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Invertibility of Sobolev mappings under minimal hypotheses

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Abstract

We prove a version of the Inverse Function Theorem for continuous weakly differentiable mappings. Namely, a nonconstant $W^{1,n}$ mapping is a local homeomorphism if it has integrable inner distortion function and satisfies a certain differential inclusion. The integrability assumption is shown to be optimal.
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1. Introduction

Throughout this paper Ω is a bounded domain in $\mathbb{R}^n$. The classical Inverse Function Theorem states that if $f:\Omega \to \mathbb{R}^n$ is continuously differentiable and the differential matrix $Df(x)$ is invertible at some point x, then f is a homeomorphism in a neighborhood of x. We are interested in a version of the Inverse Function Theorem for continuous weakly differentiable mappings. In this context the invertibility of the differential matrix is not sufficient. As an example, consider the winding mapping $f:\mathbb{R}^3 \to \mathbb{R}^3$ written in cylindrical coordinates as $f(r,\theta,z) = (r, 2\theta, z)$. Although f is Lipschitz and its Jacobian determinant $J(x,f)$ equals 2 for a.e. $x \in \mathbb{R}^n$, this mapping is not a local homeomorphism.

Let us introduce the following subset of $n \times n$ matrices.

$$\mathcal{M}(\delta) = \left\{A \in \mathbb{R}^{n\times n}\colon\ \langle A\xi, \xi\rangle \geqslant \delta |A\xi||\xi| \text{ for all } \xi \in \mathbb{R}^n\right\},$$

where $-1 \leqslant \delta \leqslant 1$. Note that $\delta = -1$ imposes no condition on the matrix. When $-1 < \delta < 0$, the set $\mathcal{M}(\delta)$ is not convex and the differential inclusion

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$$Df(x) \in \mathcal{M}(\delta) \quad \text{for a.e. } x \in \Omega, \tag{1.1}$$

cannot be integrated to yield a pointwise inequality for f.

The winding mapping does not satisfy (1.1) for any $\delta > -1$. Even so, this differential inclusion does not by itself guarantee that f is locally invertible, e.g., $f(x_1, x_2) = (x_1, 0)$. There are also such examples with strictly positive Jacobian [14, Example 18]. To quantify the invertibility of a matrix $A \in \mathbb{R}^{n\times n}$, we introduce the inner distortion $K_I(A) \in [1, \infty]$.

$$K_I(A) = \begin{cases} \frac{\|A^\sharp\|^n}{(\det A)^{n-1}}, & \det A > 0, \\ 1, & A = 0, \\ \infty, & \text{otherwise.} \end{cases} \tag{1.2}$$

Here $A^\sharp$ stands for the cofactor matrix of A and $\|\cdot\|$ is the operator norm. To shorten the notation we write $K_I(x, f) = K_I(Df(x))$ and

$$\mathscr{K}_\Omega[f] := \frac{1}{|\Omega|} \int_\Omega K_I(x, f)\,\mathrm{d}x,$$

where $|\Omega|$ is the Lebesgue measure of Ω. If $f \in W^{1,n}(\Omega, \mathbb{R}^n)$ and $K_I(x, f) < \infty$ a.e., then f has a logarithmic modulus of continuity [4,9]; that is,

$$\left|f(a) - f(b)\right|^n \leqslant \frac{C(n)\int_{2B} \|Df\|^n}{\log(e + \frac{2\operatorname{diam} B}{|a-b|})}, \quad a, b \in B,\ 2B \Subset \Omega.$$

In this paper we always take f to be its continuous representative.

If moreover $\mathscr{K}_\Omega[f] < \infty$ and f is invertible, then the inverse $h := f^{-1}$ is a $W^{1,n}$-mapping and

$$\int_\Omega K_I(x, f)\,\mathrm{d}x = \int_{f(\Omega)} \|Dh\|^n,$$

see [1, Theorem 9.1]. Thus $\mathscr{K}_\Omega[f]$ controls the modulus of continuity of f^{-1}, should it exist. Our main result addresses its existence.

Theorem 1.1. *Suppose that $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is a nonconstant mapping such that $\mathscr{K}_\Omega[f] < \infty$. If there exists $\delta > -1$ such that $Df(x) \in \mathcal{M}(\delta)$ for almost every $x \in \Omega$, then f is a local homeomorphism.*

This theorem is already known in the planar case $n = 2$ [14, Theorem 4]. The assumption $\mathscr{K}_\Omega[f] < \infty$ cannot be replaced by $\int_\Omega K_I^q(x, f)\,\mathrm{d}x < \infty$ for any $q < 1$, see [14, Example 18] or [2, Example 1].

Our proof of Theorem 1.1 is based on two results of independent interest. The first step toward proving that a mapping is a local homeomorphism is to show that it is discrete and open; that is, preimages of points are discrete sets and images of open sets are open.

Theorem 1.2. *Let $f : \Omega \to \mathbb{R}^n$ be a mapping in $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ such that $J(x, f) > 0$ a.e. If $(Df)^{-1} \in L^\infty(\Omega, \mathbb{R}^{n\times n})$, then f is discrete and open.*

The challenging Iwaniec–Šverák conjecture asserts even more: *a nonconstant mapping $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ with $\mathscr{K}_\Omega[f] < \infty$ is discrete and open.* So far this conjecture was proved only for $n = 2$ in [10]. Partial results in this direction were recently obtained in [6–8,15,19,20].

Another crucial ingredient of our proof of Theorem 1.1 is an estimate for the multiplicity $N(y, f, A) := \#(f^{-1}(y) \cap A)$ of a local homeomorphism f in terms of the integral of $K_I(\cdot, f)$ in dimensions $n \geqslant 3$. This result (Theorem 5.1) continues the line of development that began in 1967 with the celebrated Global Homeomorphism Theorem of Zorich [24].

The proof of Theorem 1.1 proceeds as follows. The differential inclusion (1.1) allows us to approximate f by mappings $f^\lambda(x) := f(x) + \lambda x$ to which Theorem 1.2 can be applied. The results of [14] yield that f^λ is a local homeomorphism. By virtue of Theorem 5.1 the mappings f^λ have uniformly bounded multiplicity, which leads to a bound for the essential multiplicity of f. This additional information suffices to show that f is discrete and open, see Proposition 2.2 below. Since f is a limit of local homeomorphisms f^λ, the conclusion follows.

Different approaches to the invertibility of Sobolev mappings were pursued in [2,3,5,16,18,22], see also references therein.

2. Background

In this section we collect necessary notation and preliminaries. An open ball with center a and radius r is denoted by $B(a,r) := \{x \in \mathbb{R}^n\colon |x-a| < r\}$. Its boundary is the sphere $S(a,r)$. If $\lambda > 0$ and $B = B(a,r)$, then $\lambda B = B(a, \lambda r)$ and $\lambda S = S(a, \lambda r)$. In addition, $\mathbb{B} = B(0,1)$, $\mathbb{B}_r = B(0,r)$, $\mathbb{S} = S(0,1)$ and $\mathbb{S}_r = S(0,r)$.

Let $\mathcal{H}^d$ stand for the d-dimensional Hausdorff measure which agrees with the Lebesgue measure when d coincides with the space dimension. The Hausdorff distance $\mathrm{d}_{\mathcal{H}}(E,F)$ between nonempty bounded sets E and F is defined as the infimum of numbers $\epsilon > 0$ such that the ϵ-neighborhood of E contains F and vice versa.

Given a continuous mapping $f : \Omega \to \mathbb{R}^n$ and a set $E \subset \Omega$, we denote by $N(y, f, E)$ the cardinality (possibly infinite) of the set $f^{-1}(y) \cap E$. If $y \in \mathbb{R}^n \setminus f(\partial\Omega)$, the local degree of f at y with respect to a domain $G \subset \Omega$ is denoted by $\deg(y, f, G)$. We write $f : A \xrightarrow{\text{hom}} B$ to indicate that f is a homeomorphism from A onto B.

Let Γ be a family of paths (parametrized curves) in $\mathbb{R}^n$, $n \geqslant 2$. The image of $\gamma \in \Gamma$ is denoted by $|\gamma|$. We let Υ_Γ be the set of all Borel functions $\rho : \mathbb{R}^n \to [0,\infty]$ such that

$$\int_\gamma \rho \, \mathrm{d}s \geqslant 1$$

for every locally rectifiable path $\gamma \in \Gamma$. The functions in Υ_Γ are called admissible for Γ. For a given weight $\omega : \mathbb{R}^n \to [0,\infty]$ we define

$$\mathsf{M}_\omega \Gamma = \inf_{\rho \in \Upsilon_\Gamma} \int \rho(x)^n \omega(x) \, \mathrm{d}x,$$

and call $\mathsf{M}_\omega \Gamma$ the weighted conformal modulus of Γ. Here it suffices to have ω defined on a Borel set containing $\bigcup_{\gamma \in \Gamma} |\gamma|$. When $\omega \equiv 1$ we obtain the conformal modulus $\mathsf{M}\Gamma$. We will also use the spherical modulus with respect to a sphere S,

$$\mathsf{M}^S \Gamma = \inf_{\rho \in \Upsilon_\Gamma} \int_S \rho(y)^n \, \mathrm{d}\mathcal{H}^{n-1}(y).$$

The reader may wish to consult the monographs [21,23] for basic properties of moduli of path families. The following generalization of the Poletsky inequality relates the moduli of Γ and of its image under f, denoted by $f\Gamma$.

Proposition 2.1. *(See [12].) Suppose that $f \in W^{1,n}(\Omega, \mathbb{R}^n)$ is a discrete and open mapping with $\mathscr{K}_\Omega[f] < \infty$. If Γ is a family of paths contained in Ω, then*

$$\mathsf{M} f\Gamma \leqslant \mathsf{M}_{K_I(\cdot, f)} \Gamma. \tag{2.1}$$

We will use the following result, which establishes the Iwaniec–Šverák conjecture under an additional assumption on the multiplicity of f.

Proposition 2.2. *Suppose that $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is a nonconstant mapping with $\mathscr{K}_\Omega[f] < \infty$. Let B be a ball such that $2B \Subset \Omega$. If*

$$\operatorname*{ess\,lim\,sup}_{r \to 0} r^{1-n} \int_{S(a,r)} N(y, f, B) \, \mathrm{d}\mathcal{H}^{n-1}(y) < \infty \tag{2.2}$$

for every $a \in \mathbb{R}^n$, then f is discrete and open in B.

This proposition is a consequence of [20, Theorem 2.2]. Although [20, Theorem 2.2] requires that

$$\operatorname*{ess\,sup}_{0<t<1} \int\limits_{\partial(tB)} \frac{\|D^\sharp f(x)\|}{|f(x)-a|^{n-1}}\,\mathrm{d}\mathcal{H}^{n-1}(x) < \infty,$$

this condition is only used to obtain (2.2).

3. Preliminary results

For the sake of brevity, the connected component of a set A that contains a point $x \in A$ will be called the x-component of A.

Proposition 3.1. *Suppose that $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is a mapping such that $\mathcal{K}_\Omega[f] < \infty$. Let $x \in \Omega$ and $y = f(x)$. If the x-component of $f^{-1}(y)$ is $\{x\}$, then f is discrete and open in some neighborhood of x.*

Proof. Pick $r > 0$ such that $B(x,r) \Subset \Omega$ and let U_j be the x-component of $(f^{-1}B(y,1/j)) \cap B(x,r)$, $j = 1, 2, \ldots.$ Since the sets $\overline{U}_j \subset \mathbb{R}^n$ are nested, compact, and connected, their intersection E is also connected. On the other hand, $x \in E \subset f^{-1}(y)$, hence $E = \{x\}$. It follows that $\operatorname{diam}(U_j) \to 0$ as $j \to \infty$. Let us fix j such that $U_j \Subset \Omega$. Note that U_j coincides with the x-component of $f^{-1}B(y,1/j)$.

We claim that f is quasilight in U_j; that is, the connected components of $f^{-1}(w) \cap U_j$ are compact for all $w \in \mathbb{R}^n$. If not, then there exists $z \in U_j$ such that the z-component of $f^{-1}(f(z))$ intersects ∂U_j at some point b. Since $f(b) = f(z) \in B(y,1/j)$, there exists $t > 0$ such that $fB(b,t) \subset B(y,1/j)$. This contradicts the definition of U_j. Therefore, f is quasilight in U_j. By [19, Theorem 1.1] f is discrete and open in U_j. $\square$

For the convenience of the reader we state two preliminary results from [21, III.3].

Lemma 3.2. *(See [21, III.3.1].) Let $f : \Omega \to \mathbb{R}^n$ be a local homeomorphism and let Q be a simply connected and locally pathwise connected set in $\mathbb{R}^n$. Suppose P is a component of $f^{-1}Q$ such that $\overline{P} \subset \Omega$. Then $f : P \overset{\text{hom}}{\longrightarrow} Q$. If in addition Q is relatively locally connected, then $f : \overline{P} \overset{\text{hom}}{\longrightarrow} \overline{Q}$.*

Lemma 3.3. *(See [21, III.3.3].) Let $f : \Omega \to \mathbb{R}^n$ be a local homeomorphism and let $A, B \subset \Omega$ be two sets such that f is homeomorphic in A and in B. If $A \cap B \neq \emptyset$ and if $fA \cap fB$ is connected, then f is homeomorphic in $A \cup B$.*

Given a sphere $S = S(a,r) \subset \mathbb{R}^n$, and a point $p \in S$, let $C_S(p,\phi)$ be the open spherical cap of S with center p and opening angle $\phi \in (0,\pi]$,

$$C_S(p,\phi) = \bigl\{ y \in S:\ \langle y-a, p-a\rangle > r^2\cos\phi \bigr\}.$$

For instance $C_S(p,\pi/2)$ is a hemisphere and $C_S(p,\pi)$ is a punctured sphere. For any $\phi \in (0,\pi]$ the cap $C_S(p,\phi)$ contains the point p.

The following topological lemma forms the main step of the proof of Zorich Global Homeomorphism Theorem, see [21, III.3].

Lemma 3.4. *Let $f : \Omega \to \mathbb{R}^n$ be a local homeomorphism, $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$. Suppose we have the following*:

(i) *$G \Subset \Omega$ such that $f : G \overset{\text{hom}}{\longrightarrow} G'$ where G' is convex*;
(ii) *$G \subset D \Subset \Omega$ and there is $a \in \partial G \cap \partial D$*;
(iii) *a ball $\mathcal{B} \subset \mathbb{R}^n$ that contains $a' = f(a)$ and such that $S = \partial\mathcal{B}$ meets G' at some point b'.*

Let $b = f^{-1}(b') \cap G$ and denote by $C^*_S(b',\phi)$ the component of $f^{-1}C_S(b',\phi)$ containing b. Then there exists $0 < \phi_0 < \pi$ such that $C^*_S(b',\phi_0) \subset D$ and the closure of $C^*_S(b',\phi_0)$ meets ∂D.

Proof. Let ϕ_0 be the supremum of all ϕ such that $C_S^*(b', \phi) \subset D$. First we observe that $\phi_0 > 0$. Indeed, since f is a local homeomorphism, there exists a neighborhood $V \subset D$ of b such that $f : V \xrightarrow{\text{hom}} f(V)$. If ϕ is sufficiently small, then $C_S(b', \phi) \Subset f(V)$, hence $C_S^*(b', \phi) \Subset V \subset D$. It remains to show that $\phi_0 < \pi$.

Suppose to the contrary that $\phi_0 = \pi$. Since $C_S^*(b', \pi) \subset D$, it follows from Lemma 3.2 that $f : \overline{C}_S^*(b', \pi) \xrightarrow{\text{hom}} \overline{C}_S(b', \pi) = S$ (here the assumption $n \geqslant 3$ is used). Since $S^* := \overline{C}_S^*(b', \pi)$ is homeomorphic to S, it separates $\mathbb{R}^n$ into two components. Let U be the bounded component of $\mathbb{R}^n \setminus S^*$. Then the boundary of $f(U)$ is contained in S which implies $f(U) = \mathscr{B}$. Moreover, $f : \overline{U} \xrightarrow{\text{hom}} \overline{\mathscr{B}}$ by Lemma 3.2. Since $b \in \overline{U} \cap \overline{G}$ and since $f(\overline{U}) \cap f(\overline{G}) = \overline{\mathscr{B}} \cap \overline{G}'$ is convex (hence connected), Lemma 3.3 yields that f is homeomorphic in $\overline{U} \cup \overline{G}$.

This leads to a contradiction. Since $\overline{U} \cup \overline{G} \subset \overline{D}$ it follows that a lies on the boundary of $\overline{U} \cup \overline{G}$. On the other hand, $f(a) = a' \in f(U)$ is an interior point of $f(\overline{U} \cup \overline{G})$. □

We shall use a geometric lemma which is essentially contained in [13].

Lemma 3.5. *Suppose we are given a ball $B(y_0, r) \subset \mathbb{R}^n$, a point $y_1 \in S(y_0, r)$ and a connected set E that contains y_0 and some point $y_2 \in S(y_0, r)$. Then there exist $q \in B(y_0, r)$ and $0 < \sigma < 2r$ such that for every $\sigma < t < 4\sigma/3$,*

(i) $y_1 \in B(q, t)$;
(ii) $S(q, t) \cap E \neq \emptyset$;
(iii) $S(q, t) \subset B(y_0, 2r) \setminus B(y_0, r/10)$.

Proof. Let α be the angle at the point $(y_0 + y_1)/2$ formed by the line segments from y_0 to $(y_0 + y_1)/2$ and from $(y_0 + y_1)/2$ to y_2. There are two cases possible.

Case 1. $0 \leqslant \alpha < \pi/2$, or, equivalently, $|y_1 - y_2| > r$. In this case we choose $q = (y_0 + y_1)/2$ and $\sigma = 3r/5$. For $\sigma < t < 4\sigma/3$ we have $B(y_0, r/10) \subset B(q, t)$ and $y_1 \in B(q, t)$. At the same time, $y_2 \notin \overline{B}(q, t)$ because

$$|y_2 - q| > \frac{\sqrt{3}}{2} r = \frac{5}{2\sqrt{3}} \sigma > \frac{4}{3} \sigma.$$

Thus, all conditions (i)–(iii) are satisfied.

Case 2. $\pi/2 \leqslant \alpha \leqslant \pi$, or, equivalently, $|y_1 - y_2| \leqslant r$. This time we choose $q = (y_1 + y_2)/2$ and $\sigma = |y_1 - y_2|/2$. Since $|y_0 - q| \geqslant (\sqrt{3}/2)r$, it follows that $\overline{B}(q, t) \cap B(y_0, r/10) = \emptyset$ provided that

$$t < \left(\frac{\sqrt{3}}{2} - \frac{1}{10} \right) r.$$

This is indeed the case, because

$$\frac{4}{3} \sigma \leqslant \frac{2}{3} r < \left(\frac{\sqrt{3}}{2} - \frac{1}{10} \right) r.$$

All conditions (i)–(iii) are met. □

4. Proof of Theorem 1.2

Let $\|(Df)^{-1}\|_\infty = L$. First we observe that the inner distortion of f is locally integrable because

$$K_I(x, f) = \left\| \left(Df(x) \right)^{-1} \right\|^n J(x, f) \leqslant L^n \|Df\|^n \quad \text{for a.e. } x \in \Omega. \tag{4.1}$$

We may assume that $\mathbb{B}_4 = B(0, 4) \Subset \Omega$. It suffices to show that f is discrete and open in $\mathbb{B}$. We will do this by proving that (2.2) holds. Without loss of generality, a in (2.2) equals 0. Fix $1 < t < 2$ and $3 < T < 4$ so that $\mathcal{H}^{n-1}(f\mathbb{S}_t) < \infty$ and $\mathcal{H}^{n-1}(f\mathbb{S}_T) < \infty$. By the area formula we have

$$\int_{\mathbb{R}^n} N(y, f, \mathbb{B}_T)\,\mathrm{d}y = \int_{\mathbb{B}_T} J(x, f)\,\mathrm{d}x < \infty.$$

Therefore, for almost every $0 < R < \infty$ we have

$$\int_{\mathbb{S}_R} N(y, f, \mathbb{B}_T)\,\mathrm{d}\mathcal{H}^{n-1}(y) < \infty \quad \text{and} \quad \mathcal{H}^{n-1}\big(f(\mathbb{S}_T) \cap \mathbb{S}_R\big) = 0. \tag{4.2}$$

We fix such $R < 1/(2L)$ so that (4.2) holds, and let

$$M := R^{1-n} \int_{\mathbb{S}_R} N(y, f, \mathbb{B}_T)\,\mathrm{d}\mathcal{H}^{n-1}(y).$$

Our goal is to prove that

$$r^{1-n} \int_{\mathbb{S}_r} N(y, f, \mathbb{B})\,\mathrm{d}\mathcal{H}^{n-1}(y) \leqslant M \quad \text{for a.e. } 0 < r < R. \tag{4.3}$$

Let $r < R$ be such that $\mathcal{H}^{n-1}(f(\mathbb{S}_t) \cap \mathbb{S}_r) = 0$, and denote by $E \subset \mathbb{S}$ the set of unit vectors v for which

$$\deg(Rv, f, \mathbb{B}_T) < \deg(rv, f, \mathbb{B}_t). \tag{4.4}$$

Let $I_v : [r, R] \to \mathbb{R}^n$ be the parametrized line segment $I_v(s) = sv$. By Proposition 3.1, either $f^{-1}(sv)$ has a nontrivial component for some $r \leqslant s \leqslant R$, or f is discrete and open in a neighborhood of $f^{-1}(I_v[r, R])$, denoted by U_v. By using the co-area formula as in [20, Lemma 2.4], we see that the former possibility only occurs for $v \in F_1$ where $\mathcal{H}^{n-1}(F_1) = 0$. The mapping f is discrete and open in the open set $U := \bigcup\{U_v \colon v \in E \setminus F_1\}$. It follows from (4.4) and basic properties of path lifting [21, Section II.3] that for each $v \in E \setminus F_1$ the segment I_v has a maximal f-lifting I_v^* starting at $\mathbb{B}_t$ and leaving $\mathbb{B}_T$.

Denote

$$\ell_f(x) := \liminf_{z \to x} \frac{|f(z) - f(x)|}{|z - x|}.$$

By our assumption on $(Df)^{-1}$ there exists a Borel null set $F \subset \Omega$ such that $\ell_f(x) \geqslant 1/L$ for $x \in \Omega \setminus F$. Let F_2 be the set of $v \in E \setminus F_1$ such that either I_v^* is unrectifiable or $\mathcal{H}^1(|I_v^*| \cap F) > 0$. Since the measure of F is zero, it follows that the family of curves $\Gamma_F := \{I_v^* \colon v \in F_2\}$ has zero weighted modulus for any locally integrable weight. In particular, $\mathsf{M}_{K_I} \Gamma_F = 0$. Since $\Gamma_F \subset U$ we can apply (2.1) and obtain $\mathsf{M}\{I_v \colon v \in F_2\} = 0$, which implies $\mathcal{H}^{n-1}(F_2) = 0$.

For $v \in E \setminus (F_1 \cup F_2)$ we have

$$\mathcal{H}^1\big(I_v^*\big) \leqslant L\mathcal{H}^1(I_v) < LR < \frac{1}{2}, \tag{4.5}$$

which contradicts the fact that I_v^* begins at $\mathbb{B}_t$ and leaves $\mathbb{B}_T$. Thus $E \subset F_1 \cup F_2$. As a consequence, $\mathcal{H}^{n-1}(E) = 0$, which means $\deg(rv, f, \mathbb{B}_t) \leqslant \deg(Rv, f, \mathbb{B}_T)$ for $\mathcal{H}^{n-1}$-a.e. $v \in \mathbb{S}$. Since $\deg(y, f, \mathbb{B}_t) = N(y, f, \mathbb{B}_t)$ for a.e. $y \in \mathbb{R}^n$ [8, Proposition 2], inequality (4.3) follows. This completes the proof of Theorem 1.2 via Proposition 2.2.

5. Multiplicity of local homeomorphisms

In 1967 Zorich [24] proved that a local homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geqslant 3$, with $K_I(\cdot, f) \in L^\infty(\mathbb{R}^n)$ must be a global homeomorphism. Martio, Rickman and Väisälä [16] gave a local version of this result. Namely, if $f : 2B \to \mathbb{R}^n$, $n \geqslant 3$, is a local homeomorphism with bounded distortion K_I, then its radius of injectivity in B is bounded from below by a constant depending only on n and $\operatorname{ess\,sup} K_I$. As a consequence, the multiplicity $N(y, f, B)$ is bounded by $C(n, \operatorname{ess\,sup} K_I)$ for all $y \in \mathbb{R}^n$.

The boundedness of K_I can be replaced by the condition

$$\exp\big(\lambda K_I^{1/(n-1)}\big) \in L^1(2B),$$

but this cannot be relaxed any further [13,17]. Surprisingly, the multiplicity bound remains true under a much weaker condition, namely $K_I \in L^1$. Example 7.2 below shows that $K_I^q \in L^1$ with $q < 1$ does not suffice. The mappings $f_j(z) = e^{jz}$ show that all results discussed here fail when $n = 2$.

Theorem 5.1. *Let $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, $n \geqslant 3$, be a local homeomorphism such that $\mathscr{K}_\Omega[f] < \infty$. If B is a ball such that $4B \Subset \Omega$, then $N(y, f, B) \leqslant C(n, \mathscr{K}_{4B}[f])$ for all $y \in \mathbb{R}^n$.*

Proof. We may assume that B is the unit ball $\mathbb{B}$. Let $x_1, \dots, x_m \in f^{-1}(y) \cap \mathbb{B}$. Moreover, let r_j be the largest radius r so that the x_j-component $U(x_j, r)$ of $f^{-1}B(y, r)$ satisfies $U(x_j, r) \subset \mathbb{B}_3$. By Lemma 3.2 f is a homeomorphism from $U(x_j, r_j)$ onto $B(y, r_j)$. We denote by s_j the largest radius s such that $\overline{B}(x_j, s) \subset \overline{U}(x_j, r_j)$. Then $f\overline{B}(x_j, s_j)$ intersects both y and $S(y, r_j)$. We notice that since $x_j \in \mathbb{B}$ and since the balls $B(x_j, s_j)$ are pairwise disjoint, there exist at most $N(n)$ indices j for which $s_j \geqslant 1$. Thus we may assume that $B(x_j, s_j) \subset \mathbb{B}_2$ for every $1 \leqslant j \leqslant m$.

We now fix $1 \leqslant j \leqslant m$ and a point $a_j \in \overline{U}(x_j, r_j) \cap \mathbb{S}_3$. We apply Lemma 3.5 with $B(y_0, r) = B(y, r_j)$, $y_1 = f(a_j)$ and $E = f(\overline{B}(x_j, s_j))$, obtaining a point q_j and a number $\sigma_j > 0$. For $\sigma_j < t < 4\sigma_j/3$ choose $w_t \in \overline{B}(x_j, s_j)$ such that $f(w_t) \in S(q_j, t)$. We apply Lemma 3.4 with $G = U(x_j, r_j)$, $D = \mathbb{B}_3$, $a = a_j$, $\mathscr{B} = B(q_j, t)$ and $b' = f(w_t)$. As a result we obtain $0 < \phi_t < \pi$ such that the spherical cap $\mathscr{C}_t := C_{S(q_j,t)}(f(w_t), \phi_t)$ satisfies $\mathscr{C}_t^* \subset \mathbb{B}_3$ and $\overline{\mathscr{C}}_t^* \cap \mathbb{S}_3$ contains some point c_t. Consequently, for every path γ joining $f(w_t)$ and $f(c_t)$ in $\mathscr{C}_t$, the maximal f-lifting γ^* of γ starting at w_t starts from $\mathbb{B}_2$ and leaves $\mathbb{B}_3$. Following [23, 10.2], we will choose a particular family Γ_t of such paths.

Let us say that a circular arc is *short* if it is contained in a half-circle. The family Γ_t will consist of all short circular arcs that connect $f(w_t)$ to $f(c_t)$ within $\mathscr{C}_t$. More precisely, let h be a Möbius transformation that maps $f(w_t)$ to infinity and $S(q_j, t) \setminus \{f(w_t)\}$ to $\mathbb{R}^{n-1}$. Observe that $h(\mathscr{C}_t)$ is the complement of a ball in $\mathbb{R}^{n-1}$. The convexity of $\mathbb{R}^{n-1} \setminus h(\mathscr{C}_t)$ implies that there exists an $(n-2)$-hemisphere V such that $h(f(c_t)) + sv \in h(\mathscr{C}_t)$ for every $s > 0$ and $v \in V$.

Introduce a family of curves $I_v : [0, \infty) \to \mathscr{C}_t$, defined by

$$I_v(s) = h^{-1}\big(h\big(f(c_t)\big) + s^{-1}v\big),$$

and denote by I_v^* the maximal f-lifting of I_v starting at w_t. Now let $0 < \ell(v) < \infty$ be the smallest number such that $I_v^*(\ell(v)) \in \mathbb{S}_3$. Let

$$\Gamma_t = \big\{I_v^*|_{[0,\ell(v)]}\colon\ v \in V_t\big\}.$$

We write $f\Gamma_t$ for the image of Γ_t under f.

There is a lower bound for the spherical modulus of $f\Gamma_t$, namely [23, Theorem 10.2]

$$\mathsf{M}^S(f\Gamma_t) \geqslant \frac{C(n)}{t}. \tag{5.1}$$

Let

$$\Gamma_j' = \{\gamma\colon\ \gamma \in f\Gamma_t \text{ for some } \sigma_j < t < 4\sigma_j/3\},$$

and let Γ_j^* be the family of the corresponding lifts γ^* starting at w_t. Then integrating (5.1) we obtain

$$\mathsf{M}\Gamma_j' \geqslant \int_{\sigma_j}^{4\sigma_j/3} \frac{C(n)}{t}\,\mathrm{d}t \geqslant C(n). \tag{5.2}$$

As observed earlier, every $\gamma \in \Gamma_j^*$ starts at $\mathbb{B}_2$ and leaves $\mathbb{B}_3$. We denote by E_j the smallest closed subset of $\overline{\mathbb{B}}_3 \setminus \mathbb{B}_2$ that contains $|\gamma| \cap (\overline{\mathbb{B}}_3 \setminus \mathbb{B}_2)$ for all $\gamma \in \Gamma_j^*$. Note that

$$E_j \subset f^{-1}\big(\overline{B}(y, 2r_j) \setminus B(y, r_j/10)\big) \tag{5.3}$$

by part (iii) of Lemma 3.5. Since the characteristic function χ_{E_j} is an admissible function for Γ_j^*, we have

$$\mathsf{M}_{K_I}\Gamma_j^* \leqslant \int\limits_{E_j} K_I(x,f)\,\mathrm{d}x. \tag{5.4}$$

The generalized Poletsky inequality $\mathsf{M}\Gamma_j' \leqslant \mathsf{M}_{K_I}\Gamma_j^*$ [14, Theorem 4.1], together with (5.2) and (5.4) yield

$$mC(n) \leqslant \sum_{j=1}^m \mathsf{M}\Gamma_j' \leqslant \sum_{j=1}^m \int\limits_{E_j} K_I(x)\,\mathrm{d}x \leqslant \left(\sup_{x\in\mathbb{B}_3\setminus\mathbb{B}_2} \sum_{j=1}^m \chi_{E_j}(x)\right) \times \int\limits_{3B} K_I(x,f)\,\mathrm{d}x. \tag{5.5}$$

Claim 1. *There exists $M = M(n, \mathscr{K}_{4B}[f])$ such that*

$$\sum_{j=1}^m \chi_{E_j}(x) \leqslant M \quad \textit{for every } x \in \mathbb{B}_3 \setminus \mathbb{B}_2. \tag{5.6}$$

By virtue of (5.5), Theorem 5.1 follows from Claim 1. In the rest of this section we prove (5.6).

Let $x \in \mathbb{B}_3 \setminus \mathbb{B}_2$ be a point covered by M of the sets E_j. After relabeling we have $x \in E_j$ for $1 \leqslant j \leqslant M$, and $r_1 \leqslant r_2 \leqslant \cdots \leqslant r_M$. Since disjoint sets have disjoint preimages, (5.3) implies $r_M \leqslant 20r_1$.

Choose $\tau > 0$ such that $B(x,\tau) \subset \mathbb{B}_3$ and f is injective in $\overline{B}(x,\tau)$. For $1 \leqslant j \leqslant M$ there exists $\gamma_j^* \in \Gamma_j^*$ which meets $B(x,\tau)$. Let w_j be the starting point of γ_j^*, and let γ_j be the subcurve of γ_j^* that begins at w_j and ends once it meets $\overline{B}(x,\tau)$.

Claim 2. *For $1 \leqslant j \leqslant M$ there is a curve τ_j that joins y to $f(w_j)$ within $\overline{B}(y,r_j)$ in such a way that the union of $|\tau_j|$ and $|f\circ\gamma_j|$ can be mapped onto a line segment by an L-biLipschitz mapping $g:\mathbb{R}^n \to \mathbb{R}^n$. Here L is a universal constant.*

Proof. Note that the image $f\circ\gamma_j$ is a short circular arc contained in the sphere $S(q,t)$ of Lemma 3.5. Part (iii) of Lemma 3.5 implies

$$\operatorname{dist}\big(y, |f\circ\gamma_j|\big) \geqslant \operatorname{dist}\big(y, S(q,t)\big) \geqslant \frac{1}{10}r_j \geqslant \frac{1}{40}\operatorname{diam}|f\circ\gamma_j|. \tag{5.7}$$

There are two cases. If $y \in B(q,t)$, then τ_j is the line segment connecting y to $f(w_j)$. By virtue of (5.7), the distance from y to $S(q,t)$ is comparable to t. Therefore, the angle between τ_j and the sphere $S(q,t)$ is bounded from below by a universal constant, and the claim follows.

Suppose that $y \notin B(q,t)$. Let $\rho_j := |f(w_j) - y|$. Note that $r_j/10 \leqslant \rho_j \leqslant r_j$. Let p be the point of the sphere $S(y,\rho_j)$ that is farthest from q, namely

$$p = y - \rho_j \frac{q-y}{|q-y|}.$$

We choose τ_j as the union of the line segment connecting y to p and the geodesic arc on $S(y,\rho_j)$ from p to $f(w_j)$. Once again, the angle between τ_j and the sphere $S(q,t)$ is bounded from below by a universal constant. $\square$

Let η_j, $1 \leqslant j \leqslant M$, be the curve obtained by concatenating $-(f\circ\gamma_j)$ with $-\tau_j$, where $-$ indicates the reversal of orientation. Note that η_j begins in $f\overline{B}(x,\tau)$, proceeds along a circular arc to $f(w_j)$, and ends at y. Its f-lifting η_j^* starting in $\overline{B}(x,\tau)$ is contained in $\overline{\mathbb{B}}_3$ and ends at x_j.

Claim 3. *There exists $\epsilon = \epsilon(n,M)$ such that $\epsilon \to 0$ as $M \to \infty$, and*

$$\min_{1\leqslant i<j\leqslant M} \mathrm{d}_{\mathcal{H}}\big(|\eta_i|, |\eta_j|\big) \leqslant \epsilon r_1/L. \tag{5.8}$$

Proof. We begin our proof of Claim 3 by observing that $|\eta_j| \subset B(y, 2r_M) \subset B(y, 40r_1)$. For $\epsilon > 0$ let $Z = \{z_1,\ldots,z_N\}$ be an $(\epsilon r_1/L)$-net in $B(y,40r_1)$, where $N = N(\epsilon,n)$. The set of all nonempty subsets of Z is an $(\epsilon r_1/L)$-net in the set of all nonempty closed subsets of $B(y,40r_1)$ equipped with the Hausdorff metric. If $M > 2^N$, then by the pigeonhole principle there exist $i < j$ such that $|\eta_i|$ and $|\eta_j|$ are within the distance $(\epsilon r_1/L)$ from the same subset of Z. Claim 3 follows. □

Fix i, j, and ϵ as in Claim 3, and let $g : \mathbb{R}^n \to \mathbb{R}^n$ be the L-biLipschitz mapping from Claim 2. By replacing f with $g \circ f$, which has a comparable distortion function K_I, we may assume that $|\eta_j|$ is a line segment. For $\delta > 0$ we denote by $W(\delta)$ the open δ-neighborhood of $|\eta_j|$. Let $W^*(\delta)$ be the x_j-component of $f^{-1}W(\delta)$.

Claim 4. *If $\delta > \epsilon r_1$, then $W^*(\delta) \cap \mathbb{S}_4 \neq \emptyset$.*

Proof. Since $\delta > \epsilon r_1$, we have $|\eta_i| \subset W(\delta)$. Suppose to the contrary that $W^*(\delta) \subset \mathbb{B}_4$. Then $W^*(\delta) \Subset \Omega$, which by Lemma 3.2 implies that $f : W^*(\delta) \to W(\delta)$ is a homeomorphism. This contradicts the fact that the f-liftings of η_i and η_j starting in $\overline{B}(x, \tau)$ end at different points, namely x_i and x_j. □

Let δ_0 be the supremum of all numbers δ such that $W^*(\delta) \subset \mathbb{B}_4$. Since f is a local homeomorphism, $\delta_0 > 0$. By Lemma 3.2, $f : W^*(\delta) \xrightarrow{\text{hom}} W(\delta)$ for every $0 < \delta < \delta_0$. By Claim 4 we have $\delta_0 \leqslant \epsilon r_1$.

Choose a point $a \in \partial W^*(\delta_0) \cap \mathbb{S}_4$. Let $a' = f(a)$. Since $a' \in \partial W(\delta_0)$, there exists $p \in |\eta_j|$ such that $|a' - p| = \delta_0$. For $\delta_0 < t < \frac{1}{2} \operatorname{diam} |\eta_j|$ choose $b'_t \in |\eta_j| \cap S(p, t)$. We apply Lemma 3.4 with $G = W^*(\delta_0)$, $D = \mathbb{B}_4$, $a = a$, $\mathscr{B} = B(p, t)$ and $b' = b'_t$. As a result we obtain $0 < \phi_t < \pi$ such that the spherical cap $\mathscr{C}_t := C_{S(p,t)}(b'_t, \phi_t)$ satisfies $\mathscr{C}^*_t \subset \mathbb{B}_4$ and $\overline{\mathscr{C}}^*_t \cap \mathbb{S}_4$ contains some point c_t. Consequently, for every path γ joining b'_t and $f(c_t)$ in $\mathscr{C}_t$, the maximal f-lifting γ^* of γ starting at $f^{-1}(b'_t) \cap |\eta^*_j|$ starts from $\mathbb{B}_3$ and leaves $\mathbb{B}_4$. Let Γ be the family of all such paths γ and let Γ^* be the family of the lifts γ^*. From [23, Theorem 10.2] we have

$$\mathsf{M}\Gamma \geqslant C(n) \int_{\epsilon r_1}^{\operatorname{diam}(\eta_j)/2} \frac{\mathrm{d}t}{t} \geqslant C(n) \log \frac{\operatorname{diam}(\eta_j)}{2\epsilon r_1}.$$

By (5.7) we have $\operatorname{diam} |\eta_j| \geqslant c r_1$ with a universal constant $c > 0$. Therefore,

$$\mathsf{M}\Gamma \geqslant C(n) \log \frac{1}{\epsilon}. \tag{5.9}$$

On the other hand, since the characteristic function $\chi_{\mathbb{B}_4 \setminus \mathbb{B}_3}$ is an admissible function for Γ_j, we obtain

$$\mathsf{M}_{K_I} \Gamma^* \leqslant \int_{\mathbb{B}_4 \setminus \mathbb{B}_3} K_I(x, f)\, \mathrm{d}x.$$

Combining this with (5.9) and using the Poletsky inequality again, we have $\epsilon \geqslant C(n, \mathscr{K}_{4B}[f])$, hence $M \leqslant C(n, \mathscr{K}_{4B}[f])$. This gives (5.6). The proof of Theorem 5.1 is complete. □

6. Proof of Theorem 1.1

Denote $f^\lambda(x) = f(x) + \lambda x$, $\lambda > 0$. Then $f^\lambda \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. Moreover, by [14, Lemma 10],

$$K_I\big(x, f^\lambda\big) \leqslant C(\delta, n) K_I(x, f) \quad \text{and} \quad \big\|\big(Df^\lambda\big)^{-1}(x)\big\| \leqslant C(\delta, \lambda) \tag{6.1}$$

for almost every $x \in \Omega$. Thus f^λ is discrete and open for every $\lambda > 0$ by Theorem 1.2. Furthermore, by [14, Lemma 13] f^λ is a local homeomorphism. (Although [14, Lemma 13] imposes a stronger condition on the distortion of f, this condition is only used to ensure that f is discrete and open.) Since $f^\lambda \to f$ locally uniformly, the following proposition implies that f is a local homeomorphism, completing the proof of Theorem 1.1.

Proposition 6.1. *Suppose that a mapping $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ with $\mathscr{K}_\Omega[f] < \infty$ can be uniformly approximated by local homeomorphisms $f_j \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ such that $\sup_j \mathscr{K}_\Omega[f_j] < \infty$. Then f is a local homeomorphism.*

Proof. By [14, Proposition 7] it suffices to show that f is discrete and open. If $n=2$, this is due to Iwaniec and Šverák [10]. Thus we assume that $n \geqslant 3$. Let $B = B(x_0, R)$ be a ball such that $8B \Subset \Omega$. We will show that

$$N(y, f, B) \leqslant C \quad \text{for a.e. } y \in \mathbb{R}^n, \tag{6.2}$$

where $C < \infty$ does not depend on y. Proposition 2.2 will then imply that f is discrete and open in B.

Applying Theorem 5.1 to f_j, we obtain

$$N(y, f_j, 2B) \leqslant C \quad \text{for every } y \in \mathbb{R}^n,$$

where C depends only on $\sup_j \mathscr{K}_\Omega[f_j]$ and n.

We fix $R < t < 2R$ so that $\mathcal{H}^{n-1}(fS(x_0,t)) < \infty$, and a point $y \in fB \setminus fS(x_0,t)$. Let $d = \operatorname{dist}(y, fS(x_0,t))$. Since $f_j \to f$ locally uniformly, there exists j_0 such that $|f_j(x) - f(x)| < d/2$ for all $j \geqslant j_0$ and all $x \in S(x_0,t)$. Consequently, the restrictions of f_j and f to $S(x_0,t)$ are homotopic via the straight-line homotopy that takes values in $\mathbb{R}^n \setminus \{y\}$. It follows that

$$\deg\big(y, f, B(x_0,t)\big) = \deg\big(y, f_j, B(x_0,t)\big) \leqslant N(y, f_j, 2B) \leqslant C$$

for all $j \geqslant j_0$. Since $N(y,f,B) \leqslant N(y,f,B(x_0,t)) = \deg(y,f,B(x_0,t))$ for almost every $y \in \mathbb{R}^n$, we conclude that (6.2) indeed holds. The proof is complete. □

7. Concluding remarks

Corollary 7.1. *Suppose that $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ is a nonconstant mapping such that $K_I(\cdot, f) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. If there exists $\delta > -1$ such that $Df(x) \in \mathcal{M}(\delta)$ for almost every $x \in \mathbb{R}^n$, then f is a homeomorphism.*

Proof. As in the proof of Theorem 1.1 we have that $f^\lambda(x) = f(x) + \lambda x$ is a local homeomorphism for all $\lambda > 0$. Since $(Df^\lambda)^{-1} \in L^\infty(\mathbb{R}^n)$, it follows from [14, Lemma 12] that

$$\liminf_{x \to a} \frac{|f^\lambda(x) - f^\lambda(a)|}{|x-a|} \geqslant \frac{\lambda}{2} > 0 \tag{7.1}$$

for all $a \in \mathbb{R}^n$. By a theorem of John [11, p. 87], f^λ is a homeomorphism. Since f is discrete and open by Theorem 1.1, we can apply [14, Proposition 7] and conclude that f is a homeomorphism. □

Sharpness of Theorem 5.1 is demonstrated by the following example which combines the ideas from [2] and [13].

Example 7.2. For any $q < 1$ there exists a sequence of mappings $f_j \in W^{1,3}(\mathbb{B}, \mathbb{R}^3)$ such that

$$\sup_j \int_{\mathbb{B}} K_I^q(x, f_j)\,\mathrm{d}x < \infty \quad \text{and} \quad N\big(0, f_j, B(0,1/4)\big) \to \infty.$$

Proof. By a version of Zorich's construction (see [9,21]) there exists a mapping $\phi \in W^{1,3}(\mathbb{R}^3, \mathbb{R}^3)$ such that $K_I(\cdot, \phi) \in L^\infty(\mathbb{R}^3)$, ϕ is a local homeomorphism outside of $\mathbb{R} \times (2\mathbb{Z}+1)^2$, and ϕ is 4-periodic in the last two variables. Therefore, it suffices for us to construct biLipschitz homeomorphisms $f_j : \mathbb{B} \to \mathbb{R}^3$ such that

(i) $\sup_j \int_{\mathbb{B}} K_I^q(x, f_j)\,\mathrm{d}x < \infty$;
(ii) $f_j(\mathbb{B}) \subset \mathbb{D} \times \mathbb{R}$ (here $\mathbb{D} \subset \mathbb{R}^2$ is the unit disc);
(iii) $f_j(\mathbb{B}_{1/4})$ contains a line segment $\{0\} \times [-L, L] \subset \mathbb{R}^2 \times \mathbb{R}$ where $L \to \infty$ as $j \to \infty$.

The compositions $\phi \circ f_j$ will be mappings with large multiplicity.

For $y \in \mathbb{R}^3$ let $s(y) = \sqrt{y_1^2 + y_2^2}$. For $\alpha > 2$ we define a mapping $x = g(y)$ by

$$x_i = s(y)^{\alpha-1} y_i, \quad i = 1, 2;$$
$$x_3 = s(y) y_3.$$

Since $s(x) = s(y)^\alpha$, the inverse mapping $y = f(x)$ outside the set $\{s(x) = 0\}$ is given by

$$y_i = s(x)^{1/\alpha - 1} x_i, \quad i = 1, 2;$$
$$y_3 = s(x)^{-1/\alpha} x_3, \quad s(x) \neq 0.$$

Let $\Omega = \{x \in \mathbb{R}^3\colon\ s(x) < 1,\ |x_3| < 1\}$ and $\Omega' = f(\Omega)$. We restrict our attention to $y \in \Omega'$, where in particular $s(y) < 1$. Elementary computations show that

$$\|Dg(y)\| \leqslant C \max\bigl(s(y), |y_3|\bigr) \quad \text{and}$$
$$J(y, g) \geqslant C s(y)^{2(\alpha-1)+1}.$$

Therefore,

$$\frac{\|Dg(y)\|^3}{J(y, g)} \leqslant C s(y)^{2(1-\alpha)-1} \max\bigl(s(y)^3, |y_3|^3\bigr). \tag{7.2}$$

Since

$$\frac{\|Dg(y)\|^3}{J(y, g)} = K_I(x, f),$$

inequality (7.2) can be used to estimate $K_I(x, f)$ as follows.

$$K_I(x, f) \leqslant C s(x)^{(2(1-\alpha)-1)/\alpha} \max\bigl(s(x)^{3/\alpha}, s(x)^{-3/\alpha}|x_3|^3\bigr) \leqslant C s(x)^{-(2\alpha+2)/\alpha},$$

where at the last step we used $|x_3| < 1$. We achieve $\int_\Omega K_I(x, f)^q \,\mathrm{d}x < \infty$ by choosing α large enough so that

$$\frac{2\alpha + 2}{\alpha} q < 2.$$

The mapping f constructed thus far is not in $W^{1,3}$, and is not even continuous. However, this can be corrected by replacing $s(y)$ with $s_j(y) = \sqrt{y_1^2 + y_2^2 + 1/j^2}$. The mapping $x = g_j(y)$ given by

$$x_i = s_j(y)^{\alpha-1} y_i, \quad i = 1, 2;$$
$$x_3 = s_j(y) y_3,$$

is biLipschitz; we denote the inverse by f_j. The computation of $\|Dg_j\|$ and $J(\cdot, g_j)$ goes through exactly as before and shows that the integral of $K_I^q(\cdot, f_j)$ is bounded independently of ϵ_j. Since $g_j(0, 0, y_3) = (0, 0, y_3/j)$, we have $f_j(0, 0, x_3) = (0, 0, j x_3)$. Thus, this mapping f_j fulfills the requirements (i)–(iii). □

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