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A minimization problem associated with elliptic systems of FitzHugh–Nagumo type ☆

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Abstract

We consider a minimization problem associated with the elliptic systems of FitzHugh–Nagumo type and prove that the minimizer of this minimization problem has not only a boundary layer, but also may oscillate in a set of positive measure.

Résumé

Nous étudions des solutions d'énergie minimale pour l'équation de FitzHugh–Nagumo. Nous prouvons que ces solutions ont plusieurs transitions rapids si la diffusion est petite.

MSC: 35J50; 93C15

Keywords: Elliptic systems; Multiple layer

1. Introduction

In this paper, we consider the following problem:

$$\begin{cases} -\varepsilon^2\Delta u = f(u) - v, & \text{in } \Omega, \\ -\Delta v + \gamma v = \delta u, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^N, ε is a parameter, γ and δ are nonnegative constants, $f(t)$ is C^1-function in R^1 satisfying the following conditions:

(f_1) There are $0 < \tau_1 < \tau_2$ such that $f(\tau_1) < 0$, $f(\tau_2) > 0$, $f'(t) < 0$ if $t \in (-\infty, \tau_1) \cup (\tau_2, +\infty)$, and $f'(t) > 0$ if $t \in (\tau_1, \tau_2)$. Moreover, $f(t) \to +\infty$ as $t \to -\infty$, $f(t) \to -\infty$ as $t \to +\infty$.

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Let $I_{-1} = (-\infty, \tau_1)$, $I_0 = (\tau_1, \tau_2)$, and $I_1 = (\tau_2, +\infty)$. By (f_1), $f(t)$ has exactly three zero points $a_i \in I_i$, $i = -1, 0, 1$. We assume that

(f_2) $\int_{a_{-1}}^{a_1} f(s)\,ds > 0$.

Typical examples satisfying (f_1) and (f_2) include $f(t) = t(a-t)(t-1)$, $a \in (0, \frac{1}{2})$; and $f_c(t) = f(t-c)$, $c > 0$.

System (1.1) is a modification of the FitzHugh–Nagumo equation which arises in studies on the physiological phenomenon of nerve conduction. This system has been studied among others by DeFigueiredo, Mitidieri, Troy [10,14,15], Lazer and McKenna [16], Reinecke and Sweers [18–21]. Existence results in [18–20] are in some sense analogies of the results for the scalar case $\delta = 0$ in [7]. Numerical results in [21] suggest that (1.1) should have other types of solutions. The aim of this paper is to prove that for suitably large $\delta > 0$, (1.1) has solutions, which either oscillate around a constant in a compact subset of Ω, or have a sharp interior layer. These solutions are local minimum of the corresponding functional. We know that for the autonomous scalar equation ($\delta = 0$), the minimizer does not have interior layer. See for example [5–7].

For each $u \in H_0^1(\Omega)$, let $G_\gamma u$ be the unique solution of the following problem:

$$\begin{cases} -\Delta v + \gamma v = u, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases}$$

Then we see (1.1) is equivalent to the following nonlocal elliptic problem:

$$\begin{cases} -\varepsilon^2 \Delta u + \delta G_\gamma u = f(u), & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{1.2}$$

The energy associated with (1.2) is

$$I(u) = \frac{1}{2} \int_\Omega \left(\varepsilon^2 |Du|^2 + \delta u G_\gamma u\right) - \int_\Omega F(u), \quad u \in H_0^1(\Omega). \tag{1.3}$$

It is easy to see from $\int_\Omega u G_\gamma u = \int_\Omega \left(|DG_\gamma u|^2 + \gamma |G_\gamma u|^2\right) \geqslant 0$, that $I(u)$ is bounded from below in $H_0^1(\Omega)$ and $I(u)$ is weakly lower semicontinuous in $H_0^1(\Omega)$. So the following problem has a minimizer:

$$\inf\left\{I(u):\ u \in H_0^1(\Omega)\right\}. \tag{1.4}$$

In this paper, we will analyse the profile of the global minimizer of (1.4) for $\varepsilon > 0$ small. Before we state our results, we give some notation.

Let $u = h_+(v)$, $v \in f(I_1)$, be the inverse function of $v = f(u)$ restricted to I_1; and let $u = h_-(v)$, $v \in f(I_{-1})$, be the inverse function of $v = f(u)$ restricted to I_{-1}.

Let

$$j(\alpha) =: \int_{h_-(\alpha)}^{h_+(\alpha)} (f(s) - \alpha)\,ds. \tag{1.5}$$

By (f_1), we see that $j'(\alpha) = h_-(\alpha) - h_+(\alpha) < 0$. Thus by (f_2), there is a unique $\alpha_0 > 0$ such that $j(\alpha_0) = 0$, $j(\alpha) > 0$ if $\alpha < \alpha_0$, and $j(\alpha) < 0$ if $\alpha > \alpha_0$.

We extend $h_+(v)$ continuously into $v \in (f(\tau_2), +\infty)$ in such a way that $h_+(v)$ is decreasing. Then since $h_+(v)$ is decreasing, it is easy to see that the following problem has a unique solution v_δ:

$$\begin{cases} -\Delta v + \gamma v = \delta h_+(v), & \text{in } \Omega, \\ v \in H_0^1(\Omega). \end{cases} \tag{1.6}$$

Moreover, by using the maximum principle, we can deduce easily that $v_{\delta_1} < v_{\delta_2}$ if $\delta_1 < \delta_2$. By the comparison theorem, it is easy to see that $\max_{x\in\Omega} v_\delta(x) \to +\infty$ as $\delta \to +\infty$. So, there is a unique $\delta_0 > 0$, such that $\max_{x\in\Omega} v_{\delta_0}(x) = \alpha_0$. It is easy to check that $\delta_0 > \gamma\alpha_0/h_+(\alpha_0)$.

Define

$$h(v) = \begin{cases} h_+(v), & \text{if } v < \alpha_0;\\ h_-(v), & \text{if } v > \alpha_0.\end{cases}$$

Consider

$$\begin{cases} -\Delta v + \gamma v \in [\delta h(v+0), \delta h(v-0)], & \text{in } \Omega,\\ v \in H_0^1(\Omega).\end{cases} \tag{1.7}$$

Then, the above problem has a solution, which is the global minimum of the corresponding functional. Besides, (1.7) has exactly one solution because $h(v)$ is decreasing. This is easy to prove but also follows from monotone operator theory as in [4]. Note that if $\delta \leqslant \delta_0$, the solution of (1.7) is the solution of (1.6) and vice versa. Let v be the solution of (1.7). It is easy to see that if $\delta > \delta_0$, the set $\{x \in \Omega\colon v(x) \geqslant \alpha_0\}$ has nonzero measure. In the following, we denote

$$S = \{x \in \Omega\colon v(x) < \alpha_0\}.$$

Note that $S = \Omega$ if $0 \leqslant \delta < \delta_0$ and $\Omega \setminus S \neq \emptyset$ if $\delta > \delta_0$.

Theorem 1.1. *Suppose that $h_-(\alpha_0) \leqslant 0$. Let u_ε be a global minimizer of* (1.4) *and let $v_\varepsilon = \delta G_\gamma u_\varepsilon$. Then $v_\varepsilon \to v$ in $C^{1,\sigma}(\Omega)$, for any $\sigma \in (0,1)$, where v is the solution of* (1.7)*. Moreover, we have*

(i) *if $0 \leqslant \delta < \delta_0$, then $u_\varepsilon \to h_+(v)$ uniformly in any compact subset of Ω as $\varepsilon \to 0$;*
(ii) *if $\delta = \delta_0$, then $\{x\colon v(x) = \alpha_0\} = \Omega \setminus S$ and the measure of the set $\{x\colon v(x) = \alpha_0\}$ is zero. Moreover, $u_\varepsilon \to h_+(v)$ uniformly in any compact subset of S as $\varepsilon \to 0$;*
(iii) *if $\delta > \delta_0$, then $\{x\colon v(x) = \alpha_0\} = \Omega \setminus S$ and the measure of the set $\{x\colon v(x) = \alpha_0\}$ is positive. Moreover, $u_\varepsilon \to h_+(v)$ uniformly in any compact subset of S as $\varepsilon \to 0$, $u_\varepsilon \to \gamma\alpha_0/\delta$ weak* in $L^\infty(\Omega \setminus S)$ as $\varepsilon \to 0$, but u_ε does not converges almost everywhere to $\gamma\alpha_0/\delta$ as $\varepsilon \to 0$ for any subsequence, and for any $\theta > 0$ small,*

$$m\{x\colon v(x) = \alpha_0, u_\varepsilon(x) \notin (h_-(\alpha_0) - \theta, h_-(\alpha_0) + \theta) \cup (h_+(\alpha_0) - \theta, h_+(\alpha_0) + \theta)\} \to 0$$

as $\varepsilon \to 0$, where mS denotes the measure of the set S.

Theorem 1.2. *Suppose that $h_-(\alpha_0) > 0$. Let u_ε be a global minimizer of* (1.4), *and let $v_\varepsilon = \delta G_\gamma u_\varepsilon$. Then $v_\varepsilon \to v$ in $C^{1,\sigma}(\Omega)$, for any $\sigma \in (0,1)$, where v is the solution of* (1.7)*. Moreover, we have*

(i) *if $0 \leqslant \delta < \delta_0$, then $u_\varepsilon \to h_+(v)$ uniformly in any compact subset of Ω as $\varepsilon \to 0$;*
(ii) *if $\delta = \delta_0$, then $\{x\colon v(x) = \alpha_0\} = \Omega \setminus S$ and the measure of the set $\{x\colon v(x) = \alpha_0\}$ is zero. Moreover, $u_\varepsilon \to h_+(v)$ uniformly in any compact subset of S as $\varepsilon \to 0$;*
(iii) *if $\delta > \delta_1 = \max(\delta_0, \gamma\alpha_0/h_-(\alpha_0))$, then the measure of the set $\{x\colon v(x) = \alpha_0\}$ is zero, and $u_\varepsilon \to h_+(v)$ uniformly in any compact subset of S as $\varepsilon \to 0$, $u_\varepsilon \to h_-(v)$ uniformly in any compact subset of $\{x\colon v(x) > \alpha_0\}$ as $\varepsilon \to 0$;*
(iv) *if $\delta_0 < \gamma\alpha_0/h_-(\alpha_0)$ and $\delta \in (\delta_0, \gamma\alpha_0/h_-(\alpha_0))$, then $\{x\colon v(x) = \alpha_0\} = \Omega \setminus S$ and the measure of the set $\{x\colon v(x) = \alpha_0\}$ is positive. Moreover, $u_\varepsilon \to h_+(v)$ uniformly in any compact subset of S as $\varepsilon \to 0$, $u_\varepsilon \to \gamma\alpha_0/\delta$ weak* in $L^\infty(\Omega \setminus S)$ as $\varepsilon \to 0$, but u_ε does not converges almost everywhere to $\gamma\alpha_0/\delta$ as $\varepsilon \to 0$ for any subsequence, and for any $\theta > 0$ small,*

$$m\{x\colon v(x) = \alpha_0, u_\varepsilon(x) \notin (h_-(\alpha_0) - \theta, h_-(\alpha_0) + \theta) \cup (h_+(\alpha_0) - \theta, h_+(\alpha_0) + \theta)\} \to 0$$

as $\varepsilon \to 0$;

(v) *if* $\delta_0 < \gamma\alpha_0/h_-(\alpha_0)$ *and* $\delta = \gamma\alpha_0/h_-(\alpha_0)$, *then* $\{x\colon v(x) = \alpha_0\} = \Omega \setminus S$ *and the measure of the set* $\{x\colon v(x) = \alpha_0\}$ *is positive. Moreover,* $u_\varepsilon \to h_+(v)$ *uniformly in any compact subset of* S *as* $\varepsilon \to 0$, $u_\varepsilon \to h_-(\alpha_0)$ *in measure in* $\Omega \setminus S$ *as* $\varepsilon \to 0$.

If $f(u) = u(a-u)(u-1)$, $0 < a < \frac{1}{2}$, then $h_-(\alpha_0) < 0$. Thus we see from Theorem 1.1 that for $\delta > \delta_0$, the minimizer of (1.4) has a boundary layer, and it oscillates wildly around the constant $\gamma\alpha_0/\delta$ in the set $\Omega \setminus S$. Moreover, for any $T \subset \Omega \setminus S$ which has positive measure, the portion in T where u_ε is close to $h_+(\alpha_0)$ has measure close to $((\gamma\alpha_0\delta^{-1} - h_-(\alpha_0))/(h_+(\alpha_0) - h_-(\alpha_0)))m(T)$, while in most of the rest part of T, u_ε is close to $h_-(\alpha_0)$. If we translate $f(t)$ to the right suitably, we see from Theorem 1.2 that for $\delta > \delta_1$, the minimizer of (1.4) not only has a boundary layer, but also has an interior layer near the measure-zero set $\{x\colon v(x) = \alpha_0\}$.

Noting that δ_0 only depends on $h_+(v)$ for $v \leqslant \alpha_0$, we can easily give examples where (f_1) and (f_2) are satisfied and $\delta_0 > \gamma\alpha_0/h_-(\alpha_0)$, and examples where (f_1) and (f_2) are satisfied and $\delta_0 < \gamma\alpha_0/h_-(\alpha_0)$. In the first case, we only need to construct f, such that $h_-(\alpha_0)$ is very close to $h_+(\alpha_0)$, while in the second case, we only need to construct f, such that $h_-(\alpha_0) > 0$ is very small.

We are not able to prove the uniform convergence of u_ε on any compact subset of Ω if $\delta = \delta_0$. It is not clear whether the convergence in (v) of Theorem 1.2 can be replaced by uniform convergence in any compact subset of $\Omega \setminus S$.

To have a better understanding of the profile of a global minimizer u_ε of (1.3), we can blow up u_ε at any point $x_0 \in \partial\Omega$ and obtain good asymptotic of u_ε near the boundary. Roughly speaking, $u_\varepsilon(x)$ depends mainly on $d(x, \partial\Omega)$ if $d(x, \partial\Omega) \leqslant R\varepsilon$ for any $R > 0$. In other words, u_ε transits from 0 to $h_+(0)$ in the inward normal direction of the boundary. See Proposition 3.5 in Section 3. On the other hand, if we blow up u_ε at a point $x_0 \in \{x\colon v(x) = \alpha_0\}$, we will encounter the following variant of the De Giorgi conjecture [9]:

$$\begin{cases} -\Delta w = f(w) - \alpha_0, & \text{in } R^N, \\ J(w, A) \leqslant J(w+\varphi, A), & \forall \varphi \in H_0^1(A), \end{cases} \tag{1.8}$$

where A is any bounded open set in R^N,

$$J(w, A) = \int_A \left(\frac{1}{2}|Dw|^2 - \big(F(w) - \alpha_0 w\big)\right).$$

Using the results in [1–3,11], we can easily classify all the bounded solutions in (1.8) if $N = 2, 3$. These solutions are either the constants $h_\pm(\alpha_0)$, or the ODE solution. See the discussion in Section 2. As an application of this result to the analysis of the behaviour of u_ε in $\{x\colon v(x) = \alpha_0\}$, we see that if $N = 2, 3$, then u_ε transits from $h_+(\alpha_0)$ to $h_-(\alpha_0)$ mainly in one direction in a neighbourhood of $x_0 \in \{x\colon v(x) = \alpha_0\}$ of order ε, although the direction can change rapidly with x_0. For other phase transition problems which lead to the De Giorgi conjecture, the readers can refer to [17,22].

Our next result shows that for some $\delta > \delta_0$, $I_\varepsilon(u)$ has a local minimizer which behaves quite well in the interior of Ω.

Theorem 1.3. *Let* $\bar\delta > \delta_0$ *be the number such that* $\max_{x\in\Omega} v_{\bar\delta}(x) = f(\tau_2)$, *where* $v_{\bar\delta}$ *is the solution of* (1.6) *with* $\delta = \bar\delta$. *Suppose that* $\delta \in (\delta_0, \bar\delta)$. *Then there is an* $\varepsilon_0 > 0$, *such that for* $\varepsilon \in (0, \varepsilon_0]$, (1.1) *has a solution* $(\bar u_\varepsilon, \bar v_\varepsilon)$, *satisfying*

(i) $\bar v_\varepsilon \to \bar v$ *in* $C^{1,\sigma}(\Omega)$, *for any* $\sigma \in (0,1)$, *where* $\bar v$ *is the solution of* (1.6);
(ii) $\bar u_\varepsilon \to h_+(\bar v)$ *uniformly in any compact subset of* Ω;
(iii) $\bar u_\varepsilon$ *is a local minimizer of* $I_\varepsilon(u)$.

Solutions of the same type as in Theorem 1.3 were obtained in [21] by using a bifurcation theorem. In the result of [21], δ is a parameter depending on ε. In [21], numerical analysis suggests that (1.1) with $f(u) = u(u-a)(1-u)$, $a \in (0, \frac{1}{2})$, have a solution which has an interior layer. Our result here shows that the number of the interior layers of the global minimizer will increase as ε tends to 0 in this case. On the other hand, since $\bar{u}_\varepsilon$ is a local minimum, we can attach a peak solution to this local minimum to get a new solution. We shall discuss this problem in a forthcoming paper. It is worth pointing out that the solution obtained by attaching a peak solution to the local minimum $\bar{u}_\varepsilon$ converges to $h_+(v)$ in $L^p(\Omega)$, $\forall p > 1$, as $\varepsilon \to 0$, but it does not converges to $h_+(v)$ uniformly in any compact subset of Ω. Thus for the solutions of (1.1), L^p convergence does not imply uniform convergence.

This paper is arranged as follows. In Section 2, we prove Theorems 1.1 and 1.2. Section 3 contains the proof of Theorem 1.3.

2. The profile of the global minimizers

Let us recall that $G_\gamma u$ is the solution of

$$\begin{cases} -\Delta v + \gamma v = u, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases}$$

It is easy to check that there is $C > 0$, such that $|G_\gamma u|_\infty \leqslant C|u|_\infty$.

Lemma 2.1. *There is a constant $C > 0$, such that for any solution $(u_\varepsilon, v_\varepsilon)$ of* (1.1)*, we have $|u_\varepsilon|_\infty, |v_\varepsilon|_\infty \leqslant C$.*

Proof. Let $x_0 \in \Omega$ be a maximum point of u_ε. Then

$$0 \leqslant -\varepsilon^2 \Delta u_\varepsilon(x_0) = f\big(u_\varepsilon(x_0)\big) - v_\varepsilon(x_0) \leqslant f\big(u_\varepsilon(x_0)\big) + C u_\varepsilon(x_0).$$

But $f(u)/u \to -\infty$, as $u \to +\infty$. Thus we see from the above relation that $u_\varepsilon(x_0) \leqslant C'$. Similarly, we can prove $\min_{x\in\Omega} u_\varepsilon \geqslant -C'$. □

Let u_ε be a minimizer of (1.4), $v_\varepsilon = \delta G_\gamma u_\varepsilon$. By Lemma 2.1, u_ε is bounded in $L^\infty(\Omega)$. From

$$-\Delta v_\varepsilon + \gamma v_\varepsilon = \delta u_\varepsilon, \quad \text{in } \Omega,$$

we see that v_ε is bounded in $W^{2,p}(\Omega)$ for and $p > 1$. Thus we assume that up to a subsequence,

$$v_\varepsilon \to v \quad \text{in } C^{1,\sigma}(\Omega), \tag{2.1}$$

for any $\sigma \in (0, 1)$.

Lemma 2.2. *Let u_ε be a minimizer of* (1.4)*, $v_\varepsilon = \delta G_\gamma u_\varepsilon$. Then*

$$u_\varepsilon \to \begin{cases} h_+(v), & \text{uniformly in any compact subset of } \{x\colon\ 0 < v(x) < \alpha_0\}; \\ h_-(v), & \text{uniformly in any compact subset of } \{x\colon\ v(x) > \alpha_0\}, \end{cases}$$

Proof. For any small $\tau > 0$, let $\eta > 0$ be small enough, such that

$$\big|v_\varepsilon(x) - v(x_0)\big| < \tau, \quad \forall x \in B_\eta(x_0).$$

Let $M > 0$ be a large constant satisfying $M \geqslant \max_{x\in\bar{\Omega}} |u_\varepsilon|$ for all $\varepsilon > 0$. Consider

$$\inf\big\{J_{\varepsilon,+}(u)\colon\ u \in H^1\big(B_\eta(x_0)\big), u = -M \text{ on } \partial B_\eta(x_0)\big\}, \tag{2.2}$$

where

$$J_{\varepsilon,+}(u) = \frac{\varepsilon^2}{2}\int\limits_{B_\eta(x_0)} |Du|^2 - \int\limits_{B_\eta(x_0)} \bigl(F(u) - (v(x_0)+2\tau)u\bigr).$$

Let $w_{\varepsilon,+}$ be a minimizer of (2.2). Then

$$-\varepsilon^2\Delta w_{\varepsilon,+} + w_{\varepsilon,+} = f(w_{\varepsilon,+}) - \bigl(v(x_0)+2\tau\bigr).$$

Thus similar to the proof of Lemma 2.1, we know that $|w_{\varepsilon,+}| \leqslant C$ for some $C > 0$, independent of $\varepsilon, \eta > 0$ small. We claim that $u_\varepsilon \geqslant w_{\varepsilon,+}$.

Let $S_\varepsilon = \{x\colon w_{\varepsilon,+} > u_\varepsilon, x \in \overline{B_\eta(x_0)}\}$. Since $w_{\varepsilon,+} < u_\varepsilon$ if $|x - x_0| = \eta$, we see $S_\varepsilon \subset B_\eta(x_0)$. Let

$$\varphi_\varepsilon = \begin{cases} w_{\varepsilon,+} - u_\varepsilon, & x \in S_\varepsilon,\\ 0, & x \in \Omega \setminus S_\varepsilon.\end{cases}$$

Then $\varphi_\varepsilon \in H_0^1(\Omega)$ and $\varphi_\varepsilon \geqslant 0$. Thus, we have

$$\begin{aligned} 0 &\leqslant I_\varepsilon(u_\varepsilon+\varphi_\varepsilon) - I_\varepsilon(u_\varepsilon)\\ &= I_\varepsilon^*(u_\varepsilon+\varphi_\varepsilon) - I_\varepsilon^*(u_\varepsilon) + \frac{\delta}{2}\int\limits_\Omega \bigl((u_\varepsilon+\varphi_\varepsilon)G_\gamma(u_\varepsilon+\varphi_\varepsilon) - u_\varepsilon G_\gamma u_\varepsilon\bigr)\\ &= I_\varepsilon^*(u_\varepsilon+\varphi_\varepsilon) - I_\varepsilon^*(u_\varepsilon) + \int\limits_\Omega \varphi_\varepsilon v_\varepsilon + \frac{\delta}{2}\int\limits_\Omega \varphi_\varepsilon G_\gamma\varphi_\varepsilon, \end{aligned} \tag{2.3}$$

where

$$I_\varepsilon^*(u) = \frac{\varepsilon^2}{2}\int\limits_{B_\eta(x_0)} |Du|^2 - \int\limits_{B_\eta(x_0)} F(u).$$

On the other hand, we have

$$\begin{aligned} 0 &\leqslant J_{\varepsilon,+}(w_{\varepsilon,+}-\varphi_\varepsilon) - J_{\varepsilon,+}(w_{\varepsilon,+})\\ &= I_\varepsilon^*(w_{\varepsilon,+}-\varphi_\varepsilon) - I_\varepsilon^*(w_{\varepsilon,+}) - \int\limits_{S_\varepsilon}\bigl(v(x_0)+2\tau\bigr)\varphi_\varepsilon\\ &= I_\varepsilon^*(u_\varepsilon) - I_\varepsilon^*(u_\varepsilon+\varphi_\varepsilon) - \int\limits_{S_\varepsilon}\bigl(v(x_0)+2\tau\bigr)\varphi_\varepsilon\\ &= I_\varepsilon(u_\varepsilon) - I_\varepsilon(u_\varepsilon+\varphi_\varepsilon) + \frac{\delta}{2}\int\limits_\Omega \varphi_\varepsilon G_\gamma\varphi_\varepsilon - \int\limits_{S_\varepsilon}\bigl(v(x_0)+2\tau-v_\varepsilon\bigr)\varphi_\varepsilon\\ &\leqslant \frac{\delta}{2}\int\limits_\Omega \varphi_\varepsilon G_\gamma\varphi_\varepsilon - \int\limits_{S_\varepsilon}\bigl(v(x_0)+2\tau-v_\varepsilon\bigr)\varphi_\varepsilon. \end{aligned} \tag{2.4}$$

Noting that $v(x_0)+2\tau-v_\varepsilon > \tau$ if $x \in B_\eta(x_0)$, we obtain

$$\tau\int\limits_{S_\varepsilon}\varphi_\varepsilon \leqslant \int\limits_{S_\varepsilon}\bigl(v(x_0)+2\tau-v_\varepsilon\bigr)\varphi_\varepsilon \leqslant \frac{\delta}{2}\int\limits_\Omega \varphi_\varepsilon G_\gamma\varphi_\varepsilon. \tag{2.5}$$

Since $|\varphi_\varepsilon| \leqslant 2C$, we have

$$|G_\gamma\varphi_\varepsilon|_{L^\infty(\Omega)} \leqslant C|\varphi_\varepsilon|_{L^p(\Omega)} \leqslant C\eta^{N/p},$$

for $p > \frac{N}{2}$. So

$$\tau \int_{S_\varepsilon} \varphi_\varepsilon \leqslant C\eta^{N/p} \int_{S_\varepsilon} \varphi_\varepsilon .$$

Thus, we see that if $\eta > 0$ small, we obtain $\varphi_\varepsilon = 0$. So we have proved that $w_{\varepsilon,+} \leqslant u_\varepsilon$.

Similarly, consider

$$\inf\bigl\{J_{\varepsilon,-}(u)\colon\ u \in H^1\bigl(B_\eta(x_0)\bigr),\ u = M \text{ on } \partial B_\eta(x_0)\bigr\}, \tag{2.6}$$

where

$$J_{\varepsilon,-}(u) = \frac{\varepsilon^2}{2} \int_{B_\eta(x_0)} |Du|^2 - \int_{B_\eta(x_0)} \bigl(F(u) - (v(x_0) - 2\tau)u\bigr).$$

Let $w_{\varepsilon,-}$ be a minimizer of (2.6). Then we have $u_\varepsilon \leqslant w_{\varepsilon,-}$.

By a result of [6,7], we know

$$w_{\varepsilon,+} \to \begin{cases} h^+(v(x_0) + 2\tau), & \text{if } v(x_0) + 2\tau < \alpha_0; \\ h^-(v(x_0) + 2\tau), & \text{if } v(x_0) + 2\tau > \alpha_0, \end{cases}$$

and

$$w_{\varepsilon,-} \to \begin{cases} h^+(v(x_0) - 2\tau), & \text{if } v(x_0) - 2\tau < \alpha_0; \\ h^-(v(x_0) - 2\tau), & \text{if } v(x_0) - 2\tau > \alpha_0, \end{cases}$$

uniformly on any compact subset of $B_\eta(x_0)$. Thus this lemma follows from $w_{\varepsilon,+} \leqslant u_\varepsilon \leqslant w_{\varepsilon,-}$. □

Lemma 2.3. *Let u_ε be a minimizer of* (1.4), $v_\varepsilon = \delta G_\gamma u_\varepsilon$. *Then*

$$m\bigl\{x\colon\ v(x) = \alpha_0, u_\varepsilon(x) \notin \bigl(h_-(\alpha_0) - \theta, h_-(\alpha_0) + \theta\bigr) \cup \bigl(h_+(\alpha_0) - \theta, h_+(\alpha_0) + \theta\bigr)\bigr\} \to 0$$

as $\varepsilon \to 0$, *where mS denotes the measure of the set S.*

Proof. Let $x_0 \in \Omega$ and let $C_r(x_0)$ be the cube with side r, centred at x_0, with sides parallel to the axes. For any small $\eta > 0$, we may assume that $\varepsilon > 0$ is small enough such that $C_{\varepsilon+\eta}(x_0) \in \Omega$. Define

$$\bar u_\varepsilon(x) = \begin{cases} u_\varepsilon(x), & x \in \Omega \setminus C_{\varepsilon+\eta}(x_0); \\ h_-(v_\varepsilon(x')) + \frac{u_\varepsilon(x'') - h_\varepsilon(x')}{\varepsilon}(|x - x_0| - |x'|), & x \in C_{\varepsilon+\eta}(x_0) \setminus C_\eta(x_0); \\ h_-(v_\varepsilon(x)), & x \in C_\eta(x_0), \end{cases}$$

where $x' = t'_{\eta,x}(x - x_0)/|x - x_0| \in \partial C_\eta(x_0)$ and $x'' = t''_{\eta+\varepsilon,x}(x - x_0)/|x - x_0| \in \partial C_{\eta+\varepsilon}(x_0)$. Then

$$\begin{aligned} 0 &\leqslant I(\bar u_\varepsilon) - I(u_\varepsilon) \\ &= \frac{1}{2}\varepsilon^2 \int_\Omega \bigl(|D\bar u_\varepsilon|^2 - |Du_\varepsilon|^2\bigr) + \frac{\delta}{2} \int_\Omega (\bar u_\varepsilon G_\gamma \bar u_\varepsilon - u_\varepsilon G_\gamma u_\varepsilon) - \int_\Omega \bigl(F(\bar u_\varepsilon) - F(u_\varepsilon)\bigr) \\ &= I_1 + I_2 - I_3. \end{aligned} \tag{2.7}$$

Noting that u_ε satisfies $-\Delta u_\varepsilon = \varepsilon^{-2}(f(u_\varepsilon) - v_\varepsilon)$, using Theorem 2.10 and Theorem 4.5 in [13], we see

$$\varepsilon\bigl|Du_\varepsilon(x)\bigr| \leqslant C|u_\varepsilon|_{L^\infty(B_\varepsilon(x))} + C\varepsilon^2\bigl|\varepsilon^{-2}(f(u_\varepsilon) - v_\varepsilon)\bigr|_{L^\infty(B_\varepsilon(x))}.$$

In particular, $\varepsilon|Du_\varepsilon| \leqslant C$ if $d(x, \partial\Omega) \geqslant 2\varepsilon$. Thus it is easy to check that $\varepsilon|D\bar u_\varepsilon| \leqslant C$. As a result,

$$I_1 = \frac{1}{2}\varepsilon^2 \int\limits_{C_{\varepsilon+\eta}(x_0)} \big(|D\bar{u}_\varepsilon|^2 - |Du_\varepsilon|^2\big) \leqslant \frac{1}{2}\varepsilon^2 \int\limits_{C_{\varepsilon+\eta}(x_0)} |D\bar{u}_\varepsilon|^2$$

$$\leqslant Cm\big(C_{\varepsilon+\eta}(x_0) \setminus C_\eta(x_0)\big) + \frac{1}{2}\varepsilon^2 \int\limits_{C_\eta(x_0)} \big|Dh_-(v_\varepsilon)\big|^2 \leqslant C\big(\varepsilon\eta^{N-1} + \varepsilon^2\eta^N\big). \tag{2.8}$$

On the other hand, we have

$$I_2 = \int\limits_\Omega (\bar{u}_\varepsilon - u_\varepsilon)v_\varepsilon + \frac{\delta}{2}\int\limits_\Omega (\bar{u}_\varepsilon - u_\varepsilon)G_\gamma(\bar{u}_\varepsilon - u_\varepsilon) = I_4 + I_5, \tag{2.9}$$

and

$$I_4 = \int\limits_{C_{\varepsilon+\eta}(x_0)} (\bar{u}_\varepsilon - u_\varepsilon)v_\varepsilon$$

$$= \mathrm{O}\big(m\big(C_{\varepsilon+\eta}(x_0) \setminus C_\eta(x_0)\big)\big) + \int\limits_{C_\eta(x_0)} (\bar{u}_\varepsilon - u_\varepsilon)v_\varepsilon$$

$$= \int\limits_{C_\eta(x_0)} \big(h_-(v_\varepsilon) - u_\varepsilon\big)v_\varepsilon + \mathrm{O}\big(\varepsilon\eta^{N-1}\big). \tag{2.10}$$

Let $G_\gamma(x, y)$ be the Green's function of $-\Delta + \gamma$ with Dirichlet boundary condition. Then $G_\gamma(x, y) \leqslant \frac{C}{|x-y|^{N-2}}$. For any $x \in C_{\varepsilon+\eta}(x_0)$, we have

$$\big|G_\gamma(\bar{u}_\varepsilon - u_\varepsilon)(x)\big| = \left|\int\limits_\Omega G_\gamma(x, y)\big(\bar{u}_\varepsilon(y) - u_\varepsilon(y)\big)\,dy\right|$$

$$= \left|\int\limits_{C_{\varepsilon+\eta}(x_0)} G_\gamma(x, y)\big(\bar{u}_\varepsilon(y) - u_\varepsilon(y)\big)\,dy\right|$$

$$\leqslant C \int\limits_{C_{\varepsilon+\eta}(x_0)} \frac{1}{|x-y|^{N-2}}\,dy \leqslant C(\varepsilon+\eta)^2.$$

So

$$I_5 = \frac{\delta}{2}\int\limits_{C_{\varepsilon+\eta}(x_0)} (\bar{u}_\varepsilon - u_\varepsilon)G_\gamma(\bar{u}_\varepsilon - u_\varepsilon) = \mathrm{O}\big((\varepsilon+\eta)^{N+2}\big). \tag{2.11}$$

For I_3, we have

$$I_3 = \int\limits_{C_{\varepsilon+\eta}(x_0)} \big(F(\bar{u}_\varepsilon) - F(u_\varepsilon)\big) = \int\limits_{C_\eta(x_0)} \big(F(\bar{u}_\varepsilon) - F(u_\varepsilon)\big) + \mathrm{O}\big(\varepsilon\eta^{N-1}\big). \tag{2.12}$$

Combining (2.7)–(2.12), we obtain

$$\int\limits_{C_\eta(x_0)} \big(\big(h_-(v_\varepsilon) - u_\varepsilon\big)v_\varepsilon - \big(F\big(h_-(v_\varepsilon) - F(u_\varepsilon)\big)\big) + \mathrm{O}\big(\varepsilon\eta^{N-1} + (\varepsilon+\eta)^{N+2}\big) \geqslant 0. \tag{2.13}$$

Thus

$$\int_{C_\eta(x_0)} \Big(\big(F\big(h_-(v_\varepsilon)\big) - h_-(v_\varepsilon)v_\varepsilon\big) - \big(F(u_\varepsilon) - u_\varepsilon v_\varepsilon\big)\Big) \leqslant \mathrm{O}\big(\varepsilon\eta^{N-1} + (\varepsilon+\eta)^{N+2}\big). \tag{2.14}$$

Since $v = 0$ on $\partial\Omega$, we see $\{x\colon\ v(x) = \alpha_0\}$ is a compact subset of Ω. Thus we can choose $C_\eta(x_j)$, $j \in J$, where J contains finite number of points, such that, $C_\eta(x_i) \cap C_\eta(x_j) = \emptyset$, $\forall i \neq j$, the set $\{\overline{C_\eta(x_j)},\ j \in J\}$ covers $\{x\colon\ v(x) = \alpha_0\}$. It is easy to see that the number of such cubes is at most C^N/η^N for some large constant $C > 0$ independing on N. Hence, from (2.14), we obtain

$$\int_{v(x)=\alpha_0} \Big(\big(F\big(h_-(v_\varepsilon)\big) - h_-(v_\varepsilon)v_\varepsilon\big) - \big(F(u_\varepsilon) - u_\varepsilon v_\varepsilon\big)\Big) \leqslant C\frac{\varepsilon\eta^{N-1} + (\varepsilon+\eta)^{N+2}}{\eta^N}.$$

So for any $\eta > 0$,

$$\int_{v(x)=\alpha_0} \Big(\big(F\big(h_-(\alpha_0)\big) - h_-(\alpha_0)\alpha_0\big) - \big(F(u_\varepsilon) - u_\varepsilon\alpha_0\big)\Big) \leqslant C\frac{\varepsilon\eta^{N-1} + (\varepsilon+\eta)^{N+2}}{\eta^N} + \mathrm{o}_\varepsilon(1).$$

That is,

$$\int_{v(x)=\alpha_0} \int_{u_\varepsilon}^{h_-(\alpha_0)} \big(f(\tau) - \alpha_0\big)\,d\tau \leqslant C\frac{\varepsilon\eta^{N-1} + (\varepsilon+\eta)^{N+2}}{\eta^N} + \mathrm{o}_\varepsilon(1). \tag{2.15}$$

Note that

$$\int_s^{h_-(\alpha_0)} \big(f(\tau) - \alpha_0\big) \geqslant c_0 > 0,$$

if $s \notin (h_-(\alpha_0) - \theta, h_-(\alpha_0) + \theta) \cup (h_+(\alpha_0) - \theta, h_+(\alpha_0) + \theta)$, and $\int_s^{h_-(\alpha_0)}(f(\tau) - \alpha_0) \geqslant 0$ for all s, (2.15) yields

$$m\big\{x\colon\ v(x) = \alpha_0, u_\varepsilon(x) \notin \big(h_-(\alpha_0) - \theta, h_-(\alpha_0) + \theta\big) \cup \big(h_+(\alpha_0) - \theta, h_+(\alpha_0) + \theta\big)\big\} \to 0 \tag{2.16}$$

as $\varepsilon \to 0$ for every $\theta > 0$ small. □

Lemma 2.4. *Let u_ε be a minimizer of* (1.4)*, $v_\varepsilon = \delta G_\gamma u_\varepsilon$. Then $v_\varepsilon \to v$ in $C^{1,\sigma}(\Omega)$ for any $\sigma \in (0,1)$, and v is a solution of* (1.7)*.*

Proof. Since u_ε is bounded in $L^\infty(\Omega)$, we may assume that up to a subsequence, there is a $u \in L^\infty(\Omega)$, such that

$$u_\varepsilon \to u, \quad \text{weak* in } L^\infty(\Omega).$$

By Lemmas 2.2 and 2.3, we see $u = h_+(v)$ if $x \in \{x\colon\ v(x) < \alpha_0\}$, $u = h_-(v)$ if $x \in \{x\colon\ v(x) > \alpha_0\}$, and $u \in [h_-(\alpha_0), h_+(\alpha_0)]$ if $x \in \{x\colon\ v(x) = \alpha_0\}$. Thus, v satisfies

$$\begin{cases} -\Delta v + \gamma v \in [\delta h(v-0), \delta h(v+0)], & \text{in } \Omega,\\ v = 0, & \text{on } \partial\Omega,\end{cases}$$

where $h(v) = h_+(v)$ if $v < \alpha_0$, $h(v) = h_-(v)$ if $v > \alpha_0$. □

Before we prove Theorems 1.1 and 1.2, we need the following lemma:

Lemma 2.5. *There is a $\delta_0 > 0$, such that if $\delta \in (0,\delta_0)$, the solution v of* (1.6) *satisfies* $\max_{x\in\Omega} v(x) < \alpha_0$*; if $\delta > \delta_0$, the solution v of* (1.6) *satisfies* $\max_{x\in\Omega} v(x) > \alpha_0$.

Proof. By the maximum principle, we can check easily that if $\delta_1 < \delta_2$, then the solutions v_{δ_1} and v_{δ_2} of (1.6) corresponding to $\delta = \delta_1$ and $\delta = \delta_2$ respectively satisfy $v_{\delta_1} < v_{\delta_2}$. On the other hand, suppose that $\max_{x\in\Omega} v_\delta \leqslant \alpha_0$ for $\delta \to +\infty$. Since

$$-\Delta v_\delta + \gamma v_\delta = \delta h^+(v_\delta) \geqslant \delta h^+(\alpha_0),$$

we see $v_\delta \geqslant c_0 \delta e$, for some constant $c_0 > 0$, where $e > 0$ is the first eigenfunction of $-\Delta + \gamma$ with Dirichlet condition. This is a contradiction.

Let

$$\delta_0 = \inf\Big\{\delta\colon \max_{x\in\Omega} v_\delta > \alpha_0\Big\}.$$

Then $\delta_0 \in (0, +\infty)$ and δ_0 is the number we need. □

Remark 2.6. It is easy to see from $-\Delta v(x_0) > 0$ at any maximum point of v that $\delta_0 > \gamma\alpha_0/h_+(\alpha_0)$.

Proof of Theorem 1.1. If $\delta \in (0, \delta_0)$, it follows from Lemma 2.5 that the solution v of (1.7) satisfies $v < \alpha_0$. Thus (i) follows from Lemma 2.2.

If $\delta = \delta_0$, then $\max_{x\in\Omega} = \alpha_0$. Suppose that $m\{x\colon v(x) = \alpha_0\} > 0$. Then we have $\delta_0 = \gamma\alpha_0/h_+(\alpha_0)$. This is a contradiction to Remark 2.6. Thus $m\{x\colon v(x) = \alpha_0\} = 0$ and (ii) follows from Lemma 2.2.

Suppose that $\delta > \delta_0$. Since $h(t) \leqslant 0$ if $t > \alpha_0$, we see that the solution v_δ of (1.7) satisfies $v_\delta(x) \leqslant \alpha_0$ for all $x \in \Omega$. Now we claim that

$$m\big\{x\colon v_\delta(x) = \alpha_0\big\} > 0.$$

Suppose that $m\{x\colon v_\delta(x) = \alpha_0\} = 0$. Then we see that v_δ is also the solution of (1.6) and $v_\delta \leqslant \alpha_0$. This is a contradiction to the definition of δ_0.

Suppose that $u_\varepsilon \to \gamma\alpha_0/\delta$ almost everywhere in $\{x : v_\delta(x) = \alpha_0\}$. Then

$$m\left\{x\colon \left|u_\varepsilon(x) - \frac{\gamma\alpha_0}{\delta}\right| \geqslant \tau\right\} \to 0$$

as $\varepsilon \to 0$, for any $\tau > 0$. This is a contradiction to Lemma 2.3 and Remark 2.6. Thus, (iii) follows from Lemmas 2.2, 2.3 and 2.4. □

Proof of Theorem 1.2. The proofs of (i) and (ii) of this theorem are exactly the same as those in Theorem 1.1.

Suppose that $\delta > \gamma\alpha_0/h_-(\alpha_0)$. We claim that

$$m\big\{x\colon v_\delta(x) = \alpha_0\big\} = 0.$$

Suppose that $m\{x\colon v_\delta(x) = \alpha_0\} > 0$. Then we have

$$\gamma\alpha_0 = \delta u(x), \qquad \text{for almost every } x \in \big\{x\colon v_\delta(x) = \alpha_0\big\}.$$

So $u(x) = \gamma\alpha_0/\delta < h_-(\alpha_0)$. This is a contradiction to $u(x) \in [h_-(\alpha_0), h_+(\alpha_0)]$ for almost every $x \in \{x\colon v_\delta(x) = \alpha_0\}$. Thus (iii) follows from Lemma 2.2.

Now we consider the case $\delta_0 < \gamma\alpha_0/h_-(\alpha_0)$.

Suppose that $\delta \in (\delta_0, \gamma\alpha_0/h_-(\alpha_0)]$. We claim that $\max_{x\in\Omega} v(x) = \alpha_0$. In fact, since $\delta h_-(\alpha_0) - \gamma\alpha_0 \leqslant 0$ and $h_-(t)$ is decreasing for $t > \alpha_0$, we see that $\delta h_-(t) - \gamma t < 0$ if $t > \alpha_0$. Suppose that $\max_{x\in\Omega} v(x) > \alpha_0$ and let $x_0 \in \Omega$ satisfy $v(x_0) = \max_{x\in\Omega} v(x) > \alpha_0$. Then v is C^2 in a small neighbourhood of x_0. But

$$0 \leqslant -\Delta v(x_0) = \delta h_-\big(v(x_0)\big) - \gamma v(x_0) < 0.$$

So we get a contradiction.

Since

$$\frac{\delta\alpha_0}{\gamma} \in \big(h_-(\alpha_0), h_+(\alpha_0)\big)$$

if $\delta \in (\delta_0, \gamma\alpha_0/h_-(\alpha_0))$ we can prove (iv) in a similar way as in the proof of (iii) of Theorem 1.1.

Finally, if $\delta = \delta_1 = \gamma\alpha_0/h_-(\alpha_0)$, then $u_\varepsilon \to h_-(\alpha_0)$ weak* in $L^\infty(\Omega \setminus S)$, which, together with Lemma 2.3, gives $u_\varepsilon \to h_-(\alpha_0)$ in measure in $\Omega \setminus S$. □

Before we close this section, we discuss briefly the local behaviour of u_ε in a small neighbourhood of $x_0 \in \{x\colon v(x) = \alpha_0\}$.

Let $w_\varepsilon(y) = u_\varepsilon(\varepsilon y + x_0)$. Then w_ε satisfies

$$-\Delta w_\varepsilon = f(w_\varepsilon) - v(\varepsilon y + x_0), \quad y \in \Omega_\varepsilon =: \{y\colon \varepsilon y + x_0 \in \Omega\}.$$

Since w_ε is bounded in $L^\infty(\Omega_\varepsilon)$, we may assume that

$$w_\varepsilon \to w, \quad \text{in } C^2_{\mathrm{loc}}\big(R^N\big).$$

We have the following result:

Proposition 2.7. *Let w be the function defined above. Then w satisfies*

$$\begin{cases} -\Delta w = f(w) - \alpha_0, & \text{in } R^N, \\ J(w, A) \leqslant J(w + \varphi, A), & \forall \varphi \in H^1_0(A), \end{cases}$$

where A is any bounded open set in R^N, $J(w, A) = \int_A(\frac{1}{2}|Dw|^2 - (F(w) - \alpha_0 w))$. If $N = 2, 3$, then either $w = h_-(\alpha_0)$, or $w = h_+(\alpha_0)$, or $w(y) = w_0(\langle a, y\rangle)$ for some $a \in S^{N-1}$, where w_0 is a solution of

$$-w_0'' = f(w_0) - \alpha_0, \quad w_0' > 0, \quad \text{in } R^1.$$

Proof. It is easy to see that

$$-\Delta w = f(w) - \alpha_0, \quad \text{in } R^N.$$

On the other hand, for any bounded open set A in R^N, and $\varphi \in H^1_0(A)$, we have

$$I(u_\varepsilon) \leqslant I(u_\varepsilon + \varphi_\varepsilon),$$

where $\varphi_\varepsilon(x) = \varphi((x - x_0)/\varepsilon)$. Thus

$$-\int_\Omega F(u_\varepsilon) \leqslant \varepsilon^2 \int_\Omega Du_\varepsilon D\varphi_\varepsilon + \frac{1}{2}\varepsilon^2 \int_\Omega |D\varphi_\varepsilon|^2 - \int_\Omega F(u_\varepsilon + \varphi_\varepsilon) + \int_\Omega \varphi_\varepsilon v_\varepsilon + \frac{\delta}{2} \int_\Omega \varphi_\varepsilon G_\gamma \varphi_\varepsilon.$$

That is,

$$-\int_A F(w_\varepsilon) \leqslant \int_A Dw_\varepsilon D\varphi + \frac{1}{2} \int_A |D\varphi|^2 - \int_A F(w_\varepsilon + \varphi) + \int_A \varphi v_\varepsilon(\varepsilon y + x_0) + \frac{\delta}{2\varepsilon^N} \int_\Omega \varphi_\varepsilon G_\gamma \varphi_\varepsilon. \tag{2.17}$$

Since $|G_\gamma \varphi_\varepsilon|_{L^\infty(\Omega)} \to 0$ as $\varepsilon \to 0$, we have

$$\left|\int_\Omega \varphi_\varepsilon G_\gamma \varphi_\varepsilon\right| \leqslant |G_\gamma \varphi_\varepsilon|_{L^\infty(\Omega)} \int_\Omega |\varphi_\varepsilon| = \mathrm{o}\big(\varepsilon^N\big).$$

Letting $\varepsilon \to 0$ in (2.17), we obtain

$$-\int_A F(w) \leqslant \int_A Dw D\varphi + \frac{1}{2} \int_A |D\varphi|^2 - \int_A F(w + \varphi) + \int_\Omega \varphi\alpha_0.$$

That is $J(w,A) \leqslant J(w+\varphi,A)$.

It is easy to see that $J(w,A) \leqslant J(w+\varphi,A)$ implies

$$\int_{B_R(0)} |Dw|^2 \leqslant CR^{N-1}, \tag{2.18}$$

for any $R > 0$, where $C > 0$ is some constant independent of R. See for example [2].

On the other hand, $J(w,A) \leqslant J(w+\varphi,A)$ implies

$$\int_{R^N} \left(|D\varphi|^2 - f'(w)\varphi^2\right) \geqslant 0, \quad \forall \varphi \in C_0^\infty\left(R^N\right), \tag{2.19}$$

which will give that the following problem have a positive solution ξ:

$$-\Delta\xi - f'(w)\xi = 0, \quad \text{in } R^N.$$

See for example [3,11]. Thus, using (2.18), we see that if $N = 2, 3$, there is a constant C_i, such that

$$\frac{\partial w}{\partial x_i} = C_i \xi.$$

See [2,3].

If $C_i = 0$, $i = 1, \ldots, N$, then $w = C$. Thus $f(C) - \alpha_0 = 0$. But from (2.19), we see $f'(C) \leqslant 0$. Thus $C = h_\pm(\alpha_0)$.

If $C_i \neq 0$ for some i, then $\partial w/\partial x_j = C_j' \partial w/\partial x_i$, $j = 1, \ldots, N$. Thus the result follows. □

Remark 2.8. The second part in Proposition 2.7 is a direct consequence of the results in [2,3,11]. This fact was observed in [12].

3. The existence of local minimizer

In Section 2, we have proved that if $\delta > \delta_0$, the global minimizer of (1.4) will either oscillate around a constant in an open set of positive measure, or have an interior jump. In this section, we shall prove that there exists a $\bar\delta > \delta_0$, such that (1.1) has a solution, which is a local minimizer of $I_\varepsilon(u)$ and just has a boundary layer.

Let $\bar\delta > 0$ be the constant, such that the solution $v_{\bar\delta}$ of (1.6) satisfies

$$f(\tau_2) = \max_{x\in\Omega} v_{\bar\delta}(x).$$

Then $\delta_0 < \bar\delta$.

Suppose that $\delta \in (\delta_0, \bar\delta)$. Let v_δ be the solution of (1.6). Then we have

$$\max_{x\in\Omega} v_\delta(x) \in \left(\alpha_0, f(\tau_0)\right).$$

Let $A = \{x \in \Omega\colon v_\delta(x) \geqslant \alpha_0\}$, where v_δ is the solution of (1.6). Then A is a compact subset of Ω. Let $\theta > 0$ be so small that $A_\theta = \{x\colon d(x,A) \leqslant \theta\} \subset \Omega$.

We denote by $g(u)$ an extension of $f(u)$, $u \geqslant \tau_2$, into $(-\infty, \tau_2)$ in such a way that $g(u) \in C^1(R^1)$ and $g(u)$ is decreasing. Let

$$\bar f(x,u) = (1 - 1_{A_\theta}) f(u) + 1_{A_\theta} g(u),$$

where $1_S = 1$ if $x \in S$, $1_S = 0$ if $x \notin S$.

Consider the following problem

$$\inf\left\{J_\varepsilon(u), u \in H_0^1(\Omega)\right\}, \tag{3.1}$$

where

$$J_\varepsilon(u)=\frac{1}{2}\int\limits_{\Omega}\left(|Du|^2+uG_\gamma u\right)-\int\limits_{\Omega}\bar F(x,u),$$

and $\bar F(x,u)=\int_0^u \bar f(x,\tau)\,d\tau$.

Let $u=k(v)$ be the inverse function of $v=g(u)$. Let $\bar u_\varepsilon$ be a minimizer of (3.1), $\bar v_\varepsilon=\delta G_\gamma \bar u_\varepsilon$. Then, $\bar u_\varepsilon$ is uniformly bounded and $\bar v_\varepsilon$ is bounded in $W^{2,p}(\Omega)$ for any $p>1$. Thus we have

$$\bar v_\varepsilon\to\bar v,\quad \text{in } C^{1,\sigma}(\Omega),$$

for any $\sigma\in(0,1)$. Similar to Lemmas 2.2 and 2.3, we have

Lemma 3.1.

$$\bar u_\varepsilon\to\begin{cases}k(\bar v), & \textit{uniformly in any compact subset of } \operatorname{int}(A_\theta);\\ h_+(\bar v), & \textit{uniformly in any compact subset of } \{x:\ 0<\bar v(x)<\alpha_0\}\cap(\Omega\setminus A_\theta);\\ h_-(\bar v), & \textit{uniformly in any compact subset of } \{x:\ \bar v(x)>\alpha_0\}\cap(\Omega\setminus A_\theta),\end{cases}$$

Lemma 3.2.

$$m\bigl\{x:\ x\in\Omega\setminus A_\theta,\bar v(x)=\alpha_0,\ \bar u_\varepsilon(x)\notin\bigl(h_-(\alpha_0)-\bar\theta,h_-(\alpha_0)+\bar\theta\bigr)\cup\bigl(h_+(\alpha_0)-\bar\theta,h_+(\alpha_0)+\bar\theta\bigr)\bigr\}\to 0$$

as $\varepsilon\to 0$, *for any* $\bar\theta>0$.

The proofs of Lemmas 3.1 and 3.2 are exactly the same as those of Lemmas 2.2 and 2.3, and thus we omit them. Define

$$\bar k(x,v)=(1-1_{A_\theta})h(v)+1_{A_\theta}k(v).$$

Then, from Lemmas 3.1 and 3.2, we have

Lemma 3.3. $\bar v$ *satisfies*

$$\begin{cases}-\Delta v+\gamma v\in[\delta\bar k(x,v+0),\delta\bar k(x,v-0)], & \textit{in } \Omega,\\ v=0, & \textit{on } \partial\Omega.\end{cases}\tag{3.2}$$

For each fixed x, $\bar k(x,v)$ is decreasing in v, thus it is easy to see that the solution of (3.2) is unique. Now we are ready to prove the following result:

Proposition 3.4. *Suppose that* $\delta\in(\delta_0,\bar\delta)$. *Let* $\bar u_\varepsilon$ *be a minimizer of* (3.1), $\bar v_\varepsilon=\delta G_\gamma\bar u_\varepsilon$. *Then*

$$\bar u_\varepsilon\to h_+(\bar v),\quad \textit{uniformly in any compact subset of } \Omega,$$

and $\bar v_\varepsilon\to\bar v$ *in* $C^{1,\sigma}(\Omega)$, *where* $\bar v$ *is the solution of* (1.6).

Proof. First we prove that $\bar v$ is the solution of (1.6). Because the solution of (3.2) is unique, to prove that $\bar v$ satisfies (1.6), we only need to prove that the solution v of (1.6) also satisfies (3.2).

Since $\delta\in(\delta_0,\bar\delta)$, we know the solution v of (1.6) satisfies $\max_{x\in\Omega}v(x)\in(\alpha_0,f(\tau_2))$. Thus, $\bar k(x,v)=k(v)=h_+(v)$ if $x\in A_\theta$. On the other hand, $v<\alpha_0$ if $x\in\Omega\setminus A_\theta$. Thus $\bar k(x,v)=h(v)=h_+(v)$ if $x\in\Omega\setminus A_\theta$. Hence, v is the solution of (3.2) and

$$\bigl\{x:\ v(x)\geqslant\alpha_0\bigr\}\cap(\Omega\setminus A_\theta)=\emptyset.$$

In view of Lemma 3.1, to prove Proposition 3.4, it remains to prove that for any $x_0 \in \partial A_\theta$,

$$\bar u_\varepsilon \to h_+(\bar v), \quad \text{uniformly in } B_{\theta/2}(x_0).$$

The proof of this claim is similar to that in Lemma 2.2. The only change here is that we need to use that minimizer of the following problem to control $\bar u_\varepsilon$:

$$\inf\left\{\frac{\varepsilon^2}{2}\int\limits_{B_\eta(x_0)} |Du|^2 - \int\limits_{B_\eta(x_0)} \big(\overline{F}(x,u) - v_0 u\big):\ u \in H^1\big(B_\eta(x_0)\big),\ u = C \text{ on } \partial B_\eta(x_0)\right\}, \tag{3.3}$$

where $v_0 \in (0, \alpha_0)$ is a constant

It is easy to check that the minimizer w_ε of (3.3) satisfies $w_\varepsilon \to h_+(v_0)$ uniformly in $B_{\eta/2}(x_0)$. Noting that $v_\varepsilon(x) < \alpha_0$ for any $x \in \partial A_\theta$, we can now prove that $\bar u_\varepsilon \to h_+(\bar v)$, uniformly in $B_{\theta/2}(x_0)$ in exactly the same way as in Lemma 2.2. □

The following result gives the asymptotic behaviour of the minimizer of (3.1) near the boundary.

Proposition 3.5. *Let $\bar u_\varepsilon$ be the minimizer of* (3.1) (*or* (1.3)). *Let $U_\varepsilon(y) = \bar u_\varepsilon(\varepsilon y + x_0)$, $x_0 \in \partial\Omega$, then $U_\varepsilon(y) \to U(y)$ as $\varepsilon \to 0$ in $C^2_{\mathrm{loc}}(R^N_+)$* (*after suitably translating and rotating the coordinate systems*), *and U is the unique solution of*

$$\begin{cases} -\Delta U = f(U), & \text{in } R^N_+,\\ 0 \leqslant U \leqslant h_+(0), & \text{in } R^N_+,\\ U = 0, & \text{on } x_N = 0,\\ U(x', x_N) \to h_+(0), & \text{as } x_N \to +\infty, \text{ uniformly for } x' \in R^{N-1}. \end{cases} \tag{3.4}$$

Proof. In fact, since U_ε satisfies

$$-\Delta U_\varepsilon = f(U_\varepsilon) - \bar v_\varepsilon(\varepsilon y + x_0),$$

U_ε is bounded in L^∞ and $\bar v_\varepsilon(\varepsilon y + x_0) \to 0$ as $\varepsilon \to 0$ uniformly for bounded y, we see that

$$U_\varepsilon(y) \to U(y) \quad \text{in } C^2_{\mathrm{loc}}\big(R^N_+\big),$$

as $\varepsilon \to 0$, and $U(y)$ satisfies

$$\begin{cases} -\Delta U = f(U), & \text{in } R^N_+,\\ U = 0, & \text{on } x_N = 0. \end{cases}$$

Now we prove $U(x', x_N) \to h_+(0)$, as $x_N \to +\infty$, uniformly for $x' \in R^{N-1}$. To prove this, we only need to prove that for any $\tau > 0$ small, there exists $R_0 > 0$ large, such that

$$\big|\bar u_\varepsilon(x + \varepsilon R \nu) - h_+(0)\big| < \tau, \tag{3.5}$$

for all $x \in \partial\Omega$, $R \geqslant R_0$, $\varepsilon \in (0, \varepsilon_R)$, where ν is the unit inward normal of $\partial\Omega$ at x, $\varepsilon_R > 0$ is a small constant depending on R.

For any $x \in \partial\Omega$, let $x_\varepsilon = x + \varepsilon R \nu$. Consider the following problem:

$$\inf\left\{\frac{\varepsilon^2}{2}\int\limits_{B_{\varepsilon R}(x_\varepsilon)} |D\overline{w}|^2 - \int\limits_{B_{\varepsilon R}(x_\varepsilon)} \big(F(\overline{w}) - \eta\overline{w}\big):\ \overline{w} \in H^1\big(B_{\varepsilon R}(x_\varepsilon)\big),\ \overline{w} = C \text{ on } \partial B_{\varepsilon R}(x_\varepsilon)\right\}, \tag{3.6}$$

where $|\eta| > 0$ is a small constant and C is a constant.

Let $w(y) = \overline{w}(\varepsilon R y + x_\varepsilon)$. Then (3.6) becomes

$$\inf\left\{ \frac{1}{R^2} \int\limits_{B_1(0)} |Dw|^2 - \int\limits_{B_1(0)} \big(F(w) - \eta w\big) \colon\ w \in H^1\big(B_1(0)\big),\ w = C, \text{on } \partial B_1(0) \right\}. \tag{3.7}$$

Let w_R be the minimizer of (3.7). Then there is a $R_0 > 0$ large, such that

$$\big| w_R(y) - h_+(\eta) \big| < \tau,$$

for all $R > R_0$, $y \in B_{1/2}(0)$. Thus, the minimizer $\overline{w}_\varepsilon$ of (3.6) satisfies

$$\big| \overline{w}_\varepsilon(y) - h_+(\eta) \big| < \tau, \tag{3.8}$$

for all $R > R_0$, $y \in B_{\varepsilon R/2}(x_\varepsilon)$.

Now for each $R > R_0$, we choose $\varepsilon_R > 0$ small, such that $\varepsilon R < \theta$ for $\varepsilon \in (0, \varepsilon_R)$, where $\theta > 0$ is a suitably small constant. Let $\overline{w}_{\varepsilon,-}$ be the minimizer of

$$\inf\left\{ \frac{\varepsilon^2}{2} \int\limits_{B_{\varepsilon R}(x_\varepsilon)} |D\overline{w}|^2 - \int\limits_{B_{\varepsilon R}(x_\varepsilon)} \big(F(\overline{w}) - \bar\eta \overline{w}\big) \colon\ \overline{w} \in H^1\big(B_{\varepsilon R}(x_\varepsilon)\big),\ \overline{w} = \overline{C} \text{ on } \partial B_{\varepsilon R}(x_\varepsilon) \right\}, \tag{3.9}$$

and let $\overline{w}_{\varepsilon,+}$ be the minimizer of

$$\inf\left\{ \frac{\varepsilon^2}{2} \int\limits_{B_{\varepsilon R}(x_\varepsilon)} |D\overline{w}|^2 - \int\limits_{B_{\varepsilon R}(x_\varepsilon)} \big(F(\overline{w}) + \bar\eta \overline{w}\big) \colon\ \overline{w} \in H^1\big(B_{\varepsilon R}(x_\varepsilon)\big),\ \overline{w} = -\overline{C}, \text{on } \partial B_{\varepsilon R}(x_\varepsilon) \right\}, \tag{3.10}$$

where $\bar\eta > 0$ is a small constant and $\bar C > 0$ is a large constant. Similar to the proof of Lemma 2.2, we know that if $\theta > 0$ is suitably small, then

$$\overline{w}_{\varepsilon,-} < u_\varepsilon < w_{\varepsilon,+}, \quad \forall y \in B_{\varepsilon R/2}(x_\varepsilon).$$

On the other hand, it follows from (3.8) that

$$\big| \overline{w}_{\varepsilon,+} - h_+(-\eta) \big|,\ \big| \overline{w}_{\varepsilon,-} - h_+(\bar\eta) \big| < \tau.$$

Thus (3.5) follows.

It remains to prove that $0 \leqslant U \leqslant h_+(0)$, in R^N_+. For any $\eta > 0$ small, we claim that

$$-\eta < \bar u_\varepsilon(x) < h_+(0) + \eta, \quad \forall x \in \big\{ x \in \Omega \colon\ d(x, \partial\Omega) \leqslant R\varepsilon \big\}. \tag{3.11}$$

Let $S_\varepsilon = \{x \colon\ u_\varepsilon(x) > h_+(0) + \eta,\ d(x, \partial\Omega) \leqslant R\varepsilon\}$. By (3.5), we know that $S_\varepsilon \cap \{x \colon\ d(x, \partial\Omega) = R\varepsilon\} = \emptyset$. Define $w_\varepsilon = u_\varepsilon$ if $x \in \Omega \setminus S_\varepsilon$, $w_\varepsilon = h_+(0) + \eta$ if $x \in S_\varepsilon$. Then $u_\varepsilon - w_\varepsilon \in H^1_0(\Omega)$. Thus we have

$$\begin{aligned}
0 &\leqslant J_\varepsilon(w_\varepsilon) - J_\varepsilon(u_\varepsilon) \\
&\leqslant \int\limits_{S_\varepsilon} \big(F(u_\varepsilon) - F\big(h_+(0) + \eta\big)\big) + \delta \int\limits_{S_\varepsilon} (u_\varepsilon - w_\varepsilon) G_\gamma u_\varepsilon + \frac{\delta}{2} \int\limits_{S_\varepsilon} (u_\varepsilon - w_\varepsilon) G_\gamma (u_\varepsilon - w_\varepsilon) \\
&\leqslant \int\limits_{S_\varepsilon} \big(F(u_\varepsilon) - F\big(h_+(0) + \eta\big)\big) + \delta |G_\gamma u_\varepsilon|_{L^\infty(S_\varepsilon)} \int\limits_{S_\varepsilon} (u_\varepsilon - w_\varepsilon) \\
&\quad + \big| G_\gamma (u_\varepsilon - w_\varepsilon) \big|_{L^\infty(\Omega)} \frac{\delta}{2} \int\limits_{S_\varepsilon} (u_\varepsilon - w_\varepsilon).
\end{aligned} \tag{3.12}$$

Because $G_\gamma u_\varepsilon$ is small near the boundary of Ω and $S_\varepsilon \subset \{x \colon\ d(x, \partial\Omega) \leqslant \tau'\}$, we see

$$\big| G_\gamma u_\varepsilon \big|_{L^\infty(S_\varepsilon)} \leqslant \tau(\varepsilon),$$

where $\tau(\varepsilon) \to 0$ as $\varepsilon \to 0$. On the other hand, we have

$$\left|G_\gamma(u_\varepsilon - w_\varepsilon)\right|_{L^\infty(\Omega)} \leqslant |u_\varepsilon - w_\varepsilon|_{L^p(\Omega)} \leqslant Cm(S_\varepsilon).$$

Thus,

$$0 \leqslant J_\varepsilon(w_\varepsilon) - J_\varepsilon(u_\varepsilon) \leqslant \int_{S_\varepsilon} \left(F(u_\varepsilon) - F\left(h_+(0) + \eta\right)\right) + \tau''(\varepsilon) \int_{S_\varepsilon} (u_\varepsilon - w_\varepsilon), \tag{3.13}$$

where $\tau''(\varepsilon) \to 0$ as $\varepsilon \to 0$.

But

$$F\left(h_+(0) + \eta\right) - F(u_\varepsilon) = \int_{u_\varepsilon}^{h_+(0)+\eta} f(s)\,ds \geqslant -f\left(h_+(0) + \eta\right)\left(u_\varepsilon - (h_+(0) + \eta)\right),$$

for any $u_\varepsilon > h_+(0) + \eta$. Thus we obtain from (3.13) that

$$-f\left(h_+(0) + \eta\right) \int_{S_\varepsilon} \left(u_\varepsilon - \left(h_+(0) + \eta\right)\right) \leqslant \tau''(\varepsilon) \int_{S_\varepsilon} \left(u_\varepsilon - \left(h_+(0) + \eta\right)\right).$$

Thus $S_\varepsilon = \emptyset$. Thus $u_\varepsilon < h_+(0) + \eta$. Similarly, $u_\varepsilon > -\eta$ if $d(x, \partial\Omega) \leqslant R\varepsilon$. Thus we have proved (3.11). Clearly, $0 \leqslant U \leqslant h_+(0)$ in R^N_+ follows from (3.11). □

Remark 3.6. The solution of (3.4) is unique and is a function of x_N only. See [7].

Proposition 3.7. *Suppose that* $\delta \in (\delta_0, \bar\delta)$. *Let* $\bar u_\varepsilon$ *be a minimizer of* (3.1). *Then* u_ε *is a local minimizer of* (1.4).

Proof. We only need to prove

$$\int_\Omega \left(\varepsilon^2 |D\varphi|^2 + \delta\varphi G_\gamma \varphi\right) - \int_\Omega f'(\bar u_\varepsilon)\varphi^2 \geqslant c_0 \int_\Omega \varphi^2, \quad \forall \varphi \in H^1_0(\Omega),$$

for some $c_0 > 0$. But

$$\int_\Omega \left(\varepsilon^2 |D\varphi|^2 + \delta\varphi G_\gamma \varphi\right) - \int_\Omega f'(\bar u_\varepsilon)\varphi^2 \geqslant \int_\Omega \varepsilon^2 |D\varphi|^2 - \int_\Omega f'(\bar u_\varepsilon)\varphi^2,$$

so the claim follows if we can prove

$$\inf_{\varphi \in H^1_0(\Omega),\ \varphi \neq 0} \frac{\int_\Omega \varepsilon^2 |D\varphi|^2 - \int_\Omega f'(\bar u_\varepsilon)\varphi^2}{\int_\Omega \varphi^2} =: \mu_\varepsilon > 0. \tag{3.14}$$

Let φ_ε is a minimizer of (3.14). We may choose φ_ε such that $\varphi_\varepsilon \geqslant 0$ and $\max_{x\in\Omega} \varphi(x) = 1$.

Suppose that $\mu_\varepsilon \to \mu \leqslant 0$. Let x_ε be a maximum point of φ_ε. Suppose that $d(x_\varepsilon, \partial\Omega)/\varepsilon \to +\infty$ as $\varepsilon \to 0$. Then $|u_\varepsilon(x_\varepsilon) - h_+(v(x_\varepsilon))|$ is small. As a result, $f'(\bar u_\varepsilon(x_\varepsilon)) \leqslant -c_0 < 0$. Since

$$-\Delta\varphi_\varepsilon - f'(\bar u_\varepsilon)\varphi_\varepsilon = \mu_\varepsilon\varphi_\varepsilon,$$

we see that $-f'(\bar u_\varepsilon(x_\varepsilon)) \leqslant \mu_\varepsilon \to 0$ as $\varepsilon \to 0$. This is impossible. So we have proved that $d(x_\varepsilon, \partial\Omega)/\varepsilon \to c < +\infty$.

Let $\bar\varphi_\varepsilon(y) = \varphi_\varepsilon(\varepsilon y + \bar x_\varepsilon)$, where $\bar x_\varepsilon \in \partial\Omega$ is the point such that $|\bar x_\varepsilon - x_\varepsilon| = d(x_\varepsilon, \partial\Omega)$. Then $\bar\varphi_\varepsilon$ is bounded in L^∞ and $\bar\varphi_\varepsilon((x_\varepsilon - \bar x_\varepsilon)/\varepsilon) = 1$. Moreover, $\bar\varphi_\varepsilon$ satisfies

$$-\Delta\bar\varphi_\varepsilon - f'\left(\bar u_\varepsilon(\varepsilon y + \bar x_\varepsilon)\right)\bar\varphi_\varepsilon = \mu_\varepsilon\bar\varphi_\varepsilon.$$

Thus, in view of the boundedness of $\bar\varphi_\varepsilon$, we may assume up to a subsequence that $\bar\varphi_\varepsilon \to \bar\varphi$ in $C^2_{\mathrm{loc}}(R^N_+)$ and $\bar\varphi$ is a bounded nontrivial solution of

$$\begin{cases} -\Delta\bar\varphi - f'(U)\bar\varphi = \mu\bar\varphi, & \text{in } R^N_+,\\ \bar\varphi = 0, & \text{on } R^{N-1}, \end{cases}$$

where U is the solution of (3.4). This is impossible. See the proof of Lemma 4.2 in [7], or the proof of Proposition 2 in [8]. □

Proof of Theorem 1.3. Theorem 1.3 follows from Propositions 3.4 and 3.7. □

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