Nonlinear scalar field equations: Existence of a positive solution with infinitely many bumps ☆

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Abstract

In this paper we consider the equation

$$(E) \quad -\Delta u + a(x)u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N,$$

where $N \geqslant 2$, $p > 1$, $p < 2^* - 1 = \frac{N+2}{N-2}$, if $N \geqslant 3$. During last thirty years the question of the existence and multiplicity of solutions to (E) has been widely investigated mostly under symmetry assumptions on a. The aim of this paper is to show that, differently from those found under symmetry assumption, the solutions found in [6] admit a limit configuration and so (E) also admits a positive solution of infinite energy having infinitely many 'bumps'.
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Résumé

Dans ce papier nous considérons l'équation

$$(E) \quad -\Delta u + a(x)u = |u|^{p-1}u \quad \text{en } \mathbb{R}^N,$$

où $N \geqslant 2$, $p > 1$, $p < 2^* - 1 = \frac{N+2}{N-2}$, si $N \geqslant 3$. Pendant les trente dernières années la question de l'existence et de la multiplicité de solutions d'(E) a été largement examinée surtout conformément aux suppositions de symétrie sur a. Le but de ce papier est de montrer que, différemment de ceux trouvés conformément à la supposition de symétrie, les solutions trouvées dans [6] admettent une configuration de limite et donc (E) admet aussi une solution positive d'énergie infinie ayant une infinité de 'bumps'.
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1. Introduction

In this paper we consider the equation

$$(E) \quad -\Delta u + a(x)u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N,$$

where $N \geqslant 2$, $p > 1$, $p < 2^* - 1 = \frac{N+2}{N-2}$, if $N \geqslant 3$, and the potential $a(x)$ is a positive function that is not required to enjoy symmetry.

During the past years there has been a considerable amount of research on this kind of questions; the interest comes, essentially, from two reasons: their specific mathematical difficulties, that make them challenging to the researchers, and, moreover, the fact that equations as (E) arise naturally in several branches of mathematical physics. Indeed the solutions of (E) can be seen as stationary states (corresponding to solitary waves) in nonlinear equations of the Klein–Gordon type

$$\frac{\partial^2 \varphi}{\partial t^2} - \Delta \varphi + \big(a(x) + \omega^2\big)\varphi - |\varphi|^{p-2}\varphi = 0 \tag{1}$$

and of Schrödinger type

$$i\frac{\partial \varphi}{\partial t} - \Delta \varphi + \big(a(x) + \omega^2\big)\varphi - |\varphi|^{p-2}\varphi = 0 \tag{2}$$

(where $\varphi = \varphi(t, x)$) is a complex function defined on $\mathbb{R} \times \mathbb{R}^N$.

Let us consider, for instance, Eq. (1): it corresponds to the Lagrangian density

$$\mathcal{L}(\varphi) = -\frac{1}{2}|\varphi_t|^2 + \frac{1}{2}|\nabla \varphi|^2 + \frac{1}{2}\big(a(x) + \omega^2\big)|\varphi|^2 - \frac{1}{p}|\varphi|^p.$$

Thus, looking for a solitary wave, of the standing wave form, means searching solutions $\varphi(x, t) = e^{i\omega t}u(x)$, with $u : \mathbb{R}^N \to \mathbb{R}$, hence one is led exactly to the equation considered in (E). Analogously searching for stationary states of (2) leads again to (E).

Furthermore, we recall that, besides the above mentioned problems, equations like (E), which are also called Euclidean scalar field equations, appear in several other context of physics: nonlinear optics, laser propagations, constructive field theory, etc.

During last thirty years the question of the existence and multiplicity of solutions to (E) has been widely investigated and most results have been obtained under symmetry assumptions on a. However, some considerable progress has been performed also in the case in which $a(x)$ is not required to fulfill symmetry properties. We just mention that, formerly, the existence of a positive solution has been shown in [7,1,2], while the existence of infinitely many changing sign solutions has been proven in [5].

We refer the interested reader either to [4] or to [6] for a more detailed description of the development of the researches as well as a quite exhaustive list of references.

Very recently, an answer to the question of the existence of infinitely many positive solutions to (E) has been given in [6] where the following result has been proved:

Theorem 1. *Let assumptions*

(h_1) $a(x) \to a_\infty > 0$ *as* $|x| \to \infty$,
(h_2) $a(x) \geqslant a_0 > 0$, $\forall x \in \mathbb{R}^N$,
(h_3) $a \in L_{loc}^{N/2}(\mathbb{R}^N)$,
(h_4) $\exists \bar{\eta} \in (0, \sqrt{a_\infty})$: $\lim_{|x| \to +\infty}(a(x) - a_\infty)e^{\bar{\eta}|x|} = +\infty$

be satisfied.

Then there exists a positive constant, $\mathcal{A} = \mathcal{A}(N, \bar{\eta}, a_0, a_\infty) \in \mathbb{R}$, such that, when

$$\big|a(x) - a_\infty\big|_{N/2, loc} := \sup_{y \in \mathbb{R}^N} \big|a(x) - a_\infty\big|_{L^{N/2}(B_1(y))} < \mathcal{A},$$

equation (E) has infinitely many positive solutions belonging to $H^1(\mathbb{R}^N)$.

The solutions, whose existence is asserted in Theorem 1, are multi-bump functions. The claim is proved just showing, by purely variational methods, that, for all $k \in \mathbb{N} \setminus \{0\}$, there exists a solution of (E) which belongs to a special class consisting of functions having exactly k bumps.

The aim of this paper is to show that (E) admits also a positive solution of infinite energy having infinitely many 'bumps'.

In order to be more precise about our achievement we need to introduce some notation.

We denote by $w(x) \in H^1(\mathbb{R}^N)$ the ground state solution of the limit equation

$$(E_\infty) \quad -\Delta u + a_\infty u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N.$$

For all function $u \in H^1(\mathbb{R}^N)$ and for fixed δ, we set

$$u^\delta(x) := (u - \delta)^+(x)$$

and

$$u_\delta(x) := u(x) - u^\delta(x)$$

and we call $u^\delta(x)$ the *emerging part* of u above δ, $u_\delta(x)$ the *submerged part* of u under δ.

Fixing δ and ρ, we say that $u \in H^1(\mathbb{R}^N)$ is *emerging around the k points* $x_1, x_2, \dots, x_k$ (*in k balls of radius ρ*) if

$$u^\delta(x) = \sum_{i=1}^{k} u_i^\delta(x)$$

where for all $i \in \{1, 2, \dots, k\}$, $u_i^\delta \geqslant 0$, $u_i^\delta \not\equiv 0$, $u_i^\delta \in H_0^1(B_\rho(x_i))$, $B_\rho(x_i) \cap B_\rho(x_j) = \emptyset$ if $i \neq j$.

Now our result can be stated as follows:

Theorem 2. *Let assumptions of Theorem 1 be satisfied.*

Then there exists a solution of (E), $\bar{u} \in H_{loc}^1(\mathbb{R}^N)$, which is emerging around an unbounded sequence of points $(\bar{x}_n)_n$, $\bar{x}_n \in \mathbb{R}^N$ $(\bar{x}_n \neq \bar{x}_m$ for $m \neq n)$.

Moreover $\bar{u}$ and $(\bar{x}_n)_n$ have the following properties:

$$\lim_{n \to \infty} \min\{|\bar{x}_n - \bar{x}_m| : m \in \mathbb{N}, \, m \neq n\} = +\infty, \tag{3}$$

$$\lim_{n \to \infty} \bar{u}(x + \bar{x}_n) = w(x) \quad \text{uniformly on all compact subsets of } \mathbb{R}^N. \tag{4}$$

We remark that, while the multi-bump solutions obtained in Theorem 1 have a variational characterization, solution $\bar{u}$ is obtained as limit, as $k \to +\infty$, of a sequence $(u_k)_k$ of multi-bump solutions to (E) given by Theorem 1. The reason of the success of our procedure is strongly related to the nature of the solutions u_k. Actually, a k-bump solution is obtained first minimizing the action functional on sets of functions emerging around a fixed k-tuple of points of $\mathbb{R}^N$, then considering the maxima of the minima when the k-tuples vary in a suitable subset of $\mathbb{R}^N$. In this argument the role played by the 'slow decay' assumption on $a(x)$ is basic, because the attractive effect of $a(x)$, due to assumption (h_4), is dominating on the repulsive disposition, due to the maximization procedure, of positive masses with respect to each other and makes the bumps not to escape at infinity.

So, relation (3) is just a consequence of the above described situation and indicates the bumps of $\bar{u}$ rarefy, as the distance from the origin increases, giving rise to a quite new phenomenon. Indeed, for instance, multi-bump positive solutions, obtained under assumptions of radial symmetry on $a(x)$ in [8], clearly do not converge as the number of the bumps increases, on the contrary the solutions are built in such a way that the bumps, as their number increases, spread out, going far away from each other and far away from the origin in a uniform way.

The paper is organized as follows: in Section 2 the variational framework of the paper is introduced and some useful facts as well as some results contained in [6] are collected; Section 3 contains some preliminary estimates; Section 4 is devoted to the proof, subdivided in several steps, of Theorem 2.

2. Notation, variational framework and useful facts

Throughout the paper we make use of the following notation:

- Given a measurable subset D of $\mathbb{R}^N$, $|D|$ denotes its Lebesgue measure.
- When D is open, $H^1(D)$ denotes the closure of $\mathcal{C}^\infty(D)$ with respect to the norm

$$\|u\|_D := \left[\int_D \left(|\nabla u|^2 + u^2 \right) dx \right]^{1/2};$$

  the norm in $H^1(\mathbb{R}^N)$ is denoted by $\| \cdot \|$.
- $L^p(D)$, $1 \leqslant p \leqslant +\infty$, denotes the Lebesgue space with norm $| \cdot |_{p,D}$

$$|u|_{p,D} := \left[\int_D |u|^p \, dx \right]^{1/p};$$

  the norm in $L^p(\mathbb{R}^N)$ is denoted by $| \cdot |_p$.
- $B_r(x)$, when x belongs to a metric space X, denotes the open ball of X having radius r and centered at x.
- $L^p_{loc}(\mathbb{R}^N)$, $1 \leqslant p < +\infty$, and $H^1_{loc}(\mathbb{R}^N)$ denote the sets of functions u such that $\forall x \in \mathbb{R}^N$ there exists an open set $D \Subset \mathbb{R}^N$, so that $x \in D$ and $u_{|D} \in L^p(D)$, respectively, $u_{|D} \in H^1(D)$.
- Given any function $u : \mathbb{R}^N \to \mathbb{R}$, supp u denotes its support (defined as in [3, Sec. 4]); if u and v are real valued functions defined in $D \subseteq \mathbb{R}^N$, we denote, as usual, by $u \vee v$ and $u \wedge v$ the functions defined by $(u \vee v)(x) := \max(u(x), v(x))$ and $(u \wedge v)(x) := \min(u(x), v(x))$ for all $x \in D$.
- $C, c, \tilde{c}, \hat{c}, c_i$ denote various positive constants.

The variational functionals, related to problems (P) and (P_∞), are denoted respectively by I and I^∞ and are defined in $H^1(\mathbb{R}^N)$ as

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + a(x)u^2 \right) dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} \, dx,$$

and

$$I^\infty(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + a_\infty u^2 \right) dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} \, dx.$$

The following lemma contains some known facts about (P_∞), see f.i. [4].

Lemma 3. (P_∞) *has a positive, ground state, solution w, unique up to translation, radially symmetric, decreasing when the radial coordinate increases and such that*

$$\lim_{|x| \to +\infty} |D^j w(x)||x|^{\frac{N-1}{2}} e^{\sqrt{a_\infty}|x|} = d_j > 0, \quad d_j \in \mathbb{R}, \ j = 0, 1. \tag{5}$$

In what follows we use the notation

$$m_\infty := I^\infty(w) \quad \text{and} \quad w_y(x) := w(x - y); \tag{6}$$

moreover, *we set*

$$\alpha(x) := a(x) - a_\infty.$$

In the rest of the paper, according to [6], the real numbers $\delta > 0$ and $\rho > 0$ are fixed in such a way that

$$\delta < \min \left\{ w(0)/3, 1, (a_0/p)^{1/(p-1)}, a_0^{1/(p-1)}/2, \left[a_\infty - \bar{\eta}^2 \right]^{1/(p-1)} \right\} \tag{7}$$

and

$$w(x) < \delta, \quad \forall x \in \mathbb{R}^N \setminus B_{\rho/2}(0).$$

Considering the sets $\mathcal{K}_k$, $k \in \mathbb{N} \setminus \{0\}$ defined as

$$\mathcal{K}_k = \begin{cases} \mathbb{R}^N & \text{when } k = 1; \\ \{(x_1, \ldots, x_k) \in (\mathbb{R}^N)^k \colon |x_i - x_j| \geqslant 3\rho, \ i, j = 1, 2, \ldots, k, \ i \neq j\} & \text{when } k > 1, \end{cases} \tag{8}$$

we set, for all $(x_1, \ldots, x_k) \in \mathcal{K}_k$,

$$S_{x_1, \ldots, x_k} = \{u \in H^1(\mathbb{R}^N) \colon u \text{ emerging around } x_1, x_2, \ldots, x_k \text{ and } I'(u)[u_i^\delta] = 0, \ \beta_i(u) = 0, \ \forall i = 1, 2, \ldots, k\},$$

where

$$\beta_i(u) = \frac{1}{|u_i^\delta|_2^2} \int_{\mathbb{R}^N} (x - x_i)\big(u_i^\delta(x)\big)^2 \, dx$$

is a barycenter type map.

The arguments developed in [6] show that Theorem 1 can be completed in the following way:

Theorem 4. *Let the assumptions of Theorem* 1 *be satisfied. Then, for all $k \in \mathbb{N} \setminus \{0\}$, there exists (at least) one solution $\bar{u}_k$ of (E) which is emerging around k points $(\bar{x}_1^k, \ldots, \bar{x}_k^k) \in \mathcal{K}_k$.*

Moreover, $\bar{u}_k$ is found as a critical point of I and is characterized as

$$I(\bar{u}_k) = \min_{S_{\bar{x}_1^k, \ldots, \bar{x}_k^k}} I(u) = \max_{\mathcal{K}_k} \min_{S_{x_1, \ldots, x_k}} I(u).$$

Furthermore, setting $d(x) = \mathrm{dist}(x, [\mathrm{supp}(\bar{u}_k)^\delta \cup \mathrm{supp}\,\alpha^-])$, the relation

$$0 < (\bar{u}_k)_\delta(x) < C\delta e^{-\bar{\eta}d(x)} \tag{9}$$

holds, with $C > 0$ depending only on $\bar{\eta}$, a_∞, N.

For what follows it is useful to remark that, for all $u \in H^1(\mathbb{R}^N)$, emerging around k points $x_1, \ldots, x_k$, the functional I can be written as

$$I(u) = I(u_\delta) + J_\delta(u^\delta) = I(u_\delta) + \sum_{i=1}^k J_\delta(u_i^\delta)$$

where the functional J_δ is defined, for all v belonging to $H^1(\mathbb{R}^N)$ and having compact support, as

$$J_\delta(v) = \frac{1}{2} \int_{\mathbb{R}^N} \big(|\nabla v|^2 + a(x)v^2\big) \, dx + \int_{\mathbb{R}^N} a(x)\delta v \, dx$$
$$- \frac{1}{p+1} \int_{\mathrm{supp}\,v} \big(\delta + |v|\big)^{p+1} \, dx + \frac{1}{p+1} \delta^{p+1} |\mathrm{supp}\,v|.$$

Next lemmas describe some important features of the functionals I and J_δ, as well as the nature of the 'natural nonsmooth constraint' $I'(u)[u_i^\delta] = 0$, $i = 1, \ldots, k$, imposed to the functions belonging to the sets $S_{x_1, \ldots, x_k}$.

Lemma 5. *Let assumptions (h_1), (h_2) and (h_3) be satisfied. The functional I is coercive, convex and, hence, weakly lower semicontinuous on the set*

$$\mathcal{H} := \{u \in H^1(\mathbb{R}^N) \colon |u(x)| \leqslant \delta, \ \forall x \in \mathbb{R}^N\}.$$

Proof. Because of the choice (7) of δ, for any function $u \in \mathcal{H}$ we have

$$
\begin{aligned}
I(u) &= \frac{1}{2} \int\limits_{\mathbb{R}^N} \left(|\nabla u|^2 + a(x)u^2 \right) dx - \frac{1}{p+1} \int\limits_{\mathbb{R}^N} |u|^{p+1} \, dx \\
&\geqslant \frac{1}{2} \int\limits_{\mathbb{R}^N} |\nabla u|^2 \, dx + a_0 \left(\frac{1}{2} - \frac{1}{p+1} \right) \int\limits_{\mathbb{R}^N} u^2 \, dx > 0,
\end{aligned}
\tag{10}
$$

thus the assertion follows. $\quad\square$

Lemma 6. *Let assumptions (h_1), (h_2) and (h_3) be satisfied. For all $u \in H^1(\mathbb{R}^N)$ such that $u^\delta \neq 0$, the function $\mathcal{I}_u : [0, +\infty) \to \mathbb{R}$ defined as*

$$
\mathcal{I}_u(t) = I\left(u_\delta + tu^\delta \right)
$$

has a unique maximum point $t_u \in (0, +\infty)$.

Lemma 7. *Let assumptions (h_1), (h_2) and (h_3) be satisfied. Let $k \in \mathbb{N} \setminus \{0\}$ and $(x_1, \ldots, x_k)$ belong to $(\mathbb{R}^N)^k$. Let $u \in H^1(\mathbb{R}^N)$ be a function emerging around the points $x_1, \ldots, x_k$, then for all $j \in \{1, \ldots, k\}$ the function*

$$
t \to I\left(u_\delta + \sum_{i \neq j} u_i^\delta + tu_j^\delta \right)
$$

has a unique maximum point $t_u^j \in (0, +\infty)$.

Lemma 8. *Let assumptions (h_1), (h_2) and (h_3) be satisfied. Then, for all $k \in \mathbb{N} \setminus \{0\}$, for all $(x_1, \ldots, x_k) \in \mathcal{K}_k$, for all $i \in \{1, \ldots, k\}$,*

$$
u \in S_{x_1, \ldots, x_k} \quad \Longrightarrow \quad \begin{cases} I(u) > 0, \\ J_\delta\left(u_i^\delta \right) > 0. \end{cases}
\tag{11}
$$

For the proofs of the above lemmas we refer the reader to [6].

We also remark that the proof of Lemma 6 shows $u_\delta + t_u u^\delta$ is the only point in the half line $\{u_\delta + tu^\delta : t \in (0, +\infty)\}$, for which

$$
I'\left(u_\delta + t_u u^\delta \right)\left[u^\delta \right] = 0.
$$

Analogously, Lemma 7 gives, for u emerging around $x_1, \ldots, x_k$, that, for any $j \in \{1, 2, \ldots, k\}$, in the half line $\{u_\delta + \sum_{i \neq j} u_i^\delta + tu_j^\delta : t \in (0, +\infty)\}$ there is just one point $u_\delta + \sum_{i \neq j} u_i^\delta + t_u^j u_j^\delta$, for which

$$
I'\left(u_\delta + \sum_{i \neq j} u_i^\delta + t_u^j u_j^\delta \right)\left[u_j^\delta \right] = 0.
$$

Given any $u \in H^1(\mathbb{R}^N)$ such that $u^\delta \neq 0$, we call $u_\delta + t_u u^\delta$ the projection of u on the set $\{u \in H^1(\mathbb{R}^N) : I'(u)[u^\delta] = 0\}$ and we set $\vartheta(u) := t_u$.

Analogously, if $u \in H^1(\mathbb{R}^N)$ is emerging around k points we call $u_\delta + \sum_{i \neq j} u_i^\delta + t_u^j u_j^\delta$ the projection of u on the set $\{u \in H^1(\mathbb{R}^N) : I'(u)[u_j^\delta] = 0\}$ and we set $\vartheta_j(u) := t_u^j$.

Remark 1. We point out that, when $a(x) = a_\infty$, the definition of $S_{x_1, \ldots, x_k}$ still makes sense and Lemmas 6, 7 and 8 hold. In this case we use the notation $S_{x_1, \ldots, x_k}^\infty$. Moreover, for all $u \in H^1(\mathbb{R}^N)$, $u^\delta \neq 0$, we denote by $u_\delta + \vartheta^\infty(u)u^\delta$, the projection of u on the set $\{u \in H^1(\mathbb{R}^N) : (I^\infty)'(u)[u^\delta] = 0\}$, and, when u is emerging around k points, $u_\delta + \sum_{i \neq j} u_i^\delta + \vartheta_j^\infty(u)u_j^\delta$ denotes the projection of u on the set $\{u \in H^1(\mathbb{R}^N) : (I^\infty)'(u)[u_j^\delta] = 0\}$.

We close this section by a lemma, whose proof can be found in [6], that gives useful information about the set of the projections (on the constraint $I'(u)[u^\delta] = 0$) of the family of ground state solutions of (P_∞).

Lemma 9. *Let assumptions (h_1), (h_2) and (h_3) be satisfied. Then $\{\vartheta(w_y) : y \in \mathbb{R}^N\}$ is a bounded set.*

3. Preliminary estimates

For all $k \in \mathbb{N} \setminus \{0\}$ and for all $(x_1, \ldots, x_k) \in \mathcal{K}_k$, we put

$$\mu(x_1, \ldots, x_k) := \min\{I(u) : u \in S_{x_1, \ldots, x_k}\};$$
$$M_{x_1, \ldots, x_k} = \{u \in S_{x_1, \ldots, x_k} : I(u) = \mu(x_1, \ldots, x_k)\};$$
$$\mu_k = \max_{\mathcal{K}_k} \mu(x_1, \ldots, x_k) = \max_{\mathcal{K}_k} \min_{S_{x_1, \ldots, x_k}} I(u).$$

Lemma 10. *Let assumptions* (h_1), (h_2), (h_3) *and* (h_4) *be satisfied. Let* $(x_1, \ldots, x_k) \in \mathcal{K}_k$ *and* $u \in M_{x_1, \ldots, x_k}$. *Then, setting* $d(x) = dist(x, [\operatorname{supp} u^\delta \cup \operatorname{supp} \alpha^-])$, *relation*

$$0 < u_\delta(x) < C\delta e^{-\bar\eta d(x)} \tag{12}$$

holds, with $C > 0$ *depending only on* $\bar\eta$, a_∞, N.

The proof of the above lemma can be found in [6, Lemma 3.4].

Lemma 11. *Let assumptions* (h_1), (h_2), *and* (h_3) *be satisfied. Let* $(x_n)_n$, $x_n \in \mathbb{R}^N$ *be a sequence such that* $\lim_{n \to +\infty} |x_n| = +\infty$ *and let* $v_n \in M_{x_n}$, *then*

$$I(v_n) \leqslant m_\infty + o(1). \tag{13}$$

Proof. Let us consider $\tilde{w}_n = (w_{x_n})_\delta + \vartheta(w_{x_n})(w_{x_n})^\delta$ (w_{x_n} as defined in (6)), then we have $\tilde{w}_n \in S_{x_n}$ and, by using Lemma 6 and Remark 1, we get

$$\mu(x_n) = I(v_n) \leqslant I(\tilde{w}_n) = I^\infty(\tilde{w}_n) + \frac{1}{2} \int_{\mathbb{R}^N} \alpha(x)(\tilde{w}_n(x))^2 dx$$

$$\leqslant I^\infty(w) + \frac{1}{2}\left| \int_{\mathbb{R}^N} \alpha(x)(\tilde{w}_n(x))^2 dx \right|$$

$$\leqslant m_\infty + \frac{1}{2}\left[\max(1, \vartheta(w_{x_n}))\right]^2 \left| \int_{\mathbb{R}^N} \alpha(x)(w_{x_n}(x))^2 dx \right|. \tag{14}$$

Now, for all $\varepsilon > 0$ an $r = r(\varepsilon) > 0$ can be found so that $\alpha(x) < \varepsilon$ as $|x| > r$; hence, for large n, the relation

$$\left| \int_{\mathbb{R}^N} \alpha(x)(w_{x_n}(x))^2 dx \right| \leqslant \sup_{B_r(0)}(w_{x_n}(x))^2 \left| \int_{B_r(0)} \alpha(x) dx \right| + \varepsilon \int_{\mathbb{R}^N} (w(x))^2 dx$$

$$\leqslant \bar{C}\varepsilon, \tag{15}$$

holds, because $\sup_{B_r(0)} w_{x_n}(x) \xrightarrow{n \to +\infty} 0$. Then, since by Lemma 9, $\sup \vartheta(w_{x_n}) \in \mathbb{R}$, (13) follows combining (14) and (15). $\quad\square$

Lemma 12. *Let assumptions* (h_1), (h_2), *and* (h_3) *be satisfied. Let* $((x_1^n, \ldots, x_k^n))_n$ *be a sequence of* k-*tuples belonging to* $\mathcal{K}_k$ *such that*

$$\lim_{n \to +\infty} (x_1^n, \ldots, x_k^n) = (x_1, \ldots, x_k).$$

Then

$$\lim_{n \to +\infty} \mu(x_1^n, \ldots, x_k^n) = \mu(x_1, \ldots, x_k). \tag{16}$$

Proof. Let us consider $u_n \in M_{x_1^n,\dots,x_k^n}$ and $u \in M_{x_1,\dots,x_k}$. Set, for all n,

$$\hat{u}_n(x) := (\hat{u}_n(x))_\delta + \sum_{i=1}^{k} \vartheta_i(\hat{u}_n)(\hat{u}_n)_i^\delta(x)$$

where, for all $i \in \{1, \dots, k\}$,

$$(\hat{u}_n)_i^\delta(x) = u_i^\delta(x + x_i - x_i^n)$$

and $(\hat{u}_n)_\delta$ is the unique positive minimizer for the minimization problem

$$\min\left\{ I(u) : u \in H^1(\mathbb{R}^N), \ |u| \leqslant \delta, \ u = \delta \text{ on } \bigcup_{i=1}^{k} \operatorname{supp}(\hat{u}_n)_i^\delta \right\}.$$

We remark that such a minimizer exists and is unique; indeed the same argument of Lemma 5 shows that the functional I is coercive and convex on the convex set $\{u \in H^1(\mathbb{R}^N) : |u| \leqslant \delta, \ u = \delta \text{ on } \bigcup_{i=1}^{k} \operatorname{supp}(\hat{u}_n)_i^\delta\}$.

Now, we have that $\hat{u}_n \in S_{x_1^n,\dots,x_k^n}$ and

$$\limsup_{n \to +\infty} \mu\left(x_1^n, \dots, x_k^n\right) = \limsup_{n \to +\infty} I(u_n) \leqslant \lim_{n \to +\infty} I(\hat{u}_n) = I(u) = \mu(x_1, \dots, x_k). \tag{17}$$

On the other hand, we set

$$\tilde{u}_n(x) = (\tilde{u}_n)_\delta(x) + \sum_{i=1}^{k} \vartheta_i(\tilde{u}_n)(\tilde{u}_n)_i^\delta(x),$$

where, for all $i \in \{1, \dots, k\}$,

$$(\tilde{u}_n)_i^\delta(x) = (u_n)_i^\delta(x + x_i^n - x_i)$$

and $(\tilde{u}_n)_\delta$ is the unique positive minimizer for the minimization problem

$$\min\left\{ I(u) : u \in H^1(\mathbb{R}^N), \ |u| \leqslant \delta, \ u = \delta \text{ on } \bigcup_{i=1}^{k} \operatorname{supp}(\tilde{u}_n)_i^\delta \right\}.$$

Then we obtain $\tilde{u}_n \in S_{x_1,\dots,x_k}$ and

$$I(u) \leqslant I(\tilde{u}_n), \qquad \lim_{n \to +\infty} \left(I(\tilde{u}_n) - I(u_n) \right) = 0.$$

Thus

$$\mu(x_1, \dots, x_k) \leqslant \liminf_{n \to +\infty} \mu\left(x_1^n, \dots, x_k^n\right)$$

holds and, together with (17), gives (16). □

Lemma 13. *Let assumptions* (h_1), (h_2), (h_3) *and* (h_4) *be satisfied, let* $(x_1, \dots, x_k) \in \mathcal{K}_k$; *then a point* $y \in \mathbb{R}^N$ *exists so that* $(x_1, \dots, x_k, y) \in \mathcal{K}_{k+1}$ *and*

$$\mu(x_1, \dots, x_k) + m_\infty < \mu(x_1, \dots, x_k, y). \tag{18}$$

Proof. The argument of this proof is similar to that used in Proposition 4.4 of [6], however we repeat it here in order to make the paper more self-contained.

Let us consider $y_n = \sigma_n \tau$, with $\sigma_n \in \mathbb{R}$, $\sigma_n \xrightarrow[n \to +\infty]{} +\infty$, $\tau \in \mathbb{R}^N$, $|\tau| = 1$ and set

$$\Sigma_n = \left\{ x \in \mathbb{R}^N : \frac{\sigma_n}{2} - 1 < (x \cdot \tau) < \frac{\sigma_n}{2} + 1 \right\}.$$

For large $n \in \mathbb{N}$, $(x_1, \dots, x_k, y_n) \in \mathcal{K}_{k+1}$; then we take $u_n \in M_{x_1,\dots,x_k,y_n}$; u_n can be written as

$$u_n(x) = (u_n)_\delta(x) + \sum_{i=1}^{k} (u_n)_i^\delta(x) + (u_n)_{k+1}^\delta(x),$$

where $(u_n)_i^\delta$ and $(u_n)_{k+1}^\delta$ are the emerging parts of u_n around x_i and y_n respectively. We remark that

$$\operatorname{supp}(u_n)_i^\delta \subset B_\rho(x_i); \qquad \beta_i(u_n) = 0, \quad \forall i \in \{1, \dots, k\}$$

and

$$\operatorname{supp}(u_n)_{k+1}^\delta \subset B_\rho(y_n); \qquad \beta_{k+1}(u_n) = 0;$$

moreover, for large n, we can assume that

$$\bigcup_{i=1}^{k} B_\rho(x_i) \subset \left\{ x \in \mathbb{R}^N : (x \cdot \tau) < \frac{\sigma_n}{2} - 1 \right\},$$

$$B_\rho(y_n) \subset \left\{ x \in \mathbb{R}^N : (x \cdot \tau) > \frac{\sigma_n}{2} + 1 \right\}.$$

Let us define $v_n(x) = \chi_n(x) u_n(x)$, where $\chi_n \in \mathcal{C}^\infty(\mathbb{R}^N, [0, 1])$, $\chi_n(x) = \chi(|(x \cdot \tau) - \frac{\sigma_n}{2}|)$ and $\chi \in \mathcal{C}^\infty(\mathbb{R}^+, [0, 1])$ is a function such that $\chi(t) = 0$ if $t \leqslant 1/2$, $\chi(t) = 1$ if $t \geqslant 1$. Let us evaluate $I(v_n)$:

$$
\begin{aligned}
I(v_n) &= I(\chi_n u_n) \\
&= \frac{1}{2} \int_{\mathbb{R}^N} \left[|\chi_n \nabla u_n + u_n \nabla \chi_n|^2 + a(x)(\chi_n u_n)^2 \right] dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |\chi_n u_n|^{p+1} dx \\
&= \frac{1}{2} \int_{\mathbb{R}^N} \chi_n^2 \left[|\nabla u_n|^2 + a(x) u_n^2 \right] dx + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \chi_n|^2 u_n^2 dx \\
&\quad + \frac{1}{4} \int_{\mathbb{R}^N} \nabla \chi_n^2 \nabla u_n^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^N} (u_n)^{p+1} dx + \frac{1}{p+1} \int_{\mathbb{R}^N} \left(1 - \chi_n^{p+1} \right)(u_n)^{p+1} dx \\
&\leqslant \mu(x_1, \dots, x_k, y_n) + \bar{c}_1 \int_{\Sigma_n} \left((u_n)_\delta \right)^2 dx + \bar{c}_2 \int_{\Sigma_n} \left((u_n)_\delta \right)^{p+1} dx \\
&\leqslant \mu(x_1, \dots, x_k, y_n) + O\left(e^{-\bar{\eta} \sigma_n} \right),
\end{aligned}
\tag{19}
$$

where last inequality is obtained using the exponential decay of u_n (see Lemma 10) and observing that, by (h_4), for large n, $\operatorname{supp} \alpha^- \cap \Sigma_n = \emptyset$.

On the other hand,

$$I(v_n) = I\left(v_n^I \right) + I\left(v_n^{II} \right), \tag{20}$$

where

$$v_n^I(x) = \begin{cases} 0 & \text{if } (x \cdot \tau) \geqslant \frac{\sigma_n}{2} - \frac{1}{2}, \\ \chi_n(x)(u_n)_\delta(x) + \sum_{i=1}^{k} (u_n)_i^\delta & \text{if } (x \cdot \tau) < \frac{\sigma_n}{2} - \frac{1}{2} \end{cases}$$

and

$$v_n^{II}(x) = \begin{cases} 0 & \text{if } (x \cdot \tau) \leqslant \frac{\sigma_n}{2} + \frac{1}{2}, \\ \chi_n(x)(u_n)_\delta(x) + (u_n)_{k+1}^\delta(x) & \text{if } (x \cdot \tau) > \frac{\sigma_n}{2} + \frac{1}{2}. \end{cases}$$

By definition, $v_n^I \in S_{x_1, \dots, x_k}$, thus

$$I\left(v_n^I \right) \geqslant \mu(x_1, \dots, x_k). \tag{21}$$

Analogously, $v_n^{II} \in S_{y_n}$; hence, considering the function $\tilde{v}_n \in S_{y_n}^\infty$ defined as $\tilde{v}_n(x) = (v_n^{II})_\delta(x) + \vartheta^\infty(v_n^{II})(v_n^{II})^\delta(x)$, we get, for large n,

$$I(v_n^{II}) \geqslant I(\tilde{v}_n) = I^\infty(\tilde{v}_n) + \frac{1}{2} \int_{\mathbb{R}^N} \alpha(x)(\tilde{v}_n(x))^2 \, dx$$

$$\geqslant m_\infty + \frac{1}{2} \left[\int_{B_\rho(y_n)} \alpha(x)(\tilde{v}_n(x))^2 \, dx - \sup_{B_{\frac{\sigma_n}{2}}(0)} (\tilde{v}_n(x))^2 \int_{\operatorname{supp}\alpha^-} |\alpha(x)| \, dx \right].$$

Now, by assumption (h_4), $\operatorname{supp}\alpha^-$ is bounded; moreover, by maximum principle,

$$\sup_{B_{\frac{\sigma_n}{2}}(0)} (\tilde{v}_n(x))^2 \leqslant \sup_{\partial B_{\frac{\sigma_n}{2}}(0)} (\tilde{v}_n(x))^2.$$

Thus, taking into account that, for large n, $\partial B_{\frac{\sigma_n}{2}}(0) \cap \operatorname{supp}\alpha^- = \emptyset$, and using Lemma 10, we get

$$I(v_n^{II}) \geqslant m_\infty + \frac{1}{2} \int_{B_\rho(y_n)} \alpha(x)(\tilde{v}_n(x))^2 \, dx - O(e^{-\bar{\eta}\sigma_n}). \tag{22}$$

Therefore, combining (19), (20), (21) and (22), we obtain

$$\mu(x_1, \ldots, x_k, y_n) \geqslant \mu(x_1, \ldots, x_k) + m_\infty + \frac{1}{2} \int_{B_\rho(y_n)} \alpha(x)(\tilde{v}_n(x))^2 \, dx - O(e^{-\bar{\eta}\sigma_n}); \tag{23}$$

but, for large n,

$$\frac{1}{2} \int_{B_\rho(y_n)} \alpha(x)(\tilde{v}_n(x))^2 \, dx - O(e^{-\bar{\eta}\sigma_n}) > 0,$$

because assumption (h_4) implies $\alpha(x)$ and $\int_{B_\rho(y_n)} \alpha(x)(\tilde{v}_n(x))^2 \, dx$ decay more slowly than $e^{-\bar{\eta}\sigma_n}$. Therefore (18) follows. $\square$

Lemma 14. *Let $(\bar{u}_k)_k$ be a sequence of solutions to (E) obtained as described in Theorem 4. For all real number $r > 0$, let us denote by $\nu(\bar{u}_k, r)$ the number of points around which $\bar{u}_k$ is emerging and that are contained in $B_r(0)$.*
   Then, for all $h \in \mathbb{N}$ there exist a real number $r_h > 0$ and a number $k_h \in \mathbb{N}$ such that $\nu(\bar{u}_k, r_h) \geqslant h$, for all $k > k_h$.

Proof. We argue by contradiction and we assume that there exist $h \in \mathbb{N}$ and sequences $(r_n)_n$, $r_n \in \mathbb{R}^+ \setminus 0$, $(k_n)_n \in \mathbb{N}$, such that $r_n \to +\infty$, $k_n \to +\infty$ as $n \to +\infty$, and $\nu(\bar{u}_{k_n}, r_n) < h$ for all $n \in \mathbb{N}$.
   Let us denote by $(\bar{x}_1^n, \ldots, \bar{x}_{k_n}^n)$ the points around which the solution $\bar{u}_{k_n}$ is emerging. Passing, if necessary, to a subsequence, we can assume that, for some $j < h$,

$$\bar{x}_i^n \xrightarrow[n \to +\infty]{} \bar{x}_i, \quad \forall i \leqslant j,$$

while

$$|\bar{x}_i^n| \xrightarrow[n \to +\infty]{} +\infty, \quad \forall i > j.$$

Let $y \in \mathbb{R}^N$ be a point, whose existence is proved by Lemma 13, such that

$$\mu(\bar{x}_1, \ldots, \bar{x}_j) + m_\infty < \mu(\bar{x}_1, \ldots, \bar{x}_j, y). \tag{24}$$

To get a contradiction we intend to show that the relations

$$\limsup_{n \to +\infty} \left[\mu(\bar{x}_1^n, \ldots, \bar{x}_{k_n}^n) - \mu(\bar{x}_{j+2}^n, \ldots, \bar{x}_{k_n}^n) \right] \leqslant \mu(\bar{x}_1, \ldots, \bar{x}_j) + m_\infty \tag{25}$$

and

$$\lim_{n \to +\infty} \left[\mu\left(\bar{x}_1^n, \ldots, \bar{x}_j^n, y, \bar{x}_{j+2}^n, \ldots, \bar{x}_{k_n}^n\right) - \mu\left(\bar{x}_{j+2}^n, \ldots, \bar{x}_{k_n}^n\right) \right] = \mu(\bar{x}_1, \ldots, \bar{x}_j, y) \tag{26}$$

hold true. Indeed from (24), (25) and (26) relation

$$\mu\left(\bar{x}_1^n, \ldots, \bar{x}_{k_n}^n\right) < \mu\left(\bar{x}_1^n, \ldots, \bar{x}_j^n, y, \bar{x}_{j+2}^n, \ldots, \bar{x}_{k_n}^n\right)$$

follows for large n, in contradiction to

$$\mu_{k_n} = I(\bar{u}_{k_n}) = \mu\left(\bar{x}_1^n, \ldots, \bar{x}_{k_n}^n\right).$$

To prove (25) let us choose

$$v_n \in M_{\bar{x}_{j+1}^n}, \qquad z_n \in M_{\bar{x}_1^n, \ldots, \bar{x}_j^n}, \qquad s_n \in M_{\bar{x}_{j+2}^n, \ldots, \bar{x}_{k_n}^n},$$

and remark that by Lemmas 11 and 12 respectively, we have

$$I(v_n) \leqslant m_\infty + o(1), \tag{27}$$
$$I(z_n) = \mu(\bar{x}_1, \ldots, \bar{x}_j) + o(1). \tag{28}$$

Let us consider now, for all n, $z_n \vee v_n \vee s_n$; then $z_n \vee v_n \vee s_n \in S_{\bar{x}_1^n, \ldots, \bar{x}_{k_n}^n}$ and

$$\begin{aligned} I(u_{k_n}) &\leqslant I(z_n \vee v_n \vee s_n) \\ &= I(z_n) + I(v_n) + I(s_n) - I(z_n \wedge v_n) - I\left((z_n \vee v_n) \wedge s_n\right). \end{aligned}$$

Hence, taking into account that $z_n \wedge v_n \leqslant \delta$ and $(z_n \vee v_n) \wedge s_n \leqslant \delta$ together Lemma 5 imply $I(z_n \wedge v_n) > 0$ and $I((z_n \vee v_n) \wedge s_n) > 0$, and using (27), (28) and (16) we get

$$\begin{aligned} \mu\left(\bar{x}_1^n, \ldots, \bar{x}_{k_n}^n\right) = I(u_{k_n}) &< I(z_n) + I(v_n) + I(s_n) \\ &\leqslant \mu(\bar{x}_1, \ldots, \bar{x}_j) + \mu\left(\bar{x}_{j+2}^n, \ldots, \bar{x}_{k_n}^n\right) + m_\infty + o(1), \end{aligned}$$

from which (25) follows.

In order to prove (26), we first argue as for proving (25). We consider

$$\tilde{u}_n \in M_{\bar{x}_1^n, \ldots, \bar{x}_j^n, y, \bar{x}_{j+2}^n, \ldots, \bar{x}_{k_n}^n}, \qquad \tilde{v}_n \in M_{\bar{x}_1^n, \ldots, \bar{x}_j^n, y}, \qquad s_n \in M_{\bar{x}_{j+2}^n, \ldots, \bar{x}_{k_n}^n};$$

then, observing that $s_n \vee \tilde{v}_n \in S_{\bar{x}_1^n, \ldots, \bar{x}_j^n, y, \bar{x}_{j+2}^n, \ldots, \bar{x}_{k_n}^n}$ and $s_n \wedge \tilde{v}_n \leqslant \delta$, and using Lemma 12, we obtain

$$\begin{aligned} \mu\left(\bar{x}_1^n, \ldots, \bar{x}_j^n, y, \bar{x}_{j+2}^n, \ldots, \bar{x}_{k_n}^n\right) = I(\tilde{u}_n) &\leqslant I(s_n \vee \tilde{v}_n) \\ &< I(s_n) + I(\tilde{v}_n) = \mu(\bar{x}_1, \ldots, \bar{x}_j, y) + \mu\left(\bar{x}_{j+2}^n, \ldots, \bar{x}_{k_n}^n\right) + o(1), \end{aligned}$$

from which we deduce

$$\limsup_{n \to +\infty} \left[\mu\left(\bar{x}_1^n, \ldots, \bar{x}_j^n, y, \bar{x}_{j+2}^n, \ldots, \bar{x}_{k_n}^n\right) - \mu\left(\bar{x}_{j+2}^n, \ldots, \bar{x}_{k_n}^n\right) \right] \leqslant \mu(\bar{x}_1, \ldots, \bar{x}_j, y). \tag{29}$$

To prove the reverse inequality, we argue analogously to Lemma 13. We set

$$\sigma_n := \min\left\{ \left|\bar{x}_i^n\right| : i = j+2, \ldots, k_n \right\}$$

thus $\sigma_n \xrightarrow[n \to +\infty]{} +\infty$ and we can assert that, for large n,

$$\bigcup_{i=j+2}^{k} B_\rho(\bar{x}_i) \subset \mathbb{R}^N \setminus B_{\frac{\sigma_n}{2}+1}(0),$$

$$\bigcup_{i=1}^{j} B_\rho(\bar{x}_i) \cup B_\rho(y) \subset B_{\frac{\sigma_n}{2}-1}(0).$$

We define $\hat{u}_n(x) = \chi_n(x)\tilde{u}_n(x)$, where $\chi_n(x) = \chi(|x| - \frac{\sigma_n}{2})$ and $\chi \in \mathcal{C}^\infty(\mathbb{R}^+, [0, 1])$ is a function such that $\chi(t) = 0$ if $|t| \leqslant 1/2$, $\chi(t) = 1$ if $|t| \geqslant 1$, and we evaluate $I(\hat{u}_n(x))$. By computations analogous to those of (19) we obtain

$$I\big(\hat{u}_n(x)\big) = I\big(\chi_n \tilde{u}_n(x)\big)$$

$$\leqslant \mu\big(\bar{x}_1^n, \ldots, \bar{x}_j^n, y, \bar{x}_{j+2}^n, \ldots, \bar{x}_{k_n}^n\big) + C_1 \int\limits_{B_{\frac{\sigma_n}{2}+1}(0) \setminus B_{\frac{\sigma_n}{2}-1}(0)} \big((\tilde{u}_n)_\delta\big)^2 \, dx$$

$$+ C_2 \int\limits_{B_{\frac{\sigma_n}{2}+1}(0) \setminus B_{\frac{\sigma_n}{2}-1}(0)} \big((\tilde{u}_n)_\delta\big)^{p+1} \, dx$$

$$\leqslant \mu\big(\bar{x}_1^n, \ldots, \bar{x}_j^n, y, \bar{x}_{j+2}^n, \ldots, \bar{x}_{k_n}^n\big) + O\big(e^{-\bar{\eta}\sigma_n}\big). \tag{30}$$

On the other hand $\hat{u}_n(x)$, for large n, can be written as $\hat{u}_n(x) = \hat{z}_n(x) + \check{z}_n(x)$, with

$$\hat{z}_n(x) = \begin{cases} 0 & \text{if } |x| \geqslant \frac{\sigma_n}{2} - \frac{1}{2}, \\ \chi_n(x)(\tilde{u}_n)_\delta(x) + (\tilde{u}_n)^\delta(x) & \text{if } |x| < \frac{\sigma_n}{2} - \frac{1}{2} \end{cases}$$

and

$$\check{z}_n(x) = \begin{cases} 0 & \text{if } |x| \leqslant \frac{\sigma_n}{2} + \frac{1}{2}, \\ \chi_n(x)(\tilde{u}_n)_\delta(x) + (\tilde{u}_n)^\delta(x) & \text{if } |x| > \frac{\sigma_n}{2} + \frac{1}{2}. \end{cases}$$

Hence

$$I(\hat{u}_n) = I(\hat{z}_n) + I(\check{z}_n) \geqslant \mu\big(x_1^n, \ldots, x_j^n, y\big) + \mu\big(x_{j+2}^n, \ldots, x_{k_n}^n\big). \tag{31}$$

Combining (30) and (31), and passing to the limit, we get the desired inequality

$$\liminf_{n \to +\infty}\big[\mu\big(x_1^n, \ldots, x_j^n, y, x_{j+2}^n, \ldots, x_{k_n}^n\big) - \mu\big(x_{j+2}^n, \ldots, x_{k_n}^n\big)\big]$$

$$\geqslant \lim_{n \to +\infty}\big[\mu\big(x_1^n, \ldots, x_j^n, y\big) - O\big(e^{-\eta\sigma_n}\big)\big] = \mu(x_1, \ldots, x_j, y)$$

that, together (29), gives (26). $\quad\square$

4. Proof of Theorem 2

This section is devoted to the proof, divided in three propositions, of Theorem 2. Proposition 15 is the main step in which the existence of a solution to (E) having infinitely many bumps is shown. Propositions 16 and 17 are devoted to prove relations (3) and (4) respectively.

Proposition 15. *Let assumptions of Theorem 2 hold.*
Then there exists a solution of (E), $\bar{u} \in H^1_{loc}(\mathbb{R}^N)$, having infinitely many emerging parts around an unbounded sequence $(\bar{x}_n)_n$ of points of $\mathbb{R}^N$, such that $|\bar{x}_n - \bar{x}_m| \geqslant 3\rho$ if $n \neq m$.

Proof. Let $(\bar{u}_k)_k$ be a sequence of solutions to (E) obtained as described in Theorem 4.

Lemma 14 states that, for all $h \in \mathbb{N}$, $r_h \in \mathbb{R}^+ \setminus \{0\}$ and $k_h \in \mathbb{N}$ exist so that $\nu(\bar{u}_k, r_h) \geqslant h$, for all $k > k_h$. On the other hand, since the points around which any $\bar{u}_k$ is emerging have interdistances greater or equal than 3ρ, a number $H(\geqslant h)$ exists so that $\nu(\bar{u}_k, r_h) \leqslant H$, for all $k \in \mathbb{N}$. Thus, $\lim_{h \to +\infty} r_h = +\infty$ and, without any loss of generality, we can assume $r_h \leqslant r_{h+1}$, for all $h \in \mathbb{N}$.

Therefore, we can build, for all h, a subsequence $(\bar{u}_{k_n}^h)_n$ of $(\bar{u}_k)_k$ in such a way that

$$\forall h \in \mathbb{N}, \quad \big(\bar{u}_{k_n}^{h+1}\big)_n \text{ is a subsequence of } \big(\bar{u}_{k_n}^h\big)_n,$$

$$\forall h \in \mathbb{N}, \ \forall n \in \mathbb{N}, \quad \nu\big(\bar{u}_{k_n}^h, r_h\big) = \hat{h}, \quad h \leqslant \hat{h} \leqslant H,$$

and for all $h \in \mathbb{N}$, the sequences of $\hat{h}$-tuples $((\bar{x}_1^n)^h, \ldots, (\bar{x}_{\hat{h}}^n)^h)_n$, consisting of the $\hat{h}$ points around which $\bar{u}_{k_n}^h$ is emerging and that are contained in $B_{r_h}(0)$, are converging as n goes to $+\infty$.

Let us now consider the sequence $(v_n)_n$, where

$$v_n := \bar{u}_{k_n}^n.$$

By construction, $(v_n)_n$ is a subsequence of $(\bar{u}_{k_n}^h)_n$ for all h; moreover the sequences of points, around which the functions of $(v_n)_n$ are emerging, are converging as n goes to infinity and their limit points have interdistances greater or equal than 3ρ and make up an unbounded numerable subset of $\mathbb{R}^N$, we denote it by L.

Our aim is to show that $\|v_n\|_{B_{r_h}(0)}$ is bounded for all $h \in \mathbb{N}$. Indeed, if this is true, considering that $r_h \xrightarrow[h \to +\infty]{} +\infty$ and that, for all n, v_n solves (E), we can infer that, up to a subsequence, $(v_n)_n$ uniformly converges on every compact set of $\mathbb{R}^N$ and that the limit function is a solution of (E) which has at least h emerging parts around points belonging to $\bar{B}_{r_h}(0)$.

Since L is numerable, we can set $L = \{\bar{x}_n : n \in \mathbb{N}\}$, where $\bar{x}_n \neq \bar{x}_m$ if $n \neq m$. Since $|\bar{x}_n| \xrightarrow[n \to +\infty]{} +\infty$, $L \cap B_{r_h + \frac{4}{3}\rho}(0)$ is finite and, by definition of $(v_n)_n$, it contains at least h points. Thus, we can assume

$$L \cap B_{r_h + \frac{4}{3}\rho}(0) = \{\bar{x}_1, \dots, \bar{x}_{\bar{h}}\}, \quad \bar{h} \geqslant h,$$

and we set

$$D_h := B_{r_h}(0) \setminus \bigcup_{i=1}^{\bar{h}} B_{\frac{4}{3}\rho}(\bar{x}_i).$$

Now, $v_n(x)$, for all n, is a solution of (E) on D_h and, for large n, the distance between D_h and the points around which v_n is emerging is greater than some $\bar{\rho} > \rho$. Hence, for large n, $0 \leqslant v_n(x) \leqslant \delta$ in D_h, $(|v_n|_{q,D_h})_n$ with $q \geqslant 1$, is bounded and, as a consequence of the fact that v_n solves (E), $(v_n)_n$ turns out to be bounded in $H^1(D_h)$ and, by standard regularity results, in $C^{1+\epsilon}(B_{\frac{5}{3}\rho}(\bar{x}_i) \setminus \bar{B}_{\frac{4}{3}\rho}(\bar{x}_i))$ for all $i \in \{1, \dots, \bar{h}\}$.

To complete the argument let us now show that $(v_n)_n$ is bounded in $H^1(B_{\frac{5}{3}\rho}(\bar{x}_i))$, for all $i \in \{1, \dots, \bar{h}\}$. To this end we argue by contradiction. We assume that for some $j \in \{1, \dots, \bar{h}\}$, $(\|v_n\|_{B_{\frac{5}{3}\rho}(\bar{x}_j)})_n$ is not bounded and we consider, for all n, the function $z_n \in H^1(\mathbb{R}^N)$ such that $\Delta z_n = 0$ in $B_{\frac{4}{3}\rho}(\bar{x}_j) \setminus \bar{B}_\rho(\bar{x}_j^n)$ and

$$z_n(x) = \begin{cases} v_n, & \forall x \notin B_{(4/3)\rho}(\bar{x}_j), \\ \vartheta_n(v_1(x + \bar{x}_j^1 - \bar{x}_j^n))^\delta, & \forall x \in \bar{B}_\rho(\bar{x}_j^n) \end{cases}$$

where $\vartheta_n \in \mathbb{R}^+ \setminus \{0\}$ denotes, for all n, $\vartheta_j(v_1(x + \bar{x}_j^1 - \bar{x}_j^n))$. We remark that $\bar{x}_j^n \xrightarrow[n \to +\infty]{} \bar{x}_j$ implies $(\vartheta_n)_n$ is bounded; moreover, by definition, z_n is emerging around the same points of v_n. We also point out that $(z_n)_n$ is bounded in $H^1(B_{\frac{5}{3}\rho}(\bar{x}_j))$, because $(v_n)_n$ is bounded in $C^{1+\epsilon}(B_{\frac{5}{3}\rho}(\bar{x}_j) \setminus \bar{B}_{\frac{4}{3}\rho}(\bar{x}_j))$.

Thus, we easily infer the sequence $\|(v_n)_\delta\|_{B_{\frac{5}{3}\rho}(\bar{x}_j)}$ is bounded. Indeed, in the opposite case we would have

$$\lim_{n \to +\infty}\left(I(v_n) - I(z_n)\right) \geqslant \lim_{n \to +\infty}\left(\frac{1}{2}\int_{B_{\frac{5}{3}\rho}(\bar{x}_j)} |\nabla(v_n)_\delta|^2\,dx + a_0\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{B_{\frac{5}{3}\rho}(\bar{x}_j)} ((v_n)_\delta)^2\,dx - C\right)$$
$$= +\infty$$

and, by the max–min characterization of v_n,

$$I(z_n) \geqslant I(v_n),$$

that is impossible.

Therefore, up to a subsequence,

$$\left\|(v_n)_j^\delta\right\|_{B_{\frac{5}{3}\rho}(\bar{x}_j)} \xrightarrow[n \to +\infty]{} +\infty$$

must be true. Consequently,

$$\left|(v_n)_j^\delta\right|_{p+1, B_{\frac{5}{3}\rho}(\bar{x}_j)} \xrightarrow[n \to +\infty]{} +\infty$$

has to hold too, because otherwise we would deduce

$$\lim_{n\to+\infty}\big(I(v_n)-I(z_n)\big)\geqslant \lim_{n\to+\infty}c\big\|(v_n)_j^\delta\big\|^2_{B_{\frac{5}{3}\rho}(\bar{x}_j)}-C=+\infty,$$

that is not consistent with $I(v_n)\leqslant I(z_n)$. However, neither

$$\sup_{n\in\mathbb{N}}\big\|(v_n)_j^\delta\big\|_{B_{\frac{5}{3}\rho}(\bar{x}_j)}\big/\big|(v_n)_j^\delta\big|_{p+1,B_{\frac{5}{3}\rho}(\bar{x}_j)}<+\infty \tag{32}$$

nor, up to a subsequence,

$$\lim_{n\to+\infty}\big\|(v_n)_j^\delta\big\|_{B_{\frac{5}{3}\rho}(\bar{x}_j)}\big/\big|(v_n)_j^\delta\big|_{p+1,B_{\frac{5}{3}\rho}(\bar{x}_j)}=+\infty \tag{33}$$

can be true. Indeed, (32) brings to conclude

$$\lim_{n\to+\infty}J_\delta\big((v_n)_j^\delta\big)=-\infty$$

which is impossible, being $J_\delta((v_n)_j^\delta)>0$ for all $n\in\mathbb{N}$. From (33) we would deduce

$$\begin{aligned}
\lim_{n\to+\infty}J_\delta\big((v_n)_j^\delta\big)&=\lim_{n\to+\infty}\max_{t>0}J_\delta\big(t(v_n)_j^\delta\big)\\
&=\lim_{n\to+\infty}\max_{t>0}J_\delta\left(t\frac{(v_n)_j^\delta}{|(v_n)_j^\delta|_{p+1,B_{\frac{5}{3}\rho}(\bar{x}_j)}}\right)\geqslant\lim_{n\to+\infty}J_\delta\left(\frac{(v_n)_j^\delta}{|(v_n)_j^\delta|_{p+1,B_{\frac{5}{3}\rho}(\bar{x}_j)}}\right)\\
&\geqslant\lim_{n\to+\infty}\left[\bar{c}_1\frac{\|(v_n)_j^\delta\|^2_{B_{\frac{5}{3}\rho}(\bar{x}_j)}}{|(v_n)_j^\delta|^2_{p+1,B_{\frac{5}{3}\rho}(\bar{x}_j)}}-\frac{1}{p+1}\int_{B_{\frac{5}{3}\rho}(\bar{x}_j)}\left(\delta+\frac{(v_n)_j^\delta}{|(v_n)_j^\delta|_{p+1,B_{\frac{5}{3}\rho}(\bar{x}_j)}}\right)^{p+1}-\bar{c}_2\right]\\
&=+\infty, \tag{34}
\end{aligned}$$

that implies

$$\lim_{n\to+\infty}\big[I(v_n)-I(z_n)\big]=+\infty,$$

contradicting $I(z_n)\geqslant I(v_n)$. Consequently, $(v_n)_n$ is bounded in $H^1(B_{\frac{5}{3}\rho}(\bar{x}_i))$ for all $i\in\{1,\dots,\bar{h}\}$ and the proof is complete. $\quad\square$

Proposition 16. *Let $\bar{u}\in H^1_{loc}(\mathbb{R}^N)$ be a solution of (E), having infinitely many emerging parts around an unbounded sequence of points $(\bar{x}_n)_n$ of $\mathbb{R}^N$, as stated in Proposition 15. Then*

$$\lim_{n\to+\infty}\min\big\{|\bar{x}_n-\bar{x}_m|:\ m\in\mathbb{N},\ m\neq n\big\}=+\infty. \tag{35}$$

Proof. The argument is carried out by contradiction. We assume false the statement; therefore we can assert the existence of two subsequences $(b_n)_n$ and $(\bar{b}_n)_n$ of $(\bar{x}_n)_n$ such that, for all n, $\bar{b}_n\neq b_n$ and $(|\bar{b}_n-b_n|)_n$ is bounded.

Let us fix $R>\rho$. Then, by definition of $\bar{u}$, for all n a k_n exists so that

$$I(\bar{u}_{k_n})=\mu_{k_n},\qquad \sup_{B_R(b_n)\cup B_R(\bar{b}_n)}|\bar{u}_{k_n}-\bar{u}|<\frac{1}{n} \tag{36}$$

and $\bar{u}_{k_n}$ is a solution of (E) emerging around k_n points $(\bar{x}_1^n,\dots,\bar{x}_{k_n}^n)$. As a consequence of (36) the distances between two points around which $\bar{u}_{k_n}$ is emerging and the points b_n and $\bar{b}_n$, respectively, must go to zero as $n\to+\infty$. Without any loss of generality, we can assume

$$\lim_{n\to+\infty}\big|\bar{x}_1^n-b_n\big|=0;\qquad \lim_{n\to+\infty}\big|\bar{x}_2^n-\bar{b}_n\big|=0.$$

By Lemma 13, for all $n\in\mathbb{N}$ there exists $y_n\in\mathbb{R}^N$ such that the relation

$$\mu\big(y_n, \bar{x}_2^n, \ldots, \bar{x}_{k_n}^n\big) > \mu\big(\bar{x}_2^n, \ldots, \bar{x}_{k_n}^n\big) + m_\infty \tag{37}$$

holds. Now, if we show that

$$\limsup_{n\to+\infty}\big[\mu\big(\bar{x}_1^n, \ldots, \bar{x}_{k_n}^n\big) - \mu\big(\bar{x}_2^n, \ldots, \bar{x}_{k_n}^n\big)\big] < m_\infty, \tag{38}$$

we are done, because combining (36), (37) and (38) we obtain, for large n,

$$\mu_{k_n} = I(\bar{u}_{k_n}) = \mu\big(\bar{x}_1^n, \ldots, \bar{x}_{k_n}^n\big) < \mu\big(y_n, \bar{x}_2^n, \ldots, \bar{x}_{k_n}^n\big) \leqslant \mu_{k_n}$$

which is impossible.

The argument for proving (38) is analogous to that of Proposition 5.1 of [6], nevertheless we include it here to make the paper self-contained.

We set $w_{1,n} := w(x - \bar{x}_1^n)$ and we consider the projection of it on the constraint $I'(u)[u^\delta] = 0$ that is $\tilde{w}_{1,n} := (w_{1,n})_\delta + \vartheta(w_{1,n})(w_{1,n})^\delta$. Let $s_n(x)$ belong to $M_{\bar{x}_2^n, \ldots, \bar{x}_{k_n}^n}$. Then

$$\tilde{w}_{1,n} \in S_{\bar{x}_1^n}, \qquad \tilde{w}_{1,n} \vee s_n \in S_{\bar{x}_1^n, \ldots, \bar{x}_{k_n}^n}$$

and

$$\begin{aligned}
\mu_{\bar{x}_1^n, \ldots, \bar{x}_{k_n}^n} = I(\bar{u}_{k_n}) &\leqslant I(\tilde{w}_{1,n} \vee s_n) \\
&\leqslant I(\tilde{w}_{1,n}) + I(s_n) - I(\tilde{w}_{1,n} \wedge s_n) = \mu_{\bar{x}_2^n, \ldots, \bar{x}_{k_n}^n} + I(\tilde{w}_{1,n}) - I(\tilde{w}_{1,n} \wedge s_n).
\end{aligned} \tag{39}$$

Now, from the argument of (14) and (15) of Lemma 11, we have

$$I(\tilde{w}_{1,n}) \leqslant m_\infty + o(1). \tag{40}$$

On the other hand

$$(\tilde{w}_{1,n} \wedge s_n)_\delta = (w_{1,n} \wedge s_n)_\delta = w_{1,n} \wedge s_n,$$

and, since $(|\bar{x}_1^n - \bar{x}_2^n|)_n$ is bounded, a $\gamma \in \mathbb{R}^+$ exists so that, for all n,

$$\mathrm{supp}(s_n)_2^\delta \subset B_\rho\big(\bar{x}_2^n\big) \subset B_{\rho+\gamma}\big(\bar{x}_1^n\big),$$

thus we infer

$$\begin{aligned}
I(\tilde{w}_{1,n} \wedge s_n) &\geqslant a_0\left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^N} (w_{1,n} \wedge s_n)^2\, dx \\
&\geqslant C \int_{\mathrm{supp}(s_n)_2^\delta} \big(w_{1,n}(x)\big)^2\, dx \geqslant C\Big(\inf_{B_{\rho+\gamma}(0)} w\Big)^2 \big|\mathrm{supp}(s_n)_2^\delta\big|.
\end{aligned} \tag{41}$$

Clearly, (39), (40) and (41) give (38) once the relation

$$\liminf_{n\to+\infty}\big|\mathrm{supp}(s_n)_2^\delta\big| > 0 \tag{42}$$

is proved. Arguing by contradiction, we assume

$$\liminf_{n\to+\infty}\big|\mathrm{supp}(s_n)_2^\delta\big| = 0.$$

Then, we deduce that $\|(s_n)_2^\delta\|/|(s_n)_2^\delta|_{p+1}$ cannot be bounded. Otherwise, in fact, up to a subsequence, the contrasting relations:

$$\frac{(s_n)_2^\delta}{|(s_n)_2^\delta|_{p+1}} \rightharpoonup \bar{s} \quad \text{weakly in } H^1\big(\mathbb{R}^N\big),$$

$$\frac{(s_n)_2^\delta}{|(s_n)_2^\delta|_{p+1}} \to \bar{s} \quad \text{strongly in } L^{p+1}\big(\mathbb{R}^N\big),$$

$$\bar{s}(x) = 0 \quad \text{a.e. in } \mathbb{R}^N,$$

would be true simultaneously.

Therefore, up to a subsequence, we have

$$\lim_{n\to+\infty} \frac{\|(s_n)_2^\delta\|}{|(s_n)_2^\delta|_{p+1}} = +\infty,$$

and, consequently, arguing as in (34) we get

$$\lim_{n\to+\infty} J_\delta\big((s_n)_2^\delta\big) = +\infty. \tag{43}$$

Now, set

$$\hat{s}_n(x) := \left[(s_n)_\delta + \sum_{i=3}^{k_n} (s_n)_i^\delta \right] \vee \tilde{w}_{2,n},$$

where $w_{2,n}(x) = w(x - \bar{x}_2^n)$, $\tilde{w}_{2,n} = (w_{2,n})_\delta + \vartheta(w_{2,n})(w_{2,n})^\delta \in S_{\bar{x}_2^n}$. Then $\hat{s}_n \in S_{\bar{x}_2^n,\dots,\bar{x}_{k_n}^n}$ and, since $s_n \in M_{\bar{x}_2^n,\dots,\bar{x}_{k_n}^n}$,

$$I(s_n) \leqslant I(\hat{s}_n). \tag{44}$$

On the other hand, considering (43) and observing that, similarly to (40), $\lim_{n\to+\infty} I(\tilde{w}_{2,n}) = m_\infty$, we get for large n

$$\begin{aligned}
I(\hat{s}_n) - I(s_n) &\leqslant I\big((s_n)_\delta\big) + \sum_{i=3}^{k_n} J_\delta\big((s_n)_i^\delta\big) + I(\tilde{w}_{2,n}) - \left[I\big((s_n)_\delta\big) + \sum_{i=2}^{k_n} J_\delta\big((s_n)_i^\delta\big) \right] \\
&= I(\tilde{w}_{2,n}) - J_\delta\big((s_n)_2^\delta\big) < 0
\end{aligned}$$

that contradicts (44). Thus (42) holds true and the proof is complete. $\square$

Proposition 17. *Let $\bar{u} \in H_{loc}^1(\mathbb{R}^N)$ and $(\bar{x}_n)_n$ be as in Proposition 16. Then*

$$\lim_{n\to+\infty} \bar{u}(x + \bar{x}_n) = w(x) \tag{45}$$

uniformly on all compact sets $K \subset \mathbb{R}^N$.

Proof. Let us fix $\bar{r} > \rho$. Then, by definition of $\bar{u}$, for all n, a $k_n \in \mathbb{N}$ exists so that

$$\sup_{B_{\bar{r}}(\bar{x}_n)} |\bar{u}_{k_n} - \bar{u}| < 1/n;$$

with $\bar{u}_{k_n}$ solution of (E) emerging around k_n points $(\bar{x}_1^n, \dots, \bar{x}_{k_n}^n)$. As a consequence, there exists a point around which $\bar{u}_{k_n}$ is emerging and whose distance from $\bar{x}_n$ is infinitesimal as $n \to +\infty$. Without any loss of generality, we assume

$$\lim_{n\to+\infty} |\bar{x}_1^n - \bar{x}_n| = 0. \tag{46}$$

Moreover, from (35), the relations

$$\lim_{n\to+\infty} |\bar{x}_1^n| = +\infty, \qquad \lim_{n\to+\infty} \min\big\{ |\bar{x}_1^n - \bar{x}_j^n| : j = 2, \dots, k_n \big\} = +\infty,$$

follow. Thus, setting $d_n = \min\{|\bar{x}_1^n - \bar{x}_j^n| : j = 2, \dots, k_n\}$, we define $\xi_n(x) := \xi(|x - \bar{x}_1^n| - \frac{d_n}{2})$ where $\xi \in C^\infty(\mathbb{R}, [0,1])$ is such that $\xi(t) = 0$ if $|t| \leqslant 1/2$ and $\xi(t) = 1$ if $|t| \geqslant 1$. Since $d_n \xrightarrow[n\to+\infty]{} +\infty$, for large n, we have

$$\xi_n(x)\bar{u}_{k_n} = \hat{u}_n(x) + \check{u}_n(x),$$

where $\hat{u}_n(x) \in S_{\bar{x}_1^n}$, $\check{u}_n(x) \in S_{\bar{x}_2^n,\dots,\bar{x}_{k_n}^n}$, and

$$\operatorname{supp} \hat{u}_n \subset B_{\frac{d_n}{2}-\frac12}\big(\bar{x}_1^n\big), \qquad \operatorname{supp} \check{u}_n \subset \mathbb{R}^N \setminus B_{\frac{d_n}{2}+\frac12}\big(\bar{x}_1^n\big).$$

Then computations analogous to (19) and (30) bring to

$$I(\hat{u}_n) + I(\check{u}_n) = I(\xi_n\bar{u}_{k_n}) \leqslant I(\bar{u}_{k_n}) + O\big(e^{-\bar{\eta}d_n}\big). \tag{47}$$

Now, considering $\bar{z}_n \in M_{\bar{x}_1^n}$ and $z_n = \zeta(|x - \bar{x}_1^n| - \frac{d_n}{2})\bar{z}_n(x)$, with $\zeta \in C^\infty(\mathbb{R}, [0,1])$ such that $\zeta(t) = 1$ if $|t| \leqslant 1/4$, $\zeta(t) = 0$ if $|t| \geqslant 1/2$ we get

$$z_n \in S_{\bar{x}_1^n}, \qquad z_n + \check{u}_n \in S_{\bar{x}_1^n, \dots, \bar{x}_{k_n}^n};$$

$$I(\bar{u}_{k_n}) \leqslant I(z_n) + I(\check{u}_n) \tag{48}$$

and, making again computations analogous to those in (30) and using Lemma 11,

$$I(z_n) = I(\bar{z}_n) + O\left(e^{-\bar{\eta} d_n}\right) \leqslant m_\infty + o(1). \tag{49}$$

Thus, combining (47), (48) and (49) we deduce

$$I(\hat{u}_n) \leqslant I(\bar{u}_{k_n}) - I(\check{u}_n) + o(1) \leqslant I(z_n) + o(1) \leqslant m_\infty + o(1). \tag{50}$$

On the other hand, denoting by $\tilde{u}_n$ the projection of $\hat{u}_n$ on the constraint $(I')^\infty(u)[u^\delta] = 0$, $\tilde{u}_n(x) = (\hat{u}_n)_\delta + \vartheta^\infty(\hat{u}_n)(\hat{u}_n)^\delta$, $\tilde{u}_n$ belongs to $S_{\bar{x}_1^n}^\infty$ and by arguments quite analogous to those used to prove (22) (considering $\bar{x}_1^n$ instead of y_n and choosing $\sigma_n = |\bar{x}_1^n|$) we obtain

$$I(\hat{u}_n) \geqslant I(\tilde{u}_n) \geqslant m_\infty + o(1). \tag{51}$$

(50) and (51) imply

$$\lim_{n \to +\infty} I(\hat{u}_n) = m_\infty, \tag{52}$$

that, used in (51), gives

$$\lim_{n \to +\infty} I(\tilde{u}_n) = m_\infty. \tag{53}$$

Now, we can write (52) as

$$m_\infty + o(1) = I(\hat{u}_n) = I\left((\hat{u}_n)_\delta\right) + J_\delta\left((\hat{u}_n)^\delta\right)$$

from which, using the coercivity of I on $(\hat{u}_n)_\delta$ and the relation $J_\delta((\hat{u}_n)^\delta) > 0$, we deduce at once that $\|(\hat{u}_n)_\delta\|$ is bounded.

To show that $\|(\hat{u}_n)^\delta\|$ is bounded too, we argue by contradiction and we assume that, up to a subsequence, $\|(\hat{u}_n)^\delta\| \xrightarrow[n \to +\infty]{} +\infty$. Then, we have

$$m_\infty + o(1) \geqslant J_\delta\left((\hat{u}_n)^\delta\right) = \max_{t > 0} J_\delta\left(t(\hat{u}_n)^\delta\right)$$

$$= \max_{t > 0} J_\delta\left(t \frac{(\hat{u}_n)^\delta}{|(\hat{u}_n)^\delta|_{p+1}}\right) \geqslant J_\delta\left(\frac{(\hat{u}_n)^\delta}{|(\hat{u}_n)^\delta|_{p+1}}\right) \geqslant c_1 \frac{(\|\hat{u}_n\|^2)^\delta}{|(\hat{u}_n)^\delta|_{p+1}} - c_2,$$

where $c_1 > 0$ and $c_2 > 0$, that implies

$$\|\hat{u}_n\| \leqslant C_1 |(\hat{u}_n)^\delta|_{p+1}, \quad C_1 > 0.$$

Since, by Sobolev embedding, it is also true the relation

$$|(\hat{u}_n)^\delta|_{p+1} \leqslant C_2 \|\hat{u}_n\|, \quad C_2 > 0$$

where C_2 is independent of n because $\operatorname{supp}(\hat{u}_n)^\delta \subset B_\rho(\bar{x}_1^n)$, we obtain

$$J_\delta\left((\hat{u}_n)^\delta\right) \leqslant c_3 \|\hat{u}_n\|^2 - c_4 |(\hat{u}_n)^\delta|_{p+1}^{p+1} + c_5 \xrightarrow[n \to +\infty]{} -\infty$$

that is impossible because $J_\delta((\hat{u}_n)^\delta) > 0$.

Therefore, $\|(\hat{u}_n)^\delta\|$ is bounded. Consequently the boundedness of $\vartheta^\infty(\hat{u}_n)$ can be easily proved, hence we deduce $\|(\tilde{u}_n)\|$ is bounded and, using (53), $\vartheta(\tilde{u}_n) \xrightarrow[n \to +\infty]{} 1$. We remark also that $\hat{u}_n(x) = \bar{u}_{k_n}$ in $B_{\frac{d_n}{2}-1}(\bar{x}_1^n)$, thus, in this set it is a solution of $-\Delta u + a(x)u = u^p$.

Let now $R > 0$ be a number arbitrarily fixed. Since $d_n \xrightarrow[n \to +\infty]{} +\infty$, we can assert that

$$\bar{u}_{k_n}\left(x + \bar{x}_1^n\right) = \hat{u}_n\left(x + \bar{x}_1^n\right) \rightharpoonup w(x) \quad \text{in } H^1\left(B_R(0)\right).$$

Thus, by a bootstrap argument, we deduce $\bar{u}_{k_n}(\cdot + \bar{x}_1^n) \in C^{1+\epsilon}(K)$, and $(\bar{u}_{k_n}(\cdot + \bar{x}_1^n))_n$ is bounded in $C^{1+\epsilon}(K)$ for some $\epsilon > 0$ and for all compact sets $K \subset B_R(0)$. As a consequence, in view of (46), we get the desired conclusion. $\quad\square$

References

[1] A. Bahri, Y.Y. Li, On a min–max procedure for the existence of a positive solution for certain scalar field equations in R^N, Rev. Mat. Iberoam. 6 (1990) 1–15.

[2] A. Bahri, P.L. Lions, On the existence of a positive solution of semilinear elliptic equations in unbounded domains, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 14 (1997) 365–413.

[3] H. Brezis, Analyse fonctionelle. Theorie et applications, Collect. Math. Appl. Maîtrise, Masson, Paris, 1983.

[4] G. Cerami, Some nonlinear elliptic problems in unbounded domains, Milan J. Math. 74 (2006) 47–77.

[5] G. Cerami, G. Devillanova, S. Solimini, Infinitely many bound states for some nonlinear scalar field equations, Calc. Var. Partial Differ. Equ. 23 (2005) 139–168.

[6] G. Cerami, D. Passaseo, S. Solimini, Infinitely many positive solutions to some scalar field equations with nonsymmetric coefficients, Commun. Pure Appl. Math. 66 (2013) 372–413.

[7] P.L. Lions, The concentration compactness principle in the calculus of variations, Parts I and II, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 1 (1984) 109–145, 223–283.

[8] J. Wei, S. Yan, Infinitely many positive solutions for the nonlinear Schrödinger equations in $\mathbb{R}^N$, Calc. Var. Partial Differ. Equ. 37 (2010) 423–439.