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Regularity for degenerate two-phase free boundary problems

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Abstract

We provide a rather complete description of the sharp regularity theory to a family of heterogeneous, two-phase free boundary problems, $\mathcal{J}_\gamma \to \min$, ruled by nonlinear, p-degenerate elliptic operators. Included in such family are heterogeneous cavitation problems of Prandtl–Batchelor type, singular degenerate elliptic equations; and obstacle type systems. The Euler–Lagrange equation associated to $\mathcal{J}_\gamma$ becomes singular along the free interface $\{u = 0\}$. The degree of singularity is, in turn, dimmed by the parameter $\gamma \in [0, 1]$. For $0 < \gamma < 1$ we show that local minima are locally of class $C^{1,\alpha}$ for a sharp α that depends on dimension, p and γ. For $\gamma = 0$ we obtain a quantitative, asymptotically optimal result, which assures that local minima are Log-Lipschitz continuous. The results proven in this article are new even in the classical context of linear, nondegenerate equations.

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Keywords: Free boundary problems; Degenerate elliptic operators; Regularity theory

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $2 \leqslant p < +\infty$, $f \in L^q(\Omega)$ for $q \geqslant n$ and $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, with, say, $\varphi^+ \neq 0$. The objective of the present manuscript is to derive optimal interior regularity estimates for the archetypal class of heterogeneous non-differentiable functionals

$$\mathcal{J}_\gamma(v) := \int_\Omega \big(|\nabla v|^p + F_\gamma(v) + f(X) \cdot v\big)\, dX \to \min, \tag{1.1}$$

among competing functions $v \in W_0^{1,p}(\Omega) + \varphi$. The parameter γ in (1.1) varies continuously from 0 to 1, i.e., $\gamma \in [0, 1]$ and the non-differentiable potential F_γ is given by

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$$F_\gamma(v) := \lambda_+\big(v^+\big)^\gamma + \lambda_-\big(v^-\big)^\gamma, \tag{1.2}$$

for scalars $0 \leqslant \lambda_- < \lambda_+ < \infty$. As usual, $v^\pm := \max\{\pm v, 0\}$, and, by convention,

$$F_0(v) := \lambda_+ \chi_{\{v>0\}} + \lambda_- \chi_{\{v\leqslant 0\}}. \tag{1.3}$$

The non-differentiability of the potential F_γ impels the Euler–Lagrange equation associated to $\mathcal{J}_\gamma$ to be singular along the *a priori* unknown interface

$$\mathfrak{F}_\gamma := \big(\partial\{u_\gamma > 0\} \cup \partial\{u_\gamma < 0\}\big) \cap \Omega,$$

between the positive and negative phases of a minimum. In fact, a minimizer satisfies, in some weak sense, the following p-degenerate and singular PDE

$$\Delta_p u = \frac{\gamma}{p}\big(\lambda_+\big(u^+\big)^{\gamma-1}\chi_{\{u>0\}} - \lambda_-\big(u^-\big)^{\gamma-1}\chi_{\{u<0\}}\big) + \frac{1}{p} f(X) \quad \text{in } \Omega, \tag{1.4}$$

where $\Delta_p u$ denotes the classical p-Laplacian operator,

$$\Delta_p u := \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big).$$

The potential F_0 is actually discontinuous and that further enforces the flux balance

$$\big|\nabla u_0^+\big|^p - \big|\nabla u_0^-\big|^p = \frac{1}{p-1}(\lambda_+ - \lambda_-), \tag{1.5}$$

along the free boundary of the problem, which breaks down the continuity of the gradient through $\mathfrak{F}_0$.

A number of important mathematical problems, coming from several different contexts, are modeled by optimization setups, for which Eq. (1.1) serves as an emblematic, leading prototype. This fact has fostered massive investigations, and linear versions, $p = 2$, of the minimization problem (1.1) have indeed received overwhelming attention in the past four decades. The upper case $\gamma = 1$ is related to obstacle type problems. The linear, homogeneous, one phase obstacle problem, i.e., $p = 2$, $f(X) \equiv 0$ and $\varphi \geqslant 0$ was fully studied in the 70s by a number of leading mathematicians: Frehse, Stampacchia, Kinderlehrer, Brezis, Caffarelli, among others. It has been established that the minimum is locally of class $C^{1,1}$ and this is the optimal regularity for solution. The two-phase version of the problem, i.e., with no sign restriction on the boundary datum φ, challenged the community for over three decades. $C^{1,1}$ estimate for two-phase obstacle problems was established in [19] with the aid of the powerful *almost* monotonicity formula obtained in [5].

The lower limiting case, $\gamma = 0$, relates to jets flow and cavities problems. The linear, homogeneous, one phase version of the problem was studied in [1], where it is proven that minima are Lipschitz continuous. The two-phase version of this problem brings major new difficulties and $C^{0,1}$ local regularity of minima was proven in [2], with the aid of the revolutionary Alt–Caffarelli–Friedman monotonicity formula, developed in that very same article. Gradient estimates for two-phase cavitation type problem with bounded non-homogeneity, i.e., $p = 2$, $f \in L^\infty$, $\gamma = 0$ in (1.1), were established by Caffarelli, Jerison and Kenig with the aid of their powerful *almost* monotonicity formula, [5].

The intermediary problem $0 < \gamma < 1$ has also received great attention in the past decades. The related free boundary problem can be used, for example, to model the density of certain chemical specie, in reaction with a porous catalyst pellet. The linear, $p = 2$, one-phase, $\varphi \geqslant 0$, homogeneous, $f \equiv 0$, version of the problem (1.1) is the theme of a successful program developed in the 80s by Phillips and Alt–Phillips, [18,17,3], among others. In similar setting, Hölder continuity of the gradient of minimizers was proven by Giaquinta and Giusti [9]. Further investigations on the linear, two-phase version of this problem also require powerful monotonicity formulae in their studies, see [27].

In the mathematical analysis of variational free boundary problems as (1.1), the first major key issue to be addressed concerns the optimal regularity estimate available for a given minimum. A simple inference on the weak Euler–Lagrange equation satisfied by a minimum, Eq. (1.4) and also the flux balance (1.5) for $\gamma = 0$, revel that $\Delta_p u$ blows up along the free boundary of the problem, $\mathfrak{F}_\gamma := \partial\{u_\gamma > 0\} \cup \partial\{u_\gamma < 0\}$. Therefore, it becomes a fundamental question to understand precisely how this phenomenon affects the (lack of) smoothness properties of minima. Under such perspective, and to some extent, the theory of two-phase free boundary problems governed by non-linear, degenerate elliptic operators had hitherto been unaccessible through current literature, mainly due to the lack of monotonicity formulae in this context.

In the study of sharp smoothness properties of minima to the functional $\mathcal{J}_\gamma$, further difficulties also arise from the very complexity of the regularity theory for the governing operator Δ_p. We recall that p-harmonic functions, i.e., solutions to the homogeneous equation

$$\Delta_p h = 0 \quad \text{in } B_1,$$

are locally of class C^{1,α_p} for an exponent $0 < \alpha_p < 1$ that depends only upon dimension and p. The precise value of α_p is in general unknown – see [13] for the planar case $n = 2$, and [22] for sharp Hölder estimates for inhomogeneous problems. This fact indicates that interior estimates available for p-harmonic functions, that in turn are below quadratic, $C^{1,1}$, will compete with optimal growth along the free interface $\mathfrak{F}_\gamma$. The regularity theory for heterogeneous equations $\Delta_p \xi = f(X)$ is even further involved and, up to our knowledge, the understanding on this class of problems is not yet fully complete.

From the mathematical point of view, the exponent γ appearing in (1.1) should be comprehended as the parameter that measures the singularity of the absorption term of the related equation. For non-differentiable but continuous functionals, $\mathcal{J}_\gamma$ with $0 < \gamma \leqslant 1$, it has been conjectured that the gradient of a minimum is locally continuous, even through the singular free interface $\mathfrak{F}_\gamma$. The first result we present in this paper gives an affirmative answer to such question. Furthermore, it provides the *asymptotically* optimal $C^{1,\alpha}$ interior regularity theory available for minima of such functionals.

Theorem 1.1 *($C^{1,\alpha}$ regularity estimates). Let u be a minimizer of the problem (1.1). Assume $0 < \gamma \leqslant 1$ and $f \in L^q(\Omega)$, for some $q > n$. Then $u \in C^{1,\alpha}_{\text{loc}}$, for*

$$\alpha := \min\left\{\alpha_p^-, \frac{\gamma}{p-\gamma}, \frac{(q-n)}{(p-1)q}\right\}, \tag{1.6}$$

where the estimate indicated in (1.6) should be read as

$$\left| \begin{array}{l} \text{If } \min\left\{\frac{\gamma}{p-\gamma}, \frac{(q-n)}{(p-1)q}\right\} < \alpha_p, \quad \text{then } u \in C^{1,\min\{\frac{\gamma}{p-\gamma}, \frac{(q-n)}{(p-1)q}\}}. \\ \text{If } \min\left\{\frac{\gamma}{p-\gamma}, \frac{(q-n)}{(p-1)q}\right\} \geqslant \alpha_p, \quad \text{then } u \in C^{1,\sigma}, \text{ for any } 0 < \sigma < \alpha_p. \end{array} \right. \tag{1.7}$$

Furthermore, for any $\Omega' \Subset \Omega$, there exists a constant $C > 0$ depending only on Ω', n, p, q, $\|\varphi\|_{L^\infty(\Omega)}$, $\|f\|_{L^q(\Omega)}$, λ_+, λ_-, γ and $(\alpha_p - \alpha)$, such that

$$\|u\|_{C^{1,\alpha}(\Omega')} \leqslant C.$$

Before continuing, let us make few comments on Theorem 1.1 and its implications. The key ingredient of the regularity estimate established in Theorem 1.1 reveals how the competing forces involved in the lack of smoothness for minima of (1.1), namely

$$(\text{regularity theory for } \Delta_p) \times \left(\text{singular absorption term } \sim u^{\gamma-1}\right) \times (\text{roughness of the source } f)$$

get adjusted, via the sharp relation (1.6). Regarding the exponent α_p, one easily verifies that the function $X \mapsto |X|^{\frac{p}{p-1}}$ has bounded p-Laplacian, thus α_p is appearing in (1.6) is below the critical value $\frac{1}{p-1}$. However, nonnegative functions with bounded p-Laplacian, v, do grow as $\text{dist}^{\frac{p}{p-1}}(X, \mathfrak{F})$ away from $\mathfrak{F} = \partial\{v > 0\}$, see [10]. In this particular setting, it is possible to replace α_p by $\frac{1}{p-1}$ in (1.6). Thus, at least if $f \in L^\infty$, Theorem 1.1 revels $u \in C^{\frac{\gamma}{p-\gamma}}$, which is the precise generalization of the optimal regularity estimate obtained for the one-phase linear setting $p = 2$, see for instance [17,18].

Confronting the effect of the singular absorption term $\sim u^{\gamma-1}$ and the influence of integrability properties of the source f, we conclude that solutions to (1.1) are locally in $C^{1,\min\{\alpha_p^-, \frac{\gamma}{p-\gamma}\}}$, provided $f \in L^q$ for any

$$q \geqslant n \cdot \frac{(p-\gamma)}{p(1-\gamma)} =: q(p,n,\gamma). \tag{1.8}$$

Interestingly enough, one verifies that

$$q\big(p,n,1^-\big)=\infty \quad \text{and} \quad \lim_{\gamma\to 1}\frac{(q(p,n,\gamma)-n)p}{(p-1)q(p,n,\gamma)}=\frac{1}{p-1}. \tag{1.9}$$

Also it is revealing to compute the limit

$$\lim_{\gamma\to 0} q(p,n,\gamma)=n, \tag{1.10}$$

which leads us to the discussion of the delicate limiting case, $\gamma=0$ in the minimization problem (1.1). As mentioned earlier in this Introduction, for homogeneous, $f\equiv 0$, linear, $p=2$, jets and cavities problems, Lipschitz regularity estimates have been established in the one-phase and two-phase case, respectively in [1] and [2]. Heterogeneous, two-phase versions of the problem require the use of *almost* monotonicity formula, [5]. However, the Caffarelli–Jerison–Kenig monotonicity formula only holds if $f(X)\geqslant -C$. Thus, even for linear problems, $p=2$, Lipschitz estimates for minimizers of (1.1), $\gamma=0$, are only known if $f\in L^\infty(\Omega)$. We further point out that the integrability exponent obtained in (1.10) is a borderline condition, as it divides the regularity theory for (non-singular) Poisson equations, $\mathrm{L}u=f$, between continuity estimates when $f\in L^{n-\epsilon}$ and differentiability properties when $f\in L^{n+\epsilon}$. The optimal regularity theory for the conformal case $f\in L^n$ is rather delicate. It has been recently established by the third author, [21], that solutions to nonlinear equations $F(X,D^2u)=f(X)\in L^n$ have a universal Log-Lipschitz modulus of continuity, i.e.,

$$\big|u(X)-u(Y)\big|\lesssim |X-Y|\cdot\log|X-Y|.$$

Such regularity is optimal in the context of heterogeneous equations with L^n right-hand sides. After some heuristic inferences, it becomes reasonable to inquire whether minimizers of problem (1.1), with $\gamma=0$, also have a universal Log-Lipschitz modulus of continuity. The second main result we establish in this paper states that indeed minimizers of $\mathcal{J}_0$ with sources $f\in L^n$ also enjoy such an optimal universal modulus of continuity.

Theorem 1.2 *(Log-Lipschitz regularity for $\gamma=0$). Let u be a minimizer of the problem (1.1), with $\gamma=0$ and $f\in L^n(\Omega)$. Then u is Log-Lipschitz continuous and for any $\Omega'\Subset\Omega$, there exists a constant C that depends only on Ω', n, p, $|\varphi\|_{L^\infty(\Omega)}$, $\|f\|_{L^n(\Omega)}$, λ_+ and λ_-, such that*

$$\big|u(X)-u(Y)\big|\leqslant C|X-Y|\log|X-Y|.$$

In particular, Theorem 1.2 assures that $u\in C^{0,\tau}_{\mathrm{loc}}(\Omega)$ for any $\tau<1$. We further mention that Theorem 1.2 is sharp due to the borderline integrability condition on the source f. We leave open the key question on whether functional $\mathcal{J}_0$ has a locally Lipschitz minimizer, provided $f\in L^q(\Omega)$ for $q>n$. We highlight that this question remains open even for the linear case $p=2$. A critical analysis on the machinery employed in the proof of Log-Lipschitz estimates, Theorem 1.2, reveals that it should not be possible to access the $C^{0,1}$ regularity theory for minima of $\mathcal{J}_0$ through pure energy considerations, even if the source $f\in L^\infty$.

We further mention that Theorem 1.1 and Theorem 1.2 can be established, with minor modifications, to further involved energy functionals of the type

$$\widetilde{\mathcal{G}_\gamma}(v)=\int_\Omega G(X,\nabla v)+G_\gamma(v)+g(X,v)\,dX,$$

where G is a p-degenerate kernel with C^1 coefficients, $|G_\gamma|\lesssim F_\gamma$ and $|g(X,v)|\leqslant\tilde{g}(X)|v|^m$, where $0\leqslant m<p$ and $\tilde{g}\in L^{\tilde{q}}$, for $\tilde{q}\geqslant\max\{\frac{p}{p-m},n\}$. We have chosen to present our results in a simpler setting as to further emphasize the new ideas designed in this work.

The paper is organized as follows. In Section 2 we gather few tools that we shall use in the proofs of Theorem 1.1 and Theorem 1.2. In Section 3 we comment on existence and establish universal L^∞ bounds for minima of problem (1.1). Section 4 is devoted to the proof of Theorem 1.1 and in Section 5 we establish Log-Lip estimates for cavitation problems, proving therefore Theorem 1.2. Under the condition $f\in L^q$, $q>n$, in Section 6 we show sharp linear growth and strong nondegeneracy properties for solutions to the cavitation problem $\gamma=0$. In Section 7 we investigate stability properties for the family of free problems $\mathcal{J}_\gamma$ in terms of the singular parameter $0\leqslant\gamma\leqslant 1$. More precisely we show that local minima of functional $\mathcal{J}_\gamma$ converges to a local minima of the functional $\mathcal{J}_0$, as $\gamma\to 0$.

2. Preliminaries and some known tools

In this section we gather some preliminaries results that we will systematically use along the article. Initially, as mentioned within the Introduction, clearly one should not expect solutions to the minimization problem (1.1) to be smoother than p-harmonic functions. Therefore, the regularity theory for degenerate elliptic operators is a first key ingredient in understanding sharp estimates for minima of $\mathcal{J}_\gamma$.

There are several different strategies to establish the C^{1,α_p} regularity theory for p-harmonic functions, see for instance [6,7,14,23,25,26]. We state such result for future references.

Theorem 2.1 *($C^{1,\alpha}$ estimates for p-harmonic functions). Let $h \in W^{1,p}(B_1)$ satisfy $\Delta_p h = 0$ in B_1 in the distributional sense. Then, there exist constants $C > 0$ and $0 < \alpha_p < 1$, both depending only on dimension and p, such that*

$$\|h\|_{C^{1,\alpha_p}(B_{1/2})} \leqslant C\|h\|_{L^p(B_1)}.$$

A particularly interesting approach was suggested by Lieberman in [15], where the regularity theory for p-harmonic functions is accessed through the following leading integral oscillation decay lemma:

Lemma 2.2. *(See Lieberman [15, Lemma 5.1].) Let h be a p-harmonic function in $B_R \subset \mathbb{R}^n$. Then, for some positive constant $0 < \alpha_p < 1$, there holds*

$$\int_{B_r} \left|\nabla h(X) - (\nabla h)_r\right|^p dX \leqslant C\left(\frac{r}{R}\right)^{n+p\alpha_p} \int_{B_R} \left|\nabla h(X) - (\nabla h)_R\right|^p dX,$$

where $C = C(n, p) > 0$ is a positive constant.

In Lemma 2.2 and throughout this article we use the classical average notation

$$(\psi)_r := \fint_{B_r(X_0)} \psi \, dX := \frac{1}{|B_r(X_0)|} \int_{B_r(X_0)} \psi \, dX.$$

Local Hölder continuity for heterogeneous equations $\Delta_p \xi = f$ can be delivered by means of Harnack inequality, which will be another fundamental tool in our analysis.

Theorem 2.3 *(Harnack inequality). (E.g. [20].) Let $\xi \in W^{1,p}(B_R)$, $\xi \geqslant 0$ a.e., satisfy $\Delta_p \xi = f$ in B_R in the distributional sense, with $f \in L^q(B_R)$ and $q > \frac{n}{p}$. Then, there exists a constant $C_r > 0$ depending only on n, q, p and $R - r$ such that*

$$\sup_{B_r} \xi \leqslant C_r \left\{ \inf_{B_r} \xi + \left(r^{p-\frac{n}{q}} \|f\|_{L^q(B_R)}\right)^{\frac{1}{p-1}} \right\}$$

for all $0 < r \leqslant R$.

In the sequel, let us discuss some further inequalities that will be used in the proofs of our main results. The estimates presented herein have elementary character and are mostly known. We include them for completeness purposes and courtesy to the readers.

Lemma 2.4. *Let $\psi \in W^{1,p}(B_1)$, with $2 \leqslant p$ and $h \in W^{1,p}_\psi(B_1)$ be solution to $\Delta_p h = 0$ in B_1. Then, for a constant $c = c(n, p) > 0$, there holds*

$$\int_{B_1} \left[|\nabla \psi|^p - |\nabla h|^p\right] dX \geqslant c \int_{B_1} \left|\nabla(\psi - h)\right|^p dX.$$

Proof. For each $0 \leqslant \tau \leqslant 1$, let ϕ_τ denote the linear interpolation between ψ and h, i.e., $\psi_\tau := \tau\psi + (1-\tau)h$. From Fundamental Theorem of Calculus we have

$$\int_{B_1} \left(|\nabla\psi|^p - |\nabla h|^p\right) dX = \int_0^1 \frac{d}{d\tau}\left(\int_{B_1} |\nabla\psi_\tau|^p \, dX\right) d\tau. \tag{2.1}$$

Passing the derivative through and using the fact that $\operatorname{div}(|\nabla h|^{p-2}\nabla h)\cdot(\psi - h) = 0$ in B_1, we find

$$\begin{aligned}\int_0^1 \frac{d}{d\tau}\left(\int_{B_1} |\nabla\psi_\tau|^p \, dX\right) d\tau &= p\int_0^1 d\tau \int_{B_1} \left(|\nabla\psi_\tau|^{p-2}\nabla\psi_\tau - |\nabla h|^{p-2}\nabla h\right)\cdot\nabla(\psi - h)\, dX \\ &= p\int_0^1 \frac{1}{\tau}\, d\tau \int_{B_1} \left(|\nabla\psi_\tau|^{p-2}\nabla\psi_\tau - |\nabla h|^{p-2}\nabla h\right)\cdot\nabla(\psi_\tau - h)\, dX,\end{aligned} \tag{2.2}$$

because $\psi_\tau - h = \tau(\psi - h)$. The lemma now follows easily from the well known classical monotonicity

$$\left\langle |\xi_1|^{p-2}\xi_1 - |\xi_2|^{p-2}\xi_2, \xi_1 - \xi_2\right\rangle > c(n,p)|\xi_1 - \xi_2|^p, \tag{2.3}$$

for any pair of vectors $\xi_1, \xi_2 \in \mathbb{R}^n$. In fact, combining (2.1), (2.2) and (2.3) we reach

$$\int_{B_1} \left(|\nabla\psi|^p - |\nabla h|^p\right) \geqslant c(n,p)\int_0^1 \tau^{-1}\, d\tau \int_{B_1} \left|\nabla(\psi_\tau - h)\right|^p = c(n,p)\int_0^1 \tau^{p-1}\, d\tau \int_{B_1} \left|\nabla(\psi - h)\right|^p,$$

and the lemma follows. □

Lemma 2.5. *Let $\gamma \in (0,1)$. For any positive scalars $a > 0$, $b > 0$ there holds*

$$(a+b)^\gamma < \left(a^\gamma + b^\gamma\right). \tag{2.4}$$

Proof. In fact, just notice that, since $\gamma - 1$ is negative, we have

$$t^{\gamma-1} > (t+a)^{\gamma-1}, \quad \forall t \in (0,\infty). \tag{2.5}$$

Then, integrate (2.5) from 0 to b to obtain the desired inequality. □

Next we prove two useful asymptotic inequalities.

Lemma 2.6. *Let $0 \leqslant \mu < 1$ and suppose a real function ϕ verifies*

$$\phi(r) \leqslant A\left(r^{e_1}\phi(r)^\mu + r^{e_2}\right),$$

for r small enough. Then $\phi(r) = \mathrm{O}(r^{\min(e_2, \frac{e_1}{1-\mu})})$ as r approaches zero.

Proof. In fact, if $\phi(r) \lesssim r^{e_1}\phi(r)^\mu + r^{e_2}$, then for $\beta := \min(e_2, \frac{e_1}{1-\mu})$, there holds

$$\begin{aligned}\frac{\phi(r)}{r^\beta} &\lesssim r^{e_1-\beta}\phi(r)^\alpha + r^{e_2-\beta} \\ &\lesssim \left(\frac{\phi(r)}{r^{\frac{\beta-e_1}{\mu}}}\right)^\mu + 1 \\ &\lesssim \left(\frac{\phi(r)}{r^\beta}\right)^\mu + 1,\end{aligned}$$

since $\frac{\beta-e_1}{\alpha} \leqslant \beta$. The above readily implies $\phi(r) = \mathrm{O}(r^\beta)$ as claimed. □

Lemma 2.7. *Let $\phi(s)$ be a non-negative and non-decreasing function. Suppose that*

$$\phi(r) \leqslant C_1\left[\left(\frac{r}{R}\right)^{\alpha} + \mu\right]\phi(R) + C_2 R^{\beta} \tag{2.6}$$

for all $r \leqslant R \leqslant R_0$, with C_1, α, β positive constants and C_2, μ non-negative constants, $\beta < \alpha$. Then, for any $\sigma < \beta$, there exists a constant $\mu_0 = \mu_0(C_1, \alpha, \beta, \sigma)$ such that if $\mu < \mu_0$, then for all $r \leqslant R \leqslant R_0$ we have

$$\phi(r) \leqslant C_3\left(\frac{r}{R}\right)^{\sigma}\left[\phi(R) + C_2 R^{\sigma}\right] \tag{2.7}$$

where $C_3 = C_3(C_1, \sigma - \beta)$ is a positive constant. In turn,

$$\phi(r) \leqslant C_4 r^{\sigma}, \tag{2.8}$$

where $C_4 = C_4(C_2, C_3, R_0, \phi, \sigma)$ is a positive constant.

Proof. It suffices to show the estimate for $\sigma = \beta$. For $0 < \theta < 1$ and $R \leqslant R_0$ we have

$$\begin{aligned}\phi(\theta R) &\leqslant C_1\left[\left(\frac{\theta R}{R}\right)^{\alpha} + \mu\right]\phi(R) + C_2 R^{\beta} \\ &= \theta^{\alpha} C_1\left[1 + \mu\theta^{-\alpha}\right]\phi(R) + C_2 R^{\beta}.\end{aligned} \tag{2.9}$$

We choose $0 < \theta < 1$ such that $2C_1\theta^{\alpha} = \theta^{\delta}$ with $\beta < \delta < \alpha$. Now we take $\mu_0 > 0$ satisfying $\mu_0\theta^{-\alpha} < 1$. Thus we obtain for all $R \leqslant R_0$

$$\phi(\theta R) \leqslant \theta^{\delta}\phi(R) + C_2 R^{\beta}. \tag{2.10}$$

Inductively we get

$$\begin{aligned}\phi\left(\theta^{k+1} R\right) &\leqslant \theta^{\delta}\phi\left(\theta^{k} R\right) + C_2\theta^{k\beta} R^{\beta} \\ &\leqslant \theta^{(k+1)\delta}\phi\left(\theta^{k} R\right) + C_2\theta^{k\beta} R^{\beta}\sum_{i=1}^{k}\theta^{i(\delta-\beta)} \\ &\leqslant C_3\theta^{(k+1)\delta}\left[\phi(R) + C_2 R^{\beta}\right]\end{aligned} \tag{2.11}$$

for all $k \in \mathbb{N}$. Hence, taking k such that $\theta^{k+1}R \leqslant r \leqslant \theta^{k}R$ we obtain (2.7).

Finally, we have

$$\begin{aligned}\phi(r) &\leqslant C_3\left(\frac{r}{R_0}\right)^{\sigma}\left[\phi(R_0) + C_2 R_0^{\sigma}\right] \\ &= \left[\frac{C_3}{R_0^{\sigma}}\left(\phi(R_0) + C_2 R_0^{\sigma}\right)\right] r^{\sigma}\end{aligned} \tag{2.12}$$

which proves inequality (2.8). □

3. Existence and L^{∞} bounds of minimizers

In this section we establish existence and pointwise bounds for a minimum of the functional $\mathcal{J}_{\gamma}$. The arguments presented herein work indistinctly for the cases $0 < \gamma \leqslant 1$ and $\gamma = 0$.

Theorem 3.1 *(Existence and L^{∞} bounds). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $f \in L^q(\Omega)$, $q \geqslant n$, $\varphi \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ and $0 < \lambda_+ \neq \lambda_- < \infty$ be fixed. For each $0 \leqslant \gamma \leqslant 1$, there exists a minimizer u_{γ} to the energy functional*

$$\mathcal{J}_{\gamma}(v) := \int_{\Omega}\left(|\nabla v|^p + F_{\gamma}(v) + f(X)\cdot v\right) dX,$$

over $W_0^{1,p}+\varphi$, *where* $F_\gamma(v):=\lambda_+(v^+)^\gamma+\lambda_-(v^-)^\gamma$ *and by convention,* $F_0(v):=\lambda_+\chi_{\{v>0\}}+\lambda_-\chi_{\{v\leqslant 0\}}$. *Furthermore,* u_γ *is bounded. More precisely,*

$$\|u_\gamma\|_{L^\infty(\Omega)}\leqslant C\big(n,p,\lambda_+,\lambda_-,\|\varphi\|_{L^\infty(\partial\Omega)},\|f\|_{L^q(\Omega)}\big).$$

Proof. Let us label

$$I_0:=\min\big\{\mathcal{J}_\gamma(v):\ v\in W_0^{1,p}+\varphi\big\}.$$

Initially we show that $I_0>-\infty$. Indeed, for any $v\in W_0^{1,p}+\varphi$, by Poincaré inequality there exists a positive constant $c=c(n,p,\Omega,\|f\|_{L^q})>0$ such that

$$c\|v\|_{L^p}^p-c\|\phi\|_{L^p}^p-\|\nabla\phi\|_{L^p}^p\leqslant\|\nabla v\|_{L^p}^p. \tag{3.1}$$

By Hölder inequality, since

$$q\geqslant n>\frac{p}{p-1}, \tag{3.2}$$

we have

$$\left|\int_\Omega f(X)v\,dX\right|\leqslant\|f\|_{L^{\frac{p}{p-1}}}\|v\|_{L^p}\leqslant C_1(n,p,\Omega)\|f\|_{L^q}\|v\|_{L^p},$$

which combined with (3.1) gives

$$-C-c\|\phi\|_{L^p}^p-\|\nabla\phi\|_{L^p}^p\leqslant\|\nabla v\|_{L^p}^p-C_1(n,p,\Omega)\|f\|_{L^q}\|v\|_{L^p}.$$

Finally, we reach

$$-C-c\|\phi\|_{L^p}^p-\|\nabla\phi\|_{L^p}^p\leqslant\|\nabla v\|_{L^p}^p-C_1(n,p,\Omega)\|f\|_{L^q}\|v\|_{L^p}\leqslant\mathcal{J}_\gamma(v). \tag{3.3}$$

Let us now show existence of a minimum. Let $v_j\in W_\phi^{1,p}(\Omega)$ be a minimizing sequence. For $j\gg 1$,

$$\mathcal{J}_\gamma(v_j)\leqslant I_0+1.$$

From (3.3) and Hölder inequality we obtain

$$\int_\Omega|\nabla v_j|^p\,dX\leqslant C\|v_j\|_{L^p}+I_0+1. \tag{3.4}$$

By Poincaré inequality we estimate

$$C\|v_j\|_{L^p}\leqslant C\big(\|\nabla v_j\|_{L^p}+\|\nabla\phi\|_{L^p}+\|\phi\|_{L^p}\big). \tag{3.5}$$

Also we have

$$C\|\nabla v_j\|_{L^p}\leqslant C+\frac{1}{2}\|\nabla v_j\|_{L^p}^p. \tag{3.6}$$

Combining (3.4), (3.5) and (3.6) we reach

$$\int_\Omega|\nabla v_j|^p\,dX\leqslant C\big(\|\nabla\phi\|_{L^p}+\|\phi\|_{L^p}\big)+I_0+1. \tag{3.7}$$

Thus, using Poincaré inequality once more, we conclude that $\{v_j-\phi\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$. By reflexivity, there is a function $u\in W_\phi^{1,p}(\Omega)$ such that, up to a subsequence,

$$v_j\to u\quad\text{weakly in } W^{1,p}(\Omega),\qquad v_j\to u\quad\text{in } L^p(\Omega),\qquad v_j\to u\quad\text{a.e. in }\Omega.$$

From lower semicontinuity of norms, we readily obtain

$$\int_\Omega |\nabla u|^p \, dX \leqslant \liminf_{j\to\infty} \int_\Omega |\nabla v_j|^p \, dX.$$

By pointwise convergence we have, in the case $0 < \gamma \leqslant 1$,

$$\int_\Omega F_\gamma(u) + f(X)u \, dX \leqslant \liminf_{j\to\infty} \int_\Omega F_\gamma(v_j) + f(X)v_j \, dX.$$

For $\gamma = 0$, recalling that we are working under the regime $\lambda_+ > \lambda_-$, we have

$$\begin{aligned}\int_\Omega \lambda_- \chi_{\{u\leqslant 0\}} \, dX &= \int_{\{u\leqslant 0\}} \lambda_- \chi_{\{v_j>0\}} \, dX + \int_{\{u\leqslant 0\}} \lambda_- \chi_{\{v_j\leqslant 0\}} \, dX \\ &\leqslant \int_{\{u\leqslant 0\}} \lambda_+ \chi_{\{v_j>0\}} \, dX + \int_\Omega \lambda_- \chi_{\{v_j\leqslant 0\}} \, dX.\end{aligned}$$

Thus,

$$\int_\Omega \lambda_- \chi_{\{u\leqslant 0\}} \, dX \leqslant \liminf_{j\to\infty}\left(\int_{\{u\leqslant 0\}} \lambda_+ \chi_{\{v_j>0\}} \, dX + \int_\Omega \lambda_- \chi_{\{v_j\leqslant 0\}} \, dX\right).$$

On the other hand, since $v_j \to u$ a.e. in Ω, we have

$$\begin{aligned}\int_\Omega \lambda_+ \chi_{\{u>0\}} \, dX &= \int_{\{u>0\}} \lambda_+ \Big(\lim_{j\to\infty} \chi_{\{v_j>0\}}\Big) \, dX \\ &= \lim_{j\to\infty} \int_{\{u>0\}} \lambda_+ \chi_{\{v_j>0\}} \, dX.\end{aligned}$$

Hence,

$$\int_\Omega F_0(u) \, dX \leqslant \liminf_{j\to\infty} \int_\Omega F_0(v_j) \, dX.$$

In conclusion,

$$\mathcal{J}_\gamma(u) \leqslant \liminf_{j\to\infty} \mathcal{J}_\gamma(v_j) = I_0,$$

for $0 \leqslant \gamma \leqslant 1$, which proves the existence of a minimizer.

Let us now turn our attention to L^∞ bounds of u_γ, which hereafter in this proof we will only refer as u. Let us label

$$j_0 := \Big\lceil \sup_{\partial\Omega} \phi \Big\rceil,$$

that is, the smallest natural number above $\sup_{\partial\Omega} \phi$. For each $j \geqslant j_0$ we define the truncated function $u_j : \Omega \to \mathbb{R}$ by

$$u_j = \begin{cases} j \cdot \operatorname{sing}(u) & \text{if } |u| > j \\ u & \text{if } |u| \leqslant j, \end{cases} \tag{3.8}$$

where $\operatorname{sing}(u) = 1$ if $u \geqslant 0$ and $\operatorname{sing}(u) = -1$ else. If we denote $A_j := \{|u| > j\}$, we have, for each $j > j_0$

$$u = u_j \quad \text{in } A_j^c \quad \text{and} \quad u_j = j \cdot \operatorname{sing}(u) \quad \text{in } A_j. \tag{3.9}$$

Thus, by minimality of u, there holds, for $0 < \gamma \leqslant 1$,

$$\int_{A_j} |\nabla u|^p \, dX = \int_{\Omega} |\nabla u|^p - |\nabla u_j|^p \, dX$$
$$\leqslant \int_{A_j} f(u_j - u) \, dX + \int_{A_j} \lambda_+ \big((u_j^+)^\gamma - (u^+)^\gamma\big) \, dX + \int_{A_j} \lambda_- \big((u_j^-)^\gamma - (u^-)^\gamma\big) \, dX. \tag{3.10}$$

Notice that

$$\int_{A_j} f(u_j - u) \, dX = \int_{A_j \cap \{u>0\}} f(j-u) \, dX + \int_{A_j \cap \{u \leqslant 0\}} f(u-j) \, dX$$
$$\leqslant 2 \int_{A_j} |f| \big(|u| - j\big) \, dX.$$

Moreover, we have

$$\lambda_+ \int_{A_j} \big((u_j^+)^\gamma - (u^+)^\gamma\big) \, dX = \lambda_+ \int_{A_j \cap \{u>0\}} \big(j^\gamma - |u|^\gamma\big) \, dX + \lambda_+ \int_{A_j \cap \{u \leqslant 0\}} \big((-j)^+\big)^\gamma - \big(u^+\big)^\gamma \, dX$$
$$\leqslant 0$$

and

$$\lambda_- \int_{A_j} \big((u_j^-)^\gamma - (u^-)^\gamma\big) \, dX = \lambda_- \int_{A_j \cap \{u>0\}} \big((j)^-\big)^\gamma - \big(u^-\big)^\gamma \, dX + \lambda_- \int_{A_j \cap \{u \leqslant 0\}} \big(j^\gamma - |u|^\gamma\big) \, dX$$
$$\leqslant 0.$$

Then, we find

$$\int_{A_j} F_\gamma(u_j) - F_\gamma(u) \leqslant 0. \tag{3.11}$$

For $\gamma = 0$ it suffices to notice that $u_j > 0$ and u have the same sign. From the range of truncation we consider, it follows that $(|u| - j)^+ \in W_0^{1,p}(\Omega)$. Hence, applying Hölder inequality and Gagliardo–Nirenberg inequality, we find

$$\int_{A_j} |f| \big(|u| - j\big)^+ \, dX \leqslant \|f\|_{L^{\frac{p}{p-1}}} \big\| \big(|u| - j\big)^+ \big\|_{L^p(A_j)}$$
$$\leqslant \|f\|_{L^q} |A_j|^{1 - \frac{1}{p^*} - \frac{1}{q}} \|\nabla u\|_{L^p(A_j)},$$

where $p^* := \frac{np}{n-p}$. Young inequality gives

$$\|f\|_{L^q} |A_j|^{1 - \frac{1}{p^*} - \frac{1}{q}} \|\nabla u\|_{L^p(A_j)} \leqslant C |A_j|^{\frac{p}{p-1} - \frac{p}{q(p-1)} - \frac{p}{p^*(p-1)}} + \frac{1}{2} \|\nabla u\|^p_{L^p(A_j)}. \tag{3.12}$$

Combining (3.10) and (3.12) we obtain

$$\int_{A_j} |\nabla u|^p \, dX \leqslant C |A_j|^{1 - \frac{p}{n} + \varepsilon}, \tag{3.13}$$

where $\varepsilon = \frac{p(pq-n)}{nq(p-1)}$ and (see (3.1) and (3.7) substituting I_0 by $\mathcal{J}_\gamma(\varphi)$)

$$\|u\|_{L^1(A_{j_0})} \leqslant |A_{j_0}|^{\frac{p-1}{p}} \|u\|_{L^p(A_{j_0})} \leqslant C. \tag{3.14}$$

Boundedness of u now follows from a general machinery, see for instance [24, Chap. 2, Lemma 5.2, p. 71]. □

Remark 3.2. A consequence of L^∞ estimates for a minimum u to the functional $\mathcal{J}_\gamma$ is the universal control of u in $W^{1,p}$. In fact, we have

$$\begin{aligned}\int_\Omega |\nabla u|^p dx &\leqslant \mathcal{J}_\gamma(\varphi) - \int_\Omega F_\gamma(u)\,dX + \int_\Omega \big|f(X)\big||u|\,dX \\ &\leqslant \mathcal{J}_\gamma(\varphi) + C\big(n,p,\Omega,\|f\|_{L^q}\big) \\ &\leqslant C, \end{aligned} \tag{3.15}$$

where $C = C(n,p,\Omega,\varphi,\|f\|_{L^q}) > 0$ is a positive constant. Here we used the elementary inequality $t^\gamma \leqslant \max\{1,t\}$, for $t > 0$ and $0 \leqslant \gamma \leqslant 1$. In conclusion,

$$\|u\|_{W^{1,p}} \leqslant C. \tag{3.16}$$

We close up this section by stating the Euler–Lagrange equation associated to the functional $\mathcal{J}_\gamma$, $0 \leqslant \gamma \leqslant 1$ as well as the flux balance – also known as the free boundary condition – satisfied by a minimum u_0 to $\mathcal{J}_0$, through the free boundary. The proofs of these facts are rather standard and we omit them here.

Proposition 3.3. *Let u_γ be a minimum to the functional $\mathcal{J}_\gamma$, $0 \leqslant \gamma \leqslant 1$. Then u_γ solves*

$$\Delta_p u = \frac{\gamma}{p}\big(\lambda_+\big(u^+\big)^{\gamma-1}\chi_{\{u>0\}} - \lambda_-\big(u^-\big)^{\gamma-1}\chi_{\{u\leqslant 0\}}\big) + \frac{1}{p}f(X) \quad \text{in } \Omega, \tag{3.17}$$

in the distributional sense. Also, if u_0 is a minimum of $\mathcal{J}_0$, with $|\{u_0 = 0\}| = 0$, $f \in L^q(\Omega)$, $q > n$, $X_0 \in \mathfrak{F}^+(u_0) \cup \mathfrak{F}^-(u_0)$ a generic free boundary point and B a ball centered at X_0, then for any $\Phi \in C_0^1(B, \mathbb{R}^n)$, there holds

$$\lim_{\epsilon_1 \searrow 0} \int_{B\cap\{u_0=\epsilon_1\}} \big\langle\big((p-1)|\nabla u_0|^p - \lambda_+\big)\nu_1, \Phi\big\rangle\, d\mathcal{H}^{n-1} + \lim_{\epsilon_2 \nearrow 0} \int_{B\cap\{u_0=\epsilon_2\}} \big\langle\big((p-1)|\nabla u_0|^p - \lambda_-\big)\nu_2, \Phi\big\rangle\, d\mathcal{H}^{n-1} = 0,$$

where ν_1 and ν_2 denote the outward normal vectors on $B \cap \{u_0 = \epsilon_1\}$ and $B \cap \{u_0 = \epsilon_2\}$ respectively. In particular, the flux balance

$$\big|\nabla u_0^+\big|^p - \big|\nabla u_0^-\big|^p = \frac{1}{p-1}(\lambda_+ - \lambda_-)$$

holds along any $C^{1,\alpha}$ piece of the free boundary.

4. Sharp $C^{1,\alpha}$ estimates for minima

This section is devoted to the proof of Theorem 1.1, which assures optimal Hölder continuity estimates for the gradient of minima of the energy functional $\mathcal{J}_\gamma$, for $0 < \gamma \leqslant 1$ and $q > n$. The borderline situation $\gamma = 0$ and $f \in L^n$ will be addressed in the next section.

Hereafter in this section, $u = u_\gamma$ denotes a minimizer of the functional $\mathcal{J}_\gamma$, with $0 < \gamma \leqslant 1$. Theorem 1.1 concerns an optimal interior regularity result; therefore, in order to prove such interior estimate, we fix an arbitrary point $X_0 \in \Omega$ and $R > 0$ such that $R < \operatorname{dist}(X_0, \partial\Omega)$. We will show that $u \in C^{1,\alpha}$ at X_0, for α as in (1.6).

In the sequel we show the first main step in our strategy to obtain sharp regularity theory for minima of the energy $\mathcal{J}_\gamma$.

Lemma 4.1 *(Comparison with p-harmonic functions). Let $u \in W^{1,p}(B_R)$ and $h \in W^{1,p}(B_R)$ satisfy $\Delta_p h = 0$ in B_R in the distributional sense. Then, there exists a positive constant $C = C(n,p) > 0$ depending on dimension and p such that for each $0 < r \leqslant R$, there holds*

$$\int_{B_r} \big|\nabla u(X) - (\nabla u)_r\big|^p\,dX \leqslant C\left(\frac{r}{R}\right)^{n+p\alpha_p} \int_{B_R} \big|\nabla u(X) - (\nabla u)_R\big|^p\,dX + C\int_{B_R} \big|\nabla u(X) - \nabla h(X)\big|^p\,dX, \tag{4.1}$$

where $0 < \alpha_p < 1$ is the optimal exponent in Lemma 2.2, which, in turn, reveals the sharp $C^{1,\alpha}$ estimate stated in Theorem 2.1.

Proof. For each $r \in (0, R]$ we estimate

$$\int_{B_r} \left|\nabla u(X) - (\nabla u)_r\right|^p dX \leqslant C_p \int_{B_r} \left|\nabla u(X) - (\nabla h)_r\right|^p dX + C_p \int_{B_r} \left|(\nabla u)_r - (\nabla h)_r\right|^p dX, \tag{4.2}$$

for a constant C_p that depends only on p. Analogously, we obtain

$$\int_{B_r} \left|\nabla u(X) - (\nabla h)_r\right|^p dX \leqslant C_p \int_{B_r} \left|\nabla u(X) - \nabla h(X)\right|^p dX + C_p \int_{B_r} \left|\nabla h(X) - (\nabla h)_r\right|^p dX. \tag{4.3}$$

In the sequel, we apply Hölder inequality and estimate

$$\begin{aligned}
\int_{B_r} \left|(\nabla u)_r - (\nabla h)_r\right|^p dX &= \frac{1}{|B_r|^{p-1}} \left| \int_{B_r} \left(\nabla u(X) - \nabla h(X)\right) dX \right|^p \\
&\leqslant \frac{1}{|B_r|^{p-1}} \left(\int_{B_r} \left|\nabla u(X) - \nabla h(X)\right| dX \right)^p \\
&\leqslant \frac{1}{|B_r|^{p-1}} \left\{ |B_r|^{1-\frac{1}{p}} \left(\int_{B_r} \left|\nabla u(X) - \nabla h(X)\right|^p dX \right)^{\frac{1}{p}} \right\}^p \\
&= \int_{B_r} \left|\nabla u(X) - \nabla h(X)\right|^p dX.
\end{aligned} \tag{4.4}$$

Combining (4.2), (4.3) and (4.4) we obtain

$$\int_{B_r} \left|\nabla u(X) - (\nabla u)_r\right|^p dX \leqslant C_p \int_{B_r} \left|\nabla h(X) - (\nabla h)_r\right|^p dX + C_p \int_{B_r} \left|\nabla u(X) - \nabla h(X)\right|^p dX. \tag{4.5}$$

Interplaying the roles of u and h in (4.5) and arguing in the bigger ball B_R, we find

$$\int_{B_R} \left|\nabla h(X) - (\nabla h)_R\right|^p dX \leqslant C_p \int_{B_R} \left|\nabla u(X) - (\nabla u)_R\right|^p dX + C_p \int_{B_R} \left|\nabla u(X) - \nabla h(X)\right|^p dX. \tag{4.6}$$

Now, in view of Lemma 2.2 and (4.5) we can further estimate

$$\begin{aligned}
\int_{B_r} \left|\nabla u(X) - (\nabla u)_r\right|^p dX \leqslant{}& C(n,p) \left(\frac{r}{R}\right)^{n+p\alpha_p} \int_{B_R} \left|\nabla h(X) - (\nabla h)_R\right|^p dX \\
&+ C(n,p) \int_{B_R} \left|\nabla u(X) - \nabla h(X)\right|^p dX.
\end{aligned} \tag{4.7}$$

Hence, combining (4.6) and (4.7) we readily obtain

$$\begin{aligned}
\int_{B_r} \left|\nabla u(X) - (\nabla u)_r\right|^p dX \leqslant{}& C(n,p) \left(\frac{r}{R}\right)^{n+p\alpha_p} \int_{B_R} \left|\nabla u(X) - (\nabla u)_R\right|^p dX \\
&+ C(n,p) \left[1 + \left(\frac{r}{R}\right)^{n+p\alpha_p}\right] \int_{B_R} \left|\nabla u(X) - \nabla h(X)\right|^p dX,
\end{aligned} \tag{4.8}$$

which finally implies

$$\int_{B_R} \left|\nabla u(X) - (\nabla u)_r\right|^p dX \leqslant C \left(\frac{r}{R}\right)^{n+p\alpha_p} \int_{B_R} \left|\nabla u(X) - (\nabla u)_R\right|^p dX + C \int_{B_R} \left|\nabla u(X) - \nabla h(X)\right|^p dX, \tag{4.9}$$

and the proof of Lemma 4.1 is concluded. □

We have now gathered all the tools and ingredients we need to establish local Hölder continuity of the gradient of a minimum of the energy functional $\mathcal{J}_\gamma$, $0 < \gamma \leqslant 1$.

Proof of Theorem 1.1. We start off the proof by denoting, for writing convenience, $B_R := B_R(X_0)$ and $u = u_\gamma$ a given minimum of the functional $\mathcal{J}_\gamma$, $0 < \gamma \leqslant 1$. Let h be the p-harmonic function in B_R that agrees with u on the boundary, i.e.,

$$\Delta_p h = 0 \quad \text{in } B_R \quad \text{and} \quad h - u \in W_0^{1,p}(B_R).$$

By Lemma 4.1 we have

$$\int_{B_r} \left|\nabla u(X) - (\nabla u)_r\right|^p dX \leqslant C\left(\frac{r}{R}\right)^{n+p\alpha_p} \int_{B_R} \left|\nabla u(X) - (\nabla u)_R\right|^p dX + C \int_{B_R} \left|\nabla u(X) - \nabla h(X)\right|^p dX. \tag{4.10}$$

On the other hand, by the minimality of u we have

$$\int_{B_R} \left(|\nabla u|^p - |\nabla h|^p\right) dX \leqslant \int_{B_R} \left(F_\gamma(h) - F_\gamma(u)\right) dX + \int_{B_R} f(X)(h-u)\, dX. \tag{4.11}$$

Invoking Lemma 2.4, there exists a constant $C_3 = C_3(p,n) > 0$ such that

$$C_3 \int_{B_R} \left(|\nabla u|^p - |\nabla h|^p\right) dX \geqslant \int_{B_R} \left|\nabla(u-h)\right|^p dX. \tag{4.12}$$

Moreover, we have

$$\int_{B_R} F_\gamma(h) - F_\gamma(u)\, dX = \lambda_+ \int_{B_R} \left[\left(h^+\right)^\gamma - \left(u^+\right)^\gamma\right] dX + \lambda_- \int_{B_R} \left[\left(h^-\right)^\gamma - \left(u^-\right)^\gamma\right] dX$$

with

$$\begin{aligned} \int_{B_R} \left[\left(h^+\right)^\gamma - \left(u^+\right)^\gamma\right] dX &\leqslant \int_{\{h^+>u^+\}} \left[\left(h^+\right)^\gamma - \left(u^+\right)^\gamma\right] dX \\ &= \int_{\{h^+>u^+\}\cap\{u^+>0\}} \left[\left(h^+\right)^\gamma - \left(u^+\right)^\gamma\right] dX + \int_{\{h^+>u^+\}\cap\{u^+=0\}} \left(h^+ - u^+\right)^\gamma dX. \end{aligned} \tag{4.13}$$

Notice furthermore that

$$\int_{\{h^+>u^+\}\cap\{u^+=0\}} \left(h^+ - u^+\right)^\gamma dX \leqslant \int_{\{h^+>u^+\}\cap\{u^+=0\}} (h-u)^\gamma\, dX. \tag{4.14}$$

By Lemma 2.5 there holds

$$\begin{aligned} \int_{\{h^+>u^+\}\cap\{u^+>0\}} \left[\left(h^+\right)^\gamma - \left(u^+\right)^\gamma\right] dX &\leqslant \int_{\{h^+>u^+\}\cap\{u^+>0\}} \left(h^+ - u^+\right)^\gamma dX \\ &= \int_{\{h^+>u^+\}\cap\{u^+>0\}} (h-u)^\gamma\, dX \\ &\leqslant \int_{B_R} |h-u|^\gamma\, dX. \end{aligned} \tag{4.15}$$

Analogously, we obtain

$$\int_{B_R} \left[(h^-)^\gamma - (u^-)^\gamma \right] dX \leqslant \int_{\{h^- > u^-\} \cap \{u^- > 0\}} \left[(h^-)^\gamma - (u^-)^\gamma \right] dX + \int_{\{h^- > u^-\} \cap \{u^- = 0\}} (u-h)^\gamma \, dX, \tag{4.16}$$

with

$$\begin{aligned} \int_{\{h^- > u^-\} \cap \{u^- > 0\}} \left[(h^-)^\gamma - (u^-)^\gamma \right] dX &\leqslant \int_{\{h^- > u^-\} \cap \{u^- > 0\}} (h^- - u^-)^\gamma \, dX \\ &= \int_{\{h^- > u^-\} \cap \{u^- > 0\}} (u-h)^\gamma \, dX \\ &\leqslant \int_{B_R} |h-u|^\gamma \, dX. \end{aligned} \tag{4.17}$$

Hence, we find

$$\int_{B_R} F_\gamma(h) - F_\gamma(u) \, dX \leqslant C \int_{B_R} |h-u|^\gamma \, dX, \tag{4.18}$$

where $C = C(\lambda_+, \lambda_-)$ is a positive constant.

Combining (4.12), (4.11) and employing Hölder inequality followed by Poincaré inequality and (4.18) we obtain

$$\begin{aligned} \int_{B_R} \left| \nabla(u-h) \right|^p dX &\leqslant C_3 \int_{B_R} F_\gamma(h) - F_\gamma(u) \, dX \\ &\leqslant C_4 \int_{B_R} |u-h|^\gamma \, dX \\ &\leqslant C_5 \left(\int_{B_R} \left| \nabla(u-h) \right|^p dX \right)^{\gamma/p} |B_R|^{1+\gamma/n-\gamma/p}, \end{aligned}$$

where C_4 and C_5 depend on p, n, λ_+ and λ_-. Thus, by Young inequality we reach the following estimate

$$\begin{aligned} \int_{B_R} F_\gamma(h) - F_\gamma(u) \, dX &\leqslant C(p,\gamma) \left[C(p,n,\lambda_+,\lambda_-) \right]^{p/(p-\gamma)} |B_R|^{1+1/n(p\gamma/(p-\gamma))} + \frac{1}{4} \left\| \nabla(u-h) \right\|_{L^p}^p \\ &\leqslant C(p) \left[C(p,n,\lambda_+,\lambda_-) \right]^{p/(p-1)} |B_R|^{1+1/n(p\gamma/(p-\gamma))} + \frac{1}{4} \left\| \nabla(u-h) \right\|_{L^p}^p, \end{aligned} \tag{4.19}$$

where $C(p,\gamma) = (\frac{4\gamma}{p})^{\frac{\gamma}{p-\gamma}} (\frac{p-\gamma}{p})$ and $C(p) = (\frac{4}{p})^{\frac{1}{p-1}}$. Hölder inequality and Poincaré inequality yield

$$\begin{aligned} \int_{B_R} f(X)(h-u) \, dX &\leqslant \|f\|_{L^q} |B_R|^{\frac{p-1}{p} - \frac{1}{q}} \|u-h\|_{L^p} \\ &\leqslant \|f\|_{L^q} |B_R|^{\frac{p-1}{p} - \frac{1}{q} + \frac{1}{n}} \left\| \nabla(u-h) \right\|_{L^p}. \end{aligned} \tag{4.20}$$

Thus, applying Young inequality once more, we reach

$$\begin{aligned} \int_{B_R} f(X)(h-u) &\leqslant C(p) \left(\|f\|_{L^q} \right)^{\frac{p}{p-1}} |B_R|^{\frac{p}{p-1}(\frac{p-1}{p} - \frac{1}{q} + \frac{1}{n})} \left\| \nabla(u-h) \right\|_{L^p}^p + \frac{1}{4} \left\| \nabla(u-h) \right\|_{L^p} \\ &= C(p) \left(\|f\|_{L^q} \right)^{\frac{p}{p-1}} |B_R|^{1+\frac{1}{n}[\frac{(q-n)p}{(p-1)q}]} + \frac{1}{4} \left\| \nabla(u-h) \right\|_{L^p}. \end{aligned} \tag{4.21}$$

Replacing (4.19) and (4.21) in (4.10) we easily obtain

$$\begin{aligned}\int_{B_r} \left|\nabla u - (\nabla u)_r\right|^p dX \leqslant{}& C(n,p,\alpha_p)\left(\frac{r}{R}\right)^{n+p\alpha_p} \int_{B_R} \left|\nabla u(X) - (\nabla u)_R\right|^p dX \\ &+ C(n,p,\alpha_p)C(n,p,\lambda_+,\lambda_-)|B_R|^{1+1/n(p\gamma/(p-\gamma))} \\ &+ C(n,p,\alpha_p)\left(\|f\|_{L^q}\right)^{\frac{p}{p-1}}|B_R|^{1+\frac{1}{n}[\frac{(q-n)p}{(p-1)q}]} \\ \leqslant{}& C\left(\frac{r}{R}\right)^{n+p\alpha_p} \int_{B_R} \left|\nabla u(X) - (\nabla u)_R\right|^p dX + CR^{n+p\gamma/(p-\gamma)} + CR^{n+p\frac{(q-n)}{(p-1)q}},\end{aligned}$$

where $C = C(n, p, \lambda_+, \lambda_-, \alpha_p, \|f\|_{L^q})$ is a positive constant. In view of Lemma 2.7 and $W^{1,p}$ bounds of u we conclude

$$⨍_{B_r(X_0)} \left|\nabla u - (\nabla u)_r\right|^p dX \leqslant C\left(n, p, \lambda_+, \lambda_-, \|f\|_{L^q(\Omega)}, \operatorname{dist}(X_0, \partial\Omega)\right) \cdot r^\alpha, \tag{4.22}$$

for α entitled in (1.6). Finally, Campanato's Embedding Theorem (see for instance [16]) gives the desired Hölder continuity of the gradient of u. The proof of Theorem 1.1 is complete. □

Remark 4.2. It is important to notice that the estimates from Campanato's Embedding Theorem are not uniform as γ goes to zero. In fact, an inspection of the proof of such theorem (see for instance [16, Theorem 1.54]) reveals that estimate (4.22) implies

$$\left|\nabla u(X) - \nabla u(Y)\right| \leqslant \frac{2^n \cdot C(n, p, \lambda_+, \lambda_-, \|f\|_{L^q(\Omega)}, \operatorname{dist}(X_0, \partial\Omega))}{2^\alpha - 1}|X - Y|^\alpha.$$

This is the reason why the constant in Theorem 1.1 does depend upon γ, even though the universal constant appearing in (4.22) does not depend upon γ.

5. Log-Lipschitz estimates

In this section we address sharp regularity for jets and cavities type problems, i.e., $\gamma = 0$, with sources in the conformal threshold case $f \in L^n(\Omega)$, where n is the dimension of the ambient. Hereafter $u = u_0$ denotes a minimizer of the energy functional

$$\mathcal{J}_0(v) := \int_\Omega \left(|\nabla v|^p + \lambda_+ \chi_{\{v>0\}} + \lambda_- \chi_{\{v\leqslant 0\}} + f(X) \cdot v\right) dX, \tag{5.1}$$

for scalars $0 \leqslant \lambda_- < \lambda_+ < \infty$. Existence and pointwise bounds for u_0 has been assured by Theorem 3.1.

Proof of Theorem 1.2. We start off by fixing an arbitrary point $X_0 \in \Omega$ and $R > 0$ such that $R < \operatorname{dist}(X_0, \partial\Omega)$. As before, we denote $B_R := B_R(X_0)$. We follow the initial steps of the proof of Theorem 1.1. Let h be the p-harmonic function in B_R that agrees with u on the boundary, i.e.,

$$\Delta_p h = 0 \quad \text{in } B_R \quad \text{and} \quad h - u \in W_0^{1,p}(B_R).$$

By Lemma 4.1 we have

$$\int_{B_r} \left|\nabla u(X) - (\nabla u)_r\right|^p dX \leqslant C\left(\frac{r}{R}\right)^{n+p\alpha_p} \int_{B_R} \left|\nabla u(X) - (\nabla u)_R\right|^p dX + C\int_{B_R} \left|\nabla u(X) - \nabla h(X)\right|^p dX. \tag{5.2}$$

On the other hand, by the minimality of u we have

$$\int_{B_R} \left(|\nabla u|^p - |\nabla h|^p\right) dX \leqslant \int_{B_R} \left(F_0(h) - F_0(u)\right) dX + \int_{B_R} f(X)(h - u)\, dX. \tag{5.3}$$

Readily one verifies that

$$\int_{B_R} \big(F_0(h) - F_0(u)\big)\, dX \leqslant C(\lambda_+, \lambda_-)|B_R|. \tag{5.4}$$

As before, applying Hölder inequality and afterwards Poincaré inequality we obtain

$$\begin{aligned} \int_{B_R} f(X)(h-u)\, dX &\leqslant \|f\|_{L^n} |B_R|^{\frac{p-1}{p} - \frac{1}{n}} \|u-h\|_{L^p} \\ &\leqslant \|f\|_{L^n} |B_R|^{\frac{p-1}{p} - \frac{1}{n} + \frac{1}{n}} \big\|\nabla(u-h)\big\|_{L^p}. \end{aligned} \tag{5.5}$$

Therefore, with the aid of Young inequality we estimate

$$\begin{aligned} \int_{B_R} f(X)(h-u)\, dX &\leqslant C(p)\big(\|f\|_{L^n}\big)^{\frac{p}{p-1}} |B_R|^{\frac{p}{p-1}(\frac{p-1}{p})} \big\|\nabla(u-h)\big\|_{L^p}^p + \frac{1}{4}\big\|\nabla(u-h)\big\|_{L^p} \\ &= C(p)\big(\|f\|_{L^n}\big)^{\frac{p}{p-1}} |B_R| + \frac{1}{4}\big\|\nabla(u-h)\big\|_{L^p}. \end{aligned} \tag{5.6}$$

Taking into account (5.2) and replacing (5.4) and (5.6) into (5.3) we reach

$$\begin{aligned} \int_{B_r} \big|\nabla u - (\nabla u)_r\big|^p\, dX &\leqslant C(n,p)\left(\frac{r}{R}\right)^{n+p\alpha_p} \int_{B_R} \big|\nabla u(X) - (\nabla u)_R\big|^p\, dX \\ &\quad + C(n,p)\big[C(\lambda_+, \lambda_-)\big]|B_R| + C(n,p)C\big(n, p, \lambda_+, \lambda_-, \|f\|_{L^n}\big)|B_R| \\ &\leqslant C\left(\frac{r}{R}\right)^{n+p\alpha_p} \int_{B_R} \big|\nabla u(X) - (\nabla u)_R\big|^p\, dX + CR^n, \end{aligned}$$

where $C = C(n, p, \lambda_+, \lambda_-, \|f\|_{L^n})$ is a positive constant. In view of Lemma 2.7 we obtain

$$\int_{B_r(X_0)} \big|\nabla u - (\nabla u)_r\big|^p\, dX \leqslant Cr^n, \tag{5.7}$$

which shows that the gradient of u lies in BMO space and for any fixed subdomain $\Omega' \Subset \Omega$, there holds

$$\|\nabla u\|_{\mathrm{BMO}(\Omega')} \leqslant C\big(\Omega', n, p, \lambda_+, \lambda_-, \|f\|_{L^n}\big).$$

From Fefferman–Stein BMO Characterization Theorem, see [8], there exist vector fields $\Gamma_0, \Gamma_1, \dots, \Gamma_n \in L^\infty(\Omega')$, such that

$$\nabla u(X) = \Gamma_0(X) + \sum_{i=1}^{n} \mathcal{R}_j(\Gamma_j),$$

where $\mathcal{R}_j$ denotes the classical Riesz transform,

$$\mathcal{R}_j(f) := f * K_j \quad \text{for } K_j(X_j) := \frac{c_n X_j}{|X|^{n+1}}.$$

It now follows by a similar reasoning employed in the Appendix of [11] that

$$\big|\nabla u(X)\big| \leqslant -\log|X - X_0|, \quad \text{for } X \in B_\rho(X_0),\ \rho \ll 1.$$

Finally, by Morrey's type estimate, we obtain, for $s > n$,

$$
\begin{aligned}
\left|u(X)-u(X_0)\right| &\leqslant C|X-X_0|^{1-\frac{n}{r}}\cdot\left(\int\limits_{B_r(X_0)}\left|\nabla u(Z)\right|^s dZ\right)^{1/p}\\
&\leqslant C|X-X_0|^{1-\frac{n}{s}}\left(\int\limits_0^{|X-X_0|}|\log Z|^s\cdot|Z|^{n-1}dZ\right)^{1/s}\\
&\leqslant C|X-X_0|\cdot\left|\log|X-X_0|\right|,
\end{aligned}
$$

and the proof of Theorem 1.2 is concluded. □

6. Lower gradient bounds

From this section on, we aim towards gradient estimates to minimizers of heterogeneous p-jet flow functional (5.1). We remark once more that even for equations with no free boundaries, say $\lambda_- = \lambda_+$, it is not possible to obtain pointwise control of the gradient of u_0, under the borderline condition $f \in L^n$. In this case, as proven in Theorem 1.2, the best control available is of logarithm order. Therefore, from this section on, we shall assume the source function $f(X)$, appearing in functional (5.1) is q-integrable, for $q > n$. Under such natural hypothesis, our next theorem shows that u_0^+ grows linearly away from the free boundary $\mathfrak{F}^+ := \partial\{u > 0\} \cap \Omega$.

Theorem 6.1. *Let u_0 be a local minimizer to $\mathcal{J}_0$, with $f \in L^q(\Omega)$, $q > n$, $\Omega' \Subset \Omega$ and $X_0 \in \{u_0 > 0\} \cap \Omega'$. There exists a constant $c_0 > 0$ depending only on n, p, λ_+ and $\|f\|_{L^q(\Omega)}$ such that*

$$u(X_0) \geqslant c_0 \operatorname{dist}(X_0, \mathfrak{F}^+).$$

Proof. Let us fix $X_0 \in \{u_0 > 0\} \cap \Omega'$. It suffices to show such estimate for points $X_0 \in \{u_0 > 0\} \cap \Omega'$ such that

$$0 < \operatorname{dist}(X_0, \mathfrak{F}^+) \ll \delta(n, p, \lambda_+, \|f\|_{L^q(\Omega)}),$$

for $\delta(n, p, \lambda_+, \|f\|_{L^q(\Omega)})$ to be regulated *a posteriori*. Let us denote $d := \operatorname{dist}(X_0, \mathfrak{F}^+)$ and if we define

$$v(X) := \frac{1}{d}u_0(X_0 + dX),$$

one easily verifies that v is a local minimizer to

$$\mathcal{J}_0^d(\xi) := \int\limits_{B_1}\left(|\nabla\xi|^p + \lambda_+\chi_{\{\xi>0\}} + d\cdot f(X_0 + d\cdot X)\cdot\xi(X)\right)dX,$$

in $W_0^{1,p}(B_1) + v$. The thesis of Theorem 6.1 is equivalent to proving that $v(0)$ is universally bounded away from zero. Clearly $v \geqslant 0$ in B_1. By Harnack inequality (see Theorem 2.3), we have

$$
\begin{aligned}
v(X) &\leqslant C(n,p)\left\{v(0) + \left\|d\cdot f(X_0 + d\cdot X)\right\|_{L^q(B_1)}^{\frac{1}{p-1}}\right\}\\
&\leqslant C(n,p)\left\{v(0) + \left(d^{1-\frac{n}{q}}\cdot\|f\|_q\right)^{\frac{1}{p-1}}\right\}, \quad \forall X \in B_{3/5}.
\end{aligned}
\tag{6.1}
$$

In the sequel, we choose a nonnegative, smooth radially symmetric cut-off function ψ satisfying

$$\phi \equiv 0 \quad \text{in } B_{1/10} \quad \text{and} \quad \phi \equiv 1 \quad \text{in } B_1 \setminus B_{1/2}$$

and define the test function g in B_1 by

$$g(X) := \min\left\{v, C(n,p)\left\{v(0) + \left(d^{1-\frac{n}{q}}\cdot\|f\|_q\right)^{\frac{1}{p-1}}\right\}\cdot\psi(X)\right\}.$$

Notice that $g \in W^{1,p}$ and from Harnack inequality, estimate (6.1), g agrees with v in $B_1 \setminus B_{1/2}$. Let us label the set

$$B_{1/2} \supset \Pi := \left\{Y \in B_{1/2}\colon C(n,p)\left\{v(0) + \left(d^{1-\frac{n}{q}}\cdot\|f\|_q\right)^{\frac{1}{p-1}}\right\}\cdot\psi(Y) < v(Y)\right\} \supset B_{1/10}.$$

From the minimality of v, we estimate

$$\int_{\Pi} \lambda_+(1-\chi_{\{g>0\}}) + d\cdot f(X_0+d\cdot X)\cdot\big[v(X)-g(X)\big]\,dX \leqslant \int_{\Pi}\big(|\nabla g|^p - |\nabla v|^p\big)\,dX. \tag{6.2}$$

The right-hand side of (6.2) is readily estimated as

$$\begin{aligned}\int_{\Pi}\big(|\nabla g|^p-|\nabla v|^p\big)\,dX &\leqslant \Big[C(n,p)\Big\{v(0)+\big(d^{1-\frac{n}{q}}\cdot\|f\|_q\big)^{\frac{1}{p-1}}\Big\}\cdot\|\psi\|_\infty\Big]^p\\ &\leqslant Cv(0)^p + C\big[d^{1-\frac{n}{q}}\cdot\|f\|_q\big]^{\frac{p}{p-1}}.\end{aligned} \tag{6.3}$$

We now turn our efforts towards estimating the left-hand side of (6.2) by below. Readily we obtain

$$\begin{aligned}\int_{\Pi}\lambda_+(1-\chi_{\{g>0\}})\,dX &= \int_{\Pi}\lambda_+\chi_{\{g=0\}}\,dX\\ &\geqslant \lambda_+|B_{1/10}|.\end{aligned} \tag{6.4}$$

Invoking once more Harnack inequality (6.1) and the fact that $\Pi\subset B_{1/2}$, we estimate

$$\int_{\Pi} d\cdot f(X_0+d\cdot X)\cdot\big[v(X)-g(X)\big]\,dX \leqslant C\big(d^{1-\frac{n}{q}}\cdot\|f\|_q\big)\cdot\Big\{v(0)+\big(d^{1-\frac{n}{q}}\cdot\|f\|_q\big)^{\frac{1}{p-1}}\Big\}. \tag{6.5}$$

Combining (6.3), (6.4) and (6.5) we reach

$$C\Big\{v(0)^p+\big(d^{1-\frac{n}{q}}\cdot\|f\|_q\big)v(0)\Big\} \geqslant \lambda_+|B_{1/10}| - C\big[d^{1-\frac{n}{q}}\cdot\|f\|_q\big]^{\frac{p}{p-1}}. \tag{6.6}$$

Therefore, choosing $0<d\leqslant\delta(n,p,\lambda_+,\|f\|_{L^q(\Omega)})\ll 1$, appropriately, we conclude

$$v(0)\geqslant c\big(n,p,\lambda_+,\|f\|_q\big)>0,$$

and the proof of Theorem 6.1 follows. □

Next we iterate linear growth established in Theorem 6.1 as we obtain a stronger non-degeneracy property for u_0 near the free boundary.

Theorem 6.2. *Let u_0 be a local minimizer to $\mathcal{J}_0$, with $f\in L^q(\Omega)$, $q>n$, $\Omega'\Subset\Omega$ and $X_0\in\{u_0\geqslant 0\}\cap\Omega'$. There exists a constant $\underline{c}>0$ depending on n, p, λ_+ and $\|f\|_{L^q(\Omega)}$, such that*

$$\sup_{B_r(X_0)} u_0^+ \geqslant \underline{c}\cdot r,$$

for any $0<r\leqslant \operatorname{dist}(\partial\Omega',\partial\Omega)$.

Proof. By continuity, it suffices to show u_0 is strongly non-degenerated, i.e., the thesis of Theorem 6.2 holds within the positivity set

$$\Omega_0^+ := \{u_0>0\}\cap\Omega'.$$

We will obtain such a result by iterating linear growth estimate. More precisely we will initially show that there exists a $\delta_0>0$ that depends only on n, Ω', p, λ_+ and $\|f\|_q$ such that if $X\in\{u_0>0\}\cap\Omega'$, there holds

$$\sup_{B_{d(X)}(X_0)} u_0 \geqslant (1+\delta_0)u_0(X_0), \tag{6.7}$$

where $d(X):=\operatorname{dist}(X,\mathfrak{F}^+)$. In order to verify (6.7), let us assume, for the purpose of contradiction, that no such a δ_0 exists. If so, it would be possible to find sequences $\delta_j=\mathrm{o}(1)$ and $X_j\in\{u_0>0\}\cap\Omega'$ satisfying

$$\sup_{B_{d_j}(X_j)} u_0 \leqslant (1+\delta_j)u_0(X_j), \quad \text{for } d_j:=\operatorname{dist}\big(X_j,\mathfrak{F}^+\big)=\mathrm{o}(1). \tag{6.8}$$

Let us consider the following normalized sequence of functions $\varrho_j : B_1 \to \mathbb{R}$ defined by

$$\varrho_j(Z) := \frac{u_0(X_j + d_j Z)}{u_0(X_j)}.$$

Clearly, $\varrho_j(0) = 1$, and from (6.8),

$$0 \leqslant \varrho_j \leqslant 1 + \delta_j \quad \text{in } B_1. \tag{6.9}$$

In addition, ϱ_j satisfies

$$\Delta_p \varrho_j = \frac{d_j^p}{u_0(X_j)^{p-1}} \cdot f(X_j + d_j Z), \tag{6.10}$$

in the distributional sense in B_1. Taking into account the linear growth established in Theorem 6.1 and Eq. (6.10), we reach

$$|\Delta_p \varrho_j| \leqslant C d_j \cdot f(X_j + d_j Z), \quad \text{in } B_1. \tag{6.11}$$

From Harnack inequality, we deduce the sequence $\{\varrho_j\}_{j\in\mathbb{N}}$ is locally equicontinuous in B_1; thus, up to a subsequence, $\varrho_j \to \varrho$ locally uniformly in B_1. Harnack inequality further reveals that for any $|X| \leqslant r < 1$, there holds

$$0 \leqslant [1 + \delta_j] - \varrho_j(X) \leqslant C_r\left([1 + \delta_j] - \varrho_j(0) + d_j^{1-\frac{n}{q}} \cdot \|f\|_q\right) = C_r \cdot \mathrm{o}(1). \tag{6.12}$$

Letting $j \to \infty$ in the above estimate, we deduce the limiting blow-up function $\varrho \equiv 1$ in B_1.

We now show that such a conclusion drives us to an inconsistency. To this end, let $Y_j \in \mathfrak{F}^+$ be such that $d_j = |X_j - Y_j|$. Up to subsequence, there would hold

$$1 + \mathrm{o}(1) = \varrho_j\left(\frac{Y_j - X_j}{d_j}\right) = 0,$$

which clearly gives a contradiction for $j \gg 1$. We have shown the validity of estimate (6.7).

To finish up the proof of Theorem 6.2, we employ a Caffarelli's polygonal type of argument. That is, we construct a polygonal along which u_0 grows linearly. Starting from $X_0 = X$, we find a sequence of points $\{X_n\}_{n\geqslant 0}$ such that:

1. $u_0(X_n) \geqslant (1 + \delta_0)^n u_0(X_0)$;
2. $|X_n - X_{n-1}| = \mathrm{dist}(X_{n-1}, \mathfrak{F}^+)$;
3. $u_0(X_n) - u_0(X_{n-1}) \geqslant c|X_n - X_{n-1}|$. In particular, $u_0(X_n) - u_0(X_0) \geqslant c|X_n - X_0|$.

Since $u(x_n) \to \infty$ as $n \to \infty$ this process must be finite, that is, there exists a last X_{n_0} in the ball $B_r(X_0)$. For such a last point,

$$|X_{n_0} - X_0| \geqslant c_p r.$$

Finally,

$$\sup_{B_r(X)} u_0 \geqslant u_0(X_{n_0}) \geqslant u_0(X_0) + c|X_n - X_0| \geqslant \underline{c} \cdot r,$$

and the proof is concluded. □

7. Stability for free boundary problems

In this section we show the stability of the family of free boundary problems obtained by the minimization of the non-differentiable functionals

$$\mathcal{J}_\gamma(v) := \int_\Omega \left(|\nabla v|^p + \lambda_+ (v^+)^\gamma + \lambda_- (v^-)^\gamma + f(X) \cdot v\right) dX \to \min, \tag{7.1}$$

as $\gamma = \mathrm{o}(1)$. The ultimate goal of this section is to show that any limit point u_0 of $\{u_\gamma\}_{\gamma=\mathrm{o}(1)}$ is a minimizer to the p-degenerate cavitation functional

$$\mathcal{J}_0(v) := \int_\Omega \big(|\nabla v|^p + \lambda_+ \chi_{\{v>0\}} + \lambda_- \chi_{\{v\leqslant 0\}} + f(X)\cdot v\big)\, dX. \tag{7.2}$$

Initially we show compactness of $\{u_\gamma\}_{0<\gamma\leqslant 1}$ in the $W^{1,p}$ topology.

Proposition 7.1. *Let u_{γ_j} be a sequence of minima to the functional $\mathcal{J}_{\gamma_j}$, $f \in L^n$ and assume $u_{\gamma_j} \to v$ a.e., $\gamma_j \to \gamma_0$. Then for any $0 < E < \infty$, $u_{\gamma_j} \to v$ in the $W^{1,E}_{\mathrm{loc}}(\Omega)$ topology.*

Proof. It follows from Proposition 3.3 and a.e. convergence that $\Delta_p u_{\gamma_j} \rightharpoonup \Delta_p v$ in the sense of measures. Thus, from truncation arguments, see for instance [4],

$$\nabla u_{\gamma_j} \to \nabla v \quad \text{a.e. in } \Omega. \tag{7.3}$$

From Theorem 1.2, for any $\Omega' \Subset \Omega$, there exists a constant $C(n, p, \lambda_+, \lambda_-, \Omega', \|f\|_n)$, independent of γ_j, such that,

$$\|\nabla u_{\gamma_j}\|_{\mathrm{BMO}(\Omega')} \leqslant C\big(n, p, \lambda_+, \lambda_-, \Omega', \|f\|_n\big). \tag{7.4}$$

Thus, from John–Nirenberg's Theorem, for $1 \leqslant E < \infty$ fixed,

$$\|\nabla u_{\gamma_j}\|_{L^{E+1}(\Omega')} \leqslant C\big(n, p, \lambda_+, \lambda_-, \Omega', \|f\|_n\big). \tag{7.5}$$

Finally, combining (7.3), (7.5) and classical arguments, see for instance [12], we deduce

$$\nabla u_{\gamma_j} \to \nabla v \quad \text{in } L^E\big(\Omega'\big),$$

and the proposition follows. □

Theorem 7.2. *Let $u_0 := \lim_{\gamma_j} u_{\gamma_j}$ as $\gamma_j \to 0$. Then u_0 is a local minimizer of $\mathcal{J}_0$.*

Proof. Let B_r be a ball in Ω. Given an arbitrary $W^{1,p}$ function ψ that agrees with u_0 on ∂B_r, we have to show that

$$\mathcal{J}_0(B_r, u_0) \leqslant \mathcal{J}_0(B_r, \psi).$$

By density we may further assume that ψ is bounded. Let us define the interpolated function

$$\psi_{\gamma_j,h} := \begin{cases} u_0 + \frac{|X|-r}{h}(u_{\gamma_j} - u_0) & \text{in } B_{r+h} \setminus B_r \\ \psi & \text{in } B_r. \end{cases}$$

One simply verifies that

$$|\nabla \psi_{\gamma_j,h}|^p \leqslant C_p \left\{ |\nabla u_0|^p + \frac{1}{h^p}|u_{\gamma_j} - u_0|^p + |\nabla u_{\gamma_j} - \nabla u_0|^p \right\} \quad \text{in } B_{r+h} \setminus B_r. \tag{7.6}$$

In the above estimate, we have used the classical facts:

$$\nabla\big(|X|\big) = \frac{X}{|X|} \quad \text{and} \quad \left(\frac{|X|-r}{h}\right)^p \leqslant 1 \quad \text{in } B_{r+h} \setminus B_r. \tag{7.7}$$

By L^∞ bounds, Theorem 3.1, there exists a constant $C_1 > 0$, independent of γ_j, such that $\|u_{\gamma_j}\|_\infty < C_1$. Thus, if we denote

$$H^\pm_{\gamma_j}(t) := \big(t^\pm\big)^{\gamma_j},$$

we have

$$H^\pm_{\gamma_j}(\psi_{\gamma_j,h}) \leqslant (3C_1)^{\gamma_j} \quad \text{in } B_{r+h} \setminus B_r, \tag{7.8}$$

and

$$H^{\pm}_{\gamma_j}(\psi_{\gamma_j,h}) \leqslant \left(\|\psi\|_{L^\infty(B_r)}\right)^{\gamma_j}\chi_{\{u_{\gamma_j}\gtrless 0\}} \quad \text{in } B_r. \tag{7.9}$$

We can estimate

$$\begin{aligned}
\mathcal{J}_{\gamma_j}(B_{r+h},\psi_{\gamma_j,h}) = &\int_{B_{r+h}\setminus B_r} |\nabla\psi_{\gamma_j,h}|^p + \lambda_+ H^+_{\gamma_j}(\psi_{\gamma_j,h}) + \lambda_- H^-_{\gamma_j}(\psi_{\gamma_j,h})\,dX \\
&+ \int_{B_{r+h}\setminus B_r} f(X)\left[u_0 + \frac{|X|-r}{h}(u_{\gamma_j}-u_0)\right]dX + \mathcal{J}_{\gamma_j}(B_r,\psi) \\
\leqslant{}& C_p\int_{B_{r+h}\setminus B_r} |\nabla u_0|^p\,dX + C_p\int_{B_{r+h}\setminus B_r} |\nabla u_{\gamma_j}-\nabla u_0|^p\,dX \\
&+ \left[2\lambda_+(3C_1)^{\gamma_j} + 3C_1\right]C_p|B_{r+h}\setminus B_r| + \frac{C_p}{h^p}\int_{B_{r+h}\setminus B_r}|u_{\gamma_j}-u_0|^p\,dX \\
&+ \mathcal{J}_0(B_r,\psi) + \left(\|\psi\|^{\gamma_j}_{L^\infty(B_r)} - 1\right)\int_{B_r}\lambda_+\chi_{\{\psi>0\}} + \lambda_-\chi_{\{\psi\leqslant 0\}}\,dX \\
&+ |B_{r+h}\setminus B_r|^{1-\frac{1}{q}}\|f\|_{L^q(\Omega)}.
\end{aligned} \tag{7.10}$$

By pointwise convergence $u_{\gamma_j}\to u_0$ we have

$$\lim_{j\to\infty}\int_{B_{r+h}\setminus B_r} |u_{\gamma_j}-u_0|^p\,dx = 0 \tag{7.11}$$

and by Proposition 7.1

$$\lim_{j\to\infty}\int_{B_{r+h}\setminus B_r} |\nabla u_{\gamma_j}-\nabla u_0|^p\,dx = 0. \tag{7.12}$$

From the minimality property of u_{γ_j},

$$\mathcal{J}_{\gamma_j}(B_{r+h},\psi_{\gamma_j,h}) \geqslant \mathcal{J}_{\gamma_j}(B_{r+h},u_{\gamma_j}) \geqslant \mathcal{J}_{\gamma_j}(B_r,u_{\gamma_j}) + \int_{B_{r+h}\setminus B_r} f(X)u_{\gamma_j}\,dX. \tag{7.13}$$

Furthermore, it follows from Proposition 7.1

$$\int_{B_r}|\nabla u_0|^p\,dX = \lim_{j\to\infty}\int_{B_r}|\nabla u_{\gamma_j}|^p\,dX. \tag{7.14}$$

By the pointwise convergence $u_{\gamma_j}\to u_0$ and Fatou's Lemma (see the proof of Theorem 3.1), we conclude

$$\int_{B_r}\lambda_+\chi_{\{u_0>0\}} + \lambda_-\chi_{\{u_0\leqslant 0\}}\,dX \leqslant \liminf_{j\to\infty}\int_{B_r}\lambda_+(u_{\gamma_j})^{\gamma_j}\chi_{\{u_{\gamma_j}>0\}} + \lambda_-(u_{\gamma_j})^{\gamma_j}\chi_{\{u_{\gamma_j}\leqslant 0\}}\,dX, \tag{7.15}$$

and

$$\lim_{j\to\infty}\int_{B_r} f(X)u_{\gamma_j}\,dX = \lim_{j\to\infty}\int_{B_r} f(X)u_0\,dX. \tag{7.16}$$

Finally, combining (7.10)–(7.16) we reach

$$\mathcal{J}_0(B_r, u_0) \leqslant \liminf_{j\to\infty} \mathcal{J}_{\gamma_j}(B_{r+h}, u_{\gamma_j}) \leqslant \mathcal{J}_0(B_r, \psi) + C_p \int_{B_{r+h}\setminus B_r} |\nabla u_0|^p \, dX + (2\lambda_+ + 3C_1)C_p|B_{r+h}\setminus B_r| + |B_{r+h}\setminus B_r|^{1-\frac{1}{q}} \|f\|_{L^q(\Omega)}. \tag{7.17}$$

Letting $h \to 0$, we finish the proof of Theorem 7.2. □

Conflict of interest statement

The authors declare that there is no conflict of interest.

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