $\int_{H^+} u^* d\lambda = 0$ then $\tilde{u}$ must be constant.*

Lemma 3.3. *Let $\tilde{u}$ be as in Eq. (9) and assume that $u(S)$ is contained in a compact subset of M. Then*

$$\sup_{z \in S} |\nabla \tilde{u}(z)| < +\infty.$$

Proof. We prove the lemma indirectly. Then using Hofer's lemma we can find sequences $\varepsilon_k \searrow 0$, $z_k \in S$ so that

- $\varepsilon_k R_k := \varepsilon_k |\nabla \tilde{u}(z_k)| \to +\infty$,
- $|\nabla \tilde{u}(z)| \leqslant 2R_k$ whenever $|z - z_k| \leqslant \varepsilon_k$.

Writing $z_k = s_k + i t_k$, we have to consider the following situations:

1. $-t_k R_k \to -\infty$,
   (a) $R_k(1 - t_k) \to l \in [0, +\infty)$,
   (b) $R - k(1 - t_k) \to +\infty$,
2. $-t_k R_k \to -l \in (-\infty, 0]$, then $R_k(1 - t_k) \to +\infty$.

Rescaling in the same way as in the proof of Lemma 3.1, i.e.

$$\tilde{u}_k(z) = \big(a(z_k + R_k^{-1}z) - a(z_k), u(z_k + R_k^{-1}z)\big) \quad \text{for case 1(b)},$$

$$\tilde{u}_k(z) = \big(a(s_k + R_k^{-1}z) - a(s_k), u(s_k + R_k^{-1}z)\big) \quad \text{for case 2}$$

and

$$\tilde{u}_k(z) = \big(a(s_k + i - R_k^{-1}z) - a(s_k + i), u(s_k + i - R_k^{-1}z)\big) \quad \text{for case 1(a)},$$

we obtain C_{loc}^∞-bounds uniform in k, where we have to use the usual elliptic regularity estimates for $\tilde{u} \mapsto \partial_s\tilde{u} + \tilde{J}(\tilde{u})\partial_t\tilde{u}$ to obtain bounds for the higher derivatives. Again by the Ascoli–Arzela theorem a subsequence of $(\tilde{u}_k)$ converges to some nonconstant map

$$\widetilde{w} = (\beta, w) : \Omega \to \mathbf{R} \times M,$$

where $\Omega = \mathbf{C}$ in case 1b and $\Omega = H^+$ in cases 1(a) and 2. In all these cases we have

$$\partial_s\widetilde{w} + \tilde{J}(\widetilde{w})\partial_t\widetilde{w} = 0$$

and

$$\big|\nabla\widetilde{w}(z)\big| \leqslant 2.$$

In case 2, we have $\widetilde{w}(\partial H^+) \subset \mathbf{R} \times \mathcal{L}$, while we have $\widetilde{w}(\partial H^+) \subset \{0\} \times \mathcal{D}^*$ in case 1(a). Denote by Ω_k the domains of definition of the rescaled maps $\tilde{u}_k$, which are the same as in the proof of Lemma 3.1. We claim that

- $E(\widetilde{w}) \leqslant E(\tilde{u})$,
- $\int_\Omega w^* d\lambda = 0$.

We then have derived a contradiction, because $\widetilde{w}$ would have to be constant (Lemma 3.2 for case 1(a), [2] for case 2 and [12,1] for case 1(b)). So let us prove the claim above.

Considering case 1(b) first, we take $\phi \in \Sigma$ and define $\phi_k \in \Sigma$ by

$$\phi_k(s) := \phi\big(s - a(z_k)\big).$$

Then

$$\int\limits_{B_{R_k\varepsilon_k}(0)\cap\Omega_k} \tilde{u}_k^* \, d(\phi\lambda) = \int\limits_{B_{\varepsilon_k}(z_k)\cap(\mathbf{R}\times[0,1])} \tilde{u}^* \, d(\phi_k\lambda) \leqslant \int\limits_{\mathbf{R}\times[0,1]} \tilde{u}^* \, d(\phi_k\lambda) \leqslant E(\tilde{u})$$

Now choose any compact subset K of Ω and find $k_0 \in \mathbf{N}$ so that for all $k \geqslant k_0$

$$K \subset B_{R_k\varepsilon_k}(0) \cap \Omega_k.$$

Then

$$\int\limits_K \tilde{u}_k^* \, d(\phi\lambda) \leqslant E(\tilde{u}) \quad \forall k \geqslant k_0$$

and therefore

$$\int\limits_K \widetilde{w}^* \, d(\phi\lambda) \leqslant E(\tilde{u}).$$

Since this holds for all compact subsets K of Ω we obtain

$$\int\limits_\Omega \widetilde{w}^* d(\phi\lambda) \leqslant E(\tilde{u})$$

and finally taking the supremum over all $\phi \in \Sigma$:

$$E(\widetilde{w}) \leqslant E(\tilde{u}).$$

Now let K be any compact subset of Ω. Then for k large enough we have $K \subset B_{R_k \varepsilon_k}(0) \cap \Omega_k$ and

$$\int\limits_K w^* d\lambda \leqslant \left| \int\limits_K w^* d\lambda - \int\limits_K u_k^* d\lambda \right| + \int\limits_{B_{R_k \varepsilon_k}(0) \cap \Omega_k} u_k^* d\lambda$$

$$\leqslant \left| \int\limits_K w^* d\lambda - \int\limits_K v_k^* d\lambda \right| + \int\limits_{B_{\varepsilon_k}(z_k) \cap (\mathbf{R} \times [0,1])} u^* d\lambda.$$

The first term converges to zero for $k \to +\infty$, but the second one also does because of

$$\int\limits_{\mathbf{R} \times [0,1]} u^* d\lambda = \int\limits_{\mathbf{R} \times [0,1]} \tilde{u}^* d(\phi_0 \lambda) \leqslant E(\tilde{u}) < +\infty$$

where $\phi_0 \equiv 1 \in \Sigma$. This implies finally

$$\int\limits_\Omega w^* d\lambda = 0$$

because the integral vanishes over any compact subset of Ω.

In the cases 2 and 1(a) the proof of the claim above is essentially the same up to some minor modifications. We have to define

$$\phi_k(s) = \phi\big(s - a(s_k)\big) \quad \text{in case 2}$$

and

$$\Phi_k(s) = \phi\big(s - a(s_k + i)\big) \quad \text{in case 1(a).}$$

Moreover we have to replace $B_{\varepsilon_k R_k}(0)$ by $B_{\varepsilon_k R_k}(i R_k t_k)$ in case 2 and $B_{\varepsilon_k R_k}(i R_k (1 - t_k))$ in case 1(a) respectively. $\quad\square$

Proposition 3.4. *Let $\tilde{u}$ be a solution of Eq. (9). Then every sequence $(s_k')_{k \in \mathbf{N}} \subset \mathbf{R}$ satisfying $s_k' \to +\infty$ or $s_k' \to -\infty$ has a subsequence $(s_k)_{k \in \mathbf{N}}$, so that there is a point $p \in \mathcal{L}$ with*

$$\tilde{u}(s_k, t) \overset{k \to \infty}{\longrightarrow} (0, p)$$

in $C^\infty([0, 1])$.

Proof. Take any sequence (s_k') as above and define

$$\tilde{u}_k : S \to \mathbf{R} \times M$$

by

$$\tilde{u}_k(s, t) := \big(a(s + s_k', t) - a(s_k', 0), u(s + s_k', t)\big).$$

Since $\tilde{J}$ does not depend on the $\mathbf{R}$-component of $\mathbf{R} \times M$, we have

$$\partial_s \tilde{u}_k + \tilde{J}(\tilde{u}_k) \partial_t \tilde{u}_k = 0.$$

Moreover with $\tilde{u}_k = (a_k, u_k)$:

$$a_k(0, 0) = 0,$$

$$\tilde{u}_k(s, 0) \in \mathbf{R} \times \mathcal{L}$$

and

$$\tilde{u}_k(s, 1) \in \{0\} \times \mathcal{D}^*.$$

Lemma 3.3 provides a gradient bound for the maps $\tilde{u}_k$ which is uniform in k. By elliptic regularity we obtain uniform C_{loc}^∞-bounds and a subsequence of $(\tilde{u}_k)$ converges in C_{loc}^∞ to some

$$\widetilde{w} = (\beta, w) : S \to \mathbf{R} \times M$$

satisfying

$$\partial_s \widetilde{w} + \tilde{J}(\widetilde{w}) \partial_t \widetilde{w} = 0,$$

$$\widetilde{w}(s, 0) \in \mathbf{R} \times \mathcal{L},$$

$$\widetilde{w}(s, 1) \in \{0\} \times \mathcal{D}^*,$$

$$\beta(0, 0) = 0,$$

$$E(\widetilde{w}) < +\infty$$

and

$$\sup_{z \in S} \left| \nabla \widetilde{w}(z) \right| < +\infty.$$

We know that for each $R > 0$

$$\int_{[-R,R] \times [0,1]} u_k^* \, d\lambda \to \int_{[-R,R] \times [0,1]} w^* \, d\lambda$$

as $k \to \infty$. But

$$\int_{[-R,R] \times [0,1]} u_k^* \, d\lambda = \int_{[-R+s_k, R+s_k] \times [0,1]} u^* \, d\lambda \overset{k \to \infty}{\longrightarrow} 0,$$

where (s_k) is a suitable subsequence of (s_k'). This holds because $u^* d\lambda$ is a non-negative integrand and $\int_S u^* \, d\lambda \leqslant E(\tilde{u}) < +\infty$. Hence

$$\int_{[-R,R] \times [0,1]} w^* \, d\lambda = 0$$

for every $R > 0$ and therefore

$$\int_S w^* \, d\lambda = 0.$$

Lemma 3.1 implies now that $\widetilde{w}$ must be constant, i.e. $\widetilde{w} = (0, w_0)$, where $w_0 \in \mathcal{L}$ might depend on the sequence s_k' that we chose to define $\tilde{u}_k$. Hence $u(s + s_k, t) \to w_0$ in C_{loc}^∞, in particular $u(s_k, t) \to w_0$ in $C^\infty([0, 1])$. Moreover $a(s + s_k, t) - a(s_k, 0) \to 0$ in C_{loc}^∞. Choosing $t = 1$ we see from the boundary condition $a(s, 1) \equiv 0$ that $a(s_k, 0) \to 0$ and therefore $a(s_k, t) \to 0$ in $C^\infty([0, 1])$. $\square$

3.2. Existence of an asymptotic limit and exponential decay estimates

Proposition 3.4 implies that the ends of a finite energy strip $\tilde{u}$ approach the Legendrian knot $\{0\} \times \mathcal{L} \subset \mathbf{R} \times M$. We will go one step further and show that a solution of Eq. (9) has well-defined asymptotic limits. We will also show that the convergence to these asymptotic limits is of exponential nature. The special coordinates derived in Proposition 2.1 will be particularly helpful.

Proposition 3.5. *Let $\tilde{u}$ be a finite energy strip as in Eq.* (9). *Then there are points p_+, $p_- \in \mathcal{L}$ so that*

$$\tilde{u}(s,t) \stackrel{s \to \pm\infty}{\longrightarrow} (0, p_\pm)$$

in $C^\infty([0,1])$.

Before we start with the proof of Proposition 3.5, let us choose convenient coordinates. We will also confine ourselves to the 'positive end' $s \to +\infty$ since the negative end is treated in the same way. By Proposition 3.4 we can find a sequence $s_k \to +\infty$, so that $\tilde{u}(s_k, t)$ converges to some point $(0, p_+) \in \{0\} \times \mathcal{L}$ in $C^\infty([0,1])$ as $k \to \infty$ and we may describe $\tilde{u}(s,t)$ by the coordinates provided by Proposition 2.1 if $|s|$ is large enough. This is because $\tilde{u}(s,t)$ remains near the set $\{0\} \times \mathcal{L}$ for large $|s|$. Moreover, our assumptions imply that the 'ends' of u stay away from the singular points. We introduce the following change of coordinates away from the singular points:

$$\mathbf{R} \times S^1 \times \mathbf{R}^2 \ni (\tau, \theta, x, y) \mapsto \left(\tau, \theta, x - \frac{a(\theta)}{b(\theta)} y, y\right). \tag{10}$$

We recall (Proposition 2.1) that the spanning surface $\mathcal{D}$ near its boundary is parameterized by

$$\left\{(\theta, x, y) \in S^1 \times \mathbf{R}^2 \,\middle|\, \begin{pmatrix} x \\ y \end{pmatrix} = t \cdot \begin{pmatrix} a(\theta) \\ b(\theta) \end{pmatrix}, \ t \in [0,1]\right\}$$

for suitable functions $a, b : S^1 \to \mathbf{R}$, and the singular points correspond to the zeros of b.

After this coordinate change we may replace $\mathbf{R} \times \mathcal{L}$ by $\mathbf{R}^2 \times \{0\} \times \{0\}$, the set $\{0\} \times \mathcal{D}^{**}$ corresponds to $\{0\} \times \mathbf{R} \times \{0\} \times \mathbf{R}^\pm$ with $\pm = \text{sign}(b)$ and we may assume that the point $(0, p_+)$ corresponds to 0. Moreover, the contact form $\lambda = dy + x\,d\theta$ changes to

$$\hat{\lambda} = dy + \left(x + \frac{a(\theta)}{b(\theta)} y\right) d\theta, \tag{11}$$

so that the contact structure at the point (θ, x, y) is generated by

$$\frac{\partial}{\partial \theta} - \left(x + \frac{a(\theta)}{b(\theta)} y\right)\frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial x} \tag{12}$$

and the Reeb vector field changes to

$$X_{\hat{\lambda}} = \frac{\partial}{\partial y} - \frac{a(\theta)}{b(\theta)} \frac{\partial}{\partial x}.$$

Our differential equation (9) has the following form:

$$v = (\tau, \theta, x, y) : [s_0, \infty) \times [0,1] \to \mathbf{R}^4,$$

$$\partial_s v + M(v)\partial_t v = 0, \tag{13}$$

$$v(s,0) \in L_0 = \mathbf{R}^2 \times \{0\} \times \{0\},$$

$$v(s,1) \in L_1 = \{0\} \times \mathbf{R} \times \{0\} \times \mathbf{R}.$$

The number s_0 is chosen in such a way that $\tilde{u}(s,t)$ lies in the domain in $\mathbf{R} \times M$ where the above coordinates exist. The map M is smooth and bounded with values in $GL(\mathbf{R}^4)$, so that all the derivatives are bounded too and $M^2 = -\mathrm{Id}$. Because the almost complex structure $\tilde{J}$ is compatible with the 2-form ω constructed in Section 2.3, we have in addition

$$M^T \Omega M = \Omega \quad \text{and} \quad \Omega M > 0,$$

where Ω is a smooth bounded map with bounded derivatives and values in $GL(\mathbf{R}^4)$ so that $\Omega^T = -\Omega$. We also note that

$$\langle v, \Omega(q)w \rangle = 0 \tag{14}$$

for $q, v, w \in L_0$ or $q, v, w \in L_1$ since the boundary conditions $\mathbf{R} \times \mathcal{L}$ and $\{0\} \times \mathcal{D}^{**}$ are Lagrangian with respect to the 2-form ω (here $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean product on $\mathbf{R}^4$). Proposition 3.4 implies that

$$\sup_{[s,\infty) \times [0,1]} \{ |\tau|, |x|, |y| \} \to 0 \tag{15}$$

as $s \to \infty$, while we only know that

$$\big| \theta(s_k, \cdot) \big|_{C^0([0,1])} \to 0 \tag{16}$$

as $k \to \infty$. Moreover,

$$\sup_{[s,\infty) \times [0,1]} \big| \partial^\alpha v \big| \to 0 \tag{17}$$

as $s \to \infty$ for all multi indices α with $|\alpha| \geqslant 1$. Our proof of Proposition 3.5 consists of showing that the component $\theta(s,t)$ converges to zero as well uniformly in t, and it will lead also to the following exponential decay estimates:

Theorem 3.6. *There exist numbers $\rho, s' > 0$ so that we have the following estimate for each multi index $\alpha \in \mathbf{N}^2$, $|\alpha| \geqslant 0$ and $s \geqslant s'$:*

$$\sup_{t \in [0,1]} \big| \partial^\alpha v(s,t) \big| \leqslant c_\alpha e^{-\rho(s-s')},$$

where c_α are suitable positive constants.

Proof of Proposition 3.5. In the following we always assume $s \geqslant s_0$ so that our boundary value problem (9) can be written in coordinates as (13). While we proceed with the proof, it will be necessary to successively choose a larger constant s_0. We will still denote this constant by s_0.

We consider the following family of inner products on $L^2([0,1], \mathbf{R}^4)$:

$$(\gamma_1, \gamma_2)_s := \int_0^1 \big\langle \gamma_1(t), \Omega(v(s,t)) M(v(s,t)) \gamma_2(t) \big\rangle dt, \tag{18}$$

where $s \geqslant s_0$ and where $\langle \cdot, \cdot \rangle$ denotes the Euclidean product on $\mathbf{R}^4$. We will in future write $M(s,t)$ and $\Omega(s,t)$ instead of $M(v(s,t))$ and $\Omega(v(s,t))$. In view of (17) we have for all multi indices $\alpha \in \mathbf{N}^2$ with $|\alpha| \geqslant 1$

$$\big| \partial^\alpha \Omega(s,t) \big|, \big| \partial^\alpha M(s,t) \big| \to 0 \tag{19}$$

uniformly in t as s tends to $+\infty$. Then the norms $\| \cdot \|_s$ on $L^2([0,1], \mathbf{R}^4)$ induced by the products (18) are all uniformly equivalent to the usual L^2 norm $\| \cdot \|$, i.e. there are positive constant c_0, c_1 independent of s so that

$$c_0 \| \cdot \| \leqslant \| \cdot \|_s \leqslant c_1 \| \cdot \|. \tag{20}$$

Define the following dense subspace of $L^2([0, 1], \mathbf{R}^4)$:

$$H_L^{1,2}\big([0, 1], \mathbf{R}^4\big) := \big\{\gamma \in H^{1,2}\big([0, 1], \mathbf{R}^4\big) \mid \gamma(0) \in L_0, \ \gamma(1) \in L_1\big\}, \tag{21}$$

where

$$L_0 := \mathbf{R}^2 \times \{0\} \times \{0\} \quad \text{and} \quad L_1 := \{0\} \times \mathbf{R} \times \{0\} \times \mathbf{R}.$$

In view of the Sobolev embedding theorem this definition makes sense. We consider the following family of unbounded linear operators on L^2 with domain of definition $H_L^{1,2}$:

$$L^2\big([0, 1], \mathbf{R}^4\big) \supset H_L^{1,2}\big([0, 1], \mathbf{R}^4\big) \to L^2\big([0, 1], \mathbf{R}^4\big)$$
$$\big(A(s)\gamma\big)(t) := -M(s, t)\dot{\gamma}(t).$$

Since the proof of Proposition 3.5 requires some work, we break up the proof into several lemmas. The following straightforward lemma summarizes some properties of the operators $A(s)$:

Lemma 3.7. *The adjoint operator $A(s)^*$ of $A(s)$ with respect to the L^2-product* (18) *has the same domain of definition as $A(s)$ and is given by*

$$\big(A(s)^*\gamma\big)(t) = \big(A(s)\gamma\big)(t) - \big(\Theta(s)\gamma\big)(t),$$

where $\Theta(s): L^2([0, 1], \mathbf{R}^4) \to L^2([0, 1], \mathbf{R}^4)$ is the following zero-order operator:

$$\big(\Theta(s)\gamma\big)(t) := M(s, t)\Omega^{-1}(s, t)\partial_t\Omega(s, t)\gamma(t).$$

Moreover, $\Theta(s)(H^{1,2}) \subset H^{1,2}$, $\Theta(s)$ is antisymmetric and

$$\big\|\partial_s^k\Theta(s)\big\|_{\mathcal{L}(L^2, L^2)} \to 0 \quad \text{as } s \to \infty, \tag{22}$$

where $k \geqslant 0$.

Our differential equation (9) can then be written as

$$\partial_s v(s, t) = \big(A(s)v(s)\big)(t), \tag{23}$$

with $v(s) := v(s, \cdot)$. The kernel Λ of the operators $A(s)$ consists of the constant paths with image in $L_0 \cap L_1$, which is a 1-dimensional set. Let

$$P_s : L^2\big([0, 1], \mathbf{R}^4\big) \to \Lambda$$

be the orthogonal projection with respect to the inner product (18) and let

$$Q_s := \text{Id} - P_s.$$

Since the kernels of the operators $A(s)$ all agree, we have the following important property:

The operators $\partial_s Q_s, \ \partial_{ss} Q_s$ have image in Λ. \hfill (24)

The following estimate is crucial:

Lemma 3.8. *There are constants $s_0, \delta > 0$ so that for all $s \geqslant s_0$ and $\gamma \in H_L^{1,2}([0, 1], \mathbf{R}^4)$ the following inequality holds*:

$$\big\|A(s)\gamma\big\|_s \geqslant \delta\|Q_s\gamma\|_s.$$

Proof. Proceeding indirectly, we assume that there are sequences $\delta_k \searrow 0$, $s_k \nearrow \infty$ and $\gamma_k \in H_L^{1,2}([0,1], \mathbf{R}^4)$ so that

$$\left\| A(s_k)\gamma_k \right\|_{s_k} < \delta_k \| Q_{s_k}\gamma_k \|_{s_k}.$$

Consider now

$$\alpha_k = \frac{Q_{s_k}\gamma_k}{\| Q_{s_k}\gamma_k \|_{s_k}},$$

so that $0 < c_1^{-1} \leqslant \|\alpha_k\|_{L^2} \leqslant c_0^{-1}$ and

$$\|\dot{\alpha}_k\|_{L^2} \leqslant c \left\| A(s_k)\alpha_k \right\|_{s_k} < \delta_k.$$

Here we have used that the norms $\|\cdot\|_{L^2}$ and $\|\cdot\|_{s_k}$ are equivalent (20) and that the norm $\|\cdot\|_{s_k}$ is $M(s_k)$-invariant. Because the embedding $H^{1,2}([0,1], \mathbf{R}^4) \hookrightarrow L^2([0,1], \mathbf{R}^4)$ is compact, a subsequence of (α_k) converges in L^2 to some α. In view of $\dot{\alpha}_k \xrightarrow{L^2} 0$ the convergence is actually of quality $H^{1,2}$, therefore $\alpha \in H^{1,2}([0,1], \mathbf{R}^4)$ and $\dot{\alpha} = 0$, i.e. $\alpha \equiv$ const. Now $H_L^{1,2} \subset H^{1,2}$ is closed and $\alpha_k \in H_L^{1,2}([0,1], \mathbf{R}^4)$, hence $\alpha \in \Lambda = L_0 \cap L_1$.

On the other hand, we have $(\alpha_k, \alpha)_{s_k} = 0$ which leads to the contradiction

$$0 < \frac{2}{c_1} \leqslant \|\alpha_k\|_{L^2}^2 + \|\alpha\|_{L^2}^2 \leqslant \frac{1}{c_0^2}\big(\|\alpha_k\|_{s_k}^2 + \|\alpha\|_{s_k}^2\big)$$

$$= \frac{1}{c_0^2}\big(\|\alpha_k - \alpha\|_{s_k}^2\big) \leqslant \frac{c_1^2}{c_0^2}\|\alpha_k - \alpha\|_{L^2}^2 \to 0. \qquad \square$$

Let us introduce some notation which will also be useful for deriving the crucial exponential decay estimate. Fix some integer $N \geqslant 1$ and introduce the vector

$$V(s) := \big(v(s), \partial_s v(s), \ldots, \partial_s^{N-1} v(s)\big),$$

which is an element in the N-fold Cartesian product of $H_L^{1,2}([0,1], \mathbf{R}^4)$, which we will denote by $(H_L^{1,2})^N$. Applying the operator $A(s)$ to each component we obtain an operator

$$A(s) : \big(H_L^{1,2}\big)^N \to \big(L^2\big)^N$$

with

$$\ker A(s) = \Lambda^N.$$

The vector V satisfies the following partial differential equation:

$$\partial_s V(s) = A(s)V(s) + \hat{\Delta}(s)\partial_t V(s), \tag{25}$$

where

$$\hat{\Delta}(s,t) := \begin{pmatrix} 0 & 0 & 0 & \ldots & \ldots & 0 \\ \Delta_{11}(s,t) & 0 & 0 & \ldots & \ldots & 0 \\ \Delta_{22}(s,t) & \Delta_{12}(s,t) & 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ \Delta_{N-1,N-1}(s,t) & \Delta_{N-2,N-1}(s,t) & \Delta_{N-3,N-1}(s,t) & \ldots & \Delta_{1,N-1}(s,t) & 0 \end{pmatrix}$$

with

$$\Delta_{lk}(s,t) := \binom{k}{l} \partial_s^l \big(-M\big(v(s,t)\big)\big).$$

The following rather remarkable estimate is essential for the proofs of Proposition 3.5 and Theorem 3.6. The choices of the inner products in (18) and Lemma 3.7 are crucial for the proof.

Lemma 3.9. *There are numbers $s_0, \delta > 0$ so that the function*

$$g(s) := \frac{1}{2} \| Q_s V(s) \|_s^2$$

satisfies

$$g''(s) \geqslant \frac{1}{2} \delta^2 g(s).$$

Proof. We have

$$g(s) = \frac{1}{2} \int_0^1 \langle Q_s V(s)(t), \Omega(s,t) M(s,t) Q_s V(s)(t) \rangle \, dt,$$

therefore, using $(\Omega M)^T = \Omega M$,

$$g'(s) = \big(\partial_s [Q_s V(s)], Q_s V(s) \big)_s + \frac{1}{2} \int_0^1 \langle Q_s V(s)(t), \partial_s [\Omega(s,t) M(s,t)] Q_s V(s)(t) \rangle \, dt$$

and

$$
\begin{aligned}
g''(s) = {} & \big\| \partial_s [Q_s V(s)] \big\|_s^2 + \big(\partial_{ss} [Q_s V(s)], Q_s V(s) \big)_s \\
& + 2 \int_0^1 \langle \partial_s [Q_s V(s)(t)], \partial_s [\Omega(s,t) M(s,t)] Q_s V(s)(t) \rangle \, dt \\
& + \frac{1}{2} \int_0^1 \langle Q_s V(s)(t), \partial_{ss} [\Omega(s,t) M(s,t)] Q_s V(s)(t) \rangle \, dt \\
=: {} & T_1 + \cdots + T_4 \\
\geqslant {} & T_2 + T_3 + T_4.
\end{aligned}
$$

We can estimate

$$|T_4| \leqslant \varepsilon(s) \| Q_s V(s) \|_s^2, \tag{26}$$

where $0 < \varepsilon(s) \overset{s \to \infty}{\longrightarrow} 0$ is a suitable function. From now on, we will write $\varepsilon(s)$ for any positive function which decays to zero as $s \to \infty$.

Now let us estimate T_3. We have to consider $\partial_s P_s$ first. If $e \in L_0 \cap L_1$ then $P_s \gamma$ is given by

$$P_s \gamma = \frac{(\gamma, e)_s}{\| e \|_s^2} \cdot e \tag{27}$$

and

$$\big| (\partial_s P_s) \gamma \big| = \left| \frac{\frac{\partial}{\partial s}(\gamma, e)_s}{\| e \|_s^2} - 2 \frac{(\gamma, e)_s}{\| e \|_s^3} \frac{\partial}{\partial s} \| e \|_s \cdot e \right| \leqslant \varepsilon(s) \| \gamma \|_s, \tag{28}$$

since there are positive constants c_0, c_1 so that $c_0 \leqslant \|e\|_s \leqslant c_1$ for all s. Moreover,

$$\big|\partial_s\big[\Omega(s)M(s)\big]\gamma\big| = \big|\big[D\Omega(v(s))\partial_s v(s)\big]M(s)\gamma + \Omega(v(s))DM(v(s))\big[\partial_s v(s), \gamma\big]\big| \leqslant c\big|\partial_s v(s)\big||\gamma|, \tag{29}$$

where $c > 0$ is some constant. Using (28), (29),

- $\|Q_s \partial_s V(s)\|_{C^0([0,1])} \overset{s\to\infty}{\longrightarrow} 0$, which follows from (17),
- $\|V(s)\|_{L^2}$ is bounded uniformly in s and
- $\partial_s Q_s + \partial_s P_s = 0$,

we obtain

$$
\begin{aligned}
|T_3| &= \left| \int_0^1 \big\langle Q_s \partial_s V(s)(t) - [\partial_s P_s]V(s)(t), \partial_s\big[\Omega(s,t)M(s,t)\big]Q_s V(s)(t)\big\rangle dt \right| \\
&\leqslant c\big\|Q_s\partial_s V(s) - [\partial_s P_s]V(s)\big\|_{C^0([0,1])}\big\|\partial_s v(s)\big\|_{L^2}\big\|Q_s V(s)\big\|_{L^2} \\
&= \varepsilon(s)\big\|A(s)v(s)\big\|_{L^2}\big\|Q_s V(s)\big\|_{L^2} \\
&\leqslant \varepsilon(s)\big\|A(s)V(s)\big\|_{L^2}\big\|Q_s V(s)\big\|_{L^2}. \tag{30}
\end{aligned}
$$

We are now left with T_2. Shortening the notation, we write $\partial_s Q_s \gamma$ instead of $(\partial_s Q_s)\gamma$ and $Q_s\hat\Delta(s)\partial_t V(s)$ instead of $Q_s(\hat\Delta(s)\partial_t V(s))$ etc. We calculate

$$
\begin{aligned}
\partial_s(Q_s V(s)) &= \partial_s Q_s V(s) + Q_s A(s)V(s) + Q_s\hat\Delta(s)\partial_t V(s) \\
&= \partial_s Q_s V(s) + A(s)Q_s V(s) - P_s A(s)V(s) + Q_s\hat\Delta(s)\partial_t V(s)
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_{ss}(Q_s V(s)) &= \partial_{ss} Q_s V(s) + \partial_s Q_s \partial_s V(s) + A(s)\partial_s V(s) - \partial_s M(s)\partial_t V(s) \\
&\quad + \partial_s Q_s\hat\Delta(s)\partial_t V(s) + Q_s\partial_s\hat\Delta(s)\partial_t V(s) + Q_s\hat\Delta(s)\partial_{st} V(s) \\
&\quad - \partial_s P_s A(s)V(s) + P_s\partial_s M(s)\partial_t V(s) - P_s A(s)\partial_s V(s).
\end{aligned}
$$

We write the term $\hat\Delta(s)\partial_{st} V(s)$ as $\tilde\Delta(s)\partial_t V(s)$, where

$$
\tilde\Delta(s,t) := \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
0 & \Delta_{11}(s,t) & 0 & \ldots & 0 \\
0 & \Delta_{22}(s,t) & \Delta_{12}(s,t) & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \Delta_{N-1,N-1}(s,t) & \Delta_{N-2,N-1}(s,t) & \ldots & \Delta_{1,N-1}(s,t)
\end{pmatrix}.
$$

Inserting this into T_2 we obtain with (24)

$$
\begin{aligned}
T_2 &= \big(A(s)\partial_s V(s), Q_s V(s)\big)_s - \big(\partial_s M(s)\,\partial_t V(s), Q_s V(s)\big)_s \\
&\quad + \big(Q_s\partial_s\hat\Delta(s)\partial_t V(s), Q_s V(s)\big)_s + \big(Q_s\tilde\Delta(s)\partial_t V(s), Q_s V(s)\big)_s \\
&=: T_{21} + T_{22} + T_{23} + T_{24}.
\end{aligned}
$$

We estimate

$$|T_{22}| = \big|\big(\partial_s M(s)M(s)A(s)Q_s V(s), Q_s V(s)\big)_s\big| \leqslant \varepsilon(s)\big\|A(s)Q_s V(s)\big\|_s\big\|Q_s V(s)\big\|_s.$$

The expressions T_{23}, T_{24} are estimated similarly, so that

$$|T_{22}|, |T_{23}|, |T_{24}| \leqslant \varepsilon(s)\big\|A(s)Q_s V(s)\big\|_s\big\|Q_s V(s)\big\|_s. \tag{31}$$

Using $\partial_t V(s) = M(s)A(s)Q_s V(s)$, Eq. (25) and Lemma 3.7, we continue with T_{21}:

$$
\begin{aligned}
T_{21} &= \big(\partial_s V(s), A(s)Q_s V(s)\big)_s - \big(\partial_s V(s), \Theta(s)Q_s V(s)\big)_s \\
&= \big\|A(s)Q_s V(s)\big\|_s^2 + \big(\hat{\Delta}(s)\partial_t V(s), A(s)Q_s V(s)\big)_s \\
&\quad - \big(A(s)Q_s V(s), \Theta(s)Q_s V(s)\big)_s - \big(\hat{\Delta}(s)\partial_t V(s), \Theta(s)Q_s V(s)\big)_s \\
&\geqslant \big\|A(s)Q_s V(s)\big\|_s^2 - \varepsilon(s)\big\|A(s)Q_s V(s)\big\|_s^2 - \varepsilon(s)\big\|A(s)Q_s V(s)\big\|_s\big\|Q_s V(s)\big\|_s \\
&\geqslant \frac{1}{2}\big\|A(s)Q_s V(s)\big\|_s^2 - \varepsilon(s)\big\|A(s)Q_s V(s)\big\|_s\big\|Q_s V(s)\big\|_s \quad \text{for large } s.
\end{aligned}
$$

Using Lemma 3.8, inequalities (26), (30), (31) and the above estimate for T_{21}, we obtain

$$
\begin{aligned}
g''(s) &\geqslant T_{21} - |T_{22}| - |T_{23}| - |T_{24}| - |T_3| - |T_4| \\
&\geqslant \frac{1}{2}\big\|A(s)Q_s V(s)\big\|_s^2 - \varepsilon(s)\big\|A(s)Q_s V(s)\big\|_s\big\|Q_s V(s)\big\|_s - \varepsilon(s)\big\|Q_s V(s)\big\|_s^2 \\
&= \big\|A(s)Q_s V(s)\big\|_s\left(\frac{1}{2}\big\|A(s)Q_s V(s)\big\|_s - \varepsilon(s)\big\|Q_s V(s)\big\|_s\right) - \varepsilon(s)\big\|Q_s V(s)\big\|_s^2 \\
&\geqslant \big\|A(s)Q_s V(s)\big\|_s\big\|Q_s V(s)\big\|_s\left(\frac{\delta}{2} - \varepsilon(s)\right) - \varepsilon(s)\big\|Q_s V(s)\big\|_s^2 \\
&\geqslant \left(\frac{\delta^2}{3} - \varepsilon(s)\right)\big\|Q_s V(s)\big\|_s^2 \quad \text{where } s \text{ is so large that } \frac{\delta}{2} - \varepsilon(s) \geqslant \frac{\delta}{3} \\
&\geqslant \frac{\delta^2}{4}\big\|Q_s V(s)\big\|_s^2 \quad s \text{ so large that } \frac{\delta^2}{3} - \varepsilon(s) \geqslant \frac{\delta^2}{4} \\
&= \frac{\delta^2}{2}g(s).
\end{aligned}
$$

This completes the proof of Lemma 3.9. $\quad\square$

Lemma 3.10. *Let s_0, δ be as in Lemma 3.9. Then we have for all $s \geqslant s_1 \geqslant s_0$*

$$
g(s) \leqslant g(s_1)e^{-\frac{\delta}{\sqrt{2}}(s-s_1)}.
$$

Proof. Defining $h(s) := g(s) - g(s_1)e^{-\frac{\delta}{\sqrt{2}}(s-s_1)}$, we observe that $h(s_1) = 0$ and $h''(s) \geqslant \frac{\delta^2}{2}h(s)$ in view of Lemma 3.9, hence h cannot have a local maximum with $h > 0$. On the other hand, we also have $h(s) \to 0$ as $s \to \infty$ in view of $g''(s) \to 0$ and Lemma 3.9. We conclude $h \leqslant 0$ which proves the lemma. $\quad\square$

We now have to estimate $|P_s v(s)|$ and $|P_s \partial_s v(s)|$, the components of $v(s)$ and $\partial_s v(s)$ along $\Lambda = \ker A(s)$.

Lemma 3.11. *If $s \geqslant s_0$ then*

$$
\big|P_s \partial_s v(s)\big| \leqslant \varepsilon(s)\big\|Q_s v(s)\big\|_s,
$$

where $0 < \varepsilon(s) \to 0$ as $s \to \infty$.

Proof. We compute using (20), (23), (27) and Lemma 3.7

$$
\begin{aligned}
\big|P_s \partial_s v(s)\big| &\leqslant c\big|(\partial_s v(s), e)_s\big| = c\big|(A(s)v(s), e)_s\big| = c\big|(A(s)Q_s v(s), e)_s\big| \\
&= c\big|(Q_s v(s), -\Theta(s)e)_s\big| \leqslant \varepsilon(s)\big\|Q_s v(s)\big\|_s. \quad\square
\end{aligned}
$$

Proof of Proposition 3.5 (*continued*). We will show now that

$$\big|P_s v(s)\big| \to 0$$

as $s \to \infty$. Then we are done because

$$\big\|v(s)\big\|_{L^2} \leqslant \big|P_s v(s)\big| + \big\|Q_s v(s)\big\|_{L^2} \overset{s\to\infty}{\longrightarrow} 0$$

and

$$\big\|\partial_s v(s)\big\|_{L^2} \overset{s\to\infty}{\longrightarrow} 0,$$

i.e. $v(s)$ converges to zero in $H^{1,2}([0,1])$ and therefore also in $C^0([0,1])$ by the Sobolev embedding theorem.

In view of Eqs. (20), (27) we have to show that

$$\big|(v(s), e)_s\big| \to 0$$

as $s \to \infty$. We know already that

$$\big|(v(s_k), e)_{s_k}\big| \leqslant \big\|v(s_k)\big\|_{s_k} \|e\|_{s_k} \leqslant c\big\|v(s_k)\big\|_{L^2} \overset{k\to\infty}{\longrightarrow} 0.$$

We estimate for $s \geqslant s_k$, combining Lemma 3.10 and Lemma 3.11, with c being a generic constant independent of k and s

$$\big|(v(s), e)_s - (v(s_k), e)_{s_k}\big| = \left|\int_{s_k}^s \frac{d}{d\sigma}(v(\sigma), e)_\sigma \, d\sigma\right|$$

$$\leqslant c\int_{s_k}^s \big\|\partial_\sigma v(\sigma)\big\|_\sigma \, d\sigma + \int_{s_k}^s \int_0^1 \big|\langle v(\sigma, t), \partial_\sigma[\Omega(v(\sigma, t))M(v(\sigma, t))] \cdot e\rangle\big| \, dt \, d\sigma$$

$$\leqslant c\big\|\partial_s v(s_k)\big\|_{s_k} \int_{s_k}^s e^{-\frac{\delta}{2\sqrt{2}}(\sigma - s_k)} \, d\sigma + c\int_{s_k}^s \big\|v(\sigma)\big\|_{L^2([0,1])} \big\|\partial_\sigma v(\sigma)\big\|_{L^2([0,1])} \, d\sigma$$

$$\leqslant c\big\|\partial_s v(s_k)\big\|_{s_k} \big(1 - e^{-\frac{\delta}{2\sqrt{2}}(s - s_k)}\big)$$

$$\leqslant c\big\|\partial_s v(s_k)\big\|_{s_k},$$

which converges to zero if $k \to \infty$. This completes the proof of Proposition 3.5. □

Proof of Theorem 3.6. We saw earlier that Lemmas 3.10 and 3.11 imply

$$\big\|\partial_s v(s)\big\|_{L^2} \leqslant ce^{-\frac{\delta}{2\sqrt{2}}(s - s_0)}$$

for all $s \geqslant s_0$, where $c, s_0 > 0$ are suitable constants. In view of $\partial_s v(s, t) + M(v(s, t))\partial_t v(s, t) = 0$ we also have

$$\big\|\partial_t v(s)\big\|_{L^2} \leqslant ce^{-\frac{\delta}{2\sqrt{2}}(s - s_0)}$$

for a suitable positive constant c. Note that $v(s) = -\int_s^{+\infty} \partial_s v(\sigma) \, d\sigma$ so that

$$\big\|v(s)\big\|_{L^2} \leqslant \int_s^{+\infty} \big\|\partial_s v(\sigma)\big\|_{L^2} \, d\sigma \leqslant c\int_s^{+\infty} e^{-\frac{\delta}{2\sqrt{2}}(\sigma - s_0)} \, d\sigma = \frac{2c\sqrt{2}}{\delta} e^{-\frac{\delta}{2\sqrt{2}}(s - s_0)}.$$

Hence we know already that $\|\partial^\alpha v(s)\|_{L^2}$ decays exponentially fast with rate at least $\rho = -\frac{\delta}{2\sqrt{2}}$ whenever $|\alpha| \leqslant 1$. Because of the Sobolev embedding theorem we obtain exponential decay for $\sup_{0 \leqslant t \leqslant 1} |v(s)|$ as well. We have to use induction to obtain the same decay behavior for the higher derivatives of v. Recalling that we defined

$$V(s) = \bigl(v(s), \partial_s v(s), \ldots, \partial_s^{N-1} v(s)\bigr), \quad N \geqslant 1,$$

we know that $\|Q_s \partial_s^k v(s)\|_{L^2}$ exhibits the desired exponential decay for any integer k. Assume that $\|V(s)\|_{L^2}$ decays exponentially with rate ρ as above (we know that this is true for $N = 2$). We claim that then $\|\partial_s V(s)\|_{L^2}$ and $\|\partial_t V(s)\|_{L^2}$ have to decay exponentially with the same rate as well. Applying Q_s to Eq. (25) and multiplying with $M(s)$ we obtain

$$\partial_t V(s) = M(s) Q_s \partial_s V(s) + M(s) P_s A(s) V(s) - M(s) Q_s \hat{\Delta}(s) \partial_t V(s),$$

which implies

$$\bigl\|\partial_t V(s)\bigr\|_{L^2} \leqslant c \bigl\|Q_s \partial_s V(s)\bigr\|_{L^2} + c \bigl|P_s A(s) V(s)\bigr| + \varepsilon(s) \bigl\|\partial_t V(s)\bigr\|_{L^2},$$

i.e. for s so large that $\varepsilon(s) \leqslant 1/2$

$$\bigl\|\partial_t V(s)\bigr\|_{L^2} \leqslant 2c \bigl\|Q_s \partial_s V(s)\bigr\|_{L^2} + 2c \bigl|P_s A(s) V(s)\bigr|. \tag{32}$$

The expression $\|Q_s \partial_s V(s)\|_{L^2}$ decays exponentially by Lemma 3.10 and the other also does because of

$$\bigl|P_s A(s) V(s)\bigr| \leqslant c \bigl|\bigl(A(s) V(s), e\bigr)_s\bigr| \leqslant c \bigl|\bigl(V(s), \Theta(s) e\bigr)_s\bigr| \leqslant \varepsilon(s) \bigl\|V(s)\bigr\|_{L^2},$$

where $e \in \Lambda^N$. This proves our claim, i.e. we have now shown exponential decay for $\|\partial_t^k \partial_s^l v(s)\|_{L^2}$, where $k \in \{0, 1\}$ and $l \geqslant 0$ is an arbitrary integer.

Eq. (25) yields

$$\partial_t V(s) = \bigl(\mathrm{Id} + M(s) \hat{\Delta}(s)\bigr)^{-1} M(s) \partial_s V(s)$$

(the inverse makes sense if s is sufficiently large), and differentiating the above identity successively by t shows by induction that $\|\partial_t^k V(s)\|_{L^2}$ decays exponentially for arbitrary integers k. The desired decay for the C^0 norm then follows from the Sobolev embedding theorem. $\quad\square$

3.3. An asymptotic formula

We need to know more about the asymptotic behavior of the solutions than merely the apriori estimate in Theorem 3.6. The aim is to prove the asymptotic formula (Theorem 1.1):

Theorem 3.12. *For sufficiently large s_0 and $s \geqslant s_0$ we have the following asymptotic formula for non constant solutions v of* (13) *having finite energy*:

$$v(s, t) = e^{\int_{s_0}^s \alpha(\tau)\,d\tau}\bigl(e(t) + r(s, t)\bigr), \tag{33}$$

where $\alpha : [s_0, \infty) \to \mathbf{R}$ is a smooth function satisfying $\alpha(s) \to \lambda < 0$ as $s \to \infty$ with λ being an eigenvalue of the selfadjoint operator

$$A_\infty : L^2\bigl([0, 1], \mathbf{R}^4\bigr) \supset H_L^{1,2}\bigl([0, 1], \mathbf{R}^4\bigr) \to L^2\bigl([0, 1], \mathbf{R}^4\bigr)$$
$$\gamma \mapsto -M_\infty \dot{\gamma}, \quad M_\infty := \lim_{s \to \infty} M\bigl(v(s, t)\bigr)$$

(see (21) *for the definition of the domain of A_∞). Moreover, $e(t)$ is an eigenvector of A_∞ belonging to the eigenvalue λ with $e(t) \neq 0$ for all $t \in [0, 1]$, and r is a smooth function so that r and all its derivatives converge to zero uniformly in t as $s \to \infty$.*

Remark. The above theorem is of course also valid for the negative end, $s \to -\infty$, of a solution. We have the same formula as in (33), but the function $\alpha(s)$ will converge to a positive eigenvalue of the operator $A_\infty = -M_\infty \frac{d}{dt}$, where $M_\infty = \lim_{s \to -\infty} M(v(s, t))$.

The first step in the proof is the following proposition. The steps from Proposition 3.13 below to Theorem 1.1 are very similarly to the corresponding results in [2] or [13].

Proposition 3.13. *There is a number $s_0 > 0$ so that*

$$\|v(s)\|_s = e^{\int_{s_0}^s \alpha(\tau)\, d\tau} \|v(s_0)\|_{s_0}$$

for all $s \geqslant s_0$, where α has the properties stated above in Theorem 1.1.

Before we can continue with the proof, we need some information about the spectra of the selfadjoint operators $A(s) - \frac{1}{2}\Theta(s)$.

Theorem 3.14. *For each $L > 0$ there are numbers $d, s_1 > 0$ and a sequence $r_n \in [nL, (n+1)L]$, $n \in \mathbf{Z}$ so that*

$$[r_n - d, r_n + d] \cap \sigma\left(A(s) - \frac{1}{2}\Theta(s)\right) = \emptyset$$

for all $s \geqslant s_1$.

Proof. Let us review the strategy of the proof: We want to view $A(s) - \frac{1}{2}\Theta(s)$ as a perturbation of A_∞, the operator obtained for $s \to \infty$. There are theorems about the spectrum of selfadjoint operators in a Hilbert space perturbed by bounded symmetric operators. The trouble here is that $A_\infty - A(s) + \frac{1}{2}\Theta(s)$ is not a bounded operator. We fix this by introducing operators B_∞ and $B(s)$, all having the same first order term, and which are unitary equivalent to the operators A_∞ and $A(s)$ so that it suffices to study the spectra of B_∞ and $B(s)$.

We would like to find a smooth map

$$T : [s_0, \infty) \times [0, 1] \to \mathrm{GL}(\mathbf{R}^4)$$

so that $T(s, .)$ converges in $C^\infty([0, 1])$ to some $T_\infty \in \mathrm{GL}(\mathbf{R}^4)$ satisfying the following conditions:

- $T^t T = \Omega M$,
- $T M = J_0 T$,
- $T^t J_0 T = -\Omega$,

with corresponding conditions for T_∞ as $s \to \infty$. Here T^t denotes the transpose of T and J_0 is multiplication by i on $\mathbf{C}^2$ if we identify $\mathbf{R}^4$ with $\mathbf{C}^2$. Actually two of the above conditions imply the third one. We may view the map T as a unitary trivialization of the hermitian vector bundle

$$\big(([s_0, \infty) \times [0, 1]) \times \mathbf{R}^4, \Omega, M\big) \overset{\sim}{\to} \big(([s_0, \infty) \times [0, 1]) \times \mathbf{R}^4, -J_0, J_0\big).$$

The construction of T is Gram–Schmidt orthogonalization with respect to the hermitian bundle metric

$$h = \langle \cdot, \Omega M . \rangle + i \langle \cdot, \Omega . \rangle.$$

We define $T(s, t)$ by mapping the generator

$$\sigma(s, t) := \frac{\partial}{\partial \theta} - \left(x(s, t) + \frac{a(\theta(s, t))}{b(\theta(s, t))} y(s, t)\right) \frac{\partial}{\partial y}$$

of the contact structure (12) in $u(s, t)$ onto $(0, 1) \in \mathbf{C}^2$. Consequently, the maps

$$\Phi_s : \left(L^2([0, 1], \mathbf{R}^4), (\cdot, \cdot)_s\right) \to \left(L^2([0, 1], \mathbf{R}^4), (\cdot, \cdot)_{L^2}\right),$$

$$\gamma \mapsto T(s, \cdot)\gamma,$$

$$\Phi_\infty : \left(L^2([0, 1], \mathbf{R}^4), (\cdot, \cdot)_s|_{s \to \infty}\right) \to \left(L^2([0, 1], \mathbf{R}^4), (\cdot, \cdot)_{L^2}\right),$$

$$\gamma \mapsto T_\infty \gamma$$

are isometries. They map $H_L^{1,2}([0, 1], \mathbf{R}^4)$ onto

$$H_{L_s}^{1,2}([0, 1], \mathbf{R}^4) := \left\{\gamma \in H^{1,2}([0, 1], \mathbf{R}^4) \;\middle|\; \begin{array}{l} \gamma(0) \in T(s, 0) \cdot L_0 \\ \gamma(1) \in T(s, 1) \cdot L_1 \end{array}\right\}$$

and

$$H_{L_\infty}^{1,2}([0, 1], \mathbf{R}^4) := \left\{\gamma \in H^{1,2}([0, 1], \mathbf{R}^4) \;\middle|\; \begin{array}{l} \gamma(0) \in T_\infty \cdot L_0 \\ \gamma(1) \in T_\infty \cdot L_1 \end{array}\right\}$$

respectively. We consider the following operators

$$\widetilde{B}(s) : L^2([0, 1], \mathbf{R}^4) \supset H_{L_s}^{1,2}([0, 1], \mathbf{R}^4) \to L^2([0, 1], \mathbf{R}^4),$$

$$\widetilde{B}(s) := \Phi_s \circ \left(A(s) - \frac{1}{2}\Theta(s)\right) \circ \Phi_s^{-1},$$

$$B_\infty : L^2([0, 1], \mathbf{R}^4) \supset H_{L_\infty}^{1,2}([0, 1], \mathbf{R}^4) \to L^2([0, 1], \mathbf{R}^4),$$

$$B_\infty := \Phi_\infty \circ A_\infty \circ \Phi_\infty^{-1},$$

where we equip $L^2([0, 1], \mathbf{R}^4)$ with the ordinary L^2-inner product $(\cdot, \cdot)_{L^2}$. Unitary equivalent selfadjoint operators have the same spectrum, hence

$$\sigma\left(\widetilde{B}(s)\right) = \sigma\left(A(s) - \frac{1}{2}\Theta(s)\right)$$

and

$$\sigma(B_\infty) = \sigma(A_\infty).$$

It remains to investigate the spectra of $\widetilde{B}(s)$ and B_∞. First we note that the operators $\widetilde{B}(s)$ and B_∞ are selfadjoint with respect to the standard L^2-product. Let us compute them. We obtain

$$\widetilde{B}(s) = -J_0 \frac{\partial}{\partial t} + J_0 \frac{\partial T(s)}{\partial t} T(s)^{-1} - \frac{1}{2} T(s)\Theta(s)T(s)^{-1}, \tag{34}$$

where the operator $-J_0 \frac{\partial}{\partial t}$ is selfadjoint and the operator

$$S(s) : L^2([0, 1], \mathbf{R}^4) \to L^2([0, 1], \mathbf{R}^4)$$

$$\gamma \mapsto J_0 \frac{\partial T(s)}{\partial t} T(s)^{-1}\gamma - \frac{1}{2} T(s)\Theta(s)T(s)^{-1}\gamma$$

is symmetric. We note that $S(s)$ converges to zero as $s \to \infty$ in the operator norm. The operator B_∞ is simply given by $-J_0 \frac{\partial}{\partial t}$.

Summarizing, we have introduced coordinates so that the operators $A(s) - \frac{1}{2}\Theta(s)$ and A_∞ correspond to operators with the same first order term on the same Hilbert space $(L^2([0, 1], \mathbf{R}^n), (\cdot, \cdot)_{L^2})$, but they all have different domains of definition. We have to fix this without changing anything that we have achieved so far.

We can find a smooth map

$$C : [s_0, \infty) \times [0, 1] \to \mathrm{Sp}(4) \cap \mathrm{O}(4) = \mathrm{U}(2)$$

having the following properties:

- $C(s, .) \to \mathrm{Id}$ in $C^\infty([0, 1])$ as $s \to \infty$,
- $C(s, 0)T(s, 0)L_0 = T_\infty L_0$,
- $C(s, 1)T(s, 1)L_1 = T_\infty L_1$.

The operators

$$B(s) : L^2\big([0, 1], \mathbf{R}^4\big) \supset H^{1,2}_{L_\infty}\big([0, 1], \mathbf{R}^4\big) \to L^2\big([0, 1], \mathbf{R}^4\big)$$

$$\big(B(s)\gamma\big)(t) := C(s, t)^{-1}\big(\widetilde{B}(s)C(s)\gamma\big)(t)$$

have the form

$$B(s) = B_\infty + \Delta(s, t),$$

where $\gamma \mapsto \Delta(s)\gamma$ is a symmetric zero order perturbation with $\|\Delta(s)\| \to 0$ as $s \to \infty$ in the operator norm. They are unitary equivalent to $\widetilde{B}(s)$ hence the spectra are the same. The spectrum of the operator B_∞, which has domain of definition $H^{1,2}_{L_\infty}([0, 1], \mathbf{R}^4)$ consists of all integer multiples of $\pi/2$. Moreover, the spectrum consists of eigenvalues only since the resolvent of B_∞ is a compact operator. Every eigenvalue has multiplicity one. Verifying this is a straight forward computation which we leave to the reader. Let us summarize our discussion as follows:

Proposition 3.15. *The spectrum of the operator*

$$A_\infty : L^2\big([0, 1], \mathbf{R}^4\big) \supset H^{1,2}_L\big([0, 1], \mathbf{R}^4\big) \to L^2\big([0, 1], \mathbf{R}^4\big),$$

$$\gamma \mapsto -M_\infty \dot{\gamma}, \quad M_\infty := \lim_{s \to \infty} M\big(v(s, t)\big)$$

consists of all integer multiples of $\frac{\pi}{2}$. The resolvent of the operator A_∞ is a compact operator on $L^2([0, 1], \mathbf{R}^4)$. All the points in the spectrum are eigenvalues of multiplicity one.

In order to control the spectra of the perturbations $B(s)$ we will need the following perturbation result (see [2]) which follows from a result of T. Kato (see [16]):

Theorem 3.16. *Let $T : H \supset D(T) \to H$ be a selfadjoint operator in a Hilbert space H and let $A_0 : H \to H$ be a linear, bounded and symmetric operator. Then the following holds:*

-

$$\mathrm{dist}\big(\sigma(T), \sigma(T + A_0)\big) := \max\Big\{ \sup_{\lambda \in \sigma(T)} \mathrm{dist}\big(\lambda, \sigma(T + A_0)\big), \sup_{\lambda \in \sigma(T+A_0)} \mathrm{dist}\big(\lambda, \sigma(T)\big) \Big\}$$

$$\leqslant \|A_0\|_{\mathcal{L}(H)}.$$

- *Assume further that the resolvent $(T - \lambda_0)^{-1}$ of T exists and is compact for some $\lambda_0 \notin \sigma(T)$.*
  Then $(T - \lambda)^{-1}$ is compact for every $\lambda \notin \sigma(T)$ and $\sigma(T)$ consists of isolated eigenvalues $\{\mu_k\}_{k \in \mathbf{Z}}$ with finite multiplicities $\{m_k\}_{k \in \mathbf{Z}}$.
  If we assume that $\sup_{k \in \mathbf{Z}} m_k \leqslant M < \infty$ and that for each $L > 0$ there is a number $m_T(L) \in \mathbf{N}$ so that every interval $I \subset \mathbf{R}$ of length L contains at most $m_T(L)$ points of $\sigma(T)$ (counted with multiplicity) then for each

$L > 0$ *there is also a number* $m_{T+A_0}(L) \in \mathbf{N}$ *so that every interval* $I \subset \mathbf{R}$ *of length* L *contains at most* $m_{T+A_0}(L)$ *points of* $\sigma(T + A_0)$.

We find for all $L > 0$ some $m \in \mathbf{N}$ so that every interval $I \subseteq \mathbf{R}$ of length L contains at most m points of the spectrum of B_∞.

Moreover by Theorem 3.16,

$$\text{dist}\big(\sigma(B_\infty), \sigma(B(s))\big) \to 0 \tag{35}$$

as $s \to \infty$.

Define now the intervals

$$I_n := \big[nL, (n+1)L\big]; \quad n \in \mathbf{Z}.$$

Then each I_n contains at most m points of $\sigma(B_\infty)$, so there is a closed subinterval $J_n \subset I_n$ of length $\frac{L}{m+1}$ that does not contain any point of $\sigma(B_\infty)$. Because of (35) there is a closed interval $J_n' \subseteq J_n \subseteq I_n$ of length $\frac{L}{2(m+1)}$ which does not contain any point of $\sigma(B(s))$ whenever $s \geqslant s_1$ where s_1 is sufficiently large (this s_1 does not depend on n).

So we found a sequence $r_n \in I_n$ and a positive constant d, so that

$$[r_n - d, r_n + d] \cap \sigma\big(B(s)\big) = \emptyset$$

for all large s. This completes the proof of Theorem 3.14. $\quad\square$

Proof of Proposition 3.13. This result has an analogue in [2] and [13,14]. However, there are some different features due to the boundary condition and the degeneracy of the problem. We assume first that $\|v(s, .)\|_{C^0([0,1))} \neq 0$ if s is sufficiently large. As in the references cited above, it is very easy to state the correct function α so that we have the proposed formula for $\|v(s)\|_s$. Indeed, we have to take

$$\alpha(s) := \frac{\frac{d}{ds}\|v(s)\|_s^2}{2\|v(s)\|_s^2}.$$

We define now

$$\xi(s, t) := \frac{v(s, t)}{\|v(s)\|_s}$$

and note that

$$\partial_s \xi(s, t) + M(s, t)\partial_t \xi(s, t) + \alpha(s)\xi(s, t) = 0. \tag{36}$$

We define

$$\Gamma_1(s, t) := -\frac{1}{2} M(s, t)\Omega^{-1}(s, t)\partial_s(\Omega M)(s, t)$$

and the covariant derivative

$$\nabla_s \xi(s) := \partial_s \xi(s) + \Gamma_1(s)\xi(s)$$

so that for all smooth $u_1, u_2 : \mathbf{R} \to L^2([0, 1], \mathbf{R}^4)$

$$\frac{d}{ds}\big(u_1(s), u_2(s)\big)_s = \big(\nabla_s u_1(s), u_2(s)\big)_s + \big(u_1(s), \nabla_s u_2(s)\big)_s,$$

hence

$$0 = \big(\nabla_s \xi(s), \xi(s)\big)_s. \tag{37}$$

The partial differential equation for ξ can be written in the form

$$A(s)\xi(s) = \nabla_s \xi(s) + \alpha(s)\xi(s) - \Gamma_1(s)\xi(s) \tag{38}$$

which implies

$$\alpha(s) = \big(\xi(s), \Gamma_1(s)\xi(s)\big)_s + \big(\xi(s), A(s)\xi(s)\big)_s. \tag{39}$$

We define

$$\Gamma_2(s,t) := -M(s,t)\Omega^{-1}(s,t)\partial_s\Omega(s,t)M(s,t).$$

and

$$\Gamma_3(s,t) := \Omega^{-1}(s,t)\partial_s\Omega(s,t).$$

Computing the adjoint operators Γ_1^* and Γ_2^* with respect to the inner product (18) yields

$$\Gamma_1^* = \Gamma_1 \quad \text{and} \quad \Gamma_2^* = \Gamma_3.$$

Introducing the operator

$$\Gamma_4(s)\xi(s) := \big(\nabla_s M(s)\big)M(s)\xi(s) := -M(s)\nabla_s\big(M(s)\xi(s)\big) - \nabla_s\xi(s) = \frac{1}{2}(\Gamma_3 - \Gamma_2),$$

we find that

$$\Gamma_4^*(s) = \frac{1}{2}(\Gamma_3^* - \Gamma_2^*) = -\Gamma_4.$$

A simple calculation shows also that

$$\partial_t\nabla_s - \nabla_s\partial_t = \partial_t\Gamma_1. \tag{40}$$

Using now the partial differential equation (38), Lemma 3.7, Eqs. (22), (37), (40) and the fact that $\|\Gamma_k(s)\xi(s)\|_s^2 \to 0$, $k = 1, \ldots, 4$, as $s \to \infty$ we estimate the derivative of α as follows:

$$\begin{aligned}
\alpha'(s) = {} & \big(\nabla_s(A(s)\xi(s)), \xi(s)\big)_s + \big(A(s)\xi(s), \nabla_s\xi(s)\big)_s \\
& + \big(\nabla_s\big(\Gamma_1(s)\xi(s)\big), \xi(s)\big)_s + \big(\Gamma_1(s)\xi(s), \nabla_s\xi(s)\big)_s \\
=: {} & T_1 + T_2 + T_3 + T_4.
\end{aligned}$$

We have

$$|T_4| \leqslant \varepsilon(s)\big\|\nabla_s\xi(s)\big\|_s$$

and

$$|T_3| \leqslant \big|\big((\nabla_s\Gamma_1(s))\xi(s), \xi(s)\big)_s\big| + \big|\big(\Gamma_1(s)\nabla_s\xi(s), \xi(s)\big)_s\big| \leqslant \varepsilon(s) + \varepsilon(s)\big\|\nabla_s\xi(s)\big\|_s.$$

Inserting (38) and using (37), we obtain

$$\begin{aligned}
T_2 = {} & \big\|\nabla_s\xi(s)\big\|_s^2 - \big(\Gamma_1(s)\xi(s), \nabla_s\xi(s)\big)_s \\
\geqslant {} & \big\|\nabla_s\xi(s)\big\|_s^2 - \varepsilon(s)]\big\|\nabla_s\xi(s)\big\|_s.
\end{aligned}$$

We now take care of the term T_1:

$$
\begin{aligned}
T_1 &= -\big(M(s)\nabla_s\big(M(s)A(s)\xi(s)\big), \xi(s)\big)_s - \big(\Gamma_4(s)A(s)\xi(s), \xi(s)\big)_s \\
&= \big(-M(s)\nabla_s\partial_t\xi(s), \xi(s)\big)_s + \big(A(s)\xi(s), \Gamma_4(s)\xi(s)\big)_s \\
&= \big(A(s)\nabla_s\xi(s), \xi(s)\big)_s + \big(M(s)\partial_t\Gamma_1(s)\xi(s), \xi(s)\big)_s + \big(A(s)\xi(s), \Gamma_4(s)\xi(s)\big)_s \\
&= \big(\nabla_s\xi(s), A(s)\xi(s)\big)_s - \big(\nabla_s\xi(s), \Theta(s)\xi(s)\big)_s \\
&\quad + \big(M(s)\partial_t\Gamma_1(s)\xi(s), \xi(s)\big)_s + \big(\nabla_s\xi(s), \Gamma_4(s)\xi(s)\big)_s \\
&\quad + \alpha(s)\big(\xi(s), \Gamma_4(s)\xi(s)\big)_s - \big(\Gamma_1(s)\xi(s), \Gamma_4(s)\xi(s)\big)_s \\
&=: T_{11} + \cdots + T_{16}.
\end{aligned}
$$

The term T_{11} is identical with T_2 which we estimated above. The expressions $|T_{12}|$, $|T_{14}|$ can be estimated from above by $\varepsilon(s)\|\nabla_s\xi(s)\|_s$ while $|T_{13}|$ and $|T_{16}|$ tend to zero as $s \to \infty$. The term T_{15} vanishes since Γ_4 is skew-adjoint. Summarizing, we got the following inequality for the derivative of α:

$$
\begin{aligned}
\alpha'(s) &\geqslant T_1 + T_2 - |T_3| - |T_4| \\
&\geqslant 2\|\nabla_s\xi(s)\|_s^2 - \varepsilon(s)\|\nabla_s\xi(s)\|_s - \varepsilon(s).
\end{aligned}
\tag{41}
$$

We assume now that the function α is not bounded from above and we wish to derive a contradiction. Then we can find a sequence $s_k \to \infty$ so that $\alpha(s_k) \to \infty$. If we had $\alpha(s) \geqslant \eta > 0$ for all large s and some positive number η then we would obtain

$$
\|v(s)\|_{L^2} \geqslant c\|v(s)\|_s \geqslant e^{\eta(s-s_0)}\|v(s_0)\|_{s_0} \to \infty
$$

in contradiction to the fact that $|v(s,\cdot)| \to 0$ uniformly in t as $s \to \infty$. Because of Theorem 3.14 we may pick $\eta > 0$ in such a way that there is a positive number d so that $\eta - d > 0$ and

$$
[\eta - d, \eta + d] \cap \sigma\left(A(s) - \frac{1}{2}\Theta(s)\right) = \emptyset.
$$

Then we can find a sequence $s_k' \to \infty$ so that $\alpha(s_k') < \eta$. We may also assume that $s_k' < s_{k+1} < s_{k+1}'$ and $\alpha(s_k) > \eta$. Hence, if α is not bounded from above then it must oscillate. Let $\hat{s}_k$ be the smallest number with $\hat{s}_k > s_k$ and $\alpha(\hat{s}_k) = \eta$. Since the operators $A(s) - \frac{1}{2}\Theta(s)$ are selfadjoint we have for every θ in the resolvent set

$$
\left\|\left(A(s) - \frac{1}{2}\Theta(s) - \theta\,\mathrm{Id}\right)^{-1}\right\|_s = \frac{1}{\mathrm{dist}(\theta, \sigma(A(s) - \frac{1}{2}\Theta(s)))}.
\tag{42}
$$

Recalling the differential equation (38) for ξ, we obtain (ε_k being a suitable sequence of positive numbers converging to zero)

$$
\begin{aligned}
1 &= \|\xi(\hat{s}_k)\|_{\hat{s}_k} \\
&\leqslant \left\|\left(A(\hat{s}_k) - \frac{1}{2}\Theta(\hat{s}_k) - \eta\,\mathrm{Id}\right)^{-1}\right\|_{\hat{s}_k} \left\|\nabla_s\xi(\hat{s}_k) - \Gamma_1(\hat{s}_k)\xi(\hat{s}_k) - \frac{1}{2}\Theta(\hat{s}_k)\xi(\hat{s}_k)\right\|_{\hat{s}_k} \\
&\leqslant \frac{1}{d}\left\|\nabla_s\xi(\hat{s}_k) - \Gamma_1(\hat{s}_k)\xi(\hat{s}_k) - \frac{1}{2}\Theta(\hat{s}_k)\xi(\hat{s}_k)\right\|_{\hat{s}_k} \\
&\leqslant \frac{1}{d}\|\nabla_s\xi(\hat{s}_k)\|_{\hat{s}_k} + \varepsilon_k,
\end{aligned}
$$

i.e. for sufficiently large k

$$
0 < \frac{d}{2} \leqslant \|\nabla_s\xi(\hat{s}_k)\|_{\hat{s}_k}.
\tag{43}
$$

We now insert this into inequality (41) and obtain that for sufficiently large k

$$\alpha'(\hat{s}_k) > 0,$$

which would imply $\alpha(s) < \eta$ for $s < \hat{s}_k$ close to $\hat{s}_k$ in contradiction to the definition of $\hat{s}_k$. Hence α must be bounded from above.

Let us show now that α cannot be unbounded from below either. Pick a sequence r_n as in Theorem 3.14. Assuming in the contrary that α is not bounded from below we can find s_n so that $\alpha(s_n) = r_n$ and $\alpha'(s_n) < 0$. In the same way as we derived (43), we also obtain here

$$0 < \frac{d}{2} \leqslant \left\| \nabla_s \xi(s_n) \right\|_{s_n}$$

for all large n and

$$\alpha'(s_n) > 0$$

which is a contradiction. Therefore α must also be bounded from below.

There exists a sequence $s_k \to \infty$ so that $\|\nabla_s \xi(s_k)\|_{s_k} \to 0$. Otherwise we had $\|\nabla_s \xi(s)\|_s \geqslant \eta > 0$ for a suitable η and all large s which would imply $\alpha'(s) \geqslant \frac{1}{2}\eta^2$ for all large s and $\alpha(s) \to \infty$ as $s \to \infty$ which is not true.

Because α is bounded, we can find a subsequence (which we also denote by $(s_k)_{k\in\mathbf{N}}$) so that

$$\lim_{k\to\infty} \alpha(s_k) = \lambda$$

exists. We claim that $\lambda \in \sigma(A_\infty)$. If we had $\lambda \notin \sigma(A_\infty)$ then $\varepsilon := \inf_{\mu\in\sigma(A_\infty)} |\lambda - \mu| > 0$ because $\sigma(A_\infty)$ is closed and therefore

$$|\mu' - \lambda| \geqslant \varepsilon - |\mu - \mu'| \quad \forall \mu \in \sigma(A_\infty), \ \mu' \in \sigma\left(A(s) - \frac{1}{2}\Theta(s) \right)$$

which implies

$$\mathrm{dist}\left(\lambda, \sigma\left(A(s) - \frac{1}{2}\Theta(s) \right) \right) \geqslant \varepsilon - \sup_{\mu'\in\sigma(A(s)-\frac{1}{2}\Theta(s))} \mathrm{dist}\left(\mu', \sigma(A_\infty)\right) > \varepsilon/2$$

if s is sufficiently large, by Theorem 3.14, i.e.

$$\alpha(s_k) \notin \sigma\left(A(s_k) - \frac{1}{2}\Theta(s_k) \right)$$

for k sufficiently large.

Then

$$
\begin{aligned}
1 &= \left\| \xi(s_k) \right\|_{s_k} \\
&= \left\| \left(A(s_k) - \frac{1}{2}\Theta(s_k) - \alpha(s_k)\,\mathrm{Id} \right)^{-1} \left(\nabla_s \xi(s_k) - \frac{1}{2}\Theta(s_k) - \Gamma_1(s_k)\xi(s_k) \right) \right\|_{s_k} \\
&\leqslant \frac{4}{\varepsilon} \left\| \nabla_s \xi(s_k) \right\|_{s_k} + \varepsilon_k,
\end{aligned}
$$

where k is chosen so large that $|\lambda - \alpha(s_k)| < \varepsilon/4$ and $\varepsilon_k \searrow 0$ is a suitable sequence. But this contradicts $\|\nabla_s \xi(s_k)\|_{s_k} \to 0$, hence $\lambda \in \sigma(A_\infty)$.

Let us show that indeed

$$\lim_{s\to\infty} \alpha(s) = \lambda.$$

Take now a sequence $s_k \to \infty$ and assume that there are subsequences (s_k'), (s_k'') which converge to different limits λ' and λ''. By our previous discussion we have

$$\lambda', \lambda'' \in \sigma(A_\infty)$$

and we assume that $\lambda' < \lambda''$. We may also assume that $s_k' < s_k'' < s_{k+1}'$. It is a consequence of Theorem 3.14 that there are $d > 0$ and $\nu \in (\lambda', \lambda'')$ so that

$$\mathrm{dist}\left(\nu, \sigma\left(A(s) - \frac{1}{2}\Theta(s)\right)\right) \geqslant d$$

whenever s is sufficiently large. Let now s be any number with $\alpha(s) = \nu$. Then we estimate as before:

$$
\begin{aligned}
1 &= \big\|\xi(s)\big\|_s \\
&= \left\|\left(A(s) - \frac{1}{2}\Theta(s) - \nu\,\mathrm{Id}\right)^{-1}\left(\nabla_s \xi(s) - \frac{1}{2}\Theta(s) - \Gamma_1(s)\xi(s)\right)\right\|_s \\
&\leqslant \frac{1}{d}\big\|\nabla_s \xi(s)\big\|_s + \varepsilon(s),
\end{aligned}
$$

where $\varepsilon(s)$ is a suitable positive function tending to zero as $s \to \infty$. Using inequality (41), we obtain $\alpha'(s) > 0$ for all large enough s with $\alpha(s) = \nu$, but this is a contradiction since it prohibits α from oscillating between λ' and λ''. Hence the limit

$$\lambda = \lim_{s \to \infty} \alpha(s) \in \sigma(A_\infty)$$

exists and it is indeed an eigenvalue because the operator A_∞ has compact resolvent so that the spectrum consists of eigenvalues only. Moreover, $\lambda \leqslant 0$ since otherwise $\|v(s)\|_{L^2} \to \infty$. Let us show that $\lambda < 0$

We know that there are $\rho, s_0 > 0$ so that for all $s \geqslant s_0$:

$$\big\|v(s)\big\|_s \leqslant c e^{-\rho(s-s_0)}$$

which follows from Theorem 3.6. Using Proposition 3.13, we see that the function

$$e^{\rho(s-s_0)}\big\|v(s)\big\|_s = \big\|v(s_0)\big\|_{s_0} e^{\rho(s-s_0)+\int_{s_0}^s \alpha(\tau)\,d\tau}$$

remains bounded for all $s \geqslant s_0$. This means that the function

$$f(s) := \rho(s - s_0) + \int_{s_0}^s \alpha(\tau)\,d\tau$$

has to be bounded as well. Now

$$f'(s) = \rho + \alpha(s) \to \rho + \lambda$$

as $s \to \infty$. Boundedness of f implies then $\rho + \lambda \leqslant 0$.

It remains to take care of the case for which $\|v(s)\|_s = 0$ for some s. Then $v(s,t) = 0$ for all $t \in [0,1]$ and a simple application of the similarity principle implies that v is constant (see [2,15]) in contradiction to our assumptions. This completes the proof of Proposition 3.13. $\quad\square$

The following three lemmas are versions of lemmas in [2] and [13]. The proof of Theorem 1.1 is then very similar to the corresponding version in [2]. For the convenience of the reader, we sketch the path until the proof of

Theorem 1.1. The proofs of the corresponding results in [2] and [13] can almost be carried over verbatim; we will indicate the necessary modifications.

Lemma 3.17. *For every $\beta = (\beta_1, \beta_2) \in \mathbf{N}^2$ and $j \in \mathbf{N}$ we have*

$$\sup_{(s,t)\in[s_0,\infty)\times[0,1]} \left|\partial^\beta \xi(s,t)\right| < \infty,$$

$$\sup_{s_0 \leqslant s < \infty} \left|\frac{d^j \alpha}{ds^j}(s)\right| < \infty,$$

where $\xi(s,t) = v(s,t)/\|v(s)\|_s$ and $\alpha(s) = (A(s)\cdot\xi(s) + \Gamma_1(s)\cdot\xi(s), \xi(s))_s$ (here, we adopt the convention $0 \in \mathbf{N}$).

Proof. This is actually a version of Lemma 3.10. from [2]. The proof remains essentially the same. There are two minor modifications: The operator $T_\infty(t)$ in [2] should be replaced by the t-independent operator T_∞ that we introduced in the proof of Theorem 3.14. Moreover, the estimate for $|\alpha'(s)|$ in [2] has to be replaced by

$$\left|\alpha'(s)\right| \leqslant c' \left\|\partial_s \xi(s)\right\|^2_{L^2([0,1])} + c'' \left\|\partial_s \xi(s)\right\|_{L^2([0,1])} + c''',$$

which follows from the estimates that lead us to inequality (41). We then get for $p > 2$ and $\delta_2 > 0$

$$\|\alpha'\|^p_{L^p([s^*-\delta_2,s^*+\delta_2])} \leqslant 4^{p-1}(c')^p \int_{s^*-\delta_2}^{s^*+\delta_2} \left(\int_0^1 \left|\partial_s \xi(s,t)\right|^2 dt\right)^p ds$$

$$+ 4^{p-1}(c'')^p \int_{s^*-\delta_2}^{s^*+\delta_2} \left(\int_0^1 \left|\partial_s \xi(s,t)\right|^2\right)^{p/2} ds + 2\cdot 4^{p-1}(c''')^p \delta_2$$

$$\leqslant 4^{p-1}(c')^p \|\partial_s \xi\|^{2p}_{L^{2p}(Q_{\delta_2})} + 4^{p-1}(c'')^p \|\partial_s \xi\|^p_{L^{2p}(Q_{\delta_2})} + 2\cdot 4^{p-1}(c''')^p \delta_2,$$

where $Q_{\delta_2} := [s^* - \delta_2, s^* + \delta_2] \times [0,1]$; but this estimate works as well as the original one in [2].   □

Lemma 3.18. *Let*

$$E \subseteq H^{1,2}_L\big([0,1],\mathbf{R}^4\big) \subseteq L^2\big([0,1],\mathbf{R}^4\big)$$

be the eigenspace of A_∞ belonging to $\lambda \in \sigma(A_\infty)$.
  Then

$$\inf_{e \in E} \left\|\xi(s) - e\right\|_{H^{1,2}([0,1],\mathbf{R}^4)} \to 0$$

as $s \to \infty$.

Proof. This is a modification of Lemma 3.6. in [13]. The proof is very similar to [13], replace $\partial_s \xi$ in the estimates by the covariant derivative $\nabla_s \xi$.   □

Lemma 3.19. *There exists $e \in E$ such that $\xi(s) \to e$ in $H^{1,2}([0,1],\mathbf{R}^4)$ as $s \to \infty$.*

Proof. This is essentially Lemma 3.12. in [2]. Using the L^2-product

$$(u_1, u_2) := \int_0^1 \big\langle u_1(t), \Omega_\infty M_\infty u_2(t)\big\rangle dt$$

instead, the proof in [2] can be carried over.   □

Proof of Theorem 1.1. By Proposition 3.13 we have

$$
\begin{aligned}
v(s,t) &= \big\|v(s)\big\|_s \xi(s,t) \\
&= e^{\int_{s_0}^{s} \alpha(\tau)\,d\tau} \|v(s_0)\|_{s_0} \xi(s,t) \\
&= e^{\int_{s_0}^{s} \alpha(\tau)\,d\tau} \big[\tilde{e}(t) + r(s,t)\big]
\end{aligned}
$$

with

$$
r(s,t) := \big\|v(s_0)\big\|_{s_0} \big(\xi(s,t) - e(t)\big),
$$

$$
\tilde{e}(t) := \big\|v(s_0)\big\|_{s_0} e(t) \in E,
$$

where $e(t)$ is the eigenvector given by Lemma 3.19. Recall from the proof of Theorem 3.14 that the operator A_∞ is unitary equivalent to the operator $B_\infty = -i\frac{d}{dt}$ acting on a suitable closed subspace of $H^{1,2}([0,1], \mathbf{R}^4)$. Eigenvectors of B_∞ are of the form $\hat{e}(t) = e^{i\lambda t}\hat{e}(0)$, hence they are nowhere zero and so are eigenvectors of A_∞. The proof that r converges to zero in C^∞ is the same as in [2], so we omit the details. $\quad\square$

3.4. Proof of Theorem 1.2

We will need later the following simple observation concerning the function α which appears in the asymptotic formula, Theorem 1.1:

Proposition 3.20. *All derivatives of the function α as in* (13) *converge to zero as $|s| \to \infty$.*

Proof. We have $\|\partial_s^k \xi(s)\|_{L^2([0,1])} \to 0$ for $k \geqslant 1$ and $s \to \infty$ because $\partial_s^k \xi(s,t)$ equals up to multiplication with a constant the derivative $\partial_s^k r(s,t)$, where r is the remainder in the asymptotic formula, Theorem 1.1. Recall Eq. (39):

$$
\alpha(s) = \big(\xi(s), \Gamma_1(s)\xi(s)\big)_s + \big(\xi(s), A(s)\xi(s)\big)_s.
$$

Differentiating with respect to s, we obtain the assertion of the proposition. $\quad\square$

We denote by E the eigenspace of the asymptotic operator A_∞ belonging to the eigenvalue λ. Let e be the generator of E such that $\xi(s) \to e$ as $s \to \infty$ (see Lemma 3.19). Let

$$
\pi_s : \big(L^2([0,1], \mathbf{R}^4), (\cdot, \cdot)_s\big) \to E,
$$

$$
\pi_s v := \frac{(v,e)_s}{\|e\|_s^2} \cdot e
$$

be the orthogonal projection onto the space E and let $Q_s := \mathrm{Id} - \pi_s$. The following lemma is similar to Lemma 3.8.

Lemma 3.21. *There are constants $s_0, \delta > 0$ so that for all $s \geqslant s_0$ and $\gamma \in H_L^{1,2}([0,1], \mathbf{R}^4)$ the following inequality holds*:

$$
\big\|\big(A(s) - \alpha(s)\big)Q_s \gamma\big\|_s \geqslant \delta\|Q_s \gamma\|_s.
$$

Proof. Proceeding indirectly, we assume that there are sequences $\delta_k \searrow 0$, $s_k \nearrow \infty$ and $\gamma_k \in H_L^{1,2}([0,1], \mathbf{R}^4)$ so that

$$
\frac{\|(A(s_k) - \alpha(s_k))Q_{s_k}\gamma_k\|_{s_k}}{\|Q_{s_k}\gamma_k\|_{s_k}} < \delta_k.
$$

With

$$\eta_k := \frac{Q_{s_k}\gamma_k}{\|Q_{s_k}\gamma_k\|_{s_k}}$$

we get $\delta_1 \geqslant \|\eta_k\|_{L^2([0,1])} \geqslant \delta_0 > 0$ for some δ_0, δ_1 and

$$\left\|\left(A(s_k) - \alpha(s_k)\right)\eta_k\right\|_{s_k} < \delta_k \to 0 \tag{44}$$

We estimate

$$\|\partial_t\eta_k\|_{L^2([0,1])} \leqslant c\left\|A(s_k)\eta_k\right\|_{s_k} < c\left(\delta_k + |\alpha(s_k)|\right) \leqslant 2c|\lambda|,$$

for sufficiently large k. The sequence η_k is therefore bounded in $H^{1,2}([0,1], \mathbf{R}^4)$ which is compactly embedded into $L^2([0,1], \mathbf{R}^4)$. Hence we may assume that after passing to a suitable subsequence $\eta_k \to \eta$ in L^2. We estimate

$$\begin{aligned}
\left\|(A_\infty - \lambda)\eta_k\right\|_{s_k} &\leqslant \left\|\left(A(s_k) - \alpha(s_k)\right)\eta_k\right\|_{s_k} + \left\|\left(A_\infty - A(s_k)\right)\eta_k\right\|_{s_k} + c\left\|\left(\alpha(s_k) - \lambda\right)\eta_k\right\|_{s_k} \\
&\leqslant \delta_k + \left\|M_\infty - M(s_k)\right\|_{L^\infty([0,1])}\|\partial_t\eta_k\|_{s_k} + c|\alpha(s_k) - \lambda| \\
&\to 0
\end{aligned}$$

and

$$\left\| - M_\infty \partial_t\eta_k - \lambda\eta \right\|_{L^2([0,1])} \leqslant c\left\|(A_\infty - \lambda)\eta_k\right\|_{s_k} + |\lambda|\|\eta - \eta_k\|_{L^2([0,1])},$$

which converges to zero, hence $\partial_t\eta_k$ converges in L^2 to $\lambda M_\infty \eta$ which is then the weak derivative $\partial_t\eta$ of η. We conclude $A_\infty \eta = \lambda\eta$, i.e. $Q_{s_k}\eta = 0$ for all k. This leads to the contradiction

$$1 = (\eta_k, \eta_k)_{s_k} \leqslant \left|(\eta_k, \eta_k - \eta)_{s_k}\right| + \left|(\eta_k, \eta)_{s_k}\right| \leqslant \text{const}\|\eta_k - \eta\|_{L^2([0,1])} \to 0$$

and completes the proof of the lemma. $\quad\square$

Our aim is now to estimate $\xi(s,t) - e(t)$, $\alpha(s) - \lambda$ and all its derivatives in absolute value from above by $ce^{-\delta s}$. For an integer $N \geqslant 1$ we introduce the vector

$$V(s,t) := \left(\partial_s^k\left(\xi(s,t) - e(t)\right)\right)_{0 \leqslant k \leqslant N-1}$$

and we want derive a PDE satisfied by V. Using Eq. (36), which is

$$\partial_s\xi(s) = A(s)\xi(s) - \alpha(s)\xi(s),$$

and $A_\infty e = \lambda e$ we obtain

$$\partial_s\left(\xi(s) - e\right) = \left(A(s) - A_\infty\right)\left(\xi(s) - e\right) + \left(A(s) - A_\infty\right)e + \left(\lambda - \alpha(s)\right)e.$$

Differentiating successively with respect to s and viewing π_s, Q_s, $A(s) - \alpha(s)$ as operators on N-tuples in $(H_L^{1,2})^N$ we obtain the following differential equation for V:

$$\partial_s V(s) = \left(A(s) - \alpha(s)\right)V(s) - \tilde{\alpha}(s)V(s) + H(s) + E(s), \tag{45}$$

where H and its derivatives decay like $e^{-|\lambda|s}$, the vector $E(s)$ is given by

$$E(s) = \left(\partial_s^k\left(\alpha(s) - \lambda\right) \cdot e\right)_{0 \leqslant k \leqslant N-1}$$

and

$$\tilde{\alpha}(s) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \alpha_{11} & 0 & 0 & \cdots & 0 \\ \alpha_{22} & \alpha_{12} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{N-1,N-1} & \alpha_{N-2,N-1} & \alpha_{N-3,N-1} & \cdots & 0 \end{pmatrix}$$

with

$$\alpha_{lk} = \binom{k}{l} \frac{d^l \alpha}{ds^l}, \quad 1 \leqslant l, k \leqslant N - 1.$$

We also note that

$$\pi_s E(s) = E(s).$$

We define now the function

$$g(s) := \frac{1}{2} \|\pi_s V(s) - V(s)\|_s^2$$

and we denote by $\Gamma(s)$ a matrix whose entries are zero order operators such that $\|D^\alpha \Gamma(s)\| \leqslant c e^{-|\lambda|s}$ in the operator norm. We will always use this notation if we are not concerned with the explicit structure of $\Gamma(s)$. We compute

$$g'(s) = \big(\partial_s\big(Q_s V(s)\big), Q_s V(s)\big)_s + \big(Q_s V(s), \Gamma(s) Q_s V(s)\big)_s.$$

We continue with the second derivative

$$\begin{aligned} g''(s) &= \big(\partial_{ss}\big(Q_s V(s)\big), Q_s V(s)\big)_s + \|Q_s V(s)\|_s^2 \\ &\quad + \big(Q_s V(s), \Gamma(s) Q_s V(s)\big)_s + \big(\partial_s\big(Q_s V(s)\big), \Gamma(s) Q_s V(s)\big)_s \\ &\geqslant \big(Q_s\big(\partial_{ss} V(s)\big), Q_s V(s)\big)_s + \big(\partial_s\big(Q_s V(s)\big), \Gamma(s) Q_s V(s)\big)_s + \big(Q_s V(s), \Gamma(s) Q_s V(s)\big)_s \\ &=: T_1 + T_2 + T_3, \end{aligned}$$

and we note that

$$|T_2|, |T_3| \leqslant c e^{-|\lambda|s} \|Q_s V(s)\|_s. \tag{46}$$

We note that $\Gamma(s)$ here is different than in the equation for $g'(s)$, but we use the same symbol since we only care about the exponential decay. We have also used that V and its derivatives are bounded by Lemma 3.17 and that the operators $\partial_s Q_s, \partial_{ss} Q_s$ have range in E^N, hence the ranges of these operators are orthogonal to the range of Q_s. We will also use the facts that $\partial_s Q_s - Q_s \partial_s$ has range in E^N and that $\tilde{\alpha}(s) Q_s - Q_s \tilde{\alpha}(s) = 0$. Differentiating (45) yields

$$\begin{aligned} \partial_{ss} V(s) &= -\partial_s\big(M(v)\big) M(v) A(s) V(s) - \alpha'(s) V(s) + \big(A(s) - \alpha(s)\big) \partial_s V(s) \\ &\quad - \tilde{\alpha}'(s) V(s) - \tilde{\alpha}(s) \partial_s V(s) + H'(s) + E'(s). \end{aligned}$$

We evaluate

$$\begin{aligned} T_1 &= \big(-\partial_s\big(M(v)\big) M(v) A(s) V(s), Q_s V(s)\big)_s + \big(\big(A(s) - \alpha(s)\big) \partial_s V(s), Q_s V(s)\big)_s \\ &\quad - \big(\alpha'(s) Q_s V(s), Q_s V(s)\big)_s - \big(\tilde{\alpha}'(s) Q_s V(s), Q_s V(s)\big)_s + \big(H'(s), Q_s V(s)\big)_s \\ &= T_{11} + \cdots + T_{15}. \end{aligned}$$

If $\varepsilon(s)$ denotes a function which converges to zero with all derivatives as $s \to \infty$ then we can estimate

$$|T_{13}|, |T_{14}| \leqslant \varepsilon(s) \|Q_s V(s)\|_s^2$$

and

$$|T_{11}|, |T_{15}| \leqslant c e^{-|\lambda|s} \|Q_s V(s)\|_s.$$

We continue with the term T_{12}:

$$T_{12} = \big(\big(A(s) - \alpha(s)\big) \pi_s \big(\partial_s V(s)\big), Q_s V(s)\big)_s + \big(\big(A(s) - \alpha(s)\big) Q_s \big(\partial_s V(s)\big), Q_s V(s)\big)_s. \tag{47}$$

We compute

$$\big(A(s) - \alpha(s)\big)\pi_s\big(\partial_s V(s)\big) = \big[-M(v)\partial_t - \alpha(s)\big]\frac{(\partial_s V(s), e)_s}{\|e\|_s^2} \cdot e$$

$$= \frac{(\partial_s V(s), e)_s}{\|e\|_s^2}\big(A(s) - \alpha(s)\big)e$$

$$= \frac{(\partial_s V(s), e)_s}{\|e\|_s^2}\big((A(s) - A_\infty)e + \big(\lambda - \alpha(s)\big)e\big),$$

so that

$$\big|\big(\big(A(s) - \alpha(s)\big)\pi_s\big(\partial_s V(s)\big), Q_s V(s)\big)_s\big|$$

$$= \left|\frac{(\partial_s V(s), e)_s}{\|e\|_s^2}\big((A(s) - A_\infty)e, Q_s V(s)\big)_s\right|$$

$$\leqslant ce^{-|\lambda|s}\|Q_s V(s)\|_s. \tag{48}$$

In a similar fashion, we obtain

$$\big|\big(\big(A(s) - \alpha(s)\big)\pi_s V(s), Q_s V(s)\big)_s\big| \leqslant ce^{-|\lambda|s}\|Q_s V(s)\|_s. \tag{49}$$

We evaluate now the second term in Eq. (47) using the differential equation (45), Eq. (49) and Proposition 3.20:

$$\big(\big(A(s) - \alpha(s)\big)Q_s\big(\partial_s V(s)\big), Q_s V(s)\big)_s$$

$$= \big(Q_s\big(\partial_s V(s)\big), \big(A(s) - \alpha(s)\big)Q_s V(s)\big)_s + \big(Q_s\big(\partial_s V(s)\big), \Theta(s)Q_s V(s)\big)_s$$

$$= \big(Q_s\big(A(s) - \alpha(s)\big)V(s), \big(A(s) - \alpha(s)\big)Q_s V(s)\big)_s - \big(\tilde{\alpha}(s)Q_s V(s), \big(A(s) - \alpha(s)\big)Q_s V(s)\big)_s$$

$$\quad + \big(H(s), \big(A(s) - \alpha(s)\big)Q_s V(s)\big)_s + \big(Q_s\big(\partial_s V(s)\big), \Theta(s)Q_s V(s)\big)_s$$

$$= \big\|\big(A(s) - \alpha(s)\big)Q_s V(s)\big\|_s^2 + \big(Q_s\big(A(s) - \alpha(s)\big)\pi_s V(s), \big(A(s) - \alpha(s)\big)Q_s V(s)\big)_s$$

$$\quad - \big(\tilde{\alpha}(s)Q_s V(s), \big(A(s) - \alpha(s)\big)Q_s V(s)\big)_s + \big(H(s), \big(A(s) - \alpha(s)\big)Q_s V(s)\big)_s$$

$$\quad + \big(Q_s\big(\partial_s V(s)\big), \Theta(s), Q_s V(s)\big)_s$$

$$\geqslant \big\|\big(A(s) - \alpha(s)\big)Q_s V(s)\big\|_s^2 - ce^{-|\lambda|s}\big\|\big(A(s) - \alpha(s)\big)Q_s V(s)\big\|_s$$

$$\quad - \varepsilon(s)\|Q_s V(s)\|_s\big\|\big(A(s) - \alpha(s)\big)Q_s V(s)\big\|_s - ce^{-|\lambda|s}\|Q_s V(s)\|_s.$$

Using Lemma 3.21 we obtain now for large s:

$$g''(s) \geqslant \big\|\big(A(s) - \alpha(s)\big)Q_s V(s)\big\|_s^2 - ce^{-|\lambda|s}\big\|\big(A(s) - \alpha(s)\big)Q_s V(s)\big\|_s$$

$$\quad - \varepsilon(s)\|Q_s V(s)\|_s\big\|\big(A(s) - \alpha(s)\big)Q_s V(s)\big\|_s - ce^{-|\lambda|s}\|Q_s V(s)\|_s - \varepsilon(s)\|Q_s V(s)\|_s^2$$

$$= \big\|\big(A(s) - \alpha(s)\big)Q_s V(s)\big\|_s\big(\big\|\big(A(s) - \alpha(s)\big)Q_s V(s)\big\|_s - \varepsilon(s)\|Q_s V(s)\|_s\big)$$

$$\quad - ce^{-|\lambda|s}\big\|\big(A(s) - \alpha(s)\big)Q_s V(s)\big\|_s - ce^{-|\lambda|s}\|Q_s V(s)\|_s - \varepsilon(s)\|Q_s V(s)\|_s^2$$

$$\geqslant \big(\delta - \varepsilon(s)\big)\big\|\big(A(s) - \alpha(s)\big)Q_s V(s)\big\|_s\|Q_s V(s)\|_s - ce^{-|\lambda|s}(1 + \delta)\|Q_s V(s)\|_s - \varepsilon(s)\|Q_s V(s)\|_s^2$$

$$\geqslant \tilde{\delta}^2 g(s) - ce^{-|\lambda|s}$$

for a suitable positive number $\tilde{\delta}$. We remark that $g(s)$ converges to zero as $s \to \infty$ since the remainder in the asymptotic formula and all its derivatives do. We introduce now the function

$$\beta(s) := g(s) + \frac{ce^{-|\lambda|s}}{|\lambda|^2 - \tilde{\delta}^2},$$

which also tends to zero as $s \to \infty$. We have

$$\beta''(s) \geqslant \tilde{\delta}^2 \beta(s).$$

Defining $\gamma(s) := \beta(s) - \beta(s_0)e^{-\tilde{\delta}(s-s_0)}$ we get $\gamma''(s) \geqslant \tilde{\delta}^2 \gamma(s)$, $\gamma(s) \to 0$ as $s \to \infty$ and $\gamma(s_0) = 0$ which implies that γ is a non positive function. Therefore,

$$g(s) \leqslant \beta(s_0)e^{-\tilde{\delta}(s-s_0)} + \frac{ce^{-|\lambda|s}}{||\lambda|^2 - \tilde{\delta}^2|} \leqslant ce^{-\hat{\delta}s} \tag{50}$$

for suitable positive constants $c, \hat{\delta}$. We now have to show exponential decay for $|\pi_s V(s)|$ and all the derivatives of $\alpha(s) - \lambda$. The proof will be by induction with respect to N, the length of the vector $V(s)$. We start by establishing the desired estimates for the case $N = 1$. We claim that

$$\left| \left(\partial_s \xi(s), \xi(s) \right)_s \right| \leqslant ce^{-|\lambda|s},$$

which follows easily from $\|\xi(s)\|_s = 1$ since

$$0 = \left(\partial_s \xi(s), \xi(s) \right)_s + \left(\xi(s), \Gamma_1(s)\xi(s) \right)_s,$$

where $\|\Gamma_1(s)\|$ has the above exponential decay. We conclude that $|(\partial_s \xi(s), \pi_s \xi(s))_s|$ also decays exponentially since $|(\partial_s \xi(s), Q_s \xi(s))_s|$ does. We calculate

$$\begin{aligned}
\left(\partial_s \xi(s), \pi_s \xi(s) \right)_s &= \frac{(e, \xi(s))_s}{\|e\|_s^2} \left(\partial_s \xi(s), e \right)_s \\
&= \frac{(e, \xi(s))_s}{\|e\|_s^2} \left(A(s)\xi(s), e \right)_s - \alpha(s) \frac{(e, \xi(s))_s^2}{\|e\|_s^2} \\
&= \frac{(e, \xi(s))_s}{\|e\|_s^2} \left(\left(\xi(s), \Theta(s)e \right)_s + \left(\xi(s), (A(s) - A_\infty)e \right)_s + \left(\xi(s), \lambda e \right)_s - \alpha(s)\left(\xi(s), e \right)_s \right)
\end{aligned}$$

and

$$\left(\lambda - \alpha(s) \right) = \frac{\|e\|_s^2}{(e, \xi(s))_s^2} \left(\partial_s \xi(s), \pi_s \xi(s) \right)_s + \frac{1}{(\xi(s), e)_s} \left(\xi(s), (A_\infty - A(s))e - \Theta(s)e \right)_s.$$

Recalling that $\inf_s (\xi(s), e)_s > 0$ we conclude that for suitable constants $c, \hat{\delta} > 0$

$$\left| \lambda - \alpha(s) \right| \leqslant ce^{-\hat{\delta}s}. \tag{51}$$

We compute

$$\begin{aligned}
\pi_s \partial_s \xi(s) &= \pi_s \left(A(s) - \alpha(s) \right)\xi(s) \\
&= \frac{1}{\|e\|_s^2} \left((A(s) - \alpha(s))\xi(s), e \right)_s \cdot e \\
&= -\alpha(s) \frac{(\xi(s), e)_s}{\|e\|_s^2} \cdot e + \frac{1}{\|e\|_s^2} \left(\xi(s), A(s)e \right)_s \cdot e + \frac{1}{\|e\|_s^2} \left(\xi(s), \Theta(s)e \right)_s \cdot e \\
&= -\alpha(s) \frac{(\xi(s), e)_s}{\|e\|_s^2} \cdot e + \frac{1}{\|e\|_s^2} \left(\xi(s), (A(s) - A_\infty)e \right)_s \cdot e \\
&\quad + \frac{1}{\|e\|_s^2} \left(\xi(s), \lambda e \right)_s \cdot e + \frac{1}{\|e\|_s^2} \left(\xi(s), \Theta(s)e \right)_s \cdot e
\end{aligned}$$

so that with (51) and (50)

$$\|\partial_s \xi(s)\|_{L^2([0,1])} \leqslant \left\| Q_s \left(\partial_s \xi(s) \right) \right\|_{L^2([0,1])} + \left\| \pi_s \left(\partial_s \xi(s) \right) \right\|_{L^2([0,1])} \leqslant ce^{-\hat{\delta}s}. \tag{52}$$

Now

$$\pi_s\big(\xi(s) - e\big) = \frac{(\xi(s), e)_s}{\|e\|_s^2} \cdot e - e = -\int_s^\infty \frac{d}{d\sigma} \frac{(\xi(\sigma), e)_\sigma}{\|e\|_\sigma^2} \cdot e \, d\sigma,$$

and the integrand has exponential decay by our previous estimates. Since we have already shown that $\|Q_s \xi(s)\|_s$ decays exponentially, we obtain

$$\big\|\xi(s) - e\big\|_{L^2([0,1])} \leqslant c e^{-\hat{\delta} s} \tag{53}$$

for suitable constants $c, \hat{\delta} > 0$. We can now complete the proof by induction as follows: Differentiating equation (39) for $\alpha(s)$ we obtain

$$\begin{aligned}
\alpha'(s) = {} & \big(\partial_s \xi(s), \Gamma_1(s)\xi(s)\big)_s + \big(\xi(s), \Gamma_1(s)\partial_s\xi(s)\big)_s + \big(\xi(s), \Gamma_1^2(s)\xi(s)\big)_s \\
& + \big(\xi(s), \partial_s \Gamma_1(s)\xi(s)\big)_s + 2\big(\partial_s\xi(s), A(s)\xi(s)\big)_s + \big(\partial_s\xi(s), \Theta(s)\xi(s)\big)_s \\
& + \big(\xi(s), \partial_s M(v) \cdot M(v)A(s)\xi(s)\big)_s.
\end{aligned}$$

All the terms containing Γ_1, Θ or derivatives of $M(v)$ already decay at an exponential rate and will continue to do so if differentiated. We will summarize all those by $H(s)$. Substituting $A(s)\xi(s) = \partial_s\xi(s) + \alpha(s)\xi(s)$ and using $(\nabla_s\xi(s), \xi(s))_s = 0$, we then obtain

$$\begin{aligned}
\alpha'(s) &= 2\big\|\partial_s\xi(s)\big\|_s^2 + 2\alpha(s)\big(\partial_s\xi(s), \xi(s)\big)_s + H(s) \\
&= 2\big\|\partial_s\xi(s)\big\|^2 + H(s).
\end{aligned}$$

Hence exponential decay of all derivatives $\|\partial_s^l\xi(s)\|_{L^2([0,1])}$ up to order $k \geqslant 1$, implies exponential decay of the derivative $\alpha^{(k+1)}(s)$. Denoting exponentially decaying expressions by $H(s)$, the PDE for $V(s)$ yields

$$\begin{aligned}
\pi_s\big(\partial_s V(s)\big) &= \pi_s\big(A(s) - \alpha(s)\big)V(s) - \tilde{\alpha}(s)\pi_s V(s) + H(s) + E(s) \\
&= \frac{1}{\|e\|_s^2}\big[\big(\lambda - \alpha(s)\big)\big(V(s), e\big)_s e + \big(V(s), \big(A(s) - A_\infty\big)e\big)_s e\big] - \tilde{\alpha}(s)\pi_s V(s) + H(s) + E(s),
\end{aligned}$$

i.e. exponential decay of $(\|\partial_s^k\xi(s)\|_{L^2([0,1])})_{0\leqslant k\leqslant N-1}$ and $(\frac{d^k}{ds^k}(\lambda - \alpha(s)))_{0\leqslant k\leqslant N-1}$ implies exponential decay of $\|\pi_s\partial_s^N\xi(s)\|_{L^2([0,1])}$ and therefore of $\|\partial_s^N\xi(s)\|_{L^2([0,1])}$ in view of (50). By iteration we obtain exponential decay for all derivatives of α and the L^2-norms of all s-derivatives of $\xi(s,t)$. Using the PDE for V, (45), we also obtain exponential decay of $\|\partial_s^k\partial_t^l\xi(s)\|_{L^2([0,1])}$, and the Sobolev embedding theorem finally implies the assertion of the theorem.

3.5. The asymptotic formula in local coordinates

We will express the asymptotic formula in Theorem 1.1 in coordinates near $\{0\} \times \mathcal{L}$ for later reference.

Recall that we have used Proposition 2.1 and the modification (10) to derive the following coordinates on suitable neighborhoods $V_\pm$ of the points $p_\pm \in \mathcal{L}$:

$$\psi_\pm : \mathbf{R}^4 \supset B_\varepsilon(0) \xrightarrow{\sim} V_\pm \subset \mathbf{R} \times M, \tag{54}$$

$$\psi_\pm(0) = p_\pm,$$

$$\psi_\pm\big(\mathbf{R}^2 \times \{0\} \times \{0\}\big) = (\mathbf{R} \times \mathcal{L}) \cap V_\pm,$$

$$\psi_\pm\big(\{0\} \times \mathbf{R} \times \{0\} \times \mathbf{R}^\pm\big) = (\{0\} \times \mathcal{D}) \cap V_\pm.$$

Using the coordinates (τ, θ, x, y) for $\mathbf{R}^4$, the contact form on $\{0\} \times \mathbf{R}^3$ is then given by

$$\hat{\lambda}_\pm = \psi_\pm^* \lambda = dy + \big(x + q(\theta)y\big)\,d\theta, \; q(\theta) := \frac{a(\theta)}{b(\theta)}$$

with Reeb vector field

$$X_{\hat{\lambda}_\pm} = \frac{\partial}{\partial y} - q(\theta)\frac{\partial}{\partial x}$$

(recall that the functions a, b determine how the surface $\mathcal{D}$ is wrapping itself around the knot $\mathcal{L}$, see Proposition 2.1). Then $v_\pm(s, t) := (\psi_\pm^{-1} \circ \tilde{u}_0)(s, t)$ is the representative of the pseudoholomorphic curve $\tilde{u}_0$ in the above coordinates near the ends and the results of this Section 3 (exponential decay estimates, asymptotic formula etc.) all refer to the maps $v_\pm$. We will now compute the eigenvector $e(t)$ in Theorem 1.1 explicitly in the above coordinates.

The vectors

$$\hat{e}_1(\theta, x, y) := \big(0, 1, -q'(\theta)y + q(\theta)\big(x + q(\theta)y\big), -\big(x + q(\theta)y\big)\big)$$

and

$$\hat{e}_2(\theta, x, y) := (0, 0, 1, 0)$$

generate the contact structure $\ker \hat{\lambda}(\theta, x, y)$ so that the almost complex structure $\hat{J}$ induced by $\tilde{J}$ is given by

$$\hat{J}(\tau, \theta, x, y)\hat{e}_1(\theta, x, y) = -\hat{e}_2(\theta, x, y), \qquad \hat{J}(\tau, \theta, x, y)\hat{e}_2(\theta, x, y) = \hat{e}_1(\theta, x, y), \tag{55}$$
$$\hat{J}(\tau, \theta, x, y)(1, 0, 0, 0) = \big(0, 0, -q(\theta), 1\big), \qquad \hat{J}(\tau, \theta, x, y)\big(0, 0, -q(\theta), 1\big) = (-1, 0, 0, 0),$$

or

$$\hat{J}(\tau, \theta, x, y) = \begin{pmatrix} 0 & -(x + q(\theta)y) \\ 0 & yq'(\theta) \\ -q(\theta) & -1 + yq'(\theta)((x + q(\theta)y)q(\theta) - yq'(\theta)) \\ 1 & -(x + q(\theta)y)yq'(\theta) \end{pmatrix}$$
$$\begin{pmatrix} 0 & -1 \\ 1 & q(\theta) \\ (x + q(\theta)y)q(\theta) - yq'(\theta) & q(\theta)((x + q(\theta)y)q(\theta) - yq'(\theta)) \\ -(x + q(\theta)y) & -(x + q(\theta)y)q(\theta) \end{pmatrix}.$$

Theorem 1.1 then provides the following formula for the map $v_\pm(s, t)$ if $|s|$ is large:

$$v_\pm(s, t) = e^{\int_{s_0}^s \alpha_\pm(\tau)\,d\tau}\big[e_\pm(t) + r_\pm(s, t)\big],$$

where $e_\pm(t)$ is an eigenvector of the operator

$$A_{\pm\infty} : L^2\big([0, 1], \mathbf{R}^4\big) \supset H_L^{1,2}\big([0, 1], \mathbf{R}^4\big) \to L^2\big([0, 1], \mathbf{R}^4\big)$$

$$(A_{\pm\infty}\gamma)(t) = -\hat{J}(0, 0, 0, 0)\dot{\gamma}(t)$$

corresponding to some eigenvalue $\lambda_\pm$, and we saw earlier in Proposition 3.15 that $\lambda_\pm$ are integer multiples of $\pi/2$. In fact, we will mostly be concerned with the case where $\lambda_+ = -\frac{\pi}{2}$ and $\lambda_- = \frac{\pi}{2}$. The subscript '$L$' refers to the boundary conditions $\gamma(0) \in \mathbf{R}^2 \times \{0\} \times \{0\}$ and $\gamma(1) \in \{0\} \times \mathbf{R} \times \{0\} \times \mathbf{R}$. The matrix $\hat{J}(0, 0, 0, 0)$ is given by

$$\hat{J}(0, 0, 0, 0) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & q(0) \\ -q(0) & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

so that we have to solve the following system of differential equations for $e_\pm = (e_1, \ldots, e_4) : [0, 1] \to \mathbf{R}^4$

$$\dot{e}_1(t) = -\lambda_\pm e_4(t),$$
$$\dot{e}_2(t) = \lambda_\pm \big(e_3(t) + q(0)e_4(t)\big),$$
$$\dot{e}_3(t) = -\lambda_\pm \big(q(0)e_1(t) + e_2(t)\big),$$
$$\dot{e}_4(t) = \lambda_\pm e_1(t),$$

with the boundary condition

$$e_3(0) = e_4(0) = 0, \qquad e_1(1) = e_3(1) = 0.$$

If λ is an integer multiple of π, we have

$$e(t) = \kappa\big(0, \cos(\lambda t), -\sin(\lambda t), 0\big), \quad \kappa \neq 0 \tag{56}$$

Otherwise, if $\lambda_\pm$ is an odd integer multiple of $\pi/2$ then

$$e_\pm(t) = -\kappa_\pm\big(\cos(\lambda_\pm t), -q_\pm(0)\cos(\lambda_\pm t), 0, \sin(\lambda_\pm t)\big) \tag{57}$$

for some constants $\kappa_\pm \neq 0$. The asymptotic formula of Theorem 1.1 then looks as follows:

$$v_\pm(s, t) = -\kappa_\pm e^{\int_{s_0}^s \alpha_\pm(\tau)\,d\tau}\big(\cos(\lambda_\pm t), -q_\pm(0)\cos(\lambda_\pm t), 0, \sin(\lambda_\pm t)\big) + e^{\int_{s_0}^s \alpha_\pm(\tau)\,d\tau}\varepsilon_\pm(s, t). \tag{58}$$

In the following we will denote by $\varepsilon(s, t)$ any $\mathbf{R}^4$– or real–valued function which converges to zero with all its derivatives uniformly in t as $s \to \pm\infty$ if we are not interested in the particular function. In order to simplify notation we will often drop the subscript $\pm$. Using Proposition 3.20 we obtain the following asymptotic formulas for the derivatives of $v(s, t)$

$$\partial_s v(s, t) = e^{\int_{s_0}^s \alpha(\tau)\,d\tau}\big[-\kappa\big(\lambda\cos(\lambda t), -\lambda q(0)\cos(\lambda t), 0, \lambda\sin(\lambda t)\big) + \varepsilon(s, t)\big], \tag{59}$$

$$\partial_t v(s, t) = e^{\int_{s_0}^s \alpha(\tau)\,d\tau}\big[-\kappa\big(-\lambda\sin(\lambda t), \lambda q(0)\sin(\lambda t), 0, \lambda\cos(\lambda t)\big) + \varepsilon(s, t)\big]. \tag{60}$$

We will sometimes use the coordinates given by Proposition 2.1 without making the boundary conditions 'flat' as in (10). In this case the appropriate versions of (58) and (59) are the following. If $\lambda_\pm$ is an odd integer multiple of $\pi/2$ we have:

$$v_\pm(s, t) = -\kappa_\pm e^{\int_{s_0}^s \alpha_\pm(\tau)\,d\tau}\big(\cos(\lambda_\pm t), -q_\pm(0)\cos(\lambda_\pm t), q_\pm(0)\sin(\lambda_\pm t), \sin(\lambda_\pm t)\big)$$
$$+ e^{\int_{s_0}^s \alpha_\pm(\tau)\,d\tau}\varepsilon_\pm(s, t) \tag{61}$$

and

$$\partial_s v_\pm(s, t) = e^{\int_{s_0}^s \alpha_\pm(\tau)\,d\tau} \cdot \big[-\kappa_\pm\big(\lambda_\pm\cos(\lambda_\pm t), -\lambda_\pm q_\pm(0)\cos(\lambda_\pm t),$$
$$\lambda_\pm q_\pm(0)\sin(\lambda_\pm t), \lambda_\pm\sin(\lambda_\pm t)\big) + \varepsilon_\pm(s, t)\big]. \tag{62}$$

For $\lambda_\pm \in \mathbf{Z}\pi$ we have

$$v_\pm(s, t) = \kappa_\pm e^{\int_{s_0}^s \alpha_\pm(\tau)\,d\tau}\big(0, \cos(\lambda_\pm t), -\sin(\lambda_\pm t), 0\big) + e^{\int_{s_0}^s \alpha_\pm(\tau)\,d\tau}\varepsilon_\pm(s, t) \tag{63}$$

and

$$\partial_s v_\pm(s, t) = e^{\int_{s_0}^s \alpha_\pm(\tau)\,d\tau}\big[\kappa_\pm\big(0, \lambda_\pm\cos(\lambda_\pm t), -\lambda_\pm\sin(\lambda_\pm t), 0\big) + \varepsilon_\pm(s, t)\big]. \tag{64}$$

Acknowledgements

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