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Ann. I. H. Poincaré – AN 32 (2015) 201–224

ANNALES DE L'INSTITUT HENRI POINCARÉ
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# Compactness and bubble analysis for 1/2-harmonic maps

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Received 10 October 2012; received in revised form 30 September 2013; accepted 12 November 2013

Available online 1 December 2013

**Abstract**

In this paper we study compactness and quantization properties of sequences of $1/2$-harmonic maps $u_k : \mathbb{R} \to \mathcal{S}^{m-1}$ such that $\|u_k\|_{\dot H^{1/2}(\mathbb{R},\mathcal{S}^{m-1})} \leqslant C$. More precisely we show that there exist a weak $1/2$-harmonic map $u_\infty : \mathbb{R} \to \mathcal{S}^{m-1}$, a finite and possible empty set $\{a_1, \dots, a_\ell\} \subset \mathbb{R}$ such that up to subsequences

$$\left|(-\Delta)^{1/4}u_k\right|^2 dx \rightharpoonup \left|(-\Delta)^{1/4}u_\infty\right|^2 dx + \sum_{i=1}^{\ell} \lambda_i \delta_{a_i}, \quad \text{in Radon measure,}$$

as $k \to +\infty$, with $\lambda_i \geqslant 0$.

The convergence of $u_k$ to $u_\infty$ is strong in $\dot W^{1/2,p}_{loc}(\mathbb{R} \setminus \{a_1, \dots, a_\ell\})$, for every $p \geqslant 1$. We quantify the loss of energy in the weak convergence and we show that in the case of non-constant $1/2$-harmonic maps with values in $\mathcal{S}^1$ one has $\lambda_i = 2\pi n_i$, with $n_i$ a positive integer.


*MSC:* 58E20; 35J20; 35B65; 35J60; 35S99

*Keywords:* Fractional harmonic maps; Nonlinear elliptic PDE's; Regularity of solutions; Commutator estimates

## 1. Introduction

In the paper [9] Rivière and the author started the investigation of the following 1-dimensional quadratic Lagrangian

$$L(u) = \int_{\mathbb{R}} \left|(-\Delta)^{1/4}u(x)\right|^2 dx, \tag{1}$$

where $u : \mathbb{R} \to \mathcal{N}$, $\mathcal{N}$ is a smooth $k$-dimensional sub-manifold of $\mathbb{R}^m$ which is at least $C^2$, compact and without boundary. We observe that (1) is a simple model of Lagrangian which is invariant under the trace of conformal maps that keep invariant the half-space $\mathbb{R}^2_+$: the Möbius group.

0294-1449/$ – see front matter 
http://dx.doi.org/10.1016/j.anihpc.2013.11.003

Precisely let $\phi : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ in $W^{1,2}(\mathbb{R}^2_+, \mathbb{R}^2_+)$ be a conformal map of degree 1, i.e. it satisfies

$$\begin{cases} \left|\dfrac{\partial \phi}{\partial x}\right| = \left|\dfrac{\partial \phi}{\partial y}\right|, \\ \left\langle \dfrac{\partial \phi}{\partial x}, \dfrac{\partial \phi}{\partial y}\right\rangle = 0, \\ \det \nabla\phi \geqslant 0 \quad \text{and} \quad \nabla\phi \neq 0. \end{cases} \tag{2}$$

Here $\langle \cdot,\cdot\rangle$ denotes the standard Euclidean inner product in $\mathbb{R}^m$.

We denote by $\tilde{\phi}$ the restriction of $\phi$ to $\mathbb{R}$. Then we have $L(u \circ \tilde{\phi}) = L(u)$.

Moreover $L(u)$ in (1) coincides with the semi-norm $\|u\|^2_{\dot{H}^{1/2}(\mathbb{R})}$ and the following identity holds

$$\int_{\mathbb{R}} \left|(-\Delta)^{1/4} u(x)\right|^2 dx = \inf\left\{ \int_{\mathbb{R}^2_+} |\nabla \tilde{u}|^2\, dx:\ \tilde{u} \in W^{1,2}\left(\mathbb{R}^2, \mathbb{R}^m\right),\ \text{trace}\ \tilde{u} = u \right\}. \tag{3}$$

The Lagrangian $L$ extends to maps $u$ in the following function space

$$\dot{H}^{1/2}(\mathbb{R}, \mathcal{N}) = \left\{ u \in \dot{H}^{1/2}\left(\mathbb{R}, \mathbb{R}^m\right):\ u(x) \in \mathcal{N},\ \text{a.e.}\right\}.$$

The operator $(-\Delta)^{1/4}$ on $\mathbb{R}$ is defined by means of the Fourier transform as follows

$$\widehat{(-\Delta)^{1/4} u} = |\xi|^{1/2}\hat{u}$$

(given a function $f$, $\hat{f}$ denotes the Fourier transform of $f$).

We denote by $\pi_{\mathcal{N}}$ the orthogonal projection from $\mathbb{R}^m$ onto $\mathcal{N}$ which happens to be a $C^\ell$ map in a sufficiently small neighborhood of $\mathcal{N}$ if $\mathcal{N}$ is assumed to be $C^{\ell+1}$. We now introduce the notion of $1/2$-harmonic map into a manifold.

**Definition 1.1.** A map $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ is called a weak $1/2$-harmonic map into $\mathcal{N}$ if for any $\phi \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m) \cap L^\infty(\mathbb{R}, \mathbb{R}^m)$ there holds

$$\frac{d}{dt} L\left(\pi_{\mathcal{N}}(u + t\phi)\right)_{|_{t=0}} = 0.$$

In short we say that a weak $1/2$-harmonic map is a *critical point of $L$ in $\dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ for perturbations in the target*.

We next give some geometric motivations related to the study of the problem (1).

First of all variational problems of the form (1) appear as a simplified model of renormalization area in hyperbolic spaces, see for instance [2]. There are also some geometric connections which are being investigated in the paper [11] between $1/2$-harmonic maps and the so-called free boundary sub-manifolds and optimization problems of eigenvalues. With this regards we refer the reader also to the papers [14,15]. Finally $1/2$-harmonic maps into the circle $\mathcal{S}^1$ might appear for instance in the asymptotics of equations in phase-field theory for fractional reaction–diffusion such as

$$\epsilon^2(-\Delta)^{1/2} u + u\left(1 - |u|^2\right) = 0,$$

where $u$ is a complex-valued "wave function".

In this paper we consider the case $\mathcal{N} = \mathcal{S}^{m-1}$. It can be shown (see [9]) that every weak $1/2$-harmonic map satisfies the following Euler–Lagrange equation

$$(-\Delta)^{1/2} u \wedge u = 0 \quad \text{in}\ \mathcal{D}'(\mathbb{R}). \tag{4}$$

One of the main achievements of the paper [9] is the rewriting of Eq. (4) in a more "tractable" way in order to be able to investigate regularity and compactness property of weak $1/2$-harmonic maps. Precisely in [9] the following two results have been proved:

**Proposition 1.1.** *A map $u$ in $\dot{H}^{1/2}(\mathbb{R}, \mathcal{S}^{m-1})$ is a weak 1/2-harmonic map if and only if it satisfies the following equation*

$$(-\Delta)^{1/4}\big(u \wedge (-\Delta)^{1/4}u\big) = T\big(\omega(u), u\big), \tag{5}$$

*where $\omega(u) : \mathbb{R}^m \to \bigwedge^2 \mathbb{R}^m$, $v \mapsto \omega(u)v := u \wedge v$ and in general for arbitrary positive integers $n$, $m$, $\ell$, for every linear operator $Q \in \dot{H}^{1/2}(\mathbb{R}^n, \mathcal{M}_{\ell\times m}(\mathbb{R}))$[1] and $u \in \dot{H}^{1/2}(\mathbb{R}^n, \mathbb{R}^m)$, $T$ is the operator defined by*

$$T(Q, u) := (-\Delta)^{1/4}\big[Q(-\Delta)^{1/4}u\big] - Q(-\Delta)^{1/2}u + (-\Delta)^{1/4}Q(-\Delta)^{1/4}u. \tag{6}$$

Eq. (5) has been completed by the following "structure equation" which is a consequence of the fact that $u \in \mathcal{S}^{m-1}$ almost everywhere:

**Proposition 1.2.** *All maps in $\dot{H}^{1/2}(\mathbb{R}, \mathcal{S}^{m-1})$ satisfy the following identity*

$$(-\Delta)^{1/4}\big(u \cdot (-\Delta)^{1/4}u\big) = S\big(\omega(u), u\big) - \mathcal{R}\big((-\Delta)^{1/4}u \cdot \mathcal{R}(-\Delta)^{1/4}u\big), \tag{7}$$

*where $\omega(u) : \mathbb{R}^m \to \mathbb{R}$, $v \mapsto \omega(u)v := u \cdot v$ and in general for arbitrary positive integers $n$, $m$, $\ell$, for every $Q \in \dot{H}^{1/2}(\mathbb{R}^n, \mathcal{M}_{\ell\times m}(\mathbb{R}))$, and $u \in \dot{H}^{1/2}(\mathbb{R}^n, \mathbb{R}^m)$, $S$ is the operator given by*

$$S(Q, u) := (-\Delta)^{1/4}\big[Q(-\Delta)^{1/4}u\big] - \mathcal{R}(Q\nabla u) + \mathcal{R}\big[(-\Delta)^{1/4}Q\mathcal{R}(-\Delta)^{1/4}u\big] \tag{8}$$

*and $\mathcal{R}$ is the Fourier multiplier of symbol $m(\xi) = -i\frac{\xi}{|\xi|}$.*

We call the operators $T$, $S$ *three-term commutators* and in [9] the following estimates have been established: for every $u \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)$ and $Q \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{M}_{\ell\times m}(\mathbb{R}))$ we have

$$\big\|T(Q, u)\big\|_{\dot{H}^{-1/2}(\mathbb{R})} \leqslant C\|Q\|_{\dot{H}^{1/2}(\mathbb{R})}\|u\|_{\dot{H}^{1/2}(\mathbb{R})}, \tag{9}$$

$$\big\|S(Q, u)\big\|_{\dot{H}^{-1/2}(\mathbb{R})} \leqslant C\|Q\|_{\dot{H}^{1/2}(\mathbb{R})}\|u\|_{\dot{H}^{1/2}(\mathbb{R})}, \tag{10}$$

and

$$\big\|\mathcal{R}\big((-\Delta)^{1/4}u \cdot \mathcal{R}(-\Delta)^{1/4}u\big)\big\|_{\dot{H}^{-1/2}(\mathbb{R})} \leqslant C\|u\|^2_{\dot{H}^{1/2}(\mathbb{R})}. \tag{11}$$

What has been discovered is a sort of "gain of regularity" in the r.h.s. of Eqs. (5) and (7) in the sense that, under the assumptions $u \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)$ and $Q \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{M}_{\ell\times m}(\mathbb{R}))$ each term individually in $T$ and $S$ – like for instance $(-\Delta)^{1/4}[Q(-\Delta)^{1/4}u]$ or $Q(-\Delta)^{1/2}u \dots$ – is not in $\dot{H}^{-1/2}$ but the special linear combination of them constituting $T$ and $S$ is in $\dot{H}^{-1/2}$. The same phenomenon appears in dimension 2 in the context of harmonic maps, for the Jacobians $J(a, b) := \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$ (with $a, b \in \dot{H}^1(\mathbb{R}^2)$) which satisfy as a direct consequence of Wente's theorem (see [5,25])

$$\big\|J(a, b)\big\|_{\dot{H}^{-1}(\mathbb{R}^2)} \leqslant C\|a\|_{\dot{H}^1(\mathbb{R}^2)}\|b\|_{\dot{H}^1(\mathbb{R}^2)} \tag{12}$$

whereas, individually, the terms $\frac{\partial a}{\partial x}\frac{\partial b}{\partial y}$ and $\frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$ are not in $\dot{H}^{-1}(\mathbb{R}^2)$.

The estimates (9) and (10) imply in particular that if $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{S}^{m-1})$ is a 1/2-harmonic map then

$$\big\|(-\Delta)^{1/4}u\big\|_{L^2(\mathbb{R})} \leqslant C\big\|(-\Delta)^{1/4}u\big\|^2_{L^2(\mathbb{R})}, \tag{13}$$

where the constant $C$ is independent of $u$.

From the inequality (13) it follows that if $C\|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})} < 1$ then the solution is constant. This the so-called *bootstrap test* and it is the key observation to prove Morrey-type estimates and to deduce Hölder regularity of 1/2-harmonic maps, see [9].

We mention here that since the paper [9] several extensions have been considered. The regularity of solutions to nonlocal linear Schrödinger systems with applications to 1/2-harmonic maps with values into general manifolds have

[1] $\mathcal{M}_{\ell\times m}(\mathbb{R})$ denotes, as usual, the space of $\ell \times m$ real matrices.

been studied by Rivière and the author in [10]. $n/2$-harmonic maps in odd dimension $n$ has been considered in [23] and [6] respectively in the case of values into the $(m-1)$-dimensional sphere and into general manifolds and the case of $\alpha$-harmonic maps in $\dot{W}^{\alpha,p}(\mathbb{R}^n, \mathcal{S}^{m-1})$, with $\alpha p = n$, has been recently studied by Schikorra and the author in [12]. Finally Schikorra [24] has also studied the partial regularity of weak solutions to nonlocal linear systems with an antisymmetric potential in the supercritical case (namely where $\alpha p < n$) under a crucial monotonicity assumption on the solutions which allows to reduce to the critical case.

In this paper we address to the issue of understanding the behavior of sequences $u_k$ of weak $1/2$-harmonic maps. We observe that as in the case of harmonic maps the bootstrap test (13) implies that if the energy is *small* then the system behaves locally like a linear system of the form $(-\Delta)^{1/2}u = 0$ (namely the r.h.s. is "dominated" by the l.h.s. of the equation). As a consequence we obtain that any sequence $u_k$ of weak $1/2$-harmonic maps with uniformly bounded energy weakly converges to a weak $1/2$-harmonic map $u_\infty$ and strongly converges to $u_\infty$ away from a finite (possibly empty) set $\{a_1, \ldots, a_\ell\} \subset \mathbb{R}$.

Namely we have (up to a subsequence)

$$\left|(-\Delta)^{1/4}u_k\right|^2 dx \rightharpoonup \left|(-\Delta)^{1/4}u_\infty\right|^2 dx + \sum_{i=1}^{\ell} \lambda_i \delta_{a_i}, \quad \text{in Radon measure,}$$

as $k \to +\infty$, with $\lambda_i \geqslant 0$. It remains the question to understand how the convergence at the concentration points $a_i$ fails to be strong. A careful analysis shows that the loss of energy during the weak convergence is not only concentrated at the points $a_i$ but it is also quantized: this amount of energy is given by the sum of energies of non-constant $1/2$-harmonic maps (the so-called *bubbles*). More precisely we get the following result:

**Theorem 1.1.** *Let $u_k \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{S}^{m-1})$ be a sequence of $1/2$-harmonic maps such that $\|u_k\|_{\dot{H}^{1/2}} \leqslant C$. Then it holds:*

1. *There exist $u_\infty \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{S}^{m-1})$ and a possibly empty set $\{a_1, \ldots, a_\ell\}$, $\ell \geqslant 1$, such that up to subsequences*

$$u_n \to u_\infty \quad \text{in } \dot{W}^{1/2,p}_{loc}\big(\mathbb{R} \setminus \{a_1, \ldots, a_\ell\}\big),\ p \geqslant 2 \text{ as } k \to +\infty \tag{14}$$

*and*

$$(-\Delta)^{1/2}u_\infty \wedge u_\infty = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}). \tag{15}$$

2. *There is a family $\tilde{u}^{i,j}_\infty \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{S}^{m-1})$ of $1/2$-harmonic maps ($i \in \{1, \ldots, \ell\}$, $j \in \{1, \ldots, N_i\}$), such that up to subsequences*

$$\left\|(-\Delta)^{1/4}\Big(u_k - u_\infty - \sum_{i,j} \tilde{u}^{i,j}_\infty\Big)\right\|_{L^2_{loc}(\mathbb{R})} \to 0, \quad \text{as } k \to +\infty. \tag{16}$$

Theorem 1.1 says that for every $i$, $\lambda_i = \sum_{j=1}^{N_i} L(\tilde{u}^{i,j}_\infty)$. Therefore there is no dissipation of energy in the region between $u_\infty$ and the bubbles and between the bubbles themselves (the so-called *neck-regions*).

We would like now to mention a result obtained in the paper [11] on the characterization of $1/2$ harmonic maps $u : \mathcal{S}^1 \to \mathcal{S}^1$ which permits us to deduce that in the case of $1/2$ harmonic maps with values in $\mathcal{S}^1$ one has $\lambda_i = 2\pi n_i$, with $n_i$ a positive integer and also to provide a simple example showing that the quantization may actually occur, namely the set $\{a_1, \ldots, a_\ell\}$ may be nonempty.

**Theorem 1.2.** *(See [11].) $u : \mathcal{S}^1 \to \mathcal{S}^1$ is a weak $1/2$-harmonic map if and only if its harmonic extension $\tilde{u} : D^2 \to \mathbb{R}^2$ is holomorphic or anti-holomorphic.*

We remark that because of the invariance of the Lagrangian (1) with respect to the trace of conformal transformations we can study without restrictions the problem in $\mathcal{S}^1$ instead of $\mathbb{R}$.

From Theorem 1.2 it follows that $1/2$-harmonic maps $u : \mathcal{S}^1 \to \mathcal{S}^1$ with $deg(u) = 1$ coincide with the trace of Möbius transformations of the disk $D^2 \subseteq \mathbb{R}^2$. Moreover every non-constant weak $1/2$-harmonic map $u : \mathcal{S}^1 \to \mathcal{S}^1$ satisfies

$$\int_{\mathcal{S}^1} \left|(-\Delta)^{1/4}u\right|^2 dx = 2\pi k < +\infty,$$

where $k$ is a positive integer which coincides with $|deg(u)|$.

Let us consider now the following sequence of $1/2$-harmonic maps

$$u_n : \mathcal{S}^1 \to \mathcal{S}^1, \quad u_n(z) = \frac{z - a_n}{1 - \bar{a}_n z},$$

with $|a_n| < 1$ and $a_n \to 1$ as $n \to +\infty$. In this case we have $u_n \to -1$ in $C^\infty_{loc}(\mathbb{R} \setminus \{1\})$, thus the set of concentration points is nonempty. Theorem 1.1 yields the existence of one bubble $\tilde{u}_\infty$ such that

$$\left\|(-\Delta)^{1/4}(u_n - \tilde{u}_\infty)\right\|_{L^2_{loc}} \to 0, \quad \text{as } n \to +\infty.$$

We explain now the method we have used to prove the main Theorem 1.1.

In order to get the quantization of the energy we exploit a "functional analysis" method introduced by Lin and Rivière in [20] in the context of harmonic maps in non-conformal dimensions, i.e., in dimension $n \geqslant 3$. Such a method consists in the use of the interpolation Lorentz spaces in the special case where the r.h.s. of the equation can be written as a linear combination of Jacobians.

This technique has been recently applied in [18,19] and in [3] for the quantization analysis respectively of linear Schrödinger systems with antisymmetric potential in 2-dimension, of bi-harmonic maps in 4-dimensions, and of Willmore surfaces. We refer the reader to the papers [21,20] for an overview of the bubbling and quantization issues in the literature.

We describe briefly the key steps to get the quantization analysis.

1. First of all we will make use of a general result proved in [3] which permits to split the domain (in our case $\mathbb{R}$) into *the converging region* (which is the complement of small neighborhoods of the $a^i$), *bubbles domains* and *neck-regions* (which are unions of degenerating annuli).

2. We prove that the $L^2$ norm of $(-\Delta)^{1/4}u_k$ in the neck regions is arbitrary small (see Theorem 3.1). Thanks to the duality of the Lorentz spaces $L^{2,1} - L^{2,\infty}$ this is reduced in estimating the $L^{2,\infty}$, $L^{2,1}$ norms of $u_k$ in these regions. Precisely we first show that the $L^{2,\infty}$ norm of the $u_k$ is arbitrary small in *degenerating annuli*, i.e. annuli whose conformal class is not a priori bounded (see Lemma 3.2) and as far as the $L^{2,1}$ norm is concerned, we use the following improved estimate on the operators $T$ and $S$ which is proved in Appendix A.

**Theorem 1.3.** *Let $u, Q \in \dot{H}^{1/2}(\mathbb{R}^n)$. Then $T(Q,u), S(Q,u) \in \mathcal{H}^1(\mathbb{R}^n)$ and*

$$\left\|T(Q,u)\right\|_{\mathcal{H}^1(\mathbb{R}^n)} \leqslant C\|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)}\|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}, \tag{17}$$

$$\left\|S(Q,u)\right\|_{\mathcal{H}^1(\mathbb{R}^n)} \leqslant C\|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)}\|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}. \tag{18}$$

We recall here some definitions. We denote by $L^{2,\infty}(\mathbb{R}^n)$ the space of measurable functions $f$ such that

$$\sup_{\lambda>0} \lambda\left|\left\{x \in \mathbb{R}^n\colon \left|f(x)\right| \geqslant \lambda\right\}\right|^{1/2} < +\infty,$$

and $L^{2,1}(\mathbb{R}^n)$ is the space of measurable functions satisfying

$$\int_0^{+\infty} \left|\left\{x \in \mathbb{R}\colon \left|f(x)\right| \geqslant \lambda\right\}\right|^{1/2} d\lambda < +\infty.$$

We denote by $\mathcal{H}^1(\mathbb{R}^n)$ the Hardy space which is the space of $L^1$ functions $f$ on $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{R}^n} \sup_{t>0} |\phi_t * f|(x)\, dx < +\infty,$$

where $\phi_t(x) := t^{-n}\phi(t^{-1}x)$ and $\phi$ is some function in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$.

Finally for every $s \in \mathbb{R}$ and $q > 1$ we denote by $\dot{W}^{s,q}(\mathbb{R}^n)$ the fractional Sobolev space

$$\left\{f \in \mathcal{S}'(\mathbb{R}^n):\ \mathcal{F}^{-1}\left[|\xi|^s \mid \mathcal{F}[f]\right] \in L^q(\mathbb{R}^n)\right\}.$$

For more properties on the Lorentz spaces, Hardy space $\mathcal{H}^1$ and fractional Sobolev spaces we refer to [16] and [17].

In a forthcoming paper [8] we are going to investigate bubbles and quantization issues in the case of nonlocal Schrödinger linear systems with applications to $1/2$-harmonic maps with values into manifolds. The difficulty there is to succeed in getting a uniform $L^{2,1}$ estimate as well on degenerating annuli as in the local case (see [18]).

It would be also very interesting to understand the geometric properties of the bubbles in the case of more general manifolds.

This paper is organized as follows.

Section 2 we address to the compactness issue which is the first part of Theorem 1.1. In Section 3 we prove $L^2$ estimates on degenerating annual domains. In Section 4 we prove the second part of Theorem 1.1. In Appendix A we prove Theorem 1.3.

## 2. Compactness

In this section we prove the first part of Theorem 1.1. The result is based on the following *ε-regularity* property whose proof can be found in [9,10] and in [7].

**Lemma 2.1** *(ε-regularity). Let $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{S}^{m-1})$ be a $1/2$-harmonic map. Then there exists $\varepsilon_0 > 0$ such that if*

$$\sum_{j \geqslant 0} 2^{-j/2} \left\|(-\Delta)^{1/4} u\right\|_{L^2(B(x,2^j r))} \leqslant \varepsilon_0, \tag{19}$$

*then there is $p > 2$ (independent of $u$) such that for every $x \in \mathbb{R}$, $y \in B(x, r/2)$ we have*

$$\left(r^{p/2-1} \int_{B(y,r/2)} \left|(-\Delta)^{1/4} u(x)\right|^p dx\right)^{1/p} \leqslant C \sum_{j \geqslant 0} 2^{-j/2} \left\|(-\Delta)^{1/4} u\right\|_{L^2(B(x,2^j r))}, \tag{20}$$

*where $C > 0$ depends on $\|u\|_{\dot{H}^{1/2} u(\mathbb{R})}$.*

By bootstrapping into Eqs. (5) and (7) and by localizing Theorems A.1 and A.2 (see [7]) one can show the following:

**Corollary 2.1.** *Let $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{S}^{m-1})$ be a $1/2$-harmonic map. Then for every $x \in \mathbb{R}$ there exists $r > 0$ such that the following holds*

$$r^{1/2-1/p} \left\|(-\Delta)^{1/4} u\right\|_{L^p(B(x,r))} \leqslant C \|u\|_{\dot{H}^{1/2}(\mathbb{R})}, \tag{21}$$

*for every $p \geqslant 2$ and*

$$r^{1/2} \left\|(-\Delta)^{1/4} u\right\|_{L^\infty(B(x,r))} \leqslant C \|u\|_{\dot{H}^{1/2}(\mathbb{R})}. \tag{22}$$

We will use also the localized version of the following result whose proof can be found in [1, p. 78]:

**Lemma 2.2.** *Let $0 < \alpha < 1$ and $g \in L^p(\mathbb{R})$, $1 < p < \infty$. Then there is a constant $C > 0$ independent of $g$, such that*

$$\left\|(-\Delta)^{-\frac{\alpha\theta}{2}} g\right\|_{L^r(\mathbb{R}^n)} \leqslant C \left\|(-\Delta)^{-\frac{\alpha}{2}} g\right\|^{\theta}_{L^s(\mathbb{R}^n)} \|g\|^{1-\theta}_{L^p(\mathbb{R}^n)},$$

*for $0 < \theta < 1$, $1 \leqslant s \leqslant \infty$, $\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{p}$.*

Next we show that if for some $x \in \mathbb{R}$ and $\rho > 0$ we have

$$\sum_{j \geqslant 0} 2^{-j/2} \left\|(-\Delta)^{1/4} u\right\|_{L^2(B(x,2^j \rho))} \leqslant \varepsilon_0,$$

where $\varepsilon_0 > 0$ is the small constant appearing in the $\varepsilon$-regularity Lemma 2.1, then $u \in \dot{W}^{\frac{1}{q}+\frac{1}{2},2}_{loc}(B(x,\rho))$, for all $q \geqslant 2$, i.e. $(-\Delta)^{\frac{1}{2q}+\frac{1}{4}} u \in L^2_{loc}(B(x,\rho))$.

**Proposition 2.1.** *Let $x \in \mathbb{R}$ and $\rho > 0$ be such that*

$$\sum_{j\geqslant 0} 2^{-j/2} \left\| (-\Delta)^{1/4} u \right\|_{L^2(B(x,2^j\rho))} \leqslant \varepsilon_0$$

*with $\varepsilon_0 > 0$ given in Lemma 2.1. Then there exists $C > 0$ (independent of $x$, $\rho$ and dependent on $\|u\|_{\dot{H}^{1/2}(\mathbb{R})}$) such that for all $q > 2$ the following holds*

$$\left\| (-\Delta)^{\frac{1}{2q}+\frac{1}{4}} u \right\|_{L^2(B(y,\rho/2))} \leqslant C \sum_{j\geqslant 0} 2^{-j/2} \left\| (-\Delta)^{1/4} u \right\|_{L^2(B(x,2^j\rho))}, \tag{23}$$

*for all $y \in B(x, \rho/2)$.*

**Proof.** We set $v = (-\Delta)^{1/4} u$. From Lemma 2.1 it follows that there exists $q > 2$ (independent of $u$) such that for every $y \in B(x, \rho/2)$

$$\|v\|_{L^q(B(y,\rho/4))} \leqslant C \sum_{j\geqslant 0} 2^{-j/2} \left\| (-\Delta)^{1/4} u \right\|_{L^2(B(x,2^j\rho))}. \tag{24}$$

By bootstrapping into Eqs. (5) and (7) one gets

$$\left\| (-\Delta)^{1/4} v \right\|_{L^{\frac{2q}{q+2}}(B(y,\rho/4))} \leqslant C \sum_{j\geqslant 0} 2^{-j/2} \left\| (-\Delta)^{1/4} u \right\|_{L^2(B(x,2^j\rho))}. \tag{25}$$

Now we set $f := (-\Delta)^{1/4} v$. By applying Lemma 2.2 in $B(y, \rho/4)$ with $g := v$, $p = q$, $r = 2$, $s = \frac{2q}{q+2}$, $\alpha = \frac{1}{2}$ and $\theta = \frac{q-2}{q}$ we obtain

$$\left\| (-\Delta)^{-\frac{q-2}{4q}} f \right\|_{L^2(B(y,\rho/4))} \leqslant C \|f\|^{\theta}_{L^{\frac{2q}{q+2}}(B(y,\rho/4))} \|v\|^{1-\theta}_{L^q(B(y,\rho/4))}. \tag{26}$$

In particular we get that $(-\Delta)^{\frac{1}{2q}} v \in L^2(B(y, \rho/4))$ and hence $(-\Delta)^{\frac{1}{2q}+\frac{1}{4}} u \in L^2(B(y, \rho/4))$ with

$$\left\| (-\Delta)^{\frac{1}{2q}+\frac{1}{4}} u \right\|_{L^2(B(y,\rho/4))} \leqslant C \sum_{j\geqslant 0} 2^{-j/2} \left\| (-\Delta)^{1/4} u \right\|_{L^2(B(x,2^j\rho))}. \tag{27}$$

From Corollary 2.1 it follows that the above arguments actually hold for every $q \geqslant 2$. This concludes the proof. □

We show now a singular point removability type result for 1/2-harmonic maps.

**Proposition 2.2** *(Singular point removability). Let $u \in \dot{H}^{\frac{1}{2}}(\mathbb{R}, \mathcal{S}^{m-1})$ be a 1/2-harmonic map in $\mathcal{D}'(\mathbb{R} \setminus \{a_1, \dots, a_\ell\})$. Then*

$$u \wedge (-\Delta)^{\frac{1}{2}} u = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

**Proof.** The fact that

$$u \wedge (-\Delta)^{\frac{1}{2}} u = 0 \quad \text{in } \mathcal{D}'\big(\mathbb{R} \setminus \{a_1, \dots, a_\ell\}\big)$$

implies that

$$(-\Delta)^{1/4}\big(u \wedge (-\Delta)^{1/4} u\big) = T(u\wedge, u) \quad \text{in } \mathcal{D}'\big(\mathbb{R} \setminus \{a_1, \dots, a_\ell\}\big),$$

where $T(u\wedge, u) \in \dot{H}^{-\frac{1}{2}}(\mathbb{R})$ and

$$\left\| T(u\wedge, u) \right\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R})} \leqslant C \|u\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R})}. \tag{28}$$

The distribution $\phi := (-\Delta)^{1/4}(u \wedge (-\Delta)^{1/4}u) - (T(u\wedge, u))$ is of order $p = 1$ and supported in $\{a_1, \ldots, a_\ell\}$. Therefore by Schwartz Theorem [4] one has

$$\phi = \sum_{|\alpha| \leqslant 1} c_\alpha \partial^\alpha \delta_{a_i}.$$

Since $\phi \in \dot{H}^{-\frac{1}{2}}(\mathbb{R})$, then the above implies that $c_\alpha = 0$ and thus

$$(-\Delta)^{1/4}\big(u \wedge (-\Delta)^{1/4}u\big) = T(u\wedge, u) \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

We conclude the proof of Proposition 2.2. □

The proof of the **first part of Theorem 1.1** concerning the compactness of uniformly bounded $1/2$-harmonic maps is contained in the following lemma.

**Lemma 2.3.** *Let $u_k \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{S}^{m-1})$ be a sequence of $1/2$-harmonic maps such that $\|u_k\|_{\dot{H}^{1/2}} \leqslant C$. Then there exist a subsequence $u_{k'}$ of $u_k$, a function $u_\infty \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{S}^{m-1})$ and $\{a_1, \ldots, a_\ell\}$, $\ell \geqslant 1$, such that*

$$u_{k'} \to u_\infty \quad \text{as } k' \to +\infty \text{ in } \dot{H}^{\frac{1}{2}}_{loc}\big((\mathbb{R} \setminus \{a_1, \ldots, a_\ell\})\big) \text{ for } p \geqslant 2, \tag{29}$$

*and*

$$(-\Delta)^{1/2}u_\infty \wedge u_\infty = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}). \tag{30}$$

**Proof. 1.** First of all there exist a subsequence $u_{k'}$ of $u_k$, a function $u_\infty \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{S}^{m-1})$ such that $u_{k'} \rightharpoonup u_\infty$ as $k' \to +\infty$.

**2.** Suppose first that for some $\rho > 0$ and for all $k \geqslant 1$, $\sum_{j\geqslant 0} 2^{-j/2}\|(-\Delta)^{1/4}u_k\|_{L^2(B(x,2^j\rho))} \leqslant \varepsilon_0$ with $\varepsilon_0 > 0$ given in Lemma 2.1. Then from Lemma 2.1 and the Rellich–Kondrachov Theorem (the embedding $\dot{W}^{\frac{1}{q}+\frac{1}{2},2}(B(x,\rho/4)) \hookrightarrow \dot{W}^{\frac{1}{2},t}(B(x,\rho/4))$ is compact for all $t < \frac{2q}{q-2}$) it follows that

$$u_{k'} \to u_\infty \quad \text{as } k' \to +\infty \text{ in } \dot{H}^{1/2}\big(B(x,\rho/4), \mathcal{S}^{m-1}\big)$$

for all $x \in \mathbb{R}$. In particular we have also

$$(-\Delta)^{\frac{1}{2}}u_{k'} \to (-\Delta)^{\frac{1}{2}}u_\infty \quad \text{as } k' \to +\infty \text{ in } \dot{H}^{-1/2}\big(B(x,\rho/4), \mathcal{S}^{m-1}\big).$$

Hence

$$(-\Delta)^{\frac{1}{2}}u_{k'} \wedge u_{k'} \to (-\Delta)^{\frac{1}{2}}u_\infty \wedge u_\infty \quad \text{as } k' \to +\infty \text{ in } \mathcal{D}'\big(B(x,\rho/4)\big),$$

and

$$(-\Delta)^{\frac{1}{2}}u_\infty \wedge u_\infty = 0 \quad \text{in } \mathcal{D}'\big(B(x,\rho/4)\big).$$

**3. Claim 1.** *There are only finitely many points $\{a_1, \ldots, a_\ell\}$ such that*

$$(-\Delta)^{\frac{1}{2}}u_\infty \wedge u_\infty = 0 \quad \text{in } \mathcal{D}'\big(\mathbb{R} \setminus \{a_1, \ldots, a_\ell\}\big). \tag{31}$$

**Proof of Claim 1.** We associate to every $x$ the number $\rho_x^k > 0$ such that

$$\big\|(-\Delta)^{1/4}u_k\big\|_{L^2(B(x,\rho_x^k))} = \frac{\varepsilon_0}{8}$$

where $\varepsilon_0$ is as in Lemma 2.1.

For every $M > 0$ and $k \geqslant 1$ we set

$$I_k^M := \left\{x:\ \rho_x^k < \frac{1}{M}\right\}$$

and

$$\mathcal{F}_k^M := \left\{B\left(x, \frac{1}{M}\right),\ x \in I_k^M\right\}.$$

By the Vitali–Besicovitch Covering Theorem (see for instance [13]), we can find an at most countable family of points $(x_j^{k,M})_{j\in J_k^M}$, $x_j^{k,M} \in I_k^M$ and $I_k^M \subseteq \bigcup_{j\in J_k^M} B(x_j^{k,M}, \frac{1}{M})$. Moreover every $x \in \mathcal{F}_k^M$ is contained in at most $N$ balls, $N$ being a number depending only on the dimension of the space.

Now we observe that

$$\begin{aligned} C &\geqslant \|u_k\|^2_{\dot H^{1/2}(\mathbb{R})} = \left\|(-\Delta)^{1/4}u_k\right\|^2_{L^2(\mathbb{R})} \\ &\geqslant N^{-1}\sum_{j\in J_k^M}\left\|(-\Delta)^{1/4}u_k\right\|^2_{L^2(B(x_j^{k,M},\frac{1}{M}))} \\ &\geqslant N^{-1}\sum_{j\in J_k^M}\frac{\varepsilon_0^2}{64} = N^{-1}\left|J_k^M\right|\frac{\varepsilon_0^2}{64}. \end{aligned}$$

Thus $|J_k^M| < +\infty$ for every $k$ and $M$ and this implies that for $k$ and $M$ large enough $|J_k^M| = C$, with $C$ independent of $k$ and $M$. In particular there exists $m_0 > 0$ such that

$$I_k^M \subseteq \bigcup_{j=1}^{m_0} B\left(x_j^{k,M}, \frac{1}{M}\right).$$

By definition we have $I_k^{M+1} \subseteq I_k^M$ for all $k$ and $M$. By using a *diagonal procedure* we can subtract a subsequence $k' \to +\infty$ such that $x_j^{k',M} \to x_j^{\infty,M}$ for all $M > 0$ and $j$ and

$$I_\infty^M \subseteq \bigcup_{j=1}^{m_0} B\left(x_j^{\infty,M}, \frac{1}{M}\right).$$

Now we let $M \to +\infty$ and get

$$I_\infty^0 \subseteq J_{\infty,0} := \left\{x_j^{\infty,0}\right\}_{j=1,\dots m_0}. \qquad \square$$

**Claim 2.** *If $x \notin J_{\infty,0}$ then there exists $\tilde r > 0$ such that*

$$u_\infty \wedge (-\Delta)^{\frac{1}{2}}u_\infty = 0 \quad \text{in } \mathcal{D}'\big(B(x,\tilde r)\big).$$

**Proof of Claim 2.** We assume that $x_j^{\infty,0} \neq \infty$ for all $j = 1,\dots,m_0$. Let $\gamma = dist(x, J_{\infty,0})$ and $K > 0$ be such that $2K^{-1} < \gamma$. Let $\tilde M > 0$ be such that for all $M \geqslant \tilde M$ and for all $j = 1,\dots,n_0$ we have

$$\left|x_j^{\infty,0} - x_j^{\infty,M}\right| < \frac{1}{4K}. \tag{32}$$

Let $\bar k > 0$ be such that for all $k' \geqslant \bar k$ and for all $j = 1,\dots,m_0$ we have

$$\left|x_j^{\infty,M} - x_j^{k',M}\right| < \frac{1}{4K}. \tag{33}$$

By combining (32) and (33) we get

$$\begin{aligned}\left|x-x_j^{k',M}\right| &\geqslant \left|x-x_j^{\infty,0}\right| \\ &\quad -\left|x_j^{\infty,0}-x_j^{\infty,M}\right|-\left|x_j^{\infty,\tilde M}-x_j^{k',M}\right| \\ &\geqslant \frac{2}{K}-\frac{1}{4K}-\frac{1}{4K}=\frac{3}{2K}>\frac{1}{M}.\end{aligned}$$

Therefore $x\notin\bigcup_{j=1}^{m_0}B(x_j^{k',M},\frac{1}{M})$, and $x\notin I_{k'}^M$ for all $k'\geqslant\bar k$. In particular $\rho_{k',x}\geqslant M^{-1}$ and (up to subsequences) $\rho_{k',x}\to\rho_{\infty,x}>0$. Now let $0<r<\rho_{\infty,x}$. We observe that $B(x,r)\subseteq B(x,\rho_{k',x})$, for $k'$ large and $\|(-\Delta)^{1/4}u_{k'}\|_{L^2(B(x,r))}\leqslant\frac{\varepsilon_0}{8}$.

Let $j_0\geqslant 3$ be such that $2^{-j_0/2}(\int_{\mathbb R}|(-\Delta)^{1/4}u_{k'}|^2\,dx)^{1/2}\leqslant\frac{\varepsilon}{8}$.

We now estimate $\sum_{h\geqslant 0}2^{-h/2}\|(-\Delta)^{1/4}u_{k'}\|_{L^2(B(x,2^h(2^{-2j_0}r)))}$.

$$\begin{aligned}&\sum_{h\geqslant 0}2^{-h/2}\left\|(-\Delta)^{1/4}u_{k'}\right\|_{L^2(B(x,2^h(2^{-2j_0}r)))}\\ &\quad=\sum_{h=0}^{j_0}2^{-h/2}\left\|(-\Delta)^{1/4}u_{k'}\right\|_{L^2(B(x,2^h(2^{-2j_0}r)))}+\sum_{h=j_0+1}^{\infty}2^{-h/2}\left\|(-\Delta)^{1/4}u_{k'}\right\|_{L^2(B(x,2^h(2^{-2j_0}r)))}\\ &\quad\leqslant\frac{\varepsilon_0}{8}\left(\sum_{h=0}^{\infty}2^{-h/2}\right)+2^{-(j_0+1)/2}\left(\int_{\mathbb R}\left|(-\Delta)^{1/4}u_{k'}\right|^2dx\right)^{1/2}\\ &\quad\leqslant\frac{\sqrt2}{\sqrt2-1}\frac{\varepsilon}{8}+\frac{\varepsilon}{16}<\varepsilon_0.\end{aligned}$$

By applying Step 2 we get

$$u_\infty\wedge(-\Delta)^{1/2}u_\infty=0\quad\text{in }\mathcal D'\big(B(x,\tilde r)\big)$$

for $\tilde r<2^{-2j_0}r$. This concludes the proof of Claim 2 by setting $a_i:=x_i^\infty$ and $\ell=m_0$. □

Now we apply Proposition 2.2 and we get that

$$u_\infty\wedge(-\Delta)^{1/2}u_\infty=0\quad\text{in }\mathcal D'(\mathbb R).$$

We can conclude the proof of Lemma 2.3 and the first part of Theorem 1.1. □

## 3. $L^{2,\infty}$ and $L^2$ estimates on degenerating annuli

The goal of this section is to prove some energy estimates of $1/2$-harmonic maps in *degenerating annuli*, i.e. annuli whose conformal class is not a priori bounded. Such estimates are crucial in the next section in order to get the quantization analysis in the neck regions.

The main result of this section is

**Theorem 3.1.** *There exists $\tilde\delta>0$ such that for any $1/2$-harmonic map $u\in\dot H^{\frac12}(\mathbb R,\mathcal S^{m-1})$, for any $\delta<\tilde\delta$ and $\lambda,\Lambda>0$ with $\lambda<(2\Lambda)^{-1}$ satisfying*

$$\sup_{\rho\in[\lambda,(2\Lambda)^{-1}]}\left(\int_{B(0,2\rho)\setminus B(0,\rho)}\left|(-\Delta)^{1/4}u\right|^2dx\right)^{1/2}\leqslant\delta,\tag{34}$$

*we have*

$$\int_{B(0,\Lambda^{-1})\setminus B(0,\lambda)}\left|(-\Delta)^{1/4}u\right|^2dx\leqslant C\sup_{\rho\in[\lambda,(2\Lambda)^{-1}]}\left(\int_{B(0,2\rho)\setminus B(0,\rho)}\left|(-\Delta)^{1/4}u\right|^2dx\right)^{1/2}.\tag{35}$$

The proof of Theorem 3.1 consists in three steps:

(1) First we show that we can control in degenerating annuli the $L^q$ norm of $(-\Delta)^{1/4}u$ for some $q > 2$ by

$$\sup_{\rho\in[\lambda,(2\Lambda)^{-1}]}\left(\int_{B(0,2\rho)\setminus B(0,\rho)} \left|(-\Delta)^{1/4}u\right|^2 dx\right)^{1/2}. \tag{36}$$

(2) Then we estimate the $L^{2,\infty}$ norm of $(-\Delta)^{1/4}u$ in degenerating annuli in terms of (36).

(3) Finally we use the global $L^{2,1}$ estimates obtained in Appendix A (see Theorem 1.3) and the duality $L^{2,1} - L^{2,\infty}$ in order to conclude.

**Lemma 3.1** *($L^q$-estimates). There exists $\tilde{\delta} > 0$ such that for every $1/2$-harmonic map $u \in \dot{H}^{\frac{1}{2}}(\mathbb{R})$, if*

$$\sup_{\rho\in[\lambda,(2\Lambda)^{-1}]}\left(\int_{B(0,2\rho)\setminus B(0,\rho)} \left|(-\Delta)^{1/4}u\right|^2 dx\right)^{1/2} \leqslant \delta, \tag{37}$$

*for some $\delta < \tilde{\delta}$, $\lambda, \Lambda > 0$ with $2\lambda < (4\Lambda)^{-1}$, then there exist $q > 2$ and $\theta \in (0,1)$ (independent of $u$, $\lambda$, $\Lambda$) such that*

$$\sup_{\rho\in[2\lambda,(4\Lambda)^{-1}]}\left(\rho^{q/2-1}\int_{B(0,2\rho)\setminus B(0,\rho)} \left|(-\Delta)^{1/4}u\right|^q dx\right)^{1/q}$$
$$\leqslant C\left[\sup_{\rho\in[\lambda,(2\Lambda)^{-1}]}\left(\int_{B(0,2\rho)\setminus B(0,\rho)} \left|(-\Delta)^{1/4}u\right|^2 dx\right)^{1/2}\right]^{\theta}.$$

**Proof.** We choose $\delta = \frac{\varepsilon_0}{8}$ where $\varepsilon_0 > 0$ is the constant appearing in the $\varepsilon$-regularity Lemma 2.1.

**Step 1.** There exists $p > 2$ (independent of $\lambda$, $\Lambda$, $u$) such that

$$\sup_{\rho\in[2\lambda,(4\Lambda)^{-1}]}\left(\rho^{p/2-1}\int_{B(0,2\rho)\setminus B(0,\rho)} \left|(-\Delta)^{1/4}u\right|^p dx\right)^{1/p} \leqslant C\|u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R})}. \tag{38}$$

**Proof of Step 1.** Let $r > 0$ be such that

$$\left(\int_{B(0,2r)\setminus B(0,r)} \left|(-\Delta)^{1/4}u\right|^2 dx\right)^{1/2} < \delta.$$

**Claim.** *There exists $p > 2$ (independent of $u$, $\delta$ and $r$) such that*

$$\left(r^{p/2-1}\int_{B(0,\frac{3}{2}r)\setminus B(0,\frac{5}{4}r)} \left|(-\Delta)^{1/4}u\right|^p dx\right)^{1/p} \leqslant C\|u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R})}. \tag{39}$$

Let $y \in B(0,\frac{3}{2}r) \setminus B(0,\frac{5}{4}r)$ (we clearly have $dist(y, \partial(B(0,2r)\setminus B(0,r))) \geqslant 1/4$).

Let $j_0 \geqslant 3$ such that $2^{-j_0/2}(\int_{\mathbb{R}} |(-\Delta)^{1/4}u|^2\,dx)^{1/2} \leqslant \delta$, and $B(y, 2^{-j_0}r) \subset (B(0,2r)\setminus B(0,r))$ for all $y \in B(0,\frac{3}{2}r)\setminus B(0,\frac{5}{4}r)$.

**Estimate of** $\sum_{h\geqslant 0} 2^{-h/2}\|(-\Delta)^{1/4}u\|_{L^2(B(0,2^h(2^{-2j_0}r)))}$**.**

$$\begin{aligned}
&\sum_{h\geqslant 0} 2^{-h/2}\big\|(-\Delta)^{1/4}u\big\|_{L^2(B(0,2^h(2^{-2j_0}r)))}\\
&\quad=\sum_{h=0}^{j_0} 2^{-h/2}\big\|(-\Delta)^{1/4}u\big\|_{L^2(B(0,2^h(2^{-2j_0}r)))}+\sum_{h=j_0+1}^{\infty} 2^{-h/2}\big\|(-\Delta)^{1/4}u\big\|_{L^2(B(0,2^h(2^{-2j_0}r)))}\\
&\quad\leqslant \delta\left(\sum_{h=0}^{\infty} 2^{-h/2}\right)+2^{-(j_0+1)/2}\left(\int_{\mathbb{R}}\big|(-\Delta)^{1/4}u\big|^2\,dx\right)^{1/2}\\
&\quad\leqslant \delta\frac{\sqrt{2}}{\sqrt{2}-1}+\frac{\delta}{2}<\varepsilon_0.
\end{aligned}$$

Lemma 2.1 implies the existence of $p>2$ such that

$$\left(r^{p/2-1}\int_{B(y,2^{-(j_0+1)}r)}\big|(-\Delta)^{1/4}u\big|^p\,dx\right)^{1/p}\leqslant C_{j_0}\|u\|_{\dot H^{\frac12}(\mathbb{R})}\tag{40}$$

for some $C_{j_0}>0$ depending on $j_0$ and on $\|u\|_{H^{1/2}(\mathbb{R})}$.

By covering the annulus $B(0,\frac32 r)\setminus B(0,\frac54 r)$ by a finite number of balls $B(y,2^{-(j_0+1)}r)$ we finally get

$$\left(r^{p/2-1}\int_{B(0,\frac32 r)\setminus B(0,\frac54 r)}\big|(-\Delta)^{1/4}u\big|^p\,dx\right)^{1/p}\leqslant \tilde C_{j_0}\|u\|_{\dot H^{\frac12}(\mathbb{R})},$$

and the **proof of the Claim** is concluded.

Hence

$$\sup_{\rho\in[2\lambda,(4\Lambda)^{-1}]}\left(\rho^{p/2-1}\int_{B(0,2\rho)\setminus B(0,\rho)}\big|(-\Delta)^{1/4}u\big|^p\,dx\right)^{1/p}\leqslant C\|u\|_{\dot H^{\frac12}(\mathbb{R})}.\tag{41}$$

We thus conclude the **proof of Step 1**. □

**Step 2.** There exists $q>2$ (independent of $\lambda$, $\Lambda$, $u$ and dependent on $p$) such that

$$\begin{aligned}
&\sup_{\rho\in[2\lambda,(4\Lambda)^{-1}]}\left(\rho^{q/2-1}\int_{B(0,2\rho)\setminus B(0,\rho)}\big|(-\Delta)^{1/4}u\big|^q\,dx\right)^{1/q}\\
&\quad\leqslant C\sup_{\rho\in[\lambda,(2\Lambda)^{-1}]}\left(\int_{B(0,2\rho)\setminus B(0,\rho)}\big|(-\Delta)^{1/4}u\big|^2\,dx\right)^{1/2}.
\end{aligned}\tag{42}$$

**Proof of Step 2.** Let us take $q^{-1}=(1-\theta)p^{-1}+\theta 2^{-1}$, with $\theta\in(0,1)$. Then by Hölder's Inequality and by using (40) we get

$$\begin{aligned}
&\left(\rho^{q/2-1}\int_{B(0,2\rho)\setminus B(0,\rho)}\big|(-\Delta)^{1/4}u\big|^p\,dx\right)^{1/q}\\
&\quad\leqslant\left(\rho^{p/2-1}\int_{B(0,2\rho)\setminus B(0,\rho)}\big|(-\Delta)^{1/4}u\big|^p\,dx\right)^{\frac{1-\theta}{p}}\left(\int_{B(0,2\rho)\setminus B(0,\rho)}\big|(-\Delta)^{1/4}u\big|^2\,dx\right)^{\frac{\theta}{2}}
\end{aligned}$$

$$\leqslant C\|u\|^{1-\theta}_{\dot H^{\frac12}(\mathbb{R})}\sup_{\rho\in[\lambda,(2\Lambda)^{-1}]}\left(\int_{B(0,2\rho)\setminus B(0,\rho)}\left|(-\Delta)^{1/4}u\right|^2dx\right)^{\frac{\theta}{2}}$$

$$\leqslant C\sup_{\rho\in[\lambda,(2\Lambda)^{-1}]}\left(\int_{B(0,2\rho)\setminus B(0,\rho)}\left|(-\Delta)^{1/4}u\right|^2dx\right)^{\frac{\theta}{2}}.$$

This concludes the **proof of Step 2 and of Lemma 3.1**. □

**Lemma 3.2** *($L^{2,\infty}$ estimates). There exists $\tilde\delta>0$ such that for any $1/2$-harmonic map $u\in\dot H^{\frac12}(\mathbb{R})$, for any $\delta<\tilde\delta$ and $\lambda,\Lambda>0$ with $\lambda<(2\Lambda)^{-1}$ satisfying*

$$\sup_{\rho\in[\lambda,(2\Lambda)^{-1}]}\left(\int_{B(0,2\rho)\setminus B(0,\rho)}\left|(-\Delta)^{1/4}u\right|^2dx\right)^{1/2}\leqslant\delta$$

*then*

$$\left\|(-\Delta)^{1/4}u\right\|_{L^{2,\infty}(B(0,(2\Lambda)^{-1}))\setminus B(0,\lambda)}\leqslant C\sup_{\rho\in[\lambda,(2\Lambda)^{-1}]}\left(\int_{B(0,2\rho)\setminus B(0,\rho)}\left|(-\Delta)^{1/4}u\right|^2dx\right)^{1/2},\tag{43}$$

*where $C$ is independent of $\lambda$, $\Lambda$.*

**Proof.** We set $f=(-\Delta)^{1/4}u$ in $B(0,(4\Lambda)^{-1})\setminus B(0,2\lambda)$ and $f=0$ otherwise.

Let $\delta<\tilde\delta/4$ where $\tilde\delta$ is the constant appearing in Theorem 3.1. From Lemma 3.1 it follows that for all $\lambda,\Lambda>0$ with $2\lambda<(4\Lambda)^{-1}$ if

$$\sup_{\rho\in[\lambda,(2\Lambda)^{-1}]}\left(\int_{B(0,2\rho)\setminus B(0,\rho)}\left|(-\Delta)^{1/4}u\right|^2dx\right)^{1/2}\leqslant\delta$$

then there exist $q>2$ and $\theta\in(0,1)$ such that

$$\sup_{\rho\in[2\lambda,(4\Lambda)^{-1}]}\left(\rho^{q/2-1}\int_{B(0,2\rho)\setminus B(0,\rho)}\left|(-\Delta)^{1/4}u\right|^q dx\right)^{1/q}$$
$$\leqslant C\left[\sup_{\rho\in[\lambda,(2\Lambda)^{-1}]}\left(\int_{B(0,2\rho)\setminus B(0,\rho)}\left|(-\Delta)^{1/4}u\right|^2dx\right)^{1/2}\right]^{\theta}.$$

We set

$$\gamma=C\left[\sup_{\rho\in[\lambda,(2\Lambda)^{-1}]}\left(\int_{B(0,2\rho)\setminus B(0,\rho)}\left|(-\Delta)^{1/4}u\right|^2dx\right)^{1/2}\right]^{\theta}.$$

We observe that for all $\rho\in[2\lambda,(4\Lambda)^{-1}]$ one has:

$$\gamma^q\geqslant\rho^{q/2-1}\int_{B(0,2\rho)\setminus B(0,\rho)}|f|^q dx$$
$$\geqslant\rho^{q/2-1}\alpha^q\left|\left\{x\in B(0,2\rho)\setminus B(0,\rho):\ |f|>\alpha\right\}\right|.$$

Let $k\in\mathbb{Z}$, then the following estimate holds

$$\alpha^2\sum_{j\geqslant k}\left|\left\{x\in B\left(0,2^{j+1}\alpha^{-2}\right)\setminus B\left(0,2^j\alpha^{-2}\right):\ |f|>\alpha\right\}\right|$$

$$
\begin{aligned}
&\leqslant \alpha^2 \sum_{j\geqslant k} \frac{(2^j\alpha^{-2})^{1-q/2}}{\alpha^q}\gamma^q = \gamma^q \sum_{j\geqslant k} 2^{j(1-q/2)} \\
&\leqslant \gamma^q 2^{1-q/2} 2^{k(1-q/2)} \big(1-2^{1-q/2}\big)^{-1} \\
&= \gamma^q 2^{k(1-q/2)} \big(2^{q/2-1}-1\big)^{-1}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\alpha^2\big|\{x\in\mathbb{R}:\ |f|>\alpha\}\big| &\leqslant \gamma^q 2^{k(1-q/2)} + \alpha^2\big|B\big(0,2^k\alpha^{-2}\big)\big| \\
&\leqslant \gamma^q 2^{k(1-q/2)}\big(2^{q/2-1}-1\big)^{-1} + \alpha^2 2 2^k\alpha^{-2}.
\end{aligned}
$$

Now we choose $k$ in such a way that $2^k=\gamma^2/2$. It follows that

$$
\alpha^2\big|\{x\in\mathbb{R}:\ |f|>\alpha\}\big| \leqslant \frac{\gamma^2}{2}\big(2^{q/2-1}-1\big)^{-1} + \gamma^2 = \frac{2^{q/2}-1}{2^{q/2}-2}\gamma^2.
$$

Hence

$$
\begin{aligned}
&\big\|(-\Delta)^{1/4}u\big\|_{L^{2,\infty}(B(0,(4\Lambda)^{-1}))\setminus B(0,2\lambda)} \\
&\quad= \sup_{\alpha>0}\Big(\alpha^2\big|\big\{x\in B\big(0,(4\Lambda)^{-1}\big)\setminus B(0,2\lambda):\ \big|(-\Delta)^{1/4}u(x)\big|>\alpha\big\}\big|\Big)^{1/2} \\
&\quad\leqslant \left(\frac{2^{q/2}-1}{2^{q/2}-2}\right)^{1/2}\gamma.
\end{aligned}
\tag{44}
$$

By combining (44) and the fact that the $L^{2,\infty}$ norms of $(-\Delta)^{1/4}u$ in the annuli $B(0,\Lambda^{-1})\setminus B(0,(4\Lambda^{-1}))$ and $B(0,2\lambda)\setminus B(0,\lambda)$ are controlled by the respective $L^2$ norms we get the estimate (43) and we conclude the proof of Lemma 3.2. □

Now we can prove Theorem 3.1.

**Proof of Theorem 3.1.** From Theorem 1.3 it follows that any 1/2-harmonic map $u\in\dot H^{1/2}(\mathbb{R},\mathcal{S}^{m-1})$ satisfies $\|(-\Delta)^{1/4}u\|_{L^{2,1}(\mathbb{R})}\leqslant C$ where $C$ depends on $\|u\|_{\dot H^{1/2}(\mathbb{R},\mathcal{S}^{m-1})}$.

Now it is enough to use the duality $L^{2,1}-L^{2,\infty}$ and Lemma 3.2 to get

$$
\begin{aligned}
\int_{B(0,(\Lambda)^{-1})\setminus B(0,\lambda)} \big|(-\Delta)^{1/4}u\big|^2\,dx &\leqslant \big\|(-\Delta)^{1/4}u\big\|_{L^{2,1}(\mathbb{R})}\big\|(-\Delta)^{1/4}u\big\|_{L^{2,\infty}(B(0,(2\Lambda)^{-1}))\setminus B(0,\lambda)} \\
&\leqslant C \sup_{\rho\in[\lambda,(2\Lambda)^{-1}]}\left(\int_{B(0,2\rho)\setminus B(0,\rho)} \big|(-\Delta)^{1/4}u\big|^2\,dx\right)^{1/2}.
\end{aligned}
$$

We can conclude the proof of Theorem 3.1. □

## 4. Bubbles and neck-regions

In the proof of the first part of Theorem 1.1 (see Lemma 2.3) we have shown (up to a subsequence) that

$$
\big|(-\Delta)^{1/4}u_k\big|^2\,dx \rightharpoonup \big|(-\Delta)^{1/4}u_\infty\big|^2\,dx + \sum_{i=1}^{\ell}\lambda_i\delta_{a_i}, \quad \text{in Radon measure.}
$$

The aim of this section is to show that for every $i\in\{1\dots\ell\}$ there exist bubbles $(\tilde u_\infty^{i,j})$, $j\in\{1,\dots,N_i\}$ such that $\lambda_i=\sum_{j=1}^{N_i}\int_{\mathbb{R}}|(-\Delta)^{1/4}\tilde u_\infty^{i,j}|^2\,dx$.

We first give the following definitions.

**Definition 4.1** *(Bubble).* A **bubble** is a **non-constant** $1/2$-harmonic map $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{S}^{m-1})$.

**Definition 4.2** *(Neck region).* A neck region for a function $f \in L^2(\mathbb{R})$ is the union of finite degenerate annuli of the type $A_k(x) = B(x, R_k) \setminus B(x, r_k)$ with $r_k \to 0$ and $\frac{R_k}{r_k} \to +\infty$ as $k \to +\infty$ satisfying the following property: for all $\delta > 0$ there exists $\Lambda > 0$ such that

$$\left(\sup_{\rho \in [\Lambda r_k, (2\Lambda)^{-1} R_k]} \int_{B(x,2\rho)\setminus B(x,\rho)} |f|^2\, dx\right)^{1/2} \leqslant \delta.$$

**Proof of the second part of Theorem 1.1.** We have to show that there is a family of bubbles $\tilde{u}^{i,j}_\infty \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{S}^{m-1})$ ($i \in \{1, \dots, \ell\}$, $j \in \{1, \dots, N_i\}$), such that up to subsequences

$$\left\| (-\Delta)^{1/4}\left(u_k - u_\infty - \sum_{i,j} \tilde{u}^{i,j}_\infty\right)\right\|_{L^2_{loc}(\mathbb{R})} \to 0, \quad \text{as } k \to \infty. \tag{45}$$

We first observe that the *bootstrap test* (13) implies that each bubble has a bounded from below energy $c_0 > 0$. Therefore for every $i$, $N_i < +\infty$.

For simplicity we assume that $\ell = 1$ and that there are at most two bubbles.

Now let us take $\delta < \tilde{\delta}$ such that $C\delta < \varepsilon_0$ (the constants $C$ and $\tilde{\delta}$ are the ones appearing in the statement of Theorem 3.1).

We also set $\gamma = \min(\frac{\delta}{2}, \frac{\varepsilon_0}{2})$.

**Step 1.** For every $k \geqslant 1$ we set

$$\rho^1_k = \inf\left\{\rho > 0\colon\ \exists x \in B(a_1, 1)\colon \int_{B(x,\rho)} \left|(-\Delta)^{1/4} u_k\right|^2 dx = \gamma^2\right\}.$$

*There are two cases*:

**Case 1.** $\liminf_{k\to+\infty} \rho^1_k > 0$.

In this case there is not concentration of the energy, namely $\lambda_1 = 0$.

**Case 2.** $\lim_{k\to+\infty} \rho^1_k = 0$. For every $k \geqslant 1$, let $x^1_k \in B(a_1, 1)$ be the point such that $\int_{B(x_{1,k},\rho^1_k)} |(-\Delta)^{1/4} u_k|^2\, dx = \gamma^2$. We have (up to subsequences) $x^1_k \to a_1$ as $k \to +\infty$ (outside any neighborhood of $a_1$ there is no concentration).

Now we choose a subsequence of $u_k$ (that we still denote by $u_k$) and a fixed radius $\alpha > 0$ such that

$$\limsup_{k\to\infty}\left[\sup_{0<r<\alpha}\left\{\int_{B(a_1,\alpha)\setminus B(a_1,r)} \left|(-\Delta)^{1/4} u_k(y)\right|^2 dy = \gamma^2\right\}\right] = 0.$$

Now we borrow the idea in [3] to split the annulus $B(x_{1,k}, \alpha) \setminus B(x_{1,k}, \rho^1_k)$ in domains of unbounded conformal class where the energy is small and domains of bounded conformal class where the energy is bounded from below.

Precisely by applying Lemma 3.2 in [3], we can find a sequence of family of radii

$$R^0_k = \alpha > R^1_k > \dots > R^{N_1}_k = \rho^1_k$$

with $\{1, \dots, N_1\} = I_0 \cup I_1$. For every $i_\ell \in I_0$ one has

$$\lim_{k\to+\infty} \log\left(\frac{{R_k}^{i_\ell}}{{R_k}^{i_\ell+1}}\right) < +\infty \quad \text{and} \quad \int_{B(x_{1,k},{R_k}^{i_\ell+1})\setminus B(x_{1,k},{R_k}^{i_\ell})} \left|(-\Delta)^{1/4} \tilde{u}_k(y)\right|^2 dy \geqslant \gamma^2, \tag{46}$$

and for every $i_\ell \in I_1$, one has

$$\begin{cases} \lim_{k\to+\infty} \log\left(\frac{{R_k}^{i_\ell}}{{R_k}^{i_\ell+1}}\right) = +\infty \quad \text{and} \\ \forall \rho \in \left({R_k}^{i_\ell}, {R_k}^{i_\ell+1}/2\right), \quad \int\limits_{B(x_{1,k},2\rho)\setminus B(x_{1,k},\rho)} \left|(-\Delta)^{1/4}\tilde{u}_k(y)\right|^2 dy \leqslant 2\gamma^2. \end{cases} \tag{47}$$

We consider the smallest annulus $A_k^{i_\ell} := B(x_{1,k}, {R_k}^{i_\ell}) \setminus B(x_{1,k}, {R_k}^{i_\ell+1})$ of the first type $i_\ell \in I_0$. For such an $i_\ell$ we define

$$r_k^{i_\ell} = \inf\left\{ r < {R_k}^{i_\ell+1}\colon \exists x \in A_k^{i_\ell}\colon \int\limits_{B(x,r)} \left|(-\Delta)^{1/4}u_k\right|^2 dx = \gamma^2 \right\}.$$

We consider the following two cases.

**Case 1.** There exists a subsequence of $r_k^{i_\ell}$ such that

$$\lim_{k\to+\infty} \frac{r_k^{i_\ell}}{{R_k}^{i_\ell}} > 0.$$

In this case there is not concentration of the energy in $A_k^{i_\ell}$ and we pass to the next $A_k^{i'_\ell}$ (if there is any).

**Case 2.** We have

$$\lim_{k\to+\infty} \frac{r_k^{i_\ell}}{{R_k}^{i_\ell}} = 0.$$

In this case we have once again concentration. Let $x_{2,k} \in A_k^{i_\ell}$ such that

$$\int\limits_{B(x_{2,k},r_k^{i_\ell})} \left|(-\Delta)^{1/4}u_k\right|^2 dx = \gamma^2,$$

and we set $\rho_k^2 = r_k^{i_\ell}$.

*We separate two sub-cases*:

*Case of two "separated" bubbles*. $\liminf_{k\to+\infty} \frac{\rho_k^1}{\rho_k^2} > 0$. In this case the following two conditions hold

$$\begin{cases} \lim_{k\to\infty} \frac{|x_{1,k} - x_{2,k}|}{\rho_k^1} = +\infty, \\ \lim_{k\to\infty} \frac{|x_{1,k} - x_{2,k}|}{\rho_k^2} = +\infty. \end{cases}$$

In this case the bubbles $\tilde{u}_{2,\infty}$ and $\tilde{u}_{1,\infty}$ are "independent".

Let us consider the two "separated" balls $B(x_{1,k}, \rho_k^1)$ and $B(x_{2,k}, \rho_k^2)$, with

$$\lim_{k\to\infty} \frac{|x_{1,k} - x_{2,k}|}{\rho_k^1 + \rho_k^2} = +\infty.$$

For every $\alpha$ we set

$$\mathcal{N}_k^1(\alpha) = B(a_1, \alpha) \setminus \left(B\left(x_{1,k}, \alpha^{-1}\rho_k^1\right) \cup B\left(x_{2,k}, \alpha^{-1}\rho_k^2\right)\right).$$

The above construction gives the existence of $\alpha$ small enough independent of $k$ such that

$$\begin{cases} \text{for } j=1,2 \text{ and for all } \rho \text{ such that } B(x_{j,k},2\rho)\setminus B(x_{j,k},2\rho)\subseteq \mathcal{N}_k^1(\alpha) \\ \displaystyle\int_{B(x_{j,k},2\rho)\setminus B(x_{j,k},\rho)} \left|(-\Delta)^{1/4}u_k\right|^2 dx \leqslant 2\gamma^2 \quad \text{and} \\ \displaystyle\int_{B(x_{j,k},\rho_k^j)} \left|(-\Delta)^{1/4}u_k\right|^2 dx = \gamma^2. \end{cases}$$

**Claim.** *The region $\mathcal{N}_k^1(\alpha)$ is a neck-region.*

**Proof of the Claim.** It is a consequence of the following general property.

**Lemma 4.1.** *Let $A_k = B(x_k, R_k)\setminus B(x_k, r_k)$ an annulus satisfying $r_k \to 0$, $\frac{R_k}{r_k}\to +\infty$ and $x_k \to x_\infty$ as $k\to+\infty$, and*

$$\sup_{r_k\leqslant\rho\leqslant\frac{R_k}{2}} \int_{B(x_k,2\rho)\setminus B(x_k,\rho)} \left|(-\Delta)^{1/4}u_k\right|^2 dx \leqslant 2\gamma^2. \tag{48}$$

*Then for all $\eta>0$ there exists $\Lambda>0$ such that*

$$\sup_{r\in[\Lambda r_k,\Lambda^{-1}R_k]} \int_{B(x_k,2r)\setminus B(x_k,r)} \left|(-\Delta)^{1/4}u_k\right|^2 dx \leqslant \eta. \tag{49}$$

**Proof.** Suppose by contradiction that there exist $\eta>0$ and two sequences $\Lambda_k\to+\infty$ as $k\to+\infty$ and $\Lambda_k r_k\leqslant\tilde r_k\leqslant(\Lambda_k)^{-1}R_k$ such that

$$\int_{B(x_k,2\tilde r_k)\setminus B(x_k,\tilde r_k)} \left|(-\Delta)^{1/4}u_k\right|^2 dx > \eta. \tag{50}$$

We define $\tilde u_k(y) = u(\tilde r_k y + x_k)$. From condition (48) and Theorem 3.1 it follows that

$$\int_{B(0,\frac{R_k}{\tilde r_k})\setminus B(0,\frac{r_k}{\tilde r_k})} \left|(-\Delta)^{1/4}\tilde u_k\right|^2 dx \leqslant 2\gamma^2. \tag{51}$$

Lemma 2.1 and Lemma 2.3 imply that $\tilde u_k \to \tilde u_\infty$ in $\dot W^{1/2,p}_{loc}(\mathbb{R}\setminus\{0\})$ for all $p\geqslant 1$, where $\tilde u_\infty$ is a nontrivial 1/2-harmonic maps ($\int_{B(0,2)\setminus B(0,1)}|(-\Delta)^{1/4}\tilde u_\infty|^2\,dx>\eta$). On the other hand the condition (48) gives

$$\int_{\mathbb{R}} \left|(-\Delta)^{1/4}\tilde u_\infty\right|^2 dx \leqslant C^2\delta^2 < \varepsilon_0^2.$$

The bootstrap test yields that $\tilde u_\infty$ is trivial which is a contradiction.

We conclude the proof of Lemma 4.1 and of the claim. □

By applying Theorem 1.1 we get that for all $\eta>0$ small enough

$$\int_{\mathcal{N}_k^1(\alpha)} \left|(-\Delta)^{1/4}u_k\right|^2 dx \leqslant \eta.$$

*Case of bubble over bubble.* $\liminf_{k\to+\infty}\frac{\rho_k^1}{\rho_k^2}=0$. We define $\tilde u_{2,k}(y)=u(\rho_k^2 y + x_{2,k})$ We have $\tilde u_{2,k}\to\tilde u_{2,\infty}$ in $W^{1/2,p}_{loc}(\mathbb{R}\setminus\{a_1\})$ for all $p\geqslant 1$ and $\int_{B(0,2)\setminus B(0,1)}|(-\Delta)^{1/4}\tilde u_{2,\infty}|^2\,dx\geqslant\eta$. Therefore $\tilde u_{2,\infty}$ is a new bubble (case *bubble over bubble*: the bubble $\tilde u_{2,\infty}$ “contains” $\tilde u_{1,\infty}$ and for $k$ large enough $x_{1,k}\in B(x_{2,k},\rho_k^2)$). For every $\alpha$ we set

$$\mathcal{N}_k^{1,2}(\alpha) = \big(B\big(x_{1,k}, \alpha\rho_k^2\big) \setminus B\big(x_{1,k}, \alpha^{-1}\rho_k^1\big)\big)$$

and

$$\mathcal{N}_k(\alpha) = \mathcal{N}_k^1(\alpha) \cup \mathcal{N}_k^{1,2}(\alpha).$$

By arguing as above one can show that $\mathcal{N}_k(\alpha)$ is a neck region.

Since we have assumed that there are at most two bubbles, the procedure stops here. Otherwise one has to continue the procedure until annuli of the type $I_0$ have been explored.

Therefore for every $\eta > 0$ we get:

1. *Case of independent bubbles*:

$$\begin{aligned}
\lim_{k\to+\infty} \int_{\mathbb{R}} \big|(-\Delta)^{1/4} u_k\big|^2\, dx &= \lim_{k\to+\infty} \int_{\mathcal{N}_k^1(\alpha)} \big|(-\Delta)^{1/4} u_k\big|^2\, dx \\
&\quad + \sum_{j=1}^{2} \lim_{k\to+\infty} \int_{B(x_{j,k},\alpha^{-1}\rho_k^j)} \big|(-\Delta)^{1/4} u_k\big|^2\, dx \\
&\quad + \lim_{k\to+\infty} \int_{\mathbb{R}\setminus B(a_1,\alpha)} \big|(-\Delta)^{1/4} u_k\big|^2\, dx \\
&\leqslant \eta + \sum_{j=1}^{2} \int_{B(0,\alpha^{-1})} \big|(-\Delta)^{1/4} \tilde{u}_\infty^j\big|^2\, dx \\
&\quad + \int_{\mathbb{R}\setminus B(a_1,\alpha)} \big|(-\Delta)^{1/4} u_\infty\big|^2\, dx. \qquad (52)
\end{aligned}$$

2. *Case of bubble over bubble*:

$$\begin{aligned}
\lim_{k\to+\infty} \int_{\mathbb{R}} \big|(-\Delta)^{1/4} u_k\big|^2\, dx &\leqslant \lim_{k\to+\infty} \int_{\mathcal{N}_k(\alpha)} \big|(-\Delta)^{1/4} u_k\big|^2\, dx \\
&\quad + \lim_{k\to+\infty} \Bigg[ \int_{B(x_{2,k},\alpha^{-1}\rho_k^2)\setminus B(x_{1,k},\alpha\rho_k^2)} \big|(-\Delta)^{1/4} u_k\big|^2\, dx \\
&\quad + \int_{B(x_{1,k},\alpha^{-1}\rho_k^1)} \big|(-\Delta)^{1/4} u_k\big|^2\, dx \Bigg] \\
&\quad + \lim_{k\to+\infty} \int_{\mathbb{R}\setminus B(a_1,\alpha)} \big|(-\Delta)^{1/4} u_k\big|^2\, dx \\
&\leqslant \eta + \int_{B(0,\alpha^{-1})\setminus B(0,\alpha)} \big|(-\Delta)^{1/4} \tilde{u}_\infty^2\big|^2\, dx \\
&\quad + \int_{B(0,\alpha^{-1})} \big|(-\Delta)^{1/4} \tilde{u}_\infty^1\big|^2\, dx + \int_{\mathbb{R}\setminus B(a_1,\alpha)} \big|(-\Delta)^{1/4} u_\infty\big|^2\, dx. \qquad (53)
\end{aligned}$$

By taking in (52) and (53) the lim for $\alpha, \eta \to 0$ we get the desired quantization estimate (45). This concludes the proof of the second part of Theorem 1.1. □

## Appendix A. Commutator estimates: Proof of Theorem 1.3

In this section we prove Theorem 1.3. To this end we shall make use of the Littlewood–Paley dyadic decomposition of unity that we recall here. Such a decomposition can be obtained as follows. Let $\phi(\xi)$ be a radial Schwartz function supported in $\{\xi \in \mathbb{R}^n\colon\ |\xi| \leqslant 2\}$, which is equal to 1 in $\{\xi \in \mathbb{R}^n\colon\ |\xi| \leqslant 1\}$. Let $\psi(\xi)$ be the function given by

$$\psi(\xi) := \phi(\xi) - \phi(2\xi).$$

$\psi$ is then a "bump function" supported in the annulus $\{\xi \in \mathbb{R}^n\colon\ 1/2 \leqslant |\xi| \leqslant 2\}$.

Let $\psi_0 = \phi$, $\psi_j(\xi) = \psi(2^{-j}\xi)$ for $j \neq 0$. The functions $\psi_j$, for $j \in \mathbb{Z}$, are supported in $\{\xi \in \mathbb{R}^n\colon\ 2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}\}$ and they realize a dyadic decomposition of the unity:

$$\sum_{j\in\mathbb{Z}} \psi_j(x) = 1.$$

We further denote

$$\phi_j(\xi) := \sum_{k=-\infty}^{j} \psi_k(\xi).$$

The function $\phi_j$ is supported on $\{\xi,\ |\xi| \leqslant 2^{j+1}\}$.

For every $j \in \mathbb{Z}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ we define the Littlewood–Paley projection operators $P_j$ and $P_{\leqslant j}$ by

$$\widehat{P_j f} = \psi_j \hat{f}, \qquad \widehat{P_{\leqslant j} f} = \phi_j \hat{f}.$$

Informally $P_j$ is a frequency projection to the annulus $\{2^{j-1} \leqslant |\xi| \leqslant 2^j\}$, while $P_{\leqslant j}$ is a frequency projection to the ball $\{|\xi| \leqslant 2^j\}$. We will set $f_j = P_j f$ and $f^j = P_{\leqslant j} f$.

We observe that $f^j = \sum_{k=-\infty}^{j} f_k$ and $f = \sum_{k=-\infty}^{+\infty} f_k$ (where the convergence is in $\mathcal{S}'(\mathbb{R}^n)$).

Given $f, g \in \mathcal{S}'(\mathbb{R}^n)$ we can split the product in the following way

$$fg = \Pi_1(f,g) + \Pi_2(f,g) + \Pi_3(f,g), \tag{54}$$

where

$$\Pi_1(f,g) = \sum_{-\infty}^{+\infty} f_j \sum_{k\leqslant j-4} g_k = \sum_{-\infty}^{+\infty} f_j g^{j-4};$$

$$\Pi_2(f,g) = \sum_{-\infty}^{+\infty} f_j \sum_{k\geqslant j+4} g_k = \sum_{-\infty}^{+\infty} g_j f^{j-4};$$

$$\Pi_3(f,g) = \sum_{-\infty}^{+\infty} f_j \sum_{|k-j|<4} g_k.$$

We observe that for every $j$ we have

$$\operatorname{supp} \mathcal{F}\big[f^{j-4} g_j\big] \subset \big\{2^{j-2} \leqslant |\xi| \leqslant 2^{j+2}\big\};$$

$$\operatorname{supp} \mathcal{F}\Bigg[\sum_{k=j-3}^{j+3} f_j g_k\Bigg] \subset \big\{|\xi| \leqslant 2^{j+5}\big\}.$$

The three pieces of the decomposition (54) are examples of paraproducts. Informally the first paraproduct $\Pi_1$ is an operator which allows high frequencies of $f$ $(\sim 2^j)$ multiplied by low frequencies of $g$ $(\ll 2^j)$ to produce high frequencies in the output. The second paraproduct $\Pi_2$ multiplies low frequencies of $f$ with high frequencies of $g$ to produce high frequencies in the output. The third paraproduct $\Pi_3$ multiplies high frequencies of $f$ with high frequencies of $g$ to produce comparable or lower frequencies in the output. For a presentation of these paraproducts we refer the reader for instance to the book [17]. The following two lemmas will be often used in the sequel.

**Lemma A.1.** *For every $f \in \mathcal{S}'$ we have*

$$\sup_{j \in Z} \left| f^j \right| \leqslant M(f).$$

**Lemma A.2.** *Let $\psi$ be a Schwartz radial function such that* $\operatorname{supp}(\psi) \subset B(0,4)$. *Then for every $s \geqslant [\frac{n}{2}] + 1$ we have*

$$\left\| (-\Delta)^s \mathcal{F}^{-1} \psi \right\|_{L^1} \leqslant C_{\psi,n} \left(1 + s^{n+1}\right) 4^{2s},$$

*where $C_{\psi,n}$ is a positive constant depending on the $C^2$ norm of $\psi$ and the dimension.*

**Lemma A.3.** *Let $f \in \dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$.*[2] *Then for all $s \geqslant [\frac{n}{2}] + 1$ and for all $j \in Z$ we have*

$$2^{-2sj} \left\| (-\Delta)^s f_j \right\|_{L^\infty} \leqslant C_{\psi,n} \left(1 + s^{n+1}\right) 4^{2s} \| f \|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)}.$$

For the proof of Lemma A.1 we refer to [9] and of Lemmas A.2 and A.3 we refer to [6].

Given $u$, $Q$ we introduce the following pseudodifferential operators

$$T(Q,u) := (-\Delta)^{1/4}\left(Q(-\Delta)^{1/4} u\right) - Q(-\Delta)^{1/2} u + (-\Delta)^{1/4} u (-\Delta)^{1/4} Q \tag{55}$$

and

$$S(Q,u) := (-\Delta)^{1/4}\left[Q(-\Delta)^{1/4} u\right] - \mathcal{R}(Q \nabla u) + \mathcal{R}\left((-\Delta)^{1/4} Q \mathcal{R}(-\Delta)^{1/4} u\right) \tag{56}$$

and $\mathcal{R}$ is the Fourier multiplier of symbol $m(\xi) = -i\frac{\xi}{|\xi|}$. We prove in this section some estimates on the operators (55) and (56).

**Proof of Theorem 1.3.** We make the proof for $n = 1$. The case $n > 1$ is analogous (for the details we refer to [7]).

- Estimate of $\|\Pi_1((-\Delta)^{1/4} Q(-\Delta)^{1/4} u)\|_{\mathcal{H}^1}$.

$$\begin{aligned} \left\| \Pi_1\left((-\Delta)^{1/4} Q(-\Delta)^{1/4} u\right) \right\|_{\mathcal{H}^1} &= \int_{\mathbb{R}^n} \left( \sum_{j=-\infty}^{\infty} 2^j Q_j^2 \left((-\Delta)^{1/4} u^{j-4}\right)^2 \right)^{1/2} dx \\ &\leqslant \int_{\mathbb{R}^n} \sup_j \left|(-\Delta)^{1/4} u^{j-4}\right| \left( \sum_j 2^j Q_j^2 \right)^{1/2} dx \\ &\leqslant \left( \int_{\mathbb{R}^n} \left(M\left((-\Delta)^{1/4} u\right)\right)^2 dx \right)^{1/2} \left( \int_R \sum_j 2^j Q_j^2 \, dx \right)^{1/2} \\ &\leqslant C \|Q\|_{\dot{H}^{1/2}(\mathbb{R})} \|u\|_{\dot{H}^{1/2}(\mathbb{R})}. \end{aligned} \tag{57}$$

- Estimate of $\|\Pi_1((-\Delta)^{1/4} Q(-\Delta)^{1/4} u)\|_{\mathcal{H}^1}$.

$$\begin{aligned} \left\| \Pi_1\left((-\Delta)^{1/4} Q(-\Delta)^{1/4} u\right) \right\|_{\mathcal{H}^1} &= \int_{\mathbb{R}^n} \left( \sum_{j=-\infty}^{\infty} \left((-\Delta)^{1/4} Q_j\right)^2 \left((-\Delta)^{1/4} u^{j-4}\right)^2 \right)^{1/2} dx \\ &\leqslant \int_{\mathbb{R}^n} \sup_j \left|(-\Delta)^{1/4} u^{j-4}\right| \left( \sum_j \left((-\Delta)^{1/4} Q_j\right)^2 \right)^{1/2} dx \end{aligned} \tag{58}$$

[2] The homogeneous Besov space $\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$ is the space of tempered distribution $u$ for which $\|u\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)} := \sup_{j \in \mathbb{Z}} \|\mathcal{F}^{-1} \times [\psi_j \mathcal{F}[u]]\|_{L^\infty(\mathbb{R}^n)}$ is finite (see for the precise definition of the Besov spaces [22]).

$$\leqslant \left(\int_{\mathbb{R}^n} \big(M\big((-\Delta)^{1/4}u\big)\big)^2\,dx\right)^{1/2}\left(\int_R \sum_j \big((-\Delta)^{1/4}Q_j\big)^2\,dx\right)^{1/2}$$
$$\leqslant C\|Q\|_{\dot H^{1/2}(\mathbb{R})}\|u\|_{\dot H^{1/2}(\mathbb{R})}.$$

- Estimate of $\|\Pi_2((-\Delta)^{1/4}Q(-\Delta)^{1/4}u)\|_{\mathcal{H}^1}$. It is as in (58).
- Estimate of $\Pi_3((-\Delta)^{1/4}(Q(-\Delta)^{1/4}u))$.
  We show that it is in $\dot B^0_{1,1}$. We observe that if $h\in \dot B^0_{\infty,\infty}$ then $(-\Delta)^{1/4}h\in \dot B^{-1/2}_{\infty\infty}$ and thus

$$\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\leqslant 1}\int_{\mathbb{R}}\sum_j\sum_{|k-j|\leqslant 3}(-\Delta)^{1/4}\big(Q_j(-\Delta)^{1/4}u_k\big)h$$
$$=\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\leqslant 1}\int_{\mathbb{R}}\sum_j\sum_{|k-j|\leqslant 3}(-\Delta)^{1/4}\big(Q_j(-\Delta)^{1/4}u_k\big)\left[(-\Delta)^{1/4}h^{j-6}+\sum_{t=j-5}^{j+6}(-\Delta)^{1/4}h_t\right]dx.$$

We have

$$\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\leqslant 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\leqslant 3}(-\Delta)^{1/4}\big(Q_j(-\Delta)^{1/4}u_k\big)h^{j-6}\,dx$$
$$=\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\leqslant 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\leqslant 3}\big(Q_j(-\Delta)^{1/4}u_k\big)(-\Delta)^{1/4}h^{j-6}\,dx$$
$$\leqslant C\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\leqslant 1}\|h\|_{\dot B^0_{\infty,\infty}}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\leqslant 3}2^{j/2}Q_j(-\Delta)^{1/4}u_k\,dx$$
$$\leqslant C\left(\int_{\mathbb{R}^n}\sum_j 2^jQ_j^2dx\right)^{1/2}\left(\int_{\mathbb{R}^n}\sum_j\big((-\Delta)^{1/4}u_j\big)^2\,dx\right)^{1/2}$$
$$\leqslant C\|Q\|_{\dot H^{1/2}(\mathbb{R})}\|u\|_{\dot H^{1/2}(\mathbb{R})}.$$

By analogous computations we get

$$\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\leqslant 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\leqslant 3}(-\Delta)^{1/4}\big(Q_j(-\Delta)^{1/4}u_k\big)\left[\sum_{t=j-5}^{j+6}(-\Delta)^{1/4}h_t\right]dx\leqslant C\|Q\|_{\dot H^{1/2}(\mathbb{R})}\|u\|_{\dot H^{1/2}(\mathbb{R})}.$$

- Estimate of $\Pi_3((-\Delta)^{1/4}Q(-\Delta)^{1/4}u-Q(-\Delta)^{1/2}u)$.

$$\big\|\Pi_3\big((-\Delta)^{1/4}Q(-\Delta)^{1/4}u-Q(-\Delta)^{1/2}u\big)\big\|_{\dot B^0_{1,1}}$$
$$=\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\leqslant 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\leqslant 3}\big[(-\Delta)^{1/4}\big(Q_j(-\Delta)^{1/4}u_k\big)-Q_j(-\Delta)^{1/2}u_k\big)\left[h^{j-6}+\sum_{t=j-5}^{j+6}h_t\right]dx. \tag{59}$$

We only estimate the terms with $h^{j-6}$, being the estimates with $h_t$ similar.

$$\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\leqslant 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\leqslant 3}\big[(-\Delta)^{1/4}\big(Q_j(-\Delta)^{1/4}u_k\big)-Q_j(-\Delta)^{1/2}u_k\big)\big[h^{j-6}\big]\,dx$$
$$=\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\leqslant 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\leqslant 3}\mathcal{F}\big[h^{j-6}\big]\mathcal{F}\big[(-\Delta)^{1/4}Q_j(-\Delta)^{1/4}u_k-Q_j(-\Delta)^{1/2}u_k\big]\,dx$$
$$=\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\leqslant 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\leqslant 3}\mathcal{F}\big[h^{j-6}\big]\left[\int_{\mathbb{R}^n}\mathcal{F}[Q_j](y)\mathcal{F}\big[(-\Delta)^{1/4}u_k\big](x-y)\big(|y|^{1/2}\right.$$
$$\left.-|x-y|^{1/2}\big)\,dy\right]dx. \tag{60}$$

Now we observe that in (60) we have $|x| \leqslant 2^{j-3}$ and $2^{j-2} \leqslant |y| \leqslant 2^{j+2}$. Thus $|\frac{x}{y}| \leqslant \frac{1}{2}$.
Hence

$$\begin{aligned}
|y|^{1/2} - |x-y|^{1/2} &= |y|^{1/2}\left[1 - \left|1 - \frac{x}{y}\right|^{1/2}\right] \\
&= |y|^{1/2}\frac{x}{y}\left[1 + \left|1 - \frac{x}{y}\right|^{1/2}\right]^{-1} \\
&= |y|^{1/2}\sum_{k=0}^{\infty}\frac{c_k}{k!}\left(\frac{x}{y}\right)^{k+1}.
\end{aligned} \tag{61}$$

We may suppose that $\sum_{k=0}^{\infty}\frac{c_k}{k!}(\frac{x}{y})^{k+1}$ is convergent if $|\frac{x}{y}| \leqslant \frac{1}{2}$, otherwise one may consider a different Littlewood–Paley decomposition by replacing the exponent $j-4$ with $j-s$, $s>0$ large enough. We introduce the following notation: for every $k \geqslant 0$ we set

$$S_k g = \mathcal{F}^{-1}\left[\xi^{-(k+1)}|\xi|^{1/2}\mathcal{F}g\right].$$

We note that if $h \in \dot{B}^s_{\infty,\infty}$ then $S_k h \in \dot{B}^{s+1/2+k}_{\infty,\infty}$ and if $h \in \dot{H}^s$ then $S_k h \in \dot{H}^{s+1/2+k}$.
Moreover if $Q \in \dot{H}^{1/2}$ then $\nabla^{k+1}(Q) \in \dot{H}^{-k-1/2}$.
We continue the estimate (60).

$$\begin{aligned}
(60) &= \sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\leqslant 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\leqslant 3}\mathcal{F}\left[h^{j-6}\right]\left[\int_{\mathbb{R}^n}\mathcal{F}[Q_j](y)\mathcal{F}\left[(-\Delta)^{1/4}u_k\right](x-y)\left(|y|^{1/2}\right.\right. \\
&\qquad \left.\left. - |x-y|^{1/2}\right)dy\right]dx \\
&= \sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\leqslant 1}\sum_{\ell=0}^{\infty}\frac{c_\ell}{\ell!}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\leqslant 3}(-i)^{\ell+1}\mathcal{F}\left[\nabla^{\ell+1}h^{j-6}\right]\mathcal{F}\left[S_\ell Q_j(-\Delta)^{1/4}u_k\right](x)\,dx \\
&\leqslant \sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\leqslant 1}\sum_{\ell=0}^{\infty}\frac{c_\ell}{\ell!}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\leqslant 3}\left[\nabla^{\ell+1}h^{j-6}\right]\left[S_\ell Q_j(-\Delta)^{1/4}u_k\right](x)\,dx \quad \text{by Lemma A.3} \\
&\leqslant C\sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\leqslant 1}\sum_{\ell=0}^{\infty}\frac{c_\ell}{\ell!}2^{-6\ell}4^{\ell+1}\|h\|_{B^0_{\infty,\infty}}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\leqslant 3}2^{(\ell+1)j}\left[S_\ell Q_j(-\Delta)^{1/4}u_k\right](x)\,dx \\
&\leqslant C\sum_{\ell=0}^{\infty}\frac{c_\ell}{\ell!}2^{-6\ell}4^{\ell+1}\left(\int_{\mathbb{R}^n}\sum_j 2^{2(\ell+1)j}|S_\ell Q_j|^2\,dx\right)^{1/2} \\
&\qquad \times\left(\int_{\mathbb{R}^n}\sum_j\left|(-\Delta)^{1/4}u_j\right|^2dx\right)^{1/2} \quad \text{by Plancherel's Theorem} \\
&\leqslant C\sum_{\ell=0}^{\infty}\frac{c_\ell}{\ell!}2^{-6\ell}4^{\ell+1}\left(\int_{\mathbb{R}^n}\sum_j 2^{2(\ell+1)j}\left|\mathcal{F}[S_\ell Q_j]\right|^2dx\right)^{1/2}\left(\int_{\mathbb{R}^n}\sum_j\left|(-\Delta)^{1/4}u_j\right|^2dx\right)^{1/2} \\
&\leqslant C\sum_{\ell=0}^{\infty}\frac{c_\ell}{\ell!}2^{-6\ell}4^{\ell+1}\left(\int_{\mathbb{R}^n}\sum_j 2^{2(\ell+1)j}2^{2(1-j)(\ell+1/2)}\left|\mathcal{F}[Q_j]\right|^2dx\right)^{1/2}\left(\int_{\mathbb{R}^n}\sum_j\left|(-\Delta)^{1/4}u_j\right|^2dx\right)^{1/2} \\
&\leqslant C\sum_{\ell=0}^{\infty}\frac{c_\ell}{\ell!}2^{-6\ell}4^{\ell+1}2^{\ell}\left(\int_{\mathbb{R}^n}\sum_j 2^j Q_j^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}^n}\sum_j\left|(-\Delta)^{1/4}u_j\right|^2dx\right)^{1/2} \\
&\leqslant C\sum_{\ell=0}^{\infty}\frac{c_\ell}{\ell!}2^{-3\ell}\|Q\|_{\dot{H}^{1/2}(\mathbb{R})}\|u\|_{\dot{H}^{1/2}(\mathbb{R})}.
\end{aligned}$$

- Estimate of $\Pi_2((-\Delta)^{1/4}(Q(-\Delta)^{1/4}u - Q(-\Delta)^{1/2}u))$.

$$
\begin{aligned}
&\big\|\Pi_2\big((-\Delta)^{1/4}\big(Q(-\Delta)^{1/4}u\big) - Q(-\Delta)^{1/2}u\big)\big\|_{\dot B^0_{1,1}} \\
&= \sup_{\|h\|_{\dot B^0_{\infty,\infty}}\leqslant 1} \int_{\mathbb{R}^n} \sum_j \sum_{|t-j|\leqslant 3} \big[(-\Delta)^{1/4}\big(Q^{j-4}(-\Delta)^{1/4}u_j\big) - (-\Delta)^{1/2}\big(Q^{j-4}u_j\big)h_t\big]\,dx \\
&= \sup_{\|h\|_{\dot B^0_{\infty,\infty}}\leqslant 1} \int_{\mathbb{R}^n} \sum_j \sum_{|t-j|\leqslant 3} \mathcal{F}\big[Q^{j-4}\big]\mathcal{F}\big[(-\Delta)^{1/4}u_j(-\Delta)^{1/4}h_t - (-\Delta)^{1/2}u_j h_t\big]\,dx \\
&= \sup_{\|h\|_{\dot B^0_{\infty,\infty}}\leqslant 1} \sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} \int_{\mathbb{R}^n} \sum_j \sum_{|t-j|\leqslant 3} (-i)^{\ell+1}\mathcal{F}\big[\nabla^{\ell+1}Q^{j-4}\big]\mathcal{F}\big[S_\ell(-\Delta)^{1/4}u_j h_t\big](x)\,dx \\
&\leqslant \sup_{\|h\|_{\dot B^0_{\infty,\infty}}\leqslant 1} \|h\|_{\dot B^0_{\infty,\infty}} \sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} \int_{\mathbb{R}^n} \sum_j \big|\nabla^{\ell+1}(Q)^{j-4}\big|\big|S_\ell(-\Delta)^{1/4}u_j\big|\,dx \\
&\leqslant \sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} \bigg(\int_{\mathbb{R}^n} \sum_j \big|2^{-(k+1/2)j}\nabla^{\ell+1}(Q)^{j-4}\big|\big|2^{(\ell+1/2)j}S_\ell(-\Delta)^{1/4}u_j\big|\,dx\bigg) \\
&\leqslant C\sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} \bigg(\int_{\mathbb{R}^n} \sum_j 2^{-2(\ell+1/2)j}\big|\nabla^{\ell+1}Q^{j-4}\big|^2\,dx\bigg)^{1/2} \\
&\quad \times \bigg(\int_{\mathbb{R}^n} \sum_j 2^{2(\ell+1/2)j}\big|S_\ell(-\Delta)^{1/4}u_j\big|^2\,dx\bigg)^{1/2} \quad \text{by Plancherel's Theorem} \\
&= C\sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} \bigg(\int_{\mathbb{R}^n} \sum_j 2^{-2(\ell+1/2)j}|\xi|^{2\ell}\big|\mathcal{F}\big[\nabla Q^{j-4}\big]\big|^2\,d\xi\bigg)^{1/2} \\
&\quad \times \bigg(\int_{\mathbb{R}^n} \sum_j 2^{2(\ell+1)j}|\xi|^{-2(\ell+1/2)}\big|\mathcal{F}\big[(-\Delta)^{1/4}u_j\big]\big|^2\,dx\bigg)^{1/2} \\
&\leqslant C\sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} 2^{-3\ell} \bigg(\int_{\mathbb{R}^n} \sum_j 2^{-j}\big|\mathcal{F}\big[\nabla Q^{j-4}\big]\big|^2 d\xi\bigg)^{1/2} \bigg(\int_{\mathbb{R}^n} \sum_j \big|\mathcal{F}\big[(-\Delta)^{1/4}u_j\big]\big|^2\,dx\bigg)^{1/2} \\
&\leqslant C\sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} 2^{-3\ell}\|Q\|_{\dot H^{1/2}(\mathbb{R})}\|u\|_{\dot H^{1/2}(\mathbb{R})}. \qquad \square
\end{aligned}
\tag{62}
$$

The proof of the following theorems and its localized version can be found in [7].

**Theorem A.1.** *Let $u, Q \in \dot W^{1/2,q}(\mathbb{R}^n)$, with $q > 2$. Then $T(Q,u), S(Q,u) \in L^{q/2}(\mathbb{R}^n)$ and*

$$\big\|T(Q,u)\big\|_{L^{q/2}} \leqslant C\big\|(-\Delta)^{1/4}Q\big\|_{L^q}\big\|(-\Delta)^{1/4}u\big\|_{L^q}; \tag{63}$$

$$\big\|S(Q,u)\big\|_{L^{q/2}} \leqslant C\big\|(-\Delta)^{1/4}Q\big\|_{L^q}\big\|(-\Delta)^{1/4}u\big\|_{L^q}. \tag{64}$$

**Theorem A.2.** *Let $Q \in \dot H^{1/2}(\mathbb{R}^n)$, $u \in \dot W^{1/2,q}(\mathbb{R}^n)$ with $q > 2$. Then $T(Q,u), S(Q,u) \in L^{\frac{2q}{q+2}}(\mathbb{R}^n)$ and*

$$\big\|T(Q,u)\big\|_{L^{\frac{2q}{q+2}}} \leqslant C\big\|(-\Delta)^{1/4}Q\big\|_{L^2}\big\|(-\Delta)^{1/4}u\big\|_{L^q}; \tag{65}$$

$$\big\|S(Q,u)\big\|_{L^{\frac{2q}{q+2}}} \leqslant C\big\|(-\Delta)^{1/4}Q\big\|_{L^2}\big\|(-\Delta)^{1/4}u\big\|_{L^q}. \tag{66}$$

**Remark A.1.** Actually Theorems A.1 and A.2 hold for the 2-terms commutators

$$\tilde{T}(Q,u) = T(Q,u) - (-\Delta)^{1/4}Q(-\Delta)^{1/4}u = (-\Delta)^{1/4}\big(Q(-\Delta)^{1/4}u\big) - Q\Delta^{1/2}u$$

and

$$\tilde{S}(Q,u) = S(Q,u) - \mathcal{R}\big((-\Delta)^{1/4}Q\mathcal{R}(-\Delta)^{1/4}u\big) = (-\Delta)^{1/4}\big[Q(-\Delta)^{1/4}u\big] - \mathcal{R}(Q\nabla u).$$

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