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Existence and blow up of solutions to certain classes of two-dimensional nonlinear Neumann problems

L'existence et l'explosion de solutions de certaines problèmes Neumann bi-dimensionnels et non linéaires

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Abstract

In this paper we study, analytically and numerically, the existence and blow up of solutions to two-dimensional boundary value problems of the form $\Delta u_\lambda = 0$ in Ω, $\partial u_\lambda/\partial \mathbf{n} = Du_\lambda + \lambda f(u_\lambda)$ on $\partial\Omega$. We place particular emphasis on $f(u) = \sinh(u) = (\mathrm{e}^u - \mathrm{e}^{-u})/2$, in which case the nonlinear flux boundary condition is frequently associated with the names of Butler and Volmer.

Résumé

Dans cet article nous étudions, analytiquement et numériquement, l'existence et l'explosion de solutions de problèmes aux limites bi-dimensionnels de la forme $\Delta u_\lambda = 0$ dans Ω, $\partial u_\lambda/\partial \mathbf{n} = Du_\lambda + \lambda f(u_\lambda)$ sur $\partial\Omega$. Nous portons une attention particulière à $f(u) = \sinh(u) = (\mathrm{e}^u - \mathrm{e}^{-u})/2$, situation dans laquelle la condition non linéaire sur le flux au bord est fréquemment associée aux noms de Butler et Volmer.

MSC: 35B30; 35B40; 35J65

Keywords: Nonlinear Neumann boundary conditions; Critical points; Blow up

1. Introduction

Let Ω be a smooth (C^∞), bounded domain in $\mathbb{R}^2$, and consider the elliptic boundary value problem

$$\begin{aligned} &\Delta u_\lambda = 0 \quad \text{in } \Omega, \\ &\frac{\partial u_\lambda}{\partial \mathbf{n}} = Du_\lambda + \lambda \sinh(u_\lambda) \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

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This is a simplified model problem, the likes of which frequently show up in connection with corrosion/oxidation modeling. Such problems also appear when modeling the electronic properties of certain semiconductor interface systems, for instance MIS (metal-insulator) or MOS (metal-oxide) semiconductor systems [1]. For the latter modeling it is often assumed that the surface electric field is controlled by the use of certain gaseous ambients. The exponential character of the flux boundary condition in general reflects the fact that the charged particles (the electrons) are thought to be regulated by a Boltzman statistics. For a discussion of some practical aspects of these problems, and some references to the applied literature we refer the reader to [1, 7], and [17]. For a discussion of related problems with "interior" exponential terms, see [2].

For $\lambda < \min\{-D, 0\}$ the solution of (1) is trivial: zero is the only solution. Our focus is thus on certain nontrivial (nonzero) solutions corresponding to $\lambda > \min\{-D, 0\}$, and in particular on the asymptotic behavior of these solutions as λ approaches zero. As it turns out there is a significant difference between the solution structure for $-D < \lambda < 0$ (if such an interval exists) and the solution structure for $0 < \lambda$. For any value of λ in the interval $-D < \lambda < 0$ we establish the existence of finitely many (and at least one) nontrivial solutions, whereas for any $\lambda > 0$ we establish the existence of infinitely many nontrivial solutions. Our existence proof is based on a variational technique, the likes of which have been used in several related contexts (see [9,14] and [16]). Maybe more surprising than the dichotomy of the solution structure is the difference in the asymptotic behavior of these solutions as λ approaches 0 from below and above, respectively.

For $\lambda > 0$ we show that any of the (infinitely many) families of solutions we construct generically contains a subsequence along which the nonlinear flux components $\lambda \sinh(u_\lambda)$ converge to a finite, nontrivial sum of delta functions $\sum_{i=1}^{K} \alpha_j \delta_{\mathbf{x}_i}$. Along the same subsequence the functions u_λ will, modulo a possible eigen-component (that can only appear for a countable set of D's) have a finite limit at all but a finite set of boundary points (generically $\{\mathbf{x}_i\}_{i=1}^{K}$). Finally, under very minimal assumptions, we derive a set of K necessary conditions for the point-mass locations $\mathbf{x}_i$.

For $\lambda < 0$ we show that any family of solutions which does not converge to 0, as $\lambda \to 0_-$, contains a subsequence which blows up pointwise almost everywhere as $\lambda \to 0_-$. We also show that along such a subsequence the nonlinear flux components $\lambda \sinh(u_\lambda)$ blow up in $H^{-1/2}(\partial\Omega)$, and even after a rescaling they converge weakly to a distribution, which is *not* supported at a finite set of points.

It is interesting to note that for $-D < \lambda < 0$ we establish an upper bound for $\|u_\lambda\|^2_{H^1(\Omega)}$ of the order $(\log \frac{1}{|\lambda|})^2$, as $\lambda \to 0_-$, whereas for the solutions we construct corresponding to $0 < \lambda$ we establish an upper bound for the "essential" part of $\|u_\lambda\|^2_{H^1(\Omega)}$ of the order $\log \frac{1}{\lambda}$. We do not claim that the solutions we variationally construct necessarily represent all solutions to the problem (1) – in certain situations (e.g. for an annulus, or for a disk and D negative) it is not hard to find an extra family of solutions for $\lambda > 0$ such that the "essential" part of $\|u_\lambda\|^2_{H^1(\Omega)}$ grows like $(\log \frac{1}{\lambda})^2$ as $\lambda \to 0_+$, and the functions u_λ as well as the boundary flux components $\lambda \sinh(u_\lambda)$ blow up almost everywhere. In the situations mentioned above these additional solutions possess higher order bifurcations, whereas the other solutions do not appear to possess any. We provide some numerical data that cast extra light on this phenomenon.

We have already in earlier papers [5,10] and [12] analyzed the special case $\frac{\partial u_\lambda}{\partial \mathbf{n}} = \lambda \sinh(u_\lambda)$ (i.e., $D = 0$). That analysis included a study of the existence structure as well as a study of the asymptotic behavior as $\lambda \to 0_+$. In that case there are no nontrivial solutions for $\lambda < 0$, and so one does not encounter the quite remarkable difference in existence structure and blow-up behavior between $\lambda \to 0_-$ and $\lambda \to 0_+$. For $D = 0$ we have presented strong numerical evidence that the fluxes $\lambda \sinh(u_\lambda)$ might occasionally (depending on the domain Ω) converge to a sum of delta functions plus a regular part, as $\lambda \to 0_+$ [12]. This should be compared to the results in this paper which (with only minor additional assumptions) show that, for $D \neq 0$, the functions $\lambda \sinh(u_\lambda)$ can only converge to a pure sum of delta functions – the limit of the fluxes $Du_\lambda + \lambda \sinh(u_\lambda)$, on the other hand, will always contain a regular part as well.

Of particular interest for D are Steklov eigenvalues, i.e., the case when $D = D_k$, for any of the countable set of nonnegative values $0 = D_1 \leqslant D_2 \leqslant \cdots$ for which the linear boundary value problem

$$\Delta\phi = 0 \quad \text{in } \Omega, \qquad \frac{\partial \phi}{\partial \mathbf{n}} = D\phi \quad \text{on } \partial\Omega, \tag{2}$$

possesses nontrivial solutions. We may think of the corresponding bound states ϕ_k as associated with impurities and defects. As stated in [1] such bound states are known to "strongly affect the electrical properties of the bulk semiconductor" – the results in the present paper (in particular the estimates of the finite dimensional projection $P_D u_\lambda$) provide some qualitative and quantitative clarification of this statement.

In the case of $D = 0$, and a domain in the shape of a disk, it is possible to give explicit (and surprisingly simple) formulas for what we believe to be all the solutions to (1), see [5]. We have not been able to derive similar explicit formulas for $D \neq 0$.

Finally we present some numerical calculations and some heuristic arguments related to the solution structure of the boundary value problem

$$\begin{aligned} &\Delta u_\lambda = 0 \quad \text{in } \Omega, \\ &\frac{\partial u_\lambda}{\partial \mathbf{n}} = D v_\lambda + \lambda f(u_\lambda) \quad \text{on } \partial\Omega, \end{aligned} \tag{3}$$

for several other functions f (odd and with $f'(0) = 1$, $f(t) > 0$ for $t > 0$).

2. The Butler–Volmer case

In this and the following two subsections we provide a very careful analysis of the existence and the asymptotic behavior of solutions to the boundary value problem (1). As already mentioned, nonlinear boundary flux conditions of this (exponential) type are frequently found in the corrosion literature; they are often associated with the names of Butler and Volmer.

When we talk about a solution to (1) we shall always mean a real function $u_\lambda \in H^1(\Omega)$ which satisfies the boundary value problem in the weak sense that

$$\int_\Omega \nabla u_\lambda \nabla v \, dx = D \int_{\partial\Omega} u_\lambda v \, d\sigma_x + \lambda \int_{\partial\Omega} \sinh(u_\lambda) v \, d\sigma_x,$$

for any $v \in H^1(\Omega)$. The fact that we restrict attention to domains Ω that are two dimensional ensures that e^v is in $L^p(\partial\Omega)$, $1 < p < \infty$, for any $v \in H^1(\Omega)$. It furthermore ensures that the mapping $v \to e^v|_{\partial\Omega}$ is compact from $H^1(\Omega)$ to $L^p(\partial\Omega)$, $1 < p < \infty$. These facts are both essential for our present analysis, in particular as far as existence of solutions to (1) is concerned. Very classical results from elliptic regularity theory ensure that any weak, finite energy solution is a classical solution to (1); indeed it is $C^\infty(\overline{\Omega})$.

Before proceeding let us present some computational results that provide intuition concerning the kind of results we might expect to be able to prove. For our computational experiments we take Ω to be the unit disk, and we first take $D = 2$ (a Steklov eigenvalue). Based on a boundary integral formulation, a collocation method, and a "continuation scheme" we now calculate what we believe to be all the nontrivial solutions to (1) (modulo rotations). For details about the numerical implementations, see [11] and [12]. In the left frame of Fig. 1 we display the $H^1(\Omega)$-norm

$$\|u_\lambda\|_{H^1(\Omega)} = \langle u_\lambda, u_\lambda \rangle_{H^1}^{1/2} = \left(\int_\Omega |\nabla u_\lambda|^2 \, dx + \int_{\partial\Omega} u_\lambda^2 \, d\sigma_x \right)^{1/2},$$

as a function of λ for all these solutions. In the right frame of Fig. 1 we display the energy

$$E_\lambda(u_\lambda) = \frac{1}{2} \int_\Omega |\nabla u_\lambda|^2 \, dx - \frac{D}{2} \int_{\partial\Omega} u_\lambda^2 \, d\sigma_x - \lambda \int_{\partial\Omega} \big(\cosh(u_\lambda) - 1\big) \, d\sigma_x, \tag{4}$$

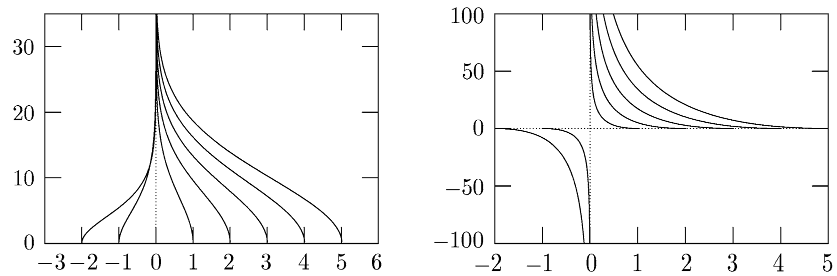

Fig. 1. Left frame: $H^1(\Omega)$-norm as a function of λ for different solutions to (1). Right frame: energies E_λ for the same solutions.

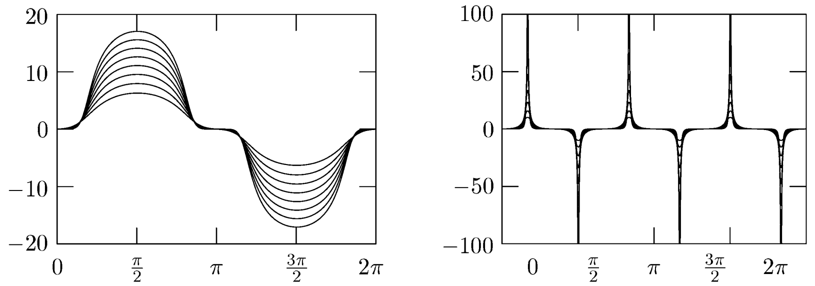

Fig. 2. Left frame: normal boundary flux component $\lambda\sinh(u_\lambda)$ for λ negative. Right frame: normal boundary flux component $\lambda\sinh(u_\lambda)$ for λ positive. In both frames $D = 2$.

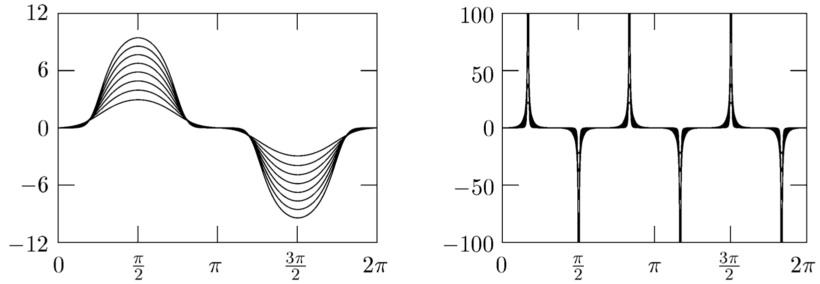

Fig. 3. Left frame: normal boundary flux component $\lambda\sinh(u_\lambda)$ for λ negative (smallest flux: $\lambda \sim -0.64 \times 10^{-1}$, largest flux $\lambda \sim -0.56 \times 10^{-4}$). Right frame: normal boundary flux component $\lambda\sinh(u_\lambda)$ for λ positive (smallest flux: $\lambda \sim 0.85 \times 10^{-1}$, largest flux $\lambda \sim 0.75 \times 10^{-4}$). In both frames $D = 1.5$.

as a function of λ for all these solutions. It is quite easy to show that there are no nontrivial solutions for $\lambda < \min\{-D, 0\}$. We note that if u_λ is a solution to (1) so is $-u_\lambda$. We furthermore note that if Ω is a disk and u_λ is a solution to (1), then so is any rotation of u_λ. We do not consider these to be essentially different solutions. In Fig. 1 there appears to be finitely many essentially different solutions for any fixed λ in the interval $-D = -2 < \lambda < 0$, whereas there appears to be infinitely many essentially different solutions for any $\lambda > 0$ (we have only "traced" solutions that bifurcate from the trivial (0) solution before $\lambda = 6$, but the same pattern would persist for "later" solutions). The $H^1(\Omega)$ norm (and also the energy) of all nontrivial solutions "blows up" as λ approaches 0.

In Fig. 2 we display the nonlinear part of the normal boundary currents $\lambda\sinh(u_\lambda)$ for 8 values of λ. The left frame corresponds to λ negative (between -0.79×10^{-1} and -0.70×10^{-4}) and the flux components are taken along the branch emanating from the trivial solution at $\lambda = -1$. The right frame corresponds to λ positive (between 0.85×10^{-1} and 0.75×10^{-4}) and the flux components are taken along the branch emanating from the trivial solution at $\lambda = 1$.

The blow-up behavior of the solutions is clearly considerably different depending on whether $\lambda \to 0_-$, or whether $\lambda \to 0_+$. The blow-up behavior is not significantly affected by whether D is a Steklov eigenvalue or not, as evidenced by Fig. 3, each frame of which depicts the nonlinear part of the normal boundary currents $\lambda\sinh(u_\lambda)$ for 8 values of λ. Only this time $D = 1.5$, and we follow solution branches emanating from the trivial solution at $\lambda = -0.5$ and $\lambda = 1.5$, respectively. Nontrivial solutions corresponding to $\lambda < 0$ (only possibly when $D > 0$) blow up pointwise almost everywhere, and the flux components $\lambda\sinh(u_\lambda)$ seem to blow up on entire intervals as $\lambda \to 0_-$, whereas it is tempting to conjecture that the corresponding flux components for $\lambda \to 0_+$ blow up at only a finite number of boundary points (provided Ω is simply connected, and D is positive).

The purpose of the next two sections is to carefully analyze the existence structure and the blow-up patterns, as partially illustrated in Figs. 1–3. For obvious reasons we separate this analysis into two different parts, one concerning $\lambda < 0$, and one concerning $\lambda > 0$.

A common tool for the two analyzes is the energy functional $E_\lambda(\cdot)$ and the associated "restricted" functional $J_\lambda(\cdot)$, defined by

$$J_\lambda(v) = \begin{cases} \inf_{t>0} E_\lambda(tv) & \text{for } \lambda < 0, \\ \sup_{t>0} E_\lambda(tv) & \text{for } \lambda > 0, \end{cases}$$

$v \in H^1(\Omega) \setminus \{0\}$. In the more general setting of (3) the expression $\cosh v - 1$ (in the energy $E_\lambda(\cdot)$) would be replaced by $F(v)$, where the nonnegative even function F is given by

$$F(t) = \int_0^t f(s)\,\mathrm{d}s.$$

We also define

$$\Sigma = \{w \in H^1(\Omega):\ \|w\|_{H^1(\Omega)} = 1\}.$$

The functional $J_\lambda(\cdot)$ is even and continuous on $H^1(\Omega) \setminus \{0\}$ (see [10] Lemma 2.4). Since $J_\lambda(\cdot)$ is homogeneous (of degree 0) it is occasionally convenient to regard it as just a functional on Σ. It is not difficult to see that

$$v \to J_\lambda(v) \quad \text{maps } H^1(\Omega) \setminus \{0\} \text{ onto the interval } [J_\lambda(1), 0] \subset (-\infty, 0] \text{ for } \lambda < 0.$$

In particular, for $\lambda < 0$ we have $J_\lambda(v) = 0$ for any v that vanishes identically on $\partial\Omega$. A simple calculation yields that

$$J_\lambda(1) = \inf_{t>0} |\partial\Omega| \left(-\frac{D}{2}t^2 - \lambda(\cosh(t) - 1) \right) = 0 \quad \text{for } \lambda < \min\{-D, 0\}, \quad \text{and}$$
$$J_\lambda(1) = \inf_{t>0} |\partial\Omega| \left(-\frac{D}{2}t^2 - \lambda(\cosh(t) - 1) \right) < 0 \quad \text{for } \min\{-D, 0\} < \lambda < 0.$$

As a consequence $J_\lambda(\cdot) = 0$ for $\lambda < \min\{-D, 0\}$. It is equally easy to show that

$$v \to J_\lambda(v) \quad \text{maps } H^1(\Omega) \setminus \{0\} \text{ into the interval } [0, \infty] \text{ for } \lambda > 0.$$

Concerning $\lambda > 0$ we see that $J_\lambda(v) = \infty$ if and only if v vanishes identically on $\partial\Omega$. For $\lambda > \max\{0, -D\}$ we also calculate $J_\lambda(1) = 0$. We thus conclude that the range of $J_\lambda(H^1(\Omega) \setminus \{0\})$ is unbounded for any $\lambda > 0$ and that the range equals the interval $[0, \infty]$ for any $\lambda > \max\{0, -D\}$.

Following the same arguments as in [10] we may show that $J_\lambda(\cdot)$ is smooth on the set $\{v:\ J_\lambda(1) \leqslant J_\lambda(v) < 0\}$ for $\min\{-D, 0\} < \lambda < 0$, and that $J_\lambda(\cdot)$ is smooth on the set $\{v:\ 0 < J_\lambda(v) < \infty\}$ for $0 < \lambda$. In each case we do this by showing that there exists a unique value $t(v) > 0$ such that $J_\lambda(v) = E_\lambda(t(v)v)$. By the chain rule,

$$J'_\lambda(v)[w] = E'_\lambda(t(v)v)[v]t'(v)[w] + E'_\lambda(t(v)v)[w]t(v) = E'_\lambda(t(v)v)[w]t(v), \tag{5}$$

where we have used the fact that $E'_\lambda(t(v)v)[v] = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=t(v)} E_\lambda(tv) = 0$, due to the definition of $t(v)$. Since $J'_\lambda(v)[v] = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=1} J_\lambda(tv) = 0$ we now arrive at the following equivalences for any v, with $J_\lambda(v) \in \mathbb{R} \setminus \{0\}$,

$$\exists\alpha \text{ such that } J'_\lambda(v)[\cdot] = \alpha\langle\cdot, v\rangle_{H^1} \iff J'_\lambda(v)[\cdot] = 0 \iff E'_\lambda(t(v)v)[\cdot] = 0.$$

$J_\lambda(v) \in \mathbb{R} \setminus \{0\}$ ensures that the positive number $t(v)$ is well defined. In other words: if v is a critical point for $J_\lambda(\cdot)$ on Σ, corresponding to a nonzero critical value, then $u = t(v)v \neq 0$ is a critical point for $E_\lambda(\cdot)$ in $H^1(\Omega)$. Conversely, if $u \neq 0$ is a critical point for $E_\lambda(\cdot)$ in $H^1(\Omega)$, with $J_\lambda(u) \neq 0$, then $v = u/\|u\|_1$ is a critical point for $J_\lambda(\cdot)$ on Σ. Such critical points are weak-, and by elliptic regularity, also strong solutions to the boundary value problem (1).

For the existence-analysis (in order to establish the existence of critical point for $J_\lambda(\cdot)$ on Σ) it is crucial that $J_\lambda(\cdot)$ has a certain compactness property. For the exponential nonlinearity we consider here this condition is a fairly direct consequence of the compactness of the mapping

$$H^1(\Omega) \ni v \to \cosh(v)|_{\partial\Omega} \in L^p(\partial\Omega).$$

Lemma 1 *((Palais–Smale condition)). Given any two sequences $v_n \in \Sigma$, $\alpha_n \in \mathbb{R}$ with $J_\lambda(v_n) \to c \neq 0$ and $J'_\lambda(v_n)[\cdot] - \alpha_n\langle\cdot, v_n\rangle_{H^1} \to 0$ in $[H^1(\Omega)]^*$, as $n \to \infty$, we may conclude that $\alpha_n \to 0$, and that there exists a subsequence (for simplicity also indexed by n) and a function $v_\infty \in \Sigma$, so that $v_n \to v_\infty$ in $H^1(\Omega)$, as $n \to \infty$.*

Proof. Since $J'_\lambda(v_n)[\cdot] - \alpha_n\langle\cdot, v_n\rangle_{H^1} \to 0$ in $[H^1(\Omega)]^*$, and since $J'_\lambda(v_n)[v_n] = 0$, it follows immediately that

$$-\alpha_n = J'_\lambda(v_n)[v_n] - \alpha_n\langle v_n, v_n\rangle_{H^1} \to 0,$$

as desired. It then also follows that

$$J_\lambda'(v_n)[\cdot] \to 0 \quad \text{in } \big[H^1(\Omega)\big]^*. \tag{6}$$

Defining $u_n = t(v_n)v_n$, we now proceed to prove that the sequence $\{u_n\}$ is bounded in $H^1(\Omega)$. We may without loss of generality suppose that the sequence $\{t(v_n)\}$ is strictly positive, since $0 < J_\lambda(v_n) < \infty$ for n sufficiently large. Our definition of $t(v_n)$ (stationarity of $E_\lambda(tv_n)$ with respect to t) implies

$$\frac{1}{2}\bigg(\int_\Omega \big|t(v_n)\nabla v_n\big|^2\,\mathrm{d}x - D\int_{\partial\Omega}\big(t(v_n)v_n\big)^2\,\mathrm{d}\sigma_x\bigg) = \frac{\lambda}{2}\int_{\partial\Omega} t(v_n)v_n \sinh\big(t(v_n)v_n\big)\,\mathrm{d}\sigma_x.$$

This in turn gives the inequality

$$\begin{aligned}|c|+1 > \big|J(v_n)\big| = \big|E_\lambda\big(t(v_n)v_n\big)\big| &= |\lambda|\int_{\partial\Omega}\bigg(\frac{u_n}{2}\sinh(u_n) - \cosh(u_n) + 1\bigg)\,\mathrm{d}\sigma_x \\ &\geqslant |\lambda|\int_{\partial\Omega}\big(\cosh(u_n) + u_n^2 - C_0\big)\,\mathrm{d}\sigma_x,\end{aligned}$$

with $C_0 = \max_{x\in\mathbb{R}}(2\cosh(x) + x^2 - \frac{x}{2}\sinh(x) - 1) < \infty$. From this it is easy to see that both $\int_{\partial\Omega} u_n^2\,\mathrm{d}\sigma_x$ and $\int_{\partial\Omega}\cosh(u_n)\,\mathrm{d}\sigma_x$ are bounded by constants depending on c, λ, and Ω. Invoking the energy convergence (the fact that $E_\lambda(u_n) \to c$) we see that $\int_\Omega |\nabla u_n|^2\,\mathrm{d}x$ is bounded by a constant with the same set of dependencies. In summary

$$0 < t(v_n) = \|u_n\|_{H^1(\Omega)} \leqslant C(c,\lambda,\Omega).$$

We may now extract a subsequence (also indexed by n) such that $u_n \rightharpoonup u_\infty$ weakly in $H^1(\Omega)$ and $t(v_n) \to b$, for some $u_\infty \in H^1(\Omega)$ and some $b \geqslant 0$. Note that $c \neq 0$ implies that $b > 0$, and the weak $H^1(\Omega)$ convergence implies that $\int_{\partial\Omega}(u_n - u_\infty)^2\,\mathrm{d}\sigma_x \to 0$. Therefore,

$$\begin{aligned}\int_\Omega |\nabla u_n - \nabla u_\infty|^2\,\mathrm{d}x &= \big(E_\lambda'(u_n) - E_\lambda'(u_\infty)\big)[u_n - u_\infty] + D\int_{\partial\Omega}(u_n - u_\infty)^2\,\mathrm{d}\sigma_x \\ &\quad + \lambda\int_{\partial\Omega}(u_n - u_\infty)\big(\sinh(u_n) - \sinh(u_\infty)\big)\,\mathrm{d}\sigma_x \\ &= E_\lambda'(u_n)[u_n - u_\infty] - E_\lambda'(u_\infty)[u_n - u_\infty] + \mathrm{o}(1).\end{aligned}$$

The last equality follows from the $L^2(\partial\Omega)$ convergence and Trudinger's inequality, that is,

$$\int_{\partial\Omega}\big(\sinh(u)\big)^2\,\mathrm{d}\sigma_x \leqslant C_1 \mathrm{e}^{C_2\|u\|^2_{H^1(\Omega)}},$$

see Lemma 2.1 of [10]. By (5), $E_\lambda'(u_n)[\cdot] = \frac{1}{t(v_n)}J_\lambda'(v_n)[\cdot]$ and so from (6), the fact that $\lim t(v_n) = b > 0$, and the weak convergence $u_n - u_\infty \rightharpoonup 0$, we now conclude that $u_n \to u_\infty$ in $H^1(\Omega)$. It follows that $v_n = \frac{1}{t(v_n)}u_n \to \frac{1}{b}u_\infty = v_\infty$ in $H^1(\Omega)$, and that $v_\infty \in \Sigma$. □

Let $\{(D_k,\phi_k)\}_{k=1}^\infty$ denote the Steklov eigenvalues and eigenvectors for (2). The eigenvalues D_k form a non-decreasing sequence $0 = D_1 \leqslant D_2 \leqslant \cdots$, with $D_k \to \infty$ as $k \to \infty$. There may be repeated values in this sequence, since each eigenvalue appears as many times as its (finite) multiplicity indicates. The ϕ_k may be selected so that $\int_{\partial\Omega}\phi_j\phi_k\,\mathrm{d}\sigma_x = \delta_{jk}$. The functions $\phi_k|_{\partial\Omega}$ now form an orthonormal basis for $L^2(\partial\Omega)$. The ϕ_k are then also orthogonal in $H^1(\Omega)$. Corresponding to any $D \in \mathbb{R}$ we define the projection operator

$$P_D u = \sum_{D_k = D}\langle u,\phi_k\rangle_{L^2(\partial\Omega)}\phi_k = \sum_{D_k = D}\frac{\langle u,\phi_k\rangle_{H^1}}{\langle\phi_k,\phi_k\rangle_{H^1}}\phi_k.$$

P_D is a projection onto the "eigenspace" V_D, associated with D. Note that we interpret this to mean that $P_D = 0$ (and $V_D = \{0\}$) if D is *not* a Steklov eigenvalue. In the case of the unit disk we have $D_{2k+1} = k$, $k \geqslant 0$ and $D_{2k} = k$,

$k \geqslant 1$, as seen in Fig.1 (where $D = 2$). In general, it is well known that the eigenvalues D_k grow according to a classical Weyl asymptotics. Indeed, under the assumption that Ω is simply connected, it is extremely easy to see that $ck \leqslant D_k \leqslant Ck$, as $k \to \infty$ (chose a conformal transformation to map Ω onto the unit disk, build a min-max characterization of the original eigenvalues and compare with the known eigenvalues for the problem (2) on the unit disk). Much more detailed results concerning the asymptotic behavior of D_k and ϕ_k have been established in [15].

It will frequently be convenient to decompose solutions to (1) as

$$u_\lambda = P_D u_\lambda + (I - P_D) u_\lambda = P_D u_\lambda + w_\lambda,$$

where D is the same as the "shift" which appears in (1). The following lemma will then play a crucial role.

Lemma 2. *Let $D \in \mathbb{R}$ be fixed, and suppose u_λ, $\lambda \neq 0$, is a solution to* (1)*, i.e., a solution to*

$$\Delta u_\lambda = 0 \quad \text{in } \Omega, \qquad \frac{\partial u_\lambda}{\partial \mathbf{n}} = D u_\lambda + \lambda \sinh(u_\lambda) \quad \text{on } \partial\Omega.$$

Let w_λ denote the function $w_\lambda = (I - P_D) u_\lambda$. There exists a constant C, depending on Ω and D, but independent of λ, and u_λ such that

$$\|P_D u_\lambda\|_{H^1(\Omega)} \leqslant C\big(\|w_\lambda\|^2_{H^1(\Omega)} + 1\big).$$

Proof. From the definition of u_λ, and integration by parts, we immediately get

$$\begin{aligned}\lambda \int_{\partial\Omega} \sinh(u_\lambda) P_D u_\lambda \,\mathrm{d}\sigma_x &= -D \int_{\partial\Omega} u_\lambda P_D u_\lambda \,\mathrm{d}\sigma_x + \int_{\partial\Omega} \frac{\partial u_\lambda}{\partial \mathbf{n}} P_D u_\lambda \,\mathrm{d}\sigma_x \\ &= -\int_{\partial\Omega} u_\lambda \frac{\partial P_D u_\lambda}{\partial \mathbf{n}} \,\mathrm{d}\sigma_x + \int_{\partial\Omega} \frac{\partial u_\lambda}{\partial \mathbf{n}} P_D u_\lambda \,\mathrm{d}\sigma_x = 0.\end{aligned}$$

For $\lambda \neq 0$ we thus conclude

$$\int_{\partial\Omega} \sinh(u_\lambda) P_D u_\lambda \,\mathrm{d}\sigma_x = 0. \tag{7}$$

By insertion of the identity

$$\sinh(u_\lambda) = \sinh(P_D u_\lambda + w_\lambda) = \sinh(P_D u_\lambda)\cosh(w_\lambda) + \cosh(P_D u_\lambda)\sinh(w_\lambda)$$

into (7), rearrangement, and use of the estimate $|\cosh(P_D u_\lambda) P_D u_\lambda| \leqslant \sinh(P_D u_\lambda) P_D u_\lambda + \mathrm{e}^{-1}$, we now obtain

$$\begin{aligned}\int_{\partial\Omega} \sinh(P_D u_\lambda) P_D u_\lambda \cosh(w_\lambda) \,\mathrm{d}\sigma_x &= -\int_{\partial\Omega} \cosh(P_D u_\lambda) P_D u_\lambda \sinh(w_\lambda) \,\mathrm{d}\sigma_x \\ &\leqslant \int_{\partial\Omega} \big|\cosh(P_D u_\lambda) P_D u_\lambda\big| \big|\sinh(w_\lambda)\big| \,\mathrm{d}\sigma_x \\ &\leqslant \int_{\partial\Omega} \big(\sinh(P_D u_\lambda) P_D u_\lambda + \mathrm{e}^{-1}\big) \big|\sinh(w_\lambda)\big| \,\mathrm{d}\sigma_x.\end{aligned}$$

Since $\cosh(w_\lambda) - |\sinh(w_\lambda)| = \mathrm{e}^{-|w_\lambda|}$ this last inequality immediately leads to

$$\int_{\partial\Omega} \sinh(P_D u_\lambda) P_D u_\lambda \,\mathrm{e}^{-|w_\lambda|} \,\mathrm{d}\sigma_x \leqslant \mathrm{e}^{-1} \int_{\partial\Omega} \big|\sinh(w_\lambda)\big| \,\mathrm{d}\sigma_x.$$

An application of Cauchy–Schwarz's inequality yields

$$\left(\int_{\partial\Omega} \big(\sinh(P_D u_\lambda) P_D u_\lambda\big)^{1/2}\, d\sigma_x\right)^2 \leqslant \int_{\partial\Omega} \sinh(P_D u_\lambda) P_D u_\lambda\, e^{-|w_\lambda|}\, d\sigma_x \cdot \int_{\partial\Omega} e^{|w_\lambda|}\, d\sigma_x$$
$$\leqslant e^{-1} \int_{\partial\Omega} \big|\sinh(w_\lambda)\big|\, d\sigma_x \cdot \int_{\partial\Omega} e^{|w_\lambda|}\, d\sigma_x \leqslant C\, e^{C\|w_\lambda\|^2_{H^1(\Omega)}}. \tag{8}$$

For the last estimate we used a version of Trudinger's inequality, asserting that for a (smooth) bounded, two dimensional domain Ω there exists a constant C such that

$$\int_{\partial\Omega} e^{|w|}\, d\sigma_x \leqslant C\, e^{C\|w\|^2_{H^1(\Omega)}}, \quad \forall w \in H^1(\Omega);$$

see for example [10] or [17]. Since $e^{|p|/2} \leqslant (\sinh(p)p)^{1/2} + C$ it follows from (8) that

$$\int_{\partial\Omega} e^{|P_D u_\lambda|/2}\, d\sigma_x \leqslant C\big(e^{C\|w_\lambda\|^2_{H^1(\Omega)}} + 1\big) \leqslant 2C\, e^{C\|w_\lambda\|^2_{H^1(\Omega)}} \leqslant e^{C\|w_\lambda\|^2_{H^1(\Omega)} + \log 2C},$$

and so an application of Jensen's inequality (applied to the convex function $p \to e^{p/2}$) leads to

$$e^{\int_{\partial\Omega} (|P_D u_\lambda|/2)(d\sigma_x/|\partial\Omega|)} \leqslant \int_{\partial\Omega} e^{|P_D u_\lambda|/2} \frac{d\sigma_x}{|\partial\Omega|} \leqslant e^{C(\|w_\lambda\|^2_{H^1(\Omega)} + 1)}.$$

By taking a logarithm on both sides we get

$$\int_{\partial\Omega} |P_D u_\lambda|\, d\sigma_x \leqslant C\big(\|w_\lambda\|^2_{H^1(\Omega)} + 1\big).$$

Since $P_D u_\lambda$ is either 0 or lies in the finite dimensional Steklov eigenspace associated with D (where all norms are equivalent) we now conclude that

$$\|P_D u_\lambda\|_{H^1(\Omega)} \leqslant C\big(\|w_\lambda\|^2_{H^1(\Omega)} + 1\big),$$

with C independent of λ and u_λ. Here we use that $\|\cdot\|_{L^1(\partial\Omega)}$ is a norm on the eigenspace V_D (since $\phi \in V_D$ vanishes identically if $\phi|_{\partial\Omega} = 0$). This completes the proof of the lemma. □

2.1. Negative λ

To consider the very simplest case first, suppose $\lambda \leqslant \min\{0, -D\}$, and suppose u_λ is a solution to (1). Green's formula then immediately gives

$$\int_\Omega |\nabla u_\lambda|^2\, dx = D \int_{\partial\Omega} u_\lambda^2\, d\sigma_x + \lambda \int_{\partial\Omega} \sinh(u_\lambda) u_\lambda\, d\sigma_x = (D+\lambda) \int_{\partial\Omega} u_\lambda^2\, d\sigma_x + \lambda \int_{\partial\Omega} (\sinh(u_\lambda) - u_\lambda) u_\lambda\, d\sigma_x \leqslant 0.$$

For the last inequality we used that $D + \lambda \leqslant 0$, that $\lambda \leqslant 0$, and that $(\sinh(x) - x)x \geqslant 0$. We may thus conclude that u_λ is a constant. However, if we additionally suppose that $D \neq 0$ or $\lambda \neq 0$, then the only constant, z, for which $Dz + \lambda \sinh(z) = 0$ is $z = 0$. In summary:

Proposition 1. *For $\lambda \leqslant \min\{0, -D\}$ the only solution to* (1) *is, with one exception, $u_\lambda = 0$. The exception is $D = \lambda = 0$, in which case any constant is a solution to* (1).

The case $\min\{0, -D\} < \lambda \leqslant 0$ is more interesting (and complicated). Except for possibly $\lambda = 0$, there are now always nontrivial solutions to (1). Corresponding to the segment $\min\{0, -D\} < \lambda < 0$ we may actually prove the following result about existence and about a general upper bound for the $H^1(\Omega)$-norm of solutions.

Proposition 2. *Suppose the shift D, appearing in* (1)*, is positive. There exist constants $C_1 > 0$ and $C_2 > 0$, depending only on D and Ω, such that*

$$\|u_\lambda\|^2_{H^1(\Omega)} \leqslant C_1\left(\log\frac{1}{|\lambda|}\right)^2 + C_2,$$

for any $-D < \lambda < 0$, and any solution u_λ to (1)*. Furthermore, there exists at least one family of solutions $U_\lambda \neq 0$, $-D < \lambda < 0$, and two constants $c_1 > 0$ and c_2 such that*

$$c_1\left(\log\frac{1}{|\lambda|}\right)^2 + c_2 \leqslant \|U_\lambda\|^2_{H^1(\Omega)} \leqslant C_1\left(\log\frac{1}{|\lambda|}\right)^2 + C_2.$$

Proof. The fact that

$$\int_\Omega |\nabla u_\lambda|^2\,\mathrm{d}x - D\int_{\partial\Omega} u_\lambda^2\,\mathrm{d}\sigma_x - \lambda\int_{\partial\Omega}\sinh(u_\lambda)u_\lambda\,\mathrm{d}\sigma_x = 0$$

for any solution u_λ to (1), and any $\lambda < 0$, may be rewritten

$$\int_\Omega |\nabla u_\lambda|^2\,\mathrm{d}x + |\lambda|\int_{\partial\Omega}\sinh(u_\lambda)u_\lambda\,\mathrm{d}\sigma_x = D\int_{\partial\Omega} u_\lambda^2\,\mathrm{d}\sigma_x. \tag{9}$$

It is well known that, given any $\epsilon > 0$, there exists a constant C_ϵ, such that

$$\|u\|^2_{L^2(\partial\Omega)} \leqslant \epsilon\|u\|^2_{H^{1/2}(\partial\Omega)} + C_\epsilon\|u\|^2_{H^{-1}(\partial\Omega)} \leqslant \epsilon\|u\|^2_{H^{1/2}(\partial\Omega)} + C_\epsilon\|u\|^2_{L^1(\partial\Omega)}. \tag{10}$$

For the last inequality we used that $L^1(\partial\Omega)$ embeds continuously into $H^{-1}(\partial\Omega)$, since $\partial\Omega$ is one-dimensional. It is also well known that

$$\|u\|^2_{H^{1/2}(\partial\Omega)} \leqslant C\|u\|^2_{H^1(\Omega)} \leqslant C\big(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^1(\partial\Omega)}\big). \tag{11}$$

By selecting ϵ sufficiently small we obtain from a combination of (9), (10) and (11), that

$$\frac{1}{2}\int_\Omega |\nabla u_\lambda|^2\,\mathrm{d}x + |\lambda|\int_{\partial\Omega}\sinh(u_\lambda)u_\lambda\,\mathrm{d}\sigma_x \leqslant C\|u_\lambda\|^2_{L^1(\partial\Omega)}. \tag{12}$$

The function $x \to g(x) = \sinh(x)x$ is convex, and so Jensen's inequality immediately asserts that

$$g\left(\int_{\partial\Omega}|u_\lambda|\frac{\mathrm{d}\sigma}{|\partial\Omega|}\right) \leqslant \int_{\partial\Omega} g\big(|u_\lambda|\big)\frac{\mathrm{d}\sigma}{|\partial\Omega|} = \int_{\partial\Omega}\sinh(u_\lambda)u_\lambda\frac{\mathrm{d}\sigma}{|\partial\Omega|}.$$

By a combination of this inequality and (12) we obtain

$$|\lambda|\,\mathrm{e}^{\int_{\partial\Omega}|u_\lambda|\frac{\mathrm{d}\sigma_x}{|\partial\Omega|}} \leqslant |\lambda|g\left(\int_{\partial\Omega}|u_\lambda|\frac{\mathrm{d}\sigma}{|\partial\Omega|}\right) + |\lambda|\cosh(1) \leqslant C\left(\int_{\partial\Omega}|u_\lambda|\,\mathrm{d}\sigma_x\right)^2 + |\lambda|\cosh(1),$$

which immediately leads to

$$\int_{\partial\Omega}|u_\lambda|\,\mathrm{d}\sigma_x \leqslant C_1\log\frac{1}{|\lambda|} + C_2,$$

for $-D < \lambda < 0$. From (12) it now follows that

$$\int_\Omega |\nabla u_\lambda|^2\,\mathrm{d}x \leqslant C_1\left(\log\frac{1}{|\lambda|}\right)^2 + C_2,$$

and so, in summary

$$\|u_\lambda\|^2_{H^1(\Omega)} \leqslant C_1\left(\log\frac{1}{|\lambda|}\right)^2 + C_2.$$

This completes the proof of the first part of this proposition. The second part is much simpler. We just note that for any $-D < \lambda < 0$ there exists exactly one positive solution to

$$\sinh z_\lambda = -\frac{D}{\lambda} z_\lambda.$$

Here we have used the fact that $-D < \lambda < 0 \Rightarrow -D/\lambda > 1$. This solution satisfies

$$c_1 \log \frac{1}{|\lambda|} + c_2 < z_\lambda,$$

for some constants $c_1 > 0$ and c_2, and so the constant function $U_\lambda(x) = z_\lambda$ is easily seen to be a nonzero solution to (1) with the desired lower bound. $\square$

Based on Fig. 1 one might expect that any sequence of solutions u_{λ_n}, $\lambda_n \to 0_-$, which does not degenerate to the 0 solution for λ_n sufficiently near 0, must contain a subsequence such that $\|u_{\lambda_n}\|^2_{H^1(\Omega)}$ is bounded from below by $c_1(\log \frac{1}{|\lambda_n|})^2$ as $\lambda_n \to 0_-$. We are not quite able to prove that, but we can establish the following weaker result. This result also shows that the only possible blow up behavior as $\lambda \to 0_-$ is blow-up almost everywhere. The family U_λ constructed in Lemma 2 does blow up everywhere.

Proposition 3. *Suppose the shift D, appearing in* (1)*, is positive. There exists a constant $c_1 > 0$ such that whenever u_{λ_n}, $-D < \lambda_n < 0$, $\lambda_n \to 0_-$, is a family of solutions to* (1) *with the property that $\|u_{\lambda_n}\|_{H^1(\Omega)}$* does not *converge to* 0 *as $\lambda_n \to 0_-$, then we may extract a subsequence, for simplicity also denoted u_{λ_n}, with*

$$c_1 \log \frac{1}{|\lambda_n|} \leqslant \|u_{\lambda_n}\|^2_{H^1(\Omega)} \quad \text{as } \lambda_n \to 0_-. \tag{13}$$

We may extract this subsequence so that u_{λ_n} converges pointwise to $\pm\infty$ almost everywhere in Ω, and so that $u_{\lambda_n}|_{\partial\Omega}$ converges pointwise to $\pm\infty$ on a set of positive one dimensional surface measure. By appropriate extraction of the subsequence we may also arrange that $w_{\lambda_n} = (I - P_D)u_{\lambda_n}$ converges pointwise to $\pm\infty$ almost everywhere in Ω, that $w_{\lambda_n}|_{\partial\Omega}$ converges pointwise to $\pm\infty$ on a set of positive one dimensional surface measure, and that

$$\exists \gamma_n \to \infty \text{ such that } \quad \frac{\lambda_n \sinh(u_{\lambda_n})}{\gamma_n} \rightharpoonup \mu \neq 0, \quad \text{weakly in } H^{-1/2}(\partial\Omega), \text{ as } \lambda_n \to 0_-.$$

Remarks. A distribution $\mu \in H^{-1/2}(\partial\Omega)$ *cannot* consist of Dirac delta functions. By comparison with Theorem 2 later in this paper the structure of the (rescaled flux-component) limit μ, for $\lambda \to 0_-$, is thus completely different from the finite sum of Dirac delta masses that generically emerges as the limit of the flux-component $\lambda \sinh(u_\lambda)$ (for the variationally constructed solutions) as $\lambda \to 0_+$. Broadly speaking the last statement of Proposition 3 means that, as λ approaches 0_-, the flux-component $\lambda \sinh(u_\lambda)$ "blows up on a thicker set" than is the case (for the variationally constructed solutions) when λ approaches 0_+. This is exactly what we evidenced in Fig. 2. Similarly we also see that w_λ blows up almost everywhere in Ω, and on a set of positive measure on $\partial\Omega$, as $\lambda \to 0_-$, whereas Theorem 2 implies that (for the variationally constructed solutions) w_λ generically only blows up at a finite number of points on $\partial\Omega$, as $\lambda \to 0_+$.

Proof of Proposition 3. In order to prove the first statement of this proposition (concerning the lower bound on $\|u_{\lambda_n}\|_{H^1(\Omega)}$) it suffices to prove that there exists a (small) constant $c_1 > 0$ such that if u_{λ_n} is a sequence of solutions to (1) with

$$\|u_{\lambda_n}\|^2_{H^1(\Omega)} \Big/ \log \frac{1}{|\lambda_n|} < c_1 \quad \text{as } \lambda_n \to 0_-, \tag{14}$$

then

$$\|u_{\lambda_n}\|_{H^1(\Omega)} \to 0 \quad \text{as } \lambda_n \to 0_-. \tag{15}$$

Now suppose (14) is satisfied for some sufficiently small positive c_1; in order to verify (15) we start by estimating $\|w_{\lambda_n}\|_{L^2(\partial\Omega)} = \|(I-P_D)u_{\lambda_n}\|_{L^2(\partial\Omega)}$

$$\|w_{\lambda_n}\|^2_{L^2(\partial\Omega)} = \sum_{k:\ D_k\neq D} \alpha^2_{k,\lambda_n} \quad \text{with } \alpha_{k,\lambda_n} = \int_{\partial\Omega} u_{\lambda_n}\phi_k \, \mathrm{d}\sigma_x. \tag{16}$$

Straightforward integration by parts gives that

$$D_k \int_{\partial\Omega} u_{\lambda_n}\phi_k \, \mathrm{d}\sigma_x = \int_{\Omega} \nabla u_{\lambda_n} \nabla\phi_k \, \mathrm{d}x = D \int_{\partial\Omega} u_{\lambda_n}\phi_k \, \mathrm{d}\sigma_x + \lambda_n \int_{\partial\Omega} \sinh u_{\lambda_n}\phi_k \, \mathrm{d}\sigma_x,$$

and so

$$\alpha_{k,\lambda_n} = \int_{\partial\Omega} u_{\lambda_n}\phi_k \, \mathrm{d}\sigma_x = \frac{\lambda_n}{D_k - D}\int_{\partial\Omega} \sinh u_{\lambda_n}\phi_k \, \mathrm{d}\sigma_x \quad \text{for } D_k \neq D, \tag{17}$$

and

$$\int_{\partial\Omega} \sinh u_{\lambda_n}\phi_k \, \mathrm{d}\sigma_x = 0 \quad \text{for } D_k = D. \tag{18}$$

From (16) and (17) it follows that

$$\|w_{\lambda_n}\|^2_{L^2(\partial\Omega)} = \sum_{k:\ D_k\neq D} \alpha^2_{k,\lambda_n} \leqslant C\lambda_n^2 \sum_{k:\ D_k\neq D} \left|\int_{\partial\Omega} \sinh u_{\lambda_n}\phi_k \, \mathrm{d}\sigma_x\right|^2 \quad \text{with } C = \Bigl(\min_{D_k\neq D} |D_k - D|\Bigr)^{-2}. \tag{19}$$

We also have the estimate

$$\sum_{k:\ D_k\neq D} \left|\int_{\partial\Omega} \sinh u_{\lambda_n}\phi_k \, \mathrm{d}\sigma_x\right|^2 = \int_{\partial\Omega} \sinh^2 u_{\lambda_n} \, \mathrm{d}\sigma_x \leqslant C_1 \, \mathrm{e}^{C_2\|u_{\lambda_n}\|^2_{H^1(\Omega)}}$$

(see for example Lemma 2.1 of [10]). Due to the assumption (14) it follows that

$$\|u_{\lambda_n}\|^2_{H^1(\Omega)} < c_1 \log\frac{1}{|\lambda_n|} \leqslant \frac{1}{C_2}\log\frac{1}{|\lambda_n|} \quad \text{as } \lambda_n \to 0_-,$$

provided $c_1 < \frac{1}{C_2}$ (incidentally, this is the only "smallness" restriction on c_1). By a combination of these last two estimates we get

$$\sum_{k:\ D_k\neq D} \left|\int_{\partial\Omega} \sinh u_{\lambda_n}\phi_k \, \mathrm{d}\sigma_x\right|^2 = \|\sinh u_{\lambda_n}\|^2_{L^2(\partial\Omega)} \leqslant \frac{C_1}{|\lambda_n|},$$

which after insertion into (19) leads to

$$\|w_{\lambda_n}\|^2_{L^2(\partial\Omega)} \leqslant C|\lambda_n|, \tag{20}$$

as $\lambda_n \to 0_-$. We easily see that $w_{\lambda_n} = (I - P_D)u_{\lambda_n}$ satisfies the boundary value problem

$$\Delta w_{\lambda_n} = 0 \quad \text{in } \Omega, \qquad \frac{\partial w_{\lambda_n}}{\partial \mathbf{n}} = D w_{\lambda_n} + \lambda_n \sinh u_{\lambda_n} \quad \text{on } \partial\Omega. \tag{21}$$

Due to the estimate

$$|\lambda_n| \|\sinh u_{\lambda_n}\|_{L^2(\partial\Omega)} \leqslant C_1|\lambda_n|^{1/2},$$

(which was proven previously) and the estimate (20) we now conclude that

$$\frac{\partial w_{\lambda_n}}{\partial \mathbf{n}} \to 0 \quad \text{in } L^2(\partial\Omega),$$

as $\lambda_n \to 0_-$. It follows, by elliptic estimates, that

$$w_{\lambda_n} \to 0 \quad \text{in } H^{3/2}(\Omega). \tag{22}$$

If $P_D = 0$ this leads to the desired conclusion (15). If $P_D \neq 0$ it still remains to show that $P_D u_{\lambda_n} \to 0$ in $H^1(\Omega)$. Since Ω is two dimensional, we may use the Trace Theorem and Sobolev's Imbedding Theorem, together with (22), to conclude that

$$(I - P_D)u_{\lambda_n} = w_{\lambda_n} \to 0 \quad \text{in } L^\infty(\partial\Omega), \tag{23}$$

as $\lambda_n \to 0_-$. From (18) we have that

$$\int_{\partial\Omega} \sinh(w_{\lambda_n} + P_D u_{\lambda_n}) P_D u_{\lambda_n} \, \mathrm{d}\sigma_x = \int_{\partial\Omega} \sinh(u_{\lambda_n}) P_D u_{\lambda_n} \, \mathrm{d}\sigma_x = 0,$$

which, due to the formula $\sinh(w + v) = \sinh w \cosh v + \sinh v \cosh w$, translates into

$$\int_{\partial\Omega} \sinh(P_D u_{\lambda_n}) \cosh w_{\lambda_n} P_D u_{\lambda_n} \, \mathrm{d}\sigma_x = -\int_{\partial\Omega} \sinh w_{\lambda_n} \cosh(P_D u_{\lambda_n}) P_D u_{\lambda_n} \, \mathrm{d}\sigma_x. \tag{24}$$

We also have the estimate

$$\begin{aligned} \left| \int_{\partial\Omega} \sinh w_{\lambda_n} \cosh(P_D u_{\lambda_n}) P_D u_{\lambda_n} \, \mathrm{d}\sigma_x \right| &\leqslant \| \sinh w_{\lambda_n} \|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} \cosh(P_D u_{\lambda_n}) |P_D u_{\lambda_n}| \, \mathrm{d}\sigma_x \\ &\leqslant \| \sinh w_{\lambda_n} \|_{L^\infty(\partial\Omega)} \left(\int_{\partial\Omega} \sinh(P_D u_{\lambda_n}) P_D u_{\lambda_n} \, \mathrm{d}\sigma_x + |\partial\Omega| \, \mathrm{e}^{-1} \right). \end{aligned}$$

By insertion of this into (24), and use of the facts that $\cosh w_{\lambda_n} \geqslant 1$, and $\sinh w_{\lambda_n} \to 0$ in $L^\infty(\partial\Omega)$ as $\lambda_n \to 0_-$ (cf. (23)), we now obtain

$$\frac{1}{2} \int_{\partial\Omega} \sinh(P_D u_{\lambda_n}) P_D u_{\lambda_n} \, \mathrm{d}\sigma_x \leqslant \| \sinh w_{\lambda_n} \|_{L^\infty(\partial\Omega)} |\partial\Omega| \mathrm{e}^{-1},$$

for $\lambda_n < 0$ sufficiently close to 0. Since

$$\|P_D u_{\lambda_n}\|^2_{L^2(\partial\Omega)} \leqslant \int_{\partial\Omega} \sinh(P_D u_{\lambda_n}) P_D u_{\lambda_n} \, \mathrm{d}\sigma_x,$$

we have therefore verified that

$$P_D u_{\lambda_n} \to 0 \quad \text{in } L^2(\partial\Omega) \text{ as } \lambda_n \to 0_-.$$

Since the range of the projection P_D is finite dimensional, all (well defined) norms are equivalent on this space, and so

$$P_D u_{\lambda_n} \to 0 \quad \text{in } H^1(\Omega) \text{ as } \lambda_n \to 0_-. \tag{25}$$

A combination of (22) and (25) now yields

$$\|u_{\lambda_n}\|_{H^1(\Omega)} \to 0 \quad \text{as } \lambda_n \to 0_-,$$

which is exactly the desired conclusion (15).

We proceed to the proof of the second statement of this proposition. Since λ_n is negative, and since $x \sinh x \geqslant 0$, we calculate

$$\int_{\Omega} |\nabla u_{\lambda_n}|^2 \, \mathrm{d}x = D \int_{\partial\Omega} u_{\lambda_n}^2 \, \mathrm{d}\sigma_x + \lambda_n \int_{\partial\Omega} u_{\lambda_n} \sinh u_{\lambda_n} \, \mathrm{d}\sigma_x \leqslant D \int_{\partial\Omega} u_{\lambda_n}^2 \, \mathrm{d}\sigma_x. \tag{26}$$

As a consequence

$$\|u_{\lambda_n}\|^2_{H^1(\Omega)} = \int_{\Omega} |\nabla u_{\lambda_n}|^2 \, \mathrm{d}x + \int_{\partial\Omega} u_{\lambda_n}^2 \, \mathrm{d}\sigma_x \leqslant (D+1) \|u_{\lambda_n}\|^2_{L^2(\partial\Omega)}. \tag{27}$$

Let u_{λ_n}, $\lambda_n \to 0_-$, be a sequence which satisfies the lower bound (13) and define

$$\tilde{u}_{\lambda_n} = u_{\lambda_n} / \|u_{\lambda_n}\|_{L^2(\partial\Omega)}.$$

Due to this definition, and (27),

$$\|\tilde{u}_{\lambda_n}\|_{L^2(\partial\Omega)} = 1 \quad \text{and} \quad \|\tilde{u}_{\lambda_n}\|_{H^1(\Omega)} \leqslant C.$$

By extraction of a subsequence, for simplicity also referred to as $\tilde{u}_{\lambda_n}$, we may obtain

$$\tilde{u}_{\lambda_n} \rightharpoonup u_0 \quad \text{weakly in } H^1(\Omega) \quad \text{and} \quad \tilde{u}_{\lambda_n} \to u_0 \quad \text{in } L^2(\partial\Omega),$$

as $\lambda_n \to 0_-$. For the second property we relied on the compactness of the trace map from $H^1(\Omega)$ to $L^2(\partial\Omega)$. Since $\tilde{u}_{\lambda_n}$, and therefore also u_0, are all harmonic in Ω we may conclude that

$$\tilde{u}_{\lambda_n} \to u_0 \quad \text{in } C^0(\Omega_c), \tag{28}$$

for any compact subdomain Ω_c of Ω. Being harmonic in Ω (and nonzero, since $\|u_0\|_{L^2(\partial\Omega)} = \lim \|\tilde{u}_{\lambda_n}\|_{L^2(\partial\Omega)} = 1$) the function u_0 is different from zero almost everywhere inside Ω (and on a set of positive surface measure on $\partial\Omega$). From the definition of $\tilde{u}_{\lambda_n}$, the fact that $\|u_{\lambda_n}\|_{L^2(\partial\Omega)} \to \infty$ (since $c_1 \log \frac{1}{|\lambda_n|} \leqslant \|u_{\lambda_n}\|^2_{H^1(\Omega)} \leqslant C\|u_{\lambda_n}\|^2_{L^2(\partial\Omega)}$) and (28) it now follows that

$$u_{\lambda_n} \to \pm\infty \quad \text{almost everywhere in } \Omega,$$

and (by extraction of a subsequence) that

$$u_{\lambda_n} \to \pm\infty \quad \text{on a set of positive one dimensional surface measure on } \partial\Omega,$$

as $\lambda_n \to 0_-$. The "limit" is $+\infty$ where u_0 is positive, $-\infty$ where u_0 is negative.

It only remains to prove the validity of the very last statements of the Proposition. As before let $w_{\lambda_n} = (I - P_D)u_{\lambda_n}$. From integration by parts, (21), and (18) it follows immediately that

$$\begin{aligned}\int_\Omega |\nabla w_{\lambda_n}|^2 \,\mathrm{d}x &= D \int_{\partial\Omega} w_{\lambda_n}^2 \,\mathrm{d}\sigma_x + \lambda_n \int_{\partial\Omega} \sinh(u_{\lambda_n}) w_{\lambda_n} \,\mathrm{d}\sigma_x = D \int_{\partial\Omega} w_{\lambda_n}^2 \,\mathrm{d}\sigma_x + \lambda_n \int_{\partial\Omega} \sinh(u_{\lambda_n}) u_{\lambda_n} \,\mathrm{d}\sigma_x \\ &\leqslant D \int_{\partial\Omega} w_{\lambda_n}^2 \,\mathrm{d}\sigma_x.\end{aligned} \tag{29}$$

Lemma 2 and (29) now yield

$$\begin{aligned}\|u_\lambda\|_{H^1(\Omega)} &= \big(\|P_D u_\lambda\|^2_{H^1(\Omega)} + \|w_\lambda\|^2_{H^1(\Omega)}\big)^{1/2} \leqslant C\big(\|w_\lambda\|_{H^1(\Omega)} + \|w_\lambda\|^2_{H^1(\Omega)} + 1\big) \\ &\leqslant C\big(\|w_\lambda\|_{L^2(\partial\Omega)} + \|w_\lambda\|^2_{L^2(\partial\Omega)} + 1\big).\end{aligned}$$

If the sequence of numbers $\|w_{\lambda_n}\|_{L^2(\partial\Omega)}$ contains a bounded subsequence then it follows immediately from this estimate that the sequence $\|u_{\lambda_n}\|_{H^1(\Omega)}$ contains a bounded subsequence. However this would contradict the estimate (13), which we have already proven, and therefore we may conclude that

$$\|w_{\lambda_n}\|_{L^2(\partial\Omega)} \to \infty \quad \text{as } \lambda_n \to 0_-. \tag{30}$$

Since w_{λ_n} is a solution to the boundary value problem (21), we arrive at the estimates

$$\begin{aligned}\big\|\lambda_n \sinh(u_{\lambda_n})\big\|_{H^{-1/2}(\partial\Omega)} &\leqslant \big\|D w_{\lambda_n} + \lambda_n \sinh(u_{\lambda_n})\big\|_{H^{-1/2}(\partial\Omega)} + \|D w_{\lambda_n}\|_{H^{-1/2}(\partial\Omega)} \\ &= \left\|\frac{\partial w_{\lambda_n}}{\partial \mathbf{n}}\right\|_{H^{-1/2}(\partial\Omega)} + \|D w_{\lambda_n}\|_{H^{-1/2}(\partial\Omega)} \leqslant C\|w_{\lambda_n}\|_{H^1(\Omega)},\end{aligned}$$

and therefore, due to (29),

$$\frac{\|\lambda_n \sinh(u_{\lambda_n})\|_{H^{-1/2}(\partial\Omega)}}{\|w_{\lambda_n}\|_{L^2(\partial\Omega)}} \leqslant C \frac{\|\lambda_n \sinh(u_{\lambda_n})\|_{H^{-1/2}(\partial\Omega)}}{\|w_{\lambda_n}\|_{H^1(\Omega)}} \leqslant C. \tag{31}$$

We now define

$$\widetilde{w}_{\lambda_n} = \frac{w_{\lambda_n}}{\|w_{\lambda_n}\|_{L^2(\partial\Omega)}}.$$

Due to this definition, and (29),

$$\|\widetilde{w}_{\lambda_n}\|_{L^2(\partial\Omega)} = 1 \quad \text{and} \quad \|\widetilde{w}_{\lambda_n}\|_{H^1(\Omega)} \leqslant C. \tag{32}$$

Because of the estimates (31), (32), and the compactness of the trace map from $H^1(\Omega)$ to $L^2(\partial\Omega)$, we may extract a subsequence, for simplicity also denoted λ_n, so that

$$\begin{aligned} &\widetilde{w}_{\lambda_n} \rightharpoonup w_0 \quad \text{weakly in } H^1(\Omega), \qquad \widetilde{w}_{\lambda_n}|\partial\Omega \to w_0|_{\partial\Omega} \quad \text{in } L^2(\partial\Omega), \\ &\text{and} \quad \frac{\lambda_n \sinh(u_{\lambda_n})}{\|w_{\lambda_n}\|_{L^2(\partial\Omega)}} \rightharpoonup \mu, \quad \text{weakly in } H^{-1/2}(\partial\Omega), \end{aligned} \tag{33}$$

as $\lambda_n \to 0_-$. The functions $\widetilde{w}_{\lambda_n}$ satisfy

$$\Delta\widetilde{w}_{\lambda_n} = 0 \quad \text{in } \Omega, \qquad \frac{\partial\widetilde{w}_{\lambda_n}}{\partial\mathbf{n}} = D\widetilde{w}_{\lambda_n} + \frac{\lambda_n \sinh(u_{\lambda_n})}{\|w_{\lambda_n}\|_{L^2(\partial\Omega)}} \quad \text{on } \partial\Omega.$$

Furthermore $\int_{\partial\Omega} \widetilde{w}_{\lambda_n}\phi \, d\sigma_x = 0$, for any ϕ that solves

$$\Delta\phi = 0 \quad \text{in } \Omega, \qquad \frac{\partial\phi}{\partial\mathbf{n}} = D\phi \quad \text{on } \partial\Omega.$$

The function w_0 is nonzero (since $\|w_0\|_{L^2(\partial\Omega)} = \lim \|\widetilde{w}_{\lambda_n}\|_{L^2(\partial\Omega)} = 1$) and it satisfies

$$\Delta w_0 = 0 \quad \text{in } \Omega, \qquad \frac{\partial w_0}{\partial\mathbf{n}} = Dw_0 + \mu \quad \text{on } \partial\Omega,$$

in a weak, variational sense. From the same argument we used in connection with u_{λ_n} it now follows that

$$w_{\lambda_n} \to \pm\infty \quad \text{almost everywhere in } \Omega,$$

and (by extraction of a subsequence) that

$$w_{\lambda_n} \to \pm\infty \quad \text{on a set of positive one dimensional surface measure on } \partial\Omega,$$

as $\lambda_n \to 0_-$. Here we use that $\|w_{\lambda_n}\|_{L^2(\partial\Omega)} \to \infty$, according to (30). We may also conclude that $\mu \neq 0$, because if μ vanished identically, then w_0 would be a Steklov eigenvector, and the orthogonality relationship for the $\widetilde{w}_{\lambda_n}$ would give that $\int_{\partial\Omega} (w_0)^2 \, d\sigma_x = \lim \int_{\partial\Omega} \widetilde{w}_{\lambda_n} w_0 \, d\sigma_x = 0$. However, this contradicts the fact that $\|w_0\|_{L^2(\partial\Omega)} = 1$. A combination of (30) and (33) now completes the proof of the proposition. □

Proposition 2 already asserts the existence of one nontrivial solution to (1) for λ in the range $-D < \lambda < 0$. There is a very useful variational characterization of a (potentially) larger class of solutions, which we shall now introduce, and which we shall also use extensively in the next section, for $\lambda > 0$. For this purpose we use the energy $E_\lambda(\cdot)$, and in particular the "restricted" functional

$$J_\lambda(v) = \inf_{t>0} E_\lambda(tv). \tag{34}$$

The functionals J_λ, $\lambda < 0$, are bounded by $-\infty < J_\lambda(1) \leqslant J_\lambda(\cdot) \leqslant 0$. We already showed in the previous section that if w^* is a critical point for $J_\lambda(\cdot)$ on

$$\Sigma = \{w \in H^1(\Omega)\colon \|w\|_{H^1(\Omega)} = 1\},$$

with $J_\lambda(w^*) < 0$, then $J_\lambda(w^*) = E_\lambda(t^*w^*)$ for some $t^* > 0$, and $u^* = t^*w^*$ is a critical point for E_λ in $H^1(\Omega)$ (see also Lemma 2.5 of [10]). Such critical points are weak- and, by elliptic regularity, also strong solutions to the boundary value problem (1).

In order to establish existence of solutions it thus suffices to find nonzero critical values (and corresponding critical points) for J_λ on Σ. To do this we employ a (by now) fairly standard result in critical point theory, cf. [9], [14] or [16]. Briefly stated this result asserts that all nonzero values of the form

$$c_k(\lambda) = \inf_{A \in \mathfrak{A}_k} \sup_{w \in A} J_\lambda(w)$$

are critical values of J_λ. The infimum in A is taken over the collection $\mathfrak{A}_k$ of subsets $A \subset \Sigma$ that are compact, even and of genus$(A) \geqslant k \geqslant 1$.[1] The most essential prerequisite in order to be able to apply this result is to verify an appropriate compactness property of the functional $J_\lambda(\cdot)$. In the present context the required property is the Palais–Smale Condition verified in Lemma 1. For more details we refer the reader to [10].

We now proceed to show that this construction, for $-D < \lambda < 0$, gives rise to at most finitely many critical values, in complete agreement with Fig. 1. We also show that these critical values tend to $-\infty$ as λ tends to 0_-, so that the $H^1(\Omega)$ norms of the corresponding critical points for $E_\lambda(\cdot)$ tend to ∞, and these therefore represent solutions that "blow up" as described in Proposition 3.

Proposition 4. *Suppose $D > 0$, and define $K^* = \max\{k\colon -D + D_k < 0\}$. Let $c_k(\lambda)$, $-D < \lambda < 0$, $k \in \mathbb{N}$, be given by*

$$c_k(\lambda) = \inf_{A \in \mathfrak{A}_k} \sup_{w \in A} J_\lambda(w),$$

with

$$J_\lambda(w) = \inf_{t>0} E_\lambda(tw) \leqslant 0,$$

and E_λ as above. Then

$$c_k(\lambda) = 0 \quad \text{for any } k \geqslant K^* + 1, \qquad -D < \lambda < 0. \tag{35}$$

Furthermore there exist positive constants a_i, b_i, $i = 1, 2$ such that

$$-a_1\left(\log\frac{1}{|\lambda|}\right)^2 - b_1 \leqslant c_k(\lambda) \leqslant -a_2 \log\frac{1}{|\lambda|} + b_2, \tag{36}$$

for $-D < \lambda < 0$, and any $1 \leqslant k \leqslant K^*$.

Proof. Suppose A is an even, compact subset of Σ, with genus $(A) \geqslant k \geqslant K^* + 1$. Then

$$A \cap \operatorname{span}\{\phi_1, \phi_2, \ldots, \phi_{K^*-1}, \phi_{K^*}\}^\perp \neq \emptyset; \tag{37}$$

otherwise the mapping

$$A \ni v \to \big(\langle \phi_1, v\rangle_{H^1}, \langle \phi_2, v\rangle_{H^1}, \ldots, \langle \phi_{K^*-1}, v\rangle_{H^1}, \langle \phi_{K^*}, v\rangle_{H^1}\big)$$

would be an odd, continuous map from A to $\mathbb{R}^{K^*} \setminus 0$, contradicting the fact that genus $(A) \geqslant K^* + 1$. From (37) (and the orthogonality of the Steklov eigenvectors ϕ_k) it follows immediately that there exists $v \in A$ ($v \neq 0$) such that

$$D_{K^*+1} \int_{\partial\Omega} v^2 \, \mathrm{d}\sigma_x \leqslant \int_\Omega |\nabla v|^2 \, \mathrm{d}x. \tag{38}$$

Since $D \leqslant D_{K^*+1}$ (and $\lambda < 0$) we get

$$E_\lambda(tv) = \frac{t^2}{2}\left[\int_\Omega |\nabla v|^2 \, \mathrm{d}x - D \int_{\partial\Omega} v^2 \, \mathrm{d}\sigma_x\right] - \lambda \int_{\partial\Omega} \big(\cosh(tv) - 1\big) \, \mathrm{d}\sigma_x \geqslant -\lambda \int_{\partial\Omega} \big(\cosh(tv) - 1\big) \, \mathrm{d}\sigma_x \geqslant 0,$$

for any $t > 0$. In other words, there exists $v \in A$ with $J_\lambda(v) = 0$, and thus it follows that

$$\sup_{w \in A} J_\lambda(w) = 0.$$

Since this identity holds for any compact, even subset of Σ with genus$(A) \geqslant K^* + 1$, it follows that

$$c_k(\lambda) = \inf_{A \in \mathfrak{A}_k} \sup_{w \in A} J_\lambda(w) = 0$$

[1] The value of genus(A) is by definition the smallest integer m, such that there exists a continuous, odd map from A into $\mathbb{R}^m \setminus \{0\}$.

for any $k \geqslant K^* + 1$. This proves the statement (35), and we now proceed with the verification of the estimates in (36). Since

$$c_1(\lambda) \leqslant c_2(\lambda) \leqslant \cdots \leqslant c_{K^*-1}(\lambda) \leqslant c_{K^*}(\lambda),$$

it suffices to verify that

$$-a_1\left(\log\frac{1}{|\lambda|}\right)^2 - b_1 \leqslant c_1(\lambda) \quad \text{and} \quad c_{K^*}(\lambda) \leqslant -a_2\log\frac{1}{|\lambda|} + b_2.$$

We start with the lower bound for $c_1(\lambda)$. The inequality

$$\sup_{w\in A} J_\lambda(w) \geqslant \inf_{w\in\Sigma} J_\lambda(w) = \inf_{w\in H^1(\Omega)} E_\lambda(w),$$

which holds for any subset $A \subset \Sigma$, immediately implies that

$$c_1(\lambda) = \inf_{A\in\mathfrak{A}_1}\sup_{w\in A} J_\lambda(w) \geqslant \inf_{w\in H^1(\Omega)} E_\lambda(w). \tag{39}$$

For the energy $E_\lambda(w)$ we have that

$$\begin{aligned} E_\lambda(w) &= \frac{1}{2}\int_\Omega |\nabla w|^2\,\mathrm{d}x - \frac{D}{2}\int_{\partial\Omega} w^2\,\mathrm{d}\sigma_x - \lambda\int_{\partial\Omega}\big(\cosh(w)-1\big)\,\mathrm{d}\sigma_x \geqslant \int_{\partial\Omega}\left[-\frac{D}{2}w^2 - \lambda\big(\cosh(w)-1\big)\right]\mathrm{d}\sigma_x \\ &\geqslant |\partial\Omega|\inf_{t\in\mathbb{R}}\left[-\frac{D}{2}t^2 - \lambda\big(\cosh(t)-1\big)\right]. \end{aligned} \tag{40}$$

A straightforward calculation (remember $-D < \lambda < 0$) gives that

$$\inf_{t\in\mathbb{R}}\left[-\frac{D}{2}t^2 - \lambda\big(\cosh(t)-1\big)\right] = -\frac{D}{2}t_*^2 - \lambda\big(\cosh(t_*)-1\big) \geqslant -\frac{D}{2}t_*^2, \tag{41}$$

where $t_* > 0$ is the unique positive solution to

$$\sinh(t_*) = -\frac{D}{\lambda}t_*.$$

This t_* satisfies, for any $\epsilon > 0$, the estimate

$$0 < t_* \leqslant (1+\epsilon)\log\frac{1}{|\lambda|} + C_{\epsilon,D},$$

where the constant $C_{\epsilon,D}$ depends on ϵ and D, but is independent of λ. After insertion into (41) we now get

$$\inf_{t\in\mathbb{R}}\left[-\frac{D}{2}t^2 - \lambda\big(\cosh(t)-1\big)\right] \geqslant -(1+\epsilon)\frac{D}{2}\left(\log\frac{1}{|\lambda|}\right)^2 - C_{\epsilon,D},$$

where the constant $C_{\epsilon,D}$ depends on ϵ and D, but is independent of λ. In combination with (39) and (40) this gives

$$c_1(\lambda) \geqslant -(1+\epsilon)|\partial\Omega|\frac{D}{2}\left(\log\frac{1}{|\lambda|}\right)^2 - C_{\epsilon,D},$$

which is a lower bound of the desired form.

We now turn our attention to the upper bound for $c_{K^*}(\lambda)$. In order to verify this bound it suffices to find a compact, even subset $A^* \subset \Sigma$ with

(i) $\mathrm{genus}(A^*) \geqslant K^*$,

(ii) $\sup_{w\in A^*} J_\lambda(w) \leqslant -a_2\log\frac{1}{|\lambda|} + b_2$.

Let $\{\phi_k\}_{k=1}^{K^*}$ be the Steklov eigenvectors corresponding to eigenvalues D_k, with $0 \leqslant D_k < D$, and define for $R > 0$

$$A_R^* = \left\{\sum_{k=1}^{K^*} s_k\phi_k\colon \left\|\sum_{k=1}^{K^*} s_k\phi_k\right\|_{H^1(\Omega)}^2 = \sum_{k=1}^{K^*}(D_k+1)s_k^2 = R^2\right\}.$$

It follows immediately from the Borsuk–Ulam Theorem that genus$(A_R^*) = K^*$. For any $w = \sum_{k=1}^{K^*} s_k\phi_k \in A_R^*$ we have that

$$\frac{\int_\Omega |\nabla w|^2\,\mathrm{d}x}{\int_{\partial\Omega} w^2\,\mathrm{d}\sigma_x} \leqslant D_{K^*},$$

so that

$$\frac{1}{2}\int\limits_\Omega |\nabla w|^2\,\mathrm{d}x - \frac{D}{2}\int\limits_{\partial\Omega} w^2\,\mathrm{d}x \leqslant -\frac{D-D_{K^*}}{2}\int\limits_{\partial\Omega} w^2\,\mathrm{d}\sigma_x = -\frac{D-D_{K^*}}{2}\sum_{k=1}^{K^*} s_k^2$$
$$= -\frac{D-D_{K^*}}{2}\frac{\sum_{k=1}^{K^*} s_k^2}{\sum_{k=1}^{K^*}(D_k+1)s_k^2}R^2 \leqslant -\frac{D-D_{K^*}}{2(D_{K^*}+1)}R^2 = -aR^2, \tag{42}$$

with $a = (D-D_{K^*})/2(D_{K^*}+1) > 0$. As a consequence of Trudingers inequality (cf. Lemma 2.1 of [10]) we also have

$$\int\limits_{\partial\Omega}\big(\cosh(w)-1\big)\,\mathrm{d}\sigma_x \leqslant \int\limits_{\partial\Omega} \mathrm{e}^{|w|}\,\mathrm{d}\sigma_x - |\partial\Omega| \leqslant C_1\,\mathrm{e}^{C_2\|w\|^2_{H^1(\Omega)}} = C_1\,\mathrm{e}^{C_2R^2}, \tag{43}$$

for any $w \in A_R^*$. A combination of (42) and (43) now gives

$$E_\lambda(w) \leqslant -aR^2 + C_1|\lambda|\,\mathrm{e}^{C_2R^2} \tag{44}$$

for any $w \in A_R^*$, with positive constants a, C_1 and C_2 independent of w, R and λ. By selecting $R = R(\lambda) = \sqrt{(1/C_2)\log(1/|\lambda|+1)}$ it follows immediately from (44) that there exist positive constants a_2 and b_2, independent of $\lambda \in (-D, 0)$, such that

$$E_\lambda(w) \leqslant -a_2\log\frac{1}{|\lambda|} + b_2 \quad \forall w \in A^*_{R(\lambda)}.$$

From the definition of $J_\lambda(\cdot)$ it now follows that

$$J_\lambda(w) \leqslant E_\lambda\big(R(\lambda)w\big) \leqslant -a_2\log\frac{1}{|\lambda|} + b_2 \quad \forall w \in A_1^*,$$

or

$$\sup_{w\in A_1^*} J_\lambda(w) \leqslant -a_2\log\frac{1}{|\lambda|} + b_2.$$

The compact, even set $A^* = A_1^* \subset \Sigma$ now satisfies the conditions (i) and (ii). This completes the proof of the upper bound for $c_{K^*}(\lambda)$, and thus the proof of this proposition. □

Remarks. From Proposition 4 it follows that $c_k(\lambda) < 0$ for $1 \leqslant k \leqslant K^*$, and $\lambda < 0$ sufficiently close to 0. As discussed earlier $c_k(\lambda)$ is thus a critical value for $J_\lambda(\cdot)$ on Σ (with corresponding critical point $w_{k,\lambda}$). Furthermore there exists $t_{k,\lambda} > 0$ such that $u_{k,\lambda} = t_{k,\lambda}w_{k,\lambda}$ is a critical point for $E_\lambda(\cdot)$ in $H^1(\Omega)$, and thus a solution to the boundary value problem (1). As a consequence of Proposition 4 these solutions satisfy

$$-a_1\left(\log\frac{1}{|\lambda|}\right)^2 - b_1 \leqslant E_\lambda(u_{k,\lambda}) \leqslant -a_2\log\frac{1}{|\lambda|} + b_2.$$

They also satisfy

$$c_1\log\frac{1}{|\lambda|} - c_2 \leqslant \|u_{k,\lambda}\|^2_{H^1(\Omega)} \leqslant C_1\left(\log\frac{1}{|\lambda|}\right)^2 + C_2,$$

in accordance with Proposition 2 and Proposition 3. From the very definition of the values $c_k(\lambda)$ it is clear that

$$c_1(\lambda) \leqslant c_2(\lambda) \leqslant \cdots \leqslant c_{K^*}(\lambda) \leqslant 0, \qquad -D < \lambda < 0.$$

One might expect that, generically, it would be true that

$$c_1(\lambda) < c_2(\lambda) < \cdots < c_{K^*}(\lambda) < 0, \qquad -D < \lambda < 0.$$

If this were they case, i.e., if all the $c_k(\lambda)$, $1 \leqslant k \leqslant K^*$, were negative, and different, then the critical points corresponding to these critical values represent K^* essentially different nontrivial solutions to the boundary value problem (1) for $-D < \lambda < 0$. □

2.2. *Positive* λ

We continue our consideration of the problem (1) with the nonlinear boundary flux $Du + \lambda \sinh(u)$, but now for $\lambda > 0$. The analysis required has many similarities to that presented in [10,12] (where we considered $D = 0$) and so in certain places we shall, for reasons of brevity, not provide all the details – but instead refer the reader to these papers. According to Fig. 1 we expect to find, for any fixed positive λ, an infinite set of essentially different solutions. Indeed we may verify this conjecture using a Lyusternik–Schnirelmann approach, similar to that described in the previous section. This variational approach also provides very precise bounds for the H^1-norms of the constructed solutions. We shall establish

Theorem 1. *Suppose $D \in \mathbb{R}$. For any fixed $\lambda > 0$, there exists an integer K_λ and an infinite set of solutions $\{u_{k,\lambda}\}_{k=K_\lambda}^{\infty}$ to the problem* (1). *There exist $\lambda^* > 0$, and K_0, such that these solutions obey the energy estimates*

$$c \leqslant E_\lambda(u_{k,\lambda}) \leqslant a_k \log\left(\frac{1}{\lambda}\right) + b_k \tag{45}$$

for $0 < \lambda < \lambda^*$, and $k \geqslant K_0$. The constants c, a_k, b_k are positive, and independent of λ.

The existence part of this theorem will be established using the auxiliary functional $J_\lambda : H^1(\Omega) \to [0, \infty]$, defined by

$$J_\lambda(v) = \sup_{t>0} E_\lambda(tv).$$

To be specific we prove the existence of infinitely many critical points, $v_{k,\lambda}$, for J_λ on the manifold $\Sigma = \{w \in H^1(\Omega);\ \int_\Omega |\nabla w|^2\,\mathrm{d}x + \int_{\partial\Omega} w^2\,\mathrm{d}\sigma_x = 1\}$, with corresponding (different) positive critical values. As seen earlier in Section 2 such critical points immediately lead to solutions $u_{k,\lambda} = t_{k,\lambda} v_{k,\lambda}$ to (1), with $E_\lambda(u_{k,\lambda}) = J_\lambda(v_{k,\lambda})$.

In order to arrive at these critical points we define, for any integer $k \geqslant 1$,

$$c_k(\lambda) = \inf_{A \in \mathfrak{A}_k} \sup_{w \in A} J_\lambda(w),$$

where $\mathfrak{A}_k$ is the collection of compact, even subsets of Σ of genus greater than or equal to k. We observe that $0 \leqslant c_1(\lambda) \leqslant c_2(\lambda) \leqslant \cdots \leqslant c_k(\lambda) \leqslant c_{k+1}(\lambda) \leqslant \cdots$.

Since the even functional $J_\lambda(\cdot)$, $\lambda > 0$, has the appropriate "smoothness properties" (on the set $\{w\colon 0 < J_\lambda(w) < \infty\}$) and satisfies the Palais–Smale Condition of Lemma 1, the same approach used in the previous section (for λ negative) here implies that any positive $c_k(\lambda)$ is a critical value for $J_\lambda(\cdot)$.

To establish the existence of nontrivial solutions to (1) it thus suffices to show that $c_K > 0$ for some $K = K_\lambda$. The bounds in Theorem 1 require estimates for the $c_k(\lambda)$ as $\lambda \to 0_+$. These estimates (as well as the existence of K_λ) are established by the following lemma.

Lemma 3. *Given $D \in \mathbb{R}$, let $K \geqslant 2$ be a fixed integer such that $D_K > D$, where D_K denotes the K'th Steklov eigenvalue for the problem* (2). *Let $c_k(\lambda)$ be as above. There exist positive constants, a_k and b_k, depending on k, D and K, but independent of λ, such that for all $k \geqslant K$ and all $0 < \lambda < D_K - D$,*

$$0 < c_k(\lambda) \leqslant a_k \log\frac{1}{\lambda} + b_k.$$

Remark. In the course of the proof of this lemma we establish a lower bound that is a bit more precise than $0 < c_k(\lambda)$. We actually show that

$$c_k(\lambda) \geqslant d(D_K - D - \lambda)^2 > 0.$$

In particular, this implies that

$$c_k(\lambda) \geqslant c > 0 \quad \text{for } 0 < \lambda < \frac{D_K - D}{2}.$$

Proof of Lemma 3. We start with the lower bound. Let A be any compact, even subset of Σ with genus$(A) \geqslant k \geqslant K$. As in (37), we know there exists $v_* \in A$ such that $\langle \phi_j, v_* \rangle_{H^1} = 0$ for $j = 1, \dots, K-1$. Here ϕ_j is the Steklov eigenvector corresponding to the eigenvalue D_j, $1 \leqslant j \leqslant K-1$. This v_* satisfies the inequality

$$D_K \int_{\partial\Omega} v_*^2 \, \mathrm{d}\sigma_x \leqslant \int_{\Omega} |\nabla v_*|^2 \, \mathrm{d}x.$$

For $0 < \lambda < D_K - D$ we thus calculate

$$\begin{aligned} E_\lambda(tv_*) &= \frac{t^2}{2}\left[\int_{\Omega} |\nabla v_*|^2 \, \mathrm{d}x - (D+\lambda)\int_{\partial\Omega} v_*^2 \, \mathrm{d}\sigma_x\right] - \lambda \int_{\partial\Omega} \left(\cosh(tv_*) - 1 - \frac{(tv_*)^2}{2}\right) \mathrm{d}\sigma_x \\ &\geqslant \frac{t^2}{2} B(K,\lambda,D) \int_{\Omega} |\nabla v_*|^2 \, \mathrm{d}x - \lambda \int_{\partial\Omega} \left(\cosh(tv_*) - 1 - \frac{(tv_*)^2}{2}\right) \mathrm{d}\sigma_x, \end{aligned} \tag{46}$$

with $B(K,\lambda,D) = \min\{1, (1 - (\lambda+D)/D_K)\}$. We also have that

$$\begin{aligned} \int_{\partial\Omega} \left(\cosh(v) - 1 - \frac{(v)^2}{2}\right) \mathrm{d}\sigma_x &\leqslant \int_{\partial\Omega} v^4 \cosh(v) \, \mathrm{d}\sigma_x \leqslant \left(\int_{\partial\Omega} v^8 \, \mathrm{d}\sigma_x\right)^{1/2} \left(\int_{\partial\Omega} \cosh(2v) \, \mathrm{d}\sigma_x\right)^{1/2} \\ &\leqslant C\left(\int_{\Omega} |\nabla v|^2 \, \mathrm{d}x\right)^2 \mathrm{e}^{C_2 \|\nabla v\|^2_{L^2(\Omega)}}, \end{aligned}$$

for any v which is orthogonal to ϕ_1 (a constant), and as a consequence

$$\int_{\partial\Omega} \left(\cosh(tv_*) - 1 - \frac{(tv_*)^2}{2}\right) \mathrm{d}\sigma_x \leqslant Ct^4 \left(\int_{\Omega} |\nabla v_*|^2 \, \mathrm{d}x\right)^2 \mathrm{e}^{C_2 \|\nabla t v_*\|^2_{L^2(\Omega)}} \leqslant C_1 t^4 \mathrm{e}^{C_2 t^2} \int_{\Omega} |\nabla v_*|^2 \, \mathrm{d}x. \tag{47}$$

For the last inequality we have used that v_* lies in Σ, so that $\|\nabla v_*\|^2_{L^2(\Omega)} \leqslant 1$. Let $t_0 > 0$ be given by

$$t_0 = c\sqrt{B(K,\lambda,D)},$$

with c chosen so small, that

$$\lambda C_1 c^2 \mathrm{e}^{C_2 c^2 B(K,\lambda,D)} = \lambda C_1 c^2 \mathrm{e}^{C_2 c^2 \min\{1,(1-(\lambda+D)/D_K)\}} < \frac{1}{4},$$

for all $0 < \lambda < D_K - D$ (C_1 being the constant from (47)). A combination of the estimates (46) and (47), with $t = t_0$ now yields

$$E_\lambda(t_0 v_*) \geqslant \frac{1}{4} t_0^2 B(K,\lambda,D) \int_{\Omega} |\nabla v_*|^2 \, \mathrm{d}x \geqslant d\left(1 - \frac{\lambda + D}{D_K}\right)^2, \tag{48}$$

where $d > 0$ depends on D_K and D, but is independent of λ (in the interval $0 < \lambda < D_K - D$). For the last inequality we have also used that

$$\int_{\Omega} |\nabla v_*|^2 \, \mathrm{d}x \geqslant d\|v_*\|^2_{H^1(\Omega)} = d > 0,$$

due to the facts that v_* is orthogonal to ϕ_1 (a constant) and that v_* lies in Σ. From (48) we immediately conclude that

$$\sup_{v \in A} J_\lambda(v) \geqslant J_\lambda(v_*) \geqslant E_\lambda(t_0 v_*) \geqslant d(D_K - D - \lambda)^2 > 0.$$

Since A is an arbitrary compact, even subset of Σ, with $\operatorname{genus}(A) \geqslant k$, it follows that

$$c_k(\lambda) = \inf_{A \in \mathfrak{A}_k} \sup_{v \in A} J_\lambda(v) \geqslant d(D_K - D - \lambda)^2 > 0,$$

as desired.

To prove the upper bound, we introduce some special functions. Let $\{\sigma_j\}_{j=1}^k$ be a set of $k(\geqslant 2)$ distinct points on $\partial\Omega$, and let ϵ and R be two positive numbers. Define

$$d_{j,\epsilon}(x) := -\log\bigl(|x - \sigma_j|^2 + \epsilon^2\bigr),$$

and then the set

$$G_{\epsilon,R} := \left\{ w = \sum_{j=1}^k \alpha_j d_{j,\epsilon}(x)\colon \int_\Omega |\nabla w|^2 \,\mathrm{d}x + \int_{\partial\Omega} w^2 \,\mathrm{d}\sigma_x = R^2 \right\}.$$

Claim 1. *Given D, $k \geqslant 2$ and $\{\sigma_j\}_{j=1}^k$ there exist $\lambda^* > 0$, and functions $\epsilon(\lambda) > 0$, $R(\lambda) > 0$, such that for $0 < \lambda < \lambda^*$*

$$\int_\Omega |\nabla v|^2 \,\mathrm{d}x - D \int_{\partial\Omega} v^2 \,\mathrm{d}\sigma_x - \lambda \int_{\partial\Omega} v \sinh(v) \,\mathrm{d}\sigma_x \leqslant 0 \quad \forall v \in G_{\epsilon(\lambda),R(\lambda)}.$$

Moreover, the functions $\epsilon(\cdot)$ and $R(\cdot)$ may be chosen so that

$$\epsilon(\lambda) = \mathrm{O}(\lambda) \quad \textit{and} \quad R(\lambda) = \mathrm{O}\bigl(\sqrt{\log(1/\lambda)}\bigr),$$

as λ approaches 0.

For $D \geqslant 0$ this claim follows directly from Lemma 3.4 in [10]; for $D < 0$ a slightly modified version of the proof of Lemma 3.4 in [10] is required. We do not reproduce the proof here, instead we proceed with the verification of the upper bounds of Lemma 3. It clearly suffices to prove each upper bound for λ sufficiently small, since $c_k(\lambda)$, for fixed k, is bounded on any finite interval $[\lambda^*, \Lambda^*]$, $\lambda^* > 0$. Now let $\epsilon = \epsilon(\lambda)$, $R = R(\lambda)$ be chosen as in the claim. For λ sufficiently small, the compact even set $G_{\epsilon,1} \subset \Sigma$ has genus k, and so

$$c_k(\lambda) = \inf_{A \in \mathfrak{A}_k} \sup_{v \in A} J_\lambda(v) \leqslant \max_{v \in G_{\epsilon,1}} J_\lambda(v) = \max_{v \in G_{\epsilon,R}} J_\lambda(v) = J_\lambda(v^*)$$

for some $v^* \in G_{\epsilon,R}$. We can also estimate

$$\begin{aligned} J_\lambda(v^*) = E_\lambda\bigl(t(v^*)v^*\bigr) &= \frac{(t(v^*))^2}{2} \int_\Omega |\nabla v^*|^2 \,\mathrm{d}x - D\frac{(t(v^*))^2}{2} \int_{\partial\Omega} (v^*)^2 \,\mathrm{d}\sigma_x - \lambda \int_{\partial\Omega} \bigl(\cosh\bigl(t(v^*)v^*\bigr) - 1\bigr) \,\mathrm{d}\sigma_x \\ &\leqslant \frac{(t(v^*))^2}{2} \left(\int_\Omega |\nabla v^*|^2 \,\mathrm{d}x + |D| \int_{\partial\Omega} (v^*)^2 \,\mathrm{d}\sigma_x \right) \leqslant \frac{(t(v^*))^2}{2} \max\bigl\{1, |D|\bigr\} R^2. \end{aligned} \tag{49}$$

We observe that the function $f(t) = E_\lambda(tv^*)$ has a strictly concave derivative

$$f'(t) = t \int_\Omega |\nabla v^*|^2 \,\mathrm{d}x - Dt \int_{\partial\Omega} (v^*)^2 \,\mathrm{d}\sigma_x - \lambda \int_{\partial\Omega} \sinh(tv^*) v^* \,\mathrm{d}\sigma_x,$$

on the interval $(0, \infty)$. To see this, we simply calculate that

$$f'''(t) = -\lambda \int_{\partial\Omega} \sinh(tv^*)(v^*)^3 \,\mathrm{d}\sigma_x < 0.$$

Claim 1 asserts that $f'(1) \leqslant 0$. Since we also have $f'(0) = 0$ and $f'(t(v^*)) = 0$, the concavity of f' now implies that $t(v^*) \leqslant 1$. The estimate (49) now yields

$$c_k(\lambda) \leqslant J_\lambda(v^*) \leqslant C_k \log\left(\frac{1}{\lambda}\right),$$

for λ sufficiently small, as desired. □

With this lemma we have established the existence of solutions to (1) for any $\lambda > 0$, indeed we have already shown that any positive $c_k(\lambda)$ corresponds to a nontrivial solution, $u_{k,\lambda}$, with $E_\lambda(u_{k,\lambda}) = c_k(\lambda)$. The energy estimates in Theorem 1 follow directly from the upper bound in Lemma 3, and from the remark following the statement of Lemma 3. That our process actually leads to infinitely many essentially different (energy-different) solutions follows from the fact that $c_k(\lambda) \to \infty$ as $k \to \infty$ for any fixed $\lambda > 0$ (see [9]). This verifies Theorem 1.

As before, let $\{D_k\}$ and $\{\phi_k\}$ be the Steklov eigenvalues and normalized eigenfunctions. We remind the reader of the definition of the bounded linear projection operator P_D:

$$P_D(\cdot) = \sum_{k:\ D_k = D} \langle \cdot, \phi_k \rangle_{L^2(\partial\Omega)} \phi_k = \sum_{k:\ D_k = D} \frac{\langle \cdot, \phi_k \rangle_{H^1(\Omega)}}{\langle \phi_k, \phi_k \rangle_{H^1(\Omega)}} \phi_k.$$

We now have the following H^1 bounds concerning the solutions, whose existence are assured by Theorem 1.

Proposition 5. *Let $\lambda^* < 1$ be a fixed positive number, and let u_λ, $0 < \lambda < \lambda^*$, be a family of solutions to* (1) (*with shift D*) *whose energies satisfy the estimate*

$$E_\lambda(u_\lambda) \leqslant a \log \frac{1}{\lambda} + b, \tag{50}$$

for some positive constants a and b. Set $w_\lambda = (I - P_D)u_\lambda$, so that $u_\lambda = w_\lambda + P_D u_\lambda$. There exist positive constants C_1 and C_2, depending on a, b, D and λ^*, but otherwise independent of u_λ and λ, such that

$$\|w_\lambda\|^2_{H^1(\Omega)} \leqslant C_1 \log \frac{1}{\lambda} + C_2, \quad 0 < \lambda < \lambda^*,$$

and

$$\|P_D u_\lambda\|_{H^1(\Omega)} \leqslant C_1 \log \frac{1}{\lambda} + C_2, \quad 0 < \lambda < \lambda^*.$$

Proof. Integration by parts combined with the upper bound (50) gives

$$\lambda \int_{\partial\Omega} \left(\frac{u_\lambda}{2} \sinh(u_\lambda) - \cosh(u_\lambda) + 1 \right) \mathrm{d}\sigma_x = E_\lambda(u_\lambda) \leqslant a \log \frac{1}{\lambda} + b.$$

Since $|u|\,\mathrm{e}^{|u|} \leqslant C_1(\frac{u}{2}\sinh(u) - \cosh(u) + 1) + C_2$, it follows that

$$\lambda \int_{\partial\Omega} |u_\lambda|\,\mathrm{e}^{|u_\lambda|}\,\mathrm{d}\sigma_x \leqslant C_1 \log \frac{1}{\lambda} + C_2.$$

A simple convexity argument (see [10] or [12]) now gives

$$\lambda \int_{\partial\Omega} |\sinh(u_\lambda)|\,\mathrm{d}\sigma_x \leqslant \lambda \int_{\partial\Omega} \cosh(u_\lambda)\,\mathrm{d}\sigma_x \leqslant C, \tag{51}$$

for some constant C. Testing Eqs. (21) for w_λ against the eigenfunctions, ϕ_k, and integrating by parts, we obtain

$$(D_k - D) \int_{\partial\Omega} w_\lambda \phi_k \,\mathrm{d}\sigma_x = \int_{\partial\Omega} \lambda \sinh(u_\lambda)\phi_k \,\mathrm{d}\sigma_x. \tag{52}$$

As in the proof of Proposition 3 we therefore have

$$\|w_\lambda\|^2_{L^2(\partial\Omega)} = \sum_{k:\ D_k \neq D} \alpha^2_{k,\lambda},$$

with

$$\alpha_{k,\lambda} = \int_{\partial\Omega} w_\lambda \phi_k \,\mathrm{d}\sigma_x = \frac{1}{D_k - D} \int_{\partial\Omega} \lambda \sinh(u_\lambda)\phi_k \,\mathrm{d}\sigma_x.$$

If $D \neq 0$ it follows immediately that there exists a constant C such that

$$\|w_\lambda\|^2_{L^2(\partial\Omega)} = \sum_{k:\ D_k \neq D} |D_k - D|^{-2} \left| \int_{\partial\Omega} \lambda \sinh(u_\lambda)\phi_k \, d\sigma_x \right|^2 \leqslant C\left(\sum_{k\neq 1} |D_k|^{-2} \left| \int_{\partial\Omega} \lambda \sinh(u_\lambda)\phi_k \, d\sigma_x \right|^2 + \left| \int_{\partial\Omega} \lambda \sinh(u_\lambda) \, d\sigma_x \right|^2 \right).$$

Simply take $C = \max\{|D|^{-2}, \max_{D_k \neq D} |1 - \frac{D}{D_k}|^{-2}\}$. If $D = 0$ it follows similarly that

$$\|w_\lambda\|^2_{L^2(\partial\Omega)} = \sum_{k:\ D_k \neq D} |D_k - D|^{-2} \left| \int_{\partial\Omega} \lambda \sinh(u_\lambda)\phi_k \, d\sigma_x \right|^2 = \sum_{k\neq 1} |D_k|^{-2} \left| \int_{\partial\Omega} \lambda \sinh(u_\lambda)\phi_k \, d\sigma_x \right|^2.$$

Due to (51), $|\int_{\partial\Omega} \lambda \sinh(u_\lambda) \, d\sigma_x| \leqslant C$, and so in both cases ($D \neq 0$ and $D = 0$) we have

$$\|w_\lambda\|^2_{L^2(\partial\Omega)} \leqslant C\left(\sum_{k\neq 1} |D_k|^{-2} \left| \int_{\partial\Omega} \lambda \sinh(u_\lambda)\phi_k \, d\sigma_x \right|^2 + 1 \right), \tag{53}$$

with C depending on a, b, D and λ^*, but otherwise independent of w_λ and λ. Now let W_λ denote the solution to

$$\Delta W_\lambda = 0 \quad \text{in } \Omega, \qquad \frac{\partial W_\lambda}{\partial \mathbf{n}} = F_\lambda = \lambda \sinh(u_\lambda) - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \lambda \sinh(u_\lambda) \, d\sigma_x \quad \text{on } \partial\Omega,$$

with $\int_{\partial\Omega} W_\lambda \, d\sigma_x = 0$. Duality and elliptic regularity estimates immediately give that

$$\|W_\lambda\|_{L^2(\partial\Omega)} \leqslant C\|F_\lambda\|_{H^{-1}(\partial\Omega)} \leqslant C\|F_\lambda\|_{L^1(\partial\Omega)} \leqslant C\big\|\lambda \sinh(u_\lambda)\big\|_{L^1(\partial\Omega)}. \tag{54}$$

Here we have also used the fact that $\partial\Omega$ is one-dimensional to obtain that $L^1(\partial\Omega)$ continuously embeds into $H^{-1}(\partial\Omega)$. The function W_λ is constructed exactly so that

$$\|W_\lambda\|^2_{L^2(\partial\Omega)} = \sum_{k\neq 1} |D_k|^{-2} \left| \int_{\partial\Omega} \lambda \sinh(u_\lambda)\phi_k \, d\sigma_x \right|^2,$$

and a combination of (53) and (54) with the L^1-bound (51) thus gives

$$\|w_\lambda\|^2_{L^2(\partial\Omega)} \leqslant C\big(\|W_\lambda\|^2_{L^2(\partial\Omega)} + 1\big) \leqslant C\big(\big\|\lambda \sinh(u_\lambda)\big\|^2_{L^1(\partial\Omega)} + 1\big) \leqslant C. \tag{55}$$

From integration by parts, and the use of (21) and (52) (if D is a Steklov eigenvalue) we get that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla w_\lambda|^2 \, dx - \frac{D}{2}\int_{\partial\Omega} w_\lambda^2 \, d\sigma_x - \lambda \int_{\partial\Omega} \big(\cosh(u_\lambda) - 1\big) \, d\sigma_x \\
&\quad = \frac{1}{2}\int_\Omega |\nabla u_\lambda|^2 \, dx - \frac{D}{2}\int_{\partial\Omega} u_\lambda^2 \, d\sigma_x - \lambda \int_{\partial\Omega} \big(\cosh(u_\lambda) - 1\big) \, d\sigma_x \\
&\quad = E_\lambda(u_\lambda) \leqslant a \log\frac{1}{\lambda} + b.
\end{aligned}$$

By a combination of this estimate with (51) and (55) it follows that

$$\int_\Omega |\nabla w_\lambda|^2 \, dx \leqslant C_1 \log\frac{1}{\lambda} + C_2. \tag{56}$$

The estimates (55) and (56) give the desired H^1 bound for w_λ.

The function $x \to g(x) = \cosh(x)$ is convex, and so Jensen's inequality, in combination with (51), asserts that

$$g\left(\int_{\partial\Omega} |u_\lambda| \frac{d\sigma}{|\partial\Omega|}\right) \leqslant \int_{\partial\Omega} g(|u_\lambda|) \frac{d\sigma}{|\partial\Omega|} \leqslant \frac{C}{\lambda}.$$

In other words

$$\|u_\lambda\|_{L^1(\partial\Omega)} \leqslant |\partial\Omega| \log\frac{1}{\lambda} + C.$$

By a combination of this estimate with (55) it follows that

$$\begin{aligned}\|P_D u_\lambda\|_{L^1(\partial\Omega)} &\leqslant \|P_D u_\lambda - u_\lambda\|_{L^1(\partial\Omega)} + \|u_\lambda\|_{L^1(\partial\Omega)} = \|w_\lambda\|_{L^1(\partial\Omega)} + \|u_\lambda\|_{L^1(\partial\Omega)} \\ &\leqslant |\partial\Omega|^{1/2} \|w_\lambda\|_{L^2(\partial\Omega)} + \|u_\lambda\|_{L^1(\partial\Omega)} \leqslant |\partial\Omega| \log\frac{1}{\lambda} + C. \end{aligned} \tag{57}$$

We note that $P_D u_\lambda$ lies in the finite dimensional eigenspace $V_D \subset H^1(\Omega)$ spanned by the "harmonically extended" eigenfunctions ($V_D = \{0\}$ if D is not an eigenvalue). We also note that $\|\cdot\|_{L^1(\partial\Omega)}$ is a norm on V_D. Since all norms are equivalent on V_D it now follows from (57) that

$$\|P_D u_\lambda\|_{H^1(\partial\Omega)} \leqslant C_1 \log\frac{1}{\lambda} + C_2,$$

as desired. □

We may also establish a lower bound that applies to the solutions constructed in Theorem 1.

Proposition 6. *There exists a constant* $c_1 > 0$ *such that whenever* u_{λ_n}, $\lambda_n \to 0_+$, *is a sequence of solutions to* (1) *with the property that* $\|u_{\lambda_n}\|_{H^1(\Omega)}$ does not *converge to* 0 *as* $\lambda_n \to 0_+$, *then we may extract a subsequence, for simplicity also denoted* u_{λ_n}, *with*

$$c_1 \log\frac{1}{\lambda_n} \leqslant \|u_{\lambda_n}\|^2_{H^1(\Omega)} \quad \text{as } \lambda_n \to 0_+.$$

Proof. It suffices to prove that there exists a constant $c_1 > 0$ such that if u_{λ_n}, $\lambda_n \to 0_+$ is a family of solutions to (1) with

$$\|u_{\lambda_n}\|^2_{H^1(\Omega)} \Big/ \log\frac{1}{\lambda_n} < c_1 \quad \text{as } \lambda_n \to 0_+,$$

then

$$\|u_{\lambda_n}\|_{H^1(\Omega)} \to 0 \quad \text{as } \lambda_n \to 0_+.$$

The argument to show this proceeds exactly as in the proof of Proposition 3. □

Having thus provided asymptotic bounds for the H^1 norm of the variationally constructed solutions, we now continue with a more detailed *blow-up analysis* of these solutions as $\lambda \to 0_+$. As we shall see, the fact that $\lambda \sinh(u_\lambda)$ is bounded in $L^1(\partial\Omega)$, leads to a completely different blow-up pattern than that, which we saw for $\lambda \to 0_-$ in the previous section. For $D = 0$ a very detailed blow-up analysis has already been carried out in [10] and [12].

Except for $D = 0$ the analysis is somewhat more complicated (and the results are less complete) when D is a Steklov eigenvalue. Some of the analysis that follows is directly unnecessary if D is not a Steklov eigenvalue. Indeed, in that case $P_D u_\lambda = 0$ and so $u_\lambda = (I - P_D)u_\lambda = w_\lambda$. For reasons of completeness we have decided to proceed in a general framework – however, simplifications for the case when D is not an eigenvalue will be noted when appropriate.

If D is a Steklov eigenvalue, the possibility exists that the contribution of the "mode" corresponding to D becomes unbounded. That is, writing $u_\lambda = P_D u_\lambda + w_\lambda$, the norm of the component $P_D u_\lambda$ can be unbounded (growing like $C_1 \log\frac{1}{\lambda} + C_2$) as λ approaches 0_+. We have already seen in the proof of Proposition 5 that

$$\|P_D u_\lambda\|_{L^1(\partial\Omega, d\sigma/|\partial\Omega|)} \leqslant \log\frac{1}{\lambda} + C_2, \tag{58}$$

i.e., the constant C_1 in front of $\log \frac{1}{\lambda}$ may be taken to be one, if we use the measure $d\sigma/|\partial\Omega|$ in our definition of the L^1-norm. A similar estimate holds for the norm on $L^p(\partial\Omega, d\sigma/|\partial\Omega|)$, however, it is not clear (except when $D=0$, and $P_D u_\lambda = \text{const}$) that a similar estimate holds for the L^∞ norm. This in turn makes it somewhat unclear whether $\|\lambda\, e^{P_D u_\lambda}\|_{L^\infty(\partial\Omega)}$ is uniformly bounded, as $\lambda \to 0_+$, in the case when D is an eigenvalue different from 0. We are, nonetheless, able to establish a partial result in this direction. In order to state this result we first introduce some notation. By V_D we denote, as before, the eigenspace corresponding to D, i.e.,

$$V_D = \left\{\phi \in C^\infty(\overline{\Omega}):\ \Delta\phi = 0 \text{ in } \Omega,\ \frac{\partial\phi}{\partial\mathbf{n}} = D\phi \text{ on } \partial\Omega\right\},$$

and given any $\phi \in V_D \setminus \{0\}$ we introduce the set I_ϕ

$$I_\phi = \left\{x \in \partial\Omega:\ \left|\phi(x)\right| < \|\phi\|_{C^0(\partial\Omega)}\right\},$$

and the set M_ϕ

$$M_\phi = \partial\Omega \setminus I_\phi = \left\{x \in \partial\Omega:\ \left|\phi(x)\right| = \|\phi\|_{C^0(\partial\Omega)}\right\}.$$

When D is a Steklov eigenvalue different from zero, we expect I_ϕ to be almost all of $\partial\Omega$. For instance, when Ω is a disk, and D is an eigenvalue different from zero, then M_ϕ consists of a finite (even) number of equispaced points.

Lemma 4. *Suppose $D = D_k$ is a Steklov eigenvalue for the boundary value problem* (2). *Suppose u_{λ_n}, $\lambda_n \to 0_+$, is a sequence of solutions to* (1) (*corresponding to that same shift D*) *whose energies satisfy the estimate*

$$E_\lambda(u_{\lambda_n}) \leqslant a \log\frac{1}{\lambda_n} + b,$$

for some positive constants a and b. Then we have the following results

$D=0$: *There exists a constant C depending only on a, b and $|\partial\Omega|$ such that*

$$\lambda_n\, e^{|P_D u_{\lambda_n}|} = \lambda_n\, e^{\frac{1}{|\partial\Omega|}\left|\int_{\partial\Omega} u_{\lambda_n}\, d\sigma\right|} \leqslant C \quad \text{as } \lambda_n \to 0_+,$$

$D \neq 0$: *There exists a subsequence, for simplicity also denoted λ_n, and a Steklov eigenvector $\phi \in V_D \setminus \{0\}$ such that, given any $x_0 \in I_\phi \subset \partial\Omega$, we may find an open neighborhood $\omega_{x_0} \subset \partial\Omega$, of x_0, with*

$$\sup_{x \in \omega_{x_0}} \lambda_n\, e^{|P_D u_{\lambda_n}(x)|} \to 0 \quad \text{as } \lambda_n \to 0_+.$$

Proof. The result for $D=0$ was already used in [10] and [12], and it follows directly from (58). We proceed with the case $D = D_k \neq 0$. From the estimates (51) and (55) in the proof of Proposition 5 we know that any sequence of solutions

$$u_{\lambda_n} = P_D u_{\lambda_n} + w_{\lambda_n}, \quad \lambda_n \to 0_+,$$

to (1), which satisfies the energy bound assumed in the present lemma, also satisfies

$$\lambda_n \int_{\partial\Omega} e^{|P_D u_{\lambda_n} + w_{\lambda_n}|}\, d\sigma \leqslant C, \tag{59}$$

and

$$\int_{\partial\Omega} w_{\lambda_n}^2\, d\sigma \leqslant C, \tag{60}$$

with C only depending on a, b, D and $|\partial\Omega|$. Suppose $\|P_D u_{\lambda_n}\|_{C^0(\partial\Omega)} \neq 0$. The sequence

$$\phi_{\lambda_n} = \frac{P_D u_{\lambda_n}}{\|P_D u_{\lambda_n}\|_{C^0(\partial\Omega)}}$$

is bounded in V_D. Due to the finite dimensionality of V_D we may thus extract a subsequence, also denoted ϕ_{λ_n}, with

$$\phi_{\lambda_n} \to \phi \in V_D \quad \text{(in any norm) as } \lambda_n \to 0_+. \tag{61}$$

The statement of this lemma is trivial if $\|P_D u_{\lambda_n}\|_{C^0(\partial\Omega)}$ is bounded (or has a bounded subsequence). We may thus suppose that

$$\|P_D u_{\lambda_n}\|_{C^0(\partial\Omega)} \to \infty \quad \text{as } \lambda_n \to 0_+.$$

The function ϕ necessarily has $\|\phi\|_{C^0(\partial\Omega)} = 1$. We may without loss of generality suppose there exists $\bar{x} \in \partial\Omega$ with $\phi(\bar{x}) = 1$. Now suppose x_0 is in I_ϕ, and therefore $-1 < \phi(x_0) < 1$. By virtue of (61), there exist open $\partial\Omega$-neighborhoods ω_{x_0} and $\omega_{\bar{x}}$, of x_0 and $\bar{x}$, respectively, and a small positive number η, such that

$$\sup_{x\in\omega_{x_0}} \left|\phi_{\lambda_n}(x)\right| \leqslant 1 - 2\eta \leqslant 1 - \eta \leqslant \inf_{x\in\omega_{\bar{x}}} \phi_{\lambda_n}(x), \tag{62}$$

for n sufficiently large (as λ_n converges to 0). Based on (59) and (62) we conclude that

$$\begin{aligned}
\sup_{x\in\omega_{x_0}} \lambda_n \, \mathrm{e}^{|P_D u_{\lambda_n}(x)|} \int_{\omega_{\bar{x}}} \mathrm{e}^{w_{\lambda_n}} \, \mathrm{d}\sigma &= \sup_{x\in\omega_{x_0}} \lambda_n \, \mathrm{e}^{\|P_D u_{\lambda_n}\|_{C^0(\partial\Omega)} |\phi_{\lambda_n}(x)|} \int_{\omega_{\bar{x}}} \mathrm{e}^{w_{\lambda_n}} \, \mathrm{d}\sigma \\
&\leqslant \lambda_n \, \mathrm{e}^{-\eta \|P_D u_{\lambda_n}\|_{C^0(\partial\Omega)}} \int_{\omega_{\bar{x}}} \mathrm{e}^{\|P_D u_{\lambda_n}\|_{C^0(\partial\Omega)} \phi_{\lambda_n} + w_{\lambda_n}} \, \mathrm{d}\sigma \\
&= \lambda_n \, \mathrm{e}^{-\eta \|P_D u_{\lambda_n}\|_{C^0(\partial\Omega)}} \int_{\omega_{\bar{x}}} \mathrm{e}^{P_D u_{\lambda_n} + w_{\lambda_n}} \, \mathrm{d}\sigma \\
&\leqslant C \, \mathrm{e}^{-\eta \|P_D u_{\lambda_n}\|_{C^0(\partial\Omega)}},
\end{aligned} \tag{63}$$

for λ_n sufficiently close to 0. Now, we also have that

$$0 < c < \int_{\omega_{\bar{x}}} \mathrm{e}^{w_{\lambda_n}} \, \mathrm{d}x, \tag{64}$$

because if this lower bound did not hold then we could find a subsequence $\lambda_{n_m} \to 0_+$ (for simplicity denoted λ_m, $m \to \infty$) so that

$$\int_{\omega_{\bar{x}}} \mathrm{e}^{w_{\lambda_m}} \, \mathrm{d}\sigma \leqslant |\omega_{\bar{x}}|/2m,$$

and as a consequence

$$\left|\left\{x \in \omega_{\bar{x}} \colon \, \mathrm{e}^{w_{\lambda_m}} \leqslant 1/m\right\}\right| \geqslant |\omega_{\bar{x}}|/2.$$

This would imply that

$$\left|\left\{x \in \omega_{\bar{x}} \colon \, w_{\lambda_m} \leqslant -\log m\right\}\right| \geqslant |\omega_{\bar{x}}|/2,$$

and thus

$$\int_{\omega_{\bar{x}}} |w_{\lambda_m}|^2 \, \mathrm{d}\sigma \geqslant \frac{|\omega_{\bar{x}}|}{2} (\log m)^2, \quad \text{as } m \to \infty$$

as $m \to \infty$ ($\lambda_m \to 0_+$). We have now arrived at a contradiction to the estimate (60). A combination of (63) and (64), with the fact that $\|P_D u_{\lambda_n}\|_{C^0(\partial\Omega)}$ converges to ∞, immediately leads to the assertion of this lemma. □

As already noted in the proof of Proposition 5, any sequence of solutions to (1) ($\lambda_n \to 0_+$) that satisfies the energy bound (50) also satisfies the estimate

$$\lambda_n \int_{\partial\Omega} \mathrm{e}^{\pm u_{\lambda_n}} \, \mathrm{d}\sigma \leqslant C.$$

We may thus extract a subsequence, $\{u_{\lambda_n}\}$, which in addition to the conclusion of Lemma 4 has

$$\frac{\lambda_n}{2}\,\mathrm{e}^{u_{\lambda_n}}\Big|_{\partial\Omega} \to \mu_+ \quad \text{and} \quad \frac{\lambda_n}{2}\,\mathrm{e}^{-u_{\lambda_n}}\Big|_{\partial\Omega} \to \mu_- \tag{65}$$

for two nonnegative measures μ_+ and μ_-. The convergence is in the sense of measures (i.e., weak* in the dual of $C^0(\partial\Omega)$). With the present blow-up analysis we seek to characterize the limiting behavior of u_{λ_n} modulo its potential "eigenfunction part", i.e., we study the limiting behavior of $w_{\lambda_n} = (I - P_D)u_{\lambda_n}$ (if D is not a Steklov eigenvalue then $w_{\lambda_n} = u_{\lambda_n}$). As already observed, w_λ solves

$$\begin{aligned}
&\Delta w_{\lambda_n} = 0 \quad \text{in } \Omega,\\
&\frac{\partial w_{\lambda_n}}{\partial \mathbf{n}} = D w_{\lambda_n} + \lambda_n \sinh u_{\lambda_n} \quad \text{on } \partial\Omega,
\end{aligned} \tag{66}$$

or in its distributional formulation

$$\int_\Omega w_{\lambda_n}\Delta v\,\mathrm{d}x + \int_{\partial\Omega} D w_{\lambda_n} v\,\mathrm{d}\sigma_x = -\int_{\partial\Omega} \lambda_n \sinh u_{\lambda_n} v\,\mathrm{d}\sigma_x,$$

for all $v \in C^2(\overline{\Omega})$ with $\frac{\partial v}{\partial \mathbf{n}} = 0$ on $\partial\Omega$. As we have already seen, w_{λ_n} and u_{λ_n} satisfy the estimates

$$\|w_{\lambda_n}\|_{L^2(\partial\Omega)} \leqslant C \quad \text{and} \quad \|\lambda_n \sinh u_{\lambda_n}\|_{L^1(\partial\Omega)} \leqslant C.$$

It now follows quite easily from an application of standard elliptic regularity theory to the boundary value problem (66), and duality, that

$$\|w_{\lambda_n}\|_{H^{1-\epsilon}(\Omega)} \leqslant C_\epsilon,$$

for any $\epsilon > 0$. The constant C_ϵ is independent of $\lambda_n \to 0_+$. By extraction of a subsequence (for simplicity also denoted λ_n) we may thus achieve convergence in $H^{1-\epsilon}(\Omega)$ for any $\epsilon > 0$, i.e., we may achieve

$$w_{\lambda_n} \to w_0 \quad \text{in } H^{1-\epsilon}(\Omega), \tag{67}$$

for any $\epsilon > 0$. The limit w_0 is a solution to the problem

$$\begin{aligned}
&\Delta w_0 = 0 \quad \text{in } \Omega,\\
&\frac{\partial w_0}{\partial \mathbf{n}} = D w_0 + (\mu_+ - \mu_-) \quad \text{on } \partial\Omega,
\end{aligned} \tag{68}$$

in the distributional sense that

$$\int_\Omega w_0\Delta v\,\mathrm{d}x + \int_{\partial\Omega} D w_0 v\,\mathrm{d}\sigma_x = -\int_{\partial\Omega} v\,\mathrm{d}(\mu_+ - \mu_-),$$

for all $v \in C^2(\overline{\Omega})$ with $\frac{\partial v}{\partial \mathbf{n}} = 0$ on $\partial\Omega$. We also note that, for $y \in \Omega$,

$$w_0(y) = -\int_{\partial\Omega} N(x,y)\big(D w_0(x)\,\mathrm{d}\sigma_x + \mathrm{d}(\mu_+ - \mu_-)_x\big) + \frac{1}{|\partial\Omega|}\int_{\partial\Omega} w_0\,\mathrm{d}\sigma_x,$$

where $N(\cdot, y)$ is the "standard" Neumann function, satisfying

$$\Delta N(\cdot, y) = \delta_y \quad \text{in } \Omega, \qquad \frac{\partial N(\cdot, y)}{\partial \mathbf{n}} = \frac{1}{|\partial\Omega|} \quad \text{on } \partial\Omega.$$

Our blow-up analysis, more specifically, concerns the structure of the measures μ_+ and μ_-. We now define what it means to be a *regular* and a *singular* point with respect to the measure $\nu = \mu_+ + \mu_-$.

Definition 1. We call a point $x_0 \in \partial\Omega$ a regular point if there exists a function $\psi \in C^0(\partial\Omega)$ such that $0 \leqslant \psi \leqslant 1$, $\psi \equiv 1$ in a neighborhood of x_0, and

$$\int_{\partial\Omega} \psi\,\mathrm{d}\nu < \frac{\pi}{2}.$$

A point on $\partial\Omega$ is called *singular* if it is *not regular*. We denote by S the set of all *singular* points. Note that it follows from this definition (and the finiteness of the measure ν) that S is a finite set. We also introduce the notion of nondegeneracy for a Steklov eigenspace.

Definition 2. A nonzero Steklov eigenvalue D_k for the boundary value problem (2) is said to have a nondegenerate eigenspace, if any eigenfunction ϕ in $V_{D_k} \setminus \{0\}$ attains its extremal values at only a finite number of points (on $\partial\Omega$), i.e., if M_ϕ consists of a finite number of points, for any $\phi \in V_{D_k} \setminus \{0\}$.

The blow-up analysis follows along the same lines as in [10] and [12], and depends crucially on the following inequality. Suppose g is a non-trivial, smooth function on $\partial\Omega$ (with $\int_{\partial\Omega} g\,\mathrm{d}\sigma = 0$) and suppose v is a classical solution to

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \mathbf{n}} = g & \text{on } \partial\Omega, \end{cases}$$

normalized by $\int_{\partial\Omega} v\,\mathrm{d}\sigma = 0$. Then, for any $0 < \delta < \pi$, there exists a constant C_δ, independent of g, such that

$$\int_{\partial\Omega} \exp\left(\frac{(\pi-\delta)|v(x)|}{\|g\|_{L^1}}\right) \mathrm{d}\sigma_x \leqslant C_\delta. \tag{69}$$

This is a special version of an inequality, first proved in [4]. For a proof of this special version, see [10].

Lemma 5. *Suppose u_{λ_n}, $\lambda_n \to 0_+$ is a sequence of solutions to* (1) (*corresponding to the shift D*) *whose energies satisfy the estimate*

$$E_{\lambda_n}(u_{\lambda_n}) \leqslant a \log \frac{1}{\lambda_n} + b,$$

for some positive constants a and b. Suppose the sequence is selected so that (65) *holds for the two regular Borel measures $\mu_\pm$. If D is a nonzero Steklov eigenvalue we additionally suppose the sequence has been selected according to the conclusion of Lemma* 4. *Let $x_0 \in \partial\Omega$ be a regular point with respect to the measure $\nu = \mu_+ + \mu_-$ in the sense of Definition* 1. *Then we have the following results*

D is not a Steklov eigenvalue: *There exist $r_0 > 0$ and $C < \infty$ such that*

$$\sup_{y \in \partial\Omega \cap B(x_0, r_0)} |u_{\lambda_n}(y)| \leqslant \sup_{y \in \overline{\Omega} \cap B(x_0, r_0)} |u_{\lambda_n}(y)| \leqslant C.$$

As a consequence $\lambda_n \mathrm{e}^{\pm u_{\lambda_n}} \to 0$ uniformly on $\partial\Omega \cap B(x_0, r_0)$ as $\lambda_n \to 0_+$.

$D = 0$: *There exist $r_0 > 0$ and $C < \infty$ such that*

$$\sup_{y \in \partial\Omega \cap B(x_0, r_0)} |w_{\lambda_n}(y)| \leqslant \sup_{y \in \overline{\Omega} \cap B(x_0, r_0)} |w_{\lambda_n}(y)| \leqslant C.$$

D is a Steklov eigenvalue $\neq 0$: *Suppose x_0 also lies in I_ϕ where ϕ is the Steklov eigenvector arising in the conclusion of Lemma* 4. *There exist $r_0 > 0$ and $C < \infty$ such that*

$$\sup_{y \in \partial\Omega \cap B(x_0, r_0)} |w_{\lambda_n}(y)| \leqslant \sup_{y \in \overline{\Omega} \cap B(x_0, r_0)} |w_{\lambda_n}(y)| \leqslant C.$$

Furthermore $\lambda_n \mathrm{e}^{\pm u_{\lambda_n}} \to 0$ uniformly on $\partial\Omega \cap B(x_0, r_0)$ as $\lambda_n \to 0_+$.

Proof. The proof of this lemma when D is not a Steklov eigenvalue is simpler than when D is such an eigenvalue (when D is not an eigenvalue $P_D u_\lambda = 0$ and $u_\lambda = w_\lambda$). We shall thus only consider the case where D is a Steklov eigenvalue. A proof for $D = 0$ has already been given in [10]. The proof for $D \neq 0$ has considerable overlap with that proof, however, for the convenience of the reader we present the details here. Let $x_0 \in \partial\Omega$ be a regular point, with $x_0 \in I_\phi$, where ϕ is the Steklov eigenvector arising in the conclusion of Lemma 4. Let ψ be a smooth function with the properties described in the definition of a regular point; by a simple regularization procedure we may arrange that $\psi \in C^\infty(\partial\Omega)$. We decompose the function $w_{\lambda_n} = w_1 + w_2 + w_3 + C^*$, into three harmonic functions and a constant (for simplicity of notation we drop the λ_n label on the functions w_i, $1 \leqslant i \leqslant 3$, and the constant C^*). The harmonic functions are normalized by $\int_{\partial\Omega} w_1 \,\mathrm{d}\sigma = \int_{\partial\Omega} w_2 \,\mathrm{d}\sigma = \int_{\partial\Omega} w_3 \,\mathrm{d}\sigma = 0$. In addition these harmonic functions satisfy the Neumann boundary conditions

$$\frac{\partial w_1}{\partial \mathbf{n}} = \lambda_n \psi \sinh(u_{\lambda_n}) - \frac{\lambda_n}{|\partial\Omega|} \int_{\partial\Omega} \psi \sinh(u_{\lambda_n}) \,\mathrm{d}\sigma,$$

$$\frac{\partial w_2}{\partial \mathbf{n}} = \lambda_n (1-\psi) \sinh(u_{\lambda_n}) - \frac{\lambda_n}{|\partial\Omega|} \int_{\partial\Omega} (1-\psi) \sinh(u_{\lambda_n}) \,\mathrm{d}\sigma,$$

$$\frac{\partial w_3}{\partial \mathbf{n}} = D w_{\lambda_n} - \frac{D}{|\partial\Omega|} \int_{\partial\Omega} w_{\lambda_n} \,\mathrm{d}\sigma.$$

By (55)

$$\left| \frac{1}{|\partial\Omega|} \int_{\partial\Omega} w_{\lambda_n} \,\mathrm{d}\sigma \right| \leqslant \frac{1}{|\partial\Omega|^{1/2}} \|w_{\lambda_n}\|_{L^2(\partial\Omega)} \leqslant C,$$

and

$$\left\| w_{\lambda_n} - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} w_{\lambda_n} \,\mathrm{d}\sigma \right\|_{L^2(\partial\Omega)} \leqslant \|w_{\lambda_n}\|_{L^2(\partial\Omega)} \leqslant C.$$

It follows immediately that the constant $C^* = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} w_{\lambda_n} \,\mathrm{d}\sigma$ is bounded independently of λ_n. From elliptic regularity theory it follows that $\|w_3\|_{H^{3/2}(\Omega)}$ is bounded independently of λ_n. Using Sobolev's imbedding theorem we conclude that

$$\|w_3\|_{C^0(\overline{\Omega})} \leqslant C, \tag{70}$$

independently of λ_n. Again, by elliptic regularity

$$\|w_2\|_{L^p(\Omega)} \leqslant C_p \left\| \frac{\partial w_2}{\partial \mathbf{n}} \right\|_{L^1(\partial\Omega)} \leqslant C_p,$$

for any $1 < p < \infty$. Suppose that $r_1 > 0$ is picked sufficiently small so that $\psi \equiv 1$ on $B(x_0, r_1) \cap \partial\Omega$. Since

$$\frac{\partial w_2}{\partial \mathbf{n}} = -\frac{\lambda_n}{|\partial\Omega|} \int_{\partial\Omega} (1-\psi) \sinh(u_{\lambda_n}) \,\mathrm{d}\sigma, \quad \text{a bounded constant,}$$

on $B(x_0, r_1) \cap \partial\Omega$, local elliptic estimates give that

$$\begin{aligned} \|w_2\|_{C^0(B(x_0,r_1/2)\cap\partial\Omega)} &\leqslant \|w_2\|_{C^0(B(x_0,r_1/2)\cap\overline{\Omega})} \\ &\leqslant C\left(\left| \frac{\lambda_n}{|\partial\Omega|} \int_{\partial\Omega} (1-\psi) \sinh(u_{\lambda_n}) \,\mathrm{d}\sigma \right| + \|w_2\|_{L^2(\Omega)} \right) \leqslant C. \end{aligned} \tag{71}$$

Lastly, since

$$\left|\sinh(x)\right| = \cosh(x) - \mathrm{e}^{-|x|}$$

the convergence in measure of $\lambda_n \cosh u_{\lambda_n}$ towards $\mu_+ + \mu_- = \nu$ means that

$$\lambda_n \int_{\partial\Omega} \psi \left|\sinh(u_{\lambda_n})\right| \mathrm{d}\sigma \to \int_{\partial\Omega} \psi \,\mathrm{d}\nu < \frac{\pi}{2}.$$

Therefore, for δ sufficiently small,

$$\left\|\frac{\partial w_1}{\partial \mathbf{n}}\right\|_{L^1(\partial\Omega)} = \int\limits_{\partial\Omega} \left|\frac{\partial w_1}{\partial \mathbf{n}}\right| \mathrm{d}\sigma < \pi - 2\delta,$$

for all sufficiently small λ_n. We may thus apply (69) with $v = w_1$ and $g = \partial w_1/\partial \mathbf{n}$ to obtain

$$\int\limits_{\partial\Omega} \mathrm{e}^{p^*|w_1|}\,\mathrm{d}\sigma \leqslant C_{p^*}, \tag{72}$$

for some $1 < p^* = (\pi-\delta)/(\pi-2\delta) \leqslant (\pi-\delta)/\|\partial w_1/\partial \mathbf{n}\|_{L^1(\partial\Omega)}$.

Suppose r_1 is chosen sufficiently small that

$$\sup_{x\in\partial\Omega\cap B(x_0,r_1/2)} \lambda_n\,\mathrm{e}^{|P_D u_{\lambda_n}(x)|} \leqslant C, \tag{73}$$

which is possible since $x_0 \in I_\phi$. Using (70), (71), (72), (73) and the boundedness of the constant C^*, we can now estimate

$$\begin{aligned}\lambda_n^{p^*} \int\limits_{\partial\Omega\cap B(x_0,r_1/2)} \psi^{p^*}\,\mathrm{e}^{p^*|u_{\lambda_n}|}\,\mathrm{d}\sigma &\leqslant C \int\limits_{\partial\Omega\cap B(x_0,r_1/2)} \lambda_n^{p^*}\,\mathrm{e}^{p^*(|P_D u_{\lambda_n}|+|w_1|+|w_2|+|w_3|+|C^*|)}\,\mathrm{d}\sigma\\ &\leqslant C\int\limits_{\partial\Omega} \mathrm{e}^{p^*|w_1|}\,\mathrm{d}\sigma \leqslant C,\end{aligned}$$

for some fixed $p^* > 1$, and all λ_n sufficiently small. Therefore, Sobolev's imbedding theorem yields

$$\begin{aligned}\left\|\frac{\partial w_1}{\partial \mathbf{n}}\right\|_{H^{-s^*}(\partial\Omega\cap B(x_0,r_1/2))} &\leqslant C\left\|\frac{\partial w_1}{\partial \mathbf{n}}\right\|_{L^{p^*}(\partial\Omega\cap B(x_0,r_1/2))}\\ &\leqslant C\left(\left\|\lambda_n\psi\sinh(u_{\lambda_n})\right\|_{L^{p^*}(\partial\Omega\cap B(x_0,r_1/2))} + \int\limits_{\partial\Omega}\lambda_n\left|\sinh(u_{\lambda_n})\right|\mathrm{d}\sigma\right) \leqslant C,\end{aligned}$$

for some fixed $s^* < \frac{1}{2}$. Interior elliptic estimates and duality now give

$$\|w_1\|_{H^{3/2-s*}(\Omega\cap B(x_0,r_1/4))} \leqslant C\left(\left\|\frac{\partial w_1}{\partial \mathbf{n}}\right\|_{H^{-s^*}(\partial\Omega\cap B(x_0,r_1/2))} + \|w_1\|_{L^2(\Omega\cap B(x_0,r_1/2))}\right) \leqslant C.$$

Sobolev's imbedding theorem then yields the local C^0 bound

$$\|w_1\|_{C^0(\partial\Omega\cap B(x_0,r_1/4))} \leqslant \|w_1\|_{C^0(\overline{\Omega}\cap B(x_0,r_1/4))} \leqslant C. \tag{74}$$

A combination of (70), (71), and (74) with the fact that the constant C^* is bounded leads to the desired estimate for w_{λ_n}, with $r_0 = r_1/4$. Once this uniform estimate is proven, it follows immediately from the estimate

$$\lambda_n\,\mathrm{e}^{|u_{\lambda_n}|} \leqslant \lambda_n\,\mathrm{e}^{|P_D u_{\lambda_n}|+|w_{\lambda_n}|},$$

and the fact that

$$\sup_{x\in\omega_{x_0}} \lambda_n\,\mathrm{e}^{|P_D u_{\lambda_n}(x)|} \to 0 \quad \text{as } \lambda_n \to 0_+,$$

that $\lambda_n\,\mathrm{e}^{|u_{\lambda_n}|}$ converges uniformly to zero on $\partial\Omega\cap B(x_0,r_0)$, for r_0 sufficiently small. □

Based on this lemma we are now able to establish the following theorem characterizing the possible limits of $\lambda_n\sinh(u_{\lambda_n})$. The proof of this theorem entirely parallels that of Theorem 1 in [12], and we refer the interested reader to that paper for the details.

Theorem 2. *Suppose u_{λ_n}, $\lambda_n \to 0_+$, is a sequence of solutions to* (1) (*corresponding to shift D*) *whose energies satisfy the estimate*

$$E_\lambda(u_{\lambda_n}) \leqslant a \log \frac{1}{\lambda_n} + b,$$

for some positive constants a and b. Suppose this sequence has been selected so that

$$\lambda_n \,\mathrm{e}^{\pm u_{\lambda_n}} \to 2\mu_\pm \quad \textit{weak}^* \textit{ in } C^0(\partial\Omega)^*,$$

i.e., in the sense of measures, and

$$w_{\lambda_n} \to w_0 \quad \textit{in } H^{1-\epsilon}(\Omega), \quad \textit{as } \lambda_n \to 0_+,$$

for any $\epsilon > 0$ (see (65) and (67)). If D is a nonzero Steklov eigenvalue we additionally suppose λ_n has been selected according to the conclusion of Lemma 4. *Let S denote the set of singular points relative to the measure $\nu = \mu_+ + \mu_-$. Then we have the following results concerning the limits of $\lambda_n \,\mathrm{e}^{\pm u_\lambda}|_{\partial\Omega}$*

D is not a Steklov eigenvalue: *The nonlinear boundary terms $\lambda_n \,\mathrm{e}^{\pm u_\lambda}|_{\partial\Omega}$ converge in the sense of measures towards $2\mu_\pm = \sum_{\mathbf{x}_i \in S} 2\alpha_i^\pm \delta_{\mathbf{x}_i}$, with $\alpha_i^\pm \geqslant 0$.*

$D = 0$: *The nonlinear boundary terms $\lambda_n \mathrm{e}^{\pm u_\lambda}|_{\partial\Omega}$ converge in the sense of measures towards $2\mu_\pm = \sum_{\mathbf{x}_i \in S} 2\alpha_i^\pm \delta_{\mathbf{x}_i} + d_\pm \,\mathrm{e}^{\pm w_0}$, with $\alpha_i^\pm \geqslant 0$, $d_\pm \geqslant 0$, and $d_+ \cdot d_- = 0$.*

D is a Steklov eigenvalue $\neq 0$: *Suppose additionally that D has a nondegenerate eigenspace in the sense of Definition* 2. *Then there exists a finite set of points $M \subset \partial\Omega$ such that the nonlinear boundary terms $\lambda_n \,\mathrm{e}^{\pm u_\lambda}|_{\partial\Omega}$ converge in the sense of measures towards $2\mu_\pm = \sum_{\mathbf{x}_i \in S\cup M} 2\alpha_i^\pm \delta_{\mathbf{x}_i}$, with $\alpha_i^\pm \geqslant 0$.*

In all cases we have that $\alpha_i^+ + \alpha_i^- > 0$ for points $\mathbf{x}_i \in S$ (or $\mathbf{x}_i \in S \cup M$), and w_0 is C^∞ in $\overline{\Omega} \setminus S$ (or $\overline{\Omega} \setminus (S \cup M)$). S (or $S \cup M$) may possibly be empty; *an empty sum of delta functions should be interpreted as* 0.

Remarks. Concerning the structure of the limit of $\lambda_n \,\mathrm{e}^{\pm u_{\lambda_n}}$ and w_{λ_n}:

(1) When $D = 0$ it was proven in [12] that (since $E_{\lambda_n}(u_{\lambda_n})$ blow up as λ_n approaches 0_+) the sets S, corresponding to our variationally constructed solutions, are nonempty and the measure $\mu_+ + \mu_- = \lim \lambda_n|\sinh(u_{\lambda_n})|$ as well as the measure $\mu_+ - \mu_- = \lim \lambda_n \sinh(u_{\lambda_n})$ has nonzero delta functions at *all* the points in S. A similar result is easy to verify when D is not a Steklov eigenvalue. We have no proof that the sets S, corresponding to our variationally constructed solutions, are nonempty when D is a nonzero Steklov eigenvalue, even though we suspect this to be true.

(2) As reported in [12] there is numerical evidence to support the presence of the "regular part" $d_\pm \,\mathrm{e}^{\pm w_0}$ in the case $D = 0$, i.e., there is numerical evidence of cases in which not both d_+ and d_- is zero.

(3) When D is a nonzero Steklov eigenvalue, Theorem 2 at present leaves open the possibility that there may be delta functions in μ_+ or μ_- at points (in M) outside of S. In the next theorem we verify that the point-mass locations for the measures $\mu_+ + \mu_-$ and $\mu_+ - \mu_-$ are the same, and that the absolute weight of any point-mass in these two measures is greater than or equal to 2π. As a corollary to the last statement it follows that $S \cup M = S$.

(4) When D is not a positive Steklov eigenvalue, it is not difficult to see that w_{λ_n} stays uniformly bounded near points in $\overline{\Omega} \setminus S$, and that given any point $\mathbf{x} \in S$ one may find points $x_n \in \overline{\Omega}$, $x_n \to \mathbf{x}$. so that $|w_{\lambda_n}(x_n)| \to \infty$ as $\lambda_n \to 0_+$ (for the details of a proof of this see for example [10] Lemma 4.8). When D is a positive Steklov eigenvalue (with a nondegenerate eigenspace) similar arguments may be used to prove that w_{λ_n} stays uniformly bounded away from a finite set of boundary points. In many interesting cases the contribution $P_D u_{\lambda_n}$ vanishes or stays uniformly bounded, and so the variationally defined sequence of solutions, u_{λ_n}, blows up "pointwise" in the same places as w_{λ_n}. It is this behavior which is markedly different than that which we have seen for $\lambda_n \to 0_-$ (cf. Proposition 3) where we may either find a subsequence along which u_{λ_n} and w_{λ_n} blow up almost everywhere in Ω, or the entire original sequence u_{λ_n} converges to zero in $H^1(\Omega)$ (and thus, by elliptic estimates, also in $C^0(\overline{\Omega})$).

We can also obtain information about the strength and locations of the boundary singularities in the limit w_0, similar to that which we have already obtained for the case $D = 0$ in [12].

There are two related approaches to establish such results that we know of. One is complex analytic, and was originally introduced in [13] to study "interior" blow-up for positive solutions (to a conceptually related problem). This approach was the basis for the analysis for the case $D=0$, presented in [10]. The other approach is based on a clever Pohozaev-like integral identity, and its asymptotic limit on shrinking neighborhoods of the singularities; it was originally introduced in [3] and was also used in [6]. Both approaches may be adapted to the present case. Since we have already demonstrated how to apply the first approach on a very related problem (the case $D=0$) we will here for completeness give a fairly detailed outline of how to apply the second approach. According to Theorem 2 all the point-mass locations of the measures $\lim \lambda_n e^{\pm u_{\lambda_n}} = 2\mu_\pm$ lie inside the finite set of singular points S, or inside the finite set $S \cup M$. In both cases we shall use $\{\mathbf{x}_i\}_{i=1}^K$ to denote the set of point-mass locations (of $\mu_+ + \mu_-$). Let $\{\alpha_i^\pm\}_{i=1}^K$ be the coefficients (weights) associated with the point masses of the measures $\mu_\pm$; we shall use the notation $\alpha_i = \alpha_i^+ - \alpha_i^-$ for the coefficients associated with the measure $\mu_+ - \mu_-$. For any fixed $\mathbf{x}_i$, let (r,θ), $0<r$, $-\pi/2 \leqslant \theta < 3\pi/2$, be a polar coordinate system around $\mathbf{x}_i$, selected so that $\theta = -\pi/2$ lines up with the outward normal to Ω, and define

$$\phi_i(r,\theta) = \frac{1}{\pi}\left(-\log r + Dr\sin\theta \log r + Dr\cos\theta\left(\theta - \frac{\pi}{2}\right)\right). \tag{75}$$

Using "complex notation" ($z = r\cos\theta + \mathrm{i}r\sin\theta$) this definition is equivalent to

$$\phi_i(z) = -\frac{1}{\pi}\mathrm{Re}\left(\log z + Dz\left(\mathrm{i}\log z + \frac{\pi}{2}\right)\right),$$

with $\log z = \log r + \mathrm{i}\theta$.

Theorem 3. *Let the situation be as in Theorem 2. If $D=0$ suppose $d_\pm = 0$. The weights of the point-masses of the measures $\mu_\pm = \sum_{i=1}^K \alpha_i^\pm \delta_{\mathbf{x}_i}$ and $\mu_+ - \mu_- = \sum_{i=1}^K \alpha_i \delta_{\mathbf{x}_i}$ satisfy*

$$\frac{\alpha_i^2}{2\pi} = \left(\alpha_i^+ + \alpha_i^-\right), \quad 1 \leqslant i \leqslant K. \tag{76}$$

Moreover, the point-mass locations $\{\mathbf{x}_i\}_{i=1}^K$ satisfy the conditions

$$\frac{\partial}{\partial \tau_x}\left(w_0(x) - \alpha_i\phi_i(x)\right)|_{x=\mathbf{x}_i} = 0, \quad 1 \leqslant i \leqslant K, \tag{77}$$

where w_0 is the limit of the sequence w_{λ_n}, τ_x is the tangent to $\partial\Omega$ at the point x, and the functions ϕ_i, $1 \leqslant i \leqslant K$, are as defined in (75).

Proof. Let $\mathbb{H}$ denote the upper halfplane $\mathbb{H} = \{(y_1,y_2)\colon\ y_2>0\}$ and let $B(R) = B(0,R)$ denote the disk of radius R centered at the origin. Given a point-mass location $\mathbf{x}_i \in \partial\Omega$, let $\Phi : \mathbb{H} \cap B(R) \to \Omega$ be a local conformal straightening of the boundary. By appropriate selection of Φ and an orthonormal coordinate system we may arrange that $\Phi(0) = \mathbf{x}_i$ is also the origin, and that $\nabla\Phi(0) = I$. By selecting R sufficiently small we may assume that $\{\mathbf{x}_j\}_{j=1}^K \cap \Phi(\partial\mathbb{H} \cap B(R)) = \mathbf{x}_i = 0$. Defining the function $v_{\lambda_n} = w_{\lambda_n} \circ \Phi$

$$\begin{cases} \Delta v_{\lambda_n} = 0 & \text{in } \mathbb{H} \cap B(R), \\ \frac{\partial v_{\lambda_n}}{\partial \mathbf{n}_y} = h(y)Dv_{\lambda_n} + \lambda_n h(y)\sinh(v_{\lambda_n}) & \text{on } \partial\mathbb{H} \cap B(R), \end{cases} \tag{78}$$

where $h(y) = |\det(\nabla\Phi(y))|^{1/2}$. We have arranged that $h(0)=1$. Moreover, we can choose $h'(0) = h_{y_1}(0)$ arbitrarily: in complex notation, Φ can be written $\Phi(z) = z + \frac{\gamma}{2}z^2 + \mathrm{O}(z^3)$ with $z = y_1 + \mathrm{i}y_2$. When written like this, $h'(0)$ can be computed to be $\mathrm{Re}(\gamma)$. Now use the change of variable $\Psi(z) = \frac{1}{k}(e^{kz} - 1)$, for k real, to form a new conformal map $\widehat{\Phi} = \Phi \circ \Psi$ (which again locally "straightens" the boundary). $\widehat{\Phi}$ is conformal from a half-ball (which may be smaller) and can be expanded as $\widehat{\Phi}(z) = z + \frac{\hat{\gamma}}{2}z^2 + \mathrm{O}(z^3)$ where $\hat{\gamma} = \gamma + k$.

Choosing $0 < \epsilon < R$, we multiply (78) by $\frac{\partial}{\partial y_1}v_{\lambda_n}$. Let $\Omega_\epsilon = \mathbb{H} \cap B(\epsilon)$, and write the outward unit normal vector $\mathbf{n} = (n_1,n_2)$. Integration by parts gives

$$\frac{1}{2}\int_{\Omega_\epsilon} \frac{\partial}{\partial y_1}\left(\nabla v_{\lambda_n}\right)^2 \mathrm{d}y = \int_{\Omega_\epsilon} \nabla v_{\lambda_n} \nabla \frac{\partial}{\partial y_1} v_{\lambda_n}\, \mathrm{d}y$$

$$= \int_{\partial\mathbb{H}\cap B(\epsilon)} \left(Dhv_{\lambda_n}\frac{\partial}{\partial y_1}v_{\lambda_n} + \lambda_n h\sinh(v_{\lambda_n})\frac{\partial}{\partial y_1}v_{\lambda_n}\right)\mathrm{d}y_1 + \int_{\partial B(\epsilon)\cap\mathbb{H}} \frac{\partial}{\partial \mathbf{n}}v_{\lambda_n}\frac{\partial}{\partial y_1}v_{\lambda_n}\,\mathrm{d}\sigma_y$$
$$= \int_{\partial\mathbb{H}\cap B(\epsilon)} \left(\frac{D}{2}h\frac{\partial}{\partial y_1}(v_{\lambda_n})^2 + \lambda_n h\frac{\partial}{\partial y_1}\cosh(v_{\lambda_n})\right)\mathrm{d}y_1 + \int_{\partial B(\epsilon)\cap\mathbb{H}} \frac{\partial}{\partial \mathbf{n}}v_{\lambda_n}\frac{\partial}{\partial y_1}v_{\lambda_n}\,\mathrm{d}\sigma_y.$$

A second integration by parts with respect to y_1 gives

$$\frac{1}{2}\int_{\partial B(\epsilon)\cap\mathbb{H}} \left(\nabla v_{\lambda_n}\right)^2 n_1\,\mathrm{d}\sigma_y = -\frac{D}{2}\int_{\partial\mathbb{H}\cap B(\epsilon)} \frac{\partial h}{\partial y_1}v_{\lambda_n}^2\,\mathrm{d}y_1 - \lambda_n\int_{\partial\mathbb{H}\cap B(\epsilon)} \frac{\partial h}{\partial y_1}\cosh(v_{\lambda_n})\,\mathrm{d}y_1$$
$$+ \int_{\partial B(\epsilon)\cap\mathbb{H}} \frac{\partial}{\partial \mathbf{n}}v\frac{\partial}{\partial y_1}v\,\mathrm{d}\sigma_y + \left(\frac{D}{2}h(v_{\lambda_n})^2 + \lambda_n h\cosh(v_{\lambda_n})\right)\Bigg|_{y_1=-\epsilon,\,y_2=0}^{y_1=\epsilon,\,y_2=0}.$$

Now we can take this limit as $\lambda_n \to 0_+$, to get the identity

$$\frac{1}{2}\int_{\partial B(\epsilon)\cap\mathbb{H}} \left(\nabla v_0\right)^2 n_1\,\mathrm{d}\sigma_y = -\frac{D}{2}\int_{\partial\mathbb{H}\cap B(\epsilon)} \frac{\partial h}{\partial y_1}(v_0)^2\,\mathrm{d}y_1 - \left(\alpha_i^+ + \alpha_i^-\right)\left(\frac{\partial h}{\partial y_1}(0)\Big/h(0)\right)$$
$$+ \int_{\partial B(\epsilon)\cap\mathbb{H}} \frac{\partial}{\partial \mathbf{n}}v_0\frac{\partial}{\partial y_1}v_0\,\mathrm{d}\sigma_y + \left(\frac{D}{2}h(y)\big(v_0(y)\big)^2\right)\Bigg|_{y_1=-\epsilon,\,y_2=0}^{y_1=\epsilon,\,y_2=0}, \tag{79}$$

for $v_0 = \lim v_{\lambda_n}$. Here we have used that $\lambda_n h\cosh(v_{\lambda_n})$ converge uniformly to zero away from a finite set of points (which are disjoint from the points $(\pm\epsilon, 0)$, for $0 < \epsilon$ sufficiently small). We now decompose v_0 into a regular and a singular part by first performing this decomposition on w_0. Using "complex notation" ($z = x_1 + \mathrm{i}x_2$) we define

$$w_s = -\frac{\alpha_i}{\pi}\,\mathrm{Re}\left(\log z + Dz\left(\mathrm{i}\log z + \frac{\pi}{2}\right)\right),$$

where the logarithm $\log z$ is chosen to have its "cut" along the negative imaginary axis (which coincides with the outward normal to Ω). It is not hard to see that $w_r = w_0 - w_s$ satisfies $\Delta w_r = 0$ in $\Omega \cap B(\epsilon)$, with $\frac{\partial}{\partial \mathbf{n}}w_r - Dw_r \in H^{3/2-t}(\partial\Omega \cap B(\epsilon))$ for ϵ sufficiently small, and any $t > 0$. In other words

$$w_0 = w_s + w_r,$$

where w_r is in $H^{3-t}(\Omega \cap B(\epsilon))$ for any $t > 0$. In particular, w_r is in $C^{1,\beta}$ in an $\overline{\Omega}$ neighborhood of 0 for any $\beta < 1$. Define $v_s = w_s \circ \Phi$ and $v_r = w_r \circ \Phi$. We thus have the decomposition

$$v_0 = v_s + v_r$$

with v_r locally in $C^{1,\beta}$, for any $\beta < 1$, and

$$v_s(z) = w_s\big(\Phi(z)\big), \qquad \Phi(z) = z + \frac{a + \mathrm{i}b}{2}z^2 + S(z),$$

with $|S(z)| \leqslant C|z|^3$, $|\frac{\mathrm{d}}{\mathrm{d}z}S(z)| \leqslant C|z|^2$, and $a = h'(0)$. We easily calculate that

$$\log\Phi(z) + D\Phi(z)\left(\mathrm{i}\log\Phi(z) + \frac{\pi}{2}\right) = \log z + \frac{a + \mathrm{i}b}{2}z + Dz\left(\mathrm{i}\log z + \frac{\pi}{2}\right) + \mathrm{O}\big(|z^2\log z|\big),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\log\Phi(z) + D\Phi(z)\left(\mathrm{i}\log\Phi(z) + \frac{\pi}{2}\right)\right) = \frac{1}{z} + \frac{a + \mathrm{i}b}{2} + D\left(\mathrm{i} + \mathrm{i}\log z + \frac{\pi}{2}\right) + \mathrm{O}\big(|z\log z|\big),$$

and so in polar coordinates (with $z = y_1 + \mathrm{i}y_2 = r\cos\theta + \mathrm{i}r\sin\theta$)

$$v_s(y) = -\frac{\alpha_i}{\pi}\left(\log r + \frac{a}{2}r\cos\theta - \frac{b}{2}r\sin\theta - Dr\sin\theta\log r - Dr\cos\theta\left(\theta - \frac{\pi}{2}\right)\right) + \mathrm{O}\big(r^2|\log r|\big), \tag{80}$$

and

$$\nabla_y v_s(y) = -\frac{\alpha_i}{\pi}\left[\left(r^{-1}\cos\theta, r^{-1}\sin\theta\right) + \frac{1}{2}(a,-b) - D\left(\theta - \frac{\pi}{2}, 1+\log r\right)\right] + \mathrm{O}\big(r|\log r|\big), \tag{81}$$

with $a = h'(0)$. We now substitute the decomposition $v_0 = v_s + v_r$ into (79), and consider the case of infinitesimally small ϵ. It is not difficult to see that some of the terms are $o(1)$; indeed we obtain (remembering that $h(0) = 1$)

$$\begin{aligned}\frac{1}{2}\int_{\partial B(\epsilon)\cap\mathbb{H}} \big(|\nabla v_s|^2 + 2\nabla v_s \nabla v_r\big)n_1\,\mathrm{d}\sigma_y &= -\big(\alpha_i^+ + \alpha_i^-\big)\frac{\partial h}{\partial y_1}(0) + \left(\frac{D}{2}h(y)\big(v_s(y)^2 + 2v_s(y)v_r(y)\big)\right)\Bigg|_{y_1=-\epsilon,\, y_2=0}^{y_1=\epsilon,\, y_2=0}\\ &+ \int_{\partial B(\epsilon)\cap\mathbb{H}} \left(\frac{\partial}{\partial\mathbf{n}}v_s\frac{\partial}{\partial y_1}v_s + \frac{\partial}{\partial\mathbf{n}}v_s\frac{\partial}{\partial y_1}v_r + \frac{\partial}{\partial\mathbf{n}}v_r\frac{\partial}{\partial y_1}v_s\right)\mathrm{d}\sigma_y + \mathrm{o}(1).\end{aligned} \tag{82}$$

Using the formula (81) we can now compute the limits

$$\lim_{\epsilon\to 0}\frac{1}{2}\int_{\partial B(\epsilon)\cap\mathbb{H}} |\nabla v_s|^2 n_1\,\mathrm{d}\sigma_y = \frac{h'(0)}{4\pi}\alpha_i^2,$$

$$\lim_{\epsilon\to 0}\int_{\partial B(\epsilon)\cap\mathbb{H}} \nabla v_s\nabla v_r n_1\,\mathrm{d}\sigma_y = -\frac{\alpha_i}{2}\frac{\partial}{\partial y_1}v_r(0),$$

$$\lim_{\epsilon\to 0}\int_{\partial B(\epsilon)\cap\mathbb{H}} \frac{\partial}{\partial\mathbf{n}}v_s\frac{\partial}{\partial y_1}v_s\,\mathrm{d}\sigma_y = \frac{3h'(0)}{4\pi}\alpha_i^2,$$

$$\lim_{\epsilon\to 0}\int_{\partial B(\epsilon)\cap\mathbb{H}} \frac{\partial}{\partial\mathbf{n}}v_s\frac{\partial}{\partial y_1}v_r\,\mathrm{d}\sigma_y = -\alpha_i\frac{\partial}{\partial y_1}v_r(0),$$

$$\lim_{\epsilon\to 0}\int_{\partial B(\epsilon)\cap\mathbb{H}} \frac{\partial}{\partial\mathbf{n}}v_r\frac{\partial}{\partial y_1}v_s\,\mathrm{d}\sigma_y = -\frac{\alpha_i}{2}\frac{\partial}{\partial y_1}v_r(0).$$

The point-boundary terms in (82) (at $y_1 = \pm\epsilon$) converge to zero as $\epsilon \to 0$. For example, by invoking (80),

$$\begin{aligned}\big|v_s(\epsilon,0)^2 - v_s(-\epsilon,0)^2\big| &= \big|v_s(\epsilon,0) + v_s(-\epsilon,0)\big|\cdot\big|v_s(\epsilon,0) - v_s(-\epsilon,0)\big|\\ &\leqslant C\big|\log(\epsilon)\big|\cdot\big|v_s(\epsilon,0) - v_s(-\epsilon,0)\big| \leqslant C\big|\log(\epsilon)\big|\epsilon,\end{aligned}$$

so that

$$\begin{aligned}&\big|h(\epsilon,0)v_s(\epsilon,0)^2 - h(-\epsilon,0)v_s(-\epsilon,0)^2\big|\\ &\quad\leqslant \big|h(\epsilon,0)\big|\cdot\big|v_s(\epsilon,0)^2 - v_s(-\epsilon,0)^2\big| + \big|h(\epsilon,0) - h(-\epsilon,0)\big|\cdot\big|v_s(-\epsilon,0)^2\big|\\ &\quad\leqslant C\big|\log(\epsilon)\big|\epsilon + C\epsilon\big|\log(\epsilon)\big|^2.\end{aligned}$$

The term involving $v_s \cdot v_r$ may be estimated similarly. In all we have therefore reduced (82) to the following limiting identity

$$\frac{h'(0)}{4\pi}\alpha_i^2 - \frac{\alpha_i}{2}\frac{\partial}{\partial y_1}v_r(0) + \big(\alpha_i^+ + \alpha_i^-\big)h'(0) = \frac{3h'(0)}{4\pi}\alpha_i^2 - \frac{3}{2}\alpha_i\frac{\partial}{\partial y_1}v_r(0),$$

or

$$\alpha_i\frac{\partial}{\partial y_1}v_r(0) = h'(0)\left(\frac{\alpha_i^2}{2\pi} - \big(\alpha_i^+ + \alpha_i^-\big)\right).$$

Since $h'(0)$ can be chosen arbitrarily, and since $\frac{\partial}{\partial y_1}v_r(0) = \frac{\partial}{\partial\tau}w_r(0)$ is independent of $h'(0)$ this identity can only be satisfied if both sides are zero. It follows that

$$\frac{\alpha_i^2}{2\pi} = \alpha_i^+ + \alpha_i^-$$

and because $\alpha_i^+ + \alpha_i^- > 0$ on $S \cup M$ (so $\alpha_i \neq 0$) it also follows that

$$\frac{\partial}{\partial \tau} w_r(0) = \frac{\partial}{\partial y_1} v_r(0) = 0.$$

These are the desired identities. □

Remarks. For $\alpha_i^{\pm} \geqslant 0$, $\alpha_i^+ + \alpha_i^- > 0$ the equations

$$\frac{(\alpha_i^+ - \alpha_i^-)^2}{2\pi} = \frac{\alpha_i^2}{2\pi} = \alpha_i^+ + \alpha_i^-$$

imply that $|\alpha_i| = |\alpha_i^+ - \alpha_i^-| \geqslant 2\pi$. The case $|\alpha_i| = 2\pi$ arrives if and only if either α_i^+ or α_i^- is zero. We suspect (but are unable to prove) that this is the case for solutions satisfying the energy bounds established in Theorem 1. Whenever $\alpha_i = \pm 2\pi$, the corresponding equations

$$\frac{\partial}{\partial \tau_x} \big(w_0(x) \mp 2\pi \phi_i(x) \big) \bigg|_{x = \mathbf{x}_i} = 0,$$

may be used to determine the potential singularities. For $D = 0$, $d_+ = d_- = 0$, when $\phi_i(x) = -\frac{1}{\pi} \log |x - \mathbf{x}_i|$, this has been done in a few cases in [12]. Just like in [3] there is a strong relation between these equations and the stationarity of an appropriate "renormalized" energy – we refer the interested reader to [5] and [11] for more details.

Corollary 1. *Let the situation be as in Theorem* 2. *Let $S \cup M$ be the finite set of point-mass locations introduced in the case when D is a nonzero Steklov eigenvalue. Then $S \cup M = S$.*

Proof. Suppose D is a nonzero Steklov eigenvalue, and let $\mathbf{x}_i$ be a point in $S \cup M$ (a point-mass location for $\mu_+ + \mu_-$ with weight $\alpha_i^+ + \alpha_i^-$). From the previous remarks it follows that $\alpha_i^+ + \alpha_i^- \geqslant |\alpha_i^+ - \alpha_i^-| \geqslant 2\pi$, and so $\int_{\partial\Omega} \psi \, \mathrm{d}\nu = \int_{\partial\Omega} \psi \, \mathrm{d}(\mu_+ + \mu_-) \geqslant 2\pi$ for any $\psi \in C^0(\partial\Omega)$, $0 \leqslant \psi \leqslant 1$, with $\psi \equiv 1$ in a neighborhood of $\mathbf{x}_i$. It follows that $\mathbf{x}_i$ is a singular point relative to $\nu = \mu_+ + \mu_-$, i.e., $\mathbf{x}_i \in S$. □

3. A general discussion

So far we have focused on solutions to

$$\Delta u_\lambda = 0 \quad \text{in } \Omega, \qquad \frac{\partial u_\lambda}{\partial \mathbf{n}} = D u_\lambda + \lambda f(u_\lambda) \quad \text{on } \partial\Omega, \tag{83}$$

with $f(x)$ having exponential growth. In the first of the following two sections we briefly discuss, and provide some numerical results for, the case of different f. As we shall see, the existence structure described in Sections 2.1 and 2.2 (finitely many solutions for any λ in the interval $-D < \lambda < 0$ and infinitely many solutions for any positive λ) is preserved for a much larger class of odd, superlinear f. However, as we shall also see, the "finite point blow-up" observed when $\lambda \to 0_+$ is very much related to exponential growth. It was already noted that the solutions for $f = \sinh$, that we constructed variationally, and that we studied asymptotically in the previous section, do not necessarily represent all solutions for $\lambda > 0$. There may be other solutions whose energy (and "essential" H^1-norm squared: $\|w_\lambda\|^2_{H^1(\Omega)}$) grow faster than $\log \frac{1}{\lambda}$ as $\lambda \to 0_+$. In Section 3.2 we discuss two different cases, both with $f = \sinh$, when such "additional" solutions are present (one case has D negative, the other has a domain with a nontrivial topology). Associated to these "high energy" solutions are secondary bifurcations, as illustrated by some of our computational examples.

3.1. Other nonlinear fluxes

For the examples in this section, we let the domain Ω be the unit ball, $\Omega = B(0,1) \subset \mathbb{R}^2$. In order to compare with the numerical results for the exponential case, that are displayed in Fig. 1, we first calculate a bifurcation plot for the case when $f(u) = u + u^3$, and $D = 2$. The left frame in Fig. 4 shows the $H^1(\Omega)$-norm, and the right frame shows the energy E_λ as a function of λ for seven different solutions to the boundary value problem (83). The Steklov eigenvalues

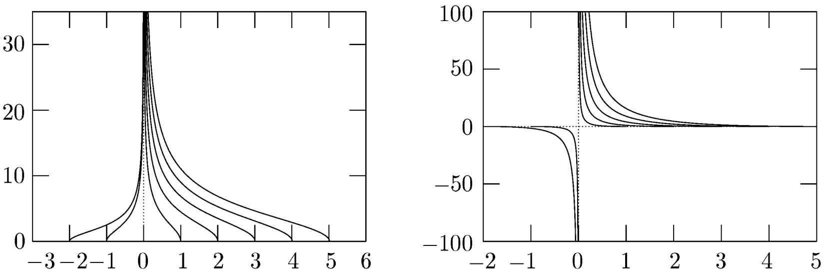

Fig. 4. Left frame: the $H^1(\Omega)$-norm as a function of λ for different solutions to (83) with $f(u) = u + u^3$. Right frame: the energies E_λ for the same solutions.

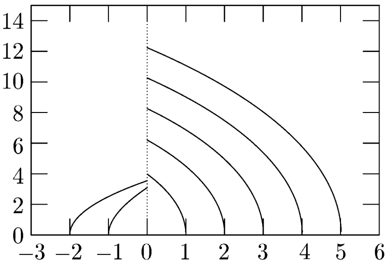

Fig. 5. The $H^1(\Omega)$-norm of $v_\lambda = \lambda^{1/2} u_\lambda$ as a function of λ for different solutions to (83), $f(u) = u + u^3$.

consist of all integers $\geqslant 0$. Fig. 4 clearly indicates an existence structure much like for the exponential case, namely: finitely many solutions for any λ in the interval $-D < \lambda < 0$, and infinitely many solutions for any positive λ. The nature of the blow-up near $\lambda = 0$ is, however, different from what we witnessed in the exponential case. To support this assertion consider the following formal scaling argument. Define $v_\lambda = \lambda^{1/2} u_\lambda$, then

$$\Delta v_\lambda = 0 \quad \text{in } \Omega, \qquad \frac{\partial v_\lambda}{\partial \mathbf{n}} = (D + \lambda) v_\lambda + v_\lambda^3 \quad \text{on } \partial\Omega.$$

It is to be expected that any of the solution "branches" for this boundary value problem as $\lambda \to 0$ approaches one of the infinitely many solutions to the boundary value problem

$$\Delta v_0 = 0 \quad \text{in } \Omega, \qquad \frac{\partial v_0}{\partial \mathbf{n}} = D v_0 + v_0^3 \quad \text{on } \partial\Omega.$$

That this "limiting behavior" is indeed what transpires is seen from the plots of the $H^1(\Omega)$ norms of the solutions v_λ, displayed in Fig. 5. The solutions $u_\lambda = \lambda^{-1/2} v_\lambda$ therefore blow up almost everywhere, as do the nonlinear flux terms $\lambda(u_\lambda + u_\lambda^3)$ (whether $\lambda \to 0_-$, or $\lambda \to 0_+$). Similar behavior would be found for any odd, superlinear polynomial. We now briefly turn to the case of a bounded f, and the case of an asymptotically linear f. In both of these cases we take $D = 0$. Consider first an $f(x)$ that is odd and bounded, and let $\lambda > 0$ be fixed. If we require that f be nondecreasing, then it is not hard to see that there exists a constant $C = C(f, \lambda)$ such that $\|u_\lambda\|_{H^1(\Omega)} \leqslant C$. If we additionally require that f' be bounded, then there exists a $\lambda^* > 0$ such that the only solution for $\lambda \neq 0$ on the half-axis $\lambda < \lambda^*$ is the zero solution. For more details, see [11]. As a specific example we take $f(u) = \arctan(u)$. The left frame in Fig. 6 shows a plot of the $H^1(\Omega)$-norm for "the first" five nontrivial solutions to the corresponding boundary value problem. Secondly consider an f of the form $f(u) = cu + g(u)$, where the constant c is positive, and where the function $g(u)$ is odd and bounded. Here we can again prove a bound on the $H^1(\Omega)$-norm of u_λ, for any $\lambda > 0$, such that $c\lambda$ is not a positive integer. The bound deteriorates as $c\lambda$ approaches any positive integer. We also illustrate this with a bifurcation plot. As a specific example we take $f(u) = \frac{1}{2}(u + \sin(u))$. Note that $f'(0) = 1$, and so the bifurcation points from the trivial solution (the Steklov eigenvalues for the "linearized" boundary value problem) remain the nonnegative integers. The right frame in Fig. 6 shows a plot of the $H^1(\Omega)$-norm for "the first" five nontrivial solutions to the corresponding boundary value problem. The plot exhibits the generic behavior, but it has an interesting additional feature (which we believe is associated with the oscillatory behavior of $\sin(u)$) namely: there seem to be an infinite number of solutions to the problem when $c\lambda = \frac{\lambda}{2}$ is a positive integer. This feature is somewhat reminiscent of a feature found in connection with the so-called Gelfand–Liouville Problem (cf. [2] and [8]).

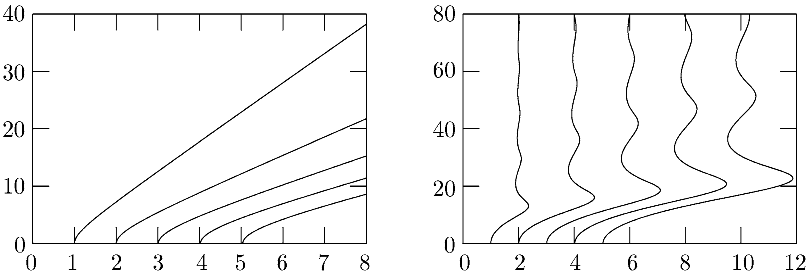

Fig. 6. The $H^1(\Omega)$-norm of "the first" five solutions to (83) with $f(u) = \arctan(u)$ (left frame) and $f(u) = \frac{1}{2}(u + \sin(u))$ (right frame). In both cases $D = 0$.

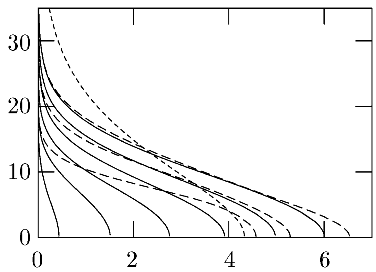

Fig. 7. $H^1(\Omega)$-norms of solutions to (1), with $D = 0$ on the annulus, $\Omega = B(0,1) \setminus \overline{B(0,1/2)}$. Only primary bifurcations are plotted.

3.2. *High energy solutions, secondary bifurcations*

In [12] we provided numerical examples of solutions to (1) (i.e., (83) with $f = \sinh$) on simply connected domains, different from a disk, in order to display families of solutions with special properties (we were looking for solutions for $D = 0$, whose flux converges to a measure with a nonzero regular part). To obtain a more "complicated" bifurcation diagram than that shown in Fig. 1 (including a family of solutions whose $H^1(\Omega)$-norm grows faster than $(\log \frac{1}{\lambda})^{1/2}$ as $\lambda \to 0_+$) we now consider (1) on a domain Ω in the shape of the annulus, $\Omega = B(0,1) \setminus \overline{B(0,1/2)}$. Fig. 7 shows the solutions bifurcating off the zero solution at the (first 10) positive Steklov eigenvalues, for $D = 0$. As usual, we plot the $H^1(\Omega)$-norm of the solutions versus λ. The solution structure at first sight appears much more complicated than that seen in the bifurcation plot for the unit disk (or any simply connected domain, for that matter). In order to make the solution structure more transparent, it is convenient to divide the solutions into three classes. One class consists of a single branch only, namely the nonconstant radial solution; it is given by

$$u_\lambda = a_\lambda \log r + b_\lambda, \quad \text{with the coefficients } a_\lambda \text{ and } b_\lambda \text{ satisfying the equations}$$
$$a_\lambda = \lambda \sinh(b_\lambda), \qquad -2a_\lambda = \lambda \sinh(-a_\lambda \log 2 + b_\lambda).$$

This branch bifurcates from the zero solution at the Steklov eigenvalue $D_6 = 3/\log 2 \approx 4.328085$. This radial solution has an $H^1(\Omega)$-norm that blows up faster than the other solutions (at a rate of $\log \frac{1}{\lambda}$ as $\lambda \to 0_+$). The remaining branches, bifurcating from the zero solution at Steklov eigenvalues with nonradial eigenfunctions, can conveniently be divided into two separate classes, A and B.

The separated energy plots look a lot cleaner, (there are no more intersections) but more importantly this separation appears to distinguish the solutions according to the form of the possible limiting measures $\mu = \lim_{\lambda_n \to 0} \partial u_{\lambda_n}/\partial \mathbf{n}$. For $\{u_{\lambda_n}\}$ on any of the branches in class A, we have

$$\mu = 2\pi \sum_{i=1}^{2N} (-1)^{i-1} \delta_{\sigma_i},$$

and for $\{u_{\lambda_n}\}$ on any of the branches in class B, we have

$$\mu = 2\pi \sum_{i=1}^{2N} (-1)^{i-1} \delta_{\sigma_i} - 2\pi \sum_{i=1}^{2N} (-1)^{i-1} \delta_{\frac{1}{2}\sigma_i}.$$

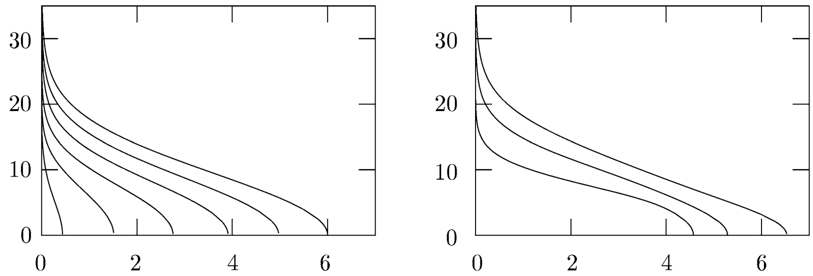

Fig. 8. Primary bifurcations on the annulus. The left frame shows the $H^1(\Omega)$-norms associated with the (first 6) solutions we have designated class A. The right frame shows the $H^1(\Omega)$-norms associated with the (first 3) solutions we have designated class B.

Here N may be any positive integer and $\{\sigma_i\}_{i=1}^{2N}$ may be any set of $2N$ distinct, equispaced points on the unit circle. Another reasonable way to think about this classification of families of solutions is through eigenvalue and eigenfunction data, rather than the limiting measure. With this in mind, we label each family by a pair (γ, ϕ) where ϕ is a Steklov eigenfunction associated with the eigenvalue γ, at which the family bifurcates from the 0 solution. For the annulus with outer radius 1 and inner radius $1/R$ for $R > 1$, these Steklov eigenfunctions and values can be found by a separation of variables approach. There are two radial eigenfunctions corresponding to the simple eigenvalues 0 and $(R+1)/\log R$, respectively. As far as nonradial eigenfunctions are concerned, they are all, modulo a rotation, of the form

$$u_m = a_m r^m \sin(m\theta) + b_m r^{-m} \sin(m\theta),$$

for some $m \geqslant 1$. The Neumann boundary conditions give rise to the two equations

$$\begin{aligned} &m a_m - m b_m = \gamma(a_m + b_m), \quad \text{and} \\ &m b_m R^{m+1} - m a_m R^{-m+1} = \gamma\big(b_m R^m + a_m R^{-m}\big). \end{aligned}$$

For each fixed $m \geqslant 1$ this 2×2 linear system possesses nontrivial solutions (a, b) for two different values of γ. For each fixed $m \geqslant 1$ we thus have two eigenvalue-eigenfunction pairs (γ_m^1, ϕ_m^1) and (γ_m^2, ϕ_m^2). Suppose the eigenvalues are ordered such that $\gamma_m^1 < \gamma_m^2$. Then the branch of solutions labelled by the pair (γ_m^1, ϕ_m^1) falls into the classification A, and the branch of solutions labeled by (γ_m^2, ϕ_m^2) falls into classification B. The eigenfunction ϕ_m^1 achieves its extremal values on the outer circle, whereas ϕ_m^2 achieves its extremal values on the inner circle. For each family of solutions the integer m coincides with the integer N, that appears in the limiting measure, i.e., it determines the number of δ-functions that emerge.

The branch of nonconstant radial solutions (whose $H^1(\Omega)$-norm blows up at the rate $\log\frac{1}{\lambda}$ as $\lambda \to 0_+$) is also interesting from the point of view of higher order bifurcations. The simplicity of this family of solutions means we can easily apply a separation of variables argument to identify an infinite (countable) set of λ-values at which we should expect secondary bifurcations. For details on this calculation, see [11]. The presence of these secondary (as well as higher order) bifurcations can be verified by computational experiments, such as that provided in Fig. 9. Based on our computational experience it appears that the limiting measures, proceeding directly along the secondary bifurcations, for the annulus $\Omega = B(0,1) \setminus \overline{B(0,1/2)}$ take the form

$$\mu = \lim_{\lambda \to 0_+} \frac{\partial u_\lambda}{\partial \mathbf{n}} = 2\pi \sum_{i=1}^{N} \delta_{\sigma_i} - 2\pi \sum_{i=1}^{N} \delta_{\frac{1}{2}\sigma_i},$$

where N is an arbitrary positive integer, and $\{\sigma_i\}_{i=1}^{N}$ is a set of N distinct, equispaced points on the unit circle. It appears that the secondary bifurcations have $H^1(\Omega)$-norms that blow up like $(\log\frac{1}{\lambda})^{1/2}$ as $\lambda \to 0_+$.

Returning to the domain $\Omega = B(0,1)$, there is a situation where we may apply a similar separation of variables argument, to predict an infinite number of secondary bifurcations. If we let $D = -1$, then we have one family of solutions to (1) that are constant, for $0 < \lambda < 1$. The $H^1(\Omega)$-norm of this family of solutions also blows up at the rate of $\log\frac{1}{\lambda}$ as $\lambda \to 0_+$. Fig. 10 shows some of the bifurcation diagram in this situation.

With the appearance of these instances of secondary bifurcations a natural question arises: do all (or most) families have secondary bifurcations, or do such bifurcations only emanate from families of solutions whose $H^1(\Omega)$-norm

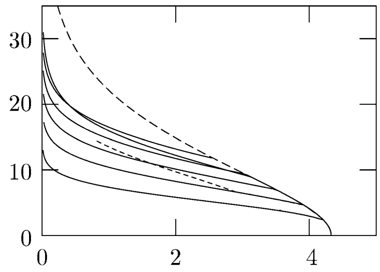

Fig. 9. The radial solution, with a collection of secondary and, what appears to be, one tertiary bifurcation.

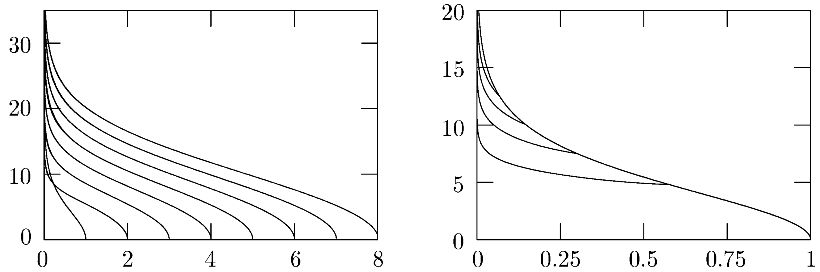

Fig. 10. Two views of solutions to (1) for $D = -1$ on the unit ball. The left frame shows the first 8 primary bifurcations from the zero solution. The right frame shows the constant solution with its first 4 secondary bifurcations.

blows up faster than $(\log \frac{1}{\lambda})^{1/2}$? The answer to the second question appears to be no (in Fig. 9, the tertiary bifurcation developing seems to disprove this). However, at the same time it appears that a large number of families possess no secondary bifurcations. We examine this phenomenon in more detail in two different situations. We first note that a necessary condition for a bifurcation at a solution U_λ, is the existence of a nonzero function h that solves

$$\begin{cases} \Delta h = 0 & \text{in } \Omega, \\ \frac{\partial h}{\partial \mathbf{n}} = (D + \lambda \cosh(U_\lambda))h & \text{on } \partial\Omega. \end{cases} \tag{84}$$

Let $V_\lambda(x) \in C^\infty(\partial\Omega)$ be the function defined by $V_\lambda(x) = D + \lambda \cosh(U_\lambda)$. Given $\lambda > 0$ we define the set of "generalized real eigenvalues" to be the values $\{\gamma(\lambda)\} \subset \mathbb{R}$ for which the equation

$$\begin{cases} \Delta h = 0 & \text{in } \Omega, \\ \frac{\partial h}{\partial \mathbf{n}} = \gamma V_\lambda(x)h & \text{on } \partial\Omega \end{cases} \tag{85}$$

admits a nonzero solution. Clearly, for a given $\lambda > 0$, the set of generalized real eigenvalues includes the value 1, if and only the necessary condition (84) for bifurcation is satisfied.

We now focus on the possibility of finding secondary bifurcations in two different situations pertaining to the unit disk. In the first situation we take U_λ to be the constant solution to (1) with $D = -1$. This is a simple case. Experimentally we know this family of solutions admits bifurcations – as evidenced by the right plot in Fig. 10. But it is instructive to see the correlation with the presence of $\gamma = 1$ as a generalized real eigenvalue for (85). The curves in Fig. 11 depict the first five nonzero, generalized real eigenvalues as a function of $\lambda \in (0, 1)$ (the smallest generalized real eigenvalue is 0). Each of these five curves seem to intersect the $\gamma = 1$ line at a value of λ, close to which we have previously noticed a secondary bifurcation appear from the constant solution. In this case it is actually not hard to show that each curve gives rise to exactly one intersection with the line $\gamma = 1$, resulting in an infinite (countable) set of potential secondary bifurcations. All the nonzero, generalized real eigenvalues converge to ∞ as $\lambda \to 1_-$ and they all converge to zero as $\lambda \to 0_+$.

For the second situation we consider one of the families of exact solutions to (1), constructed in [5]. These solutions pertain to $D = 0$, $\Omega = B(0, 1)$; the particular family we consider bifurcates from the zero solution at $\lambda = 1$, and it has the form

$$U_\lambda(x) = 2\log\left|x - \rho(\lambda)\sigma_1\right| - 2\log\left|x - \rho(\lambda)\sigma_2\right|, \tag{86}$$

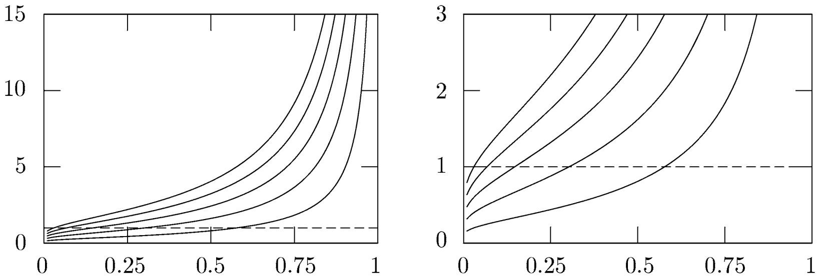

Fig. 11. Two views of the smallest five generalized real eigenvalues, as calculated from (85) with $D = -1$ and U_λ in the form of the constant solution. We expect a bifurcation whenever a curve of generalized eigenvalues crosses $\gamma = 1$. This expectation is confirmed by comparison with the bifurcation plot, shown in Fig. 10.

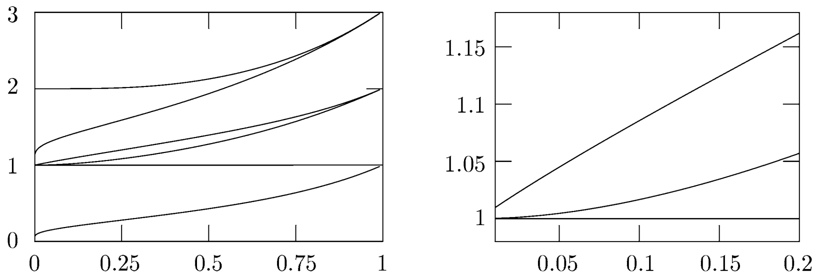

Fig. 12. Two views of the smallest five generalized real eigenvalues, as calculated from (85) with $D = 0$, using the solutions U_λ, given by (86).

with $\rho(\lambda) = (\frac{1+\lambda}{1-\lambda})^{1/2}$ and $\{\sigma_1, \sigma_2\} = \{(1,0), (-1,0)\}$ (or any two diametrically opposed points on the boundary of the unit disk). The fact that we have solutions in closed form is helpful when calculating the generalized eigenvalues. These solutions happen to provide the simplest example of a family, for which the flux converges to a pure sum of δ-functions. Fig. 12 shows the five smallest nonzero generalized real eigenvalues for the problem (85) in the case when U_λ is given by (86). We see a marked difference, when compared to Fig. 11. Two of the generalized eigenvalues start at the value 1 (when $\lambda = 1$). One of these remains at 1 for all values $0 < \lambda < 1$, the other clearly falls below. The third generalized real eigenvalue approaches 1 from above as λ nears zero, whereas all the others seem to stay strictly above 1. That the nonzero generalized real eigenvalues converge in pairs to the set of positive integers as $\lambda \to 1_-$ is consistent with the fact that the positive integers are (double) Steklov eigenvalues for the problem

$$\Delta\phi = 0 \quad \text{in } \Omega, \qquad \frac{\partial \phi}{\partial \mathbf{n}} = \gamma\phi \quad \text{on } \partial\Omega.$$

The fact that one of the generalized eigenvalues remains 1 for all values $0 < \lambda < 1$ does not mean that we should expect all elements of the family of solutions given by (86) to be a "true" bifurcation point, rather it should be seen as a reflection of the symmetry of the set of solutions. An examination of the eigenfunctions confirms what we should expect: for any given $\lambda \in (0,1)$ an eigenfunction corresponding to $\gamma = 1$ is $h = \frac{\partial}{\partial \tau} U_\lambda$. The form of this eigenfunction is consistent with the fact that $u_\lambda \circ R(\theta)$ is also a solution to the boundary value problem (1) if u_λ is a solution and $R(\theta)$ is any rotation. The fact that no other generalized eigenvalue ever equals 1 indicates, that we should expect no points of "true" secondary bifurcation from the solution given by (86). We have similar expectations for all the solutions constructed in [5]. Based on our numerical experience we are also inclined to believe that the solution classes denoted A and B (see Fig. 8) which we encountered in connection with the annulus, have no associated secondary bifurcations.

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References

[1] T. Ando, A.B. Fowler, F. Stern, Electronic properties of two-dimensional system, Rev. Modern Phys. 54 (1982) 437–621.
[2] J. Bebernes, D. Eberly, Mathematical Problems from Combustion Theory, Appl. Math. Sci., vol. 83, Springer-Verlag, Berlin, 1989.
[3] F. Bethuel, H. Brezis, F. Helein, Ginzburg–Landau Vortices, Birkhäuser, Boston, 1994.
[4] H. Brezis, F. Merle, Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)\,\mathrm{e}^u$ in two dimensions, Comm. Partial Differential Equations 16 (1991) 1223–1253.
[5] K. Bryan, M. Vogelius, Singular solutions to a nonlinear elliptic boundary value problem originating from corrosion modeling, Quart. Appl. Math. 60 (2002) 675–694.
[6] J. Davila, A linear elliptic equation with a nonlinear boundary condition, Preprint, Rutgers University, 2001.
[7] J. Deconinck, Current Distributions and Electrode Shape Changes in Electrochemical Systems, Lecture Notes in Engrg., vol. 75, Springer-Verlag, Berlin, 1992.
[8] J. Jacobsen, A globalization of the Implicit Function Theorem with applications to nonlinear elliptic equations, Contemp. Math. 289 (2001) 249–272.
[9] O. Kavian, Introduction à la théorie des points critiques et applications aux problèmes elliptiques, Math. Appl., vol. 13, Springer-Verlag, Berlin, 1993.
[10] O. Kavian, M. Vogelius, On the existence and "blow up" of solutions to a two-dimensional nonlinear boundary-value problem arising in corrosion modelling, Proc. Roy. Soc. Edinburgh Sect. A 133 (2003) 119–149. Corrigendum to same, Proc. Roy. Soc. Edinburgh Sect. A 133 (2003) 729–730.
[11] K. Medville, Ph.D. Thesis, Rutgers University, 2004.
[12] K. Medville, M.S. Vogelius, Blow-up behavior of planar harmonic functions satisfying a certain exponential Neumann boundary condition, SIAM J. Math. Anal. 36 (2005) 1772–1806.
[13] K. Nagasaki, T. Suzuki, Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearities, Asymptotic Anal. 3 (1990) 173–188.
[14] P. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Regional Conf. Ser. in Math., vol. 65, Amer. Math. Soc., Providence, RI, 1986.
[15] S.E. Shamma, Asymptotic behavior of Stekloff eigenvalues and eigenfunctions, SIAM J. Appl. Math. 20 (1971) 482–490.
[16] M. Struwe, Variational Methods, Ergeb. Math. Grenzgeb., vol. 34, Springer-Verlag, Berlin, 1996.
[17] M. Vogelius, J.-M. Xu, A nonlinear elliptic boundary value problem related to corrosion modeling, Quart. Appl. Math. 56 (1998) 479–505.

Further reading

[18] J.L. Moseley, Asymptotic solutions for a Dirichlet problem with an exponential nonlinearity, SIAM J. Math. Anal. 14 (1983) 719–735.
[19] V.H. Weston, On the asymptotic solution of a partial differential equation with an exponential nonlinearity, SIAM J. Math. Anal. 9 (1978) 1030–1053.