Eventual regularization for the slightly supercritical quasi-geostrophic equation

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Abstract

We prove that weak solutions of the slightly supercritical quasi-geostrophic equation become smooth for large time. The proof uses ideas from a recent article of Caffarelli and Vasseur and is based on an argument in the style of De Giorgi.
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Résumé

Dans cet article, nous montrons que les solutions faibles de l'équation quasi-géostrophique légèrement sur-critique deviennent régulières en temps grand. La démonstration utilise des idées d'un article récent de Caffarelli et Vasseur et repose sur un argument de type de De Giorgi.
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1. Introduction

We consider the quasi-geostrophic equation for a function $\theta : \mathbb{R}^2 \times [0, +\infty) \to \mathbb{R}$,

$$\partial_t \theta(x, t) + w \cdot \nabla \theta(x, t) + (-\Delta)^{\alpha/2} \theta(x, t) = 0,$$
$$\theta(x, 0) = \theta_0(x), \tag{1.1}$$

where $(-\Delta)^{\alpha/2} \theta = \Lambda^\alpha \theta$ is the fractional laplacian in the x variable and $w = (-R_2\theta, R_1\theta) = R^\perp \theta$ where R_i are the Riesz transforms

$$R_i \theta(x) = c\,\mathrm{PV} \int_{\mathbb{R}^2} \frac{(y_i - x_i)\theta(y)}{|y - x|^3} \, \mathrm{d}y.$$

When $\alpha > 1$ it is said that the equation is subcritical, and it is well known [5] that solutions are smooth. In the critical case $\alpha = 1$, smoothness of the solutions has been proved recently in [2] and [7].

The well-posedness of the supercritical case ($\alpha < 1$) is still open. There are some partial results assuming some initial extra regularity. In [3], Constantin and Wu showed that if the solution θ is in the Hölder class C^δ with $\delta > 1 - \alpha$

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then the solution θ is actually C^∞. In [4], Constantin and Wu showed that if the velocity w in (1.1) is $C^{1-\alpha}$ then the solution θ is Hölder continuous independently of the relation $w = R^\perp \theta$. In this paper, we prove that θ always becomes Hölder continuous for large time, and then higher regularity follows from [3].

Even though both in [7] and [2], they obtain the global well-posedness of the critical quasi-geostrophic equation, a closer look at the results and proofs reveals that they are quite different in nature. The proof in [7] is certainly simpler than the one in [2]. The result in [7] says that certain cleverly constructed modulus of continuity are preserved by the flow of the equation. In [2] a regularization technique inspired by De Giorgi's methods for elliptic PDEs is used to exploit the regularization effect of the equation. Thus even with L^2 initial data, the methods in [1] show that the solutions become immediately smooth.

In [2] the full structure of the nonlinearity in (1.1) is not used. Their result is somewhat more general. The purpose of this paper is to use the methods of [2] exploiting the exact structure of the nonlinear term in (1.1) and obtain a regularity result for the slightly supercritical case. The idea is to iteratively show that the oscillation of the function θ improves as we look at smaller parabolic cylinders, and use that information to get better local estimates for the nonlinear term $w \cdot \nabla \theta$. As it is standard, this improvement of oscillation in smaller cylinders leads to a Hölder continuity result. In order to compensate for the nonlocal dependence of w with respect to θ, we need to make a change of variables in each iterative step that *follows the flow* of the nonlocal contribution. Unfortunately this procedure works only at points (x, t) if t is not too small. So our result is not an immediate regularization, but instead an eventual regularization. More precisely, we prove that if $\alpha = 1 - \varepsilon$ with $\varepsilon \ll 1$, then for any initial data θ_0, there is a time t_0 after which the solution θ becomes smooth. This has been well known for critical QG equations for some years [6] and also for many other equations (for instance Navier–Stokes), but up to our knowledge it is new for the supercritical quasi-geostrophic equation.

Our main results are:

Theorem 1.1. *Let θ be a solution of the quasi-geostrophic equation* (1.1) *with initial data θ_0 in L^2. Assume that $\alpha = 1 - \varepsilon$ with $\varepsilon \leqslant \delta$. Then for any $T > 0$, θ is δ-Hölder continuous at time T if δ is small enough. Moreover, there is an estimate*

$$\left| \theta(x, T) - \theta(y, T) \right| \leqslant C |x - y|^\delta$$

where C and δ depend on $\|\theta_0\|_{L^2}$ and T.

Theorem 1.2. *If ε is small enough, for any $\theta_0 \in L^2(\mathbb{R}^2)$, there is a T_0 such that the solution θ of* (1.1) *is C^∞ for $t > T_0$ (T_0 depends only on ε and $\|\theta_0\|_{L^2}$, and $T_0 \to 0$ as $\varepsilon \to 0$).*

The most common way to prove eventual regularity for some equation is by combining a global regularity result for small initial data with an appropriate decay of the weak solution with respect to time. We point out that our proof of Theorem 1.2 is essentially different. Even though the decay of the L^∞ norm is used in the proof, after the L^∞ norm is under control we still need to wait an extra period of time to obtain regularity. Our proof is not based on a perturbative argument of the critical case either.

By a *solution* of (1.1), we mean a weak solution θ (a solution in the sense of distributions) for which the following level set energy inequality holds

$$\int_{\mathbb{R}^n} \theta_\lambda^2(x, t_2)\, \mathrm{d}x + 2 \int_{t_1}^{t_2} \|\theta_\lambda\|_{\dot{H}^\alpha}^2\, \mathrm{d}t \leqslant \int_{\mathbb{R}^n} \theta_\lambda^2(x, t_1)\, \mathrm{d}x \tag{1.2}$$

where $\theta_\lambda = (\theta - \lambda)_+$ and $\|.\|_{\dot{H}^\alpha}$ stands for the homogeneous Sobolev norm

$$\|f\|_{\dot{H}^\alpha}^2 = \int \left| \hat{f}(\xi) \right|^2 |\xi|^{2\alpha}\, \mathrm{d}\xi = c \iint \frac{|f(x) - f(y)|^2}{|x - y|^{2+2\alpha}}\, \mathrm{d}x\, \mathrm{d}y.$$

It can be shown that such solutions exist for any initial data θ_0 by adding a vanishing viscosity term $\nu \triangle \theta$ to the right-hand side and making $\nu \to 0$ (see [6] and also the appendix in [2]).

The methods in this paper do not require essentially the dimension to be 2. The same result would hold if $\theta : \mathbb{R}^n \to \mathbb{R}$ and $w = T\theta$ for some singular integral operator T of order zero such that $T\theta$ is divergence free and the kernel associated to T is differentiable away from the origin.

2. Preliminaries

In this section we review some results and constructions which are mostly adaptations from [2].

2.1. L^2 and L^∞ estimates

Theorem 2.1. *If θ is a solution of* (1.1) *then $\|\theta\|_{L^2}$ is decreasing in time. More precisely*

$$\big\|\theta(.,t)\big\|_{L^2(\mathbb{R}^2)} \leqslant \|\theta_0\|_{L^2(\mathbb{R}^2)}.$$

The theorem above is well known and could be derived directly from the energy inequality.
The following interesting theorem is an adaptation of a result from [2].

Theorem 2.2. *If θ is a solution of* (1.1) *then*

$$\sup_{x\in\mathbb{R}^2} \big|\theta(x,t)\big| \leqslant Ct^{-\frac{1}{\alpha}} \big\|\theta(x,0)\big\|_{L^2}.$$

The proof of Theorem 2.2 relies only on the energy inequality (1.2). The proof of Theorem 2.2 was given in [2] for the case $\alpha = 1$. The same idea works for general α and the complete proof can be found in the appendix in [4].

2.2. The extension problem

It is useful to define the fractional laplacian $(-\Delta)^{\alpha/2}$ using the extension to the upper half space as in [1]. Given the function $\theta(x,t)$, we extend it to a new variable z to obtain the unique function (that we still call θ) $\theta(x,z,t)$ satisfying the equation

$$\operatorname{div} z^\varepsilon \nabla\theta = 0 \quad \text{where } z > 0,$$

where $\nabla\theta$ refers to the gradient in the variables x and z. It can be proved that $(-\Delta)^{\frac{1-\varepsilon}{2}}\theta(x,0,t) = \lim_{z\to 0} z^\varepsilon \partial_z \theta(x,z,t)$. Given this construction it is now convenient to rewrite Eq. (1.1) for $\alpha = 1 - \varepsilon$ in terms of the new coordinates

$$\operatorname{div} z^\varepsilon \nabla\theta = 0 \quad \text{where } z > 0, \tag{2.1}$$

$$\partial_t \theta(x,0,t) + w \cdot \nabla\theta(x,0,t) + \lim_{z\to 0} z^\varepsilon \partial_z \theta(x,z,t) = 0, \tag{2.2}$$

the practical advantage with respect to (1.1) is that we replaced a nonlocal operator $(-\Delta)^{\alpha/2}$ by a local equation (in one more variable). We still have however the nonlocal contribution from $w = (-R_2\theta, R_1\theta)$.

We abuse notation by writing $\theta(x,t) = \theta(x,0,t)$.

2.3. Normalized problem

Theorem 2.2 tells us that after any small period of time t_0, the solution will be in L^∞. So we can assume that we have a solution in L^∞ from the beginning by considering $\theta(x, t + t_0)$.

Moreover, we can rescale the function θ and consider

$$\tilde{\theta} = \frac{1}{\|\theta\|_{L^\infty}} \theta\big(T^{-1/\alpha}x, T^{-1}t\big)$$

so that we reduce the problem to the case $\|\theta\|_{L^\infty} = 1$ and $T = 1$. Including the extension variable z, the scaling is $\tilde{\theta} = \frac{1}{\|\theta\|_{L^\infty}} \theta(T^{-1/\alpha}x, T^{-1/\alpha}z, T^{-1}t)$. However we will have to replace Eqs. (2.1)–(2.2) by

$$\operatorname{div} z^\varepsilon \nabla\theta = 0 \quad \text{where } z > 0, \tag{2.3}$$

$$\partial_t \theta(x,0,t) + Mw \cdot \nabla\theta(x,0,t) + \lim_{z\to 0} z^\varepsilon \partial_z \theta(x,z,t) = 0 \tag{2.4}$$

where M is some constant depending only on $\|\theta\|_{L^\infty}$ and T.

2.4. Scaling

We use the same notation as in [2] appropriately scaled in terms of α. We denote

$$B_r = \left\{ x \in \mathbb{R}^2 \colon |x| < r \right\},$$
$$B_r^* = B_r \times [0, r) = \left\{ (x, z) \in \mathbb{R}^3 \colon |x| < r \wedge 0 \leqslant z < r \right\},$$
$$Q_r = B_r \times [0, r) \times \left(1 - r^\alpha, 1\right] = \left\{ (x, z, t) \in \mathbb{R}^4 \colon |x| < r \text{ and } 0 \leqslant z < r \text{ and } 1 - r^\alpha < t \leqslant 1 \right\}.$$

The natural scaling of the equation is given by the fact that if θ solves (2.3)–(2.4), then also does $\tilde{\theta}(x, z, t) = \lambda^{-\varepsilon} \theta(x_0 + \lambda x, \lambda z, t_0 + \lambda^\alpha t)$ for any $\lambda > 0$.

On the other hand, we will use C^δ scaling, which does not preserve the equation exactly. If θ solves (2.3)–(2.4), then $\tilde{\theta}(x, z, t) = \lambda^{-\delta} \theta(x_0 + \lambda x, \lambda z, t_0 + \lambda^\alpha t)$ solves

$$\operatorname{div} z^\varepsilon \nabla \theta = 0 \quad \text{where } z > 0,$$
$$\partial_t \theta(x, 0, t) + \lambda^{\delta - \varepsilon} M w \cdot \nabla \theta(x, 0, t) + \lim_{z \to 0} z^\varepsilon \partial_z \theta(x, z, t) = 0.$$

Note that $\lambda^{\delta - \varepsilon} M \leqslant M$ if $\lambda < 1$.

2.5. Local improvement of oscillation

The following theorem is the key result that leads to Hölder continuity in [2].

Theorem 2.3. *Let θ be a solution to*

$$\operatorname{div} z^\varepsilon \nabla \theta = 0 \quad \text{where } z > 0,$$
$$\partial_t \theta(x, 0, t) + w \cdot \nabla \theta(x, 0, t) + \lim_{z \to 0} z^\varepsilon \partial_z \theta(x, z, t) = 0$$

for an arbitrary divergence free vector field w such that

$$\|w\|_{L^\infty([0,1], L^{2n/\alpha}(B_1))} \leqslant K.$$

Then

$$\underset{Q_{1/2}}{\operatorname{osc}} \theta \leqslant (1 - \eta) \underset{Q_1}{\operatorname{osc}} \theta$$

for some $\eta > 0$ depending only on K, ε and dimension (dimension is two in the quasi-geostrophic equation case).

The proof of Theorem 2.3 was given in [2] for the case $\alpha = 1$. It relies only on a local energy inequality and De Giorgi's oscillation lemma. We prove both things in Appendix A, so that the proof of Theorem 2.3 generalizes to smaller values of α. The proof in full detail is provided in [4].

In [2], the estimate in $L^\infty([0, 1], L^{2n/\alpha}(B_1))$ was replaced by an estimate in $L^\infty(BMO)$ plus a control on the mean. Their assumption reads

$$\|w\|_{L^\infty([0,1], BMO(\mathbb{R}^n))} + \sup_{[0,1]} \left| \int_{B_1} w(x, t) \, dx \right| \leqslant K.$$

This was done because the $L^{2n/\alpha}(B_1)$ norm is not invariant by the scaling of the equation. Since in this paper we will deal with scaling in a somewhat different way, we keep the sharp assumption from the proof, in $L^\infty([0, 1], L^{2n/\alpha}(B_1))$.

The value of η does depend on ε. In particular it degenerates as $\varepsilon \to 1$ (or equivalently as $\alpha = 1 - \varepsilon$ goes to zero). However, since in this paper we are interested only in the case of ε small, we can consider η to be independent of ε (say for $\varepsilon \in [0, 1/2]$).

3. Proofs

In this section we provide the proofs of the main Theorems 1.1 and 1.2.

Lemma 3.1. *If θ is a solution of* (1.1)*, then for any $t > 0$ we have the estimate*

$$\int_{\mathbb{R}^2 \setminus B_1} \frac{|\theta(x, t)|}{|x|^2} \, dx \leqslant C \|\theta_0\|_{L^2}. \tag{3.1}$$

For any $t > 1$, we have the improved estimate

$$\int_{\mathbb{R}^2 \setminus B_1} \frac{|\theta(x, t)|}{|x|^2} \, dx \leqslant C(1 + \log t) t^{-\alpha} \|\theta_0\|_{L^2}. \tag{3.2}$$

Proof. For any $R > 1$, we split the integral and use Hölder's inequality

$$\int_{\mathbb{R}^2 \setminus B_1} \frac{|\theta(x, t)|}{|x|^2} \, dx = \int_{B_R \setminus B_1} \frac{|\theta(x, t)|}{|x|^2} \, dx + \int_{\mathbb{R}^2 \setminus B_R} \frac{|\theta(x, t)|}{|x|^2} \, dx$$

$$\leqslant \|\theta\|_{L^\infty} \log R + \frac{C}{R} \|\theta\|_{L^2}.$$

The first estimate follows if we pick $R = 1$. Since the estimate holds for any R, when $t > 1$ we choose $R = t^\alpha$. Using Theorems 2.2 and 2.1 we get

$$\int_{\mathbb{R}^2 \setminus B_1} \frac{|\theta(x, t)|}{|x|^2} \, dx \leqslant \alpha \|\theta\|_{L^\infty} \log t + C t^{-\alpha} \|\theta\|_{L^2}$$

$$\leqslant C(1 + \log t) t^{-\alpha} \|\theta_0\|_{L^2}$$

which finishes the proof. $\square$

Proof of Theorem 1.1. We will prove that θ is Hölder continuous at the point $(0, T)$. There is nothing special about $x = 0$, so the proof implies the result of the theorem.

Let us choose some $t_0 < T$ (for example $t_0 = T/1000$), we have $\|\theta(-, t_0)\|_{L^\infty(\mathbb{R}^3)} \leqslant C \|\theta_0\|_{L^2}$ by Theorem 2.2. Moreover, from Lemma 3.1,

$$\int_{\mathbb{R}^2 \setminus B_1} \frac{|\theta(x, t_0)|}{|x|^2} \, dx \leqslant C \|\theta_0\|_{L^2}. \tag{3.3}$$

We normalize the problem in the following way. We consider

$$\tilde{\theta}(x, z, t) = \frac{\theta(x/(T - T_0)^{\frac{1}{\alpha}}, z/(T - T_0)^{\frac{1}{\alpha}}, (t - t_0)/(T - T_0))}{C \|\theta_0\|_{L^2}},$$

so that $|\tilde{\theta}| \leqslant 1$ in $\mathbb{R}^2 \times [0, +\infty) \times [0, 1]$,

$$\int_{\mathbb{R}^2 \setminus B_1} \frac{|\tilde{\theta}(x, 0, t)|}{|x|^2} \, dx \leqslant 1 \tag{3.4}$$

for $t \in [0, 1]$, and $\tilde{\theta}$ solves (2.3) and (2.4). Note that the constant M depends only on $\|\theta(-, t_0)\|_{L^\infty}$ and the right-hand side of (3.3) which are controlled by the L^2 norm of the original initial condition.

We stress that all estimates in the rest of this proof depend only on $\|\theta\|_{L^\infty([t_0, +\infty), \mathbb{R}^2)}$ and the right-hand side in (3.3).

From now on we will abuse notation by omitting the tilde in $\tilde{\theta}$ and we write just θ. We assume $\|\theta(-,t)\|_{L^\infty} \leqslant 1$ and (3.4) for all $t \in [0,1]$.

We write

$$\theta_r(x,z,t) = \frac{1}{r^\delta} \theta\big(rx, rz, 1 - r^\alpha(1-t)\big).$$

Assuming that $\delta \ll 1$, we will show that $\mathrm{osc}_{Q_r} \theta \leqslant C r^\delta$ for any $r < 1$, obtaining Hölder continuity at the point $(0,1)$ (and by scaling and translation, at any point (x,T) in the original equation). This is equivalent of saying that $\mathrm{osc}_{Q_1} \theta_r \leqslant C$ for any $r < 1$.

We will find a $1 > \rho > 0$ such that for $r_k = \rho^k$, $\mathrm{osc}_{Q_1} \theta_{r_k} \leqslant 1$, and the result clearly follows.

We prove that $\mathrm{osc}_{Q_1} \theta_{r_k} \leqslant 1$ by a usual iterative procedure, but since the equation is nonlocal, we must carry on some extra information in the iteration. In this case, the first step in the iterative process is a little bit different from the successive steps. We explain them separately to avoid confusion.

We stress that we need to choose $\rho > 0$ and $\delta > 0$ small. Then for $0 < \varepsilon \leqslant \delta$ the theorem will apply. The choice of ρ and δ must be made carefully. When we write a universal constant C in this proof, we mean a constant that does not depend on ρ or δ.

Step 1. We start with $|\theta| \leqslant 1$ in $\mathbb{R}^2 \times [0, +\infty) \times [0,1]$ and $w = R^\perp \theta = (-R_2\theta, R_1\theta) \in L^\infty([0,1], BMO)$.

We also know (3.4), which tells us that the contribution of the tails in the integral representation of $R_i\theta$ is bounded. Equivalently, that $R_i(\theta(1 - \chi_{B_2})) \in L^\infty$. On the other hand $R_i(\theta \chi_{B_2}) \in L^p$ for any $p < +\infty$ since $\theta \chi_{B_2}$ is bounded and compactly supported. Thus, for any $p < +\infty$, there is a constant C such that $\sup_{t \in [0,1]} \|R_i\theta(-,t)\|_{L^p(B_1)} \leqslant C$. In particular this estimate holds for $p = n/\alpha$ and we can apply Theorem 2.3 to get

$$\mathop{\mathrm{osc}}_{Q_{1/2}} \theta \leqslant 2 - 2\eta.$$

Before rescaling θ to prepare for the next iterative step, we perform a small change of variables to *follow the flow*. This is the key to make the iteration scheme succeed.

We write $w = w_1 + w_2$ where w_2 is given by the truncated integral

$$w_2(x,t) = c \int_{\mathbb{R}^2 \setminus B_2} \frac{\theta(y)(y-x)^\perp}{|y-x|^3} \, dy.$$

Note that w_2 is a continuous function in x as long as $x \in B_2$. Let $V : [0,1] \to \mathbb{R}^2$ be a solution to the following ODE

$$V(1) = 0,$$
$$\dot{V}(t) = M w_2\big(V(t), t\big).$$

We define $\tilde{\theta}(x,y,t) = \theta(x + V(t), y, t)$ and verify that $\tilde{\theta}$ satisfies the equation

$$\mathrm{div}\, z^\varepsilon \nabla \tilde{\theta} = 0 \quad \text{where } z > 0,$$
$$\partial_t \tilde{\theta}(x,0,t) + M\big(w(x,t) - w_2(V(t),t)\big) \cdot \nabla \tilde{\theta}(x,0,t) + \lim_{z \to 0} z^\varepsilon \partial_z \tilde{\theta}(x,z,t) = 0.$$

From (3.4), we get that $|w_2| < L$ for some universal constant L. If we choose ρ such that

$$L\rho^\alpha + \rho \leqslant 1/2, \tag{3.5}$$

then $(x + V(t), y, t) \in Q_{1/2}$ if $(x,y,t) \in Q_\rho$.

Now we rescale. For $m = (\sup_{Q_{1/2}} \theta + \inf_{Q_{1/2}} \theta)/2$, let

$$\theta_1(x,y,t) = (\tilde{\theta} - m)_\rho,$$
$$\mathrm{div}\, z^\varepsilon \nabla \theta_1 = 0 \quad \text{where } z > 0,$$
$$\partial_t \theta_1(x,0,t) + r^{\delta - \varepsilon} M\big(w(x,t) - \overline{w}(t)\big) \cdot \nabla \theta_1(x,0,t) + \lim_{z \to 0} z^\varepsilon \partial_z \theta_1(x,z,t) = 0$$

where

$$\overline{w}(t) = c \int_{\mathbb{R}^2 \setminus B_{2/\rho}} \frac{(\theta_1(y,t) - \theta_1(0,t))y^\perp}{|y|^3} \, dy.$$

If δ is small enough so that $\rho^{-\delta}(1 - \eta) \leqslant 1$, then we will have $|\theta_1| \leqslant 1$ in Q_1. Moreover $|\theta_1(x,t)| \leqslant \rho^{-\delta}$ where $|x| > 1$.

We define $M_1 = r^{(\delta - \varepsilon)k} M \leqslant M$. We are ready to move to the second step of the iteration.

Step k for $k > 1$. Assume that at the beginning of the kth step in the iteration we have a θ_k such that

$$\operatorname{div} z^\varepsilon \nabla \theta_k = 0 \quad \text{where } z > 0,$$
$$\partial_t \theta_k(x,0,t) + M_k\big(w(x,t) - \overline{w}(t)\big) \cdot \nabla \theta_k(x,0,t) + \lim_{z \to 0} z^\varepsilon \partial_z \theta_k(x,z,t) = 0$$

where $w = R^\perp \theta_k$ and

$$\overline{w} = c \int_{\mathbb{R}^2 \setminus B_{2/\rho}} \frac{\theta_k(y)y^\perp}{|y|^3} \, dy.$$

Moreover $|\theta_k| \leqslant 1$ in Q_1 and $\theta_k(x,t) \leqslant 2|x|^{2\delta}$ where $|x| > 1$. Recall that $M_k \leqslant M$.

Let us write $w - \overline{w} = w_1 + w_2 + w_3$ where

$$w_1(x,t) = c \int_{B_2} \frac{\theta_k(y,t)(y-x)^\perp}{|y-x|^3} \, dy, \tag{3.6}$$

$$w_2(x,t) = c \int_{B_{2/\rho} \setminus B_2} \frac{\theta_k(y,t)(y-x)^\perp}{|y-x|^3} \, dy, \tag{3.7}$$

$$w_3(x,t) = c \int_{\mathbb{R}^2 \setminus B_{2/\rho}} \theta_k(y,t)\left(\frac{(y-x)^\perp}{|y-x|^3} - \frac{y^\perp}{|y|^3}\right) dy. \tag{3.8}$$

Let us analyze each component w_i. Since we are choosing δ small enough, we can and will assume $\rho^{-\delta} < 3/2 < 2$.

For estimating w_1, we notice that we are integrating on a given compact domain B_2. Modulo a lower order correction, this is the same as applying a Riesz transform to a function with compact support. Therefore, from the L^∞ estimate of θ_k, we can apply classical Calderon–Zygmund estimates, we obtain that w_1 is in $L^\infty([0,1], L^p(B_1))$ for any $p \in (1, \infty)$. In particular for $p = 2n/\alpha$, and its norm (for this particular p) is less than a universal constant K (in this case it does not depend even on ρ).

Both w_2 and w_3 are bounded. We will estimate their L^∞ norms in Q_1, which is stronger than the norms in $L^\infty([0,1], L^{2n/\alpha}(B_1))$:

$$\left| w_2(x,t) \right| \leqslant c \int_{B_{2/\rho} \setminus B_2} \frac{2^{1+2\delta} \rho^{-2\delta}}{|y-x|^2} \, dy \quad \text{using that } \left| \theta_k(y,t) \right| \leqslant 2^{1+2\delta} \rho^{-2\delta} \text{ if } y \in B_{2/\rho}$$
$$\leqslant -C \log \rho,$$
$$\left| w_3(x,t) \right| \leqslant c \int_{\mathbb{R}^2 \setminus B_{2/\rho}} 2|y|^{2\delta} \left| \frac{(y-x)^\perp}{|y-x|^3} - \frac{y^\perp}{|y|^3} \right| dy \quad \text{using that } \left| \theta_k(y,t) \right| \leqslant 2|y|^{2\delta} \text{ where } |x| > 1$$
$$\leqslant c \int_{\mathbb{R}^2 \setminus B_{2/\rho}} 2|y|^{2\delta} \frac{C|x|}{|y|^3} \, dy \quad \text{where } |x| \leqslant 1$$
$$\leqslant C\rho.$$

Since $w_1 + w_2 + w_3 \in L^\infty([0, 1], L^{2n/\alpha}(B_1))$, we can apply Theorem 2.3 again to obtain $\operatorname{osc}_{Q_{1/2}} \theta \leqslant 2 - 2\eta$ (where η depends on ρ).

Since w_1 and w_2 are continuous in B_2, we can solve the equation as before

$$V(1) = 0,$$
$$\dot{V}(t) = M_k\big(w_2(V(t), t) + w_3(V(t), t)\big).$$

Note that from the estimates above for $|w_1|$ and $|w_2|$ we have $|\dot{V}(t)| \leqslant -C \log \rho + C\rho$. Therefore $|V(t)| \leqslant -C\rho^\alpha \log \rho + C\rho^{1+\alpha}$ if $t \in [(1 - \rho^\alpha), 1]$.

We choose ρ small such that

$$-C\rho^\alpha \log \rho + C\rho^{1+\alpha} + \rho \leqslant 1/2 \tag{3.9}$$

so as to make sure that $(x + V(t), y, t) \in Q_{1/2}$ if $(x, y, t) \in Q_\rho$. Note that there is no circular dependence of constants since the constants C above are universal.

We continue as in Step 1. We define $\tilde{\theta}_k(x, y, t) = \theta_k(x + V(t), y, t)$ and verify that $\tilde{\theta}_k$ satisfies the equation

$$\operatorname{div} z^\varepsilon \nabla \tilde{\theta}_k = 0 \quad \text{where } z > 0,$$
$$\partial_t \tilde{\theta}_k(x, 0, t) + M_k \tilde{w} \cdot \nabla \tilde{\theta}_k(x, 0, t) + \lim_{z \to 0} z^\varepsilon \partial_z \tilde{\theta}_k(x, z, t) = 0$$

with

$$\begin{aligned}
\tilde{w}(x, t) &= w(x + V(t), t) - \overline{w}(t) - \frac{\dot{V}(t)}{M_k} \\
&= w_1(x + V(t), t) + w_2(x + V(t), t) + w_3(x + V(t), t) - w_2(V(t), t) - w_3(V(t), t) \\
&= c\left(\int_{\mathbb{R}^2} \frac{\tilde{\theta}_k(y, t)(y - x)^\perp}{|y - x|^3}\, dy - \int_{\mathbb{R}^2 \setminus B_2} \frac{\tilde{\theta}_k(y, t) y^\perp}{|y|^3}\, dy \right).
\end{aligned}$$

Now we rescale. Since $\operatorname{osc}_{Q_\rho} \tilde{\theta} \leqslant 2 - 2\eta \leqslant 2\rho^\delta$, let us pick $m \in [-1 + \rho^\delta, 1 - \rho^\delta]$ such that $|\tilde{\theta} - m| \leqslant \rho^\delta$ in Q_ρ (typically $m = (\sup_{Q_{1/2}} \theta_k + \inf_{Q_{1/2}} \theta_k)/2$). Let

$$\theta_{k+1}(x, y, t) = (\tilde{\theta} - m)_\rho,$$
$$\operatorname{div} z^\varepsilon \nabla \theta_{k+1} = 0 \quad \text{where } z > 0,$$
$$\partial_t \theta_{k+1}(x, 0, t) + M_{k+1}\big(w(x, t) - \overline{w}(t)\big) \cdot \nabla \theta_{k+1}(x, 0, t) + \lim_{z \to 0} z^\varepsilon \partial_z \theta_{k+1}(x, z, t) = 0$$

where $M_{k+1} = \rho^{\delta - \varepsilon} M_k \leqslant M$, $w = R^\perp \theta$ and

$$\overline{w}(t) = c \int_{\mathbb{R}^2 \setminus B_{2/\rho}} \frac{\theta_{k+1}(y) y^\perp}{|y|^3}\, dy.$$

Then we will have $|\theta_{k+1}| \leqslant 1$ in Q_1. To make sure we obtain our desired estimates when $|x| > 1$ we must make some computations which follow:

$$\begin{aligned}
|\theta_{k+1}(x, t)| &\leqslant \rho^{-\delta} |\tilde{\theta}(\rho x, \rho^\alpha t) - m| \\
&\leqslant \rho^{-\delta}\big(m + |\theta_k(\rho x + V(\rho^\alpha t), \rho^\alpha t)|\big) \\
&\leqslant \begin{cases} \rho^{-\delta}(2 - \rho^\delta) & \text{if } |x| \leqslant \frac{1}{2\rho}, \\ \rho^{-\delta}(1 - \rho^\delta + \rho^{-2\delta}(\rho|x| + 1/2)^{2\delta}) & \text{if } |x| > \frac{1}{2\rho}. \end{cases}
\end{aligned}$$

In case $1 \leqslant |x| \leqslant \frac{1}{2\rho}$ we have $|\theta_{k+1}(x, t)| \leqslant \rho^{-\delta}(2 - \rho^\delta) \leqslant 2 \leqslant 2|x|^{2\delta}$ since ρ^δ was chosen larger than $2/3$.

In case $|x| \geqslant \frac{1}{2\rho}$, we have

$$
\begin{aligned}
\left|\theta_{k+1}(x,t)\right| &\leqslant \rho^{-\delta}\left(1 - \rho^{\delta} + 2\left(\rho|x| + 1/2\right)^{2\delta}\right) \\
&\leqslant \rho^{-\delta} - 1 + 2\rho^{-\delta}\left(2\rho|x|\right)^{2\delta} \\
&\leqslant \rho^{-\delta} - 1 + 2\rho^{\delta}2^{2\delta}|x|^{2\delta} \\
&\leqslant 2|x|^{2\delta}\left(\frac{\rho^{-\delta}}{2|x|^{2\delta}} - \frac{1}{2|x|^{2\delta}} + 2^{2\delta}\rho^{\delta}\right) \\
&\leqslant 2|x|^{2\delta}\left(\frac{(4\rho)^{\delta}}{2} - \frac{4^{\delta}\rho^{2\delta}}{2} + (4\rho)^{\delta}\right) \\
&\leqslant 2|x|^{2\delta}(4\rho)^{\delta}\left(\frac{3}{2} - \frac{\rho^{\delta}}{2}\right) < 2|x|^{2\delta}
\end{aligned}
$$

where the last inequality holds if $\rho \leqslant 1/16$, since then we would have

$$
(4\rho)^{\delta}\left(\frac{3}{2} - \frac{\rho^{\delta}}{2}\right) < \rho^{\delta/2}\left(\frac{3}{2} - \frac{\rho^{\delta}}{2}\right)
$$

which is less than 1 for any $\delta > 0$ (the polynomial $x(3/2 - x^3/2)$ has a maximum at $x = 1$).

Therefore in every case we obtained $|\theta_{k+1}| \leqslant 1$ in Q_1 and $|\theta_{k+1}| \leqslant 2|x|^{2\delta}$ when $|x| > 1$. We finish step k and are ready for the next step in the iteration.

We stress that there is no circular dependence in the choice of constants. The constant ρ is the first one which has to be chosen. It must satisfy three inequalities:

- $L\rho^{\alpha} + \rho \leqslant 1/2$ for (3.5) in the first step.
- $-C\rho^{\alpha}\log\rho + C\rho^{1+\alpha} + \rho \leqslant 1/2$ for (3.9).
- $\rho < 1/16$ in order to make the very last inequality work.

All the constants above depend only on M and (3.4), which both depend only on $\|\theta_0\|_{L^2}$.

Once we have ρ, the value of η follows from applying Theorem 2.3. So η depends on the initial choice of ρ. Once we have η and ρ, we choose δ so that $\rho^{\delta} \geqslant (1 - \eta)$ and $\rho^{-\delta} \leqslant 2$. $\quad\square$

Proof of Theorem 1.2. Using Theorem 2.2, there is a t_0 depending only on $\|\theta_0\|_{L^2}$ such that $\|\theta(-,t_0)\|_{L^\infty} \leqslant 1$. We can also pick t_0 large such that the right-hand side in Lemma 3.1 is smaller than 1. At that point, we are already in the normalized situation of the proof of Theorem 1.1, the choice of all constants from that point does not depend on $\|\theta_0\|_{L^2}$.

We consider the function θ starting at this t_0 and we apply Theorem 1.1 with $T = 1$, $M = 1$ and the right-hand side in (3.4) equal to 1. Then if $\varepsilon \leqslant \delta$ for δ small enough, we will have $\theta \in C^{\delta}$ at any time $t \geqslant t_0 + 1$. Further C^{∞} regularity follows from [3]. $\quad\square$

Appendix A

In this appendix we present the local energy inequality and a weighted version of De Giorgi's isoperimetrical lemma so that we can reproduce the proof in [2] of Theorem 2.3. The proofs use essentially the same ideas as in [2]. The main modification is that we need to use the weight z^{ε} in the upper half space, so that the Dirichlet to Neumann map for harmonic functions corresponds to the fractional laplacian $(-\Delta)^{\alpha/2}$ (see [1]).

For the local energy inequality, we can deal with a more general equation

$$
\begin{aligned}
&\operatorname{div} z^{\varepsilon}\nabla\theta = 0 \quad \text{where } z > 0, \\
&\partial_t\theta(x,0,t) + w \cdot \nabla\theta(x,0,t) + \lim_{z\to 0} z^{\varepsilon}\partial_z\theta(x,z,t) = 0
\end{aligned}
\tag{3.10}
$$

where w is a fixed divergence free vector field in $\mathbb{R}^n$ and $\theta: \mathbb{R}^n \times [0,\infty) \times [0,\infty) \to \mathbb{R}$. For proving Theorem 2.3 we do not need to use the relation between w and θ, and the dimension n is arbitrary.

Note that after restricting θ to $z = 0$, (3.10) is equivalent to

$$\partial_t \theta(x,t) + w \cdot \nabla \theta(x,t) + (-\Delta)^{\alpha/2}\theta(x,t) = 0. \tag{3.11}$$

Proposition 3.2 (*Local energy inequality*). *Let $t_1 < t_2$ and let $\theta \in L^\infty(t_1, t_2; L^2(\mathbb{R}^n))$ with $(-\Delta)^{\alpha/2}\theta \in L^2((t_1, t_2) \times \mathbb{R}^n)$ be a solution to* (1.1) *with velocity w satisfying*:

$$\|w\|_{L^\infty(t_1, t_2; L^{2n/\alpha}(B_2))} \leqslant C.$$

Then there exists a constant C_1 (depending only on C) such that for every $t \in (t_1, t_2)$ and cut-off function η compactly supported in B_2^*:

$$\int_{t_1}^{t_2} \int_{B_2^*} z^\varepsilon \big|\nabla\big(\eta[\theta^*]_+\big)\big|^2 \, dx \, dz \, dt + \int_{B_2} \big(\eta[\theta]^+\big)^2(x, t_2) \, dx$$

$$\leqslant \int_{B_2} \big(\eta[\theta]_+\big)^2(x, t_1) \, dx + C_1 \int_{t_1}^{t_2} \int_{B_2} \big(|\nabla\eta|[\theta]_+\big)^2 \, dx \, dt + C_1 \int_{t_1}^{t_2} \int_{B_2^*} z^\varepsilon \big(|\nabla\eta|[\theta]_+\big)^2 \, dx \, dz \, dt. \tag{3.12}$$

Note that the only difference with the corresponding estimate in [2] is the factor z^ε in every integral involving the extension to $z > 0$. This is a straightforward modification following [1]. This only modification applies along the proof.

Note also that the BMO norm plus an estimate on the mean is stronger than $L^{2n/\alpha}$, so in particular the estimate holds if $w \in L^\infty(BMO)$ and the mean of w in B_2 is also bounded.

Proof. We have for every $t \in (t_1, t_2)$:

$$0 = \int_0^\infty \int_{\mathbb{R}^n} \eta^2 [\theta]_+ \, \text{div}\big(z^\varepsilon \nabla \theta\big) \, dx \, dz$$

$$= \int_0^\infty \int_{\mathbb{R}^n} -z^\varepsilon \big|\nabla\big(\eta[\theta]_+\big)\big|^2 + z^\varepsilon |\nabla\eta|^2 [\theta]_+^2 \, dx \, dz + \int_{\mathbb{R}^n} \eta^2 [\theta]_+ (-\Delta)^{\alpha/2}\theta \, dx$$

where the characterization of $(-\Delta)^{\alpha/2}\theta$ as the Dirichlet to Neumann operator from [1] was used.

As in [2], we use Eq. (1.1) which leads to

$$\int_{t_1}^{t_2} \int_0^\infty \int_{\mathbb{R}^n} z^\varepsilon \big|\nabla\big(\eta[\theta]_+\big)\big|^2 \, dx \, dz \, dt + \int_{\mathbb{R}^n} \eta^2 \frac{[\theta]_+^2(t_2)}{2} \, dx$$

$$\leqslant \int_{\mathbb{R}^n} \eta^2 \frac{[\theta]_+^2(t_1)}{2} \, dx + \int_{t_1}^{t_2} \int_0^\infty \int_{\mathbb{R}^n} z^\varepsilon |\nabla\eta|^2 [\theta]_+^2 \, dx \, dz \, dt + \left| \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \eta \nabla\eta \cdot w[\theta]_+^2 \, dx \, dt \right|.$$

To dominate the last term, we use Sobolev embedding and the variational characterization of the equation $\text{div}\, z^\varepsilon \nabla U = 0$:

$$\big\|\eta[\theta]_+\big\|_{L^{\frac{2n}{n-2\alpha}}(\mathbb{R}^n)}^2 \leqslant C \big\|\eta[\theta]_+\big\|_{H^{\alpha/2}}^2 = C \int_{\mathbb{R}^n} (\eta\theta_+)(-\Delta)^{\alpha/2}\theta_+ \, dx$$

$$= \min_{v(x,0,t)=\eta(x,0,t)\theta(x,0,t)} \int_{\mathbb{R}^n \times (0,+\infty)} z^\varepsilon |\nabla v|^2 \, dx \, dz$$

$$\leqslant \int_{B_2^*} z^\varepsilon \big|\nabla(\eta\theta_+)\big|^2 \, dx \, dz.$$

Recall that η is supported inside B_2. Now we continue in the standard way as in [2]. For some small $\tilde{\varepsilon}$, we write

$$\left| \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \nabla \eta^2 \cdot w \frac{\theta_+^2}{2} \, dx \, dt \right| \leqslant \tilde{\varepsilon} \int_{t_1}^{t_2} \|\eta \theta_+\|_{L^{\frac{2n}{n-\alpha}}} \, dt + \frac{C}{\tilde{\varepsilon}} \int_{t_1}^{t_2} \|\nabla \eta \cdot w \theta_+\|_{L^{\frac{2n}{n+\alpha}}}^2 \, dt.$$

The first term is absorbed by the left-hand side in (3.12). The second is bounded using Hölder's inequality:

$$\frac{C}{\tilde{\varepsilon}} \int_{t_1}^{t_2} \|\nabla \eta \cdot w \theta_+\|_{L^{\frac{2n}{n+\alpha}}}^2 \, dt \leqslant \frac{C}{\tilde{\varepsilon}} \|w\|_{L^\infty(t_1,t_2;L^{2n/\alpha}(B_2))} \int_{t_1}^{t_2} \int_{B_2} |\nabla \eta \, \theta_+|^2 \, dx \, dt$$

which finishes the proof. $\quad\square$

Now we show the De Giorgi isoperimetrical lemma with the weight z^ε. This is a property of H^1 functions independently of the equation.

Proposition 3.3 (*De Giorgi isoperimetrical lemma*). *Letting w be a function in $H^1(B_1^*, z^\varepsilon)$, we have the estimate*

$$\left(\int_{\{w \leqslant 0\}} z^\varepsilon \, dX \right) \left(\int_{\{w \geqslant 1\}} z^\varepsilon \, dX \right) \leqslant C \left(\int_{\{0 < w < 1\}} z^\varepsilon \, dX \right)^{1/2} \left(\int_{B_1^*} |\nabla w| z^\varepsilon \, dX \right)^{1/2} \quad (3.13)$$

(*recall the notation $X = (X', X_{n+1}) = (x, z)$ with $X \in \mathbb{R}^{n+1}$ and $x = X' \in \mathbb{R}^n$*).

This estimate can also be written as

$$\left| \{w \leqslant 0\} \right| \left| \{w \geqslant 1\} \right| \leqslant C \left| \{0 < w < 1\} \right|^{1/2} \|w\|_{\dot{H}^1(z^\varepsilon)}$$

where the measures of the sets are computed with respect to the weight z^ε.

Note that (3.13) is not scale invariant. If we replace B_1^* by B_r^*, the constant C would depend on r.

Proof. We can consider $\tilde{w} = \max(0, \min(1, w))$, so it is no loss of generality to assume $w(X) \in [0, 1]$ for every X, so that $|\nabla w| = 0$ a.e. in $\{w \geqslant 0\}$ and $\{w \leqslant 0\}$.

Let X be a point such that $w(X) = 0$ and Y be such that $w(Y) = 1$. Letting $\theta = \frac{Y-X}{|Y-X|}$, we compute

$$Y_{n+1}^\varepsilon \leqslant Y_{n+1}^\varepsilon \int_0^{|Y-X|} \left| \nabla w(X + t\theta) \right| dt.$$

For a fixed value of X we write

$$Z = X + t\theta, \quad (3.14)$$

$$Y = X + m\theta, \quad (3.15)$$

$$dt \, dY = \frac{m^{n-1}}{t^{n-1}} \, dm \, dZ. \quad (3.16)$$

In order to estimate the second factor in the left-hand side, we integrate in Y:

$$\left(\int_{\{w(Y)=1\}} Y_{n+1}^\varepsilon \, dY \right) \leqslant \int_{\{w(Y)=1\}} \int_0^{|Y-X|} Y_{n+1}^\varepsilon \left| \nabla w(X + t\theta) \right| dt \, dY$$

$$\leqslant \int_{\{0 < w(Z) < 1\}} \frac{|\nabla w(Z)|}{|X - Z|^n} \int_{m_0}^{m_1} m^{n-1} [X + m\theta]_{n+1}^\varepsilon \, dm \, dZ$$

$$\leqslant C \int_{\{0 < w(Z) < 1\}} \frac{|\nabla w(Z)|}{|X - Z|^n} [Y^*]_{n+1}^\varepsilon \, dZ$$

where Y^* is the point on the line $X + m\theta$ such that Y_{n+1} is maximum. Note that $\theta = \frac{Z-X}{|Z-X|}$.

Now we integrate in X:

$$\left(\int\limits_{\{w(X)=0\}} X_{n+1}^{\varepsilon}\,\mathrm{d}X\right)\left(\int\limits_{\{w(Y)=1\}} Y_{n+1}^{\varepsilon}\,\mathrm{d}Y\right) \leqslant C \int\limits_{\{w(X)=0\}}\int\limits_{\{0<w(Z)<1\}} \frac{|\nabla w(Z)|}{|X-Z|^n}[Y^*]_{n+1}^{\varepsilon} X_{n+1}^{\varepsilon}\,\mathrm{d}Z\,\mathrm{d}X.$$

The point Y^* depends on X and Z. In every case, Z is on the line segment joining X and Y^*, so either $X_{n+1} \leqslant Z_{n+1}$ or $Y_{n+1}^* \leqslant Z_{n+1}$. On the other hand $\max(X_{n+1}, Y_{n+1}^*) \leqslant 1$ since $X, Y^* \in B_1^*$. Thus $[Y^*]_{n+1}^{\varepsilon} X_{n+1}^{\varepsilon} \leqslant Z_{n+1}^{\varepsilon}$. Therefore

$$\left(\int\limits_{\{w(X)=0\}} X_{n+1}^{\varepsilon}\,\mathrm{d}X\right)\left(\int\limits_{\{w(Y)=1\}} Y_{n+1}^{\varepsilon}\,\mathrm{d}Y\right) \leqslant C \int\limits_{\{w(X)=0\}}\int\limits_{\{0<w(Z)<1\}} \frac{|\nabla w(Z)|}{|X-Z|^n} Z_{n+1}^{\varepsilon}\,\mathrm{d}Z\,\mathrm{d}X$$

$$\leqslant C \int\limits_{\{0<w(Z)<1\}} |\nabla w(Z)| Z_{n+1}^{\varepsilon}\,\mathrm{d}Z$$

$$\leqslant C\left(\int\limits_{\{0<w<1\}} Z_{n+1}^{\varepsilon}\,\mathrm{d}Z\right)^{1/2}\left(\int\limits_{B_1^*} |\nabla w(Z)| Z_{n+1}^{\varepsilon}\,\mathrm{d}Z\right)^{1/2}.$$

The last inequality follows by Cauchy–Schwartz since the support of ∇w is included in $\{0 < w < 1\}$ (we are assuming $0 \leqslant w \leqslant 1$ in B_1^*). This finishes the proof. $\quad\square$

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