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A critical fractional equation with concave–convex power nonlinearities [☆]

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Abstract

In this work we study the following fractional critical problem

$$(P_\lambda) = \begin{cases} (-\Delta)^s u = \lambda u^q + u^{2_s^*-1}, \quad u>0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a regular bounded domain, $\lambda > 0$, $0 < s < 1$ and $n > 2s$. Here $(-\Delta)^s$ denotes the fractional Laplace operator defined, up to a normalization factor, by

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy, \qquad x \in \mathbb{R}^n.$$

Our main results show the existence and multiplicity of solutions to problem (P_λ) for different values of λ. The dependency on this parameter changes according to whether we consider the concave power case $(0 < q < 1)$ or the convex power case $(1 < q < 2_s^* - 1)$. These two cases will be treated separately.

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1. Introduction

In recent years, considerable attention has been given to nonlocal diffusion problems, in particular to the ones driven by the fractional Laplace operator. One of the reasons for this comes from the fact that this operator naturally arises in several physical phenomena like flames propagation and chemical reactions of liquids, in population dynamics and geophysical fluid dynamics, or in mathematical finance (American options). It also provides a simple model to describe certain jump Lévy processes in probability theory. In all these cases, the nonlocal effect is modeled by the singularity at infinity. For more details and applications, see [6,9,20,27,49,50] and the references therein.

In this paper we focus our attention on critical nonlocal fractional problems. To be more precise, we consider the following critical problem with convex–concave nonlinearities

$$(P_\lambda) = \begin{cases} (-\Delta)^s u = \lambda u^q + u^{2_s^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a regular bounded domain, $\lambda > 0$, $n > 2s$, $0 < q < 2_s^* - 1$ and

$$2_s^* = \frac{2n}{n-2s} \tag{1.1}$$

is the fractional critical Sobolev exponent. Here $(-\Delta)^s$ is the fractional Laplace operator defined, up to a normalization factor, by the Riesz potential as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} \, dy, \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $s \in (0,1)$ is a fixed parameter (see [46, Chapter 5] or [22,45] for further details).

One can also define a fractional power of the Laplacian using spectral decomposition. The same problem considered here but for this spectral fractional Laplacian has been treated in [7]. Some related problems involving this operator have been studied in [11,14,19,48]. As in [7] the purpose of this paper is to study the existence of weak solutions for (P_λ). Previous works related to the operator defined in (1.2), or by a more general kernel, can be found in [16,23,30,34,35,37,38,41–43].

Problems similar to (P_λ) have been also studied in the local setting with different elliptic operators. As far as we know, the first example in this direction was given in [25] for the p-Laplacian operator. Other results, this time for the Laplacian (or essentially the classical Laplacian) operator can be found in [1,4,10,18]. More generally, the case of fully nonlinear operators has been studied in [17].

It is worth noting here that the problem (P_λ), with $\lambda = 0$, has no solution whenever Ω is a star-shaped domain. This has been proved in [24,36] using a Pohozaev identity for the operator $(-\Delta)^s$. This fact motivates the perturbation term λu^q, $\lambda > 0$, in our work.

We now summarize the main results of the paper. First, in Section 2 we look at the problem (P_λ) in the concave case $q < 1$ and prove the following.

Theorem 1.1. *Assume $0 < q < 1$, $0 < s < 1$, and $n > 2s$. Then, there exists $0 < \Lambda < \infty$ such that problem (P_λ)*

(1) *has no solution for $\lambda > \Lambda$;*
(2) *has a minimal solution for any $0 < \lambda < \Lambda$; moreover, the family of minimal solutions is increasing with respect to λ;*
(3) *if $\lambda = \Lambda$ there exists at least one solution;*
(4) *for $0 < \lambda < \Lambda$ there are at least two solutions.*

The convex case is treated in Section 3. The existence result for problem (P_λ) is given by:

Theorem 1.2. *Assume $1 < q < 2_s^* - 1$, $0 < s < 1$, and $n > 2s$. Then, problem (P_λ) admits at least one solution provided that either*

- *$n > \frac{2s(q+3)}{q+1}$ and $\lambda > 0$, or*
- *$n \leqslant \frac{2s(q+3)}{q+1}$ and λ is sufficiently large.*

Theorem 1.1 corresponds to the nonlocal version of the main result of [4], while Theorem 1.2 may be seen as the nonlocal counterpart of the results obtained for the standard Laplace operator in [13, Subsections 2.3, 2.4 and 2.5] (see also [25, Theorems 3.2 and 3.3] for the case of the p-Laplacian operator). Note, in particular, that when $s = 1$ one has $2s(q+3)/(q+1) = 2(q+3)/(q+1) < 4$, due to the choice of $q > 1$.

We will denote by $H^s(\mathbb{R}^n)$ the usual fractional Sobolev space endowed with the so-called *Gagliardo norm*

$$\|g\|_{H^s(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} + \left(\int\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \right)^{1/2}, \tag{1.3}$$

while $X_0^s(\Omega)$ is the function space defined as

$$X_0^s(\Omega) = \{u \in H^s(\mathbb{R}^n)\colon\ u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}. \tag{1.4}$$

We refer to [40,41] for a general definition of $X_0^s(\Omega)$ and its properties and to [2,22,28] for an account of the properties of $H^s(\mathbb{R}^n)$.

In $X_0^s(\Omega)$ we can consider the following norm

$$\|v\|_{X_0^s(\Omega)} = \left(\int\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \right)^{1/2}.$$

We also recall that $(X_0^s(\Omega), \|\cdot\|_{X_0^s(\Omega)})$ is a Hilbert space, with scalar product

$$\langle u, v\rangle_{X_0^s(\Omega)} = \int\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy. \tag{1.5}$$

See for instance [40, Lemma 7].

Observe that by [22, Proposition 3.6] we have the following identity

$$\|u\|_{X_0^s(\Omega)} = \|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^n)}. \tag{1.6}$$

This leads us to establish as a definition that the solutions to our problem in this variational framework are those functions satisfying the relationship (1.8) below.

In our context, the Sobolev constant is given by

$$S(n,s) := \inf_{v\in H^s(\mathbb{R}^n)\setminus\{0\}} Q_{n,s}(v) > 0, \tag{1.7}$$

where

$$Q_{n,s}(v) := \frac{\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}{(\int_{\mathbb{R}^n} |v(x)|^{2_s^*}\,dx)^{2/2_s^*}}, \qquad v \in H^s(\mathbb{R}^n),$$

is the associated Rayleigh quotient. The constant $S(n,s)$ is well defined, as can be seen in [2, Theorem 7.58].

1.1. Variational formulation of the problem

Let us start describing the notion of solution in this context. In order to present the weak formulation of (P_λ) and taking into account that we are looking for positive solutions, we will consider the following Dirichlet problem

$$(P_\lambda^+) = \begin{cases} (-\Delta)^s u = \lambda (u_+)^q + (u_+)^{2_s^*-1} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $u_+ := \max\{u, 0\}$ denotes the positive part of u. With this at hand, we can now give the following.

Definition 1.3. We say that $u \in X_0^s(\Omega)$ is a weak solution of (P_λ^+) if for every $\varphi \in X_0^s(\Omega)$, one has

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy = \lambda \int_\Omega (u_+)^q \varphi\,dx + \int_\Omega (u_+)^{2_s^*-1}\varphi\,dx. \tag{1.8}$$

In the sequel we will omit the term *weak* when referring to solutions that satisfy the conditions of Definition 1.3. The crucial observation here is that, by the Maximum Principle [45, Proposition 2.2.8], if u is a solution of (P_λ^+) then u is strictly positive in Ω and, therefore, it is also a solution of (P_λ).

To find solutions of (P_λ^+), we will use a variational approach. Hence, we will associate a suitable functional to our problem. More precisely, the Euler–Lagrange functional related to problem (P_λ^+) is given by $\mathcal{J}_{s,\lambda} : X_0^s(\Omega) \to \mathbb{R}$ defined as follows

$$\mathcal{J}_{s,\lambda}(u) = \frac{1}{2}\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \frac{\lambda}{q+1}\int_\Omega (u_+)^{q+1}\,dx - \frac{1}{2_s^*}\int_\Omega (u_+)^{2_s^*}\,dx.$$

Note that $\mathcal{J}_{s,\lambda}$ is $\mathcal{C}^1$ and that its critical points correspond to solutions of (P_λ^+).

In both cases, $q < 1$ and $q > 1$, we will use the Mountain Pass Theorem (MPT) by Ambrosetti and Rabinowitz (see [5]). In order to do that, we will show that $\mathcal{J}_{s,\lambda}$ satisfies a compactness property and has suitable geometrical features. The fact that the functional has the suitable geometry is easy to check. Observe that the embedding $X_0^s(\Omega) \hookrightarrow L^{2_s^*}(\mathbb{R}^n)$ is not compact (see [2]). This is even true when the nonlocal operator has a more general kernel (see [41, Lemma 9-b)]). Hence, the difficulty to apply MPT lies on proving a local Palais–Smale (PS for short) condition at level $c \in \mathbb{R}$ ($(\mathrm{PS})_c$). Moreover, since the PS condition does not hold globally, we have to prove that the Mountain Pass critical level of $\mathcal{J}_{s,\lambda}$ lies below the threshold of application of the $(\mathrm{PS})_c$ condition.

In the concave setting, $q < 1$, the idea is to prove the existence of at least two positive solutions for an admissible small range of λ. For that we are using a contradiction argument, inspired by [4]. The proof is divided into several steps: we first show that we have a solution that is a local minimum for the functional $\mathcal{J}_{s,\lambda}$. In the next step, in order to find a second solution, we suppose that this local minimum is the only critical point of the functional, and then we prove a local $(\mathrm{PS})_c$ condition for c under a critical level related with the best fractional critical Sobolev constant given in (1.7). Also we find a path under this critical level localizing the Sobolev minimizers at the possible concentration on Dirac Deltas. These Deltas are obtained by the concentration–compactness result in [34, Theorem 1.5] inspired in the classical result by P.-L. Lions in [32,33]. Applying the MPT given in [5] and its refined version given in [26], we will reach a contradiction.

In the convex case $q > 1$ we also apply the MPT to obtain the existence of at least one solution for (P_λ^+) for suitable values of λ depending on the dimension n. As before, we prove a local $(\mathrm{PS})_c$ condition in an appropriate range related with the constant $S(n,s)$ defined on (1.7). The strategy to obtain a solution follows the ideas given in [13] (see also [47,51]) adapted to our nonlocal functional framework.

The linear case $q = 1$, when the right hand side of the equation is equal to $\lambda u + |u|^{2_s^*-2}u$, was treated in [37,38, 41–43]. In these works the authors studied also nonlinearities more general than those given by the power critical function as well as the existence of solutions not necessarily positive.

2. The critical and concave case $0 < q < 1$

This section is devoted to the study of problem (P_λ) in the case of the exponent $0 < q < 1$. We point out that the result of Theorem 1.1 in the subcritical case could be obtained by the arguments given in this paper. However, in this subcritical case the PS condition is easier to prove – it is indeed satisfied for any energy level – and the separation of solutions, presented in Lemma 2.3 below, is not needed. This approach has been carried out in [8] where the authors obtain the equivalent to Theorem 1.1 for a related problem using a technique developed in [3].

We begin with the following result that uses, in its proof, a standard comparison method as well as some ideas given in [4, Lemma 3.1 and Lemma 3.4].

Lemma 2.1. *Let $0 < q < 1$ and let Λ be defined by*

$$\Lambda := \sup\{\lambda > 0: \text{ problem } (P_\lambda) \text{ has solution}\}. \tag{2.1}$$

Then, $0 < \Lambda < \infty$ and the critical concave problem (P_λ) has at least one solution for every $0 < \lambda \leqslant \Lambda$. Moreover, for $0 < \lambda < \Lambda$ we get a family of minimal solutions increasing with respect to λ.

By Lemma 2.1 we easily deduce statements (1)–(3) of Theorem 1.1. Hence, in the sequel we focus on proving statement (4) of that theorem, that is on the existence of a second solution for (P_λ).

First we prove a regularity result which will be useful in certain parts of this section:

Proposition 2.2. *Let u be a positive solution to the problem*

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

and assume that $|f(x,t)| \leqslant C(1+|t|^p)$, for some $1 \leqslant p \leqslant 2_s^* - 1$ and $C > 0$. Then $u \in L^\infty(\Omega)$.

Proof. The proof uses standard techniques for the fractional Laplacian, in particular the following inequality: if φ is a convex and differentiable function, then

$$(-\Delta)^s \varphi(u) \leqslant \varphi'(u)(-\Delta)^s u.$$

Let us define, for $\beta \geqslant 1$ and $T > 0$ large,

$$\varphi(t) = \varphi_{T,\beta}(t) = \begin{cases} 0, & \text{if } t \leqslant 0, \\ t^\beta, & \text{if } 0 < t < T, \\ \beta T^{\beta-1}(t-T) + T^\beta, & \text{if } t \geqslant T. \end{cases}$$

Observe that $\varphi(u) \in X_0^s(\Omega)$ since φ is Lipschitz with constant $K = \beta T^{\beta-1}$ and, therefore,

$$\begin{aligned} \|\varphi(u)\|_{X_0^s(\Omega)} &= \left(\int\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\varphi(u(x)) - \varphi(u(y))|^2}{|x-y|^{n+2s}} \, dx \, dy \right)^{1/2} \\ &\leqslant \left(\int\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{K^2 |u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \right)^{1/2} = K \|u\|_{X_0^s(\Omega)}. \end{aligned}$$

By (1.6) and the Sobolev embedding theorem given in [2, Theorem 7.58], we have

$$\int\limits_\Omega \varphi(u)(-\Delta)^s \varphi(u) = \|\varphi(u)\|^2_{X_0^s(\Omega)} \geqslant S(n,s) \|\varphi(u)\|^2_{L^{2_s^*}(\Omega)}, \tag{2.2}$$

where $S(n,s)$ is defined in (1.7). On the other hand, since φ is convex, and $\varphi(u)\varphi'(u) \in X_0^s(\Omega)$,

$$\int\limits_\Omega \varphi(u)(-\Delta)^s \varphi(u) \leqslant \int\limits_\Omega \varphi(u)\varphi'(u)(-\Delta)^s u \leqslant C \int\limits_\Omega \varphi(u)\varphi'(u)\big(1 + u^{2_s^*-1}\big).$$

From (2.2) and the previous inequality we get the following basic estimate:

$$\|\varphi(u)\|^2_{L^{2_s^*}(\Omega)} \leqslant C \int\limits_\Omega \varphi(u)\varphi'(u)\big(1 + u^{2_s^*-1}\big). \tag{2.3}$$

Since $u\varphi'(u) \leqslant \beta\varphi(u)$ and $\varphi'(u) \leqslant \beta(1+\varphi(u))$, the above estimate (2.3) becomes

$$\left(\int\limits_\Omega \big(\varphi(u)\big)^{2_s^*} \right)^{2/2_s^*} \leqslant C\beta \left(1 + \int\limits_\Omega \big(\varphi(u)\big)^2 + \int\limits_\Omega \big(\varphi(u)\big)^2 u^{2_s^*-2} \right). \tag{2.4}$$

It is important to point out here that since $\varphi(u)$ grows linearly, both sides of (2.4) are finite.

Claim. *Let β_1 be such that $2\beta_1 = 2_s^*$. Then $u \in L^{\beta_1 2_s^*}$.*

To see this, we take R large to be determined later. Then, Hölder's inequality with $p=\beta_1=2_s^*/2$ and $p'=2_s^*/(2_s^*-2)$ gives

$$\int_\Omega (\varphi(u))^2 u^{2_s^*-2} = \int_{\{u\leqslant R\}} (\varphi(u))^2 u^{2_s^*-2} + \int_{\{u>R\}} (\varphi(u))^2 u^{2_s^*-2}$$
$$\leqslant \int_{\{u\leqslant R\}} (\varphi(u))^2 R^{2_s^*-2} + \left(\int_\Omega (\varphi(u))^{2_s^*}\right)^{2/2_s^*} \left(\int_{\{u>R\}} u^{2_s^*}\right)^{(2_s^*-2)/2_s^*}.$$

By the Monotone Convergence Theorem, we may take R so that

$$\left(\int_{\{u>R\}} u^{2_s^*}\right)^{(2_s^*-2)/2_s^*} \leqslant \frac{1}{2C\beta_1}.$$

In this way, the second term above is absorbed by the left hand side of (2.4) to get

$$\left(\int_\Omega (\varphi(u))^{2_s^*}\right)^{2/2_s^*} \leqslant 2C\beta_1\left(1+\int_\Omega (\varphi(u))^2 + \int_{\{u\leqslant R\}} (\varphi(u))^2 R^{2_s^*-2}\right). \tag{2.5}$$

Using that $\varphi_{T,\beta_1}(u)\leqslant u^{\beta_1}$ in the right hand side of (2.5) and then letting $T\to\infty$ in the left hand side, since $2\beta_1=2_s^*$, we obtain

$$\left(\int_\Omega u^{2_s^*\beta_1}\right)^{2/2_s^*} \leqslant 2C\beta_1\left(1+\int_\Omega u^{2_s^*} + R^{2_s^*-2}\int_\Omega u^{2_s^*}\right) < \infty.$$

This proves the claim.

We now go back to inequality (2.4) and we use as before that $\varphi_{T,\beta}(u)\leqslant u^\beta$ in the right hand side and then we take $T\to\infty$ in the left hand side. Then,

$$\left(\int_\Omega u^{2_s^*\beta}\right)^{2/2_s^*} \leqslant C\beta\left(1+\int_\Omega u^{2\beta} + \int_\Omega u^{2\beta+2_s^*-2}\right).$$

Since $\int_\Omega u^{2\beta}\leqslant |\Omega| + \int_\Omega u^{2\beta+2_s^*-2}$, we get the following recurrence formula

$$\left(\int_\Omega u^{2_s^*\beta}\right)^{2/2_s^*} \leqslant 2C\beta\big(1+|\Omega|\big)\left(1+\int_\Omega u^{2\beta+2_s^*-2}\right).$$

Therefore,

$$\left(1+\int_\Omega u^{2_s^*\beta}\right)^{\frac{1}{2_s^*(\beta-1)}} \leqslant C_\beta^{\frac{1}{2(\beta-1)}}\left(1+\int_\Omega u^{2\beta+2_s^*-2}\right)^{\frac{1}{2(\beta-1)}}, \tag{2.6}$$

where $C_\beta = 4C\beta(1+|\Omega|)$.

For $m\geqslant 1$ we define β_{m+1} inductively so that $2\beta_{m+1}+2_s^*-2 = 2_s^*\beta_m$, that is

$$\beta_{m+1}-1 = \frac{2_s^*}{2}(\beta_m-1) = \left(\frac{2_s^*}{2}\right)^m(\beta_1-1).$$

Hence, from (2.6) it follows that

$$\left(1+\int_\Omega u^{2_s^*\beta_{m+1}}\right)^{\frac{1}{2_s^*(\beta_{m+1}-1)}} \leqslant C_{\beta_{m+1}}^{\frac{1}{2(\beta_{m+1}-1)}}\left(1+\int_\Omega u^{2_s^*\beta_m}\right)^{\frac{1}{2_s^*(\beta_m-1)}},$$

with $C_{m+1} := C_{\beta_{m+1}} = 4C\beta_{m+1}(1+|\Omega|)$. Then, defining for $m\geqslant 1$

$$A_m := \left(1 + \int_\Omega u^{2_s^* \beta_m}\right)^{\frac{1}{2_s^*(\beta_m - 1)}},$$

by the Claim proved before, and using a limiting argument, we conclude that there exists $C_0 > 0$, independent of $m > 1$, such that

$$A_{m+1} \leqslant \prod_{k=2}^{m+1} C_k^{\frac{1}{2(\beta_k - 1)}} A_1 \leqslant C_0 A_1.$$

This implies that $\|u\|_{L^\infty(\Omega)} \leqslant C_0 A_1$. □

Coming back to the proof of Theorem 1.1, as we said in the Introduction, to find the existence of the second solution, we first show that the minimal solution $u_\lambda > 0$ given by Lemma 2.1 is a local minimum for the functional $\mathcal{J}_{s,\lambda}$. For that, following the ideas given in [18] we establish a separation lemma in the topology of the class

$$\mathcal{C}_s(\Omega) := \left\{ w \in \mathcal{C}^0(\overline{\Omega}) \colon \|w\|_{\mathcal{C}_s(\Omega)} := \left\| \frac{w}{\delta^s} \right\|_{L^\infty(\Omega)} < \infty \right\}, \tag{2.7}$$

where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. Then we have the following.

Lemma 2.3. *Assume $0 < \lambda_1 < \lambda_0 < \lambda_2 < \Lambda$. Let u_{λ_1}, u_{λ_0} and u_{λ_2} be the corresponding minimal solutions to (P_λ), for $\lambda = \lambda_1, \lambda_0$ and λ_2 respectively. If*

$$Z = \left\{ u \in \mathcal{C}_s(\Omega) \;\middle|\; u_{\lambda_1} \leqslant u \leqslant u_{\lambda_2} \right\},$$

then there exists $\varepsilon > 0$ such that

$$\{u_{\lambda_0}\} + \varepsilon B_1 \subset Z,$$

with $B_1 = \{w \in \mathcal{C}^0(\overline{\Omega})\colon \|\frac{w}{\delta^s}\|_{L^\infty(\Omega)} < 1\}$.

Proof. Let u be an arbitrary solution of (P_λ) for $0 < \lambda < \Lambda$. Then, by Hopf's Lemma (see [15, Proposition 2.7] and [35, Lemma 3.2]) there exists a positive constant c such that

$$u(x) \geqslant c\delta(x)^s, \quad x \in \Omega. \tag{2.8}$$

On the other hand by [35, Proposition 1.1] we get that there exists a positive constant C such that

$$u(x) \leqslant C\delta(x)^s, \quad x \in \Omega. \tag{2.9}$$

Thus, by (2.8) and (2.9) we finish the proof. □

Using this previous result we now obtain a local minimum of the functional $\mathcal{J}_{s,\lambda}$ in the $\mathcal{C}_s(\Omega)$-topology. This is the first step in order to get a local minimum in $X_0^s(\Omega)$. That is,

Lemma 2.4. *For all $\lambda \in (0, \Lambda)$ the minimal solution u_λ is a local minimum of the functional $\mathcal{J}_{s,\lambda}$ in the $\mathcal{C}_s$-topology.*

Proof. The proof follows in a similar way as in [4] (see also Lemma 3.3 of [18]). In our case we have to consider the nonlocal operator $(-\Delta)^s$ instead of $(-\Delta)$ and the space $\mathcal{C}_s(\Omega)$ instead of $\mathcal{C}_0^1(\Omega)$. We omit the details. □

To prove that we already have a minimum in the space $X_0^s(\Omega)$ we show that the result obtained by Brezis and Nirenberg in [13] is also valid in our context.

Proposition 2.5. *Let $z_0 \in X_0^s(\Omega)$ be a local minimum of $\mathcal{J}_{s,\lambda}$ in $\mathcal{C}_s(\Omega)$; by this we mean that there exists $r_1 > 0$ such that*

$$\mathcal{J}_{s,\lambda}(z_0) \leqslant \mathcal{J}_{s,\lambda}(z_0 + z), \quad \forall z \in \mathcal{C}_s(\Omega) \text{ with } \|z\|_{\mathcal{C}_s(\Omega)} \leqslant r_1. \tag{2.10}$$

Then, z_0 is also a local minimum of $\mathcal{J}_{s,\lambda}$ in $X_0^s(\Omega)$, that is, there exists $r_2 > 0$ so that

$$\mathcal{J}_{s,\lambda}(z_0) \leqslant \mathcal{J}_{s,\lambda}(z_0 + z), \quad \forall z \in X_0^s(\Omega) \text{ with } \|z\|_{X_0^s(\Omega)} \leqslant r_2.$$

Proof. We follow the ideas given in [18, Theorem 5.1]. Let z_0 be as in (2.10) and set, for $\varepsilon > 0$,

$$B_\varepsilon(z_0) = \{z \in X_0^s(\Omega): \|z - z_0\|_{X_0^s(\Omega)} \leqslant \varepsilon\}.$$

Now, we argue by contradiction and we suppose that for every $\varepsilon > 0$ we have

$$\min_{v \in B_\varepsilon(z_0)} \mathcal{J}_{s,\lambda}(v) < \mathcal{J}_{s,\lambda}(z_0). \tag{2.11}$$

We pick $v_\varepsilon \in B_\varepsilon(z_0)$ such that $\min_{v \in B_\varepsilon(z_0)} \mathcal{J}_{s,\lambda}(v) = \mathcal{J}_{s,\lambda}(v_\varepsilon)$. The existence of v_ε comes from a standard argument of weak lower semi-continuity. We want to prove that

$$v_\varepsilon \to z_0 \quad \text{in } \mathcal{C}_s(\Omega) \text{ as } \varepsilon \searrow 0, \tag{2.12}$$

because this would imply that there are $z \in \mathcal{C}_s(\Omega)$, arbitrarily close to z_0 in the metric of $\mathcal{C}_s(\Omega)$ (in fact, $z = v_\varepsilon$ for some ε), such that

$$\mathcal{J}_{s,\lambda}(z) < \mathcal{J}_{s,\lambda}(z_0).$$

This contradicts our hypothesis (2.10).

Let $0 < \varepsilon \ll 1$. Note that the Euler–Lagrange equation satisfied by v_ε involves a Lagrange multiplier ξ_ε such that

$$\langle \mathcal{J}'_{s,\lambda}(v_\varepsilon), \varphi \rangle = \xi_\varepsilon \langle v_\varepsilon, \varphi \rangle_{X_0^s(\Omega)}, \quad \forall \varphi \in X_0^s(\Omega). \tag{2.13}$$

As a consequence, since v_ε is a minimum of $\mathcal{J}_{s,\lambda}$ in $B_\varepsilon(z_0)$, we have

$$\xi_\varepsilon = \frac{\langle \mathcal{J}'_{s,\lambda}(v_\varepsilon), v_\varepsilon \rangle}{\|v_\varepsilon\|^2_{X_0^s(\Omega)}} \leqslant 0. \tag{2.14}$$

By (2.13) we easily get that v_ε satisfies

$$\begin{cases} (-\Delta)^s v_\varepsilon = \dfrac{1}{1 - \xi_\varepsilon} f_\lambda(v_\varepsilon) =: f_\lambda^\varepsilon(v_\varepsilon) & \text{in } \Omega, \\ v_\varepsilon = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $f_\lambda(t) := \lambda(t_+)^q + (t_+)^{2_s^*-1}$.

Since $v_\varepsilon > 0$ and

$$\|v_\varepsilon\|_{X_0^s(\Omega)} \leqslant C,$$

by Proposition 2.2 there exists a constant $C_1 > 0$ independent of ε such that $\|v_\varepsilon\|_{L^\infty(\Omega)} \leqslant C_1$. Moreover, by (2.14), it follows that $\|f_\lambda^\varepsilon(v_\varepsilon)\|_{L^\infty(\Omega)} \leqslant C$. Therefore, by [35, Proposition 1.1] (see also [44, Proposition 5]), we get that $\|v_\varepsilon\|_{\mathcal{C}^{0,s}(\overline{\Omega})} \leqslant C_2$, for some C_2 independent of ε. Here $\mathcal{C}^{0,s}$ denotes the space of Hölder continuous functions with exponent s.

Thus, by the Ascoli–Arzelá Theorem there exists a subsequence, still denoted by v_ε, such that $v_\varepsilon \to z_0$ uniformly as $\varepsilon \searrow 0$. Moreover, by [35, Theorem 1.2], we obtain that for a suitable positive constant C

$$\left\| \frac{v_\varepsilon - z_0}{\delta^s} \right\|_{L^\infty(\Omega)} \leqslant C \sup_\Omega \left| f_\lambda^\varepsilon(v_\varepsilon) - f_\lambda(z_0) \right|.$$

Since the latter tends to zero as $\varepsilon \searrow 0$, (2.12) is proved. □

Lemma 2.4 and Proposition 2.5 provide us with the existence of a positive local minimum in $X_0^s(\Omega)$ of $\mathcal{J}_{s,\lambda}$ that will be denoted by u_0. We now make a translation as in [4] in order to simplify the calculations.

For $0 < \lambda < \Lambda$, we consider the functions

$$g_\lambda(x,t) = \begin{cases} \lambda(u_0+t)^q - \lambda u_0^q + (u_0+t)^{2_s^*-1} - u_0^{2_s^*-1}, & \text{if } t \geqslant 0, \\ 0, & \text{if } t < 0, \end{cases} \tag{2.15}$$

and

$$G_\lambda(x,\xi) = G_\lambda(\xi) = \int_0^\xi g_\lambda(x,t)\,dt. \tag{2.16}$$

The associated energy functional $\widetilde{\mathcal{J}}_{s,\lambda} : X_0^s(\Omega) \to \mathbb{R}$ is given by

$$\widetilde{\mathcal{J}}_{s,\lambda}(u) = \frac{1}{2}\|u\|^2_{X_0^s(\Omega)} - \int_\Omega G_\lambda(x,u)\,dx. \tag{2.17}$$

Since $u \in X_0^s(\Omega)$, $\widetilde{\mathcal{J}}_{s,\lambda}$ is well defined. We define the translate problem

$$(\widetilde{P}_\lambda) = \begin{cases} (-\Delta)^s u = g_\lambda(x,u) & \text{in } \Omega \subset \mathbb{R}^n, \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega. \end{cases}$$

We know that if $\tilde{u} \not\equiv 0$ is a critical point of $\widetilde{\mathcal{J}}_{s,\lambda}$ then it is a solution of $(\widetilde{P}_\lambda)$ and, by the Maximum Principle [45, Proposition 2.2.8], this implies that $\tilde{u} > 0$. Therefore $u = u_0 + \tilde{u} > 0$ will be a second solution of (P_λ^+) and consequently a second one of (P_λ). Hence, in order to prove statement (4) of Theorem 1.1, it is enough to study the existence of a non-trivial critical point for $\widetilde{\mathcal{J}}_{s,\lambda}$.

First we have

Lemma 2.6. *$u = 0$ is a local minimum of $\widetilde{\mathcal{J}}_{s,\lambda}$ in $X_0^s(\Omega)$.*

Proof. The proof follows along the lines of [4, Lemma 4.2], see also [7, Lemma 3.4], so we omit the details. □

2.1. The Palais–Smale condition for $\widetilde{\mathcal{J}}_{s,\lambda}$

In this subsection assuming that we have a unique critical point, we prove that the functional $\widetilde{\mathcal{J}}_{s,\lambda}$ satisfies a local Palais–Smale condition (see Lemma 2.10). The main tool for proving this fact is an extension of the concentration–compactness principle by Lions in [32,33] for nonlocal fractional operators, given in [34, Theorem 1.5]. We will also need some technical results related to the behavior of the fractional Laplacian of a product. We start with the following.

Lemma 2.7. *Let ϕ be a regular function that satisfies*

$$|\phi(x)| \leqslant \frac{\widetilde{C}}{1+|x|^{n+s}}, \quad x \in \mathbb{R}^n \tag{2.18}$$

and

$$|\nabla\phi(x)| \leqslant \frac{\widetilde{C}}{1+|x|^{n+s+1}}, \quad x \in \mathbb{R}^n, \tag{2.19}$$

for some $\widetilde{C} > 0$. Let $B : X_0^{s/2}(\Omega) \times X_0^{s/2}(\Omega) \to \mathbb{R}$ be the bilinear form defined by

$$B(f,g)(x) := 2\int_{\mathbb{R}^n} \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{n+s}}\,dy. \tag{2.20}$$

Then, for every $s \in (0,1)$, there exist positive constants C_1 and C_2 such that for $x \in \mathbb{R}^n$ one has

$$|(-\Delta)^{s/2}\phi(x)| \leqslant \frac{C_1}{1+|x|^{n+s}},$$

and

$$\left|B(\phi,\phi)(x)\right| \leqslant \frac{C_2}{1+|x|^{n+s}}.$$

Proof. Let

$$I(x) := \int_{\mathbb{R}^n} \frac{|\phi(x)-\phi(y)|}{|x-y|^{n+s}}\,dy.$$

For any $x \in \mathbb{R}^n$, it is clear that

$$\left|(-\Delta)^{s/2}\phi(x)\right| \leqslant 2I(x).$$

Also, since $|\phi(x)| \leqslant \widetilde{C}$, we have

$$\left|B(\phi,\phi)(x)\right| \leqslant 2\widetilde{C}I(x).$$

Hence, it suffices to prove that

$$I(x) \leqslant \frac{C}{1+|x|^{n+s}}, \quad \forall x \in \mathbb{R}^n, \tag{2.21}$$

for a suitable positive constant C.

Since ϕ is a regular function, for $|x|<1$ we obtain that

$$\begin{aligned} I(x) &\leqslant \|\nabla\phi\|_{L^\infty(\mathbb{R}^n)} \int_{|y|<2} \frac{dy}{|x-y|^{n+s-1}} + C\int_{|y|\geqslant 2} \frac{dy}{|y|^{n+s}} \\ &\leqslant C \leqslant \frac{C}{1+|x|^{n+s}}. \end{aligned} \tag{2.22}$$

Let now $|x| \geqslant 1$. Then

$$I(x) := I_{A_1}(x) + I_{A_2}(x) + I_{A_3}(x), \tag{2.23}$$

where

$$I_{A_i}(x) := \int_{A_i} \frac{|\phi(x)-\phi(y)|}{|x-y|^{n+s}}\,dy, \quad i=1,2,3,$$

with

$$A_1 := \left\{y\colon |x-y| \leqslant \frac{|x|}{2}\right\}, \qquad A_2 := \left\{y\colon |x-y| > \frac{|x|}{2},\ |y| \leqslant 2|x|\right\}$$

and

$$A_3 := \left\{y\colon |x-y| > \frac{|x|}{2},\ |y| > 2|x|\right\}.$$

Therefore, since for $|x| \geqslant 1$ and $y \in A_1$, $|\phi(x)-\phi(y)| \leqslant |\nabla\phi(\xi)||x-y|$ with $\frac{|x|}{2} \leqslant |\xi| \leqslant \frac{3}{2}|x|$, by (2.19), we obtain that

$$I_{A_1}(x) \leqslant \frac{C}{|x|^{n+s+1}} \int_{A_1} \frac{dy}{|x-y|^{n+s-1}} \leqslant C|x|^{-(n+2s)}. \tag{2.24}$$

Using now that, for any $x, y \in \mathbb{R}^n$ we have the inequality

$$\left|\phi(x)\right| + \left|\phi(y)\right| \leqslant \frac{C}{1+\min\{|x|^{n+s}, |y|^{n+s}\}},$$

we get

$$I_{A_2}(x) \leqslant \frac{C}{|x|^{n+s}} \int\limits_{A_2} \frac{dy}{(1+|y|^{n+s})} \leqslant C|x|^{-(n+s)}, \tag{2.25}$$

and

$$I_{A_3}(x) \leqslant \frac{C}{|x|^{n+s}} \int\limits_{A_3} \frac{dy}{|y|^{n+s}} \leqslant C|x|^{-(n+2s)}. \tag{2.26}$$

Note that the last estimate follows from the fact that $(x, y) \in A_3$ implies $|x-y| \geqslant |y|/2$. Then, by (2.23)–(2.26), we get that

$$I(x) \leqslant C|x|^{-(n+s)} \leqslant \frac{C}{1+|x|^{n+s}}, \quad |x| \geqslant 1. \tag{2.27}$$

Hence, by (2.22) and (2.27), we conclude (2.21). □

To establish the next auxiliary results we consider a radial, nonincreasing cut-off function $\phi \in C_0^\infty(\mathbb{R}^n)$ and

$$\phi_\varepsilon(x) := \phi(x/\varepsilon). \tag{2.28}$$

Now we get the following.

Lemma 2.8. *Let $\{z_m\}$ be a uniformly bounded sequence in $X_0^s(\Omega)$ and ϕ_ε the function defined in (2.28). Then,*

$$\lim_{\varepsilon\to 0}\lim_{m\to\infty}\left|\int\limits_{\mathbb{R}^n} z_m(x)(-\Delta)^{s/2}\phi_\varepsilon(x)(-\Delta)^{s/2}z_m(x)\,dx\right| = 0. \tag{2.29}$$

Proof. First of all note that, as a consequence of the fact that $\{z_m\}$ is uniformly bounded in the reflexive space $X_0^s(\Omega)$, say by M, we get that there exists $z \in X_0^s(\Omega)$, such that, up to a subsequence,

$$\begin{aligned} &z_m \rightharpoonup z \quad \text{weakly in } X_0^s(\Omega),\\ &z_m \to z \quad \text{strongly in } L^r(\Omega),\ 1 \leqslant r < 2_s^*,\\ &z_m \to z \quad \text{a.e. in } \Omega. \end{aligned} \tag{2.30}$$

Also it is clear that

$$\left|(-\Delta)^{s/2}\phi_\varepsilon(x)\right| = \varepsilon^{-s}\left|\left((-\Delta)^{s/2}\phi\right)\left(\frac{x}{\varepsilon}\right)\right| \leqslant C\varepsilon^{-s}. \tag{2.31}$$

Therefore defining

$$I_1 := \left|\int\limits_{\mathbb{R}^n} z_m(x)(-\Delta)^{s/2}\phi_\varepsilon(x)(-\Delta)^{s/2}z_m(x)\,dx\right|,$$

from (2.31) and the fact that $\|z_m\|_{X_0^s(\Omega)} < M$, we get

$$\begin{aligned} I_1 &\leqslant \left\|(-\Delta)^{s/2}z_m\right\|_{L^2(\mathbb{R}^n)}\left\|z_m(-\Delta)^{s/2}\phi_\varepsilon\right\|_{L^2(\Omega)}\\ &\leqslant M\left\|(z_m-z)(-\Delta)^{s/2}\phi_\varepsilon\right\|_{L^2(\Omega)} + M\left\|z(-\Delta)^{s/2}\phi_\varepsilon\right\|_{L^2(\Omega)}\\ &\leqslant C\varepsilon^{-s}\|z_m-z\|_{L^2(\Omega)} + M\left\|z(-\Delta)^{s/2}\phi_\varepsilon\right\|_{L^2(\Omega)}. \end{aligned} \tag{2.32}$$

Since $\|z\|_{X_0^s(\Omega)} \leqslant M$ then $\|z\|_{L^{2_s^*}(\Omega)} \leqslant C$, that is $z^2 \in L^{\frac{n}{n-2s}}(\Omega)$. Hence, for every $\rho > 0$ there exists $\eta \in C_0^\infty(\Omega)$ such that

$$\left\|z^2-\eta\right\|_{L^{\frac{n}{n-2s}}(\Omega)} \leqslant \rho. \tag{2.33}$$

Then, by (2.31), (2.33) and Hölder's inequality with $p = n/n-2s$ we obtain that

$$\begin{aligned}\big\|z(-\Delta)^{s/2}\phi_\varepsilon\big\|^2_{L^2(\Omega)} &\leqslant \int_{\mathbb{R}^n}\big|z^2(x)-\eta(x)\big|\big|(-\Delta)^{s/2}\phi_\varepsilon(x)\big|^2\,dx+\int_{\mathbb{R}^n}\big|\eta(x)\big|\big|(-\Delta)^{s/2}\phi_\varepsilon(x)\big|^2\,dx\\
&\leqslant \big\|z^2-\eta\big\|_{L^{\frac{n}{n-2s}}(\Omega)}\big\|(-\Delta)^{s/2}\phi_\varepsilon\big\|^2_{L^{\frac{n}{s}}(\mathbb{R}^n)}+\|\eta\|_{L^\infty(\Omega)}\big\|(-\Delta)^{s/2}\phi_\varepsilon\big\|^2_{L^2(\mathbb{R}^n)}\\
&\leqslant \rho\varepsilon^{-2s}\bigg(\int_{\mathbb{R}^n}\bigg|\big((-\Delta)^{s/2}\phi\big)\bigg(\frac{x}{\varepsilon}\bigg)\bigg|^{\frac{n}{s}}dx\bigg)^{\frac{2s}{n}}+C\varepsilon^{-2s}\int_{\mathbb{R}^n}\bigg|\big((-\Delta)^{s/2}\phi\big)\bigg(\frac{x}{\varepsilon}\bigg)\bigg|^2dx\\
&\leqslant \rho\bigg(\int_{\mathbb{R}^n}\big|(-\Delta)^{s/2}\phi(z)\big|^{\frac{n}{s}}dz\bigg)^{\frac{2s}{n}}+C\varepsilon^{n-2s}\int_{\mathbb{R}^n}\big|(-\Delta)^{s/2}\phi(z)\big|^2dz\\
&\leqslant C\rho+C\varepsilon^{n-2s}.\end{aligned}\tag{2.34}$$

Hence, using (2.30), from (2.32), (2.34) and the fact that $n>2s$, it follows that

$$\lim_{\varepsilon\to0}\lim_{m\to\infty}I_1\leqslant\lim_{\varepsilon\to0}C\big(\rho+\varepsilon^{n-2s}\big)^{\frac12}=C\rho^{\frac12}.$$

Since $\rho>0$ is fixed but arbitrarily small, we conclude the proof of Lemma 2.8. □

Also, we have the following.

Lemma 2.9. *With the same assumptions of Lemma 2.8 we have that*

$$\lim_{\varepsilon\to0}\lim_{m\to\infty}\bigg|\int_{\mathbb{R}^n}(-\Delta)^{s/2}z_m(x)B(z_m,\phi_\varepsilon)(x)\,dx\bigg|=0,\tag{2.35}$$

where B is defined in (2.20).

Proof. Let

$$I_2:=\bigg|\int_{\mathbb{R}^n}(-\Delta)^{s/2}z_m(x)B(z_m,\phi_\varepsilon)(x)\,dx\bigg|.$$

Since $\|z_m\|_{X_0^s(\Omega)}\leqslant M$, then

$$\begin{aligned}I_2&\leqslant M\big\|B(z_m,\phi_\varepsilon)\big\|_{L^2(\mathbb{R}^n)}\\
&\leqslant M\big\|B(z_m-z,\phi_\varepsilon)\big\|_{L^2(\mathbb{R}^n)}+M\big\|B(z,\phi_\varepsilon)\big\|_{L^2(\mathbb{R}^n)},\end{aligned}\tag{2.36}$$

where z is, as in Lemma 2.8, the weak limit of the sequence $\{z_m\}$ in $X_0^s(\Omega)$. We estimate each of the summands in the previous inequality. Let

$$\psi(x):=\frac{1}{1+|x|^{n+s}}\quad\text{and}\quad\psi_\varepsilon(x):=\psi\bigg(\frac{x}{\varepsilon}\bigg).\tag{2.37}$$

By Lemma 2.7 applied to ϕ, we note that

$$B(\phi_\varepsilon,\phi_\varepsilon)(x)=\varepsilon^{-s}B(\phi,\phi)\bigg(\frac{x}{\varepsilon}\bigg)\leqslant C\varepsilon^{-s}\psi\bigg(\frac{x}{\varepsilon}\bigg)=C\frac{\varepsilon^{-s}}{1+|\frac{x}{\varepsilon}|^{n+s}}\leqslant C\varepsilon^{-s}.\tag{2.38}$$

Therefore, by Cauchy–Schwarz inequality and (2.38), it follows that

$$\begin{aligned}\big\|B(z_m-z,\phi_\varepsilon)\big\|^2_{L^2(\mathbb{R}^n)}&\leqslant\int_{\mathbb{R}^n}B(z_m-z,z_m-z)(x)B(\phi_\varepsilon,\phi_\varepsilon)(x)\,dx\\
&\leqslant C\varepsilon^{-s}\int_{\mathbb{R}^n}B(z_m-z,z_m-z)(x)\,dx\end{aligned}\tag{2.39}$$

$$
\begin{aligned}
&= C\varepsilon^{-s}\|z_m - z\|^2_{X_0^{\frac{s}{2}}(\Omega)} \\
&= C\varepsilon^{-s}\int_{\mathbb{R}^n} (z_m - z)(x)(-\Delta)^{s/2}(z_m - z)(x)\,dx \\
&\leqslant C\varepsilon^{-s}\|z_m - z\|_{L^2(\Omega)}\big\|(-\Delta)^{s/2}(z_m - z)\big\|_{L^2(\mathbb{R}^n)} \\
&\leqslant C\varepsilon^{-s}\|z_m - z\|_{L^2(\Omega)}.
\end{aligned} \tag{2.40}
$$

On the other hand, for a suitable function f, we have that

$$
\begin{aligned}
\int_{\mathbb{R}^n} z^2(x)(-\Delta)^{s/2} f(x)\,dx &= \int_{\mathbb{R}^n} f(x)(-\Delta)^{s/2} z^2(x)\,dx \\
&= \int_{\mathbb{R}^n} f(x)\big(2z(x)(-\Delta)^{s/2} z(x) - B(z,z)(x)\big)\,dx.
\end{aligned} \tag{2.41}
$$

Then, arguing as in (2.39) and applying (2.41) with $f := \psi_\varepsilon(x)$, from (2.38) we get that

$$
\begin{aligned}
\big\|B(z,\phi_\varepsilon)\big\|^2_{L^2(\mathbb{R}^n)} &\leqslant \int_{\mathbb{R}^n} B(z,z)(x)B(\phi_\varepsilon,\phi_\varepsilon)(x)\,dx \\
&\leqslant C\varepsilon^{-s}\int_{\mathbb{R}^n} B(z,z)(x)\psi_\varepsilon(x)\,dx \\
&\leqslant C\varepsilon^{-s}\int_{\mathbb{R}^n} \big(-z^2(x)(-\Delta)^{s/2}\psi_\varepsilon(x) + 2z(x)\psi_\varepsilon(x)(-\Delta)^{s/2}z(x)\big)\,dx \\
&:= I_{2,1} + I_{2,2}.
\end{aligned} \tag{2.42}
$$

We estimate now $I_{2,1}$ and $I_{2,2}$ separately. Let $\rho > 0$. By Lemma 2.7 applied to ψ and (2.31), it follows that

$$
\begin{aligned}
|I_{2,1}| &\leqslant C\varepsilon^{-2s}\int_{\mathbb{R}^n} z^2(x)\left|\big((-\Delta)^{s/2}\psi\big)\left(\frac{x}{\varepsilon}\right)\right|dx \\
&\leqslant C\varepsilon^{-2s}\int_{\mathbb{R}^n} z^2(x)\psi\left(\frac{x}{\varepsilon}\right)dx \\
&\leqslant C\varepsilon^{-2s}\int_{\mathbb{R}^n} \big(z^2 - \eta\big)(x)\psi_\varepsilon(x)\,dx + \varepsilon^{-2s}\int_{\mathbb{R}^n} \eta(x)\psi_\varepsilon(x)\,dx,
\end{aligned} \tag{2.43}
$$

where $\eta \in C_0^\infty(\Omega)$ is the function that satisfies (2.33). Then from (2.43) we obtain

$$
\begin{aligned}
|I_{2,1}| &\leqslant C\rho\varepsilon^{-2s}\|\psi_\varepsilon\|_{L^{\frac{n}{2s}}(\mathbb{R}^n)} + C\varepsilon^{-2s}\|\eta\|_{L^\infty(\mathbb{R}^n)}\|\psi_\varepsilon\|_{L^1(\mathbb{R}^n)} \\
&\leqslant C\rho\|\psi\|_{L^{\frac{n}{2s}}(\mathbb{R}^n)} + C\varepsilon^{n-2s}\|\eta\|_{L^\infty(\mathbb{R}^n)}\|\psi\|_{L^1(\mathbb{R}^n)}.
\end{aligned} \tag{2.44}
$$

On the other hand,

$$
|I_{2,2}| \leqslant C\varepsilon^{-s}\big\|(-\Delta)^{s/2}z\big\|_{L^2(\mathbb{R}^n)}\|z\psi_\varepsilon\|_{L^2(\Omega)} \leqslant C\varepsilon^{-s}\|z\psi_\varepsilon\|_{L^2(\Omega)}. \tag{2.45}
$$

Therefore, by (2.33), we get

$$
\begin{aligned}
|I_{2,2}|^2 &\leqslant C\varepsilon^{-2s}\left(\int_{\Omega} \big|\big(z^2 - \eta\big)(x)\big|\big|\psi_\varepsilon(x)\big|^2\,dx + \int_{\mathbb{R}^n} \eta\big|\psi_\varepsilon(x)\big|^2\,dx\right) \\
&\leqslant C\varepsilon^{-2s}\big(\rho\|\psi_\varepsilon\|^2_{L^{\frac{n}{s}}(\mathbb{R}^n)} + \|\eta\|_{L^\infty(\mathbb{R}^n)}\|\psi_\varepsilon\|^2_{L^2(\mathbb{R}^n)}\big) \\
&\leqslant C\rho\|\psi\|^2_{L^{\frac{n}{s}}(\mathbb{R}^n)} + C\varepsilon^{n-2s}\|\eta\|_{L^\infty(\mathbb{R}^n)}\|\psi\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{2.46}
$$

Then, by (2.44) and (2.46), it follows from (2.42) that

$$\|B(z,\phi_\varepsilon)\|^2_{L^2(\mathbb{R}^n)} \leqslant C\big(\rho+\rho^{\frac{1}{2}}\big)+C\big(\varepsilon^{n-2s}+\varepsilon^{\frac{n-2s}{2}}\big). \tag{2.47}$$

Hence, from (2.30), (2.40) and (2.47), since $n>2s$, we obtain

$$\lim_{\varepsilon\to 0}\lim_{m\to\infty}\big(\|B(z_m-z,\phi_\varepsilon)\|^2_{L^2(\mathbb{R}^n)}+\|B(z,\phi_\varepsilon)\|^2_{L^2(\mathbb{R}^n)}\big)\leqslant \lim_{\varepsilon\to 0} C\big(\rho^{\frac{1}{2}}+\varepsilon^{\frac{n-2s}{2}}\big)=C\rho^{\frac{1}{2}}.$$

Thus, since ρ is an arbitrary positive value,

$$\lim_{\varepsilon\to 0}\lim_{m\to\infty}\big(\|B(z_m-z,\phi_\varepsilon)\|^2_{L^2(\mathbb{R}^n)}+\|B(z,\phi_\varepsilon)\|^2_{L^2(\mathbb{R}^n)}\big)=0. \tag{2.48}$$

Finally, by (2.36) and (2.48), we conclude that

$$\lim_{\varepsilon\to 0}\lim_{m\to\infty}|I_2|=0. \qquad \square$$

Now we can prove the principal result of this subsection:

Lemma 2.10. *If $u=0$ is the only critical point of $\widetilde{\mathcal{J}}_{s,\lambda}$ in $X_0^s(\Omega)$, then $\widetilde{\mathcal{J}}_{s,\lambda}$ satisfies the $(\mathrm{PS})_{c_1}$ condition, provided $c_1<c^*$, where c^* is defined as*

$$c^*=\frac{s}{n}S(n,s)^{\frac{n}{2s}}. \tag{2.49}$$

Here $S(n,s)$ denotes the Sobolev constant defined in (1.7).

Proof. Let $\{u_m\}$ be a Palais–Smale sequence for $\widetilde{\mathcal{J}}_{s,\lambda}$ verifying

$$\widetilde{\mathcal{J}}_{s,\lambda}(u_m)\to c_1<c^* \quad\text{and}\quad \widetilde{\mathcal{J}}'_{s,\lambda}(u_m)\to 0. \tag{2.50}$$

Then, since there exists $M>0$ such that $\|u_m\|_{X_0^s(\Omega)}\leqslant M$, and, by hypothesis $u=0$ is the unique critical point of $\widetilde{\mathcal{J}}_{s,\lambda}$, it follows that

$$\begin{aligned} &u_m \rightharpoonup 0 \quad \text{weakly in } X_0^s(\Omega),\\ &u_m\to 0 \quad \text{strongly in } L^r(\Omega),\ 1\leqslant r<2_s^*,\\ &u_m\to 0 \quad \text{a.e. in } \Omega. \end{aligned} \tag{2.51}$$

Also, since u_0 is a critical point of $\mathcal{J}_{s,\lambda}$, we have that

$$\begin{aligned} \mathcal{J}_{s,\lambda}(z_m)&=\widetilde{\mathcal{J}}_{s,\lambda}(u_m)+\mathcal{J}_{s,\lambda}(u_0)+\lambda\int_\Omega\left(\frac{(u_0+(u_m)_+)^{q+1}}{q+1}+u_0^q\big(u_m-(u_m)_+\big)-\frac{(u_0+u_m)_+^{q+1}}{q+1}\right)dx\\ &\quad+\int_\Omega\left(\frac{(u_0+(u_m)_+)^{2_s^*}}{2_s^*}+u_0^{2_s^*-1}\big(u_m-(u_m)_+\big)-\frac{(u_0+u_m)_+^{2_s^*}}{2_s^*}\right)dx\\ &\leqslant \widetilde{\mathcal{J}}_{s,\lambda}(u_m)+\mathcal{J}_{s,\lambda}(u_0), \end{aligned} \tag{2.52}$$

where

$$z_m=u_m+u_0. \tag{2.53}$$

Moreover, for every $\varphi\in X_0^s(\Omega)$,

$$\begin{aligned} \langle\mathcal{J}'_{s,\lambda}(z_m),\varphi\rangle&=\langle\widetilde{\mathcal{J}}'_{s,\lambda}(u_m),\varphi\rangle+\int_\Omega\big(\lambda\big(u_0+(u_m)_+\big)^q+\big(u_0+(u_m)_+\big)^{2_s^*-1}\big)\varphi\,dx\\ &\quad-\int_\Omega\big(\lambda(u_0+u_m)_+^q+(u_0+u_m)_+^{2_s^*-1}\big)\varphi\,dx. \end{aligned} \tag{2.54}$$

Then, by (2.50), (2.51) and (2.54) we obtain that

$$\mathcal{J}'_{s,\lambda}(z_m) \to 0. \tag{2.55}$$

From (2.52) and (2.55) we get that the sequence $\{z_m\}$ is uniformly bounded in $X_0^s(\Omega)$. As a consequence, and the fact that $u = 0$ is the unique critical point of $\mathcal{J}_{s,\lambda}$, up to a subsequence, we get that

$$\begin{aligned} &z_m \rightharpoonup u_0 \quad \text{weakly in } X_0^s(\Omega),\\ &z_m \to u_0 \quad \text{strongly in } L^r(\Omega),\ 1 \leqslant r < 2_s^*,\\ &z_m \to u_0 \quad \text{a.e. in } \Omega. \end{aligned} \tag{2.56}$$

Following [29] it is easy to prove that $X_0^s(\Omega)$ could also be defined as the closure of $C_0^\infty(\Omega)$ with respect to the $X_0^s(\Omega)$-norm (see also [24]). Hence, applying [34, Theorem 1.5] we have that there exist an index set $I \subseteq \mathbb{N}$, a sequence of points $\{x_k\}_{k\in I} \subset \Omega$, and two sequences of nonnegative real numbers $\{\mu_k\}_{k\in I}$, $\{\nu_k\}_{k\in I}$, such that

$$\left|(-\Delta)^{s/2}(z_m)_+\right|^2 \to \mu \geqslant \left|(-\Delta)^{s/2}u_0\right|^2 + \sum_{k\in I}\mu_k\delta_{x_k}. \tag{2.57}$$

Moreover,

$$\left|(z_m)_+\right|^{2_s^*} \to \nu = |u_0|^{2_s^*} + \sum_{k\in I}\nu_k\delta_{x_k}, \tag{2.58}$$

in the sense of measures, with

$$\nu_k \leqslant S(n,s)^{-\frac{2_s^*}{2}}\mu_k^{\frac{2_s^*}{2}} \quad \text{for every } k \in I. \tag{2.59}$$

Here δ_{x_k} denotes the Dirac Delta at x_k, while $S(n,s)$ is the constant given in (1.7). We fix $k_0 \in I$, and we consider $\phi \in C_0^\infty(\mathbb{R}^n)$ a nonincreasing cut-off function satisfying

$$\phi = 1 \quad \text{in } B_1(x_{k_0}) \quad \text{and} \quad \phi = 0 \quad \text{in } B_2(x_{k_0})^c. \tag{2.60}$$

Set now

$$\phi_\varepsilon(x) = \phi(x/\varepsilon), \quad x \in \mathbb{R}^n. \tag{2.61}$$

Taking the derivative of the identity given in (1.6), see also [39, Lemma 16], for any $u, \varphi \in X_0^s(\Omega)$ we obtain that

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy = \int_{\mathbb{R}^n}\varphi(x)(-\Delta)^s u(x)\,dx. \tag{2.62}$$

Then, using $\phi_\varepsilon(z_m)_+$ as a test function in (2.55), by (2.62), and the fact that

$$\int_{\mathbb{R}^n}\big(\phi_\varepsilon(z_m)_+\big)(-\Delta)^s z_m\,dx \geqslant \int_{\mathbb{R}^n}\big(\phi_\varepsilon(z_m)_+\big)(-\Delta)^s(z_m)_+\,dx,$$

we have that

$$0 \geqslant \lim_{m\to\infty}\left(\int_{\mathbb{R}^n}\big(\phi_\varepsilon(z_m)_+\big)(-\Delta)^s(z_m)_+\,dx - \left(\lambda\int_{B_{2\varepsilon}(x_{k_0})}\big((z_m)_+\big)^{q+1}\phi_\varepsilon\,dx + \int_{B_{2\varepsilon}(x_{k_0})}\big((z_m)_+\big)^{2_s^*}\phi_\varepsilon\,dx\right)\right).$$

Hence,

$$\begin{aligned} &\lim_{m\to\infty}\Bigg(\int_{\mathbb{R}^n}(z_m)_+(x)(-\Delta)^{s/2}(z_m)_+(x)(-\Delta)^{s/2}\phi_\varepsilon(x)\,dx\\ &\qquad - 2\int_{\mathbb{R}^n}(-\Delta)^{s/2}(z_m)_+(x)\int_{\mathbb{R}^n}\frac{(\phi_\varepsilon(x)-\phi_\varepsilon(y))((z_m)_+(x)-(z_m)_+(y))}{|x-y|^{n+s}}\,dx\,dy\Bigg)\\ &\quad \leqslant \lim_{m\to\infty}\left(\lambda\int_{B_{2\varepsilon}(x_{k_0})}\big((z_m)_+\big)^{q+1}\phi_\varepsilon\,dx + \int_{B_{2\varepsilon}(x_{k_0})}\big((z_m)_+\big)^{2_s^*}\phi_\varepsilon\,dx - \int_{B_{2\varepsilon}(x_{k_0})}\big((-\Delta)^{s/2}(z_m)_+\big)^2\phi_\varepsilon\,dx\right). \end{aligned}$$

Therefore, by (2.56), (2.57) and (2.58) we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{m\to\infty}\Bigg(\int_{\mathbb{R}^n}(z_m)_+(x)(-\Delta)^{s/2}(z_m)_+(x)(-\Delta)^{s/2}\phi_\varepsilon(x)\,dx\\
&\qquad -2\int_{\mathbb{R}^n}(-\Delta)^{s/2}(z_m)_+(x)\int_{\mathbb{R}^n}\frac{(\phi_\varepsilon(x)-\phi_\varepsilon(y))((z_m)_+(x)-(z_m)_+(y))}{|x-y|^{n+s}}\,dx\,dy\Bigg)\\
&\quad\leqslant \lim_{\varepsilon\to 0}\Bigg(\lambda\int_{B_{2\varepsilon}(x_{k_0})}u_0^{q+1}\phi_\varepsilon\,dx+\int_{B_{2\varepsilon}(x_{k_0})}\phi_\varepsilon\,d\nu-\int_{B_{2\varepsilon}(x_{k_0})}\phi_\varepsilon\,d\mu\Bigg).
\end{aligned}\tag{2.63}$$

Since ϕ is a regular function with compact support it is clear that it satisfies the hypothesis of Lemma 2.7. Therefore, by Lemma 2.8 and Lemma 2.9 applied to the sequence $\{(z_m)_+\}$, it follows that the left hand side of (2.63) goes to zero. That is, we obtain that

$$\lim_{\varepsilon\to 0}\Bigg(\int_{B_{2\varepsilon}(x_{k_0})}\phi_\varepsilon\,d\nu+\lambda\int_{B_{2\varepsilon}(x_{k_0})}u_0^{q+1}\phi_\varepsilon\,dx-\int_{B_{2\varepsilon}(x_{k_0})}\phi_\varepsilon\,d\mu\Bigg)=\nu_{k_0}-\mu_{k_0}\geqslant 0.$$

Thus, from (2.59), we have that either $\nu_{k_0}=0$ or

$$\nu_{k_0}\geqslant S(n,s)^{\frac{n}{2s}}.\tag{2.64}$$

Suppose now that $\nu_{k_0}\neq 0$. By (2.52), (2.55) and (2.64) we obtain that

$$\begin{aligned}
c_1+\mathcal{J}_{s,\lambda}(u_0)&\geqslant \lim_{m\to\infty}\Big(\mathcal{J}_{s,\lambda}(z_m)-\frac{1}{2}\big\langle\mathcal{J}'_{s,\lambda}(z_m),z_m\big\rangle\Big)\\
&\geqslant \lambda\Big(\frac{1}{2}-\frac{1}{q+1}\Big)\int_\Omega u_0^{q+1}\,dx+\frac{s}{n}\int_\Omega u_0^{2_s^*}\,dx+\frac{s}{n}\nu_{k_0}\\
&\geqslant \mathcal{J}_{s,\lambda}(u_0)+\frac{s}{n}S(n,s)^{\frac{n}{2s}}\\
&=\mathcal{J}_{s,\lambda}(u_0)+c^*.
\end{aligned}$$

This is a contradiction with (2.50). Since k_0 was arbitrary, we deduce that $\nu_k=0$ for all $k\in I$. As a consequence, we obtain that $(u_m)_+\to 0$ in $L^{2_s^*}(\Omega)$. Note that, since u_m is equal to zero outside Ω, indeed we have that $(u_m)_+\to 0$ in $L^{2_s^*}(\mathbb{R}^n)$. This implies convergence of $\lambda((u_m)_+)^q+((u_m)_+)^{2_s^*-1}$ in $L^{\frac{2n}{n+2s}}(\mathbb{R}^n)$. Finally, using the continuity of the inverse operator $(-\Delta)^{-s}$, we obtain strong convergence of u_m in $X_0^s(\Omega)$. $\square$

2.2. *Proof of statement (4) of Theorem 1.1*

In Lemma 2.10 we have proved that if $u\equiv 0$ is the only critical point of the functional $\widetilde{\mathcal{J}}_{s,\lambda}$, then $\widetilde{\mathcal{J}}_{s,\lambda}$ verifies the Palais–Smale condition at any level $c_1<c^*$, where c^* is the critical level defined in (2.49).

Now, we want to show that we can obtain a local $(\mathrm{PS})_c$-sequence for $\widetilde{\mathcal{J}}_{s,\lambda}$ under the critical level c^*. For this, assume, without loss of generality, that $0\in\Omega$. By [21] (see also [11,31]) the infimum in (1.7) is attained at the function

$$u_\varepsilon(x)=\frac{\varepsilon^{(n-2s)/2}}{(|x|^2+\varepsilon^2)^{(n-2s)/2}},\qquad \varepsilon>0,\tag{2.65}$$

that is

$$\big\|(-\Delta)^{s/2}u_\varepsilon\big\|^2_{L^2(\mathbb{R}^n)}=\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy=S(n,s)\|u_\varepsilon\|^2_{L^{2_s^*}(\mathbb{R}^n)}.\tag{2.66}$$

Also, let us introduce a cut-off function $\phi_0\in C^\infty(\mathbb{R})$, nonincreasing and satisfying

$$\phi_0(t)=\begin{cases}1 & \text{if } 0\leqslant t\leqslant\frac{1}{2},\\ 0 & \text{if } t\geqslant 1.\end{cases}$$

For a fixed $r>0$ small enough such that $\overline{B}_r\subset\Omega$, set $\phi(x)=\phi_r(x)=\phi_0(\frac{|x|}{r})$ and consider the family of nonnegative truncated functions

$$\eta_\varepsilon(x)=\frac{\phi u_\varepsilon(x)}{\|\phi u_\varepsilon\|_{L^{2_s^*}(\Omega)}}\in X_0^s(\Omega). \tag{2.67}$$

Then, we have the following.

Lemma 2.11. *There exists $\varepsilon>0$ small enough such that*

$$\sup_{t\geqslant 0}\widetilde{\mathcal{J}}_{s,\lambda}(t\eta_\varepsilon)<c^*. \tag{2.68}$$

Proof. We follow the proof of [4, Lemma 4.4] (see also [18, Lemma 3.9]).

Assume $n\geqslant 4s$. Since

$$(a+b)^p\geqslant a^p+b^p+\mu a^{p-1}b,\quad \text{for some } \mu>0 \text{ and every } a,b\geqslant 0,\ p>1, \tag{2.69}$$

then the function G_λ defined in (2.16), satisfies

$$G_\lambda(u)\geqslant\frac{1}{2_s^*}(u_+)^{2_s^*}+\frac{\mu}{2}(u_+)^2u_0^{2_s^*-2}. \tag{2.70}$$

Therefore,

$$\widetilde{\mathcal{J}}_{s,\lambda}(t\eta_\varepsilon)\leqslant\frac{t^2}{2}\|\eta_\varepsilon\|^2_{X_0^s(\Omega)}-\frac{t^{2_s^*}}{2_s^*}-\frac{t^2}{2}\mu\int_\Omega u_0^{2_s^*-2}\eta_\varepsilon^2\,dx.$$

Since $u_0\geqslant a_0>0$ in $\operatorname{supp}(\eta_\varepsilon)$ we get, for any $t\geqslant 0$ and $\varepsilon>0$ small enough,

$$\widetilde{\mathcal{J}}_{s,\lambda}(t\eta_\varepsilon)\leqslant\frac{t^2}{2}\|\eta_\varepsilon\|^2_{X_0^s(\Omega)}-\frac{t^{2_s^*}}{2_s^*}-\frac{t^2}{2}\widetilde{\mu}\|\eta_\varepsilon\|^2_{L^2(\Omega)}. \tag{2.71}$$

Moreover, since $\|u_\varepsilon\|_{L^{2_s^*}(\mathbb{R}^n)}$ is independent of ε, by [41, Proposition 21] we have

$$\begin{aligned}\|\eta_\varepsilon\|^2_{X_0^s(\Omega)}&=\frac{\|\phi u_\varepsilon\|^2_{X_0^s(\Omega)}}{\|\phi u_\varepsilon\|^2_{L^{2_s^*}(\Omega)}}\\&\leqslant\frac{\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}{\|\phi u_\varepsilon\|^2_{L^{2_s^*}(\Omega)}}+O\big(\varepsilon^{n-2s}\big)\\&=S(n,s)+O\big(\varepsilon^{n-2s}\big).\end{aligned} \tag{2.72}$$

Furthermore, by [18, Lemma 3.8] (see also [41, Proposition 22]) it follows that

$$\|\eta_\varepsilon\|^2_{L^2(\Omega)}\geqslant\begin{cases}C\varepsilon^{2s} & \text{if } n>4s,\\ C\varepsilon^{2s}\log(1/\varepsilon) & \text{if } n=4s.\end{cases} \tag{2.73}$$

Therefore, from (2.71), (2.72) and (2.73), we get

$$\widetilde{\mathcal{J}}_{s,\lambda}(t\eta_\varepsilon)\leqslant\frac{t^2}{2}\big(S(n,s)+C\varepsilon^{n-2s}\big)-\frac{t^{2_s^*}}{2_s^*}-\frac{t^2}{2}\widetilde{C}\varepsilon^{2s}:=g(t), \tag{2.74}$$

with $\widetilde{C}>0$. Since $\lim_{t\to\infty}g(t)=-\infty$, then $\sup_{t\geqslant 0}g(t)$ is attained at some $t_{\varepsilon,\lambda}:=t_\varepsilon\geqslant 0$. If $t_\varepsilon=0$, then

$$\sup_{t\geqslant 0}\widetilde{\mathcal{J}}_{s,\lambda}(t\eta_\varepsilon)\leqslant\sup_{t\geqslant 0}g(t)=g(0)=0$$

for any $0<\lambda<\Lambda$ and (2.68) is trivially verified. Now, we suppose that $t_\varepsilon>0$. Differentiating the above function $g(t)$, we obtain that

$$0 = g'(t_\varepsilon) = t_\varepsilon\big(S(n,s) + C\varepsilon^{n-2s}\big) - t_\varepsilon^{2_s^*-1} - t_\varepsilon \widetilde{C}\varepsilon^{2s}, \tag{2.75}$$

which implies

$$t_\varepsilon \leqslant \big(S(n,s) + C\varepsilon^{n-2s}\big)^{\frac{1}{2_s^*-2}}. \tag{2.76}$$

Also we have, for $\varepsilon > 0$ small enough,

$$t_\varepsilon \geqslant c > 0. \tag{2.77}$$

Indeed from (2.75) we get

$$t_\varepsilon^{2_s^*-2} = S(n,s) + C\varepsilon^{n-2s} - \widetilde{C}\varepsilon^{2s} \geqslant c > 0,$$

provided ε is small enough. Moreover, the function

$$t \mapsto \frac{t^2}{2}\big(S(n,s) + C\varepsilon^{n-2s}\big) - \frac{t^{2_s^*}}{2_s^*}$$

is increasing on $[0, (S(n,s) + C\varepsilon^{n-2s})^{\frac{1}{2_s^*-2}}]$. Whence, by (2.76) and (2.77), we obtain

$$\sup_{t\geqslant 0} g(t) = g(t_\varepsilon) \leqslant \frac{s}{n}\big(S(n,s) + C\varepsilon^{n-2s}\big)^{\frac{n}{2s}} - \overline{C}\varepsilon^{2s},$$

for some $\overline{C} > 0$. Therefore, by (2.74), for $n > 4s$, we get that

$$\sup_{t\geqslant 0} \widetilde{\mathcal{J}}_{s,\lambda}(t\eta_\varepsilon) \leqslant g(t_\varepsilon) \leqslant \frac{s}{n}S(n,s)^{\frac{n}{2s}} + C\varepsilon^{n-2s} - \overline{C}\varepsilon^{2s} < \frac{s}{n}S(n,s)^{\frac{n}{2s}} = c^*. \tag{2.78}$$

If $n = 4s$ the same conclusion follows.

The last case $2s < n < 4s$ follows by using the estimate (2.69) which gives

$$G_\lambda(u) \geqslant \frac{1}{2_s^*}(u_+)^{2_s^*} + \frac{\mu}{2_s^*-1}u_0(u_+)^{2_s^*-1}. \tag{2.79}$$

Then, (2.79) jointly with the inequality (3.28) of [18], instead of (2.73), and arguing in a similar way as above, finish the proof. □

To complete the existence of the second solution, that is statement (4) in Theorem 1.1, in view of the previous results, we look for a path with energy below the critical level c^*. Let us fix $\lambda \in (0, \Lambda)$. We consider $M_\varepsilon > 0$ large enough so that $\widetilde{\mathcal{J}}_{s,\lambda}(M_\varepsilon\eta_\varepsilon) < \widetilde{\mathcal{J}}_{s,\lambda}(0)$. Note that such M_ε exists, since $\lim_{t\to\infty}\widetilde{\mathcal{J}}_{s,\lambda}(t\eta_\varepsilon) = -\infty$. Also, by Lemma 2.6, there exists $\alpha > 0$ such that if $\|u\|_{X_0^s(\Omega)} = \alpha$, then $\widetilde{\mathcal{J}}_{s,\lambda}(u) \geqslant \widetilde{\mathcal{J}}_{s,\lambda}(0)$. We define

$$\Gamma_\varepsilon = \big\{\gamma \in \mathcal{C}\big([0,1], X_0^s(\Omega)\big):\ \gamma(0) = 0,\ \gamma(1) = M_\varepsilon\eta_\varepsilon\big\},$$

and the minimax value

$$c_\varepsilon = \inf_{\gamma\in\Gamma_\varepsilon}\sup_{0\leqslant t\leqslant 1}\widetilde{\mathcal{J}}_{s,\lambda}\big(\gamma(t)\big). \tag{2.80}$$

By the arguments above, $c_\varepsilon \geqslant \widetilde{\mathcal{J}}_{s,\lambda}(0)$. Also, by Lemma 2.11, for $\varepsilon \ll 1$ we obtain that

$$c_\varepsilon \leqslant \sup_{0\leqslant t\leqslant 1}\widetilde{\mathcal{J}}_{s,\lambda}(tM_\varepsilon\eta_\varepsilon) = \sup_{t\geqslant 0}\widetilde{\mathcal{J}}_{s,\lambda}(t\eta_\varepsilon) < c^*.$$

Therefore, by Lemma 2.10 and the MPT [5] if $c_\varepsilon > \widetilde{\mathcal{J}}_{s,\lambda}(0)$, or the corresponding refinement given in [26] if the minimax level is equal to $\widetilde{\mathcal{J}}_{s,\lambda}(0)$, we obtain the existence of a non-trivial solution of $(\widetilde{P}_\lambda)$, provided $u \equiv 0$ is its unique solution. Of course this is a contradiction. Thus, $\widetilde{\mathcal{J}}_{s,\lambda}$ admits a critical point $\tilde{u}$ different from the trivial function. As a consequence, $u = u_0 + \tilde{u}$ is a solution, different of u_0, of problem (P_λ). This concludes the proof of Theorem 1.1.

3. The critical and convex case $q > 1$

In this section we discuss the problem (P_λ) in the convex setting $q > 1$. Here, we argue essentially as in [37,38, 41–43], where the authors studied the linear case $q = 1$ using again variational techniques. With respect to the case $q = 1$, there are some extra difficulties to prove the $(\mathrm{PS})_c$ condition and to obtain the estimates of the Mountain Pass critical value. First of all it is easy to check the good geometry of the functional. That is we have the following.

Proposition 3.1. *Assume $\lambda > 0$ and $1 < q < 2_s^* - 1$. Then, there exist $\alpha > 0$ and $\beta > 0$ such that*

a) *for any $u \in X_0^s(\Omega)$ with $\|u\|_{X_0^s(\Omega)} = \alpha$ one has that $\mathcal{J}_{s,\lambda}(u) \geqslant \beta$,*
b) *there exists a positive function $e \in X_0^s(\Omega)$ so that $\|e\|_{X_0^s(\Omega)} > \alpha$ and $\mathcal{J}_{s,\lambda}(e) < \beta$.*

Proof. a) By the Sobolev embedding theorem, since $q + 1 < 2_s^*$, it can be easily seen that

$$\mathcal{J}_{s,\lambda}(u) \geqslant g\big(\|u\|_{X_0^s(\Omega)}\big),$$

where $g(t) = C_1 t^2 - \lambda C_2 t^{q+1} - C_3 t^{2_s^*}$, for some positive constants C_1, C_2 and C_3. Therefore, there will exist $\alpha > 0$ such that $\beta := g(\alpha) > 0$. Then, $\mathcal{J}_{s,\lambda}(u) \geqslant \beta$ for $u \in X_0^s(\Omega)$ with $\|u\|_{X_0^s(\Omega)} = \alpha$.

b) Fix a positive function $u_0 \in X_0^s(\Omega)$ such that $\|u_0\|_{X_0^s(\Omega)} = 1$ and consider $t > 0$. Since $2_s^* > 2$, it follows that

$$\lim_{t\to\infty} \mathcal{J}_{s,\lambda}(tu_0) = -\infty.$$

Then, there exists t_0 large enough, such that for $e := t_0 u_0$, we get that $\|e\|_{X_0^s(\Omega)} > \alpha$ and $\mathcal{J}_{s,\lambda}(e) < \beta$. $\square$

By a similar argument, it follows that

$$\lim_{t\to 0^+} \mathcal{J}_{s,\lambda}(tu_0) = 0. \tag{3.1}$$

Let us check now that we have the compactness properties of $\mathcal{J}_{s,\lambda}$.

3.1. The Palais–Smale condition for $\mathcal{J}_{s,\lambda}$

In this subsection we show that the functional $\mathcal{J}_{s,\lambda}$ satisfies the Palais–Smale condition in a suitable energy range involving the best fractional critical Sobolev constant $S(n,s)$ given in (1.7), that is we prove the following.

Proposition 3.2. *Let $\lambda > 0$ and $1 < q < 2_s^* - 1$. Then, the functional $\mathcal{J}_{s,\lambda}$ satisfies the $(\mathrm{PS})_{c_2}$ condition provided $c_2 < c^*$, where c^* is given in (2.49).*

Proof. Let $\{u_m\}$ be a $(\mathrm{PS})_{c_2}$-sequence for $\mathcal{J}_{s,\lambda}$ in $X_0^s(\Omega)$, that is

$$\mathcal{J}_{s,\lambda}(u_m) \to c_2 \tag{3.2}$$

and

$$\mathcal{J}'_{s,\lambda}(u_m) \to 0. \tag{3.3}$$

First of all we get that $\{u_m\}$ is bounded in $X_0^s(\Omega)$. Indeed by (3.2) and (3.3), there exists $M > 0$ such that

$$\|u_m\|_{X_0^s(\Omega)} \leqslant M. \tag{3.4}$$

In order to prove our result we proceed by steps.

Claim 1. *There exists $u_\infty \in X_0^s(\Omega)$ such that $\langle \mathcal{J}'_{s,\lambda}(u_\infty), \varphi\rangle = 0$ for any $\varphi \in X_0^s(\Omega)$.*

Proof. By (3.4) and the fact that $X_0^s(\Omega)$ is a reflexive space, up to a subsequence, still denoted by u_m, there exists $u_\infty \in X_0^s(\Omega)$ such that $u_m \rightharpoonup u_\infty$ weakly in $X_0^s(\Omega)$, that is

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u_m(x)-u_m(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy$$
$$\to \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u_\infty(x)-u_\infty(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy \quad \text{for any } \varphi\in X_0^s(\Omega). \tag{3.5}$$

Moreover, we have

$$u_m \rightharpoonup u_\infty \quad \text{weakly in } L^{2_s^*}(\Omega), \tag{3.6}$$
$$u_m \to u_\infty \quad \text{strongly in } L^r(\Omega),\ 1\leqslant r<2_s^*, \tag{3.7}$$
$$u_m \to u_\infty \quad \text{a.e. in } \Omega. \tag{3.8}$$

Hence, taking the limit when $m\to\infty$, by (3.3), (3.5)–(3.8) we conclude

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u_\infty(x)-u_\infty(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy = \lambda\int_\Omega \big((u_\infty)_+\big)^q\varphi\,dx + \int_\Omega \big((u_\infty)_+\big)^{2_s^*-1}\varphi\,dx,$$

for any $\varphi\in X_0^s(\Omega)$. □

Claim 2. *The following equality holds:*

$$\mathcal{J}_{s,\lambda}(u_m) = \mathcal{J}_{s,\lambda}(u_\infty) + \frac{1}{2}\|u_m-u_\infty\|^2_{X_0^s(\Omega)} - \frac{1}{2_s^*}\int_\Omega \big|(u_m)_+(x)-(u_\infty)_+(x)\big|^{2_s^*}dx + o(1).$$

Proof. First of all, we observe that by (3.4) and the Sobolev embedding theorem, the sequence u_m is bounded in $X_0^s(\Omega)$ and in $L^{2_s^*}(\Omega)$. Hence, since (3.7) and (3.8) hold true, by the Brezis–Lieb Lemma (see [12, Theorem 1]), we get

$$\|u_m\|^2_{X_0^s(\Omega)} = \|u_m-u_\infty\|^2_{X_0^s(\Omega)} + \|u_\infty\|^2_{X_0^s(\Omega)} + o(1), \tag{3.9}$$
$$\int_\Omega \big|(u_m)_+\big|^{2_s^*}dx = \int_\Omega \big|(u_m)_+(x)-(u_\infty)_+(x)\big|^{2_s^*}dx + \int_\Omega \big|(u_\infty)_+\big|^{2_s^*}dx + o(1) \tag{3.10}$$

and

$$\big\|(u_m)_+\big\|_{L^{q+1}(\Omega)} \to \big\|(u_\infty)_+\big\|_{L^{q+1}(\Omega)}. \tag{3.11}$$

Therefore, by (3.9), (3.10) and (3.11) we deduce that

$$\begin{aligned}\mathcal{J}_{s,\lambda}(u_m) &= \frac{1}{2}\|u_m-u_\infty\|^2_{X_0^s(\Omega)} + \frac{1}{2}\|u_\infty\|^2_{X_0^s(\Omega)} - \frac{\lambda}{q+1}\int_\Omega \big((u_\infty)_+\big)^{q+1}dx \\ &\quad - \frac{1}{2_s^*}\int_\Omega \big|(u_m)_+(x)-(u_\infty)_+(x)\big|^{2_s^*}dx - \frac{1}{2_s^*}\int_\Omega \big((u_\infty)_+\big)^{2_s^*}dx + o(1) \\ &= \mathcal{J}_{s,\lambda}(u_\infty) + \frac{1}{2}\|u_m-u_\infty\|^2_{X_0^s(\Omega)} - \frac{1}{2_s^*}\int_\Omega \big|(u_m)_+(x)-(u_\infty)_+(x)\big|^{2_s^*}dx + o(1),\end{aligned}$$

which gives the desired assertion. □

Claim 3. *The following estimate holds:*

$$\begin{aligned}\|u_m-u_\infty\|^2_{X_0^s(\Omega)} &= \int_\Omega \big|(u_m)_+(x)-(u_\infty)_+(x)\big|^{2_s^*}dx + o(1) \\ &\leqslant \int_\Omega \big|(u_m)(x)-(u_\infty)(x)\big|^{2_s^*}dx + o(1).\end{aligned}$$

Proof. Note that, as a consequence of (3.6) and (3.10), we get

$$\begin{aligned}
&\int_\Omega \big(((u_m)_+)^{2_s^*-1}(x) - ((u_\infty)_+)^{2_s^*-1}(x)\big)\big(u_m(x)-u_\infty(x)\big)\,dx \\
&\quad = \int_\Omega ((u_m)_+)^{2_s^*}\,dx - \int_\Omega ((u_\infty)_+)^{2_s^*-1}u_m\,dx - \int_\Omega ((u_m)_+)^{2_s^*-1}u_\infty\,dx + \int_\Omega ((u_\infty)_+)^{2_s^*}\,dx \\
&\quad = \int_\Omega ((u_m)_+)^{2_s^*}\,dx - \int_\Omega ((u_\infty)_+)^{2_s^*}\,dx + o(1) \\
&\quad = \int_\Omega \big|(u_m)_+(x) - (u_\infty)_+(x)\big|^{2_s^*}\,dx + o(1).
\end{aligned} \tag{3.12}$$

Furthermore, (3.7) and (3.11) give

$$\begin{aligned}
&\int_\Omega \big(((u_m)_+)^{q}(x) - ((u_\infty)_+)^{q}(x)\big)\big(u_m(x)-u_\infty(x)\big)\,dx \\
&\quad = \int_\Omega ((u_m)_+)^{q+1}\,dx - \int_\Omega ((u_\infty)_+)^{q}u_m\,dx - \int_\Omega ((u_m)_+)^{q}u_\infty\,dx + \int_\Omega ((u_\infty)_+)^{q+1}\,dx \\
&\quad = o(1).
\end{aligned} \tag{3.13}$$

Then, by (3.3), Claim 1, (3.12) and (3.13), we conclude that

$$\begin{aligned}
o(1) &= \langle \mathcal{J}'_{s,\lambda}(u_m), u_m - u_\infty\rangle \\
&= \langle \mathcal{J}'_{s,\lambda}(u_m) - \mathcal{J}'_{s,\lambda}(u_\infty), u_m - u_\infty\rangle \\
&= \|u_m - u_\infty\|^2_{X_0^s(\Omega)} - \lambda \int_\Omega \big(((u_m)_+)^{q}(x) - ((u_\infty)_+)^{q}(x)\big)\big(u_m(x)-u_\infty(x)\big)\,dx \\
&\quad - \int_\Omega \big(((u_m)_+)^{2_s^*-1}(x) - ((u_\infty)_+)^{2_s^*-1}(x)\big)\big(u_m(x)-u_\infty(x)\big)\,dx \\
&= \|u_m - u_\infty\|^2_{X_0^s(\Omega)} - \int_\Omega \big|\big((u_m)_+\big)(x) - \big((u_\infty)_+\big)(x)\big|^{2_s^*}\,dx + o(1). \qquad \square
\end{aligned}$$

Now, we can finish the proof of Proposition 3.2.

By Claim 3 we know that

$$\frac{1}{2}\|u_m - u_\infty\|^2_{X_0^s(\Omega)} - \frac{1}{2_s^*}\int_\Omega \big|\big((u_m)_+\big)(x) - \big((u_\infty)_+\big)(x)\big|^{2_s^*}\,dx = \frac{s}{n}\|u_m - u_\infty\|^2_{X_0^s(\Omega)} + o(1). \tag{3.14}$$

Then, by (3.2), Claim 2 and (3.14) we obtain

$$\mathcal{J}_{s,\lambda}(u_\infty) + \frac{s}{n}\|u_m - u_\infty\|^2_{X_0^s(\Omega)} = \mathcal{J}_{s,\lambda}(u_m) + o(1) = c_2 + o(1). \tag{3.15}$$

On the other hand, by (3.4), up to a subsequence, we can assume that

$$\|u_m - u_\infty\|^2_{X_0^s(\Omega)} \to L \geqslant 0, \tag{3.16}$$

and then, as a consequence of Claim 3,

$$\int_\Omega |u_m(x) - u_\infty(x)|^{2_s^*}\,dx \to \tilde{L} \geqslant L.$$

By the definition of $S(n,s)$ given in (1.7), we have

$$L \geqslant S(n,s)\tilde{L}^{2/2_s^*} \geqslant S(n,s)L^{2/2_s^*},$$

so that

$$L = 0 \quad \text{or} \quad L \geqslant S(n,s)^{\frac{n}{2s}}.$$

We now prove that the case $L \geqslant S(n,s)^{\frac{n}{2s}}$ cannot occur. Indeed taking $\varphi = u_\infty \in X_0^s(\Omega)$ as a test function in Claim 1, we have that

$$\|u_\infty\|^2_{X_0^s(\Omega)} = \lambda \int_\Omega \big((u_\infty)_+\big)^{q+1}\,dx + \int_\Omega \big((u_\infty)_+\big)^{2_s^*}\,dx.$$

That is,

$$\mathcal{J}_{s,\lambda}(u_\infty) = \lambda\left(\frac{1}{2} - \frac{1}{q+1}\right)\big\|\big((u_\infty)_+\big)\big\|^{q+1}_{L^{q+1}(\Omega)} + \frac{s}{n}\big\|\big((u_\infty)_+\big)\big\|^{2_s^*}_{L^{2_s^*}(\Omega)} \geqslant 0, \tag{3.17}$$

thanks to the positivity of λ and the fact that $q > 1$. Therefore, if $L \geqslant S(n,s)^{\frac{n}{2s}}$, then, by (3.15), (3.16) and (3.17) we get

$$c_2 = \mathcal{J}_{s,\lambda}(u_\infty) + \frac{s}{n}L \geqslant \frac{s}{n}L \geqslant \frac{s}{n}S(n,s)^{\frac{n}{2s}},$$

which contradicts the fact that $c_2 < c^*$, for the c^* given in (2.49). Thus $L = 0$ and so, by (3.16), we obtain that

$$\|u_m - u_\infty\|_{X_0^s(\Omega)} \to 0. \qquad \square$$

Remark 3.3. Note that the proof of Proposition 3.2 could be also obtained by the concentration–compactness theory of Subsection 2.1. This simply means that the arguments performed in the last part of the proof of Lemma 2.10 can be adapted to the convex setting.

3.2. Proof of Theorem 1.2

By Proposition 3.1 and (3.1) we get that $\mathcal{J}_{s,\lambda}$ satisfies the geometric features required by the MPT (see [5]). Moreover, by Proposition 3.2 the functional $\mathcal{J}_{s,\lambda}$ verifies the Palais–Smale condition at any level c, provided $c < c^*$.

Now, as in the concave case, we find a path with energy below the critical level c^*. That is, we have the following.

Lemma 3.4. *Let $\lambda > 0$, c^* be as in (2.49) and η_ε be the nonnegative function defined in (2.67). Then, there exists $\varepsilon > 0$ small enough such that*

$$\sup_{t \geqslant 0} \mathcal{J}_{s,\lambda}(t\eta_\varepsilon) < c^*,$$

provided

- $n > \frac{2s(q+3)}{q+1}$ *and* $\lambda > 0$ *or*
- $n \leqslant \frac{2s(q+3)}{q+1}$ *and* $\lambda > \lambda_s$*, for a suitable* $\lambda_s > 0$.

Proof. Let $n > \frac{2s(q+3)}{q+1}$.

First of all note that since $q > 1$ we get that $n > 2s(1 + \frac{1}{q})$. Therefore, denoting by $N := -(n - (n-2s)(q+1)) > 0$, for some positive constants c and $\widetilde{C}$, it follows that

$$\begin{aligned}\int_{\mathbb{R}^n} \eta_\varepsilon(x)^{q+1}\,dx &= C\int_{|x|<r} u_\varepsilon^{q+1}\,dx\\ &= C\varepsilon^{(\frac{n-2s}{2})(q+1)}\int_{|x|<r}\frac{dx}{(|x|^2+\varepsilon^2)^{\frac{(n-2s)(q+1)}{2}}}\\ &= C\varepsilon^{-(\frac{n-2s}{2})(q+1)}\int_0^r\frac{\rho^{n-1}}{(1+(\frac{\rho}{\varepsilon})^2)^{\frac{(n-2s)(q+1)}{2}}}\,d\rho\\ &= C\varepsilon^{n-(\frac{n-2s}{2})(q+1)}\int_0^{r/\varepsilon}\frac{t^{n-1}}{(1+t^2)^{\frac{(n-2s)(q+1)}{2}}}\,dt\\ &\geqslant C\varepsilon^{n-(\frac{n-2s}{2})(q+1)}\int_1^{r/\varepsilon} t^{n-1-(n-2s)(q+1)}\,dt\\ &= \frac{C\varepsilon^{n-(\frac{n-2s}{2})(q+1)}}{N}\left(1-\left(\frac{\varepsilon}{r}\right)^N\right)\\ &\geqslant \widetilde{C}\varepsilon^{n-(\frac{n-2s}{2})(q+1)}. \end{aligned}\tag{3.18}$$

Then, by (2.72) and (3.18) for any $t \geqslant 0$ and $\varepsilon > 0$ small enough we obtain

$$\begin{aligned}\mathcal{J}_{s,\lambda}(t\eta_\varepsilon) &= \frac{t^2}{2}\|\eta_\varepsilon\|^2_{X_0^s(\Omega)} - \frac{t^{2_s^*}}{2_s^*} - \lambda\frac{t^{q+1}}{q+1}\int_\Omega \eta_\varepsilon^{q+1}\,dx\\ &\leqslant \frac{t^2}{2}\big(S(n,s)+C\varepsilon^{n-2s}\big) - \frac{t^{2_s^*}}{2_s^*} - \widetilde{C}\lambda\frac{t^{q+1}}{q+1}\varepsilon^{n-(\frac{n-2s}{2})(q+1)} =: g(t).\end{aligned}\tag{3.19}$$

It is clear that

$$\lim_{t\to\infty} g(t) = -\infty,$$

therefore $\sup_{t\geqslant 0} g(t)$ is attained at some $t_{\varepsilon,\lambda} := t_\varepsilon \geqslant 0$. As we comment in the proof of Lemma 2.11 we could suppose $t_\varepsilon > 0$. Differentiating $g(t)$ and equaling to zero, we obtain that

$$0 = g'(t_\varepsilon) = t_\varepsilon\big(S(n,s)+C\varepsilon^{n-2s}\big) - t_\varepsilon^{2_s^*-1} - \widetilde{C}\lambda t_\varepsilon^{q}\varepsilon^{n-(\frac{n-2s}{2})(q+1)}.\tag{3.20}$$

Hence,

$$t_\varepsilon < \big(S(n,s)+C\varepsilon^{n-2s}\big)^{\frac{1}{2_s^*-2}}.$$

Moreover, we have that for $\varepsilon > 0$ small enough

$$t_\varepsilon \geqslant c > 0.\tag{3.21}$$

Indeed, from (3.20) it follows that

$$t_\varepsilon^{2_s^*-2} + \widetilde{C}\lambda t_\varepsilon^{q-1}\varepsilon^{n-(\frac{n-2s}{2})(q+1)} = S(n,s)+C\varepsilon^{n-2s} \geqslant c > 0, \quad \text{for } \varepsilon > 0 \text{ small enough.}$$

Also, since the function

$$t \mapsto \frac{t^2}{2}\big(S(n,s)+C\varepsilon^{n-2s}\big) - \frac{t^{2_s^*}}{2_s^*}$$

is increasing on $[0, (S(n,s)+C\varepsilon^{n-2s})^{\frac{1}{2_s^*-2}}]$, by (3.19) and (3.21) we obtain

$$\sup_{t\geqslant 0} g(t) = g(t_\varepsilon) \leqslant \frac{s}{n}\big(S(n,s) + C\varepsilon^{n-2s}\big)^{\frac{n}{2s}} - \overline{C}\varepsilon^{n-(\frac{n-2s}{2})(q+1)}$$
$$\leqslant \frac{s}{n} S(n,s)^{\frac{n}{2s}} + C\varepsilon^{n-2s} - \overline{C}\varepsilon^{n-(\frac{n-2s}{2})(q+1)}, \tag{3.22}$$

for some $\overline{C} > 0$. Finally, from our hypothesis on n, we conclude from (3.22) that

$$\sup_{t\geqslant 0} \mathcal{J}_{s,\lambda}(t\eta_\varepsilon) \leqslant g(t_\varepsilon) < \frac{s}{n} S(n,s)^{\frac{n}{2s}}.$$

Consider now the case $n \leqslant \frac{2s(q+3)}{q+1}$. Arguing exactly as in the previous case, we get that

$$\big(S(n,s) + C\varepsilon^{n-2s}\big) = t_{\varepsilon,\lambda}^{2_s^*-2} + \widetilde{C}\lambda t_{\varepsilon,\lambda}^{q-1}\varepsilon^{n-(\frac{n-2s}{2})(q+1)}, \tag{3.23}$$

with $t_{\varepsilon,\lambda} > 0$ the point where $\sup_{t\geqslant 0} g(t)$ is attained. We claim that

$$t_{\varepsilon,\lambda} \to 0 \quad \text{when } \lambda \to +\infty. \tag{3.24}$$

To see this, assume that $\overline{\lim}_{\lambda\to\infty} t_{\varepsilon,\lambda} = \ell > 0$. Then, passing to the limit when $\lambda \to +\infty$ in (3.23) we would get $(S(n,s) + C\varepsilon^{n-2s}) = +\infty$, which is a contradiction and (3.24) follows. If we take now β the positive number given in Proposition 3.1, by (3.24) we obtain that

$$0 \leqslant \sup_{t\geqslant 0} \mathcal{J}_{s,\lambda}(t\eta_\varepsilon) \leqslant g(t_{\varepsilon,\lambda})$$
$$= \frac{t_{\varepsilon,\lambda}^2}{2}\big(S(n,s) + C\varepsilon^{n-2s}\big) - \frac{t_{\varepsilon,\lambda}^{2_s^*}}{2_s^*} - \widetilde{C}\lambda \frac{t_{\varepsilon,\lambda}^{q+1}}{q+1}\varepsilon^{n-(\frac{n-2s}{2})(q+1)}$$
$$\leqslant \frac{t_{\varepsilon,\lambda}^2}{2}\big(S(n,s) + C\varepsilon^{n-2s}\big) - \frac{t_{\varepsilon,\lambda}^{2_s^*}}{2_s^*} \to 0,$$

when $\lambda \to \infty$. Then,

$$\lim_{\lambda\to+\infty} \sup_{t\geqslant 0} \mathcal{J}_{s,\lambda}(t\eta_\varepsilon) = 0,$$

which easily yields the desired conclusion for the case $n \leqslant \frac{2s(q+3)}{q+1}$. □

We conclude now the proof of Theorem 1.2. In order to do so, we define

$$\Gamma_\varepsilon = \big\{\gamma \in C\big([0,1], X_0^s(\Omega)\big) \colon \gamma(0) = 0,\ \gamma(1) = M_\varepsilon\eta_\varepsilon\big\}$$

for some $M_\varepsilon > 0$ big enough such that $\mathcal{J}_{s,\lambda}(M_\varepsilon\eta_\varepsilon) < 0$. Observe that for every $\gamma \in \Gamma_\varepsilon$ the function $t \to \|\gamma(t)\|_{X_0^s(\Omega)}$ is continuous in $[0,1]$. Therefore, for the α given in Proposition 3.1, since $\|\gamma(0)\|_{X_0^s(\Omega)} = 0 < \alpha$ and $\|\gamma(1)\|_{X_0^s(\Omega)} = \|M_\varepsilon\eta_\varepsilon\|_{X_0^s(\Omega)} > \alpha$ for M_ε sufficiently large, there exists $t_0 \in (0,1)$ such that $\|\gamma(t_0)\|_{X_0^s(\Omega)} = \alpha$. As a consequence,

$$\sup_{0\leqslant t\leqslant 1} \mathcal{J}_{s,\lambda}\big(\gamma(t)\big) \geqslant \mathcal{J}_{s,\lambda}\big(\gamma(t_0)\big) \geqslant \inf_{\|v\|_{X_0^s(\Omega)}=\alpha} \mathcal{J}_{s,\lambda}(v) \geqslant \beta > 0,$$

where β is the positive value given in Proposition 3.1. Hence,

$$c_\varepsilon = \inf_{\gamma\in\Gamma_\varepsilon} \sup_{0\leqslant t\leqslant 1} \mathcal{J}_{s,\lambda}\big(\gamma(t)\big) > 0.$$

Then, by Lemma 3.4, Proposition 3.2 and the MPT given in [5] we conclude that the functional $\mathcal{J}_{s,\lambda}$ admits a critical point $u \in X_0^s(\Omega)$, provided $n > \frac{2s(q+3)}{q+1}$ and $\lambda > 0$ or $n \leqslant \frac{2s(q+3)}{q+1}$ and $\lambda > \lambda_s$, for a suitable $\lambda_s > 0$. Moreover, since $\mathcal{J}_{s,\lambda}(u) = c_\epsilon \geqslant \beta > 0$ and $\mathcal{J}_{s,\lambda}(0) = 0$, the function u is not the trivial one. This concludes the proof of Theorem 1.2.

Remark 3.5. Some of the results obtained in Section 2 and Section 3 are true for integrodifferential operators more general than the fractional Laplacian, such as, for instance, the ones considered in [40,41].

Conflict of interest statement

We, the authors, certify that there is no conflict of interest with any financial organization regarding the material discussed in the manuscript.

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