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A relaxation process for bifunctionals of displacement-Young measure state variables: A model of multi-material with micro-structured strong interface

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Abstract

The gradient displacement field of a micro-structured strong interface of a three-dimensional multi-material is regarded as a gradient-Young measure so that the stored strain energy of the material is defined as a bifunctional of displacement-Young measure state variables. We propose a new model by computing a suitable variational limit of this bifunctional when the thickness and the stiffness of the strong material are of order ε and $\frac{1}{\varepsilon}$ respectively. The stored strain energy functional associated with the model in pure displacements living in a Sobolev space is obtained as the marginal map of the limit bifunctional. We also obtain a new asymptotic formulation in terms of Young measure state variable when considering the other marginal map.

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1. Introduction

In [1] and [10] a variational model of multi-material with a very rigid interface is obtained by identifying the classical Γ-limit of the stored strain energy functional when the magnitude order ε of the interface thickness goes to zero and the stiffness of the material occupying the interface grows as $\frac{1}{\varepsilon}$. In this paper we assume that the thin structure is occupied by a material which undergoes reversible solid/solid phase transformation, while the strain of the soft material occupying the complementary set of the layer can be high. As the main mechanical features are high strain of the soft material and oscillations of gradient displacement in the layer of hight stiffness, we deal with

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the asymptotic analysis of the problem by means of a new variational convergence where competing objects are pairs (u,ν) of displacements/gradient Young measures state variables. The advantage of using the second argument lies in the fact that ν encodes the gradient oscillations of u restricted to the layer. We obtain a new formulation $(\bar{u},\bar{\nu}) \in \operatorname{arg\,min} \mathcal{F}(u,\nu) - L(u)$ of the problem by identifying the limit $(u,\nu) \mapsto \mathcal{F}(u,\nu)$ of the stored strain energy functional $u \mapsto F_\varepsilon(u)$ rewritten as a bifunctional $(u,\mu) \mapsto \mathcal{F}_\varepsilon(u,\mu)$ (we write $L(u)$ for the exterior loading).

Let Ω be the reference configuration occupied by the material and $S \times]0,\varepsilon[$ the thin inclusion. The limit energy functional of F_ε obtained in [10] is of the form $u \mapsto F(u) := \int_\Omega Qf(\nabla u)\,dx + \int_S Qg_0(\hat{\nabla}\gamma_S(u))\,d\hat{x}$ for all Sobolev-functions u with smooth trace $\gamma_S(u)$ on the two-dimensional interface S, where Qf and Qg_0 denote the quasiconvexifications of f and g_0. As a straightforward consequence of our formulation we find the stored strain energy F as to be the marginal map $u \mapsto \inf_\nu \mathcal{F}(u,\nu)$ of the energy functional $\mathcal{F}$ when the Young measure $\nu = \nu_{\hat{x}} \otimes d\hat{x}$ is then regarded as an internal state variable. By comparing the two variational formulations $\bar{u} \in \operatorname{arg\,min}(F - L(u))$ and $(\bar{u},\bar{\nu}) \in \operatorname{arg\,min}(\mathcal{F} - L(u))$, we obtain an integral representation with respect to the probability measure $\bar{\nu}_{\hat{x}}$ on the set $\mathbf{M}^{3\times 2}$ of 3×2-matrices, of the significant macroscopic quantities $\hat{\nabla}\gamma_S(\bar{u})$ and $Qg_0(\hat{\nabla}\gamma_S(\bar{u}))$. In some sense we may think the variable $\bar{\nu}$ as the microscopic description of $\hat{\nabla}\gamma_S(\bar{u})$ and $Qg_0(\hat{\nabla}\gamma_S(\bar{u}))$.

Another way for obtaining a variational formulation of the problem is to consider the marginal map G of $\mathcal{F} - L$ when the displacement field u is now regarded as an internal variable. We show that the energy functional $\nu \mapsto G(\nu) := \inf_u(\mathcal{F}(u,\nu) - L(u))$ is a variational limit of $\mu \mapsto \inf_u(\mathcal{F}_\varepsilon(u,\mu) - L(u))$ so that $\bar{\nu} \in \operatorname{arg\,min} G$ is a new formulation of the problem in terms of gradient Young measures parametrized on the interface S. By comparing it with the formulation $(\bar{u},\bar{\nu}) \in \operatorname{arg\,min}(\mathcal{F} - L)$, we show that $\bar{u}$ is a solution of the nonlinear Dirichlet problem $\min(\int_{\Omega\setminus S} Qf(\nabla u)\,dx - L(u))$ subjected to the boundary condition $\bar{u}(\hat{x}) = \hat{\nabla}^{-1}(\int_{\mathbf{M}^{3\times 2}} \hat{\lambda}\,d\bar{\nu}_{\hat{x}})$ on the interface S. Consequently, one may think the surface energy $\int_S Qg_0(\hat{\nabla}\gamma_S(u))\,d\hat{x}$ obtained in [10] as a relaxation of the boundary condition above (notice the analogy with the relaxation of boundary conditions in BV-spaces).

This paper illustrates, in the modeling of multi-materials, the following general strategy: in order to capture various convergence phenomena on minimizing sequences regarded as Sobolev variables of a problem $(\mathcal{P}_\varepsilon)$, one defines a suitable measure state variable μ connected to the Sobolev state variable u (Young measure, concentration measure...), an energy bifunctional $(u,\mu) \mapsto \mathcal{F}_\varepsilon$ modeling $(\mathcal{P}_\varepsilon)$, a suitable variational convergence process, and identify its limit $\mathcal{F}$. We recover the limit energy in terms of Sobolev variables as the marginal functional of $\mathcal{F}$ when μ is regarded as an internal variable. This idea as already been used in the framework of relaxation theory for a reduction dimension problem in [12], and for control problems in [22]. We also obtain a new asymptotic formulation in terms of measure state variable by considering the marginal functional of $\mathcal{F}$ when u is an internal variable.

The paper is organized as follows. Section 2 provides the abstract setting of the problem. We introduce a suitable variational convergence for bifunctionals and establish the variational convergence of their marginal maps. In Section 3, after a brief exposition of the mechanical setting, according to Section 2, we set up notation and terminology for the problem and prove the variational convergence of the bifunctional $\mathcal{F}_\varepsilon$ to the bifunctional $\mathcal{F}$ (Theorem 2). The two last sections are devoted to the asymptotic analysis of the marginal maps and their consequences (Theorem 3, Corollaries 1 and 2). For making the paper as self contained as possible, we repeat the material from Young measures without proofs in Appendix A.

2. A variational convergence of sequences of functionals defined on topological product spaces

2.1. The abstract setting

Given three first countable topological spaces, X, Y, $\hat{Y}$, a map $\mathcal{A}$ from Y to $\hat{Y}$, and extended real-valued functionals $\mathcal{F}_n : X \times Y \to \mathbb{R} \cup \{+\infty\}$, $\mathcal{F} : X \times \hat{Y} \to \mathbb{R} \cup \{+\infty\}$, our purpose is to define a variational convergence of the sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ toward the functional $\mathcal{F}$ so that, under a suitable convergence process associated with $\mathcal{A}$ and some compactness hypotheses, the following implication holds true when $n \to +\infty$:

$$\mathcal{F}_n \to \mathcal{F} \Rightarrow \begin{cases} \inf_{X\times Y} \mathcal{F}_n \to \min_{X\times \hat{Y}} \mathcal{F}; \\ \operatorname{arg\,min}_{X\times Y} \mathcal{F}_n \ni (x_n, y_n) \to (x, \hat{y}) \in \operatorname{arg\,min}_{X\times\hat{Y}} \mathcal{F} \text{ at least for a subsequence.} \end{cases}$$

We begin by introducing a new weak notion of convergence between elements of Y and $\hat{Y}$, and next, between elements of $X \times Y$ and $X \times \hat{Y}$.

Definition 1. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in Y and $\hat{y}$ in $\hat{Y}$. We say that y_n $\mathcal{A}$-converges to $\hat{y}$ and we write

$$y_n \overset{\mathcal{A}}{\rightharpoonup} \hat{y}$$

iff there exists y in Y such that $y_n \to y$ and $\hat{y} = \mathcal{A}(y)$.

Let $((x_n, y_n))_{n\in\mathbb{N}}$ be a sequence in $X \times Y$ and $(x, \hat{y})$ in $X \times \hat{Y}$. We say that (x_n, y_n) converges to $(x, \hat{y})$ and we write

$$(x_n, y_n) \overset{I\times\mathcal{A}}{\rightharpoonup} (x, \hat{y})$$

iff x_n converges to x, and y_n $\mathcal{A}$-converges to $\hat{y}$.

We introduce now the variational convergence associated with the previous convergence.

Definition 2. We say that $\mathcal{F}_n$ $\Gamma_{X,Y,\hat{Y}}$-converges to $\mathcal{F}$ and we write

$$\mathcal{F}_n \overset{\Gamma_{X,Y,\hat{Y}}}{\longrightarrow} \mathcal{F}$$

iff for all $(x, \hat{y})$ in $X \times \hat{Y}$, both following assertions hold:

(i) $\forall (x_n, y_n) \in X \times Y$ s.t. $(x_n, y_n) \overset{I\times\mathcal{A}}{\rightharpoonup} (x, \hat{y})$, $\mathcal{F}(x, \hat{y}) \leqslant \liminf_{n\to+\infty} \mathcal{F}_n(x_n, y_n)$,

(ii) $\exists (x_n, y_n) \in X \times Y$ s.t. $(x_n, y_n) \overset{I\times\mathcal{A}}{\rightharpoonup} (x, \hat{y})$, $\mathcal{F}(x, \hat{y}) \geqslant \limsup_{n\to+\infty} \mathcal{F}_n(x_n, y_n)$.

Note that this convergence is closely related to the Γ-convergence. When $X = \{0\}$ or, which is equivalent, when $\mathcal{F}_n$ and $\mathcal{F}$ do not depend on x, we denote it briefly by $\Gamma_{Y,\hat{Y}}$. When $Y = \{0\}$ and $\hat{Y} = \{\hat{0}\}$ i.e. when $\mathcal{F}_n$ and $\mathcal{F}$ do not depend on y and $\hat{y}$, we will write it simply Γ_X and our definition agrees with the classical Γ-convergence (for more details on Γ-convergence, see [5,16]). Note also that when $Y = \hat{Y}$ and $\mathcal{A}$ is the identity map, $\Gamma_{X,Y,Y}$ is the $\Gamma_{X\times Y}$-convergence. The proposition below expresses the variational nature of the $\Gamma_{X,Y,\hat{Y}}$-convergence.

Proposition 1. *Let us assume that $(\mathcal{F}_n)_{n\in\mathbb{N}}$ $\Gamma_{X,Y,\hat{Y}}$-converges to $\mathcal{F}$ and let $((x_n, y_n))_{n\in\mathbb{N}}$ be a sequence of $X \times Y$ satisfying*

$$\mathcal{F}_n(x_n, y_n) \leqslant \inf_{(x,y)\in X\times Y} \mathcal{F}_n(x, y) + \frac{1}{n}.$$

Assume furthermore that $\{(x_n, y_n)\colon n \in \mathbb{N}\}$ is relatively compact for the convergence $\overset{I\times\mathcal{A}}{\rightharpoonup}$ defined above. Then any cluster point $(\bar{x}, \bar{\hat{y}}) \in X \times \hat{Y}$ is a minimizer of $\mathcal{F}$ and

$$\lim_{n\to+\infty} \inf\bigl\{\mathcal{F}_n(x, y)\colon (x, y) \in X \times Y\bigr\} = \mathcal{F}(\bar{x}, \bar{\hat{y}}).$$

Proof. The proof is similar to that of Theorem 12.1.1 in [6] and left to the reader. □

2.2. The variational convergence of marginal functionals

Let us consider the following marginal functionals $F_n, F : X \to \mathbb{R} \cup \{+\infty\}$, $G_n : Y \to \mathbb{R} \cup \{+\infty\}$, and $G : \hat{Y} \to \mathbb{R} \cup \{+\infty\}$ defined by:

$$F_n(x) = \inf_{y\in Y} \mathcal{F}_n(x, y), \qquad F(x) = \inf_{\hat{y}\in\hat{Y}} \mathcal{F}(x, \hat{y}),$$
$$G_n(y) = \inf_{x\in X} \mathcal{F}_n(x, y), \qquad G(\hat{y}) = \inf_{x\in X} \mathcal{F}(x, \hat{y}).$$

The variational convergence of the functionals $\mathcal{F}_n$ yields the variational convergence of their marginal maps, precisely:

Theorem 1. *Let us assume that* $(\mathcal{F}_n)_{n\in\mathbb{N}}$ $\Gamma_{X,Y,\hat{Y}}$*-converges to* $\mathcal{F}$*. Assume furthermore that the following inf-compactness property holds*: *for every sequence* $((x_n, y_n))_{n\in\mathbb{N}}$ *satisfying* $\sup_{n\in\mathbb{N}} \mathcal{F}_n(x_n, y_n) < +\infty$*, there exists a subsequence* $(x_{\sigma(n)}, y_{\sigma(n)})$ *in* $X \times Y$ *and* $(x, \hat{y})$ *in* $X \times \hat{Y}$ *such that* $(x_{\sigma(n)}, y_{\sigma(n)}) \overset{I\times\mathcal{A}}{\rightharpoonup} (x, \hat{y})$*. Then*

(i) $F_n \xrightarrow{\Gamma_X} F$,
(ii) $G_n \xrightarrow{\Gamma_{Y,\hat{Y}}} G$.

Proof. Proof of assertion (i). On account of Theorem 12.1.1 in [6], we are going to establish that for any subsequence of F_n, one can extract a subsequence which Γ_X-converges to F. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence converging to x in X and consider a sequence $(y_n)_{n\in\mathbb{N}}$ in Y such that

$$\mathcal{F}_n(x_n, y_n) - \frac{1}{n} \leqslant \inf_{y\in Y} \mathcal{F}_n(x_n, y) := F_n(x_n). \tag{1}$$

We can assume that $\sup_{n\in\mathbb{N}} F_n(x_n) < +\infty$ (otherwise there is nothing to prove) so that $\sup_{n\in\mathbb{N}} \mathcal{F}_n(x_n, y_n) < +\infty$. Thus, according to the inf-compactness assumption, there exist a subsequence $(x_{\sigma(n)}, y_{\sigma(n)})$ and $\hat{y}$ in $\hat{Y}$ such that $x_{\sigma(n)}$ converges to x, and $y_{\sigma(n)}$ $\mathcal{A}$-converges to $\hat{y}$. Furthermore, since $(\mathcal{F}_n)_{n\in\mathbb{N}}$ $\Gamma_{X,Y,\hat{Y}}$-converges to $\mathcal{F}$, one has

$$\mathcal{F}(x, \hat{y}) \leqslant \liminf_{n\to+\infty} \mathcal{F}_{\sigma(n)}(x_{\sigma(n)}, y_{\sigma(n)}).$$

Since from (1) one has $F_{\sigma(n)}(x_{\sigma(n)}) \geqslant \mathcal{F}_{\sigma(n)}(x_{\sigma(n)}, y_{\sigma(n)}) - \frac{1}{\sigma(n)}$, we deduce

$$F(x) \leqslant \mathcal{F}(x, \hat{y}) \leqslant \liminf_{n\to+\infty} F_{\sigma(n)}(x_{\sigma(n)}).$$

From now on, to shorten notation, we write n instead of $\sigma(n)$. Let $(\hat{y}_p)_{p\in\mathbb{N}}$ be a sequence in $\hat{Y}$ satisfying

$$F(x) = \inf_{\hat{y}\in\hat{Y}} \mathcal{F}(x, \hat{y}) = \lim_{p\to+\infty} \mathcal{F}(x, \hat{y}_p). \tag{2}$$

Combining (i) and (ii) of Definition 2, for every fixed p there exists a sequence $((x_n^p, y_{p,n}))_{n\in\mathbb{N}}$ in $X \times Y$ such that

$$\begin{cases} x_n^p \to x, \\ y_{p,n} \overset{\mathcal{A}}{\rightharpoonup} \hat{y}_p \end{cases}$$

and satisfying

$$\mathcal{F}(x, \hat{y}_p) = \lim_{n\to+\infty} \mathcal{F}_n\big(x_n^p, y_{p,n}\big). \tag{3}$$

From (2) and (3), we obtain

$$F(x) = \lim_{p\to+\infty} \lim_{n\to+\infty} \mathcal{F}_n\big(x_n^p, y_{p,n}\big).$$

Then, using a standard diagonalization argument, there exists a map $n \mapsto p(n)$ satisfying $p(n) \to +\infty$ whenever $n \to +\infty$ for which one has

$$F(x) = \lim_{n\to+\infty} \mathcal{F}_n\big(x_n^{p(n)}, y_{p(n),n}\big) \geqslant \limsup_{n\to+\infty} F_n\big(x_n^{p(n)}\big).$$

The sequence defined by $x_n = x_n^{p(n)}$ then satisfies assertion (ii) of Definition 2 and the proof of (i) is complete. The proof of assertion (ii) is very similar and left to the reader. □

2.3. A concrete example

In this section, we present a concrete example entering within the general framework described above. It is the main subject of the paper which will be treated in details in the next section. We will deal with a second example in a forthcoming paper where we will take into account the concentration gradient phenomenon. For the analysis of concentration effects we refer the reader to [18] and [25].

We denote the sets of 3×3 and 3×2 matrices with real numbers entries by $\mathbf{M}^{3\times3}$ and $\mathbf{M}^{3\times2}$ respectively. Considering the space $\mathbf{M}^{3\times3}$ as the product $\mathbf{M}^{3\times2} \times \mathbb{R}^3$, we will denote by $\hat{\lambda}$ the first coordinate in $\mathbf{M}^{3\times2}$ of any element λ of $\mathbf{M}^{3\times3}$. We write $P_{\mathbf{M}^{3\times2}}$ to denote the projection mapping $\mathbf{M}^{3\times3}$ to $\mathbf{M}^{3\times2}$. In what follows, we use notation of Theorem 4 in Appendix A.

Let Ω, B be two open bounded subsets of $\mathbb{R}^3$, and $S \subset \mathbb{R}^2$ such that $B = S \times (0,1)$, we define the sets of Young measures $\mathcal{Y}_{3\times3}(B)$ and $\mathcal{Y}_{3\times2}(S)$ as follow:

$$\mu \in \mathcal{Y}_{3\times3}(B) \quad \Leftrightarrow \quad \mu \in \mathbf{M}^+\big(B \times \mathbf{M}^{3\times3}\big) \quad \text{and} \quad P_B\#\mu = \mathcal{L},$$
$$\nu \in \mathcal{Y}_{3\times2}(S) \quad \Leftrightarrow \quad \nu \in \mathbf{M}^+\big(S \times \mathbf{M}^{3\times2}\big) \quad \text{and} \quad P_S\#\nu = \hat{\mathcal{L}}$$

where $P_B\#\mu$ (resp. $P_S\#\nu$) denotes the image of the measure μ (resp. ν) by the projection $P_B : B \times \mathbf{M}^{3\times3} \to B$ (resp. $P_S : S \times \mathbf{M}^{3\times2} \to S$) and $\mathcal{L}$ (resp. $\hat{\mathcal{L}}$) the Lebesgue measure on B (resp. S). For every probability measure $\mathbf{P}$ on $\mathbf{M}^{3\times3}$ or $\mathbf{M}^{3\times2}$, we write $\mathrm{bar}(\mathbf{P})$ for its barycenter, i.e. $\mathrm{bar}(\mathbf{P}) = \int \lambda \, d\mathbf{P}(\lambda)$.

The map $\mathcal{A}$ is defined by:

$$\mathcal{A} : \mathcal{Y}_{3\times3}(B) \to \mathcal{Y}_{3\times2}(S),$$
$$\mu = \mu_x \otimes dx \mapsto \nu = \left(\int_0^1 \hat{\mu}_{\hat{x},s} \, ds \right) \otimes d\hat{x}$$

where $\hat{\mu}_x = P_{\mathbf{M}^{3\times2}}\#\mu_x$, $x \in \Omega$, and $\int_0^1 \hat{\mu}_{\hat{x},s} \, ds$ is the probability measure parametrized by $\hat{x} \in S$, which acts on all $\varphi \in C_0(\mathbf{M}^{3\times2})$ as follows:

$$\left\langle \int_0^1 \hat{\mu}_{\hat{x},s} \, ds, \varphi \right\rangle := \int_0^1 \int_{\mathbf{M}^{3\times2}} \varphi(\hat{\lambda}) \, d\hat{\mu}_{\hat{x},s} \, ds.$$

Given a sequence $(\mu_\varepsilon)_{\varepsilon>0}$ in the space $\mathcal{Y}_{3\times3}(B)$ equipped with the narrow convergence (see Appendix A) and ν in $\mathcal{Y}_{3\times2}(S)$, according to Definition 1 one has

$$\mu_\varepsilon \overset{\mathcal{A}}{\rightharpoonup} \nu \quad \Leftrightarrow \quad \exists \mu \in \mathcal{Y}_{3\times3}(B) \quad \text{s.t.} \begin{cases} \mu_\varepsilon \overset{\text{nar}}{\rightharpoonup} \mu, \\ \nu = \mathcal{A}(\mu) \end{cases}$$

and, for $(v_\varepsilon, \mu_\varepsilon)$ in $L^p(\Omega, \mathbb{R}^3) \times \mathcal{Y}_{3\times3}(B)$, and (v, ν) in $L^p(\Omega, \mathbb{R}^3) \times \mathcal{Y}_{3\times2}(S)$:

$$(v_\varepsilon, \mu_\varepsilon) \overset{I\times\mathcal{A}}{\rightharpoonup} (v, \nu) \quad \Leftrightarrow \quad \begin{cases} v_\varepsilon \to v \text{ in } L^p(\Omega, \mathbb{R}^3), \\ \mu_\varepsilon \overset{\mathcal{A}}{\rightharpoonup} \nu. \end{cases}$$

Given two locally Lipschitz functions $f, g : \mathbf{M}^{3\times3} \to \mathbb{R}^+$ satisfying a growth condition of order p, we consider the integral functional $\mathcal{F}_\varepsilon$ defined by

$$\mathcal{F}_\varepsilon : L^p\big(\Omega, \mathbb{R}^3\big) \times \mathcal{Y}_{3\times3}(B) \to \mathbb{R} \cup \{+\infty\},$$
$$\mathcal{F}_\varepsilon(u, \mu) := \begin{cases} \int_{\Omega_\varepsilon} f(\nabla u) \, dx + \int_{B\times\mathbf{M}^{3\times3}} g(\hat{\lambda} \mid \frac{1}{\varepsilon}\lambda_3) \, d\mu + I_\varepsilon(u, \mu) & \text{if } u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$I_\varepsilon(u, \mu) := \begin{cases} 0 & \text{if } \mu = \delta_{\nabla v(x)} \otimes dx, \ v = 1_B r_\varepsilon u, \\ +\infty & \text{otherwise.} \end{cases}$$

and $r_\varepsilon u$ is defined by $r_\varepsilon u(\hat{x}, x_3) = u(\hat{x}, \varepsilon x_3)$.

Let $\hat{\nabla}\mathcal{Y}_{3\times2}(S)$ denote the subset of $\mathcal{Y}_{3\times2}(S)$ made up of all Young measures generated by gradients of $W^{1,p}(S, \mathbb{R}^3)$-Sobolev functions, γ_S the trace operator from $W^{1,p}(\Omega \setminus S, \mathbb{R}^3)$ into $L^p(S, \mathbb{R}^3)$ and define the function $g_0 : \mathbf{M}^{3\times2} \to \mathbb{R}$ for all $\hat{\lambda}$ in $\mathbf{M}^{3\times2}$ by $g_0(\hat{\lambda}) := \inf_{\xi\in\mathbb{R}^3} g(\hat{\lambda} \mid \xi)$. Setting $X = L^p(\Omega, \mathbb{R}^3)$, $Y = \mathcal{Y}_{3\times3}(B)$ and

$\hat{Y} := \mathcal{Y}_{3\times 2}(S)$, in the next section we prove that the sequence $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$ $\Gamma_{X,Y,\hat{Y}}$-converges to the functional $\mathcal{F}$, defined by

$$\mathcal{F} : L^p(\Omega, \mathbb{R}^3) \times \mathcal{Y}_{3\times 2}(S) \to \mathbb{R} \cup \{+\infty\},$$

$$\mathcal{F}(u, \nu) := \begin{cases} \int_\Omega Qf(\nabla u)\,dx + \int_{S\times \mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\,d\nu + I(u,\nu) & \text{if } u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3),\ \gamma_S u \in W^{1,p}(S, \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$I(u, \nu) := \begin{cases} 0 & \text{if } \nu \in \hat{\nabla}\mathcal{Y}_{3\times 2}(S),\ \operatorname{bar}(\nu_{\hat{x}}) = \hat{\nabla}\gamma_S(u)(\hat{x}) \text{ a.e. in } S, \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 1 applied to the sequence of the two marginal maps sheds new light on the mechanical problem.

3. A model in terms of displacement-Young measures: Analysis of microstructures of the strong material

In the three-dimensional Euclidean space $\mathbb{E}^3$ referred to the orthonormal frame $(0; e_1, e_2, e_3)$, we consider a domain Ω with a C^1 boundary Γ. Let $\Omega^\pm = \Omega \cap [\pm x_3 > 0]$, the interior $S = \{\partial\Omega^+ \cap \partial\Omega^-\}^\circ$ of the common part of the boundaries of $\Omega^\pm$ is assumed to have a positive $\mathcal{H}^2$-measure and, to shorten the proofs, included in the plane $[x_3 = 0]$. The set Ω is the physical reference configuration of the assembly of two materials. More precisely, given a small dimensionless parameter ε and a global characteristic length h (for example the diameter of Ω), the set $B_\varepsilon = \{x + \varepsilon z\colon\ 0 < z < h,\ x \in S\}$ is the reference configuration of a strong material (whose stiffness is of order $\frac{1}{\varepsilon}$) while $\Omega_\varepsilon = \Omega \setminus B_\varepsilon$ is the reference configuration of a material with stiffness of order 1. (see Fig. 1). The structure is clamped on a part Γ_0 of Γ with a positive $\mathcal{H}^2$-measure, the complementary part Γ_ψ of Γ_0 is submitted to surface loads ψ and we assume that $\mathcal{H}^1(\Gamma_0 \cap \bar{S}) > 0$. Obviously one can there consider other type of boundary conditions (e.g. a combination of some components of the stress vector and of the displacement). Moreover the structure is submitted to applied body forces Φ.

Let $p \geqslant 1$, we say that a Borel function $W : \mathbf{M}^{3\times 3} \to \mathbb{R}$ satisfies a (C_p) condition if

$$\begin{cases} \exists \alpha, \beta, C \in \mathbb{R}^+ \text{ s. t. } \forall(\lambda, \lambda') \in \mathbf{M}^{3\times 3},\ \alpha|\lambda|^p \leqslant W(\lambda) \leqslant \beta(1 + |\lambda|^p), \\ |W(\lambda) - W(\lambda')| \leqslant C|\lambda - \lambda'|(1 + |\lambda|^{p-1} + |\lambda'|^{p-1}). \end{cases} \tag{4}$$

We say that a quasiconvex function $\phi : \mathbf{M}^{3\times 3} \to \mathbb{R}$ (resp. $\phi : \mathbf{M}^{3\times 2} \to \mathbb{R}$) *satisfies a growth condition of order* p if there exists $\gamma \geqslant 0$ such that

$$\forall \lambda \in \mathbf{M}^{3\times 3},\quad |\phi(\lambda)| \leqslant \gamma(1 + |\lambda|^p) \qquad \big(\text{resp. } \forall \hat{\lambda} \in \mathbf{M}^{3\times 2},\quad |\phi(\hat{\lambda})| \leqslant \gamma(1 + |\hat{\lambda}|^p)\big).$$

The soft and the strong materials are modeled as hyperelastic and the bulk energy densities f, g of the two materials occupying Ω_ε and B_ε satisfy a (C_p) condition with $p > 1$. To shorten notation, we assume that (C_p) is satisfied with the same constants α, β and C. We make the assumption that the strain of the soft material can be high and that the thin structure B_ε is occupied by a material which undergoes reversible solid–solid phase transformation as for instance crystalline solids. In this context, the densities f and g are not convex and g entails a multi-well structure. It is worth pointing out that the assumed growth condition violates the mechanical principle which asserts that it needs infinite amount of energy to squeeze a small piece of material down to a point. We also do not take into account preservation of orientation and injectivity conditions on the deformation fields so that the model presented in this section is a first attempt to account large purely elastic deformation. We hope to deal with this much more complex situation in a future work (for some results where these constraints are taken into account, we refer the reader to [20,3,4]).

We assume that the global characteristic length h is equal to 1 and the stored strain energy associated with a displacement field u is given by the functional

$$F_\varepsilon(u) := \int_{\Omega_\varepsilon} f(\nabla u)\,dx + \frac{1}{\varepsilon}\int_{B_\varepsilon} g(\nabla u)\,dx.$$

The equilibrium configuration of the structure is given by the displacement field $\bar{u}_\varepsilon$, solution-more generally ε-approximate solution-of the problem

$$\inf\big\{F_\varepsilon(u) - L(u)\colon\ u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)\big\}$$

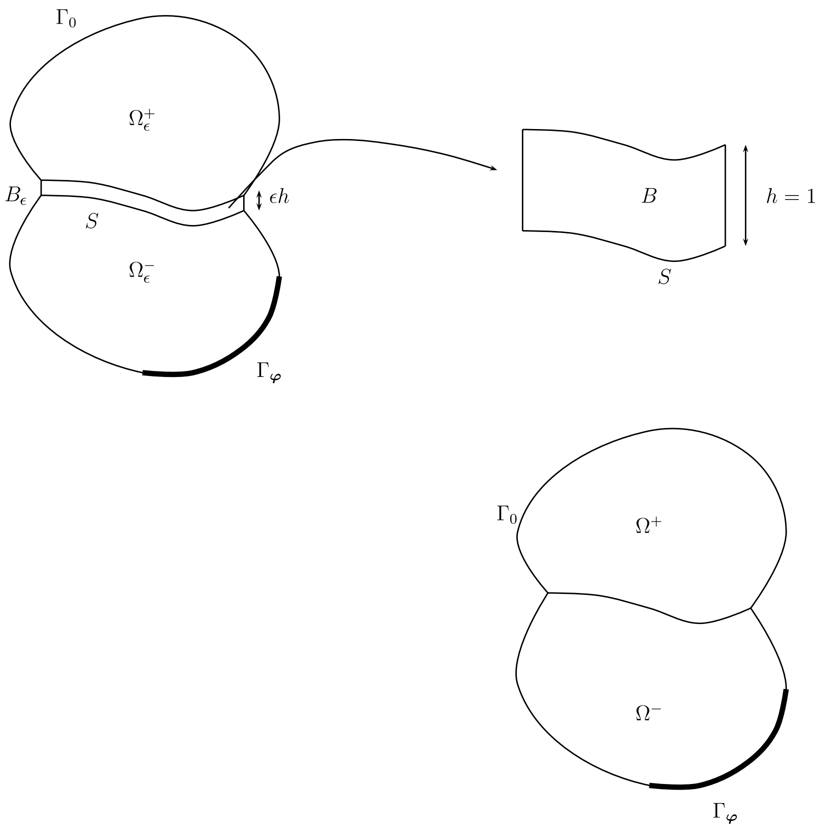

Fig. 1. Bonded assembly – left: the physical configuration – right: the rescaled layer – below: the limit configuration.

where $L(u) = \int_\Omega \Phi\, dx + \int_{\Gamma_\psi} \psi\, d\mathcal{H}^2$. We want to analyze the behavior of $\bar{u}_\varepsilon$ when ε tends to zero and to identify the variational problem whose limit is a solution. But since the material in the layer B_ε possesses a fine micro-structure, the gradient minimizing sequence $(\nabla \bar{u}_\varepsilon)_{\varepsilon>0}$ develops oscillations we would like to integrate into the variational problem. This is why we write the strain energy $\frac{1}{\varepsilon}\int_{B_\varepsilon} g(\nabla u)\, dx$ in terms of Young measures so that the limit problem also accounts for a two-dimensional microstructure (for existence of microstructures see [9] and for microstructures in thin films, we refer the reader to [11,17,21] and references therein).

Since the behavior of the displacement is radically different in Ω_ε and B_ε, in a first stage, it is convenient to write the energy functional F_ε in terms of two arguments, one u, the displacement on Ω_ε, the other v, the displacement on B_ε occupied by the strong material. On the other hand, in order to work in a fixed space for the variable v, the change of scale $(\hat{x}, x_3) = (\hat{x}, \varepsilon y_3)$ transforming $(\hat{x}, x_3) \in B_\varepsilon$ into $(\hat{x}, y_3) \in B := S \times (0,1)$ leads to consider the following functional

$$\mathcal{G}_\varepsilon : L^p(\Omega, \mathbb{R}^3) \times L^p(B, \mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\},$$

$$\mathcal{G}_\varepsilon(u, v) := \begin{cases} \int_{\Omega_\varepsilon} f(\nabla u)\, dx + \int_B g(\hat{\nabla} v | \frac{1}{\varepsilon}\frac{\partial v}{\partial x_3})\, dx + I_\varepsilon(u, v) & \text{if } u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

where, for all $(u, v) \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3) \times W^{1,p}(B, \mathbb{R}^3)$

$$I_\varepsilon(u, v) := \begin{cases} 0 & \text{if } 1_B r_\varepsilon u = v, \\ +\infty & \text{otherwise} \end{cases}$$

and $r_\varepsilon u(\hat{x}, y_3) := u(\hat{x}, \varepsilon y_3)$. Now we write $\mathcal{G}_\varepsilon$ in terms of pairs of displacements-Young measures by defining the functional $\mathcal{F}_\varepsilon$ as follows

$$\mathcal{F}_\varepsilon : L^p(\Omega, \mathbb{R}^3) \times \mathcal{Y}_{3\times 3}(B) \to \mathbb{R} \cup \{+\infty\},$$

$$\mathcal{F}_\varepsilon(u,\mu) := \begin{cases} \int_{\Omega_\varepsilon} f(\nabla u)\,dx + \int_{B\times \mathbf{M}^{3\times 3}} g(\hat{\lambda} \mid \frac{1}{\varepsilon}\lambda_3)\,d\mu + I_\varepsilon(u,\mu) & \text{if } u \in W^{1,p}_{\Gamma_0}(\Omega,\mathbb{R}^3),\\ +\infty & \text{otherwise,}\end{cases}$$

where

$$I_\varepsilon(u,\mu) := \begin{cases} 0 & \text{if } \mu = \delta_{\nabla v(x)} \otimes dx,\ v = 1_B r_\varepsilon u,\\ +\infty & \text{otherwise.}\end{cases}$$

Clearly $\mathcal{F}_\varepsilon$ is one way of writing $\mathcal{G}_\varepsilon$ and from the strict variational point of view, it is equivalent to identify the variational limit of F_ε and that of $\mathcal{F}_\varepsilon$ in the spirit of the previous section. Indeed we have

$$\begin{aligned}\inf_{u\in W^{1,p}_{\Gamma_0}(\Omega,\mathbb{R}^3)} \big(F_\varepsilon(u) - L(u)\big) &= \inf_{(u,v)\in L^p(\Omega,\mathbb{R}^3)\times L^p(B,\mathbb{R}^3)} \big(\mathcal{G}_\varepsilon(u,v) - L(u)\big)\\ &= \inf_{(u,\mu)\in L^p(\Omega,\mathbb{R}^3)\times \mathcal{Y}_{3\times 3}(B)} \big(\mathcal{F}_\varepsilon(u,\mu) - L(u)\big).\end{aligned}$$

Nevertheless, we want to point out that the last formulation has the advantage to encode the gradient oscillations of ε-minimizers in the layer B_ε thanks to the Young measure state variable.

In order to apply Proposition 1 we begin by establishing the following compactness lemma

Lemma 1 *(Compactness). Let $((u_\varepsilon, v_\varepsilon, \mu_\varepsilon))_{\varepsilon>0}$ be a sequence in $L^p(\Omega,\mathbb{R}^3) \times L^p(B,\mathbb{R}^3) \times \mathcal{Y}_{3\times 3}(B)$ satisfying $\sup_{\varepsilon>0} \mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon) = \sup_{\varepsilon>0} \mathcal{F}_\varepsilon(u_\varepsilon,\mu_\varepsilon) < +\infty$. Then there exist $(u,v) \in W^{1,p}_{\Gamma_0}(\Omega,\mathbb{R}^3) \times W^{1,p}(B,\mathbb{R}^3)$, $\nu \in \hat{\nabla}\mathcal{Y}_{3\times 2}(S)$ and a subsequence not relabeled such that*

(i) *$u_\varepsilon \to u$ weakly in $W^{1,p}_{\Gamma_0}(\Omega,\mathbb{R}^3)$ and strongly in $L^p(\Omega,\mathbb{R}^3)$, $v_\varepsilon \to v$ weakly in $W^{1,p}(B,\mathbb{R}^3)$ and strongly in $L^p(B,\mathbb{R}^3)$;*
(ii) *$\frac{\partial v_\varepsilon}{\partial y_3} \to 0$ strongly in $L^p(B,\mathbb{R}^3)$, $\frac{\partial v}{\partial y_3} = 0$ and $v \in W^{1,p}(S,\mathbb{R}^3)$;*
(iii) *$\gamma_S(u) = v$ on S where γ_S denotes the trace operator from $W^{1,p}(\Omega \setminus S,\mathbb{R}^3)$ into $L^p(S)$;*
(iv) *$(u_\varepsilon,\mu_\varepsilon) \overset{I\times\mathcal{A}}{\rightharpoonup} (u,\nu)$, with $\nu = \mathcal{A}(\mu)$ where $\mu = (\hat{\mu}_x \otimes \delta_{0_{\mathbb{R}^3}}) \otimes dx$ and $\hat{\mu} := \hat{\mu}_x \otimes dx$ is the Young measure generated by $(\hat{\nabla} v_\varepsilon)_{\varepsilon>0}$. Moreover u and ν are connected as follows*: $\mathrm{bar}(\nu_{\hat{x}}) = \hat{\nabla}\gamma_S(u)(\hat{x})$ *for a.e. $\hat{x} \in S$.*

Proof. Assertions (i) and (ii) are straightforward consequences of the coerciveness condition in (4), Poincaré's inequality and the isometry between $W^{1,p}(S,\mathbb{R}^3)$ and $\{v \in W^{1,p}(B,\mathbb{R}^3)\colon \frac{\partial v}{\partial y_3} = 0\}$. We are going to establish assertion (iii). For a.e. $x \in B$ we have

$$v_\varepsilon(\hat{x}, y_3) = u_\varepsilon(\hat{x}, \varepsilon y_3) = u_\varepsilon(\hat{x}, 0) + \int_0^{\varepsilon y_3} \frac{\partial u_\varepsilon}{\partial y_3}(\hat{x}, s)\,ds$$

where $u_\varepsilon(\hat{x},0)$ must be taken in the trace sense. Let $Q_\rho(\hat{x}_0) = \hat{Q}_\rho(\hat{x}_0) \times (0,\rho)$ be the cylinder of $\mathbb{R}^3$ where $\hat{Q}_\rho(\hat{x}_0)$ is the ball of $\mathbb{R}^2$ centered at $\hat{x}_0 \in S$ with $\rho > 0$ small enough so that $Q_\rho(\hat{x}_0) \subset B$. Integrating the previous equality over $Q_\rho(\hat{x}_0)$ yields

$$\fint_{Q_\rho(\hat{x}_0)} v_\varepsilon\,dx = \fint_{Q_\rho(\hat{x}_0)} u_\varepsilon(\hat{x},0)\,dx + \fint_{Q_\rho(\hat{x}_0)} \int_0^{\varepsilon y_3} \frac{\partial u_\varepsilon}{\partial y_3}(\hat{x}, s)\,ds\,dx.$$

Letting $\varepsilon \to 0$, according to the continuity of the trace operator and to (i), (ii), we obtain

$$\fint_{\hat{Q}_\rho(\hat{x}_0)} v\,d\hat{x} = \fint_{\hat{Q}_\rho(\hat{x}_0)} \gamma_S(u)\,d\hat{x} + \limsup_{\varepsilon\to 0} I_{\rho,\varepsilon} \tag{5}$$

where $I_{\rho,\varepsilon} := \fint_{Q_\rho(\hat{x}_0)} \int_0^{\varepsilon y_3} \frac{\partial u_\varepsilon}{\partial y_3}(\hat{x},s)\,ds\,dx$. Let us estimate $I_{\rho,\varepsilon}$. An easy calculation using Hölder's inequality gives

$$
\begin{aligned}
|I_{\rho,\varepsilon}| &\leqslant \fint_{Q_\rho(\hat{x}_0)} (\varepsilon y_3)^{1-\frac{1}{p}} \left(\int_0^{\varepsilon y_3} \left| \frac{\partial u_\varepsilon}{\partial y_3}(\hat{x}, s) \right|^p ds \right)^{\frac{1}{p}}, d\hat{x}\, dy_3 \\
&\leqslant (\varepsilon\rho)^{1-\frac{1}{p}} \fint_{\hat{Q}_\rho(\hat{x}_0)} \left(\int_0^{\varepsilon\rho} \left| \frac{\partial u_\varepsilon}{\partial y_3}(\hat{x}, s) \right|^p ds \right)^{\frac{1}{p}} d\hat{x} \\
&\leqslant (\varepsilon\rho)^{1-\frac{1}{p}} \left(\int_{\hat{Q}_\rho(\hat{x}_0)} \int_0^{\varepsilon\rho} \left| \frac{\partial u_\varepsilon}{\partial y_3}(\hat{x}, s) \right|^p ds\, d\hat{x} \right)^{\frac{1}{p}} \\
&\leqslant (\varepsilon\rho)^{1-\frac{1}{p}} \left(\int_{B_\varepsilon} |\nabla u_\varepsilon|^p\, dx \right)^{\frac{1}{p}}.
\end{aligned}
$$

But since $\sup_{\varepsilon>0} \mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon) = \sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon) < +\infty$, from the coerciveness property satisfied by g one has $\int_{B_\varepsilon} |\nabla u_\varepsilon|^p\, dx \leqslant \frac{\varepsilon}{\alpha}$ so that the previous estimate yields $|I_{\rho,\varepsilon}| \leqslant C(\rho, \alpha)\varepsilon$ where $C(\rho, \alpha)$ is a positive constant depending only on ρ and α. From this estimate, (5) becomes

$$
\fint_{\hat{Q}_\rho(\hat{x}_0)} v\, d\hat{x} = \fint_{\hat{Q}_\rho(\hat{x}_0)} \gamma_S(u)\, d\hat{x}
$$

for all $\rho > 0$. Letting $\rho \to 0$ finally gives $\gamma_S(u)(x_0) = v(x_0)$ for a.e. x_0 in S.

It remains to establish assertion (iv). Since $\mathcal{F}_\varepsilon(u_\varepsilon, \mu_\varepsilon) < +\infty$ we have $\mu_\varepsilon = \delta_{\nabla v_\varepsilon(x)} \otimes dx$ and $\sup_{\varepsilon>0} \int_B |\nabla v_\varepsilon|^p\, dx < +\infty$ so that the Young measures μ_ε and $\delta_{\hat{\nabla} v_\varepsilon(x)} \otimes dx$ are tight. According to the Prokhorov compactness theorem (Theorem 5 of Appendix A), there exist a subsequence that we do not relabel, and $\mu \in \nabla\mathcal{Y}_{3\times3}(B)$, $\hat{\mu} \in \mathcal{Y}_{3\times2}(B)$ such that

$$
\mu_\varepsilon = \delta_{\nabla v_\varepsilon(x)} \otimes dx \overset{\text{nar}}{\rightharpoonup} \mu, \qquad \delta_{\hat{\nabla} v_\varepsilon(x)} \otimes dx \overset{\text{nar}}{\rightharpoonup} \hat{\mu}. \tag{6}
$$

On the other hand, from assertion (ii)

$$
\delta_{\frac{\partial v_\varepsilon}{\partial x_3}(x)} \otimes dx \overset{\text{nar}}{\rightharpoonup} \delta_{0_{\mathbb{R}^3}} \otimes dx. \tag{7}
$$

Combining (6) and (7) one easily deduce that $\mu_x = \hat{\mu}_x \otimes \delta_{0_{\mathbb{R}^3}}$, and setting $\nu = \nu_{\hat{x}} \otimes d\hat{x}$ where $\nu_{\hat{x}} := \int_0^1 \hat{\mu}_{\hat{x},s}\, ds$, we finally obtain that $\mu_\varepsilon \overset{\mathcal{A}}{\rightharpoonup} \nu$.

We have to prove that ν belongs to $\hat{\nabla}\mathcal{Y}_{3\times2}(S)$. According to the Kinderlehrer–Pedregal characterization theorem (Theorem 6 of Appendix A), it is equivalent to establish the three following assertions:

$(\mathrm{KP})_1$ there exists $w \in W^{1,p}(S, \mathbb{R}^3)$ such that $\mathrm{bar}(\nu_{\hat{x}}) = \hat{\nabla} w(\hat{x})$ for a.e. $x \in S$;
$(\mathrm{KP})_2$ $\int_{\mathbf{M}^{3\times2}} |\hat{\lambda}|^p\, d\nu_{\hat{x}} < +\infty$ for a.e. $x \in S$;
$(\mathrm{KP})_3$ for all quasiconvex function ϕ *satisfying a growth condition of order* p,

$$
\phi\big(\mathrm{bar}(\nu_{\hat{x}})\big) \leqslant \int_{\mathbf{M}^{3\times2}} \phi(\hat{\lambda})\, d\nu_{\hat{x}} \quad \text{for a.e. } \hat{x} \in S.
$$

Proof of (KP)$_1$: From assertion (i) and classical properties on Young measures, we have $\hat{\nabla} v(x) = \int_{\mathbf{M}^{3\times2}} \hat{\lambda}\, d\hat{\mu}_x$ for a.e. $x \in B$ and since $\frac{\partial v}{\partial x_3} = 0$,

$$
\hat{\nabla} v(\hat{x}) = \int_0^1 \int_{\mathbf{M}^{3\times2}} \hat{\lambda}\, d\hat{\mu}_{\hat{x},s}\, ds = \int_{\mathbf{M}^{3\times2}} \hat{\lambda}\, d\nu_{\hat{x}}
$$

for a.e. $\hat{x}$ in S so that v is the suitable Sobolev-function w.

Proof of (KP)₂: From the definition of $\nu_{\hat{x}}$ and the lower semicontinuity property for Young measures (Proposition 5 of Appendix A) we have

$$\begin{aligned}\int_S \int_{\mathbf{M}^{3\times 2}} |\hat{\lambda}|^p \, d\nu_{\hat{x}} \, d\hat{x} &= \int_B \int_{\mathbf{M}^{3\times 2}} |\hat{\lambda}|^p \, d\hat{\mu}_x \, dx \\ &= \int_B \int_{\mathbf{M}^{3\times 3}} |\hat{\lambda}|^p (d\hat{\mu}_x \otimes \delta_{0_{\mathbb{R}^3}}) \, dx \\ &= \int_{B\times\mathbf{M}^{3\times 3}} |\hat{\lambda}|^p \, d\mu \leqslant \int_{B\times\mathbf{M}^{3\times 3}} |\lambda|^p \, d\mu \\ &\leqslant \liminf_{\varepsilon\to 0} \int_B |\nabla v_\varepsilon|^p \, dx \leqslant \sup_{\varepsilon>0} \mathcal{F}_\varepsilon(u_\varepsilon, \mu_\varepsilon) < +\infty\end{aligned}$$

which proves that $\int_{\mathbf{M}^{3\times 2}} |\hat{\lambda}|^p \, d\nu_{\hat{x}}$ is finit for a.e. $\hat{x}$ in S.

Proof of (KP)₃: Let $\phi : \mathbf{M}^{3\times 2} \to \mathbb{R}$ be a quasiconvex function satisfying a growth condition of order p and define the function $\tilde{\phi} : \mathbf{M}^{3\times 3} \to \mathbb{R}$ by $\tilde{\phi}(\lambda) = \phi(\hat{\lambda})$. It is easy to check that $\tilde{\phi}$ is quasiconvex and clearly satisfies the same growth condition. Since $\mu \in \nabla\mathcal{Y}_{3\times 3}(B)$ we have for a.e. x in B

$$\begin{aligned}\phi\big(\hat{\nabla} v(\hat{x})\big) = \tilde{\phi}\big(\nabla v(x)\big) &\leqslant \int_{\mathbf{M}^{3\times 3}} \tilde{\phi}(\lambda) \, d\mu_x \\ &= \int_{\mathbf{M}^{3\times 3}} \phi(\hat{\lambda}) \, d\hat{\mu}_x \otimes \delta_{0_{\mathbb{R}^3}} = \int_{\mathbf{M}^{3\times 2}} \phi(\hat{\lambda}) \, d\hat{\mu}_x\end{aligned}$$

so that for a.e. x in S

$$\phi\big(\hat{\nabla} v(\hat{x})\big) = \int_0^1 \phi\big(\hat{\nabla} v(\hat{x})\big) \, ds \leqslant \int_0^1 \int_{\mathbf{M}^{3\times 2}} \phi(\hat{\lambda}) \, d\hat{\mu}_{\hat{x},s} \, ds = \int_{\mathbf{M}^{3\times 2}} \phi(\hat{\lambda}) \, d\nu_{\hat{x}}. \qquad \square$$

Remark 1. In the proof above we established $\mathcal{A}(\nabla\mathcal{Y}_{3\times 3}(B)) \subset \hat{\nabla}\mathcal{Y}_{3\times 2}(S)$. In fact it is easy to check that $\mathcal{A}(\nabla\mathcal{Y}_{3\times 3}(B)) = \hat{\nabla}\mathcal{Y}_{3\times 2}(S)$.

Consider the functional defined in Section 2.3:

$$\mathcal{F} : L^p\big(\Omega, \mathbb{R}^3\big) \times \mathcal{Y}_{3\times 2}(S) \to \mathbb{R} \cup \{+\infty\},$$

$$\mathcal{F}(u, \nu) := \begin{cases} \int_\Omega Qf(\nabla u) \, dx + \int_{S\times\mathbf{M}^{3\times 2}} g_0(\hat{\lambda}) \, d\nu + I(u, \nu) & \text{if } u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3),\ \gamma_S u \in W^{1,p}(S, \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

We establish the $\Gamma_{X,Y,\hat{Y}}$-convergence of the sequence $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$ to the functional $\mathcal{F}$ by means of the two next propositions.

Proposition 2 *(Lower bound). Let (u, ν) be any pair in $L^p(\Omega, \mathbb{R}^3) \times \mathcal{Y}_{3\times 2}(S)$. Then, for all sequence $((u_\varepsilon, \mu_\varepsilon))_{\varepsilon>0}$ in $L^p(\Omega, \mathbb{R}^3) \times \mathcal{Y}_{3\times 3}(B)$ converging to (u, ν), we have*

$$\mathcal{F}(u, \nu) \leqslant \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, \mu_\varepsilon).$$

Proof. One can assume $\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, \mu_\varepsilon) < +\infty$ otherwise there is nothing to prove. Consequently, from Lemma 1 we have

$$u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3);$$

$$\nu \in \hat{\nabla}\mathcal{Y}_{3\times 2}(S); \qquad \mathrm{bar}(\nu_{\hat{x}}) = \hat{\nabla}\gamma_S(u)(\hat{x}) \quad \text{for a.e. } \hat{x} \in S.$$

This proves that $I(u,\nu)=0$ and it suffices to establish the two following estimates:

$$\int_{S\times\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\,d\nu \leqslant \liminf_{\varepsilon\to 0}\int_B g\left(\hat{\nabla}v_\varepsilon \,\middle|\, \frac{1}{\varepsilon}\frac{\partial v_\varepsilon}{\partial x_3}\right)dx; \tag{8}$$

$$\int_\Omega Qf(\nabla u)\,dx \leqslant \liminf_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)\,dx. \tag{9}$$

Proof of (8): From the lower semicontinuity property for Young measures (see Proposition 5 of Appendix A) and assertion (iv) of Lemma 1, it follows that

$$\begin{aligned}\liminf_{\varepsilon\to 0}\int_{B\times\mathbf{M}^{3\times 3}} g\left(\hat{\lambda}\,\middle|\,\frac{1}{\varepsilon}\lambda_3\right)d\mu_\varepsilon &\geqslant \liminf_{\varepsilon\to 0}\int_{B\times\mathbf{M}^{3\times 3}} g_0(\hat{\lambda})\,d\mu_\varepsilon\\ &\geqslant \int_{B\times\mathbf{M}^{3\times 3}} g_0(\hat{\lambda})\,d\mu\\ &= \int_{B\times\mathbf{M}^{3\times 3}} g_0(\hat{\lambda})\,d(\hat{\mu}_x\otimes\delta_{0^3_{\mathbb{R}}})\otimes dx\\ &= \int_{S\times\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\,d\nu_{\hat{x}}\otimes d\hat{x} = \int_{S\times\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\,d\nu.\end{aligned}$$

Proof of (9): For fixed $\eta>\varepsilon$ we have

$$\int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)\,dx \geqslant \int_{\Omega_\eta} f(\nabla u_\varepsilon)\,dx \geqslant \int_{\Omega_\eta} Qf(\nabla u_\varepsilon)\,dx$$

and, since $w\mapsto\int_{\Omega_\eta} Qf(\nabla w)\,dx$ is lower semicontinuous for the weak convergence in $W^{1,p}(\Omega_\eta,\mathbb{R}^3)$, we deduce

$$\liminf_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)\,dx \geqslant \int_{\Omega_\eta} Qf(\nabla u)\,dx.$$

We end the proof by letting $\eta\to 0$. □

For establishing the upper bound in the definition of our variationel convergence, we need to prove the following relaxation result

Lemma 2. *Let (u,ν) in $W^{1,p}_{\Gamma_0}(\Omega,\mathbb{R}^3)\times\hat{\nabla}\mathcal{Y}_{3\times 2}(S)$ with $\hat{\nabla}\gamma_S(u)(\hat{x})=\mathrm{bar}(\nu_{\hat{x}})$ for a.e. x in S and a sequence $((u_n,v_n))_{n\in\mathbb{N}}$ in $W^{1,p}_{\Gamma_0}(\Omega,\mathbb{R}^3)\times W^{1,p}(S,\mathbb{R}^3)$ such that u_n weakly converges to u in $W^{1,p}_{\Gamma_0}(\Omega,\mathbb{R}^3)$, $\delta_{\hat{\nabla}v_n(\hat{x})}\otimes dx \overset{\mathrm{nar}}{\rightharpoonup} \nu$ in $\mathcal{Y}_{3\times 2}(S)$ and*

$$\lim_{n\to+\infty}\int_\Omega f(\nabla u_n)\,dx = \int_\Omega \mathcal{Q}f(\nabla u)\,dx,$$

$$\lim_{n\to+\infty}\int_S g_0(\hat{\nabla}v_n)\,d\hat{x} = \int_{S\times\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\,d\nu. \tag{10}$$

Then there exists a sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$ satisfying all the conditions fulfilled by $(u_n)_{n\in\mathbb{N}}$ and furthermore which satisfies $\gamma_S(\tilde{u}_n)=v_n$.

Proof. Such a sequence $((u_n,v_n))_{n\in\mathbb{N}}$ exists, consult for instance [6, Theorem 11.2.1 and Theorem 11.4.2]. Note that one may assume $(|\nabla u_n|^p)_{n\in\mathbb{N}}$ uniformly integrable. Indeed, consider the sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$ whose gradients generate the same Young measure μ and such that $(|\nabla\tilde{u}_n|^p)_{n\in\mathbb{N}}$ is uniformly integrable (see Proposition 7 of Appendix A). By using Propositions 5, 6 of Appendix A and standard lower semicontinuity results in Sobolev spaces we have

$$\begin{aligned}\int_\Omega \mathcal{Q}f(\nabla u)\,dx &= \lim_{n\to+\infty}\int_\Omega f(\nabla u_n)\,dx\\ &\geqslant \int_{\Omega\times\mathbf{M}^{3\times 3}} f(\lambda)\,d\mu\\ &= \lim_{n\to+\infty}\int_\Omega f(\nabla\tilde{u}_n)\,dx\\ &\geqslant \int_\Omega \mathcal{Q}f(\nabla u)\,dx\end{aligned}$$

so that

$$\lim_{n\to+\infty}\int_\Omega f(\nabla\tilde{u}_n)\,dx = \lim_{n\to+\infty}\int_\Omega f(\nabla u_n)\,dx = \int_\Omega \mathcal{Q}f(\nabla u)\,dx$$

which proves the thesis. In what follows, we still denote by $(u_n)_{n\in\mathbb{N}}$ the sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$. We are going to modify the function u_n near S so that the constraint $\gamma_S(u_n) = v_n$ holds. The function v_n will be indifferently considered as a $W^{1,p}(S,\mathbb{R}^3)$-function or a $W^{1,p}(\Omega,\mathbb{R}^3)$-function with $\frac{\partial v_n}{\partial x_3} = 0$.

By coerciveness of g_0, $\int_S |\hat{\nabla} v_n|^p\,d\hat{x}$ is bounded, thus v_n strongly converges in $L^p(S,\mathbb{R}^3)$ to a function v which classically satisfies $\hat{\nabla} v(\hat{x}) = \mathrm{bar}(\nu_{\hat{x}})$ for a.e. $\hat{x}$ in S. Then $\gamma_S(u) = v$ on S. Let $\eta > 0$ intended to go to 0 and set $\Sigma_\eta := S\times(-\frac{\eta}{2},\frac{\eta}{2})$. We are going to modify u_n on Σ_η in order that the trace on S of the new function be equal to v_n, and in such a way to decrease $\lim_{n\to+\infty}\int_\Omega f(\nabla u_n)\,dx$. Let $\varphi\in\mathcal{C}_c^\infty(\mathbb{R})$ satisfying

$$\begin{aligned}&\varphi_\eta = 1 \quad \text{on } \Omega\setminus\Sigma_{2\eta}, \qquad \varphi_\eta = 0 \quad \text{on } \Sigma_\eta, \quad 0\leqslant\varphi_\eta\leqslant 1,\\ &|\nabla\varphi_i|\leqslant\frac{1}{\eta}\end{aligned}$$

and define

$$u_{n,\eta} = \varphi_\eta(u_n - v_n) + v_n. \tag{11}$$

Clearly $u_{n,\eta}$ belongs to $W^{1,p}_{\Gamma_0}(\Omega,\mathbb{R}^3)$ and $\gamma_s(u_{n,\eta}) = v_n$. Moreover

$$\begin{aligned}\int_\Omega f(\nabla u_{n,\eta})\,dx &= \int_{\Sigma_\eta} f(\nabla u_{n,\eta})\,dx + \int_{\Sigma_{2\eta}\setminus\Sigma_\eta} f(\nabla u_{n,\eta})\,dx + \int_{\Omega\setminus\Sigma_{2\eta}} f(\nabla u_{n,\eta})\,dx\\ &\leqslant \int_{\Sigma_\eta} f(\nabla v_n)\,dx + \int_{\Sigma_{2\eta}\setminus\Sigma_\eta} f(\nabla u_{n,\eta})\,dx + \int_\Omega f(\nabla u_n)\,dx.\end{aligned}$$

Thus, from the growth condition in (4),

$$\int_\Omega f(\nabla u_{n,\eta})\,dx \leqslant C\left(\eta + \frac{1}{\eta^p}\int_{\Sigma_{2\eta}} |u_n - v_n|^p\,dx + \int_{\Sigma_{2\eta}} \left(|\nabla u_n|^p\right)dx\right) + \int_\Omega f(\nabla u_n)\,dx$$

where, from now on, C denotes various positive constants depending only on β, p and Ω. Letting $n\to+\infty$, from (10) we obtain

$$\limsup_{n\to+\infty}\int_\Omega f(\nabla u_{n,\eta})\,dx \leqslant C\left(\eta + \frac{1}{\eta^p}\int_{\Sigma_{2\eta}} |u_-v_|^p\,dx + \sup_{n\in\mathbb{N}}\int_{\Sigma_{2\eta}} \left(|\nabla u_n|^p\right)dx\right) + \int_\Omega Qf(\nabla u)\,dx.$$

But since $\gamma_S(u) = v$ on S, the following Poincaré inequality holds

$$\int_{\Sigma_{2\eta}} |u - v|^p \, dx \leqslant \eta^p \int_{\Sigma_{2\eta}} \left| \frac{\partial u}{\partial x_3} \right|^p dx$$

so that, letting $\eta \to 0$, from the uniform integrability of $(|\nabla u_n|)_{n\in\mathbb{N}}$

$$\limsup_{\eta\to+\infty} \limsup_{n\to\infty} \int_\Omega f(\nabla u_{n,\eta}) \, dx \leqslant \int_\Omega \mathcal{Q}f(\nabla u) \, dx.$$

We conclude by a standard diagonalization argument: there exists $n \mapsto \eta(n)$ such that

$$\limsup_{n\to+\infty} \int_\Omega f(\nabla u_{n,\eta(n)}) \, dx \leqslant \int_\Omega \mathcal{Q}f(\nabla u) \, dx.$$

It is easily checked that sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$ defined by $\tilde{u}_n = u_{n,\eta(n))}$ strongly converges to u in $L^p(\Omega, \mathbb{R}^3)$ which completes the proof. $\square$

Proposition 3 *(Upper bound). For all $(u, \nu) \in L^p(\Omega, \mathbb{R}^3) \times \mathcal{Y}_{3\times2}(S)$ there exists a sequence $((u_\varepsilon, \mu_\varepsilon))_{\varepsilon>0}$ in $L^p(\Omega, \mathbb{R}^3) \times \mathcal{Y}_{3\times3}(B)$ converging to (u, ν) and satisfying*

$$\mathcal{F}(u, \nu) \geqslant \limsup_{\varepsilon\to0} \mathcal{F}_\varepsilon(u_\varepsilon, \mu_\varepsilon).$$

Proof. One can assume $\mathcal{F}(u, \nu) < +\infty$ so that $(u, \nu) \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3) \times \hat{\nabla}\mathcal{Y}_{3\times2}(S)$. Classically, there exists $v_n \in W^{1,p}(S, \mathbb{R}^3)$ such that $(|\hat{\nabla} v_n|^p)_{n\in\mathbb{N}}$ is uniformly integrable and $\delta_{\hat{\nabla} v_n(\hat{x})} \otimes d\hat{x} \overset{\text{nar}}{\rightharpoonup} \nu$ in $\mathcal{Y}_{3\times2}(S)$ (Proposition 7 of Appendix A). Note that we also have

$$(\delta_{\hat{\nabla} v_n(\hat{x})} \otimes \delta_{0_{\mathbb{R}^3}}) \otimes dx \overset{\text{nar}}{\rightharpoonup} (\nu_{\hat{x}} \otimes \delta_{0_{\mathbb{R}^3}}) \otimes dx \tag{12}$$

in $\mathcal{Y}_{3\times3}(B)$ when $n \to +\infty$. Since g_0 satisfies a growth condition of order p, we have (Proposition 6 of Appendix A)

$$\lim_{n\to+\infty} \int_S g_0(\hat{\nabla} v_n) \, d\hat{x} = \int_{S\times\mathbf{M}^{3\times2}} g_0(\hat{\lambda}) \, d\nu.$$

But, according to a classical interchange argument between infimum and integrals (see [2])

$$\int_S g_0(\hat{\nabla} v_n) \, d\hat{x} = \inf_{\xi\in\mathcal{D}(S,\mathbb{R}^3)} \int_S g(\hat{\nabla} v_n \mid \xi) \, d\hat{x}.$$

Let ξ_n in $\mathcal{D}(S, \mathbb{R}^3)$ satisfying

$$\left| \int_S g(\hat{\nabla} v_n \mid \xi_n) \, d\hat{x} - \int_S g_0(\hat{\nabla} v_n) \, d\hat{x} \right| \leqslant \frac{1}{n},$$

then

$$\lim_{n\to+\infty} \int_S g(\hat{\nabla} v_n \mid \xi_n) \, d\hat{x} = \int_{S\times\mathbf{M}^{3\times2}} g_0(\hat{\lambda}) \, d\nu. \tag{13}$$

Consider the function $v_{n,\varepsilon}$ in $W^{1,p}(B, \mathbb{R}^3)$ defined by

$$v_{n,\varepsilon}(x) = v_n(\hat{x}) + \varepsilon x_3 \xi_n(\hat{x}).$$

For fixed n, we first claim that

$$\mu_{n,\varepsilon} := \delta_{\nabla v_{n,\varepsilon}(x)} \otimes dx \overset{\text{nar}}{\rightharpoonup} (\delta_{\hat{\nabla} v_n(\hat{x})} \otimes \delta_{0_{\mathbb{R}^3}}) \otimes dx \tag{14}$$

in $\mathcal{Y}_{3\times 3}(B)$ when $\varepsilon \to 0$. Indeed, setting $\tilde{v}_n(x) := v_n(\hat{x})$, since $\nabla v_{n,\varepsilon} - \nabla \tilde{v}_n \to 0$ strongly in $L^p(B, \mathbf{M}^{3\times 3})$, $\nabla v_{n,\varepsilon}$ and $\nabla \tilde{v}_n$ generates the same Young measure $(\delta_{\hat{\nabla} v_n(\hat{x})} \otimes \delta_{0_{\mathbb{R}^3}}) \otimes dx$ in $\mathcal{Y}_{3\times 3}(B)$.

On the other hand, according to the classical relaxation theory in Sobolev spaces, there exists $u_n \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)$ strongly converging to u in $L^p(\Omega, \mathbb{R}^3)$ such that

$$\lim_{n\to+\infty} \int_\Omega f(\nabla u_n)\,dx = \int_\Omega Qf(\nabla u)\,dx \tag{15}$$

and we can modify u_n near S in such a way that $\gamma_S(u_n) = v_n$ (see Lemma 2 of Section 3). Let $\eta > \varepsilon$ and consider the function $u_{n,\varepsilon,\eta}$ defined for all x in Ω by

$$u_{n,\varepsilon,\eta}(\hat{x}, x_3) := \theta(x_3) v_{n,\varepsilon}\left(\hat{x}, \frac{x_3}{\varepsilon}\right) + \big(1 - \theta(x_3)\big) u_n(\hat{x}, x_3)$$

where θ is a $\mathcal{C}^1$-function satisfying $0 \leqslant \theta \leqslant 1$, $|\frac{\partial\theta}{\partial x_3}| \leqslant \frac{1}{\eta-\varepsilon}$ and

$$\theta = \begin{cases} 1 & \text{in } B_\varepsilon, \\ 0 & \text{in } \Omega \setminus B_\eta. \end{cases}$$

To shorten notation, we do not indicate the dependence on η and ε for θ. Note that $v_{n,\varepsilon,\eta} \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)$ and $r_\varepsilon u_{n,\varepsilon,\eta} = v_{n,\varepsilon}$ on B. From the local Lipschitz property in (4) satisfied by g one can easily establish

$$\lim_{\varepsilon\to 0} \int_B g\left(\hat{\nabla} v_{n,\varepsilon} \,\Big|\, \frac{1}{\varepsilon} \frac{\partial v_{n,\varepsilon}}{\partial x_3}\right) dx = \int_S g(\hat{\nabla} v_n \mid \xi_n)\,d\hat{x}. \tag{16}$$

Let us write

$$\int_{\Omega_\varepsilon} f(\nabla u_{n,\varepsilon,\eta})\,dx = \int_{\Omega\setminus B_\eta} f(\nabla u_n)\,dx + \int_{B_\eta\setminus B_\varepsilon} f(\nabla u_{n,\varepsilon,\eta})\,dx. \tag{17}$$

We claim that $\lim_{\eta\to 0} \lim_{\varepsilon\to 0} \int_{B_\eta\setminus B_\varepsilon} f(\nabla u_{n,\varepsilon,\eta})\,dx. = 0$. Indeed since

$$\hat{\nabla} u_{n,\varepsilon,\eta} = \theta(\hat{\nabla} v_n + x_3 \hat{\nabla}\xi_n) + (1-\theta)\hat{\nabla} u_n,$$
$$\frac{\partial u_{n,\varepsilon,\eta}}{\partial x_3} = \frac{\partial\theta}{x_3}(v_n - u_n) + x_3\xi_n \frac{\partial\theta}{x_3} + \theta\xi_n + (1-\theta)\frac{\partial u_n}{x_3},$$

the following estimate holds:

$$\left| \int_{B_\eta\setminus B_\varepsilon} f(\nabla u_{n,\varepsilon,\eta})\,dx \right| \leqslant C\left(\int_{B_\eta\setminus B_\varepsilon} h\left(\xi_n, \hat{\nabla}\xi_n, \hat{\nabla} u_n, \hat{\nabla} v_n \frac{\partial u_n}{\partial x_3}\right) dx \right.$$
$$\left. + \int_{B_\eta\setminus B_\varepsilon} \left|\frac{\partial\theta}{x_3}\right|^p |v_n - u_n|^p\,dx + \int_{B_\eta\setminus B_\varepsilon} \left|x_3\xi_n \frac{\partial\theta}{x_3}\right|^p dx \right) \tag{18}$$

where h is Lebesgue integrable and does not depend on ε and η. Clearly the first term in (18) tend to 0 when ε then η go to 0. We estimate the two last terms. Since $\gamma_S(u_n) = v_n$ on S, Poincaré's inequality yields

$$\int_{B_\eta\setminus B_\varepsilon} \left|\frac{\partial\theta}{x_3}\right|^p |v_n - u_n|^p\,dx \leqslant \frac{1}{(\eta-\varepsilon)^p} \int_{B_\eta} |v_n - u_n|^p\,dx$$
$$\leqslant \frac{\eta^p}{(\eta-\varepsilon)^p} \int_{B_\eta} \big|\nabla(v_n - u_n)\big|^p\,dx$$

which tends to 0 when ε then η goes to 0. On the other hand

$$\int_{B_\eta \setminus B_\varepsilon} \left| x_3 \xi_n \frac{\partial \theta}{x_3} \right|^p dx \leqslant \frac{\eta^p}{(\eta-\varepsilon)^p} \int_{B_\eta} |\xi_n|^p \, dx$$

which tends to 0 for the same reason. Therefore, combining (16), (17) and (18) we obtain

$$\lim_{\eta\to 0} \lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_{n,\varepsilon,\eta}, \mu_{n,\varepsilon}) = \int_\Omega f(\nabla u_n)\, dx + \int_S g(\hat{\nabla} v_n \mid \xi_n)\, d\hat{x}.$$

Then, using a standard diagonalization argument, there exists a map $\varepsilon \mapsto \eta(\varepsilon)$ satisfying $\eta(\varepsilon) \to 0$ whenever $\varepsilon \to 0$ and, setting $u_{n,\varepsilon} := u_{n,\varepsilon,\eta(\varepsilon)}$,

$$\lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_{n,\varepsilon}, \mu_{n,\varepsilon}) = \int_\Omega f(\nabla u_n)\, dx + \int_S g(\hat{\nabla} v_n \mid \xi_n)\, d\hat{x} \tag{19}$$

for all fixed $n \in \mathbb{N}$. Moreover, going back to the expression of $u_{n,\varepsilon}$ we have

$$\int_\Omega |u_{n,\varepsilon} - u_n|^p \, dx \leqslant \int_{B_{\eta(\varepsilon)}} |v_n + x_3\xi_n - u_n|^p \, dx,$$

thus $u_{n,\varepsilon}$ strongly converges to u_n in $L^p(\Omega, \mathbb{R}^3)$ when $\varepsilon \to 0$.

Collecting (12), (13), (14), (15) and (19), we deduce the following convergence scheme where the first arrow indicates a convergence with respect to ε and the second to n:

$$\mu_{n,\varepsilon} \overset{\text{nar}}{\rightharpoonup} (\delta_{\hat{\nabla} v_n(\hat{x})} \otimes \delta_{0_{\mathbb{R}^3}}) \otimes dx \overset{\text{nar}}{\rightharpoonup} (\nu_{\hat{x}} \otimes \delta_{0_{\mathbb{R}^3}}) \otimes dx;$$

$$u_{n,\varepsilon} \to u_n \to u \quad \text{strongly in } L^p\big(\Omega, \mathbb{R}^3\big);$$

$$\mathcal{F}_\varepsilon(u_{n,\varepsilon}, \mu_{n,\varepsilon}) \to \int_\Omega f(\nabla u_n)\, dx + \int_S g(\hat{\nabla} v_n \mid \xi_n)\, d\hat{x} \to \int_\Omega Qf(\nabla u)\, dx + \int_S g_0(\hat{\lambda})\, d\nu.$$

The conclusion of Proposition 3 then follows by using a standard diagonalization argument[1] in the product space $L^p(\Omega, \mathbb{R}^3) \times \mathcal{Y}_{3\times 3}(B) \times \mathbb{R}$ and noticing that $\mathcal{A}((\nu_{\hat{x}} \otimes \delta_{0_{\mathbb{R}^3}}) \otimes dx) = \nu$. □

On account of Definition 2, Proposition 1, Lemma 1 and Propositions 2, 3 above, we can state the main theorem of this section.

Theorem 2. *The sequence of functionals $(\mathcal{F}_\varepsilon - L)_{\varepsilon>0}$ $\Gamma_{X,Y,\hat{Y}}$-converges to the functional $\mathcal{F} - L$. In addition, if $(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon) \in L^p(\Omega, \mathbb{R}^3) \times \mathcal{Y}_{3\times 3}(B)$ is a ε-minimizer of $\mathcal{F}_\varepsilon - L$, i.e. satisfies*

$$\mathcal{F}_\varepsilon(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon) - L(\bar{u}_\varepsilon) \leqslant \varepsilon + \inf_{(u,\mu) \in L^p(\Omega,\mathbb{R}^3)\times \mathcal{Y}_{3\times 3}(B)} \big(\mathcal{F}_\varepsilon(u,\mu) - L(u)\big),$$

then there exists a subsequence of $((\bar{u}_\varepsilon, \bar{\mu}_\varepsilon))_{\varepsilon>0}$ converging to $(\bar{u}, \bar{\nu})$ which is a minimizer of $\inf_{(u,\nu)\in L^p(\Omega,\mathbb{R}^3)\times \mathcal{Y}_{3\times 2}(S)} (\mathcal{F}(u,\nu) - L(u))$. Moreover $\gamma_S(\bar{u})$ belongs to $W^{1,p}(S, \mathbb{R}^3)$ and, for a.e. $\hat{x}$ in S $\operatorname{bar}(\bar{\nu}_{\hat{x}}) = \hat{\nabla}\gamma_S(\bar{u})(\hat{x})$.

Proof. The proof is a straightforward consequence of Lemma 1, Propositions 2 and 3. Indeed, L is easily seen to be a continuous perturbation of $\mathcal{F}_\varepsilon$, so that $\mathcal{F}_\varepsilon \overset{\Gamma_{X,Y,\hat{Y}}}{\longrightarrow} \mathcal{F} \Longrightarrow \mathcal{F}_\varepsilon - L \overset{\Gamma_{X,Y,\hat{Y}}}{\longrightarrow} \mathcal{F} - L$. □

[1] Such an argument is valid because the set $\mathcal{Y}_{3\times 3}(B)$ endowed with the narrow topology is first countable (see [13, Proposition 2.3.1]).

4. The Young measure considered as an internal variable: The formulation in terms of displacements

In this section, we derive the classical model obtained in [1] in the context of the linear elasticity or in [10] in a more general setting, from the model obtained in Section 3. We show that the stored strain energy functional associated with the classical model is the marginal map of the functional limit $\mathcal{F}$ obtained in the previous section when we consider the Young measure ν, which represents the fine microstructure of the layer, as an internal variable. In some sense the formulation in terms of displacement can be regarded as the macroscopic version of the model suggested in Section 3 (see Corollary 1). With the notations of Sections 2, 3 we define the two functionals

$$H_\varepsilon, H : L^p\big(\Omega, \mathbb{R}^3\big) \to \mathbb{R} \cup \{+\infty\}$$

by

$$H_\varepsilon(u) := \begin{cases} \int_{\Omega_\varepsilon} f(\nabla u)\,dx + \int_B g(\hat{\nabla} r_\varepsilon u \mid \frac{1}{\varepsilon}\frac{\partial r_\varepsilon u}{\partial x_3})\,dx & \text{if } u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3), \\ +\infty & \text{otherwise} \end{cases}$$

and

$$H(u) := \begin{cases} \int_\Omega Qf(\nabla u)\,dx + \int_S Qg_0(\hat{\nabla}\gamma_S(u))\,d\hat{x} & \text{if } u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3),\ \gamma_S(u) \in W^{1,p}(S, \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

Clearly, H_ε is the marginal map associated with the functional $\mathcal{F}_\varepsilon$, i.e., for all $u \in L^p(\Omega, \mathbb{R}^3)$ we have

$$H_\varepsilon(u) = \inf_{\mu \in \mathcal{Y}_{3\times 3}(B)} \mathcal{F}_\varepsilon(u, \mu).$$

On the other hand H is the stored strain energy functional obtained in [10] in nonlinear elasticity, or in [1] in the linear elasticity framework where we have to replace ∇u with the linearized strain tensor $e(u) = \frac{1}{2}(\nabla u +^t \nabla u)$ and the quasiconvexifications of f and g_0 with their convexifications. In the next proposition, we establish that H is the marginal map associated with the limit functional $\mathcal{F}$.

Proposition 4. *The functional H is the marginal map associated with the functional $\mathcal{F}$ when the Young measure ν is considered as an internal variable. More precisely, for all $u \in L^p(\Omega, \mathbb{R}^3)$ we have*

$$H(u) = \inf_{\nu \in \mathcal{Y}_{3\times 2}(S)} \mathcal{F}(u, \nu).$$

Proof. We begin by introducing various sets. For every $\hat{A}$ in $\mathbf{M}^{3\times 2}$ we define the set $\operatorname{adm}(\hat{A})$ of probability measures on $\mathbf{M}^{3\times 2}$ by:

$$\mathbf{P} \in \operatorname{adm}(\hat{A}) \iff \begin{cases} \hat{A} = \int_{\mathbf{M}^{3\times 2}} \hat{\lambda}\, d\mathbf{P}; \\ \int_{\mathbf{M}^{3\times 2}} |\hat{\lambda}|^p\, d\mathbf{P} < +\infty; \\ \phi(\hat{A}) \leqslant \int_{\mathbf{M}^{3\times 2}} \phi(\hat{\lambda})\, d\mathbf{P} \end{cases}$$

for all quasiconvex function ϕ *satisfying a growth condition of order* p. On the other hand, for each fixed $u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)$ such that $\gamma_S(u) \in W^{1,p}(S, \mathbb{R}^3)$ we define the subset $\operatorname{Adm}(u)$ of $\mathcal{Y}_{3\times 2}(S)$ by:

$$\nu \in \operatorname{Adm}(u) \iff \begin{cases} \nu \in \hat{\nabla}\mathcal{Y}_{3\times 2}(S); \\ \operatorname{bar}(\nu_{\hat{x}}) = \hat{\nabla}\gamma_S(u)(\hat{x}) \text{ for a.e. } \hat{x} \in S. \end{cases}$$

The proof is based on the following localization lemma.

Lemma 3. *With the notations above we have*

(i) $$\inf\left\{ \int_{\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\, d\mathbf{P}\colon \mathbf{P} \in \operatorname{adm}(\hat{A}) \right\} = Qg_0(\hat{A}) \quad \textit{for all } \hat{A} \in \mathbf{M}^{3\times 2};$$

(ii) $$\inf_{\nu \in \operatorname{Adm}(u)} \int_{S\times \mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\, d\nu = \int_S \inf_{\mathbf{P} \in \operatorname{adm}(\hat{\nabla}\gamma_S(u)(\hat{x}))} \left(\int_{\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\, d\mathbf{P} \right) d\hat{x}.$$

Proof of Lemma 3. *Proof of (i)*: For every $\mathbf{P} \in \mathrm{adm}(\hat{A})$ one has

$$\int_{\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\, d\mathbf{P} \geqslant \int_{\mathbf{M}^{3\times 2}} Qg_0(\hat{\lambda})\, d\mathbf{P} \geqslant Qg_0(\hat{A})$$

so that $\inf\{\int_{\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\, d\mathbf{P}\colon\ \mathbf{P} \in \mathrm{adm}(\hat{A})\} \geqslant Qg_0(\hat{A})$.

For the converse inequality, set $\hat{Y} =]0,1[^2$, fix $\psi \in W_0^{1,p}(\hat{Y},\mathbb{R}^3)$ and define the probability measure $\mathbf{P}_\psi$ by

$$\mathbf{P}_\psi := \int_{\hat{Y}} \delta_{\hat{A}+\nabla\psi(\hat{y})}\, d\hat{y}$$

which acts on every continuous function $\varphi : \mathbf{M}^{3\times 2} \to \mathbb{R}$ satisfying a growth condition of order p as follows:

$$\int_{\mathbf{M}^{3\times 2}} \varphi(\hat{\lambda})\, d\mathbf{P}_\psi := \int_{\hat{Y}} \varphi\big(\hat{A} + \nabla\psi(\hat{y})\big)\, d\hat{y}.$$

Clearly $\mathbf{P}_\psi \in \mathrm{adm}(\hat{A})$, consequently

$$\mathcal{E}(\hat{A}) := \big\{\mathbf{P}_\psi\colon\ \psi \in W_0^{1,p}\big(\hat{Y},\mathbb{R}^3\big)\big\} \subset \mathrm{adm}(\hat{A}).$$

It follows that

$$\begin{aligned}\inf\bigg\{\int_{\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\, d\mathbf{P}\colon\ \mathbf{P} \in \mathrm{adm}(\hat{A})\bigg\} &\leqslant \inf\bigg\{\int_{\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\, d\mathbf{P}\colon\ \mathbf{P} \in \mathcal{E}(\hat{A})\bigg\}\\ &= \inf_{\phi \in W_0^{1,p}(\hat{Y},\mathbb{R}^3)} \int_{\hat{Y}} g_0\big(\hat{A} + \phi(\hat{y})\big)\, d\hat{y}\\ &= Qg_0(\hat{A}).\end{aligned}$$

In the last equality we have used the quasiconvex envelop formula for real-valued functions satisfying a *growth condition of order* p (see [15]).

Proof of (ii): Since $\nu \in \mathrm{Adm}(u)$ yields $\nu_{\hat{x}} \in \mathrm{adm}(\hat{\nabla}\gamma_S(u)(\hat{x}))$, clearly we have

$$\inf_{\nu \in \mathrm{Adm}(u)} \int_S \int_{\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\, d\nu_{\hat{x}}\, d\hat{x} \geqslant \int_S \inf_{\mathbf{P} \in \mathrm{adm}(\hat{\nabla}\gamma_S(u)(\hat{x}))} \bigg(\int_{\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\, d\mathbf{P}\bigg)\, d\hat{x}.$$

Conversely, for all $\eta > 0$, and $\hat{x} \in S$, let $\mathbf{P}^\eta_{\hat{x}}$ in $\mathrm{adm}(\hat{\nabla}\gamma_S(u)(\hat{x}))$ satisfying

$$\inf_{\mathbf{P} \in \mathrm{adm}(\hat{\nabla}\gamma_S(u)(\hat{x}))} \bigg(\int_{\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\, d\mathbf{P}\bigg) \geqslant \int_{\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\, d\mathbf{P}^\eta_{\hat{x}} - \eta. \tag{20}$$

We can assume that the map $\hat{x} \mapsto \mathbf{P}^\eta_{\hat{x}}$ is measurable (see [14]). Set $\nu := \mathbf{P}^\eta_{\hat{x}} \otimes d\hat{x}$. As $\mathbf{P}^\eta_{\hat{x}} \in \mathrm{adm}(\hat{\nabla}\gamma_S(u)(\hat{x}))$, the Young measure ν belongs to $\mathrm{Adm}(u)$ so that (20) yields, since η is arbitrary,

$$\int_S \inf_{\mathbf{P} \in \mathrm{adm}(\hat{\nabla}\gamma_S(u)(\hat{x}))} \bigg(\int_{\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\, d\mathbf{P}\bigg)\, d\hat{x} \geqslant \inf_{\nu \in \mathrm{Adm}(u)} \int_S \int_{\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\, d\nu_{\hat{x}}\, d\hat{x}$$

which completes the proof of Lemma 3. □

Proof of Proposition 4 continued. According to Lemma 3 we obtain

$$\inf_{\nu\in\mathcal{Y}_{3\times 2}(S)}\mathcal{F}(u,\nu)=\int_\Omega Qf(\nabla u)\,dx+\inf_{\nu\in\mathrm{Adm}(u)}\int_{S\times\mathbf{M}^{3\times 2}} g_0(\hat\lambda)\,d\nu$$
$$=\int_\Omega Qf(\nabla u)dx+\int_S \inf_{\mathbf{P}\in\mathrm{adm}(\hat\nabla\gamma_S(u)(\hat x))}\Bigl(\int_{\mathbf{M}^{3\times 2}} g_0(\hat\lambda)\,d\mathbf{P}\Bigr)d\hat x$$
$$=\int_\Omega Qf(\nabla u)\,dx+\int_S Qg_0\bigl(\hat\nabla\gamma_S(u)\bigr)\,d\hat x$$

which proves the proposition. □

Applying Proposition 4, Lemma 1 and Theorem 1 we recover the classical nonlinear model of multimaterial with strong interface obtained in [1] and [10]. Precisely

Theorem 3. *Let us equip the space $X=L^p(\Omega,\mathbb{R}^3)$ with the strong convergence. Then the sequence of functionals $(H_\varepsilon)_{\varepsilon>0}$ Γ_X-converges to the functional H. In addition, if $\bar u_\varepsilon\in L^p(\Omega,\mathbb{R}^3)$ is a ε-minimizer of $H_\varepsilon-L$, i.e. which satisfies*

$$H_\varepsilon(\bar u_\varepsilon)-L(\bar u_\varepsilon)\leqslant\varepsilon+\inf_{u\in L^p(\Omega,\mathbb{R}^3)}\bigl(H_\varepsilon-L(u)\bigr),$$

then there exists a subsequence of $(\bar u_\varepsilon)_{\varepsilon>0}$ which strongly converges in $L^p(\Omega,\mathbb{R}^3)$ and weakly in $W^{1,p}_{\Gamma_0}(\Omega,\mathbb{R}^3)$ to some $\bar u$ which is a minimizer of the classical nonlinear problem

$$\inf_{u\in L^p(\Omega,\mathbb{R}^3)}\bigl(H(u)-L(u)\bigr).$$

Let $(\bar u,\bar\nu)$ be a minimizer of $\inf_{(u,\nu)\in L^p(\Omega,\mathbb{R}^3)\times\mathcal{Y}_{3\times 2}(S)}(\mathcal{F}(u,\nu)-L(u))$. The next corollary states that $\bar u$ is a minimizer of the classical nonlinear problem and that at a.e. $\hat x$ in S, one may think $\bar\nu$ as the microscopic description of the macroscopic quantities $\hat\nabla\gamma_S(\bar u)(\hat x)$ and $Qg_0(\hat\nabla\gamma_S(\bar u)(\hat x))$.

Corollary 1. *Let $(\bar u,\bar\nu)$ be a minimizer of $\inf_{(u,\nu)\in L^p(\Omega,\mathbb{R}^3)\times\mathcal{Y}_{3\times 2}(S)}(\mathcal{F}(u,\nu)-L(u))$, then $\bar u$ is a minimizer of $\inf_{u\in L^p(\Omega,\mathbb{R}^3)}(H(u)-L(u))$. Moreover for a.e. $\hat x$ in S one has*

$$\hat\nabla\gamma_S(\bar u)(\hat x)=\int_{\mathbf{M}^{3\times 2}}\hat\lambda\,d\bar\nu_{\hat x};$$
$$Qg_0\bigl(\hat\nabla\gamma_S(\bar u)(\hat x)\bigr)=\int_{\mathbf{M}^{3\times 2}} g_0(\hat\lambda)\,d\bar\nu_{\hat x}.$$

Proof. We begin by proving the two equalities. The proof of the first one is straightforward since $\bar\nu\in\mathrm{Adm}(\bar u)$. For the same reason, for a.e. $\hat x$ in S we have

$$Qg_0\bigl(\hat\nabla\gamma_S(\bar u)(\hat x)\bigr)\leqslant\int_{\mathbf{M}^{3\times 2}} Qg_0(\hat\lambda)\,d\bar\nu_{\hat x}$$
$$\leqslant\int_{\mathbf{M}^{3\times 2}} g_0(\hat\lambda)\,d\bar\nu_{\hat x}.$$

The converse inequality is more involved and requires a localization argument. Let $\hat x_0$ be a fixed point of S, $Q_\rho(\hat x_0)=\hat Q_\rho(\hat x_0)\times(-\rho,\rho)$ the cylinder of $\mathbb{R}^3$ where $\hat Q_\rho(\hat x_0)$ is the ball of $\mathbb{R}^2$ centered at $\hat x_0\in S$. Write $\mathcal{F}^{\rho,\hat x_0}_\varepsilon$, $\mathcal{F}^{\rho,\hat x_0}$, $H^{\rho,\hat x_0}_\varepsilon$, $H^{\rho,\hat x_0}$ for the functionals $\mathcal{F}_\varepsilon$, $\mathcal{F}$, H_ε, H localized at $Q_\rho(\hat x_0)$ where the constraint $u\in W^{1,p}_{\Gamma_0}(\Omega,\mathbb{R}^3)$

is replaced by the constraint $u \in \bar{u} + W_0^{1,p}(Q_\rho(\hat{x}_0), \mathbb{R}^3)$. It is easy to check that $(\bar{u}, \bar{\nu})$ restricted to $Q_\rho(\hat{x}_0)$ is also a minimizer of the localized problem

$$\inf\{\mathcal{F}^{\rho,\hat{x}_0}(u,\nu):\ u \in L^p(Q_\rho(\hat{x}_0), \mathbb{R}^3),\ \nu \in \mathcal{Y}_{3\times 2}(\hat{Q}_\rho(\hat{x}_0))\}.$$

On the other hand, analysis similar to that in the proofs of Theorem 2 and Theorem 3 shows that $\mathcal{F}_\varepsilon^{\rho,\hat{x}_0} \xrightarrow{\Gamma_{X,Y,\hat{Y}}} \mathcal{F}^{\rho,\hat{x}_0}$ and $H_\varepsilon^{\rho,\hat{x}_0} \xrightarrow{\Gamma_X} H^{\rho,\hat{x}_0}$. From the second variational convergence process $\xrightarrow{\Gamma_X}$, there exists a sequence $(u_\varepsilon)_{\varepsilon>0}$ strongly converging to $\bar{u}$ in $L^p(Q_\rho(\hat{x}_0), \mathbb{R}^3)$ such that

$$\lim_{\varepsilon\to 0}\left(H_\varepsilon^{\rho,\hat{x}_0}(u_\varepsilon) - L(u_\varepsilon)\right) = \int_{Q_\rho(\hat{x}_0)} Qf(\nabla\bar{u})\,dx + \int_{\hat{Q}_\rho(\hat{x}_0)} Qg_0(\hat{\nabla}\gamma_S(\bar{u}))\,d\hat{x} - L(\bar{u}). \tag{21}$$

Moreover, according to the Prokhorov compactness theorem, $\mu_\varepsilon = \delta_{\nabla u_\varepsilon(x)} \otimes dx \xrightarrow{\mathcal{A}} \bar{\bar{\nu}}$ where $\bar{\bar{\nu}}$ is some Young measure in $\hat{\nabla}\mathcal{Y}_{3\times 2}(S)$. Now, from the first variational convergence process $\xrightarrow{\Gamma_{X,Y,\hat{Y}}}$, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\left(H_\varepsilon^{\rho,\hat{x}_0}(u_\varepsilon) - L(u_\varepsilon)\right) &= \lim_{\varepsilon\to 0}\left(\mathcal{F}_\varepsilon^{\rho,\hat{x}_0}(u_\varepsilon,\mu_\varepsilon) - L(u_\varepsilon)\right)\\ &\geqslant \int_{Q_\rho(\hat{x}_0)} Qf(\nabla\bar{u})\,dx + \int_{\hat{Q}_\rho(\hat{x}_0)}\left(\int_{\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\,d\bar{\bar{\nu}}_{\hat{x}}\right)d\hat{x} - L(\bar{u})\\ &\geqslant \int_{Q_\rho(\hat{x}_0)} Qf(\nabla\bar{u})\,dx + \int_{\hat{Q}_\rho(\hat{x}_0)}\left(\int_{\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\,d\bar{\nu}_{\hat{x}}\right)d\hat{x} - L(\bar{u}).\end{aligned} \tag{22}$$

Combining (21) and (22) we deduce

$$\int_{\hat{Q}_\rho(\hat{x}_0)} Qg_0(\hat{\nabla}\gamma_S(\bar{u}))\,d\hat{x} \geqslant \int_{\hat{Q}_\rho(\hat{x}_0)}\left(\int_{\mathbf{M}^{3\times 2}} g_0(\lambda)\,d\bar{\nu}_{\hat{x}}\right)d\hat{x}.$$

By choosing $\hat{x}_0$ outside a suitable $\mathcal{H}^2$-negligible subset of S (take $\hat{x}_0$ a Lebesgue point of each two integrands in each two members), dividing the two members by $\mathcal{H}^2(\hat{Q}_\rho(\hat{x}_0))$ and letting $\rho \to 0$, we obtain $Qg_0(\hat{\nabla}\gamma_S(\bar{u})(\hat{x}_0)) \geqslant \int_{\mathbf{M}^{3\times 2}} g_0(\lambda)\,d\bar{\nu}_{\hat{x}_0}$ which completes the proof of the second equality.

It remains to establish that $\bar{u}$ is a minimizer of the classical nonlinear problem. Indeed, from the second equality previously established we infer

$$\begin{aligned}\int_\Omega Qf(\nabla\bar{u})\,dx + \int_S Qg_0(\hat{\nabla}\gamma_S(\bar{u}))\,d\hat{x} - L(\bar{u}) &= \mathcal{F}(\bar{u},\bar{\nu}) - L(\bar{u})\\ &= \inf_{u\in L^p(\Omega,\mathbb{R}^3)}\left(\inf_{\nu\in\mathcal{Y}_{3\times 2}(S)} \mathcal{F}(u,\nu) - L(u)\right)\\ &= \inf_{u\in L^p(\Omega,\mathbb{R}^3)}\left(H(u) - L(u)\right)\end{aligned}$$

which completes the proof of the corollary. □

5. The displacement considered as an internal variable: The formulation in terms of Young measure

Since $\mathcal{H}^1(\Gamma_0 \cap \bar{S}) > 0$, for each $\nu \in \hat{\nabla}\mathcal{Y}_{3\times 2}(S)$ there exists a unique function u in $W^{1,p}(S, \mathbb{R}^3)$ satisfying $u = 0$ on $\Gamma_0 \cap \bar{S}$, defined for a.e. $\hat{x}$ in S by $u(\hat{x}) = \hat{\nabla}^{-1}(\mathrm{bar}(\nu_{\hat{x}}))$. For every fixed measure ν in $\hat{\nabla}\mathcal{Y}_{3\times 2}(S)$, we consider the set

$$W(\nu) := \{u \in W_{\Gamma_0}^{1,p}(\Omega, \mathbb{R}^3):\ \gamma_S(u)(\hat{x}) = \hat{\nabla}^{-1}(\mathrm{bar}(\nu_{\hat{x}})) \text{ for a.e. } \hat{x} \text{ in } S\}$$

and define the functional $G : \mathcal{Y}_{3\times 2}(S) \to \mathbb{R} \cup \{+\infty\}$ by

$$G(\nu) := \begin{cases} \inf_{u\in W(\nu)}(\int_\Omega Qf(\nabla u)\,dx - L(u)) + \int_{S\times\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\,d\nu & \text{if } \nu \in \hat{\nabla}\mathcal{Y}_{3\times 2}(S),\\ +\infty & \text{otherwise.}\end{cases}$$

It is straightforward to see that G is the marginal map of the functional $\mathcal{F}$ when u is considered as an internal variable, namely $G := \inf_{u\in L^p(\Omega,\mathbb{R}^3)} \mathcal{F}(u,.)$. It is also interesting to notice that G is a sum of a bulk and surface energy, precisely:

$$G(\nu) = \int_\Omega h\big(x, \mathrm{bar}(\nu_{\hat{x}})\big)\,dx + \int_S \bigg(\int_{\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\,d\nu_{\hat{x}}\bigg)\,d\hat{x}.$$

(Take $h := Qf \circ \nabla\bar{u}$ where $\bar{u}$ is a solution of $\inf_{u\in W(\nu)}(\int_\Omega Qf(\nabla u)\,dx - L(u))$.)

Applying Theorem 1, we deduce that $G_\varepsilon := \inf_{u\in L^p(\Omega,\mathbb{R}^3)} \mathcal{F}_\varepsilon(u,.) \xrightarrow{\Gamma_{Y,\hat{Y}}} G$. The formulation of the model in terms of Young measure is then given by the problem $\bar{\nu} \in \arg\min G$. From this formulation, we deduce that a minimizer $\bar{u}$ of the classical formulation is solution of a Dirichlet problem (in a variational form) with the following boundary condition: $\bar{u}(\hat{x}) = \hat{\nabla}^{-1}(\mathrm{bar}(\nu_{\hat{x}}))$ on S. Precisely

Corollary 2. *Let $(\bar{u},\bar{\nu})$ be a minimizer of $\inf_{(u,\nu)\in L^p(\Omega,\mathbb{R}^3)\times\mathcal{Y}_{3\times 2}(S)}(\mathcal{F}(u,\nu) - L(u))$, then $\bar{u}$ is a minimizer of the Dirichlet problem*

$$\inf\bigg\{\int_{\Omega\setminus S} Qf(\nabla u)\,dx - L(u)\colon\ u \in W^{1,p}_{\Gamma_0}\big(\Omega,\mathbb{R}^3\big),\ u(\hat{x}) = \hat{\nabla}^{-1}\big(\mathrm{bar}(\nu_{\hat{x}})\big) \text{ a.e. on } S\bigg\}.$$

Proof. On account of the $\Gamma_{Y,\hat{Y}}$-convergence of G_ε to G, there exists a sequence $\mu_\varepsilon \in \mathcal{Y}_{3\times 3}(B)$ satisfying $\mu_\varepsilon \overset{\mathcal{A}}{\rightharpoonup} \bar{\nu}$ and

$$\lim_{\varepsilon\to 0} G_\varepsilon(\mu_\varepsilon) = \inf_{u\in W(\bar{\nu})}\bigg(\int_\Omega Qf(\nabla u)\,dx - L(u)\bigg) + \int_{S\times\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\,d\bar{\nu}. \tag{23}$$

But since $G_\varepsilon(\mu_\varepsilon) < +\infty$, there exists u_ε in $W^{1,p}_{\Gamma_0}(\Omega,\mathbb{R}^3)$ such that $\mu_\varepsilon = \delta_{\nabla(r_\varepsilon u_\varepsilon)} \otimes dx$ in $\mathcal{Y}_{3\times 3}(B)$ and $G_\varepsilon(\mu_\varepsilon) = \mathcal{F}_\varepsilon(u_\varepsilon,\mu_\varepsilon)$. From Lemma 1, a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ strongly converges to some $\bar{\bar{u}}$ in $L^p(\Omega,\mathbb{R}^3)$. Then, according to the $\Gamma_{X,Y,\hat{Y}}$ convergence of $\mathcal{F}_\varepsilon - L$ to $\mathcal{F} - L$, we deduce

$$\begin{aligned}
\lim_{\varepsilon\to 0} G_\varepsilon(\mu_\varepsilon) &= \lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon,\mu_\varepsilon) - L(u_\varepsilon) \geqslant \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon,\mu_\varepsilon) - L(u_\varepsilon)\\
&\geqslant \int_\Omega Qf(\nabla\bar{\bar{u}})\,dx - L(\bar{\bar{u}}) + \int_{S\times\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\,d\bar{\nu}\\
&\geqslant \int_\Omega Qf(\nabla\bar{u})\,dx - L(\bar{u}) + \int_{S\times\mathbf{M}^{3\times 2}} g_0(\hat{\lambda})\,d\bar{\nu}.
\end{aligned} \tag{24}$$

Combining (23) and (24) we see that

$$\inf_{u\in W(\bar{\nu})}\bigg(\int_\Omega Qf(\nabla u)\,dx - L(u)\bigg) = \int_\Omega Qf(\nabla\bar{u})\,dx - L(\bar{u})$$

which ends the proof. □

Appendix A

For a general exposition of the theory of Young measures, we refer the reader to [7,8,26,27] and the references therein. In all the appendix, Ω is an open bounded subset of $\mathbb{R}^N$ and $E = \mathbb{R}^d$, $d = m \times N$ so that $\mathbb{R}^d$ is canonically isomorphic to the space $\mathbf{M}^{m\times N}$ of $m \times N$ matrices.

Definition 3. We call Young measure on $\Omega \times E$, any positive measure $\mu \in \mathbf{M}^+(\Omega \times E)$ whose image by the projection π_Ω on Ω is the Lebesgue measure $\mathcal{L}$ on Ω: for every Borel subset B of Ω

$$\pi_\Omega \# \mu(B) := \mu(B \times E) = \mathcal{L}(B).$$

We denote by $\mathcal{Y}(\Omega; E)$ the set of all Young measures on $\Omega \times E$ and equip $\mathcal{Y}(\Omega; E)$ with the narrow topology, that is the weakest topology which makes the maps

$$\mu \mapsto \int_{\Omega \times E} \varphi \, d\mu$$

continuous, where φ runs through $\mathcal{C}_b(\Omega; E)$. This topology induces the narrow convergence of Young measures defined as follows: let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of measures in $\mathcal{Y}(\Omega; E)$ and $\mu \in \mathcal{Y}(\Omega; E)$, then

$$\mu_n \overset{\text{nar}}{\rightharpoonup} \mu \quad \Longleftrightarrow \quad \begin{cases} \forall \varphi \in \mathcal{C}_b(\Omega; E), \\ \lim_{n\to+\infty} \int_{\Omega\times E} \varphi(x, \lambda)\, d\mu_n(x, \lambda) = \int_{\Omega\times E} \varphi(x, \lambda)\, d\mu(x, \lambda). \end{cases}$$

The following slicing property, is a generalization of Fubini's theorem.

Theorem 4. *Let μ be any Young measure in $\mathcal{Y}(\Omega; E)$. There exists a family of probability measure $(\mu_x)_{x\in\Omega}$ on E, unique up to equality $\mathcal{L}$-a.e. such that*

(i) *$x \mapsto \int_E \psi(x, \Lambda)\, d\mu_x$ is $\mathcal{L}$-measurable,*
(ii) *$\int_{\Omega\times E} \psi(x, \Lambda)\, d\mu(x, \Lambda) = \int_\Omega (\int_E \psi(x, \Lambda)\, d\mu_x(\Lambda))\, dx$*

for each function μ-integrable ψ. The family $(\mu_x)_{x\in\Omega}$ is called a disintegration of the Young measure μ and we write $\mu = \mu_x \otimes \mathcal{L}$.

Let us define the tightness notion for Young measures

Definition 4. A subset $\mathcal{H}$ of $\mathcal{Y}(\Omega; E)$ is said to be tight if

$$\forall \varepsilon > 0,\ \exists \mathcal{K}_\varepsilon \text{ compact subset of } E \text{ such that } \sup_{\mu\in\mathcal{H}} \mu(\Omega \times E \setminus \mathcal{K}_\varepsilon) < \varepsilon.$$

Theorem below may be considered as the parametrized version of the classical Prokhorov compactness theorem

Theorem 5 *(Prokhorov's compactness theorem). Let $(\mu^n)_{n\in\mathbb{N}}$ be a tight sequence in $\mathcal{Y}(\Omega; E)$. Then, there exists a subsequence $(\mu^{n_k})_{k\in\mathbb{N}}$ of $(\mu^n)_{n\in\mathbb{N}}$ and μ in $\mathcal{Y}(\Omega; E)$ such that*

$$\mu_{n_k} \overset{\text{nar}}{\rightharpoonup} \mu \quad \text{in } \mathcal{Y}(\Omega; E).$$

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions $u_n : \Omega \to E$ and consider the sequence of their associated Young measures $(\mu_n)_{n\in\mathbb{N}}$, $\mu_n = \delta_{u_n(x)} \otimes \mathcal{L}$. If $\mu_n \overset{\text{nar}}{\rightharpoonup} \mu$ in $\mathcal{Y}(\Omega; E)$, the Young measure μ is said to be *generated by* the sequence of functions $(u_n)_{n\in\mathbb{N}}$. In general, μ is not associated with a function.

The next proposition is a semicontinuity result related to nonnegative functions.

Proposition 5. *Let $\varphi : \Omega \times E \to [0, +\infty]$ be a $\mathcal{B}(\Omega) \otimes \mathcal{B}(E)$ measurable function such that $\lambda \mapsto \varphi(x, \lambda)$ is lsc for a.e. x in Ω. Let moreover $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of Young measures in $\mathcal{Y}(\Omega; E)$ narrowly converging to some Young measure μ in $\mathcal{Y}(\Omega; E)$. Then*

$$\int_{\Omega\times E} \varphi(x, \lambda)\, d\mu(x, \lambda) \leqslant \liminf_{n\to+\infty} \int_{\Omega\times E} \varphi(x, \lambda)\, d\mu_n(x, \lambda).$$

Let us recall the notion of uniform integrability: a sequence $(f_n)_{n\in\mathbb{N}}$, $f_n : \Omega \to \mathbb{R}$ is said to be *uniformly integrable* if

$$\lim_{R\to+\infty} \sup_{n\in\mathbb{N}} \int_{[|f_n|>R]} |f_n| = 0.$$

One may extend the set $\mathcal{C}_b(\Omega, \mathbb{R}^m)$ of test functions related to the narrow convergence as follows:

Proposition 6. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of Young measures associated with a sequence of functions $(u_n)_{n\in\mathbb{N}}$, narrowly converging to some Young measure μ. On the other hand let $\varphi : \Omega \times E \to \mathbb{R}$ be a $\mathcal{B}(\Omega)\otimes\mathcal{B}(E)$ measurable function such that $\lambda \mapsto \varphi(x,\lambda)$ is continuous for a.e. x in Ω. Assume moreover that $x \mapsto \varphi(x, u_n(x))$ is uniformly integrable. Then*

$$\int_{\Omega\times E} \varphi(x,\lambda)\,d\mu(x,\lambda) = \lim_{n\to+\infty}\int_{\Omega} \varphi\big(x,u_n(x)\big)\,dx.$$

In order to apply Proposition 6, the following result is fundamental. For a proof, we refer the reader to [18,23].

Proposition 7. *Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $W^{1,p}_{\Gamma_0}(\Omega,\mathbb{R}^m)$ whose gradients generate a $W^{1,p}$-Young measure μ. Then there exists another sequence $(v_n)_{n\in\mathbb{N}}$ in $W^{1,p}_{\Gamma_0}(\Omega,\mathbb{R}^m)$, whose gradients generate the same Young measure μ, and such that $(|\nabla v_n|^p)_{n\in\mathbb{N}}$ is uniformly integrable.*

We end this section with the following characterization theorem for $W^{1,p}$-Young measures (Young measures generated by gradients of $W^{1,p}$-functions), established by D. Kinderlehrer and P. Pedregal (see [19,23,24]).

Theorem 6. *Let $p > 1$. Then $\mu \in \mathcal{Y}(\Omega; E)$ is a $W^{1,p}$-Young measure iff there exists $u \in W^{1,p}(\Omega,\mathbb{R}^m)$ such that the three following assertions hold*:

(i) $\nabla u(x) = \int_E \lambda\,d\mu_x(\lambda)$,

(ii) *for all quasiconvex function ϕ satisfying a growth condition of order p one has*

$$\phi\big(\nabla u(x)\big) \leqslant \int_E \phi(\lambda)\,d\mu_x(\lambda) \quad \textit{for a.e. } x\in\Omega.$$

(iii) $\int_E |\lambda|^p\,d\mu(x) < +\infty$ *for a.e.$x \in \Omega$.*

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