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On the radius of analyticity of solutions to the cubic Szegő equation

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Abstract

This paper is concerned with the cubic Szegő equation

$$i\partial_t u = \Pi\left(|u|^2 u\right),$$

defined on the L^2 Hardy space on the one-dimensional torus $\mathbb{T}$, where $\Pi : L^2(\mathbb{T}) \to L^2_+(\mathbb{T})$ is the Szegő projector onto the non-negative frequencies. For analytic initial data, it is shown that the solution remains spatial analytic for all time $t \in (-\infty, \infty)$. In addition, we find a lower bound for the radius of analyticity of the solution. Our method involves energy-like estimates of the special Gevrey class of analytic functions based on the ℓ^1 norm of Fourier transforms (the Wiener algebra).

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1. Introduction

In studying the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = \pm|u|^2 u, \quad (t, x) \in \mathbb{R} \times M,$$

Burq, Gérard and Tzvetkov [2] observed that dispersion properties are strongly influenced by the geometry of the underlying manifold M. In [6], Gérard and Grellier mentioned, if there exists a smooth local in time flow map on the Sobolev space $H^s(M)$, then the following Strichartz-type estimate must hold:

$$\left\|e^{it\Delta} f\right\|_{L^4([0,1]\times M)} \lesssim \|f\|_{H^{s/2}(M)}. \tag{1.1}$$

It is shown in [1,2] that, on the two-dimensional sphere, the infimum of the number s such that (1.1) holds is $\frac{1}{4}$; however, if $M = \mathbb{R}^2$, the inequality (1.1) is valid for $s = 0$. As pointed out in [6], this can be interpreted as a lack

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of dispersion properties for the spherical geometry. Taking this idea further, it is remarked in [6] that dispersion disappears completely when M is a sub-Riemannian manifold (for instance, the Heisenberg group).

As a toy model to study non-dispersive Hamiltonian equation, Gérard and Grellier [6] introduced the *cubic Szegő equation*:

$$i\partial_t u = \Pi\big(|u|^2 u\big), \quad (t,\theta) \in \mathbb{R} \times \mathbb{T}, \tag{1.2}$$

on $L^2_+(\mathbb{T})$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the one-dimensional torus, which is identical to the unit circle in the complex plane. Notice that $L^2_+(\mathbb{T})$ is the L^2 Hardy space which is defined by

$$L^2_+(\mathbb{T}) = \left\{ u = \sum_{k\in\mathbb{Z}} \hat{u}(k)e^{ik\theta} \in L^2(\mathbb{T}):\ \hat{u}(k) = 0 \text{ for all } k < 0 \right\}.$$

Furthermore, in (1.2), the operator $\Pi : L^2(\mathbb{T}) \to L^2_+(\mathbb{T})$ is the Szegő projector onto the non-negative frequencies, i.e.,

$$\Pi\left(\sum_{k\in\mathbb{Z}} v_k e^{ik\theta}\right) = \sum_{k\geqslant 0} v_k e^{ik\theta}.$$

We mention the following existence result, proved in [6].

Theorem 1.1. *(See [6].) Given $u_0 \in H^s_+(\mathbb{T})$, for some $s \geqslant \frac{1}{2}$, then the cubic Szegő equation (1.2) has a unique solution $u \in C(\mathbb{R}, H^s_+(\mathbb{T}))$.*

Moreover, it has been shown in [6] that the Szegő equation (1.2) is completely integrable in the sense of admitting a Lax pair structure, and as a consequence, it possesses an infinite number of conservation laws. Some other interesting work concerning the Szegő equation can be found in [7,8,14,19–21].

Replacing the Fourier series by the Fourier transform, one can analogously define the Szegő equation on

$$L^2_+(\mathbb{R}) = \big\{\phi \in L^2(\mathbb{R}):\ \operatorname{supp}\hat{\phi} \subset [0,\infty)\big\}.$$

In [19], Pocovnicu constructed explicit spatially real analytic solutions for the cubic Szegő equation defined on $L^2_+(\mathbb{R})$. For the initial datum $u_0 = \frac{2}{x+i} - \frac{4}{x+2i}$, it was discovered that one of the poles of the explicit real analytic solution $u(t,x)$ approaches the real line, as $|t| \to \infty$; more precisely, the imaginary part of a pole decreases in the speed $O(\frac{1}{t^2})$. Thus, the radius of analyticity of $u(t,x)$ shrinks algebraically to zero, as $|t| \to \infty$. This phenomenon gives rise to the following questions: for analytic initial data, does the solution remain spatial analytic for all time? If so, can one estimate, from below, the radius of analyticity? In this manuscript, we attempt to answer these questions by employing the technique of the so-called Gevrey class of analytic functions.

The Gevrey classes of real analytic functions are characterized by an exponential decay of their Fourier coefficients. If we set $A := \sqrt{I - \Delta}$, they are defined by $\mathcal{D}(A^s e^{\sigma A})$, which consist of all L^2 functions u such that $\|A^s e^{\sigma A} u\|_{L^2(\mathbb{T})}$ is finite, where $s \geqslant 0$, $\sigma > 0$ (see e.g. [4,5,13]). Note, if $\sigma = 0$, then $\mathcal{D}(A^s e^{\sigma A}) = \mathcal{D}(A^s) \cong H^s(\mathbb{T})$. However, if $\sigma > 0$, then $\mathcal{D}(A^s e^{\sigma A})$ is the set of real analytic functions with the radius of analyticity bounded below by σ. Also notice, $\mathcal{D}(A^s e^{\sigma A})$ is a Banach algebra provided $s > \frac{1}{2}$ for 1D (see [4]).

The so-called method of Gevrey estimates has been extensively used in literature to establish regularity results for nonlinear evolution equations. It was first introduced for the periodic Navier–Stokes equations in [5], and studied later in the whole space in [15], moreover, it was extended to nonlinear analytic parabolic PDE's in [4], and for Euler equations in [10,12,13] (see also references therein). Recently, this method was also applied to establish analytic solutions for nonlinear wave equations [9].

In this paper, we employ a special such class based on the space W of functions with summable Fourier series. For a given function $u \in L^1(\mathbb{T})$, $u = \sum_{k\in\mathbb{Z}} \hat{u}(k)e^{ik\theta}$, $\theta \in \mathbb{T}$, then the Wiener norm of u is given by

$$\|u\|_W = \|\hat{u}\|_{\ell^1} = \sum_{k\in\mathbb{Z}} \big|\hat{u}(k)\big|. \tag{1.3}$$

Notice that W is a Banach algebra (Wiener algebra).

Based on the Wiener algebra, the following special Gevrey norm is defined in [16]:

$$\|u\|_{G_\sigma(W)} = \sum_{k\in\mathbb{Z}} e^{\sigma|k|}\left|\hat{u}(k)\right|, \quad \sigma \geqslant 0. \tag{1.4}$$

If $u \in L^1(\mathbb{T})$ is such that $\|u\|_{G_\sigma(W)} < \infty$, then we write $u \in G_\sigma(W)$.

It is known that the Gevrey class $G_\sigma(W)$ is a Banach algebra [16], and it characterizes the real analytic functions if $\sigma > 0$. In particular, a function $u \in C^\infty(\mathbb{T})$ is real analytic with uniform radius of analyticity ρ, if and only if, $u \in G_\sigma(W)$, for every $0 < \sigma < \rho$.

Now, we state the main result of this paper.

Theorem 1.2. *Assume $u_0 \in L^2_+(\mathbb{T}) \cap G_\sigma(W)$, for some $\sigma > 0$. Then the unique solution $u(t)$ of (1.2) provided by Theorem 1.1 satisfies $u(t) \in G_{\tau(t)}(W)$, for all $t \in \mathbb{R}$, where $\tau(t) = \sigma e^{-\lambda|t|}$, with some $\lambda > 0$ depending on u_0. More precisely, there exists $C_0 > 0$, specified in (2.9) below, such that $\|u(t)\|_{G_{\tau(t)}(W)} \leqslant C_0$, for all $t \in \mathbb{R}$.*

Essentially, Theorem 1.2 shows the persistency of the spatial analyticity of the solution $u(t)$ for all time $t \in (-\infty, \infty)$ provided the initial datum is analytic. Recall that $\tau(t)$ is a lower bound of the radius of spatial analyticity of $u(t)$. Thus, it implies that the radius of analyticity of $u(t)$ cannot shrink faster than exponentially, as $|t| \to \infty$.

Remark 1.3. The precise definition of λ in Theorem 1.2 is given in (2.23) below. In fact, as shown in Remark 2.2, one can prove that the radius $\rho(t)$ of real analyticity of $u(t)$ satisfies, for every $s > 1$,

$$\limsup_{t\to\infty}\left|\frac{\log\rho(t)}{t}\right| \leqslant K_s\|u_0\|^2_{H^s},$$

which is independent of the $G_\sigma(W)$ norm of u_0. The optimality of such an estimate is not known. However, let us mention the following two recent results in [8]. Firstly, if u_0 is a rational function of $e^{i\theta}$ with no poles in the closed unit disc, then so is $u(t)$, and $\rho(t)$ remains bounded from below by some positive constant for all time. Secondly, this bound is by no means uniform. Indeed, starting with

$$u_0 = e^{i\theta} + \varepsilon, \quad \varepsilon > 0,$$

one can show that

$$\rho\left(\frac{\pi}{\varepsilon}\right) = O\left(\varepsilon^2\right).$$

This phenomenon is to be compared to the one displayed by Kuksin in [11] for NLS on the torus with small dispersion coefficient.

Finally, let us mention a recent work by Haiyan Xu [21], who found a Hamiltonian perturbation of the cubic Szegő equation which admits solutions with exponentially shrinking radius of analyticity. Moreover, one can check that the method of Theorem 1.2 applies as well to this perturbation, so that the above result is optimal in the case of this equation.

By investigating the steady state of the cubic nonlinear Schrödinger equation, it is demonstrated in [16] that, by employing the Gevrey class $G_\sigma(W)$, one can obtain a more accurate estimate of the lower bound of the radius of analyticity of solutions to differential equations, compared to the estimate derived from using the regular Gevrey classes $\mathcal{D}(A^s e^{\sigma A})$ (see also the discussion in [9]). Such observation is verified again in this paper, since we find that, in studying the cubic Szegő equation, the Gevrey class method, based on $G_\sigma(W)$, provides an estimate of the lower bound of the analyticity radius of the solution, which has a substantially slower shrinking rate, than the estimate obtained from using the classes $\mathcal{D}(A^s e^{\sigma A})$. One may refer to Remark 2.4 for this comparison.

Throughout, we study the cubic Szegő equation defined on the torus $\mathbb{T}$. However, by using Fourier transforms instead of Fourier series, our techniques are also applicable to the same equation defined on the real line, and similar regularity results and estimates can be obtained as well (see also ideas from [15]).

Moreover, Theorem 1.2 is also valid under the framework of general Gevrey classes, i.e., intermediate spaces between the space of C^∞ functions and real analytic functions. Indeed, if we define Gevrey classes $G_\sigma^\gamma(W)$ based on the norm

$$\|u\|_{G_\sigma^\gamma(W)} = \sum_{k\in\mathbb{Z}} e^{\sigma|k|^\gamma}\left|\hat{u}(k)\right|, \quad \gamma \in (0,1],$$

then, $G_\sigma^\gamma(W)$ are Banach algebras, due to the elementary inequality $e^{\sigma(k+j)^\gamma} \leqslant e^{\sigma k^\gamma} e^{\sigma j^\gamma}$, for $\gamma \in (0,1]$. Thus, the proof of Theorem 1.2 works equally for $G_\sigma^\gamma(W)$, where $\gamma \in (0,1]$. For the sake of clarity, we demonstrate our technique for $\gamma = 1$, i.e., the Gevrey class of real analytic functions.

2. Proof of the main result

Before we start the proof of the main result, the following proposition should be mentioned.

Proposition 2.1. *Assume $u_0 \in H_+^s(\mathbb{T})$, for some $s > 1$. Let u be the unique global solution of (1.2), furnished by Theorem 1.1. Then,*

$$\left\|u(t)\right\|_W \leqslant C(s)\|u_0\|_{H^s}, \quad \text{for all } t \in \mathbb{R}. \tag{2.1}$$

Proof. The proof can be found in [6], we recall it here. First we define the Hankel operator $H_u : L_+^2(\mathbb{T}) \to L_+^2(\mathbb{T})$ of symbol $u \in H_+^{1/2}(\mathbb{T})$ by

$$H_u(h) = \Pi(u\bar{h}). \tag{2.2}$$

Also, the Toeplitz operator $T_b : L_+^2(\mathbb{T}) \to L_+^2(\mathbb{T})$ of symbol $b \in L^\infty(\mathbb{T})$ is defined by

$$T_b(h) = \Pi(bh).$$

In [6], it has been shown that the cubic Szegő equation admits a Lax pair (H_u, B_u), where

$$B_u = \frac{i}{2}H_u^2 - iT_{|u|^2}.$$

Thus the trace norm $Tr(|H_{u(t)}|)$ is a conserved quantity. By Peller's theorem [17,18], $Tr(|H_u|)$ is equivalent to the $B_{1,1}^1$ norm of u. In particular, for every $s > 1$,

$$\frac{1}{2}\|u\|_W \leqslant Tr\left(|H_u|\right) \leqslant C_s\|u\|_{H^s}. \tag{2.3}$$

Hence

$$\left\|u(t)\right\|_W \leqslant 2Tr\left(|H_{u(t)}|\right) = 2Tr\left(|H_{u_0}|\right) \leqslant 2C_s\|u_0\|_{H^s}.$$

The proof is complete. □

For the sake of completion, we provide a straightforward proof of (2.3) in Appendix A. We now start the proof of Theorem 1.2.

Proof. Due to the assumption on the initial datum u_0, we know that u_0 is real analytic, and hence $u_0 \in H_+^s(\mathbb{T})$, for every non-negative real number s, in particular for $s \geqslant \frac{1}{2}$. Therefore, the global existence and uniqueness of the solution $u \in C(\mathbb{R}, H_+^s(\mathbb{T}))$ are guaranteed by Theorem 1.1, for $s \geqslant \frac{1}{2}$.

Throughout, we focus on the positive time $t \geqslant 0$. By replacing t by $-t$, the same proof works for the negative time.

We shall implement the Galerkin approximation method. Recall the cubic Szegő equation is defined on the Hardy space $L_+^2(\mathbb{T})$ with a natural basis $\{e^{ik\theta}\}_{k\geqslant 0}$. Denote by P_N the projection onto the span of $\{e^{ik\theta}\}_{0\leqslant k\leqslant N}$. We let

$$u_N(t) = \sum_{k=0}^{N} \hat{u}_N(t,k)e^{ik\theta} \tag{2.4}$$

be the solution of the Galerkin system:

$$i\partial_t u_N = P_N\big(|u_N|^2 u_N\big), \tag{2.5}$$

with the initial condition $u_N(0) = P_N u_0$. We see that (2.5) is an N-dimensional system of ODE with the conservation law

$$\|u_N(t)\|_{L^2}^2 = \sum_{k=0}^{N} \big|\hat{u}_N(t,k)\big|^2,$$

and thus it has a unique solution $u_N \in C^\infty(\mathbb{R})$ on $\mathbb{R}$.

Furthermore, we observe that

$$\sum_{k=0}^{N} k\big|\hat{u}_N(t,k)\big|^2$$

is also conserved, hence $\|u_N(t)\|_{H^{1/2}}$ is a conservation law. Consequently, arguing exactly as in Section 2 of [6] with Brezis–Gallouët type estimates, we obtain that for every $s \geqslant \frac{1}{2}$ and every $T > 0$,

$$\sup_N \sup_{t\in[0,T]} \big\|u_N(t)\big\|_{H^s} < \infty.$$

By using Eq. (2.5), one concludes that the same estimate holds for the time derivative $u_N'(t)$. Now, let us fix an arbitrary $T > 0$. Since, moreover, the injection of $H^{s+\varepsilon}$ into H^s is compact, we conclude from Ascoli's theorem that, up to a subsequence, $u_N(t)$ converge to some $\tilde{u}(t)$ in every H^s, uniformly for $t \in [0,T]$. Then, it is straightforward to check, by letting $N \to \infty$, that $\tilde{u}$ is a solution of the cubic Szegő equation (1.2) on $[0,T]$ with the initial datum u_0. Since u is the unique global solution furnished by Theorem 1.1, one must have $u = \tilde{u}$ on $[0,T]$. Since H^s is contained into W for every $s > \frac{1}{2}$, $u_N(t)$ tends to $u(t)$ in W uniformly for $t \in [0,T]$. By Proposition 2.1, there exists a constant $C_1 > 0$ such that

$$\big\|u(t)\big\|_W + 1 \leqslant C_1, \quad \text{for all } t \in \mathbb{R}. \tag{2.6}$$

Consequently, there exists $N' \in \mathbb{N}$ such that

$$\big\|u_N(t)\big\|_W \leqslant \big\|u(t)\big\|_W + 1 \leqslant C_1, \quad \text{for all } N > N', \quad t \in [0,T]. \tag{2.7}$$

Also, recall that the initial condition $u_0 \in G_\sigma(W)$, i.e., $\|u_0\|_{G_\sigma(W)} < \infty$. Since $u_N(0) = P_N u_0$, one has

$$\sum_{k=0}^{N} e^{\sigma k}\big|\hat{u}_N(0,k)\big| \leqslant \|u_0\|_{G_\sigma(W)}. \tag{2.8}$$

Define

$$C_0 := \max\left\{\|u_0\|_{G_\sigma(W)}, \frac{1+\sqrt{5}}{2}eC_1\right\}, \tag{2.9}$$

where C_1 has been specified in (2.6).

Let us fix an arbitrary $N > N'$. We aim to prove

$$\sum_{k=0}^{N} e^{\tau(t)k}\big|\hat{u}_N(t,k)\big| \leqslant C_0, \quad \text{for all } t \in [0,T], \tag{2.10}$$

with $\tau(t) > 0$ that will be specified in (2.23), below.

Notice, due to (2.4) and (2.5), we infer

$$\frac{d}{dt}\hat{u}_N(t,k) = -i \sum_{\substack{n-j+m=k\\ 0\leqslant n,j,m\leqslant N}} \hat{u}_N(t,n)\overline{\hat{u}_N(t,j)}\hat{u}_N(t,m), \quad t \in [0,T],\ k = 0,1,\dots,N.$$

Then, one can easily find that

$$\frac{d}{dt}\left|\hat{u}_N(t,k)\right| \leqslant \sum_{\substack{n-j+m=k\\0\leqslant n,j,m\leqslant N}} \left|\hat{u}_N(t,n)\right|\left|\hat{u}_N(t,j)\right|\left|\hat{u}_N(t,m)\right|, \tag{2.11}$$

for $k=0,1,\dots,N$, and all $t\in[0,T]$.

In order to estimate the Gevrey norm, we consider

$$\begin{aligned}\frac{d}{dt}\left(e^{\tau(t)k}\left|\hat{u}_N(t,k)\right|\right) &= \tau'(t)ke^{\tau(t)k}\left|\hat{u}_N(t,k)\right| + e^{\tau(t)k}\frac{d}{dt}\left|\hat{u}_N(t,k)\right|\\ &\leqslant \tau'(t)ke^{\tau(t)k}\left|\hat{u}_N(t,k)\right| + e^{\tau(t)k}\sum_{\substack{n-j+m=k\\0\leqslant n,j,m\leqslant N}} \left|\hat{u}_N(t,n)\right|\left|\hat{u}_N(t,j)\right|\left|\hat{u}_N(t,m)\right|,\end{aligned}$$

for $k=0,1,\dots,N$, and $t\in[0,T]$, where (2.11) has been used in the last inequality.

Summing over all integers $k=0,1,\dots,N$ yields

$$\begin{aligned}&\frac{d}{dt}\left(\sum_{k=0}^{N} e^{\tau(t)k}\left|\hat{u}_N(t,k)\right|\right)\\
&\leqslant \tau'(t)\sum_{k=0}^{N} ke^{\tau(t)k}\left|\hat{u}_N(t,k)\right| + \sum_{k=0}^{N} e^{\tau(t)k}\left(\sum_{\substack{n-j+m=k\\0\leqslant n,j,m\leqslant N}} \left|\hat{u}_N(t,n)\right|\left|\hat{u}_N(t,j)\right|\left|\hat{u}_N(t,m)\right|\right)\\
&= \tau'(t)\sum_{k=0}^{N} ke^{\tau(t)k}\left|\hat{u}_N(t,k)\right| + \sum_{k=0}^{N}\left(\sum_{\substack{n-j+m=k\\0\leqslant n,j,m\leqslant N}} e^{\tau(t)n}\left|\hat{u}_N(t,n)\right|e^{-\tau(t)j}\left|\hat{u}_N(t,j)\right|e^{\tau(t)m}\left|\hat{u}_N(t,m)\right|\right)\\
&\leqslant \tau'(t)\sum_{k=0}^{N} ke^{\tau(t)k}\left|\hat{u}_N(t,k)\right| + \left(\sum_{k=0}^{N} e^{\tau(t)k}\left|\hat{u}_N(t,k)\right|\right)^2\left(\sum_{k=0}^{N}\left|\hat{u}_N(t,k)\right|\right),\end{aligned} \tag{2.12}$$

where the last formula is obtained by using the Young's convolution inequality and the fact $e^{-\tau j}\leqslant 1$, for $\tau,\ j\geqslant 0$.

Now, we estimate the second term on the right-hand side of (2.12). The key ingredient of the calculation is the elementary inequality $e^x\leqslant e+x^\ell e^x$, for all $x\geqslant 0$, $\ell\geqslant 0$, and we select $\ell=\frac{1}{2}$ here. Hence

$$\begin{aligned}&\left(\sum_{k=0}^{N} e^{\tau(t)k}\left|\hat{u}_N(t,k)\right|\right)^2\left(\sum_{k=0}^{N}\left|\hat{u}_N(t,k)\right|\right)\\
&\leqslant \left(\sum_{k=0}^{N} e\left|\hat{u}_N(t,k)\right| + \sum_{k=0}^{N}\tau^{\frac{1}{2}}(t)k^{\frac{1}{2}}e^{\tau(t)k}\left|\hat{u}_N(t,k)\right|\right)^2\left(\sum_{k=0}^{N}\left|\hat{u}_N(t,k)\right|\right)\\
&\leqslant 2e^2\left(\sum_{k=0}^{N}\left|\hat{u}_N(t,k)\right|\right)^3 + 2\tau(t)\left(\sum_{k=0}^{N} ke^{\tau(t)k}\left|\hat{u}_N(t,k)\right|\right)\left(\sum_{k=0}^{N} e^{\tau(t)k}\left|\hat{u}_N(t,k)\right|\right)\left(\sum_{k=0}^{N}\left|\hat{u}_N(t,k)\right|\right),\end{aligned} \tag{2.13}$$

where we have used Young's inequality and Hölder's inequality.

Thus, combining (2.12) and (2.13) yields

$$\begin{aligned}\frac{d}{dt}\left(\sum_{k=0}^{N} e^{\tau(t)k}\left|\hat{u}_N(t,k)\right|\right) &\leqslant \tau'(t)\sum_{k=0}^{N} ke^{\tau(t)k}\left|\hat{u}_N(t,k)\right| + 2e^2\left(\sum_{k=0}^{N}\left|\hat{u}_N(t,k)\right|\right)^3\\
&\quad + 2\tau(t)\left(\sum_{k=0}^{N} ke^{\tau(t)k}\left|\hat{u}_N(t,k)\right|\right)\left(\sum_{k=0}^{N} e^{\tau(t)k}\left|\hat{u}_N(t,k)\right|\right)\left(\sum_{k=0}^{N}\left|\hat{u}_N(t,k)\right|\right)\end{aligned}$$

$$\leqslant \frac{1}{2}\tau'(t)\sum_{k=0}^{N} ke^{\tau(t)k}\left|\hat{u}_N(t,k)\right| + 2e^2C_1^3$$
$$+ \left(\frac{1}{2}\tau'(t) + 2C_1\tau(t)\sum_{k=0}^{N} e^{\tau(t)k}\left|\hat{u}_N(t,k)\right|\right)\left(\sum_{k=0}^{N} ke^{\tau(t)k}\left|\hat{u}_N(t,k)\right|\right), \tag{2.14}$$

for all $t \in [0, T]$, where we have used (2.7).

Denote by $\tau_N(t)$, $t \in [0, t_N]$, the unique solution of the ODE

$$\frac{1}{2}\tau_N'(t) + 2C_1\tau_N(t)z_N(t) = 0, \quad \text{with } \tau_N(0) = \sigma, \tag{2.15}$$

where we set

$$z_N(t) := \sum_{k=0}^{N} e^{\tau_N(t)k}\left|\hat{u}_N(t,k)\right|. \tag{2.16}$$

Due to (2.15) and (2.16), we infer from (2.14) that

$$\frac{dz_N}{dt}(t) \leqslant \frac{1}{2}\tau_N'(t)\sum_{k=0}^{N} ke^{\tau_N(t)k}\left|\hat{u}_N(t,k)\right| + 2e^2C_1^3$$
$$\leqslant -2C_1z_N(t)\tau_N(t)\sum_{k=0}^{N} ke^{\tau_N(t)k}\left|\hat{u}_N(t,k)\right| + 2e^2C_1^3, \quad t \in [0, t_N]. \tag{2.17}$$

Next, we estimate $\tau_N(t)\sum_{k=0}^{N} ke^{\tau_N(t)k}|\hat{u}_N(t,k)|$ by considering the following two cases:

Case 1: $N \geqslant \frac{1}{\tau_N(t)}$. In this case, one has

$$\tau_N(t)\sum_{k=0}^{N} ke^{\tau_N(t)k}\left|\hat{u}_N(t,k)\right| \geqslant \tau_N(t)\sum_{\frac{1}{\tau_N(t)}\leqslant k\leqslant N} ke^{\tau_N(t)k}\left|\hat{u}_N(t,k)\right| \geqslant \sum_{\frac{1}{\tau_N(t)}\leqslant k\leqslant N} e^{\tau_N(t)k}\left|\hat{u}_N(t,k)\right|$$
$$= \sum_{k=0}^{N} e^{\tau_N(t)k}\left|\hat{u}_N(t,k)\right| - \sum_{0\leqslant k<\frac{1}{\tau_N(t)}} e^{\tau_N(t)k}\left|\hat{u}_N(t,k)\right|$$
$$\geqslant z_N(t) - e\sum_{0\leqslant k<\frac{1}{\tau_N(t)}}\left|\hat{u}_N(t,k)\right| \geqslant z_N(t) - eC_1, \tag{2.18}$$

where the fact (2.7) has been used.

Case 2: $N < \frac{1}{\tau_N(t)}$. In this case, in order to obtain the same estimate as (2.18), we proceed as follows:

$$\tau_N(t)\sum_{k=0}^{N} ke^{\tau_N(t)k}\left|\hat{u}_N(t,k)\right| \geqslant 0 = z_N(t) - \sum_{k=0}^{N} e^{\tau_N(t)k}\left|\hat{u}_N(t,k)\right| \geqslant z_N(t) - e\sum_{k=0}^{N}\left|\hat{u}_N(t,k)\right| \geqslant z_N(t) - eC_1.$$

We conclude from the above two cases that

$$\tau_N(t)\sum_{k=0}^{N} ke^{\tau_N(t)k}\left|\hat{u}_N(t,k)\right| \geqslant z_N(t) - eC_1,$$

and by substituting it into (2.17), one has

$$\frac{dz_N}{dt}(t) \leqslant -2C_1z_N^2(t) + 2eC_1^2z_N(t) + 2e^2C_1^3, \quad \text{for all } t \in [0, t_N]. \tag{2.19}$$

Notice that the right-hand side of (2.19) is negative when $z_N > z^* = \frac{1+\sqrt{5}}{2} e C_1$, and hence (2.19) implies that

$$z_N(t) \leqslant \max\{z_N(0), z^*\} = \max\left\{ \sum_{k=0}^{N} e^{\sigma k} |\hat{u}_N(0,k)|, \frac{1+\sqrt{5}}{2} e C_1 \right\} \leqslant C_0, \tag{2.20}$$

for all $t \in [0, t_N]$, where we have also used (2.8) and (2.9) in the above estimate. Therefore, by virtue of the uniform bound (2.20) of $z_N(t)$, the solution $\tau_N(t)$ of the initial value problem (2.15) on $[0, t_N]$ can be extended to the solution on $[0, T]$, and thus (2.20) holds for all $t \in [0, T]$, i.e.,

$$z_N(t) \leqslant C_0, \quad \text{for all } t \in [0, T], \tag{2.21}$$

and along with (2.15), we infer

$$\tau_N(t) = \sigma \exp\left(-4C_1 \int_0^t z_N(s)\, ds \right) \geqslant \sigma e^{-4C_0 C_1 t}, \quad \text{for all } t \in [0, T]. \tag{2.22}$$

Let us define

$$\tau(t) = \sigma e^{-\lambda |t|}, \quad \text{with } \lambda = 4C_0 C_1, \tag{2.23}$$

where C_0 and C_1 are specified in (2.9) and (2.6), respectively. Then, (2.22) and (2.23) show that $\tau(t) \leqslant \tau_N(t)$ on $[0, T]$, and consequently,

$$\|u_N(t)\|_{G_{\tau(t)}(W)} = \sum_{k=0}^{N} e^{\tau(t)k} |\hat{u}_N(t,k)| \leqslant \sum_{k=0}^{N} e^{\tau_N(t)k} |\hat{u}_N(t,k)| = z_N(t) \leqslant C_0, \tag{2.24}$$

for all $t \in [0, T]$, due to (2.21). Since N is an arbitrary integer larger than N', we conclude, for every fixed number N_0, for every $t \in [0, T]$,

$$\sum_{k=0}^{N_0} e^{\tau(t)k} |\hat{u}(t,k)| = \lim_{N\to\infty} \sum_{k=0}^{N_0} e^{\tau(t)k} |\hat{u}_N(t,k)| \leqslant C_0.$$

Therefore, since $N_0 \geqslant 0$ and $T > 0$ are arbitrarily selected, $\|u(t)\|_{G_{\tau(t)}(W)} \leqslant C_0$ for all $t \geqslant 0$. □

Remark 2.2. In Theorem 1.2, we found a lower bound $\tau(t)$ of the radius of spatial analyticity of $u(t)$, where $\tau(t) = \sigma e^{-\lambda|t|}$, with $\lambda = 4C_0 C_1$. By the definition of C_0 in (2.9), one has

$$\lambda = \begin{cases} 2(1+\sqrt{5}) e C_1^2, & \text{if } \|u_0\|_{G_\sigma(W)} \leqslant (1+\sqrt{5}) e C_1/2, \\ 4C_1 \|u_0\|_{G_\sigma(W)}, & \text{if } \|u_0\|_{G_\sigma(W)} > (1+\sqrt{5}) e C_1/2. \end{cases} \tag{2.25}$$

Here, we shall provide a slightly different lower bound $\tilde{\tau}(t)$ of the radius of analyticity of $u(t)$. More precisely, we can choose $\tilde{\tau}(t) = \sigma e^{-\tilde{\lambda}(t)|t|}$, where $\tilde{\lambda}(t)$ defined in (2.29) below, is almost *independent* of the Gevrey norm $\|u_0\|_{G_\sigma(W)}$ of the initial datum, for large values of $|t|$. Indeed, by (2.19), it is easy to see that

$$\frac{dz_N}{dt}(t) \leqslant -2C_1 \left(z_N(t) - \frac{eC_1}{2} \right)^2 + \frac{5}{2} e^2 C_1^3. \tag{2.26}$$

After some manipulations of (2.26), we obtain

$$\int_0^t \left(z_N(s) - \frac{eC_1}{2} \right)^2 ds \leqslant \frac{z_N(0)}{2C_1} + \frac{5e^2 C_1^2 t}{4} \leqslant \frac{\|u_0\|_{G_\sigma(W)}}{2C_1} + \frac{5e^2 C_1^2 t}{4}. \tag{2.27}$$

Note

$$\int_0^t z_N(s)\,ds = \int_0^t \left(z_N(s) - \frac{eC_1}{2}\right) ds + \frac{eC_1}{2}t$$
$$\leqslant \left[\int_0^t \left(z_N(s) - \frac{eC_1}{2}\right)^2 ds\right]^{\frac{1}{2}} \sqrt{t} + \frac{eC_1}{2}t$$
$$\leqslant \left[\frac{\|u_0\|_{G_\sigma(W)}}{2C_1} + \frac{5e^2C_1^2 t}{4}\right]^{\frac{1}{2}} \sqrt{t} + \frac{eC_1}{2}t, \tag{2.28}$$

where we have used the estimate (2.27). Thus, by (2.22) and (2.28), we may select

$$\tilde{\tau}(t) = \sigma e^{-\tilde{\lambda}(t)|t|}, \quad \text{with } \tilde{\lambda}(t) = 2C_1\left[\frac{2\|u_0\|_{G_\sigma(W)}}{C_1|t|} + 5e^2C_1^2\right]^{\frac{1}{2}} + 2eC_1^2,\ |t| > 0, \tag{2.29}$$

and then $\tilde{\tau}(t) \leqslant \tau_N(t)$. Thus, by adopting the argument in Theorem 1.2, it can be shown that $\|u(t)\|_{G_{\tilde{\tau}(t)}(W)} \leqslant C_0$ for all $t \in \mathbb{R}$. Also, we see from (2.29) that $\tilde{\lambda}(t) \to 2(1+\sqrt{5})eC_1^2$ as $|t| \to \infty$, that is, $\tilde{\lambda}(t)$ is almost *independent* of $\|u_0\|_{G_\sigma(W)}$, for large values of $|t|$, in contrast to the definition (2.25) of λ.

Remark 2.3. For analytic initial data, the Gevrey norm estimate $\|u(t)\|_{G_{\tau(t)}(W)} \leqslant C_0$, where $\tau(t) = \sigma e^{-\lambda|t|}$, can provide a growth estimate of the H^s norm of the solution $u(t)$. Indeed,

$$\|u\|_{H^s}^2 = \sum_{k\geqslant 0}(k^{2s}+1)|u_k|^2 \leqslant \sup|u_k|\left(\sum_{k\geqslant 0}|u_k|e^{\tau k}\frac{k^{2s}}{e^{\tau k}} + \sum_{k\geqslant 0}|u_k|\right).$$

Since the maximum of the function $k \mapsto \frac{k^{2s}}{e^{\tau k}}$ occurs at $k = \frac{2s}{\tau}$, we obtain

$$\|u\|_{H^s}^2 \leqslant \|u\|_W\left[e^{-2s}\left(\frac{2s}{\tau}\right)^{2s}\|u\|_{G_\tau(W)} + \|u\|_W\right].$$

It follows that

$$\|u(t)\|_{H^s}^2 \leqslant C(s)e^{2\lambda st},$$

that is to say, the H^s norm grows at most exponentially, if $s > \frac{1}{2}$, which agrees with the H^s norm estimates in Corollary 2, Section 3 of [6].

Remark 2.4. Let us set $A = \sqrt{I - \Delta}$. Recall the regular Gevrey classes of analytic functions are defined by $\mathcal{D}(A^s e^{\sigma A})$ furnished the norm $\|A^s e^{\sigma A}\cdot\|_{L^2(\mathbb{T})}$, where $s \geqslant 0$, $\sigma > 0$. It has been mentioned in the Introduction that we choose to employ the special Gevrey class $G_\sigma(W)$ in this manuscript, since it provides better estimate of the lower bound the radius of analyticity of the solution. In particular, we can do the following comparisons.

Suppose the initial condition $u_0 \in \mathcal{D}(A^s e^{\sigma A})$, $s > \frac{1}{2}$, $\sigma > 0$, and let us perform the estimates by using the regular Gevrey classes $\mathcal{D}(A^s e^{\sigma A})$. Adopting similar arguments as in [9,12], one can manage to show that

$$\left\|A^s e^{\tau_1(t)A}u(t)\right\|_{L^2}^2 \leqslant \left\|A^s e^{\sigma A}u_0\right\|_{L^2}^2 + C\int_0^{|t|}\left\|u(t')\right\|_{H^s}^4 dt', \quad s > \frac{1}{2},$$

if $\tau_1(t) = \sigma e^{-\int_0^{|t|} h(t')\,dt'}$, where $h(t) = C(\|A^p e^{\sigma A}u_0\|_{L^2}^2 + \int_0^{|t|}\|u(t')\|_{H^s}^4 dt')$. Since $\|u(t)\|_{H^s}$, $s > \frac{1}{2}$, has an upper bound that grows exponentially as $|t| \to \infty$ (see [6]), we infer that $\tau_1(t)$ might shrinks *double* exponentially, compared to the exponential shrinking rate of $\tau(t)$ established in Theorem 1.2, where the Gevrey class $G_\sigma(W)$ is used. Such advantage of employing the special Gevrey class $G_\sigma(W)$ stems from the uniform boundedness of the norm $\|u(t)\|_W$ for the solution u to the cubic Szegő equation for sufficiently regular initial data.

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Appendix A

For the sake of completion, we provide a straightforward proof of the following property of the Hankel operator.

Proposition A.1. *For any $u \in L^2_+(\mathbb{T}) \cap W$, the following double inequality holds*

$$\frac{1}{2}\|u\|_W \leqslant Tr\big(|H_u|\big) \leqslant \sum_{k=0}^{\infty}\left(\sum_{\ell=0}^{\infty}\big|\hat{u}(k+\ell)\big|^2\right)^{\frac{1}{2}}. \tag{A.1}$$

Proof. Recall the following result in the operator theory (see, e.g., [3]). Let A be an operator on a Hilbert space H, where A belongs to the trace class. If $\{e_k\}$ and $\{f_k\}$ are two orthonormal families in H, then

$$\sum_k \big|(Ae_k, f_k)\big| \leqslant Tr(|A|). \tag{A.2}$$

In order to find a lower bound of $Tr(|H_u|)$, we use the estimate (A.2) by computing $\sum_k |(H_u(e^{ik\theta}), f_k)|$ with two different orthonormal systems $\{f_k\}$ selected below. Notice that, by the definition (2.2) of the Hankel operator $H_u : L^2_+(\mathbb{T}) \to L^2_+(\mathbb{T})$, we have

$$H_u\big(e^{ik\theta}\big) = \Pi\big(ue^{-ik\theta}\big) = \Pi\left(\sum_{j\geqslant 0}\hat{u}(j)e^{i(j-k)\theta}\right) = \sum_{j\geqslant 0}\hat{u}(j+k)e^{ij\theta}. \tag{A.3}$$

If we choose $f_k = e^{ik\theta}$, $k \geqslant 0$, and use (A.3), then it follows that

$$Tr\big(|H_u|\big) \geqslant \sum_{k\geqslant 0}\big|\big(H_u\big(e^{ik\theta}\big), e^{ik\theta}\big)\big| = \sum_{k\geqslant 0}\left|\left(\sum_{j\geqslant 0}\hat{u}(j+k)e^{ij\theta}, e^{ik\theta}\right)\right| = \sum_{k\geqslant 0}\big|\hat{u}(2k)\big|.$$

However, if we select $f_k = e^{i(k+1)\theta}$, for every integer $k \geqslant 0$, then

$$Tr\big(|H_u|\big) \geqslant \sum_{k\geqslant 0}\big|\big(H_u\big(e^{ik\theta}\big), f_k\big)\big| = \sum_{k\geqslant 0}\big|\hat{u}(2k+1)\big|.$$

Summing up, we have proved

$$2Tr\big(|H_u|\big) \geqslant \sum_{k\geqslant 0}\big|\hat{u}(k)\big| = \|u\|_W.$$

We now pass to the second inequality. Recall from (2.2) that, for every $h_1, h_2 \in L^2_+$,

$$\big(H_u(h_1), h_2\big) = (u, h_1h_2) = \big(H_u(h_2), h_1\big),$$

which implies that H_u^2 is a positive self-adjoint linear operator. Moreover,

$$Tr\big(H_u^2\big) = \sum_{k,\ell\geqslant 0}\big|\hat{u}(k+\ell)\big|^2 = \sum_{n=0}^{\infty}(n+1)\big|\hat{u}(n)\big|^2 < \infty$$

as soon as $u \in H^{1/2}$. In other words, $|H_u| = \sqrt{H_u^2}$ is a positive Hilbert–Schmidt operator if $u \in L^2_+ \cap H^{1/2}$. Let $\{\rho_j\}$ be the sequence of positive eigenvalues of $|H_u|$, and let $\{\varepsilon_j\}$ be an orthonormal sequence of corresponding eigenvectors. Notice that

$$\big(H_u(\varepsilon_j), H_u(\varepsilon_{j'})\big) = \big(H_u^2(\varepsilon_{j'}), \varepsilon_j\big) = \rho_{j'}^2\delta_{jj'}.$$

We infer that the sequence $\{H_u(\varepsilon_j)/\rho_j\}$ is orthonormal. We then define the following antilinear operator on L^2_+,

$$\Omega_u(h) = \sum_j \frac{(H_u(\varepsilon_j), h)}{\rho_j}\varepsilon_j.$$

Notice that, due to the orthonormality of both systems $\{\varepsilon_j\}$ and $\{H_u(\varepsilon_j)/\rho_j\}$,

$$\|\Omega_u(h)\| \leqslant \|h\|.$$

We now observe that

$$\rho_j = (\Omega_u(H_u(\varepsilon_j)), \varepsilon_j) = \sum_{k=0}^{\infty}(\Omega_u(e^{ik\theta}), \varepsilon_j)(e^{ik\theta}, H_u(\varepsilon_j)) = \sum_{k=0}^{\infty}(\Omega_u(e^{ik\theta}), \varepsilon_j)(\varepsilon_j, H_u(e^{ik\theta}))$$
$$= \sum_{k,\ell\geqslant 0} \overline{\hat{u}(k+\ell)}(\Omega_u(e^{ik\theta}), \varepsilon_j)(\varepsilon_j, e^{i\ell\theta}),$$

and therefore, for every N,

$$Tr(|H_u|) = \sum_j \rho_j = \sum_{k,\ell\geqslant 0} \overline{\hat{u}(k+\ell)}(\Omega_u(e^{ik\theta}), e^{i\ell\theta}).$$

Apply the Cauchy–Schwarz inequality to the sum on ℓ,

$$Tr(|H_u|) \leqslant \sum_{k=0}^{\infty}\|\Omega_u(e^{ik\theta})\|\left(\sum_{\ell=0}^{\infty}|\hat{u}(k+\ell)|^2\right)^{\frac{1}{2}},$$

and the claim follows from $\|\Omega_u(e^{ik\theta})\| \leqslant \|e^{i\theta}\| = 1$. □

Using the above proposition, it is easy to derive the estimate (2.3) used in the proof of Proposition 2.1. Indeed, by the Cauchy–Schwarz inequality in the k sum, we have, for every $s > 1$,

$$\sum_{k=0}^{\infty}\left(\sum_{\ell=0}^{\infty}|\hat{u}(k+\ell)|^2\right)^{\frac{1}{2}} \leqslant \left(\sum_{k=0}^{\infty}(1+k)^{1-2s}\right)^{\frac{1}{2}}\left(\sum_{k,\ell\geqslant 0}(1+k)^{2s-1}|\hat{u}(k+\ell)|^2\right)^{\frac{1}{2}}$$
$$\leqslant \left(\frac{s}{s-1}\right)^{\frac{1}{2}}\left(\sum_{k,\ell\geqslant 0}(1+k+\ell)^{2s-1}|\hat{u}(k+\ell)|^2\right)^{\frac{1}{2}}$$
$$\leqslant C_s\|u\|_{H^s}.$$

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