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On the weak solutions to the equations of a compressible heat conducting gas

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Abstract

We consider the weak solutions to the Euler–Fourier system describing the motion of a compressible heat conducting gas. Employing the method of convex integration, we show that the problem admits infinitely many global-in-time weak solutions for any choice of smooth initial data. We also show that for any initial distribution of the density and temperature, there exists an initial velocity such that the associated initial-value problem possesses infinitely many solutions that conserve the total energy.

1. Introduction

The concept of *weak solution* has been introduced in the mathematical theory of systems of (nonlinear) hyperbolic conservation laws to incorporate the inevitable singularities in their solutions that may develop in a finite time no matter how smooth and small the data are. As is well known, however, many nonlinear problems are not well posed in the weak framework and several classes of admissible weak solutions have been identified to handle this issue. The implications of the *Second law of thermodynamics* have been widely used in the form of various entropy conditions in order to identify the physically relevant solutions. Although this approach has been partially successful when dealing with systems in the simplified $1D$-geometry, see Bianchini and Bressan [4], Bressan [5], Dafermos [10], Liu [18], among others, the more realistic problems in higher spatial dimensions seem to be out of reach of the theory mostly because the class of "entropies" is rather poor consisting typically of a single (physical) entropy. Recently, De Lellis and Székelyhidi [12] developed the so-called Baire category method from the theory of differential inclusions (cf. Bressan and Flores [6], Cellina [7], Dacorogna and Marcellini [9], Kirchheim [16], Müller and Šverák [19]) to identify a large class of weak solutions to the Euler system violating the principle of well-posedness in various directions. Besides the apparently non-physical solutions producing the kinetic energy (cf. Shnirelman [22]), a large class of data

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has been identified admitting infinitely many weak solutions that comply with a major part of the known admissibility criteria, see De Lellis and Székelyhidi [11].

In this paper, we develop the technique of [11] to examine the well-posedness of the full *Euler–Fourier system*:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = 0, \tag{1.2}$$

$$\partial_t\big(\varrho e(\varrho, \vartheta)\big) + \mathrm{div}_x\big(\varrho e(\varrho, \vartheta)\mathbf{u}\big) + \mathrm{div}_x \mathbf{q} = -p(\varrho, \vartheta)\,\mathrm{div}_x \mathbf{u}, \tag{1.3}$$

where $\varrho(t,x)$ is the mass density, $\mathbf{u} = \mathbf{u}(t,x)$ the velocity field, and $\vartheta(t,x)$ the (absolute) temperature of a compressible, heat conducting gas, see Wilcox [23]. For the sake of simplicity, we restrict ourselves to the case of perfect monoatomic gas, for which the pressure $p(\varrho, \vartheta)$ and the specific internal energy $e(\varrho, \vartheta)$ are interrelated through the constitutive equations:

$$p(\varrho, \vartheta) = \frac{2}{3}\varrho e(\varrho, \vartheta), \qquad p(\varrho, \vartheta) = a\varrho\vartheta, \quad a > 0. \tag{1.4}$$

Although the system (1.1)–(1.3) describes the motion in the absence of viscous forces, we suppose that the fluid is heat conductive, with the heat flux $\mathbf{q}$ determined by the standard Fourier law:

$$\mathbf{q} = -\kappa \nabla_x \vartheta, \quad \kappa > 0. \tag{1.5}$$

The problem (1.1)–(1.5) is supplemented with the initial data

$$\varrho(0,\cdot) = \varrho_0, \qquad (\varrho\mathbf{u})(0,\cdot) = \varrho_0 \mathbf{u}_0, \qquad \vartheta(0,\cdot) = \vartheta_0 \quad \text{in } \Omega. \tag{1.6}$$

In addition, to avoid the effect of the kinematic boundary, we consider the periodic boundary conditions, meaning the physical domain Ω will be taken the flat torus

$$\Omega = \mathbb{T}^3 = \big([0,1]|_{\{0,1\}}\big)^3.$$

The first part of the paper exploits the constructive aspect of convex integration. We present a "variable coefficients" variant of a result of De Lellis and Székelyhidi [11] (cf. Remark 3.2 below) and show the existence of *infinitely many* global-in-time[3] weak solutions to the problem (1.1)–(1.6) for any physically relevant choice of (smooth) initial data. Here, physically relevant means that the initial distribution of the density ϱ_0 and the temperature ϑ_0 are strictly positive in Ω. These solutions satisfy also the associated entropy equation; whence they comply with the Second law of thermodynamics.

Similarly to their counterparts constructed in [11], these "wild" weak solutions violate the First law of thermodynamics, specifically, the total energy at any positive time is strictly larger than for the initial data. In order to eliminate the non-physical solutions, we therefore impose the total energy conservation in the form:

$$E(t) = \int_\Omega \varrho\left(\frac{1}{2}|\mathbf{u}|^2 + e(\varrho, \vartheta)\right)(t,x)\,\mathrm{d}x = \int_\Omega \varrho_0\left(\frac{1}{2}|\mathbf{u}_0|^2 + e(\varrho_0, \vartheta_0)(x)\right)\mathrm{d}x$$
$$= E_0 \quad \text{for (a.a.) } t \in (0,T). \tag{1.7}$$

Following [15] we show that the system (1.1)–(1.6), augmented with the total energy balance (1.7), satisfies the principle of *weak-strong uniqueness*. Specifically, the weak and strong solutions emanating from the same initial data necessarily coincide as long as the latter exists. In other words, the strong solutions are unique in the class of weak solutions. This property remains valid even if we replace the internal energy equation (1.3) by the entropy *inequality*

[3] By global-in-time solutions we mean here solutions defined on $[0,T)$ for any given $T > 0$. For discussion about solutions defined on $[0,\infty)$ see Section 5.

$$\partial_t\big(\varrho s(\varrho,\vartheta)\big)+\operatorname{div}_x\big(\varrho s(\varrho,\vartheta)\mathbf{u}\big)+\operatorname{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right)\geqslant-\frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta^2},$$
$$\vartheta Ds(\varrho,\vartheta)\equiv De(\varrho,\vartheta)+p(\varrho,\vartheta)D\left(\frac{1}{\varrho}\right), \tag{1.8}$$

in the spirit of the theory developed in [14].

Although the stipulation of (1.7) obviously eliminates the non-physical energy producing solutions, we will show that for *any* initial data ϱ_0, ϑ_0 there exists an initial velocity $\mathbf{u}_0$ such that the problem (1.1)–(1.6) admits infinitely many global-in-time weak solutions that satisfy the total energy balance (1.7).

The paper is organized as follows. After a brief introduction of the concept of weak solutions in Section 2, we discuss the problem of existence of infinitely many solutions for arbitrary initial data, see Section 3. In particular, we prove a "variable coefficients" variant of a result of De Lellis and Székelyhidi [11] and employ the arguments based on Baire's category. In Section 4, we show the weak-strong uniqueness principle for the augmented system and then identify the initial data for which the associated solutions conserve the total energy. The paper is concluded by some remarks on possible extensions in Section 5.

2. Weak solutions

To simplify presentation, we may assume, without loss of generality, that

$$a=\kappa=1.$$

We say that a trio $[\varrho,\vartheta,\mathbf{u}]$ is a *weak solution* of the problem (1.1)–(1.6) in the space–time cylinder $(0,T)\times\Omega$ if:

- the density ϱ and the temperature ϑ are positive in $(0,T)\times\Omega$;
- $$\int_0^T\int_\Omega(\varrho\partial_t\varphi+\varrho\mathbf{u}\cdot\nabla_x\varphi)\,\mathrm{d}x\,\mathrm{d}t=-\int_\Omega\varrho_0\varphi(0,\cdot)\,\mathrm{d}x \tag{2.1}$$
  for any test function $\varphi\in C_c^\infty([0,T)\times\Omega)$;
- $$\int_0^T\int_\Omega(\varrho\mathbf{u}\cdot\partial_t\varphi+\varrho\mathbf{u}\otimes\mathbf{u}:\nabla_x\varphi+\varrho\vartheta\operatorname{div}_x\varphi)\,\mathrm{d}x\,\mathrm{d}t=-\int_\Omega\varrho_0\mathbf{u}_0\cdot\varphi(0,\cdot)\,\mathrm{d}x \tag{2.2}$$
  for any test function $\varphi\in C_c^\infty([0,T)\times\Omega;R^3)$;
- $$\int_0^T\int_\Omega\left(\frac{3}{2}[\varrho\vartheta\partial_t\varphi+\varrho\vartheta\mathbf{u}\cdot\nabla_x\varphi]-\nabla_x\vartheta\cdot\nabla_x\varphi-\varrho\vartheta\operatorname{div}_x\mathbf{u}\varphi\right)\mathrm{d}x\,\mathrm{d}t=-\int_\Omega\varrho_0\vartheta_0\varphi(0,\cdot)\,\mathrm{d}x \tag{2.3}$$
  for any test function $\varphi\in C_c^\infty([0,T)\times\Omega)$.

As a matter of fact, the weak solutions we construct in this paper will be rather regular with the only exception of the velocity field. In particular, the functions ϱ, ϑ, and even $\operatorname{div}_x\mathbf{u}$ will be continuously differentiable in $[0,T]\times\Omega$, and, in addition,

$$\vartheta\in L^p\big(0,T;W^{2,p}(\Omega)\big),\qquad\partial_t\vartheta\in L^p\big(0,T;L^p(\Omega)\big)\quad\text{for any }1\leqslant p<\infty.$$

Thus Eqs. (1.1), (1.3) will be in fact satisfied pointwise a.a. in $(0,T)\times\Omega$. As for the velocity field, we have

$$\mathbf{u}\in C_{\text{weak}}\big([0,T];L^2(\Omega;R^3)\big)\cap L^\infty\big((0,T)\times\Omega;R^3\big),\qquad\operatorname{div}_x\mathbf{u}\in C\big([0,T]\times\Omega\big).$$

Here and hereafter, the symbol $C_{\text{weak}}([0,T];X)$ denotes the space of continuous functions in the t-variable with respect to the weak topology of the space X.

3. Second law is not enough

Our first objective is to show the existence of infinitely many solutions to the Euler–Fourier system for arbitrary (smooth) initial data.

Theorem 3.1. *Let $T > 0$. Let the initial data satisfy*

$$\varrho_0 \in C^3(\Omega), \qquad \vartheta_0 \in C^2(\Omega), \qquad \mathbf{u}_0 \in C^3(\Omega; R^3),$$
$$\varrho_0(x) > \underline{\varrho} > 0, \qquad \vartheta_0(x) > \underline{\vartheta} > 0 \quad \text{for any } x \in \Omega. \tag{3.1}$$

Then the initial-value problem (1.1)–(1.6) admits infinitely many weak solutions in $(0,T) \times \Omega$ belonging to the class:

$$\varrho \in C^2([0,T] \times \Omega), \qquad \partial_t \vartheta \in L^p(0,T; L^p(\Omega)), \qquad \nabla_x^2 \vartheta \in L^p(0,T; L^p(\Omega; R^{3\times 3})) \quad \text{for any } 1 \leqslant p < \infty,$$
$$\mathbf{u} \in C_{\text{weak}}([0,T]; L^2(\Omega; R^3)) \cap L^\infty((0,T) \times \Omega; R^3), \qquad \operatorname{div}_x \mathbf{u} \in C^2([0,T] \times \Omega).$$

Remark 3.1. Using the maximal regularity theory for parabolic equations (see Amann [3], Krylov [17]) we observe that ϑ is a continuous function of the time variable t ranging in the interpolation space $[L^p(\Omega); W^{2,p}(\Omega)]_\alpha$ for any $1 \leqslant p < \infty$ finite and any $\alpha \in (0,1)$. Thus it is possible to show that the conclusion of Theorem 3.1 remains valid if we assume that

$$\vartheta_0 \in [L^p(\Omega); W^{2,p}(\Omega)]_\alpha \quad \text{for sufficiently large } 1 \leqslant p < \infty \text{ and } 0 < \alpha < 1, \qquad \vartheta_0 > 0 \quad \text{in } \Omega,$$

where $[\ ,\ ]_\alpha$ denotes the real interpolation. In particular, the solution $\vartheta(t,\cdot)$ will remain in the same regularity class for any $t \in [0,T]$.

The rest of this section is devoted to the proof of Theorem 3.1.

3.1. *Reformulation*

Following Chiodaroli [8], we reformulate the problem in the new variables ϱ, ϑ, and $\mathbf{w} = \varrho \mathbf{u}$ obtaining, formally

$$\partial_t \varrho + \operatorname{div}_x \mathbf{w} = 0, \tag{3.2}$$
$$\partial_t \mathbf{w} + \operatorname{div}_x \left(\frac{\mathbf{w} \otimes \mathbf{w}}{\varrho} \right) + \nabla_x (\varrho \vartheta) = 0, \tag{3.3}$$
$$\frac{3}{2}(\varrho \partial_t \vartheta + \mathbf{w} \cdot \nabla_x \vartheta) = \Delta \vartheta - \vartheta \operatorname{div}_x \mathbf{w} + \vartheta \frac{\nabla_x \varrho}{\varrho} \cdot \mathbf{w}. \tag{3.4}$$

Next, we take the following ansatz for the density:

$$\varrho(t,x) = \varrho_0(x) - h(t) \operatorname{div}_x (\varrho_0(x) \mathbf{u}_0(x)) \equiv \tilde{\varrho}(t,x),$$

with

$$h \in C^2[0,T], \qquad h(0) = 0, \qquad h'(0) = 1,$$
$$\varrho_0(x) - h(t) \operatorname{div}_x (\varrho_0(x) \mathbf{u}_0(x)) > \frac{\underline{\varrho}}{2} \quad \text{for all } t \in [0,T],\ x \in \Omega. \tag{3.5}$$

Accordingly, we write $\mathbf{w}$ in the form of its Helmholtz decomposition

$$\mathbf{w} = \mathbf{v} + \nabla_x \Psi, \qquad \operatorname{div}_x \mathbf{v} = 0, \qquad \Delta \Psi = h'(t) \operatorname{div}_x (\varrho_0 \mathbf{u}_0) = \partial_t \tilde{\varrho}, \qquad \int_\Omega \Psi \, \mathrm{d}x = 0.$$

Obviously, by virtue of the hypotheses (3.1) imposed on the initial data, we have

$$\tilde{\varrho} \in C^2([0,T] \times \Omega), \qquad \nabla_x \Psi \in C^2([0,T] \times \Omega; R^3), \qquad \tilde{\varrho}(0,\cdot) = \varrho_0, \qquad \mathbf{w}_0 = \varrho_0 \mathbf{u}_0.$$

Moreover, the equation of continuity (3.2) is satisfied pointwise in $(0,T)\times\Omega$, while the remaining two equations (3.2), (3.3) read

$$\partial_t \mathbf{v} + \operatorname{div}_x\left(\frac{(\mathbf{v}+\nabla_x\Psi)\otimes(\mathbf{v}+\nabla_x\Psi)}{\tilde{\varrho}}\right) + \nabla_x(\tilde{\varrho}\vartheta + \partial_t\Psi) = 0, \qquad \operatorname{div}_x \mathbf{v} = 0, \tag{3.6}$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0 = \varrho_0\mathbf{u}_0 - \nabla_x\Delta^{-1}\operatorname{div}_x(\varrho_0\mathbf{u}_0), \tag{3.7}$$

$$\frac{3}{2}\big(\tilde{\varrho}\partial_t\vartheta + (\mathbf{v}+\nabla_x\Psi)\cdot\nabla_x\vartheta\big) = \Delta\vartheta - \vartheta\Delta\Psi + \vartheta\frac{\nabla_x\tilde{\varrho}}{\tilde{\varrho}}\cdot(\mathbf{v}+\nabla_x\Psi), \qquad \vartheta(0,\cdot) = \vartheta_0. \tag{3.8}$$

3.2. *Internal energy and entropy equations*

For a given vector field $\mathbf{v}\in L^\infty((0,T)\times\Omega;R^3)$, the internal energy equation (3.8) is *linear* with respect to ϑ and as such admits a unique solution $\vartheta = \vartheta[\mathbf{v}]$ satisfying the initial condition $\vartheta(0,\cdot) = \vartheta_0$. Moreover, the standard L^p-theory for parabolic equations (see e.g. Krylov [17]) yields

$$\begin{aligned}
&\vartheta(t,x) > 0 \quad \text{for all } t\in[0,T],\ x\in\Omega,\\
&\partial_t\vartheta \in L^p\big(0,T;L^p(\Omega)\big), \qquad \nabla_x^2\vartheta\in L^p\big(0,T;L^p\big(\Omega;R^{3\times 3}\big)\big) \quad \text{for any } 1\leqslant p<\infty,
\end{aligned} \tag{3.9}$$

where the bounds depend only on the data and $\|\mathbf{v}\|_{L^\infty((0,T)\times\Omega;R^3)}$.

Dividing (3.8) by ϑ we deduce the *entropy equation*

$$\tilde{\varrho}\partial_t\log\left(\frac{\vartheta^{3/2}}{\tilde{\varrho}}\right) + (\mathbf{v}+\nabla_x\Psi)\cdot\nabla_x\log\left(\frac{\vartheta^{3/2}}{\tilde{\varrho}}\right) = \Delta\log(\vartheta) + \big|\nabla_x\log(\vartheta)\big|^2, \tag{3.10}$$

where we have used the identity $-\Delta\Psi = \partial_t\tilde{\varrho}$. We note that, given the regularity of the solutions in Theorem 3.1, the entropy equation (3.10) and the internal energy equation (3.8) are *equivalent*. In particular, the weak solutions we construct are compatible with the Second law of thermodynamics.

3.2.1. *Uniform bounds*

Introducing a new variable

$$Z = \log\left(\frac{\vartheta^{3/2}}{\tilde{\varrho}}\right)$$

we may rewrite (3.10) as

$$\tilde{\varrho}\partial_t Z + \left(\mathbf{v}+\nabla_x\Psi - \frac{8}{9}\nabla_x\log(\tilde{\varrho})\right)\cdot\nabla_x Z = \frac{2}{3}\Delta Z + \frac{4}{9}|\nabla_x Z|^2 + \frac{2}{3}\Delta\log(\tilde{\varrho}) + \frac{4}{9}\big|\nabla_x\log(\tilde{\varrho})\big|^2. \tag{3.11}$$

Applying the standard parabolic comparison principle to (3.11) we conclude that $|Z|$ is bounded only in terms of the initial data and the time T. Consequently, the constants $\underline{\vartheta}$, $\overline{\vartheta}$ can be taken in such a way that

$$0 < \underline{\vartheta} \leqslant \vartheta[\mathbf{v}](t,x) \leqslant \overline{\vartheta} \quad \text{for all } t\in[0,T],\ x\in\Omega. \tag{3.12}$$

We emphasize that the constants $\underline{\vartheta}$, $\overline{\vartheta}$ are *independent* of $\mathbf{v}$ – a crucial fact that will be used in the future analysis.

3.3. *Reduction to a modified Euler system*

Summing up the previous discussion, our task reduces to finding (infinitely many) solutions to the problem

$$\begin{aligned}
&\partial_t\mathbf{v} + \operatorname{div}_x\left(\frac{(\mathbf{v}+\nabla_x\Psi)\otimes(\mathbf{v}+\nabla_x\Psi)}{\tilde{\varrho}}\right) + \nabla_x\left(\tilde{\varrho}\vartheta[\mathbf{v}] + \partial_t\Psi - \frac{2}{3}\chi\right) = 0,\\
&\operatorname{div}_x\mathbf{v} = 0, \qquad \mathbf{v}(0,\cdot) = \mathbf{v}_0,
\end{aligned} \tag{3.13}$$

with a suitable spatially homogeneous function $\chi = \chi(t)$. Since $\nabla_x\chi = 0$, the specific value of χ is irrelevant in the equations but plays a role when reformulating the problem in terms of differential inclusions.

Following the strategy (and notation) of De Lellis and Székelyhidi [11], we introduce the linear system

$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0, \qquad \mathrm{div}_x \mathbf{v} = 0, \qquad \mathbf{v}(0,\cdot) = \mathbf{v}_0, \qquad \mathbf{v}(T,\cdot) = \mathbf{v}_T, \tag{3.14}$$

together with the function $\bar{e}$,

$$\bar{e}[\mathbf{v}] = \chi - \frac{3}{2}\tilde{\varrho}\vartheta[\mathbf{v}] - \frac{3}{2}\partial_t \Psi, \tag{3.15}$$

with a positive function $\chi \in C[0,T]$ determined below.

Furthermore, we introduce the space $R^{3\times3}_{\mathrm{sym},0}$ of symmetric traceless matrices, with the operator norm

$\lambda_{\max}[\mathbb{U}]$ – the maximal eigenvalue of $\mathbb{U} \in R^{3\times3}_{\mathrm{sym},0}$.

Finally, we define the set of subsolutions

$$\begin{aligned} X_0 = \Big\{ \mathbf{v} \;\Big|\; & \mathbf{v} \in L^\infty\big((0,T)\times\Omega; R^3\big) \cap C^1\big((0,T)\times\Omega; R^3\big) \cap C_{\mathrm{weak}}\big([0,T]; L^2\big(\Omega; R^3\big)\big), \\ & \mathbf{v} \text{ satisfies (3.14) with some } \mathbb{U} \in C^1\big((0,T)\times\Omega; R^{3\times3}_{\mathrm{sym},0}\big), \\ & \inf_{t\in(\varepsilon,T),\, x\in\Omega} \left\{ \bar{e}[\mathbf{v}] - \frac{3}{2}\lambda_{\max}\left[\frac{(\mathbf{v}+\nabla_x\Psi)\otimes(\mathbf{v}+\nabla_x\Psi)}{\tilde{\varrho}} - \mathbb{U}\right]\right\} > 0 \text{ for any } 0<\varepsilon<T \Big\}. \end{aligned} \tag{3.16}$$

Remark 3.2. Note that X_0 is substantially different from its analogue introduced by Chiodaroli [8] and De Lellis and Székelyhidi [11], in particular, the function $\bar{e}[\mathbf{v}]$ depends on the field $\mathbf{v}$.

As shown by De Lellis and Székelyhidi [11], we have the (pointwise) inequality

$$\frac{1}{2}|\mathbf{w}|^2 \leqslant \frac{3}{2}\lambda_{\max}[\mathbf{w}\otimes\mathbf{w} - \mathbb{U}], \quad \mathbf{w} \in R^3, \; \mathbb{U} \in R^{3\times3}_{\mathrm{sym},0},$$

where the identity holds only if

$$\mathbb{U} = \mathbf{w}\otimes\mathbf{w} - \frac{1}{3}|\mathbf{w}|^2\mathbb{I}.$$

Consequently, by virtue of (3.12), there exists a constant c depending only on the initial data $[\varrho_0, \vartheta_0, \mathbf{u}_0]$ such that

$$\sup_{t\in[0,T]} \big\|\mathbf{v}(t,\cdot)\big\|_{L^\infty(\Omega;R^3)} < c \quad \text{for all } \mathbf{v} \in X_0. \tag{3.17}$$

Next, we choose the function $\chi \in C[0,T]$ in (3.15) so large that

$$\frac{3}{2}\lambda_{\max}\left[\frac{(\mathbf{v}_0+\nabla_x\Psi)\otimes(\mathbf{v}_0+\nabla_x\Psi)}{\tilde{\varrho}}\right] < \chi - \frac{3}{2}\tilde{\varrho}\vartheta[\mathbf{v}_0] - \frac{3}{2}\partial_t\Psi \equiv \bar{e}[\mathbf{v}_0] \quad \text{for all } (t,x) \in [0,T]\times\Omega,$$

in particular, the function $\mathbf{v}_0 = \mathbf{v}_0(x)$, together with the associated tensor $\mathbb{U} \equiv 0$, belongs to the set X_0, where $\mathbf{v}_0 = \mathbf{v}_T$.

We define a topological space X as a completion of X_0 in $C_{\mathrm{weak}}([0,T]; L^2(\Omega; R^3))$ with respect to the metric d induced by the weak topology of the Hilbert space $L^2(\Omega; R^3)$. This is possible as all functions belonging to X_0 range a bounded ball of $L^2(\Omega; R^3)$, on which the weak topology is metrizable. As we have just observed, the space X_0 is non-empty as $\mathbf{v} = \mathbf{v}_0$ is in X_0.

Finally, we consider a family of functionals

$$I_\varepsilon[\mathbf{v}] = \int_\varepsilon^T \int_\Omega \left(\frac{1}{2}\frac{|\mathbf{v}+\nabla_x\Psi|^2}{\tilde{\varrho}} - \bar{e}[\mathbf{v}]\right) \mathrm{d}x\,\mathrm{d}t \quad \text{for } \mathbf{v} \in X, \; 0<\varepsilon<T. \tag{3.18}$$

As a direct consequence of the parabolic regularity estimates (3.9), we observe that

$$\bar{e}[\mathbf{v}] \to \bar{e}[\mathbf{w}] \quad \text{in } C\big([0,T]\times\Omega\big) \quad \text{whenever} \quad \mathbf{v} \to \mathbf{w} \quad \text{in } X; \tag{3.19}$$

therefore each I_ε is a compact perturbation of a convex functional; whence lower semi-continuous in X.

In order to proceed, we need the following crucial result that may be viewed as a "variable coefficients" counterpart of [11, Proposition 3].

Proposition 3.1. *Let* $\mathbf{v} \in X_0$ *such that*

$$I_\varepsilon[\mathbf{v}] < -\alpha < 0, \quad 0 < \varepsilon < T/2.$$

There is $\beta = \beta(\alpha) > 0$ *and a sequence* $\{\mathbf{v}_n\}_{n=1}^\infty \subset X_0$ *such that*

$$\mathbf{v}_n \to \mathbf{v} \quad \text{in } C_{\text{weak}}\big([0,T]; L^2\big(\Omega; R^3\big)\big), \qquad \liminf_{n\to\infty} I_\varepsilon[\mathbf{v}_n] \geqslant I_\varepsilon[\mathbf{v}] + \beta.$$

We point out that the quantity $\beta = \beta(\alpha)$ is independent of ε and $\mathbf{v}$.

Postponing the proof of Proposition 3.1 to the next section, we complete the proof of Theorem 3.1 following the line of arguments of [11]. It is worth noting that the arguments are based on points of continuity of suitable Baire-1 maps – a method introduced in Kirchheim's thesis [16]. To begin, we observe that cardinality of the space X_0 is infinite. Secondly, since each I_ε is a bounded lower semi-continuous functional on a complete metric space, the points of continuity of I_ε form a residual set in X. The set

$$\mathcal{C} = \bigcap_{m>1} \big\{\mathbf{v} \in X \;\big|\; I_{1/m}[\mathbf{v}] \text{ is continuous}\big\},$$

being an intersection of a countable family of residual sets, is residual, in particular of infinite cardinality, see De Lellis and Székelyhidi [11] for a more detailed explanation of these arguments.

Finally, we claim that for each $\mathbf{v} \in \mathcal{C}$ we have

$$I_{1/m}[\mathbf{v}] = 0 \quad \text{for all } m > 1;$$

whence

$$\frac{1}{2}\frac{|\mathbf{v} + \nabla_x \Psi|^2}{\tilde{\varrho}} = \bar{e}[\mathbf{v}] \equiv \chi - \frac{3}{2}\tilde{\varrho}\vartheta[\mathbf{v}] - \frac{3}{2}\partial_t \Psi,$$

$$\mathbb{U} = \frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\tilde{\varrho}} - \frac{1}{3}\frac{|\mathbf{v} + \nabla_x \Psi|^2}{\tilde{\varrho}}\mathbb{I} \quad \text{for a.a. } (t,x) \in (0,T) \times \Omega,$$

in other words, the function $\mathbf{v}$ is a weak solution to the problem (3.13). Indeed, assuming $I_{1/m}[\mathbf{v}] < -2\alpha < 0$, we first find a sequence $\{\mathbf{u}_n\}_{n=1}^\infty \subset X_0$ such that

$$\mathbf{u}_n \to \mathbf{v} \quad \text{in } C_{\text{weak}}\big([0,T]; L^2\big(\Omega; R^3\big)\big), \qquad I_{1/m}[\mathbf{u}_n] < -\alpha.$$

Then for each $\mathbf{u}_n$ we use Proposition 3.1 and together with standard diagonal argument we obtain a sequence $\{\mathbf{v}_n\}_{n=1}^\infty \subset X_0$ such that

$$\mathbf{v}_n \to \mathbf{v} \quad \text{in } C_{\text{weak}}\big([0,T]; L^2\big(\Omega; R^3\big)\big), \qquad \liminf_{n\to\infty} I_{1/m}[\mathbf{v}_n] \geqslant I_{1/m}[\mathbf{v}] + \beta, \quad \beta > 0,$$

in contrast with the fact that $\mathbf{v}$ is a point of continuity of $I_{1/m}$.

3.4. Proof of Proposition 3.1

The proof of Proposition 3.1 is based on a localization argument, where variable coefficients are replaced by constants. The fundamental building block is the following result proved by De Lellis and Székelyhidi [11, Proposition 3], Chiodaroli [8, Section 6, formula (6.9)]:

Lemma 3.1. *Let* $[T_1, T_2]$, $T_1 < T_2$, *be a time interval and* $B \subset R^3$ *a domain. Let* $\tilde{r} \in (0,\infty)$, $\tilde{\mathbf{V}} \in R^3$ *be* constant *fields such that*

$$0 < \underline{r} < \tilde{r} < \bar{r}, \qquad |\tilde{\mathbf{V}}| < \overline{V}.$$

Suppose that

$$\mathbf{v} \in C_{\text{weak}}\big([T_1,T_2]; L^2\big(B, R^3\big)\big) \cap C^1\big((T_1,T_2) \times \overline{B}; R^3\big)$$

satisfies the linear system

$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0, \qquad \mathrm{div}_x \mathbf{v} = 0 \quad \textit{in } (T_1, T_2) \times B$$

with the associated field $\mathbb{U} \in C^1((T_1, T_2) \times \overline{B}; R^{3\times 3}_{\mathrm{sym},0})$ *such that*

$$\frac{3}{2}\lambda_{\max}\left[\frac{(\mathbf{v} + \tilde{\mathbf{V}}) \otimes (\mathbf{v} + \tilde{\mathbf{V}})}{\tilde{r}} - \mathbb{U}\right] < e \quad \textit{in } (T_1, T_2) \times B$$

for a certain function $e \in C([T_1; T_2] \times \overline{B})$.

Then there exist sequences $\{\mathbf{w}_n\}_{n=1}^{\infty} \subset C_c^{\infty}((T_1, T_2) \times B; R^3)$, $\{\mathbb{Y}_n\}_{n=1}^{\infty} \subset C_c^{\infty}((T_1, T_2) \times B; R^{3\times 3}_{\mathrm{sym},0})$ *such that* $\mathbf{v}_n = \mathbf{v} + \mathbf{w}_n$, $\mathbb{U}_n = \mathbb{U} + \mathbb{Y}_n$ *satisfy*

$$\partial_t \mathbf{v}_n + \mathrm{div}_x \mathbb{U}_n = 0, \qquad \mathrm{div}_x \mathbf{v}_n = 0 \quad \textit{in } (T_1, T_2) \times B,$$
$$\frac{3}{2}\lambda_{\max}\left[\frac{(\mathbf{v}_n + \tilde{\mathbf{V}}) \otimes (\mathbf{v}_n + \tilde{\mathbf{V}})}{\tilde{r}} - \mathbb{U}_n\right] < e \quad \textit{in } (T_1, T_2) \times B,$$
$$\mathbf{v}_n \to \mathbf{v} \in C_{\mathrm{weak}}\big([T_1, T_2]; L^2\big(B; R^3\big)\big),$$

and

$$\liminf_{n\to\infty} \int_{T_1}^{T_2}\int_B |\mathbf{v}_n - \mathbf{v}|^2 \,\mathrm{d}x\,\mathrm{d}t \geqslant \Lambda\big(\underline{r}, \bar{r}, \overline{V}, \|e\|_{L^\infty((T_1,T_2)\times B)}\big) \int_{T_1}^{T_2}\int_B \left(e - \frac{1}{2}\frac{|\mathbf{v} + \tilde{\mathbf{V}}|^2}{\tilde{r}}\right)^2 \mathrm{d}x\,\mathrm{d}t. \tag{3.20}$$

Remark 3.3. Note that $\tilde{\mathbf{V}}$ is constant in Lemma 3.1; whence

$$\partial_t \mathbf{v} = \partial_t (\mathbf{v} + \tilde{\mathbf{V}}).$$

Remark 3.4. It is important that the constant Λ depends only on the quantities indicated explicitly in (3.20), in particular Λ is independent of $\mathbf{v}$, of the length of the time interval, and of the domain B.

3.4.1. Localization principle

The scale invariance encoded in (3.20) can be used for showing a "variable coefficients" variant of Lemma 3.1, specifically when both $\tilde{r}$ and $\tilde{\mathbf{V}}$ are sufficiently smooth functions of t and x.

Lemma 3.2. *Let* $\tilde{\varrho} \in C^1([T_1, T_2] \times \Omega)$, $\mathbf{V} \in C^1([T_1, T_2] \times \Omega; R^3)$, $T_1 < T_2$ *be functions satisfying*

$$0 < \underline{r} < \tilde{\varrho}(t, x) < \bar{r}, \qquad \big|\mathbf{V}(t, x)\big| < \overline{V} \quad \textit{for all } t, x.$$

Suppose that

$$\mathbf{v} \in C_{\mathrm{weak}}\big([T_1, T_2]; L^2\big(\Omega, R^3\big)\big) \cap C^1\big((T_1, T_2) \times \Omega; R^3\big)$$

solves the linear system

$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0, \qquad \mathrm{div}_x \mathbf{v} = 0 \quad \textit{in } (T_1, T_2) \times \Omega$$

with the associated field $\mathbb{U} \in C^1((T_1, T_2) \times B; R^{3\times 3}_{\mathrm{sym},0})$ *such that*

$$\frac{3}{2}\lambda_{\max}\left[\frac{(\mathbf{v} + \mathbf{V}) \otimes (\mathbf{v} + \mathbf{V})}{\tilde{\varrho}} - \mathbb{U}\right] < e - \delta \quad \textit{in } (T_1, T_2) \times \Omega \tag{3.21}$$

for some $e \in C([T_1; T_2] \times \overline{B})$ *and* $\delta > 0$.

Then there exist sequences $\{\mathbf{w}_n\}_{n=1}^{\infty} \subset C_c^{\infty}((T_1, T_2) \times \Omega; R^3)$, $\{\mathbb{Y}_n\}_{n=1}^{\infty} \subset C_c^{\infty}((T_1, T_2) \times \Omega; R^{3\times 3}_{\mathrm{sym},0})$ *such that* $\mathbf{v}_n = \mathbf{v} + \mathbf{w}_n$, $\mathbb{U}_n = \mathbb{U} + \mathbb{Y}_n$ *satisfy*

$$\partial_t \mathbf{v}_n + \mathrm{div}_x \mathbb{U}_n = 0, \qquad \mathrm{div}_x \mathbf{v}_n = 0 \quad \textit{in } (T_1, T_2) \times \Omega, \tag{3.22}$$
$$\frac{3}{2}\lambda_{\max}\left[\frac{(\mathbf{v}_n + \mathbf{V}) \otimes (\mathbf{v}_n + \mathbf{V})}{\tilde{\varrho}} - \mathbb{U}_n\right] < e \quad \textit{in } C\big((T_1; T_2) \times \Omega\big), \tag{3.23}$$
$$\mathbf{v}_n \to \mathbf{v} \in C_{\mathrm{weak}}\big([T_1, T_2]; L^2\big(\Omega; R^3\big)\big), \tag{3.24}$$

and

$$\liminf_{n\to\infty}\int_{T_1}^{T_2}\int_{\Omega}|\mathbf{v}_n-\mathbf{v}|^2\,\mathrm{d}x\,\mathrm{d}t \geqslant \Lambda\big(\underline{r},\bar{r},\overline{V},\|e\|_{L^\infty((T_1,T_2)\times\Omega)}\big)\int_{T_1}^{T_2}\int_{\Omega}\left(e-\frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}|^2}{\tilde{\varrho}}\right)^2\mathrm{d}x\,\mathrm{d}t. \tag{3.25}$$

Remark 3.5. The role of the positive parameter δ in (3.21) is only to say that the inequality (3.21) is strict, otherwise the conclusion of the lemma is independent of the specific value of δ.

Remark 3.6. In view of (3.24), the convergence formula (3.25) may be *equivalently* replaced by

$$\begin{aligned}&\liminf_{n\to\infty}\int_{T_1}^{T_2}\int_{\Omega}\frac{1}{2}\frac{|\mathbf{v}_n+\mathbf{V}|^2}{\tilde{\varrho}}\,\mathrm{d}x\,\mathrm{d}t\\ &\quad\geqslant\int_{T_1}^{T_2}\int_{\Omega}\frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}|^2}{\tilde{\varrho}}\,\mathrm{d}x\,\mathrm{d}t+\Lambda\big(\underline{r},\bar{r},\overline{V},\|e\|_{L^\infty((T_1,T_2)\times\Omega)}\big)\int_{T_1}^{T_2}\int_{\Omega}\left(e-\frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}|^2}{\tilde{\varrho}}\right)^2\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.26}$$

Proof of Lemma 3.2. We start with an easy observation that there exists $\varepsilon=\varepsilon(\delta,|e|)$ such that

$$\left\{\begin{aligned}\frac{3}{2}\left|\lambda_{\max}\left[\frac{(\mathbf{v}+\mathbf{V})\otimes(\mathbf{v}+\mathbf{V})}{\tilde{\varrho}}-\mathbb{U}\right]-\lambda_{\max}\left[\frac{(\mathbf{v}+\tilde{\mathbf{V}})\otimes(\mathbf{v}+\tilde{\mathbf{V}})}{\tilde{r}}-\mathbb{U}\right]\right|<\frac{\delta}{4},\\ \left|\frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}|^2}{\tilde{\varrho}}-\frac{1}{2}\frac{|\mathbf{v}+\tilde{\mathbf{V}}|^2}{\tilde{r}}\right|<\frac{\delta}{4}\end{aligned}\right\} \tag{3.27}$$

whenever

$$\frac{3}{2}\lambda_{\max}\left[\frac{(\mathbf{v}+\mathbf{V})\otimes(\mathbf{v}+\mathbf{V})}{\tilde{\varrho}}-\mathbb{U}\right]<e,\qquad|\tilde{\varrho}-\tilde{r}|<\varepsilon,\qquad|\mathbf{V}-\tilde{\mathbf{V}}|<\varepsilon.$$

For δ appearing in (3.21), we fix $\varepsilon=\varepsilon(\delta,\|e\|_{L^\infty((T_1,T_2)\times\Omega)})$ as in (3.27) and find a (finite) decomposition of the set $(T_1,T_2)\times\Omega$ such that

$$[T_1,T_2]\times\Omega=\bigcup_{i=1}^{N}\overline{Q}_i,\qquad Q_i=\big(T_1^i,T_2^i\big)\times B_i,\qquad Q_i\cap Q_j=\emptyset\quad\text{for } i\neq j,$$

$$\sup_{Q_i}\tilde{\varrho}-\inf_{Q_i}\tilde{\varrho}<\varepsilon,\qquad\sup_{Q_i}\left|\mathbf{V}-\frac{1}{|Q_i|}\int_{Q_i}\mathbf{V}\,\mathrm{d}x\,\mathrm{d}t\right|<\varepsilon,$$

where Q_i are suitable cubes and the number N depends on ε and the Lipschitz constants of $\tilde{\varrho}$, $\mathbf{V}$ in $[T_1,T_2]\times\Omega$.

Now, we apply Lemma 3.1 on each set Q_i with the choice of parameters

$$\tilde{r}=\sup_{Q_i}\tilde{\varrho},\qquad\tilde{\mathbf{V}}=\frac{1}{|Q_i|}\int_{Q_i}\mathbf{V}\,\mathrm{d}x\,\mathrm{d}t.$$

In accordance with (3.21), (3.27), we have

$$\frac{3}{2}\lambda_{\max}\left[\frac{(\mathbf{v}+\tilde{\mathbf{V}})\otimes(\mathbf{v}+\tilde{\mathbf{V}})}{\tilde{r}}-\mathbb{U}\right]<e-\frac{\delta}{2}\quad\text{in } Q_i.$$

Under these circumstances, Lemma 3.1 yields a sequence of smooth functions $\mathbf{v}_n^i$, $\mathbb{U}_n^i$, with $\mathbf{v}-\mathbf{v}_n^i$, $\mathbb{U}-\mathbb{U}_n^i$ compactly supported in Q_i, such that

$$\begin{aligned}&\partial_t\mathbf{v}_n^i+\operatorname{div}_x\mathbb{U}_n^i=0,\qquad\operatorname{div}_x\mathbf{v}_n^i=0\quad\text{in } Q_i,\\ &\frac{3}{2}\lambda_{\max}\left[\frac{(\mathbf{v}_n^i+\tilde{\mathbf{V}})\otimes(\mathbf{v}_n^i+\tilde{\mathbf{V}})}{\tilde{r}}-\mathbb{U}_n^i\right]<e-\frac{\delta}{2},\\ &\mathbf{v}_n^i\to\mathbf{v}\quad\text{in } C_{\mathrm{weak}}\big(\big[T_1^i,T_2^i\big],L^2(B_i)\big),\end{aligned} \tag{3.28}$$

and

$$\liminf_{n\to\infty}\int_{Q_i}\left|\mathbf{v}_n^i-\mathbf{v}\right|^2\mathrm{d}x\,\mathrm{d}t \geqslant \Lambda\left(\underline{r},\bar{r},\overline{V},\|e\|_{L^\infty((T_1,T_2)\times\Omega)}\right)\int_{Q_i}\left(e-\frac{1}{2}\frac{|\mathbf{v}+\tilde{\mathbf{V}}|^2}{\tilde{r}}-\frac{\delta}{2}\right)^2\mathrm{d}x\,\mathrm{d}t. \tag{3.29}$$

In view of (3.27), we replace $\tilde{r}$ by $\tilde{\varrho}$ and $\tilde{\mathbf{V}}$ by $\mathbf{V}$ in (3.28) to obtain

$$\frac{3}{2}\lambda_{\max}\left[\frac{(\mathbf{v}_n^i+\mathbf{V})\otimes(\mathbf{v}_n^i+\mathbf{V})}{\tilde{\varrho}}-\mathbb{U}_n\right]<e \quad \text{in } \overline{Q}_i.$$

As $\mathbf{v}_n$, $\mathbb{U}_n$ are compactly supported perturbations of $\mathbf{v}$, $\mathbb{U}$ in Q_i, we may define

$$\mathbf{v}_n(t,x)=\mathbf{v}_n^i(t,x), \qquad \mathbb{U}_n=\mathbb{U}_n^i \quad \text{for any } (t,x)\in\overline{Q}_i,\ i=1,\dots,N.$$

In accordance with the previous discussion, $\mathbf{v}_n$, $\mathbb{U}_n$ satisfy (3.22)–(3.24). In order to see (3.25), use (3.27) to observe that

$$\left(e-\frac{1}{2}\frac{|\mathbf{v}+\tilde{\mathbf{V}}|^2}{\tilde{r}}-\frac{\delta}{2}\right)>\left(e-\frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}|^2}{\tilde{\varrho}}-\frac{3\delta}{4}\right)>0 \quad \text{in } Q_i;$$

whence, making use of the hypothesis (3.21), specifically of the fact that

$$e-\frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}|^2}{\tilde{\varrho}}>\delta,$$

we may infer that

$$\left(e-\frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}|^2}{\tilde{\varrho}}-\frac{3\delta}{4}\right)\geqslant\frac{1}{4}\left(e-\frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}|^2}{\tilde{\varrho}}\right) \quad \text{in } Q_i.$$

Thus, summing up the integrals in (3.29) we get (3.25). □

3.4.2. Application to functionals I_ε

Fixing $\varepsilon\in(0,T/2)$ we complete the proof of Proposition 3.1. Given $e\in C([0,T]\times\Omega)$, we introduce the spaces

$$\begin{aligned} X_{0,e}=\Big\{\mathbf{v}\ \Big|\ &\mathbf{v}\in C^1\big((0,T)\times\Omega;R^3\big)\cap C_{\mathrm{weak}}\big([0,T];L^2\big(\Omega;R^3\big)\big),\\ &\mathbf{v} \text{ satisfies (3.14) with some } \mathbb{U}\in C^1\big((0,T)\times\Omega;R^{3\times3}_{\mathrm{sym},0}\big),\\ &\frac{3}{2}\lambda_{\max}\left[\frac{(\mathbf{v}+\nabla_x\Psi)\otimes(\mathbf{v}+\nabla_x\Psi)}{\tilde{\varrho}}-\mathbb{U}\right]<e \text{ for } t\in(0,T),\ x\in\Omega\Big\}, \end{aligned} \tag{3.30}$$

along with the associated functionals

$$I_{\varepsilon,e}[\mathbf{v}]=\int_\varepsilon^T\int_\Omega\left(\frac{1}{2}\frac{|\mathbf{v}+\nabla_x\Psi|^2}{\tilde{\varrho}}-e\right)\mathrm{d}x\,\mathrm{d}t \quad \text{for } \mathbf{v}\in X,\ 0<\varepsilon<T/2. \tag{3.31}$$

The following assertion is a direct consequence of Lemma 3.2.

Lemma 3.3. *Let $\mathbf{v}\in X_{0,e}$, $e\in C([0,T]\times\Omega)$, $0<\varepsilon<T/2$ be such that*

$$I_{\varepsilon,e}[\mathbf{v}]<-\alpha<0.$$

There is $\beta=\beta(\alpha,\|e\|_{L^\infty((0,T)\times\Omega)})>0$, independent of ε, and a sequence $\{\mathbf{v}_n\}_{n>0}\subset X_{0,e}$ such that

$$\mathbf{v}_n\equiv\mathbf{v} \quad \text{in } [0,\varepsilon]\times\Omega,$$

$$\mathbf{v}_n\to\mathbf{v} \quad \text{in } C_{\mathrm{weak}}\big([0,T];L^2\big(\Omega;R^3\big)\big), \qquad \liminf_{n\to\infty}I_{\varepsilon,e}[\mathbf{v}_n]\geqslant I_{\varepsilon,e}[\mathbf{v}]+\beta.$$

Remark 3.7. We have used Lemma 3.2 with (3.26), where, by virtue of Jensen's inequality,

$$\int_\varepsilon^T \int_\Omega \left(e - \frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}|^2}{\tilde{\varrho}}\right)^2 \mathrm{d}x\,\mathrm{d}t \geqslant \frac{1}{(T-\varepsilon)|\Omega|}\left(\int_\varepsilon^T \int_\Omega \left(e - \frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}|^2}{\tilde{\varrho}}\right)\mathrm{d}x\,\mathrm{d}t\right)^2 \geqslant \frac{\alpha^2}{(T-\varepsilon)|\Omega|}.$$

Finally, we show how Lemma 3.3 implies Proposition 3.1. Under the hypotheses of Proposition 3.1 and in accordance with the definition of the space X_0, we find $\delta > 0$ and a function $e \in C([0,T] \times \Omega)$ such that

$$e \leqslant \bar{e}[\mathbf{v}], \qquad e \equiv \bar{e}[\mathbf{v}] - \delta \quad \text{whenever} \quad t \in [\varepsilon, T],$$

and

$$\mathbf{v} \in X_{0,e}.$$

Thus, we have

$$I_{\varepsilon,e}[\mathbf{v}] = \int_\varepsilon^T \int_\Omega \left(\frac{1}{2}\frac{|\mathbf{v}+\nabla_x\Psi|^2}{\tilde{\varrho}} - \bar{e}[\mathbf{v}] + \delta\right)\mathrm{d}x\,\mathrm{d}t = I_\varepsilon[\mathbf{v}] + (T-\varepsilon)|\Omega|\delta < -\alpha/2 < 0$$

as soon as $\delta > 0$ was chosen small enough.

Consequently, by virtue of Lemma 3.3, there is a sequence of functions $\{\mathbf{v}_n\}_{n=1}^\infty$ and $\beta = \beta(\alpha) > 0$ such that

$$\mathbf{v}_n \in X_{0,e}, \qquad \mathbf{v}_n \equiv \mathbf{v} \quad \text{in } [0,\varepsilon] \times \Omega,$$

and

$$\mathbf{v}_n \to \mathbf{v} \quad \text{in } C_{\mathrm{weak}}\big([0,T]; L^2(\Omega; R^3)\big), \qquad \liminf_{n\to\infty} I_{\varepsilon,e}[\mathbf{v}_n] \geqslant I_{\varepsilon,e}[\mathbf{v}] + \beta = I_\varepsilon[\mathbf{v}] + \beta + (T-\varepsilon)|\Omega|\delta.$$

Moreover, in accordance with (3.19),

$$I_{\varepsilon,e}[\mathbf{v}_n] - I_\varepsilon[\mathbf{v}_n] = \int_\varepsilon^T \int_\Omega \bar{e}[\mathbf{v}_n] - \bar{e}[\mathbf{v}] + \delta\,\mathrm{d}x\,\mathrm{d}t \to (T-\varepsilon)|\Omega|\delta \quad \text{as } n \to \infty;$$

whence we may infer that

$$\liminf_{n\to\infty} I_\varepsilon[\mathbf{v}_n] \geqslant I_\varepsilon[\mathbf{v}] + \beta.$$

Finally, it remains to observe that $\mathbf{v}_n \in X_0$ for all n large enough. To this end, note that

$$\frac{3}{2}\lambda_{\max}\left[\frac{(\mathbf{v}_n + \nabla_x\Psi) \otimes (\mathbf{v}_n + \nabla_x\Psi)}{\tilde{\varrho}} - \mathbb{U}\right] = \frac{3}{2}\lambda_{\max}\left[\frac{(\mathbf{v} + \nabla_x\Psi) \otimes (\mathbf{v} + \nabla_x\Psi)}{\tilde{\varrho}} - \mathbb{U}\right] < e \leqslant \bar{e}[\mathbf{v}] = \bar{e}[\mathbf{v}_n]$$

for all $t \in [0,\varepsilon]$, while

$$\frac{3}{2}\lambda_{\max}\left[\frac{(\mathbf{v}_n + \nabla_x\Psi) \otimes (\mathbf{v}_n + \nabla_x\Psi)}{\tilde{\varrho}} - \mathbb{U}\right] < e = \bar{e}[\mathbf{v}] - \delta \leqslant \bar{e}[\mathbf{v}_n] - \delta/2 \quad \text{for all } t \in [\varepsilon, T]$$

for all n large enough. We have proved Proposition 3.1.

4. Dissipative solutions

The solutions of the Euler–Fourier system constructed in Section 3 suffer an essential deficiency, namely they do not comply with the First law of thermodynamics, meaning, they violate the total energy conservation (1.7). On the other hand, the initial data in (3.1) are smooth enough for the problem to possess a standard classical solution existing on a possibly short time interval $(0, T_{\max})$, see e.g. Alazard [1,2]. Note that the Euler–Fourier system fits also in the general framework and the corresponding existence theory developed by Serre [20,21]. As the classical solutions are unique (in their regularity class) and obviously satisfy the total energy balance (1.7), the latter can be added to (2.1)–(2.3) as an admissibility condition. The weak solutions of (1.1)–(1.6) satisfying (1.7) will be called *dissipative solutions*.

4.1. Relative entropy (energy) and weak-strong uniqueness

Following [15] we introduce the relative entropy functional

$$\mathcal{E}(\varrho,\vartheta,\mathbf{u}|r,\Theta,\mathbf{U})=\int_{\Omega}\left(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2+H_{\Theta}(\varrho,\vartheta)-\frac{\partial H_{\Theta}(r,\Theta)}{\partial\varrho}(\varrho-r)-H_{\Theta}(r,\Theta)\right)\mathrm{d}x, \tag{4.1}$$

where H_Θ is the ballistic free energy,

$$H_{\Theta}(\varrho,\vartheta)=\varrho\big(e(\varrho,\vartheta)-\Theta s(\varrho,\vartheta)\big)=\varrho\left(\frac{3}{2}\vartheta-\Theta\log\left(\frac{\vartheta^{3/2}}{\varrho}\right)\right).$$

Repeating step by step the arguments of [15] we can show that any dissipative solution of the problem (1.1)–(1.6) satisfies the *relative entropy inequality*:

$$\begin{aligned}
&\big[\mathcal{E}(\varrho,\vartheta,\mathbf{u}|r,\Theta,\mathbf{U})\big]_{t=0}^{t=\tau}+\int_0^{\tau}\int_{\Omega}\Theta\frac{|\nabla_x\vartheta|^2}{\vartheta^2}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\leqslant\int_0^{\tau}\int_{\Omega}\big(\varrho(\mathbf{U}-\mathbf{u})\cdot\partial_t\mathbf{U}+\varrho(\mathbf{U}-\mathbf{u})\otimes\mathbf{u}:\nabla_x\mathbf{U}-p(\varrho,\vartheta)\,\mathrm{div}_x\,\mathbf{U}\big)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad-\int_0^{\tau}\int_{\Omega}\big(\varrho\big(s(\varrho,\vartheta)-s(r,\Theta)\big)\partial_t\Theta+\varrho\big(s(\varrho,\vartheta)-s(r,\Theta)\big)\mathbf{u}\cdot\nabla_x\Theta\big)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\int_0^{\tau}\int_{\Omega}\left(\left(1-\frac{\varrho}{r}\right)\partial_t p(r,\Theta)-\frac{\varrho}{r}\mathbf{u}\cdot\nabla_x p(r,\Theta)\right)\mathrm{d}x\,\mathrm{d}t+\int_0^{\tau}\int_{\Omega}\frac{\nabla_x\vartheta}{\vartheta}\cdot\nabla_x\Theta\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{4.2}$$

for any trio of smooth "test" functions

$$r,\ \Theta,\ \mathbf{U},\quad r>0,\ \Theta>0.$$

We report the following result [13, Theorem 6.1].

Theorem 4.1 *(Weak-strong uniqueness). Let* $[\varrho,\vartheta,\mathbf{u}]$ *be a dissipative* (*weak*) *solution of the problem* (1.1)–(1.6), *emanating from the initial data* $[\varrho_0,\vartheta_0,\mathbf{u}_0]$ *satisfying* (3.1), *such that*

$$0<\underline{\varrho}<\varrho(t,x)<\bar{\varrho},\qquad 0<\underline{\vartheta}<\vartheta(t,x)<\bar{\vartheta},\qquad \big|\mathbf{u}(t,x)\big|<\bar{u}\quad\text{for a.a. }(t,x)\in(0,T)\times\Omega.$$

Suppose that the same problem (*with the same initial data*) *admits a classical solution* $[\tilde{\varrho},\tilde{\vartheta},\tilde{\mathbf{u}}]$ *in* $(0,T)\times\Omega$. *Then*

$$\varrho\equiv\tilde{\varrho},\qquad \vartheta\equiv\tilde{\vartheta},\qquad \mathbf{u}\equiv\tilde{\mathbf{u}}.$$

Remark 4.1. Here, "classical" means that all the necessary derivatives appearing in the equations are continuous functions in $[0,T]\times\Omega$.

Remark 4.2. The proof of Theorem 4.1 is based on taking $r=\tilde{\varrho}$, $\Theta=\tilde{\vartheta}$, $\mathbf{U}=\tilde{\mathbf{u}}$ as test functions in the relative entropy inequality (4.2) and making use of a Gronwall type argument. This has been done in detail in [13, Section 6] in the case of a viscous fluid satisfying the Navier–Stokes–Fourier system. However, the same arguments can be used to handle the inviscid case provided the solutions are uniformly bounded on the existence interval.

Remark 4.3. As the proof of Theorem 4.1 is based on the relative entropy inequality (4.2), the conclusion remains valid if we replace the internal energy *equation* (1.3) by the entropy *inequality* (1.8) as long as we require (1.7).

4.2. *Infinitely many dissipative solutions*

Apparently, the stipulation of the total energy balance (1.7) eliminates the non-physical solutions obtained in Theorem 3.1, at least in the case of *regular* initial data. As we will see, the situation changes if we consider non-smooth initial data, in particular the initial velocity field $\mathbf{u}_0$ belonging only to $L^\infty(\Omega; R^3)$. Our final goal is the following result.

Theorem 4.2. *Let $T > 0$ be given. Let the initial data ϱ_0, ϑ_0 be given, satisfying*

$$\varrho_0, \vartheta_0 \in C^2(\Omega), \qquad \varrho_0(x) > \underline{\varrho} > 0, \qquad \vartheta_0(x) > \underline{\vartheta} > 0 \quad \text{for any } x \in \Omega. \tag{4.3}$$

Then there exists a velocity field $\mathbf{u}_0$,

$$\mathbf{u}_0 \in L^\infty\big(\Omega; R^3\big),$$

such that the problem (1.1)–(1.6) admits infinitely many dissipative (weak) solutions in $(0, T) \times \Omega$, with the initial data $[\varrho_0, \vartheta_0, \mathbf{u}_0]$.

Remark 4.4. As we shall see below, the solutions obtained in the proof of Theorem 4.2 enjoy the same regularity as those in Theorem 3.1, in particular, the equation of continuity (1.1) as well as the internal energy balance (1.3) are satisfied pointwise (a.a.) in $(0, T) \times \Omega$.

Remark 4.5. In general, the initial velocity $\mathbf{u}_0$ depends on the length of the existence interval T. See Section 5 for more discussion concerning possible extension of the solutions to $[0, \infty)$.

The remaining part of this section is devoted to the proof of Theorem 4.2 that may be viewed as an extension of the results of Chiodaroli [8] and De Lellis and Székelyhidi [11] to the case of a heat conducting fluid.

4.2.1. *Suitable initial data*

Following the strategy of [11] our goal is to identify suitable initial data $\mathbf{u}_0$ for which the associated (weak) solutions of the momentum equation dissipate the kinetic energy. In contrast with [11], however, we have to find the initial data for which the kinetic energy decays *sufficiently* fast in order to compensate the associated production of heat.

The velocity field $\mathbf{v} = \varrho\mathbf{u}$ we look for will be *solenoidal*, in particular, we focus on the initial data satisfying

$$\mathrm{div}_x(\varrho_0\mathbf{u}_0) = 0.$$

This assumption simplifies considerably the ansatz introduced in Section 3.1, specifically,

$$\varrho = \tilde{\varrho} = \varrho_0(x), \qquad \mathbf{v} = \varrho\mathbf{u}, \qquad \mathrm{div}_x\,\mathbf{v} = 0, \qquad \Psi \equiv 0;$$

whence the problem reduces to solving

$$\partial_t\mathbf{v} + \mathrm{div}_x\left(\frac{\mathbf{v}\otimes\mathbf{v}}{\tilde{\varrho}}\right) + \nabla_x\left(\tilde{\varrho}\vartheta[\mathbf{v}] - \frac{2}{3}\chi\right) = 0, \qquad \mathrm{div}_x\,\mathbf{v} = 0, \qquad \mathbf{v}(0,\cdot) = \mathbf{v}_0, \qquad \mathrm{div}_x\,\mathbf{v}_0 = 0, \tag{4.4}$$

for a suitable spatially homogeneous function $\chi = \chi(t)$.

Mimicking the steps of Section 3.3 we introduce the quantity

$$\bar{e}[\mathbf{v}] = \chi - \frac{3}{2}\tilde{\varrho}\vartheta[\mathbf{v}]. \tag{4.5}$$

As the anticipated solutions satisfy

$$\frac{1}{2}\frac{|\mathbf{v}|^2}{\tilde{\varrho}} = \bar{e}[\mathbf{v}],$$

the energy of the system reads

$$E(t) = \int_\Omega \left(\frac{|\mathbf{v}|^2}{2\tilde{\varrho}} + \frac{3}{2}\tilde{\varrho}\vartheta[\mathbf{v}]\right)(t,\cdot)\,\mathrm{d}x = \chi(t)|\Omega| = \chi(t). \tag{4.6}$$

Consequently, in accordance with the construction procedure used in Section 3.3, it is enough to find a suitable *constant* χ and the initial velocity field $\mathbf{v}_0$ such that

$$\operatorname{div}_x \mathbf{v}_0 = 0, \qquad E_0 = \int_\Omega \left(\frac{|\mathbf{v}_0|^2}{2\tilde{\varrho}_0} + \frac{3}{2}\tilde{\varrho}_0 \vartheta_0 \right) \mathrm{d}x = \chi,$$

and the associated space of subsolutions X_0 defined in (3.16) (with $\nabla_x \Psi = 0$) is non-empty. This is the objective of the remaining part of this section.

4.2.2. Dissipative data for the Euler system

Similarly to (3.30), we introduce the set of subsolutions

$$X_{0,e}[T_1, T_2] = \Big\{ \mathbf{v} \ \Big| \ \mathbf{v} \in C^1\big((T_1, T_2) \times \Omega; R^3\big) \cap C_{\mathrm{weak}}\big([T_1, T_2]; L^2\big(\Omega; R^3\big)\big),$$
$$\mathbf{v} \text{ satisfies (3.14) with some } \mathbb{U} \in C^1\big((T_1, T_2) \times \Omega; R^{3\times 3}_{\mathrm{sym},0}\big),$$
$$\frac{3}{2}\lambda_{\max}\left[\frac{\mathbf{v} \otimes \mathbf{v}}{\tilde{\varrho}} - \mathbb{U}\right] < e \text{ for } t \in (T_1, T_2),\ x \in \Omega \Big\}, \tag{4.7}$$

where $e \in C([T_1, T_2] \times \Omega)$.

The following result may be seen as an extension of [11, Proposition 5]:

Lemma 4.1. *Suppose that* $\mathbf{v} \equiv \mathbf{v}_0(x)$*, together with the associated field* $\mathbb{U}_{\mathbf{v}} \equiv 0$*, belong to the set of subsolutions* $X_{0,e}[0, T]$.

Then for any $\tau \in (0, T)$ *and any* $\varepsilon > 0$*, there exist* $\overline{\tau} \in (0, T)$*,* $|\tau - \overline{\tau}| < \varepsilon$ *and* $\mathbf{w} \in X_{0,e}[\overline{\tau}, T]$*, such that*

$$\frac{1}{2}\frac{|\mathbf{w}(\overline{\tau}, \cdot)|^2}{\tilde{\varrho}} = e(\overline{\tau}, \cdot), \tag{4.8}$$
$$\mathbf{w} \equiv \mathbf{v}, \qquad \mathbb{U}_{\mathbf{w}} \equiv 0 \quad \textit{in a (left) neighborhood of } T.$$

Remark 4.6. Note that, thanks to (4.8),

$$\mathbf{w}(t, \cdot) \to \mathbf{w}(\overline{\tau}, \cdot) \quad \text{(strongly) in } L^2\big(\Omega; R^3\big) \text{ as } t \to \overline{\tau}+.$$

Remark 4.7. The result is probably not optimal; one should be able, with greater effort, to show the same conclusion with $\overline{\tau} = \tau$.

Proof of Lemma 4.1. We construct the function $\mathbf{w}$ as a limit of a sequence $\{\mathbf{w}_k\}_{k=1}^\infty \subset X_{0,e}[0, T]$,

$$\mathbf{w}_k \to \mathbf{w} \quad \text{in } C_{\mathrm{weak}}\big([0, T]; L^2\big(\Omega; R^3\big)\big),$$

where $\mathbf{w}_k$ will be obtained recursively, with the starting point

$$\mathbf{w}_0 = \mathbf{v} \equiv \mathbf{v}_0, \qquad \tau_0 = \tau, \qquad \varepsilon_0 = \varepsilon.$$

More specifically, we construct the functions $\mathbf{w}_k$, together with τ_k, ε_k, $k = 1, \ldots$ satisfying:

- $$\mathbf{w}_k \in X_{0,e}[0, T], \qquad \operatorname{supp}[\mathbf{w}_k - \mathbf{w}_{k-1}] \subset (\tau_{k-1} - \varepsilon_k, \tau_{k-1} + \varepsilon_k), \quad \text{where } 0 < \varepsilon_k < \frac{\varepsilon_{k-1}}{2}; \tag{4.9}$$
- $$d(\mathbf{w}_k, \mathbf{w}_{k-1}) < \frac{1}{2^k}, \qquad \sup_{t \in (0,T)} \left| \int_\Omega \frac{1}{\tilde{\varrho}}(\mathbf{w}_k - \mathbf{w}_{k-1}) \cdot \mathbf{w}_m \, \mathrm{d}x \right| < \frac{1}{2^k} \quad \text{for all } m = 0, \ldots, k-1, \tag{4.10}$$

  recalling that d is the metric induced by the weak topology of the Hilbert space $L^2(\Omega; R^3)$;

- there exists τ_k,

  $$\tau_k \in (\tau_{k-1} - \varepsilon_k, \tau_{k-1} + \varepsilon_k)$$

  such that

  $$\int_\Omega \frac{1}{2}\frac{|\mathbf{w}_k|^2}{\tilde{\varrho}}(\tau_k, \cdot)\,\mathrm{d}x \geqslant \int_\Omega \frac{1}{2}\frac{|\mathbf{w}_{k-1}|^2}{\tilde{\varrho}}(t, \cdot)\,\mathrm{d}x + \frac{\lambda}{\varepsilon_k^2}\alpha_k^2$$
  $$\geqslant \int_\Omega \frac{1}{2}\frac{|\mathbf{w}_{k-1}|^2}{\tilde{\varrho}}(\tau_{k-1}, \cdot)\,\mathrm{d}x + \frac{\lambda}{2\varepsilon_k^2}\alpha_k^2 \quad \text{for all } t \in (\tau_{k-1} - \varepsilon_k, \tau_{k-1} + \varepsilon_k), \tag{4.11}$$

  where

  $$\alpha_k = \int_{\tau_{k-1}-\varepsilon_k}^{\tau_{k-1}+\varepsilon_k} \int_\Omega \left(e - \frac{1}{2}\frac{|\mathbf{w}_{k-1}|^2}{\tilde{\varrho}}\right)\mathrm{d}x\,\mathrm{d}t > 0,$$

  and $\lambda > 0$ is constant independent of k.

Supposing we have already constructed $\mathbf{w}_0, \dots, \mathbf{w}_{k-1}$ we find $\mathbf{w}_k$ enjoying the properties (4.9)–(4.11). To this end, we first compute

$$\alpha_k = \int_{\tau_{k-1}-\varepsilon_k}^{\tau_{k-1}+\varepsilon_k} \int_\Omega \left(e - \frac{1}{2}\frac{|\mathbf{w}_{k-1}|^2}{\tilde{\varrho}}\right)\mathrm{d}x\,\mathrm{d}t \quad \text{for a certain } 0 < \varepsilon_k < \frac{\varepsilon_{k-1}}{2}$$

and observe that

$$\frac{\alpha_k}{2\varepsilon_k} = \frac{1}{2\varepsilon_k}\int_{\tau_{k-1}-\varepsilon_k}^{\tau_{k-1}+\varepsilon_k} \int_\Omega \left(e - \frac{1}{2}\frac{|\mathbf{w}_{k-1}|^2}{\tilde{\varrho}}\right)\mathrm{d}x\,\mathrm{d}t \to \int_\Omega \left(e - \frac{1}{2}\frac{|\mathbf{w}_{k-1}|^2}{\tilde{\varrho}}\right)(\tau_{k-1})\,\mathrm{d}x > 0 \quad \text{for } \varepsilon_k \to 0$$

as $\mathbf{w}_{k-1}$ is smooth in $(0, T)$.

Consequently, by the same token, we can choose $\varepsilon_k > 0$ so small that

$$\frac{1}{2\varepsilon_k}\int_{\tau_{k-1}-\varepsilon_k}^{\tau_{k-1}+\varepsilon_k} \int_\Omega \frac{1}{2}\frac{|\mathbf{w}_{k-1}|^2}{\tilde{\varrho}}\,\mathrm{d}x\,\mathrm{d}t + \frac{\Lambda(\tilde{\varrho}, \|e\|_{L^\infty((0,T)\times\Omega)})}{4\varepsilon_k^2}\alpha_k^2$$
$$\geqslant \int_\Omega \frac{1}{2}\frac{|\mathbf{w}_{k-1}|^2}{\tilde{\varrho}}(t, \cdot)\,\mathrm{d}x + \frac{\Lambda(\tilde{\varrho}, \|e\|_{L^\infty((0,T)\times\Omega)})}{8\varepsilon_k^2}\alpha_k^2$$
$$\geqslant \int_\Omega \frac{1}{2}\frac{|\mathbf{w}_{k-1}|^2}{\tilde{\varrho}}(\tau_{k-1}, \cdot)\,\mathrm{d}x + \frac{\Lambda(\tilde{\varrho}, \|e\|_{L^\infty((0,T)\times\Omega)})}{16\varepsilon_k^2}\alpha_k^2 \quad \text{for all } t \in (\tau_{k-1} - \varepsilon_k, \tau_{k-1} + \varepsilon_k), \tag{4.12}$$

where $\Lambda(\tilde{\varrho}, \|e\|_{L^\infty((0,T)\times\Omega)}) > 0$ is the universal constant from Lemma 3.2.

Applying Lemma 3.2 in the form specified in Remark 3.6 we obtain a function $\mathbf{w}_k \in X_{0,e}$ such that

$$\operatorname{supp}[\mathbf{w}_k - \mathbf{w}_{k-1}] \subset (\tau_{k-1} - \varepsilon_k, \tau_{k-1} + \varepsilon_k),$$
$$d(\mathbf{w}_k, \mathbf{w}_{k-1}) < \frac{1}{2^k}, \qquad \sup_{t\in(0,T)}\left|\int_\Omega \frac{1}{\tilde{\varrho}}(\mathbf{w}_k - \mathbf{w}_{k-1}) \cdot \mathbf{w}_m\,\mathrm{d}x\right| < \frac{1}{2^k}, \quad m = 0, \dots, k-1 \tag{4.13}$$

and

$$\int_{\tau_{k-1}-\varepsilon_k}^{\tau_{k-1}+\varepsilon_k} \int_\Omega \frac{1}{2}\frac{|\mathbf{w}_k|^2}{\tilde{\varrho}}\,\mathrm{d}x\,\mathrm{d}t \geqslant \int_{\tau_{k-1}-\varepsilon_k}^{\tau_{k-1}+\varepsilon_k} \int_\Omega \frac{1}{2}\frac{|\mathbf{w}_{k-1}|^2}{\tilde{\varrho}}\,\mathrm{d}x\,\mathrm{d}t + \frac{\Lambda(\tilde{\varrho}, \|e\|_{L^\infty((0,T)\times\Omega)})}{2\varepsilon_k}\alpha_k^2, \tag{4.14}$$

where we have applied Jensen's inequality to the last integral in (3.26).

Finally, the relations (4.12), (4.14) yield (4.11) with some $\tau_k \in (\tau_{k-1} - \varepsilon_k, \tau_{k-1} + \varepsilon_k)$, $\lambda = \Lambda/16$.

Now, by virtue of (4.10), there is $\mathbf{w}$ such that

$$\mathbf{w}_k \to \mathbf{w} \quad \text{in } C_{\text{weak}}\big([0,T]; L^2\big(\Omega; R^3\big)\big). \tag{4.15}$$

Moreover, (4.9) implies

(i) $\quad \tau_k \to \overline{\tau} \in (0,T), \qquad |\overline{\tau} - \tau| < \varepsilon;$

(ii) for any $\delta > 0$ there is $k = k_0(\delta)$ such that

$$\mathbf{w}(t,\cdot) = \mathbf{w}_k(t,\cdot) = \mathbf{w}_{k_0}(t,\cdot) \quad \text{for all } t \in (0, \overline{\tau} - \delta) \cup (\overline{\tau} + \delta, T),\ k \geqslant k_0. \tag{4.16}$$

In particular, (4.16) yields

$$\mathbf{w} \in X_{0,e}[\overline{\tau}, T], \quad \text{and} \quad \mathbf{w} \equiv \mathbf{v}, \qquad \mathbb{U}_{\mathbf{w}} \equiv 0 \quad \text{in a (left) neighborhood of } T.$$

Next, in view of (4.11),

$$\int_\Omega \frac{1}{2} \frac{|\mathbf{w}_{k-1}|^2}{\tilde{\varrho}}(t,\cdot)\,\mathrm{d}x \nearrow Y \quad \text{uniformly for } t \in (\tau_{k-1} - \varepsilon_k, \tau_{k-1} + \varepsilon_k), \tag{4.17}$$

therefore

$$\frac{\alpha_k}{\varepsilon_k} = \frac{1}{\varepsilon_k} \int_{\tau_{k-1}-\varepsilon_k}^{\tau_{k-1}+\varepsilon_k} \int_\Omega \left(e - \frac{1}{2} \frac{|\mathbf{w}_{k-1}|^2}{\tilde{\varrho}} \right) \mathrm{d}x\,\mathrm{d}t \to 0; \tag{4.18}$$

whence, finally,

$$\int_\Omega \frac{1}{2} \frac{|\mathbf{w}_k|^2}{\tilde{\varrho}}(\overline{\tau},\cdot)\,\mathrm{d}x \nearrow \int_\Omega e(\overline{\tau},\cdot)\,\mathrm{d}x. \tag{4.19}$$

Combining (4.19) with (4.10), (4.15) we get

$$\mathbf{w}_k(\overline{\tau},\cdot) \to \mathbf{w}(\overline{\tau},\cdot) \quad \text{in } L^2\big(\Omega; R^3\big)$$

which implies (4.8). Indeed we have

$$\int_\Omega \frac{1}{\tilde{\varrho}} |\mathbf{w}_n - \mathbf{w}_m|^2(\overline{\tau},\cdot)\,\mathrm{d}x$$
$$= \int_\Omega \frac{1}{\tilde{\varrho}} |\mathbf{w}_n|^2(\overline{\tau},\cdot)\,\mathrm{d}x - \int_\Omega \frac{1}{\tilde{\varrho}} |\mathbf{w}_m|^2(\overline{\tau},\cdot)\,\mathrm{d}x - 2\int_\Omega \frac{1}{\tilde{\varrho}} (\mathbf{w}_n - \mathbf{w}_m)\cdot \mathbf{w}_m(\overline{\tau},\cdot)\,\mathrm{d}x \quad \text{for all } n > m,$$

where, by virtue of (4.10),

$$\int_\Omega \frac{1}{\tilde{\varrho}} (\mathbf{w}_n - \mathbf{w}_m)\cdot \mathbf{w}_m(\overline{\tau},\cdot)\,\mathrm{d}x = \sum_{k=0}^{n-m-1} \int_\Omega \frac{1}{\tilde{\varrho}} (\mathbf{w}_{k+1} - \mathbf{w}_k)\cdot \mathbf{w}_m(\overline{\tau},\cdot)\,\mathrm{d}x \to 0 \quad \text{for } m \to \infty. \qquad \square$$

4.2.3. Construction of suitable initial data for the Euler–Fourier system

Fixing ϱ_0, ϑ_0 satisfying (4.3) and $\varrho = \tilde{\varrho} \equiv \varrho_0$ we can use (3.12) to deduce that there is a constant $\overline{\vartheta}$ depending only on $[\varrho_0, \vartheta_0]$ such that

$$\big|\vartheta[\mathbf{v}]\big| \leqslant \overline{\vartheta}, \quad \text{whence} \quad \frac{3}{2} p\big(\tilde{\varrho}, \overline{\vartheta}[\mathbf{v}]\big) < \overline{P} \quad \text{on the whole interval } [0,T], \tag{4.20}$$

with $\overline{P}$ independent of $\mathbf{v}$.

Next, we estimate the difference $\vartheta - \vartheta_0$ satisfying the equation

$$\tilde{\varrho}\partial_t(\vartheta - \vartheta_0) + \mathbf{v}\cdot\nabla_x(\vartheta - \vartheta_0) - \frac{2}{3}\Delta(\vartheta - \vartheta_0) = -\mathbf{v}\cdot\nabla_x\vartheta_0 + \frac{2}{3}\Delta\vartheta_0 + \frac{2}{3}\vartheta\mathbf{v}\cdot\frac{\nabla_x\tilde{\varrho}}{\tilde{\varrho}}.$$

Consequently, using (4.20) and the comparison principle, we deduce that

$$\left|\vartheta[\mathbf{v}](t,\cdot) - \vartheta_0\right| \leqslant c\left(1 + \|\mathbf{v}\|_{L^\infty((0,T)\times\Omega;R^3)}\right)t \quad \text{for all } t \in [0,T]. \tag{4.21}$$

We take $\mathbf{v}_0 \in C^1(\Omega)$, $\mathrm{div}_x\,\mathbf{v}_0 = 0$, and a constant χ_0 in such a way that

$$\frac{3}{2}\lambda_{\max}\left(\frac{\mathbf{v}_0\otimes\mathbf{v}_0}{\tilde{\varrho}}\right) < \chi_0 - \frac{3}{2}\varrho_0\vartheta_0. \tag{4.22}$$

Moreover, for any $\overline{\chi} > 2\chi_0$, $K > 0$ given, it is easy to construct a function $\chi \in C[0,T]$ such that

- $$\chi(0) = \chi(T) = \chi_0, \qquad \chi(t) > \chi_0 \quad \text{for all } t \in (0,T), \qquad \max_{t\in(0,T)}\chi(t) = \overline{\chi};$$
- there is $\tau \in (0,T)$ and $\varepsilon > 0$ such that

$$\chi(\overline{\tau}) - \chi_0 > \frac{\overline{\chi}}{2}, \qquad \chi(t) < \chi(\overline{\tau}) - K(t-\overline{\tau}) \quad \text{for all } t \in (\overline{\tau},T) \quad \text{whenever} \quad |\tau - \overline{\tau}| < \varepsilon. \tag{4.23}$$

Consequently, we have

$$\mathbf{v} \equiv \mathbf{v}_0 \in X_{0,e}[0,T] \quad (\text{with } \mathbb{U} \equiv 0)$$

provided

$$e(t,x) = \chi(t) - \frac{3}{2}\varrho_0\vartheta_0.$$

Applying Lemma 4.1, we find a function $\mathbf{w} \in X_{0,e}[\overline{\tau},T]$, with the corresponding field $\mathbb{U}_{\mathbf{w}}$, such that

$$\frac{1}{2}\frac{|\mathbf{w}(\overline{\tau},\cdot)|^2}{\tilde{\varrho}} = \chi(\overline{\tau}) - \frac{3}{2}\varrho_0\vartheta_0 > \frac{\overline{\chi}}{2} + \chi_0 - \frac{3}{2}\varrho_0\vartheta_0,$$
$$\mathbf{w} \equiv \mathbf{v}_0, \qquad \mathbb{U}_{\mathbf{w}} = 0 \quad \text{in a (left) neighborhood of } T,$$

and

$$\frac{1}{2}\frac{|\mathbf{w}|^2}{\tilde{\varrho}} < \frac{3}{2}\lambda_{\max}\left(\frac{\mathbf{w}\otimes\mathbf{w}}{\tilde{\varrho}} - \mathbb{U}_{\mathbf{w}}\right) < \chi(t) - \frac{3}{2}\varrho_0\vartheta_0$$
$$\leqslant \chi(\overline{\tau}) - \frac{3}{2}\varrho_0\vartheta_0 - K(t-\overline{\tau}), \quad t \in (\overline{\tau},T].$$

Denoting $\mathbf{w}_0 = \mathbf{w}(\overline{\tau},\cdot)$ and shifting everything to the origin $t = 0$, we infer that there is a function $\mathbf{w} \in X_{0,e}(0,T)$, with the following properties:

- $$\mathbf{w}(0,\cdot) = \mathbf{w}_0, \qquad \frac{1}{2}\frac{|\mathbf{w}_0|^2}{\tilde{\varrho}} = \chi(\overline{\tau}) - \frac{3}{2}\varrho_0\vartheta_0, \qquad \mathbf{w}(T,\cdot) = \mathbf{v}_0, \tag{4.24}$$
- $$e(t,x) = \begin{cases} \chi(\overline{\tau}) - \frac{3}{2}\varrho_0\vartheta_0 - Kt, & t \in [0, \frac{\chi(\overline{\tau})-\chi_0}{K}], \\ \chi_0 - \frac{3}{2}\varrho_0\vartheta_0 & \text{for } t \in [\frac{\chi(\overline{\tau})-\chi_0}{K}, T]. \end{cases} \tag{4.25}$$

Similarly to (3.16), we introduce the set X_0, together with the function

$$\overline{e}[\mathbf{v}] = \chi(\overline{\tau}) - \frac{3}{2}\varrho_0\vartheta[\mathbf{v}].$$

Our ultimate goal is to show that the function $\mathbf{w}$, introduced in (4.24), belongs to X_0 as long as we conveniently fix the parameters $\overline{\chi}$, K. To this end, it is enough to show that e, defined through (4.25), satisfies

$$e(t,x) < \bar{e}[\mathbf{w}] = \chi(\overline{\tau}) - \frac{3}{2}\varrho_0\vartheta[\mathbf{w}] \quad \text{for all } t \in (0,T]. \tag{4.26}$$

For $t \in (0, \frac{\chi(\overline{\tau})-\chi_0}{K}]$, this amounts to showing

$$\frac{3}{2}\big(\varrho_0\vartheta[\mathbf{w}] - \varrho_0\vartheta_0\big) < Kt, \quad t \in \left(0, \frac{\chi(\overline{\tau})-\chi_0}{K}\right],$$

which follows from (4.21) provided $K = K(\overline{\chi})$ is taken large enough.

Next, for $t \in [\frac{\chi(\overline{\tau})-\chi_0}{K}, T]$, we have to check that

$$\frac{3}{2}\big(\varrho_0\vartheta[\mathbf{w}] - \varrho_0\vartheta_0\big) < \frac{\chi(\overline{\tau})}{2}, \quad t \in \left[\frac{\chi(\overline{\tau})-\chi_0}{K}, T\right],$$

which follows from (4.20), (4.23) provided we fix $\overline{\chi} = \overline{\chi}(\varrho_0, \vartheta_0)$ large enough.

Having found a suitable subsolution we can finish the proof of Theorem 4.2 exactly as in Section 3.3.

5. Concluding remarks

- In this paper, we focused exclusively on the physically relevant $3D$-setting. The reader will have noticed that exactly the same results may be obtained also in the $2D$-case. Note, however, that the method does not apply to the $1D$-system as the conclusion of Lemma 3.1 is no longer available.
- Theorem 4.2 obviously applies to the larger class of dissipative solutions for which the internal energy balance is replaced by the entropy *inequality*

$$\partial_t\left(\varrho \log\left(\frac{\vartheta^{3/2}}{\varrho}\right)\right) + \operatorname{div}_x\left(\varrho \log\left(\frac{\vartheta^{3/2}}{\varrho}\right)\mathbf{u}\right) - \operatorname{div}_x\left(\frac{\nabla_x\vartheta}{\vartheta}\right) \geqslant \frac{|\nabla_x\vartheta|^2}{\vartheta^2}. \tag{5.1}$$

  Moreover, we could even construct dissipative solutions with an "artificial" entropy production satisfying (5.1) with *strict* inequality and, at the same time, conserving the total energy. On the other hand, a criterion based on maximality of the entropy production could be possibly used to identify a class of physically relevant solutions.
- The conclusion of Theorem 3.1 can be extended to the time interval $[0,\infty)$ by means of continuation. Indeed we can take the function h in (3.5) such that $h(T) = 0$; whence

$$\tilde{\varrho}(T,\cdot) = \varrho_0.$$

  Moreover, as pointed out in Remark 3.1

$$\vartheta(t,\cdot) \in \big[L^p(\Omega), W^{2,p}(\Omega)\big]_\alpha \quad \text{for all } t \in [0,T],$$

  therefore we can apply Theorem 3.1 recursively on the time intervals $[nT, (n+1)T]$, $n = 1, \ldots$.
  A similar extension of Theorem 4.2 seems possible but technically more complicated.

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