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Phase transitions with a minimal number of jumps in the singular limits of higher order theories

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Abstract

For a smooth $W:(0,\infty)\times\mathbb{R}^d\to\mathbb{R}$ and a family of L-periodic $W^{1,2}$-functions $\vartheta_\epsilon:\mathbb{R}\to\mathbb{R}^d$ with $\vartheta_\epsilon\rightharpoonup\vartheta$, the basic problem is to understand the weak* limit as $\epsilon\to 0$ of L-periodic minimizers of

$$\int_0^L\left(\frac{\epsilon}{2}\varphi'^2+W(\varphi,\vartheta_\epsilon)\right)ds. \tag{†}$$

It is assumed that $W(\phi,\theta)\to\infty$ as $\phi\to 0,\infty$, and that $W(\cdot,\theta)$, which has no more than three critical points counting multiplicity depending on $\theta\in\mathbb{R}^d$, is of a type that arises in the Cahn–Hilliard theory of phase separations where $d=1$. The limiting problem with $\epsilon=0$ is to minimize, over bounded L-periodic measurable functions φ,

$$\int_0^L W\big(\varphi(s),\vartheta(s)\big)\,ds. \tag{‡}$$

Minimizers of (‡) need not be unique (there may be uncountably many), they may be discontinuous and minimizers with only simple jumps may coexist with minimizers with much more complicated discontinuities. Weak* limits of minimizers of (†) as $\epsilon\to 0$ are minimizers of the relaxation of (‡). However it is shown that if, for a sequence of minimizers of (†),

$$\limsup_{k\to\infty}\sqrt{\epsilon_k}\int_0^L\left|\varphi'_{\epsilon_k}(s)\right|^2ds<\infty,\quad \epsilon_k\to 0,$$

then the weak* limit of any subsequence of $\{\varphi_{\epsilon_k}\}$ is an actual minimizer of (‡) which is continuous except at a finite number of simple jumps. Moreover, for sequences $\epsilon_k\to 0$ from a set of positive Lebesgue density, it is shown that the weak* limit of L-periodic minimizers of (†) is a minimizer of (‡) with a finite number of simple jumps. Under additional hypotheses it is shown that, for sequences from a set of full Lebesgue density, the weak* limits of L-periodic minimizers of (†) are minimizers of (‡) with a *minimal number of simple jumps*.

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1. Introduction

A crucial feature in models of phase transitions is the non-convexity of bulk energy functions with respect to the phase variable φ. For instance, in one-dimensional Cahn–Hilliard theory the Helmoltz free energy density is given by

$$W(\varphi,\vartheta)=\frac{1}{4}\left(\varphi^2-1\right)^2-\vartheta\varphi, \tag{1.1}$$

where ϑ is the chemical potential. If φ and ϑ are L-periodic, the total mesoscopic energy per period is

$$\mathcal{J}_\epsilon(\varphi):=\int_0^L\left(\frac{\epsilon}{2}\varphi'^2+\frac{1}{4}\left(\varphi^2-1\right)^2-\vartheta\varphi\right)dx,\quad \epsilon>0, \tag{1.2}$$

where $\epsilon\varphi'^2/2$ corresponds to the energy of phase interactions, $\sqrt{\epsilon}$ characterizes the width of the interfaces between phases and in practice ϵ is very small. Critical points of $\mathcal{J}_\epsilon$ satisfy

$$-\epsilon\varphi''(x)+\varphi(x)^3-\varphi(x)=\vartheta(x),\quad x\in\mathbb{R}. \tag{1.3}$$

Alternatively, (1.3) occurs as a quasi-steady, phase field model in the theory of solidification with ϑ representing temperature [1,2]. One-dimensional phase transition modelling can also give rise to higher order non-convex variational problems [10,12], see [9] for a general discussion. However we confine ourselves to studying generalizations of (1.2) and the weak* limits in L^∞_{per} of its minimizers as $\epsilon\to 0$ and two questions arise.

(1) How does one describe the weak* limits of minimizers of (1.2) and its generalizations?
(2) Is there is a "macroscopic" variational problem with minimizers that coincide with these weak* limits?

A common belief is that both issues can be resolved using Γ-convergence theory, but this is not always the case. Recall that a sequence of functionals $F_\epsilon: X\to[\alpha,\infty]$, $\alpha>-\infty$, defined on a metric space X, has Γ-limit $F: X\to[\alpha,\infty]$ if, for every φ_0 and $\varphi_\epsilon\to\varphi_0$,

$$F(\varphi_0)\leqslant\liminf_{\epsilon\to 0}F_\epsilon(\varphi_\epsilon)$$

and there exists a sequence $\overline{\varphi}_\epsilon\to\varphi_0$ so that

$$F(\varphi_0)=\lim_{\epsilon\to 0}F_\epsilon(\overline{\varphi}_\epsilon).$$

Let $\mathcal{M}F_\epsilon$ and $\mathcal{M}F$ be the sets of minimizers of F_ϵ and F respectively, and let $\mathcal{L}F$ be set of limit points of sequences $x_\epsilon\in\mathcal{M}F_\epsilon$ as $\epsilon\to 0$. It is clear that $\mathcal{L}F\subset\mathcal{M}F$, but they are not equal in general. In the mesoscopic theory of phase transitions F_ϵ would represent the total free energy and $\varphi_\epsilon\in\mathcal{M}F_\epsilon$ the corresponding stable equilibrium states. If a macroscopic theory is to be regarded as a limit of mesoscopic theory, then macroscopic stable equilibria should belong to $\mathcal{L}F$ and the Γ-limit F interpreted as an approximation to the macroscopic free energy. The validity of such an approach depends on the size of $\mathcal{M}F\setminus\mathcal{L}F$. If it is not empty, an additional selection principle is needed to identify the solutions of the macroscopic problem that are relevant to the mesoscopic theory (particularly if $\mathcal{L}F$ is small in $\mathcal{M}F$).

The following two examples related to our problem illustrate different relations between $\mathcal{M}F$ and $\mathcal{L}F$. In both, ϑ is a given, continuous L-periodic function. For $L>0$ fixed, let L^p_{per} or L^∞_{per} be the space of L-periodic functions with restrictions that are pth-power integrable or essentially bounded on $(0,L)$.

The first problem is to minimize the Ginzburg–Landau energy functional (1.2):

$$\mathcal{J}_\epsilon\left(\varphi^*\right)=\min_{\varphi\in L^4_{\mathrm{per}}}\mathcal{J}_\epsilon(\varphi),\quad \mathcal{J}_\epsilon(\varphi)=\int_0^L\left(\epsilon\varphi'^2+\frac{1}{4}\left(\varphi^2-1\right)^2-\varphi\vartheta\right)ds. \tag{1.4}$$

The second is to minimize the scaled free energy functional

$$\mathfrak{J}_\epsilon(\psi^*) = \min_{\psi \in L^4_{\text{per}}} \mathfrak{J}_\epsilon(\psi), \quad \mathfrak{J}_\epsilon(\psi) = \int_0^L \left(\sqrt{\epsilon}\psi'^2 + \frac{1}{4\sqrt{\epsilon}}(\psi^2 - 1)^2 - \psi\vartheta \right) ds. \tag{1.5}$$

Note that if ϑ in (1.4) is replaced by $\sqrt{\epsilon}\vartheta$, then (1.4) is transformed into (1.5), but there is an essential difference between the two as they stand. Let $X = L^4_{\text{per}}$ with the weak topology in which bounded sets are metrizable. The Γ-limit $\mathcal{J}$ of $\mathcal{J}_\epsilon$ is a particular case of general theory, see, for example, [6],

$$\mathcal{J}(\varphi) =: \Gamma\text{-}\lim \mathcal{J}_\epsilon(\varphi) = \int_0^L W^{**}(\varphi, \vartheta)\, ds, \tag{1.6}$$

where, for fixed $\theta \in \mathbb{R}$, $W^{**}(\cdot, \theta)$ denotes the convex envelope of $W(\cdot, \theta)$ in (1.1). Since W is bounded below, the set of minimizers of (1.6) is non-empty and there are various possibilities. For example, (1.6) may have a unique minimizer. This happens if the set of zeros of ϑ is discrete, a case that was thoroughly investigated in [1,2] (see also [10] for generalizations and further discussion). Alternatively, (1.6) may have an infinite set of minimizers, including Young measure solutions, as happens when ϑ in (1.1) vanishes on some interval. Moreover, a minimizer may be discontinuous at every point of such an interval.

On the other hand, see [4, Chapter 6], [11], we have that

$$\Gamma\text{-}\lim \mathfrak{J}_\epsilon(\psi) = \mathfrak{J}(\psi) := \begin{cases} \wp_0 \mathcal{N}(\psi) - \int_0^L \psi\vartheta\, ds & \text{if } |\psi| = 1 \\ \quad \text{almost everywhere on } [0, L) \\ \quad \text{and } \psi \text{ is piecewise constant,} \\ +\infty \quad \text{otherwise,} \end{cases} \tag{1.7}$$

where $\mathcal{N}(\psi)$ is the number of discontinuities of ψ in $[0, L)$ and

$$\wp_0 = 2\int_{-1}^{1} \sqrt{\frac{1}{4}(\phi^2 - 1)^2}\, d\phi = \frac{4}{3}. \tag{1.8}$$

Clearly elements of $\mathcal{M}\mathfrak{J}$ are piecewise constant functions ψ with a finite number of jumps at points $s_i \in [0, L)$, and $\mathcal{N}(\psi)\wp_0$ is a weighted count of jumps per period. Roughly speaking the first term in (1.7) strives to minimize the number of jumps, but this process is controlled by ϑ.

A difference between $\mathcal{M}\mathfrak{J}_\epsilon$ and $\mathcal{M}\mathcal{J}_\epsilon$ is that elements of $\mathcal{M}\mathfrak{J}_\epsilon$ have a regularity property independent of ϵ because the set

$$\{\Phi(\psi_\epsilon):\ \psi_\epsilon \in \mathcal{M}\mathfrak{J}_\epsilon,\ \epsilon \in (0, 1)\}, \quad \text{where } \Phi(\phi) = \int_0^\phi |s^2 - 1|\, ds,$$

is bounded in the Sobolev space $W^{1,1}_{\text{per}}$. By contrast, the minimizers of the Ginzburg–Landau energy $\mathcal{J}_\epsilon$ *have no regularity independent of* ϵ. An analysis of the relation between $\mathcal{M}\mathcal{J}$ and $\mathcal{L}\mathcal{J}$ is consequently more difficult and is our concern here. A special case of our results concerns $\mathcal{L}\mathcal{J}$ under the assumption that the limiting problem, (1.4) with $\epsilon = 0$, has at least one piecewise continuous minimizer, which is true when $\{\vartheta = 0\} \cap [0, L)$ has a finite number of connected components. The conclusion is that there exists a set $E \subset (0, 1)$ which is Lebesgue dense at 0 (see (5.1)) and the elements of $\mathcal{L}\mathcal{J}$ which arise from sequences in E are piecewise continuous functions with the minimal possible weighted number of jumps (see (2.3)).

Similar non-convex variational problems arise in the classical nonlinear theory of elastic rods [7] (also [5]) as follows. Suppose that $x \in \mathbb{R}$ represents the positions of material points in an elastic rod with uniform density ϱ when it is in equilibrium in the absence of external forces on a straight line. Suppose that $u(x, t)$ denotes the position of the same point when the rod is deformed in the same straight line by a force field $f(x, t)$, and that f and the stretch field u_x are L-periodic in x. If the elastic energy density is given in terms of the stretch and stretch gradient by the formula

$$\frac{\epsilon}{2}{u_{xx}}^2 + W(u_x),$$

where W is convex (a one-phase material), then u satisfies the wave equation

$$\varrho u_{tt} = -\epsilon u_{xxxx} + \big(W'(u_x)\big)_x + f, \qquad u_x(x+L,t) = u_x(x,t).$$

In the corresponding travelling-wave problem, $f(x,t) = f(x-ct)$, $u = u(x-ct)$, and the stretch variable $\varphi(s) = u'(s)$, with $\vartheta'(s) = f(s)$, satisfies

$$-\epsilon\varphi'' + \partial_\phi \underline{W}\big(\varphi(s), \vartheta(s), \gamma\big) = 0, \qquad \varphi(s+L) = \varphi(s),$$

for some constant γ, where

$$\underline{W}(\phi,\theta,\gamma) = W(\phi) - c^2\varrho\phi^2/2 + (\theta+\gamma)\phi.$$

Note that, for large c, $\underline{W}$ is non-convex in ϕ even though W is convex. The external functions are ϑ and γ. If instead of remaining straight, a compressible rod with bending stiffness is prescribed to lie on a given periodic curve in a vertical plane, the stored energy is again non-convex in the stretch, and its curvature and height are two additional external functions in the travelling-wave equation. This observation in [13] explains our interest in periodic boundary conditions and multiple external functions.

Main results. Let H^1_{per} be the Hilbert space of L-periodic continuous functions on $\mathbb{R}$ with restrictions in $W^{1,2}(0,L)$. In what follows, $d \in \mathbb{N}$, $\varphi : \mathbb{R} \to (0,\infty)$, $\vartheta : \mathbb{R} \to \mathbb{R}^d$ and $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ are L-periodic functions, while $(\phi,\theta,A,\epsilon) \in \mathbb{R}^{d+3}$. We study minimization problems for generalizations of (1.2) of the following type. Let $B : \mathbb{R} \to \mathbb{R}$ be a smooth increasing function and, for a set $E \subset (0,1]$ with a limit point at 0, let $\{\vartheta_\epsilon \colon \epsilon \in E\}$ be bounded in $(H^1_{\text{per}})^d$. Then the problem is to find $\varphi_\epsilon \in H^1_{\text{per}}$ such that

$$J_\epsilon(\varphi_\epsilon) = \inf_{\varphi \in H^1_{\text{per}}} J_\epsilon(\varphi) =: \mathfrak{E}(\epsilon), \quad \epsilon > 0, \tag{1.9}$$

where

$$J_\epsilon(\varphi) = \int_0^L \left(\frac{\epsilon}{2}\big(B(\varphi)'\big)^2 + W(\varphi,\vartheta_\epsilon)\right) ds, \quad \varphi \in H^1_{\text{per}}. \tag{1.10}$$

For the Ginzburg–Landau functional (1.2), $B(\phi) = \phi$ and W is given by (1.1), whereas the scaled Ginzburg–Landau functional (1.5) corresponds to W in (1.1) with $B(\phi) = \sqrt{2}\phi$ and $\vartheta_\epsilon = \sqrt{\epsilon}\vartheta$. A solution φ_ϵ to problem (1.9), for $\epsilon > 0$, satisfies the Euler–Lagrange equation

$$\epsilon B'\big(\varphi_\epsilon(s)\big)B(\varphi_\epsilon)''(s) - \partial_\phi W\big(\varphi_\epsilon(s), \vartheta_\epsilon(s)\big) = 0 \quad \text{on } \mathbb{R}. \tag{1.11}$$

It follows from the maximum principle and (H1) below that solutions φ_ϵ of (1.11) have $\|\varphi_\epsilon\|_{L^\infty_{\text{per}}} + \|\varphi_\epsilon^{-1}\|_{L^\infty_{\text{per}}}$ bounded by a constant independent of ϵ when $\|\vartheta_\epsilon\|_{(H^1_{\text{per}})^d} \leqslant M_E$, $\epsilon \in E$. Our goal is to give an explicit upper bound for the number of jumps of φ, a weak* limit in L^∞_{per} of minimizers φ_{ϵ_k}, for a "almost all sequences ϵ_k tending to zero" in a sense that will be made precise. Before summarizing our result we discuss the essential properties of the potentials W and exterior functions ϑ_ϵ under consideration.

Conditions (H1–3), which are formulated precisely and illustrated by a figure in Section 2, describe a function $W(\cdot,\theta)$ which is either a single or double well potential depending on $\theta \in \mathbb{R}^d$. (H1) is a coercivity condition that $W(\phi,\theta) \to \infty$ as $\phi \to \alpha, \beta$ where $\alpha, \beta \in [-\infty,\infty]$. To avoid repetition, we consider only the mixed case $\alpha > -\infty$, $\beta = \infty$ and take $\alpha = 0$. (H2) says that, for $\theta \in \mathbb{R}^d$, the function $W(\cdot,\theta)$ has no more than three critical points counting multiplicity. (H3) implies that the distance between local or global minimizers $\phi^\pm(\theta)$ of $W(\cdot,\theta)$ is bounded below by a positive constant. Therefore the only possible bifurcations of critical points of $W(\cdot,\theta)$ are from inflection points to local-minimum–local-maximum pairs. Thus W is of a type that arises in first order phase transition problems. Hypotheses (H1–3) induce a decomposition of $\mathbb{R}^d$ into disjoint sets corresponding to the number and type of critical points of W. The most important is G^0_3, which consists of those $\theta \in \mathbb{R}^d$ for which $W(\cdot,\theta)$ has two distinct global minima corresponding to two distinct coexisting stable phases (see Fig. 1).

(H4) in Section 4 means, roughly speaking, that, for some $\vartheta \in (H^1_{\text{per}})^d$ and $\Theta_0, \Theta_1 < \infty$,

$$\limsup_{\epsilon\to 0} \|\vartheta_\epsilon - \vartheta\|_{(L^1_{\text{per}})^d}/\sqrt{\epsilon} = \Theta_0 \quad \text{and} \quad \limsup_{\epsilon\to 0} \sqrt{\epsilon}\|\partial_\epsilon\vartheta_\epsilon\|_{(L^1_{\text{per}})^d} = \Theta_1.$$

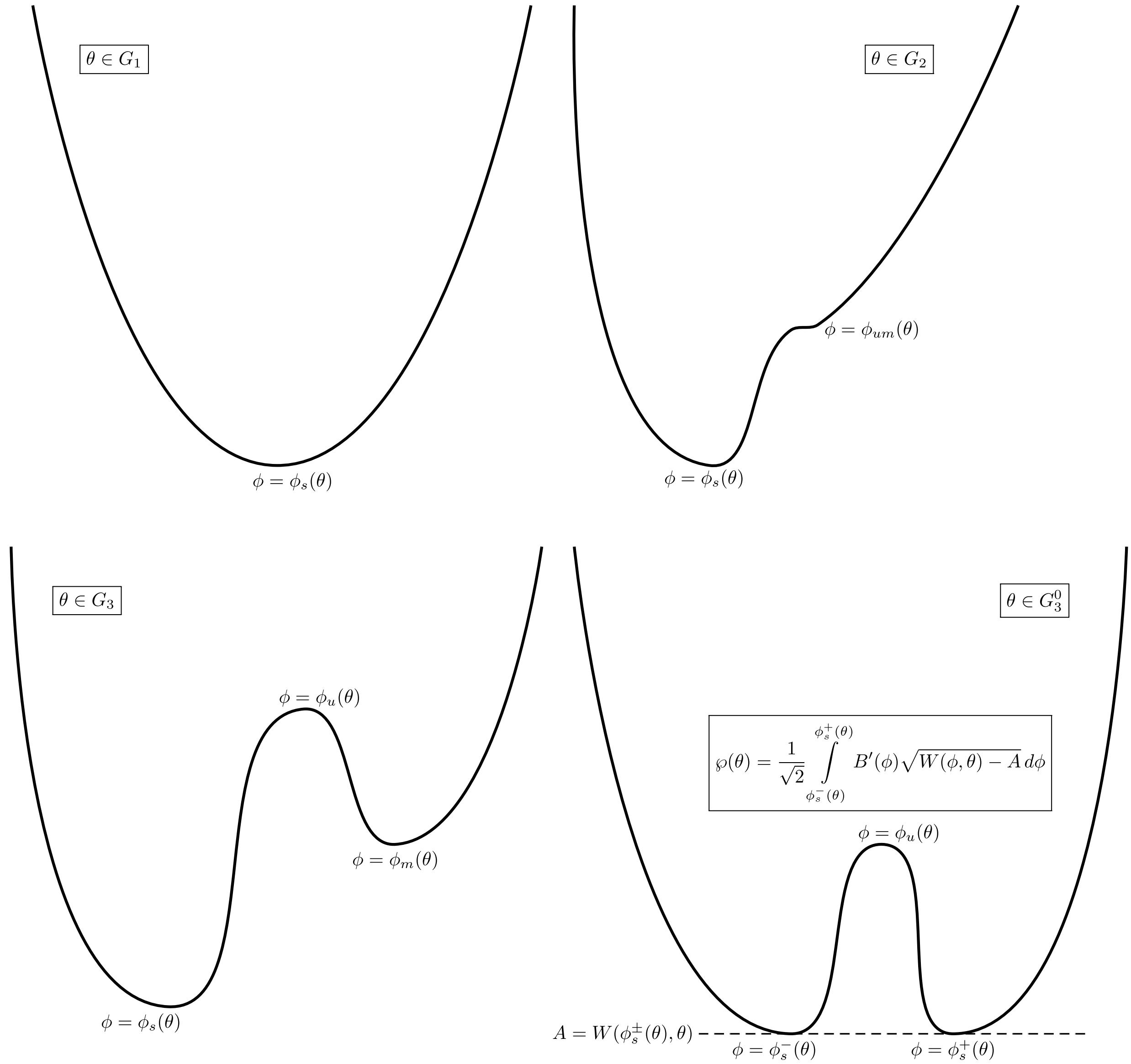

Fig. 1. Graphs of $W(\cdot, \theta)$ depending on $\theta \in \mathbb{R}^d$.

(H5) is the requirement that there exists a piecewise regular solutions to the primary variational problem:

$$\int_0^L W\big(\varphi^*, \vartheta\big)\, ds = \inf_{\varphi \in L^\infty_{\text{per}}} \int_0^L W(\varphi, \vartheta)\, ds =: \mathfrak{E}(0). \tag{1.12}$$

It holds if the critical set $\mathcal{G}_3^0(\vartheta) = \{s\colon\ \vartheta(s) \in G_3^0\}$ has a finite number of connected components.

By definition, a piecewise regular minimizer of (1.12) has finitely many jumps per period at points s_i with $\vartheta(s_i) \in G_3^0$ and to each such minimizer φ^* we can assign a weighted number of jumps $\mathcal{W}(\varphi^*)$, which is commensurate with, but not necessarily equal to, the actual number of jumps per period,

$$\mathcal{W}\big(\varphi^*\big) = \sum_{s_i \in [0,L)} \wp\big(\vartheta(s_i)\big).$$

Here $\wp : G_3^0 \to \mathbb{R}$ is given by formula (2.3a), which is similar to (1.8) in that special case. Let $\mathcal{W}_{\min}$ be the infimum of the weighted number of jumps of all piecewise regular minimizers. We will see that $\mathcal{W}_{\min}$ is attained. Let $\mathcal{N}^* < \infty$ be the maximum of the actual number of jumps of all piecewise regular minimizers with minimal weighted number of jumps.

(H6) in Section 5 is the requirement that $\mathfrak{E}$ defined by (1.9) is locally absolutely continuous on (0, 1). (In Appendix A we discuss (H6) which is often satisfied trivially; when ϑ_ϵ is independent of ϵ it follows because $\mathfrak{E}$ is concave.)

The following, Theorem 5.6, is one of the main results of the paper.

Minimal number of jumps principle. *Suppose that* (H1–6) *hold with* $\Theta_0 = \Theta_1 = 0$ *in* (H4). *Then, for any* $\delta > 0$ *there is a set* $E_\delta \subset (0, 1]$ *which is Lebesgue dense at* 0, *see* (5.1)*, with the following property. If a sequence* $\{\varphi_{\epsilon_n}\colon\ E_\delta \ni \epsilon_n \to 0\}$ *of minimizers converges weak* in* L^∞_{per} *to* φ*, then* φ *is a piecewise regular minimizer of* (1.12) *with weighted number of jumps* $\mathcal{W}(\varphi) \leqslant \mathcal{W}_{\min} + \delta$ *and actual number of jumps* $\mathcal{N}(\varphi) \leqslant \mathcal{N}^*$.

Method. As mentioned above, the passage to the limit in (1.9) is difficult because its solutions lack regularity properties independent of ϵ. To cope, we use the fact that the Euler–Lagrange equation (1.11) implies that $\mathcal{A}_\epsilon'(s) = \partial_\theta W(\varphi_\epsilon(s), \vartheta_\epsilon(s))\vartheta_\epsilon'(s)$ where the adiabatic variable $\mathcal{A}$ (see [3]) is defined by

$$\mathcal{A}_\epsilon(s) := W\big(\varphi_\epsilon(s), \vartheta_\epsilon(s)\big) - \frac{\epsilon}{2}\big(B(\varphi_\epsilon)'(s)\big)^2.$$

Combining this observation with the remark following (1.11), we conclude that solutions of (1.11) satisfy the estimates

$$M^{-1} \leqslant \varphi_\epsilon(s), \varphi(s) \leqslant M \quad \text{for } s \in \mathbb{R}, \qquad \|\mathcal{A}_\epsilon\|_{H^1_{\text{per}}} \leqslant M,$$

in which the constant M depends only on W and $\sup_\epsilon \|\vartheta_\epsilon\|_{(H^1_{\text{per}})^d}$. In particular, if periodic solutions of (1.11) converge weak* in L^∞_{per} to φ, then, after passing to a subsequence, we can assume that $(\mathcal{A}_\epsilon, \vartheta_\epsilon)$ converges weakly in $(H^1_{\text{per}})^{d+1}$ to some $(\mathcal{A}, \vartheta)$. *The main idea is to obtain a representation for weak* limits of solutions to* (1.11) *in terms of* $\mathcal{A}$ *and* ϑ. To this end we introduce the averaging operator

$$\Psi_\epsilon[\varphi_\epsilon](s) = \frac{1}{h(\epsilon)\sqrt{\epsilon}} \int_{-h(\epsilon)\sqrt{\epsilon}/2}^{h(\epsilon)\sqrt{\epsilon}/2} \varphi_\epsilon(s+t)\,dt \equiv \frac{1}{h(\epsilon)} \int_{-h(\epsilon)/2}^{h(\epsilon)/2} v_\epsilon(y, s)\,dy,$$

where $h(\epsilon) = \ln|\ln\epsilon|$ for $\epsilon \in (0, 1/e]$ and $v_\epsilon(y, s) = \varphi_\epsilon(s + \sqrt{\epsilon}y)$. By Lemma 3.7, the function v_ϵ can be approximated with an accuracy of ϵ^p, $p \in (0, 1/4)$, by a solution to the autonomous ordinary differential equation

$$\frac{1}{2}\big(B(u_\epsilon)'(y)\big)^2 - W\big(u_\epsilon(y, s), \vartheta_\epsilon(s)\big) + \mathcal{A}_\epsilon(s) = 0, \quad y \in \big(-h(\epsilon), h(\epsilon)\big).$$

This choice of h is more or less essential for the existence of such an approximation and leads to the representation

$$\Psi_\epsilon[\varphi_\epsilon](s) = \frac{1}{h(\epsilon)} \int_{-h(\epsilon)/2}^{h(\epsilon)/2} u_\epsilon(y, s)\,dy + O\big(\epsilon^p\big).$$

In Section 2.3 we show that $\Psi(s) = \lim_{\epsilon\to 0} \Psi_\epsilon[\varphi_\epsilon](s)$ exists for every $s \in \mathbb{R}$, and that the weak* limit is continuous on $[0, L)$ except at points of the critical set

$$s \in \mathcal{F}(\mathcal{A}, \vartheta) := \Big\{s \in [0, L)\colon\ \vartheta(s) \in G_3^0 \text{ and } \mathcal{A}(s) = \inf_\phi W\big(\phi, \vartheta(s)\big)\Big\}.$$

Since the weak* limit φ of φ_ϵ equals Ψ almost everywhere, we can identify φ with Ψ and thus replace the analysis of solutions to (1.11) with an analysis of the corresponding sequence $\Psi_\epsilon[\varphi_\epsilon]$. The advantage gained is that $\Psi_\epsilon[\varphi_\epsilon]$ converges pointwise to φ and the limiting behaviour of $\Psi_\epsilon[\varphi_\epsilon]$, $\epsilon \in E$, near $s \in \mathcal{F}(\mathcal{A}, \vartheta)$ can be characterized by what we call the *oscillation defect*. This oscillation defect, osc-def $E(s)$, is defined by (3.3) in terms of the family $\{\varphi_\epsilon\colon\ \epsilon \in E\}$ and not in terms of its weak* limit points. Nevertheless φ is continuous at s when osc-def $E(s) = 0$.

The relation between the oscillation defect and the limiting behaviour of the energy functional is established in Theorem 3.3 which is the first significant result of the paper. If a sequence of periodic solutions φ_ϵ, $\epsilon \in E$, of Eq. (1.11) converges weak* in L^∞_{per} to a function φ, it follows that

$$\sum_{s_i} \wp\big(\vartheta(s_i)\big) \leqslant \liminf_{E \ni \epsilon \to 0} \frac{\sqrt{\epsilon}}{2} \int_0^L \big(B\big(\varphi_\epsilon(s)\big)'\big)^2\, ds, \tag{1.13}$$

where the sum is taken over the set of points $s_i \in [0, L) \cap \mathcal{F}(\mathcal{A}, \vartheta)$ with non-zero oscillation defect. This theorem holds for solutions of the Euler–Lagrange equation, not only for minimizers of J_ϵ. However, when applied to a sequence of minimizers, Theorem 3.3 implies that if the limit on the right in (1.13) is finite, then φ is a piecewise regular minimizer and

$$\mathcal{W}(\varphi) \leqslant \liminf_{E \ni \epsilon \to 0} \frac{\sqrt{\epsilon}}{2} \int_0^L \big(B\big(\varphi_\epsilon(s)\big)'\big)^2\, ds.$$

The second significant result of the paper is Theorem 4.2 which can be regarded as the "inverse" of Theorem 3.3. In the simplest case, when $\Theta_0 = 0$ in (H4), if the primary variational problem has at least one piecewise regular minimizer φ, then Theorem 4.2 says that

$$\limsup_{\epsilon \to 0} \frac{\mathfrak{E}(\epsilon) - \mathfrak{E}(0)}{\sqrt{\epsilon}} \leqslant 2\mathcal{W}(\varphi), \quad \text{where } \mathfrak{E}(\epsilon) = J_\epsilon(\varphi_\epsilon).$$

Theorems 3.3 and 4.2 give estimates, from below and above, of the weighted number of jumps of a weak* limit of minimizers, in terms of the total energy $\mathfrak{E}(\epsilon) = J_\epsilon(\varphi_\epsilon)$ and the interfacial energy

$$\mathcal{B}(\varphi_\epsilon) = \frac{\epsilon}{2} \int_0^L \big(B\big(\varphi_\epsilon(s)\big)'\big)^2\, ds.$$

Notice that while $\mathfrak{E}(\epsilon)$ depends only on ϵ, the interfacial energy depends on the minimizer φ_ϵ, since (1.9) may have many solutions. The relation between total and interfacial energies is important in the general theory of singularly perturbed variational problems. For example, for minimizers of the scaled Ginzburg–Landau functional (1.5), the interfacial and bulk energy contributions to the total energy are approximately equal when ϵ is small. This is an example of the principle of equipartition of energy which plays a crucial role in the analysis of (1.5) and its multidimensional generalizations. However it is special, and for more general problems we need a different technique.

Our approach is based on Struwe's monotonicity method [14, Chapter II, Section 9]. Application of this method to problem (1.9), in the simplest case when $\Theta_1 = 0$ in (H4), leads to the inequality, see Lemma 5.4,

$$\mathfrak{B}(\epsilon) \leqslant \mathfrak{E}'(\epsilon) + \Lambda_1(\epsilon), \qquad \sqrt{\epsilon}\Lambda_1(\epsilon) \to 0 \quad \text{as } \epsilon \to 0, \tag{1.14}$$

in which $\mathfrak{B}(\epsilon) = \sup_{\varphi_\epsilon} \epsilon^{-1}\mathcal{B}(\varphi_\epsilon)$. In its turn, this lead to a criterion for estimating the number of jumps of a weak* limit of minimizers when $\Theta_0 = \Theta_1 = 0$ in (H4), namely, for $\delta > 0$,

$$\limsup_{k \to \infty} \sqrt{\epsilon_k}\,\mathfrak{E}'(\epsilon_k) \leqslant \mathcal{W}_{\min} + \delta, \quad \epsilon_k \to 0 \quad \Rightarrow \quad \mathcal{W}(\varphi) \leqslant \mathcal{W}_{\min} + \delta.$$

That this criterion is valid for sequences in a set which is Lebesgue dense at zero is the content of Lemma 5.5. Its proof, which depends on the energy estimate (1.14), might be regarded as the main insight in this paper.

2. Minimization problems

The *primary problem* is

$$\inf_{\varphi \in L^\infty_{\text{per}}} \int_0^L W\big(\varphi(s), \vartheta(s)\big)\, ds, \tag{2.1}$$

where $\vartheta \in (H^1_{\text{per}})^d$ is given and W satisfies the following hypotheses which are illustrated in Fig. 1.

(H1) $W \in C^3((0,\infty) \times \mathbb{R}^d)$,

$$W(\phi,\theta) \to \infty, \qquad \left|\partial_\phi W(\phi,\theta)\right| \to \infty \quad \text{as } \phi \to 0, \infty,$$

uniformly for θ in compact subsets of $\mathbb{R}^d$.

(H2) For every $\theta \in \mathbb{R}^d$, the function $W(\cdot,\theta)$ has no more than three, counting multiplicity, critical points each of which is one of the following types: a stable global minimizer $\phi_s(\theta)$ which is non-degenerate, an unstable local maximizer $\phi_u(\theta)$ which is non-degenerate, a metastable local minimizer $\phi_m(\theta)$ which is non-degenerate, or a critical inflection point $\phi_{um}(\theta)$. With this hypothesis we can write $\mathbb{R}^d$ as the union of three disjoint sets, G_1, G_2, G_3 as follows.

G_1 is the set of $\theta \in \mathbb{R}^d$ such that $W(\cdot,\theta)$ has only one critical point – the non-degenerate global minimizer $\phi_s(\theta)$;

G_2 is the set of $\theta \in \mathbb{R}^d$ such that $W(\cdot,\theta)$ has two critical points – the non-degenerate global minimizer $\phi_s(\theta)$ and a critical inflection point $\phi_{um}(\theta)$ with $W(\phi_s(\theta),\theta) < W(\phi_{um}(\theta),\theta)$.

G_3 is the set of $\theta \in \mathbb{R}^d$ such that $W(\cdot,\theta)$ has three critical points – a non-degenerate global minimizer $\phi_s(\theta)$, a non-degenerate local maximizer $\phi_u(\theta)$, and a non-degenerate local minimizer $\phi_m(\theta)$ with

$$W\big(\phi_s(\theta),\theta\big) \leqslant W\big(\phi_m(\theta),\theta\big) < W\big(\phi_u(\theta),\theta\big).$$

In its turn, G_3 contains the set G_3^0 of θ such that $W(\cdot,\theta)$ has two non-degenerate global minimizers, $\phi_s^-(\theta) < \phi_s^+(\theta)$, and one non-degenerate local maximizer $\phi_u(\theta)$. In other words, a metastable state has become stable and ϕ_m coincides with one of the points $\phi_s^\pm$.

Let $\mathrm{F} \subset \mathbb{R}^{d+1}$ be defined by

$$F = \big\{(A,\theta) \in \mathbb{R}^{d+1}\colon\ A = W\big(\phi_s^\pm(\theta),\theta\big) \text{ where } \theta \in G_3^0\big\}.$$

(H3) For θ in a compact set there exists $c > 0$ such that

$$\begin{aligned}
&\left|\phi_u(\theta) - \phi_s(\theta)\right| \geqslant c, \qquad \left|\phi_m(\theta) - \phi_s(\theta)\right| \geqslant c, \qquad \left|\phi_s^+(\theta) - \phi_s^-(\theta)\right| \geqslant c,\\
&W\big(\phi_u(\theta),\theta\big) - W\big(\phi_s(\theta),\theta\big) \geqslant c \quad \text{for } \theta \in G_3 \setminus G_3^0,\\
&W\big(\phi_u(\theta),\theta\big) - W\big(\phi_s^\pm(\theta),\theta\big) \geqslant c \quad \text{for } \theta \in G_3^0,\\
&W\big(\phi_{um}(\theta),\theta\big) - W\big(\phi_s(\theta),\theta\big) \geqslant c \quad \text{for } \theta \in G_2,\\
&\partial_\phi^2 W\big(\phi_s(\theta),\theta\big) \geqslant c \quad \text{for } \theta \in \mathbb{R} \setminus G_3^0,\\
&\partial_\phi^2 W\big(\phi_s^\pm(\theta),\theta\big) \geqslant c \quad \text{for } \theta \in G_3^0,\\
&\partial_\phi^2 W\big(\phi_u(\theta),\theta\big) < 0 \quad \text{for } \theta \in G_3,\\
&\partial_\phi^2 W\big(\phi_m(\theta),\theta\big) > 0 \quad \text{for } \theta \in G_3 \setminus G_3^0.
\end{aligned}$$

Remark 2.1. If W is a real-analytic function, then G_3^0 is a real-analytic variety. More generally, except in degenerate situations, possibly caused by symmetries, the set G_3^0 is typically the closure of a union of manifolds with dimensions $d-1$, or smaller. In particular, when $d = 1$, G_3^0 is often a discrete set of points.

To proceed we need some additional notation. Let $(\mathcal{A},\vartheta) \in (H^1_{\mathrm{per}})^{d+1}$ be given. With F, G_i, $i = 1,2,3$, and G_3^0 defined in terms of W by (H2), let

$$\begin{aligned}
&\mathcal{G}_i(\vartheta) = \big\{s \in \mathbb{R}\colon\ \vartheta(s) \in G_i\big\}, \quad i = 1,2,3,\\
&\mathcal{G}_3^0(\vartheta) = \big\{s \in \mathbb{R}\colon\ \vartheta(s) \in G_3^0\big\},\\
&\mathcal{F}(\mathcal{A},\vartheta) = \big\{s \in \mathbb{R}\colon\ \big(\mathcal{A}(s),\vartheta(s)\big) \in F\big\} \subset \mathcal{G}_3^0(\vartheta).
\end{aligned}$$

Since the prescribed functions ϑ are continuous, the sets $\mathcal{G}_1(\vartheta)$, $\mathcal{G}_3(\vartheta)$ are open and $\mathcal{G}_2(\vartheta)$, $\mathcal{G}_3^0(\vartheta)$ are closed; so the sets $\mathbb{R} \setminus \mathcal{G}_3^0(\vartheta)$ and $\mathcal{G}_3(\vartheta) \setminus \mathcal{G}_3^0(\vartheta)$ are open. Since $\mathcal{G}_3^0(\vartheta) = \{s\colon\ \vartheta(s) \in G_3^0\}$, it follows from Remark 2.1 that if ϑ is real-analytic, then $\mathcal{G}_3^0$ is a one-dimensional real-analytic variety. If it is not the whole space $\mathbb{R}$ it is a discrete set of points.

2.1. Minimizers of primary problem

For any $\vartheta \in (H^1_{\rm per})^d$, φ is a minimizer of the primary problem (2.1) if

$$\varphi(t) \in \begin{Bmatrix} \{\phi_s(\vartheta(t))\}, & t \in \mathbb{R} \setminus \mathcal{G}_3^0(\vartheta) \\ \{\phi_s^+(\vartheta(t)), \phi_s^-(\vartheta(t))\}, & t \in \mathcal{G}_3^0(\vartheta) \end{Bmatrix}$$

almost everywhere. Minimizers always exist but they may not be unique and often cannot be continuous. By (H3), the size, $\phi_s^+(\theta) - \phi_s^-(\theta)$, $\theta \in G_3^0$, of possible jumps of minimizers is bounded below by a positive constant. We are interested in minimizers that are piecewise regular in the following sense. Say that $\{S_n\}$ is a *discrete periodic* sequence if $S_{n+1} - S_n \in [a, b]$ for some $a, b > 0$ and all n, and $\{S_n\colon n \in \mathbb{Z}\} = \{S_n + L\colon n \in \mathbb{Z}\}$.

Piecewise regular minimizers. For $\vartheta \in (H^1_{\rm per})^d$, a minimizer $\varphi \in L^\infty_{\rm per}$ of the primary problem (2.1) is said to be *piecewise regular* if there exists a discrete periodic sequence $\{S_n\} \subset \mathcal{G}_3^0(\vartheta)$ with the following properties. For every n, the restriction $\varphi|_{(S_n, S_{n+1})} \in W^{1,2}(S_n, S_{n+1})$,

$$\lim_{s \to S_n \pm 0} \varphi(s) \in \bigl\{\phi_s^-\bigl(\vartheta(S_n)\bigr),\ \phi_s^+\bigl(\vartheta(S_n)\bigr)\bigr\}$$

and the function φ has a jump at S_n with magnitude $\phi_s^+(\vartheta(S_n)) - \phi_s^-(\vartheta(S_n))$.

For a piecewise regular minimizer the *actual number of jumps per period* is

$$\mathcal{N}(\varphi) := \operatorname{card} \mathcal{Q}(\varphi) < \infty \quad \text{where } \mathcal{Q}(\varphi) = [0, L) \cap \{S_n\colon n \in \mathbb{N}\}. \tag{2.2}$$

It will be convenient to estimate the number of jumps of a piecewise regular minimizer by assigning to each a number other than unity. Let $B : (0, \infty) \to \mathbb{R}$ be the smooth strictly increasing function in (1.10) and, for $\theta \in G_3^0$, let

$$\wp(\theta) = \frac{1}{\sqrt{2}} \int_{\phi_s^-(\theta)}^{\phi_s^+(\theta)} B'(\phi)\sqrt{W(\phi, \theta) - A}\, d\phi \quad \text{where } A = W\bigl(\phi_s^\pm(\theta), \theta\bigr). \tag{2.3a}$$

Then the *weighted number of jumps of φ per period* is

$$\mathcal{W}(\varphi) = \sum_{s \in \mathcal{Q}(\varphi)} \wp\bigl(\vartheta(s)\bigr) < \infty. \tag{2.3b}$$

A piecewise regular minimizer φ has a *minimal weighted number of jumps* if

$$\mathcal{W}(\varphi) = \inf_{\tilde\varphi} \mathcal{W}(\tilde\varphi) =: \mathcal{W}_{\min}, \tag{2.3c}$$

where the infimum is taken over all piecewise regular minimizers $\tilde\varphi$. Lemma 2.3 asserts that $\mathcal{W}_{\min}$ is attained.

Remark 2.2. Note that, for $\theta \in G_3^0$,

$$\wp(\theta) = \frac{1}{2} \int_0^T \bigl(B(u)'\bigr)^2\, dt$$

if u satisfies

$$A = W\bigl(u(t), \theta\bigr) - \frac{1}{2}\bigl(B(u)'\bigr)^2, \qquad u(0) = \phi_s^-(\theta), \qquad u(T) = \phi_s^+(\theta),$$

where $A = W(\phi_s^\pm(\theta), \theta)$. Note also that, for a given B, $\wp(\vartheta(s))$, $s \in \mathcal{G}_3^0(\vartheta)$, is bounded below by a positive constant $k(M)$, for any ϑ with $\|\vartheta\|_{(H^1_{\rm per})^d} \leqslant M$. To see this, suppose $(A, \theta) \in F$ and $\|\theta\| \leqslant c_L M$, where $\|\vartheta(s)\| \leqslant c_L \|\vartheta\|_{(H^1_{\rm per})^d} \leqslant c_L M$. Then, since $\{\phi^-(\theta), \phi^+(\theta)\colon \|\theta\| \leqslant c_L M\}$ is bounded and since $W \in C^3$, by Taylor's theorem,

$$\left| W(\phi, \theta) - A - \frac{1}{2}\partial_\phi^2 W\bigl(\phi^-(\theta), \theta\bigr)\bigl(\phi - \phi^-(\theta)\bigr)^2 \right| \leqslant C\bigl|\phi - \phi^-(\theta)\bigr|^3$$

for $\phi^-(\theta) \leqslant \phi \leqslant \phi^+(\theta)$, where C depends on M and W. Therefore there exists $\phi_1 > 0$, independent of θ with $\|\theta\| \leqslant c_L M$, such that $\phi^-(\theta) + \phi_1 \leqslant \phi^+(\theta)$ and

$$\sqrt{W(\phi,\theta) - A} \geqslant \frac{\sqrt{c}}{2}\big(\phi - \phi^-(\theta)\big)$$

for $\phi \in (\phi^-(\theta), \phi^-(\theta) + \phi_1)$, where c is as in (H3). Therefore

$$\int_{\phi^-(\theta)}^{\phi^+(\theta)} B'(\phi)\sqrt{W(\phi,\theta) - A}\, d\phi \geqslant \int_{\phi^-(\theta)}^{\phi^-(\theta)+\phi_1} B'(\phi)\sqrt{W(\phi,\theta) - A}\, d\phi \geqslant C,$$

where C is independent of θ with $\|\theta\| \leqslant c_L M$.

We now record an observation that will be useful later.

Lemma 2.3. *Suppose that* (H1–3) *hold and that there exists at least one piecewise regular minimizer. Then there exists a piecewise regular minimizer with a minimal weighted number of jumps. Let*

$$\mathcal{N}^* = \max_{\varphi_{\min}} \mathcal{N}(\varphi_{\min}),$$

where the maximum of the actual number of jumps is taken over all piecewise regular minimizers with minimal weighted number of jumps. Then there exists $\delta > 0$ such that $\mathcal{N}(\varphi) \leqslant \mathcal{N}^*$ if $\mathcal{W}(\varphi) \leqslant \mathcal{W}_{\min} + \delta$.

Proof. Let $\mathcal{W}(\varphi_k) \leqslant \mathcal{W}_{\min} + 1/k$ for a sequence of piecewise regular minimizers. Denote by $S_n^{(k)} \in \mathcal{G}_3^0(\vartheta)$ the points of discontinuities of φ_k. Since, by Remark 2.2, the quantities $\wp(\vartheta(S_n^{(k)}))$ are uniformly bounded away from 0, the numbers $\mathcal{N}(\varphi_k)$ of discontinuities of φ_k form a finite set of integers. Therefore, after passing to a subsequence, we can assume that $\mathcal{N}(\varphi_k) = \mathcal{N}$ for all k. Since the set $S_n^{(k)}$ is invariant with respect to the shift $s \to s + L$, we can assume that, for every n, the sequence $\{S_n^{(k)}\}$ converges as $k \to \infty$, to S_n say, where $\{S_n\}$ is invariant with respect to the shift and $S_n \in \mathcal{G}_3^0(\vartheta)$.

The piecewise regular minimizers φ_k are periodic and continuous at $s \in (S_{n-1}^{(k)}, S_n^{(k)})$ and hence coincide with one of $\phi_s(\vartheta(s))$, $\phi_s^{\pm}(\vartheta(s))$ there. However, from (H3), each of the critical points $\phi_s(\vartheta(s_0))$, $\phi_s^{\pm}(\vartheta(s_0))$ is non-degenerate, for each $s_0 \in (S_{n-1}^{(k)}, S_n^{(k)})$. Hence there is a unique continuous branch of critical points $\tilde{\varphi}(t)$ defined in a vicinity of s_0 by the system

$$\partial_\phi W\big(\tilde{\varphi}(t), \vartheta(t)\big) = 0, \qquad \tilde{\varphi}(s_0) = \varphi_k(s_0).$$

Therefore φ_k coincides with $\tilde{\varphi}$ in a neighbourhood of each point $s_0 \in (S_{n-1}^{(k)}, S_n^{(k)})$. Moreover the non-degeneracy condition (H3) implies that

$$\big|\varphi_k'(s_0)\big| = \big|\tilde{\varphi}'(s_0)\big| = \left|\frac{\partial^2_{\phi\theta} W(\varphi_k(s_0), \vartheta(s_0)) \cdot \vartheta'(s_0)}{\partial^2_\phi W(\varphi_k(s_0), \vartheta(s_0))}\right| \leqslant c(M)\big\|\vartheta'(s_0)\big\|.$$

Thus, since ϑ' is square-integrable on $(0, L)$,

$$\sum_{n\in\mathbb{Z}} \left\{ \int_{(0,L)\cap(S_{n-1}^{(k)}, S_n^{(k)})} \varphi_k'(s)^2\, ds \right\} =: \int_0^L \big(\varphi_k'\big)^2\, ds \leqslant c.$$

We can therefore assume that φ_k converges uniformly to some continuous function φ on each compact subset of the interval (S_{n-1}, S_n) and that

$$\sum_{n\in\mathbb{Z}} \left\{ \int_{(0,L)\cap(S_{n-1}, S_n)} \varphi'(s)^2\, ds \right\} =: \int_0^L \big(\varphi'\big)^2\, ds \leqslant c,$$

where φ_k converges weak* in $L^\infty_{\rm per}$ to φ. Since the set of relaxed minimizers (Section 2.2) is weakly closed, φ is a relaxed minimizer. Hence it is a piecewise regular minimizer of (2.1).

Now choose $\eta > 0$ so that the points η and $L+\eta$ do not lie in the sequence $\{S_n\}$. Then

$$\mathcal{W}(\varphi) = \sum_{S_n \in (\eta, L+\eta)} \wp\big(\vartheta(S_n)\big),$$

and for all sufficiently large k,

$$\mathcal{W}(\varphi_k) = \sum_{S_n^{(k)} \in (\eta, L+\eta)} \wp\big(\vartheta\big(S_n^{(k)}\big)\big).$$

In the limiting process some points $S_n^{(k)}$ can be lost: more precisely, for each S_n there is a finite set $P(n)$ of cardinality $p(n) \geqslant 1$ so that

$$S_i^{(k)} \to S_n \quad \text{as } k \to \infty \text{ for all } i \in P(n).$$

In other words we can split the set of sequences $\{S_n^{(k)}\} \cap (\eta, L+\eta)$ into disjoint clusters of cardinality $p(n)$ such that all sequences in a cluster converge to one point S_n. Obviously

$$\sum_{S_n \in (\eta, L+\eta)} p(n) = \mathcal{N}.$$

Since, for $i \in P(n)$,

$$\wp\big(\vartheta\big(S_i^{(k)}\big)\big) \to \wp\big(\vartheta(S_n)\big) \quad \text{as } k \to \infty \text{ for all } i \in P(n),$$

we obtain that

$$\sum_{S_n \in (\eta, L+\eta)} p(n) \wp\big(\vartheta(S_n)\big) = \lim_{k \to \infty} \sum_{S_n^{(k)} \in (\eta, L+\eta)} \wp\big(\vartheta\big(S_n^{(k)}\big)\big) = \mathcal{W}_{\min} \leqslant \sum_{S_n \in (\eta, L+\eta)} \wp\big(\vartheta(S_n)\big).$$

We conclude that φ is a piecewise regular minimizer with a minimal weighted number of jumps, that $p(n) = 1$ for all n, and that $\mathcal{N}^*$ is finite.

Suppose that for no $\delta > 0$ does $\mathcal{W}(\varphi) \leqslant \mathcal{W}_{\min} + \delta$ imply that $\mathcal{N}(\varphi) \leqslant \mathcal{N}^*$. Then there is a sequence of piecewise regular minimizers $\{\varphi_k\}$ with

$$\mathcal{W}(\varphi_k) \leqslant \mathcal{W}_{\min} + 1/k, \quad \text{and} \quad \mathcal{N}(\varphi_k) \geqslant \mathcal{N}^* + 1.$$

Now we repeat the preceding argument to obtain, in the weak* limit of φ_k, a piecewise regular minimizer φ with minimal weighted number of jumps and $p(n) = 1$ for all n. Since $p(n) = 1$ for all n, $\mathcal{N}(\varphi) \geqslant \mathcal{N}^* + 1$. But $\mathcal{N}(\varphi) \leqslant \mathcal{N}^*$, because φ is a piecewise regular minimizer with minimal weighted number of jumps. This contradiction proves the result. □

Generalized solutions. For given $\vartheta \in (H^1_{\rm per})^d$, a pair $(\mathcal{A}, \varphi) \in H^1_{\rm per} \times L^\infty_{\rm per}$ is called a *generalized solution* of

$$\partial_\phi W(\varphi, \vartheta) = 0 \tag{2.4}$$

if

(GS)(i): for all $t \in \mathbb{R} \setminus \mathcal{G}_3^0(\vartheta)$,

$$\varphi(t) \in \big\{\phi_\iota\big(\vartheta(t)\big),\ \iota = s, m, u, um\big\} \quad \text{and} \quad \mathcal{A}(t) = W\big(\varphi(t), \vartheta(t)\big);$$

(GS)(ii): for all $t \in \mathcal{G}_3^0(\vartheta)$,

$$\mathcal{A}(t) \in \big\{W\big(\phi_s^\pm\big(\vartheta(t)\big), \vartheta(t)\big), W\big(\phi_u\big(\vartheta(t)\big), \vartheta(t)\big)\big\};$$

(GS)(iii):

$$\varphi(t)=\phi_u\big(\vartheta(t)\big)\quad \text{for } t\in\mathcal{G}_3^0(\vartheta)\setminus\mathcal{F}(\mathcal{A},\vartheta),$$
$$\varphi(t)=k(t)\phi_s^-\big(\vartheta(t)\big)+\big(1-k(t)\big)\phi_s^+\big(\vartheta(t)\big)\quad \text{for } t\in\mathcal{F}(\mathcal{A},\vartheta),$$

where $k:\mathcal{F}(\mathcal{A},\vartheta)\mapsto[0,1]$ is measurable.

The continuity of $\mathcal{A}$ restricts the behaviour of φ as follows.

Lemma 2.4. *For $\vartheta\in(H^1_{\mathrm{per}})^d$, let $(\mathcal{A},\varphi)$ be a generalized solution of* (2.4). *Then φ is continuous on $\mathbb{R}\setminus\mathcal{F}(\mathcal{A},\vartheta)$.*

Proof. To simplify notation, let $\varphi_\iota(t)=\phi_\iota(\vartheta(t))$, $\iota\in\{s,u,m,um\}$, $t\in\mathbb{R}$. Let $s_0\in\mathcal{G}_1(\vartheta)$, which is open since G_1 is open and ϑ is continuous. Then $\varphi_s(t)$ is the unique critical point of $W(\cdot,\vartheta(t))$, for t in a neighbourhood of s_0. Hence, by (GS)(i), $\varphi(t)=\varphi_s(t)$ in this neighbourhood of s_0. However, by the implicit function theorem, $\varphi_s(t)$ is a continuous function of t in a neighbourhood of s_0, since $\varphi_s(s_0)$ is a non-degenerate critical point of $W(\cdot,\vartheta(s_0))$. Thus φ is continuous on $\mathcal{G}_1(\vartheta)$.

Let $s_0\in\mathcal{G}_3(\vartheta)\setminus\mathcal{G}_3^0(\vartheta)$, an open set. It follows from (GS)(i) that, for t in a neighbourhood of s_0, $\varphi(t)\in\{\varphi_\iota(t)\colon \iota=s,u,m\}$ and $\mathcal{A}(t)=W(\varphi(t),\vartheta(t))$. Since, by (H3),

$$\partial_\phi W\big(\varphi_\iota(s_0),\vartheta(s_0)\big)=0\quad\text{and}\quad \partial_\phi^2 W\big(\varphi_\iota(s_0),\vartheta(s_0)\big)\neq 0,$$

if follows from the implicit function theorem that φ_ι depends continuously on t in a neighbourhood of s_0. Since $s_0\in\mathcal{G}_3(\vartheta)\setminus\mathcal{G}_3^0(\vartheta)$, there exists $\epsilon>0$ such that, for all $|t-s_0|$ sufficiently small,

$$W\big(\varphi_s(t),\vartheta(t)\big)+\epsilon\leqslant W\big(\varphi_m(t),\vartheta(t)\big)\leqslant W\big(\varphi_u(t),\vartheta(t)\big)-\epsilon.$$

Since $\varphi(t)\in\{\varphi_\iota(t)\colon \iota=s,u,m\}$ and $\mathcal{A}(t)=W(\varphi(t),\vartheta(t))$, and since $\mathcal{A}$ is continuous, it follows that $\varphi(t)=\varphi_\iota(t)$, t in a neighbourhood of s_0, for *one choice of* ι. Thus φ is continuous at s_0 for any $s_0\in\mathcal{G}_3(\vartheta)\setminus\mathcal{G}_3^0(\vartheta)$.

Let $s_0\in\mathcal{G}_3^0(\vartheta)\setminus\mathcal{F}(\mathcal{A},\vartheta)$. Since, by (H3),

$$\partial_\phi W\big(\varphi_u(s_0),\vartheta(s_0)\big)=0\quad\text{and}\quad \partial_\phi^2 W\big(\varphi_u(s_0),\vartheta(s_0)\big)\neq 0,$$

by the implicit function theorem, $\varphi_u(t)$ depends continuously on t in a neighbourhood of s_0. Also, by (H3),

$$\partial_\phi W\big(\varphi_s^\pm(s_0),\vartheta(s_0)\big)=0\quad\text{and}\quad \partial_\phi^2 W\big(\varphi_s^\pm(s_0),\vartheta(s_0)\big)\neq 0.$$

It follows that there are continuous functions $\tilde\varphi^\pm$, defined for t in a neighbourhood of s_0 in the open set $\mathcal{G}_3(\vartheta)$ with, for some $\epsilon>0$,

$$\tilde\varphi^\pm(s_0)=\varphi_s^\pm(s_0),\qquad \partial_\phi W\big(\tilde\varphi^\pm(t),\vartheta(t)\big)=0,$$
$$\tilde\varphi^-(t)+\epsilon\leqslant\varphi_u(t)\leqslant\tilde\varphi^+(t)-\epsilon,\qquad W\big(\tilde\varphi^\pm(t),\vartheta(t)\big)+\epsilon\leqslant W\big(\varphi_u(t),\vartheta(t)\big).$$

Therefore, for t in this neighbourhood of s_0 in $\mathcal{G}_3(\vartheta)$, $\{\varphi_u(t),\tilde\varphi^\pm(t)\}$ are the three critical points of $W(\cdot,\vartheta(t))$ and, for all such t,

$$\mathcal{A}(t)\in\big\{W\big(\varphi_u(t),\vartheta(t)\big),W\big(\tilde\varphi^\pm(t),\vartheta(t)\big)\big\}.$$

However, by (GS)(iii),

$$\varphi(s_0)=\varphi_u(s_0),\qquad \mathcal{A}(s_0)=W\big(\varphi_u(s_0),\vartheta(s_0)\big),$$

where $\mathcal{A}$ is continuous at s_0. It follows that $\varphi(t)=\varphi_u(t)$ in a neighbourhood of s_0 and the continuity of φ at a point of $\mathcal{G}_3^0(\vartheta)\setminus\mathcal{F}(\mathcal{A},\vartheta)$ follows.

Next assume that $s_0\in\mathcal{G}_2(\vartheta)$. Then the function $W(\cdot,\vartheta(s_0))$ has exactly two critical points $\varphi_s(s_0)$ and $\varphi_{um}(s_0)$ with

$$W\big(\varphi_s(s_0),\vartheta(s_0)\big)<W\big(\varphi_{um}(s_0),\vartheta(s_0)\big).\tag{2.5}$$

Moreover $\mathcal{A}$ is continuous and

$$\mathcal{A}(s_0)\in\big\{W\big(\varphi_{um}(s_0),\vartheta(s_0)\big),W\big(\varphi_s(s_0),\vartheta(s_0)\big)\big\}.$$

Let $t_n \to s_0$. From (H3), $t_n \in \mathbb{R} \setminus \mathcal{G}_3^0(\vartheta)$ for all n sufficiently large; from (GS)(i),

$$\varphi(t_n) \in \{\varphi_s(t_n), \varphi_m(t_n), \varphi_u(t_n), \varphi_{um}(t_n)\};$$

and $\{\varphi(t_n)\}$ lies in a compact subset of $(0,\infty)$, by (H1). Suppose that, for some subsequence, $\varphi(t_{n_k})$ converges to φ^*. From the continuity of $\partial_\phi W$ and ϑ we conclude that φ^* is a critical point of $W(\cdot, \vartheta(s_0))$. Hence $\varphi^* \in \{\varphi_{um}(s_0), \varphi_s(s_0)\}$. Since $\mathcal{A}$ is continuous,

$$\mathcal{A}(s_0) = \lim_k \mathcal{A}(t_{n_k}) = \lim_n W\big(\varphi(t_{n_k}), \vartheta(t_{n_k})\big),$$

and since (2.5) holds, we conclude that the full sequence $\{\varphi(t_n)\}$ converges to a limit which belongs to $\{\varphi_{um}(s_0), \varphi_s(s_0)\}$. By (GS)(i) and (2.5), the value of $\mathcal{A}(s_0)$ determines the value of $\varphi(s_0)$ and thereby ensures the continuity of φ at a point of $\mathcal{G}_2(\vartheta)$. This completes the proof. □

2.2. Relaxed minimizers

For $\theta \in \mathbb{R}^d$, let $W^{**}(\cdot,\theta)$ be the convex envelope of $W(\cdot,\theta)$, defined for $\phi \in (0,\infty)$ by $W^{**}(\phi,\theta) = \sup_{(a,b)\in C(\theta)} a\phi + b$, where

$$C(\theta) = \big\{(a,b) \in \mathbb{R}^2\colon\ a\phi + b \leqslant W(\phi,\theta) \text{ for all } \phi \in (0,\infty)\big\}.$$

For a given $\vartheta \in (H^1_{\mathrm{per}})^d$, $\varphi \in L^\infty_{\mathrm{per}}$ is a *relaxed minimizer* if

$$\int_0^L W^{**}(\varphi,\vartheta)\,ds = \inf_{\psi \in L^\infty_{\mathrm{per}}} \int_0^L W^{**}(\psi,\vartheta)\,ds \tag{2.6}$$

where (2.6) is the *relaxed minimization problem*.

Lemma 2.5. *For $\vartheta \in (H^1_{\mathrm{per}})^d$ let φ be a relaxed minimizer. Then*

$$\begin{aligned} &\varphi(t) = \phi_s\big(\vartheta(t)\big) \quad \textit{for } t \in \mathbb{R}\setminus\mathcal{G}_3^0(\vartheta),\\ &\varphi(t) = k(t)\phi_s^-\big(\vartheta(t)\big) + \big(1-k(t)\big)\phi_s^-\big(\vartheta(t)\big) \quad \textit{for } t \in \mathcal{G}_3^0(\vartheta), \end{aligned} \tag{2.7}$$

almost everywhere, where $k : \mathcal{G}_3^0(\vartheta) \mapsto [0,1]$ is measurable.

Let

$$\mathcal{A}_\vartheta(t) = \inf_{\phi\in\mathbb{R}} W^{**}\big(\phi,\vartheta(t)\big) = \begin{Bmatrix} W(\phi_s(\vartheta(t)),\vartheta(t)) & \textit{for } t \in \mathbb{R}\setminus\mathcal{G}_3^0(\vartheta) \\ W(\phi_s^{\pm}(\vartheta(t)),\vartheta(t)) & \textit{for } t \in \mathcal{G}_3^0(\vartheta) \end{Bmatrix}.$$

Then $\mathcal{A}_\vartheta \in H^1_{\mathrm{per}}$ and there is a constant M depending only on $\|\vartheta\|_{(H^1_{\mathrm{per}})^d}$ such that

$$M^{-1} \leqslant \varphi(t) \leqslant M \quad \textit{for } t \in \mathbb{R} \quad \textit{and} \quad \|\mathcal{A}_\vartheta\|_{H^1_{\mathrm{per}}} \leqslant M. \tag{2.8}$$

Also $\mathcal{F}(\mathcal{A}_\vartheta,\vartheta) = \mathcal{G}_3^0(\vartheta)$ and $(\mathcal{A}_\vartheta,\varphi)$ is a generalized solution of (2.4). *In particular, after being redefined on a set of zero measure, a relaxed minimizer φ is continuous on $\mathbb{R}\setminus\mathcal{G}_3^0(\vartheta)$ and $\mathcal{A}_\vartheta$ is continuous on $\mathbb{R}$.*

Proof. Let φ be a solution to (2.6). Since, for $s \in \mathbb{R}\setminus\mathcal{G}_3^0(\vartheta)$, the function $W^{**}(\cdot,\vartheta(s))$ has a unique global minimizer $\phi_s(\vartheta(s))$, it follows that $\phi_s(\vartheta(s))$ and $\varphi(s)$ must coincide almost everywhere on $\mathbb{R}\setminus\mathcal{G}_3^0(\vartheta)$. For $s \in \mathcal{G}_3^0(\vartheta)$, the function $W^{**}(\cdot,\vartheta(s))$ takes its global minimum at every point of the interval $[\phi_s^-(\vartheta(s)), \phi_s^+(\vartheta(s))]$. Hence $\varphi(s)$ is an arbitrary point of this segment. Since φ is measurable, the measurability of k in (2.7) follows. The first part of (2.8) is immediate from (H1).

Next we show that the function

$$\mathcal{A}_\vartheta(s) = \min_{\phi\in(0,\infty)} W^{**}\big(\phi,\vartheta(s)\big) = \min_{\phi\in(0,\infty)} W\big(\phi,\vartheta(s)\big)$$

belongs to H^1_{per} and satisfies the second inequality in (2.8). Now $\|\vartheta(s)\| \leqslant c_L\|\vartheta\|_{(H^1_{\mathrm{per}})^d}$ and, by (H1), there exists $C = C(\|\vartheta\|_{(H^1_{\mathrm{per}})^d}) > 0$, such that

$$\min_{\phi\in(0,\infty)} W(\phi,\theta) = \min_{\phi\in[C^{-1},C]} W(\phi,\theta)$$

for all $\theta \in \mathbb{R}^d$ with $\|\theta\| \leqslant c_L\|\vartheta\|_{(H^1_{\mathrm{per}})^d} \leqslant k$, say. Let $a(\theta) = \min_{\phi\in(0,\infty)} W(\phi,\theta)$ and let $K(M)$ be the Lipschitz constant of W on $[C^{-1},C]\times[-k,k]^d$. Then

$$\begin{aligned} W(\phi,\theta_1) &= W(\phi,\theta_2) + W(\phi,\theta_1) - W(\phi,\theta_2) \\ &\leqslant W(\phi,\theta_2) + K(M)\big(\|\theta_1-\theta_2\|\big). \end{aligned}$$

Hence

$$a(\theta_1) \leqslant a(\theta_2) + K(M)\|\theta_1-\theta_2\|.$$

Interchanging the indices 1 and 2, we find that a is Lipschitz continuous with Lipschitz constant $K(M)$ on the ball of radius c and centre 0 in $\mathbb{R}^d$. Since $\mathcal{A}_\vartheta(s) = a(\vartheta(s))$, it follows that $\mathcal{A}_\vartheta \in H^1_{\mathrm{per}}$ and the second part of (2.8) holds. From the definition of $\mathcal{A}_\vartheta$, $\mathcal{F}(\mathcal{A}_\vartheta,\vartheta) = \mathcal{G}^0_3(\vartheta)$, and it follows from the first part of the proof that $(\mathcal{A}_\vartheta,\varphi)$ is a generalized solution of (2.4). The continuity of φ on $\mathbb{R}\setminus\mathcal{G}^0_3(\vartheta)$ follows from Lemma 2.4. Finally $\mathcal{A}_\vartheta \in H^1_{\mathrm{per}}$ is continuous. □

The following criterion ensures that certain relaxed minimizers coincide almost everywhere with piecewise regular minimizers of (2.1).

Lemma 2.6. *For $\vartheta \in (H^1_{\mathrm{per}})^d$, let φ be a relaxed minimizer. Suppose that $\{a_n\}$ is a discrete periodic sequence with, for each point $s_0 \in \mathcal{G}^0_3(\vartheta)\cap(a_n,a_{n+1})$,*

$$\begin{aligned} &\textit{either}\quad \lim_{\beta\to 0}\operatorname*{ess\,sup}_{|s-s_0|\leqslant\beta}\big|\varphi(s)-\phi_s^-\big(\vartheta(s_0)\big)\big| = 0, \\ &\textit{or}\quad \lim_{\beta\to 0}\operatorname*{ess\,sup}_{|s-s_0|\leqslant\beta}\big|\varphi(s)-\phi_s^+\big(\vartheta(s_0)\big)\big| = 0. \end{aligned} \tag{2.9}$$

Then, after being redefined on a set of zero measure, φ is a piecewise regular minimizer of (2.1) *with jumps at $\{S_n\}\subset\{a_n\}$. Moreover, if $\|\vartheta\|_{(H^1_{\mathrm{per}})^d} \leqslant M$,*

$$\int_0^L \varphi'(s)^2\,ds := \sum_{n\in\mathbb{Z}}\left\{\int_{(0,L)\cap(S_{n-1},S_n)} \varphi'(s)^2\,ds\right\} \leqslant c(M) < \infty. \tag{2.10}$$

Proof. Let $\omega(t)$ be a standard mollifying kernel with $\operatorname{supp}\omega \subset [-1,1]$ and define the mollified function

$$\bar\varphi_\tau(s) = \frac{1}{\tau}\int_{\mathbb{R}} \omega\left(\frac{t-s}{\tau}\right)\varphi(t)\,dt, \quad \varphi \in L^\infty_{\mathrm{per}}.$$

By Lemma 2.5, the relaxed minimizer φ is continuous on $\mathbb{R}\setminus\mathcal{G}^0_3(\vartheta)$. Hence, by (2.9), for any $s \in (a_n,a_{n+1})$, $\lim_{\tau\to 0}\bar\varphi_\tau(s)$ exists and equals either $\phi_s(\vartheta(s))$ or $\phi_s^\pm(\vartheta(s))$. Hence the function φ can be redefined on a set of zero measure in such a way that it is everywhere the pointwise limit of a sequence of continuous functions on every interval (a_n,a_{n+1}). Once redefined, at every point of these intervals its value coincides with one of the stable critical point $\phi_s(\vartheta(s))$, $\phi_s^\pm(\vartheta(s))$ and

$$\lim_{\beta\to 0}\operatorname*{ess\,sup}_{|s-s_0|\leqslant\beta}\big|\varphi(s)-\varphi(s_0)\big| = 0 \quad \text{for any } s_0 \in (a_n,a_{n+1}). \tag{2.11}$$

In particular this function (also denoted by φ) is a minimizer of the primary problem (2.1). To show that φ is continuous on this interval it is sufficient to prove that the set of mollifying functions $\bar\varphi_\tau$ is equi-continuous on each subinterval $[\sigma,\gamma]\subset(a_n,a_{n+1})$, for $n\in\mathbb{Z}$. If this is false, then there exist $\delta>0$ and sequences $\{s_k\}$, $\{t_k\}\subset[\sigma,\gamma]$, $\{\tau_k\}$ so that

$$\big|\bar\varphi_{\tau_k}(s_k)-\bar\varphi_{\tau_k}(t_k)\big| \geqslant \delta, \qquad |s_k-t_k|\to 0 \quad\text{and}\quad \tau_k\to 0 \quad \text{as } n\to\infty. \tag{2.12}$$

Without loss of generality we can assume that $s_k, t_k \to t^* \in [\sigma, \gamma]$ as $n \to \infty$. It follows from (2.11) that there is $\beta > 0$ so that

$$\operatorname*{ess\,sup}_{|t-t^*|\leqslant\beta} \bigl|\varphi(t) - \varphi\bigl(t^*\bigr)\bigr| < \delta/3.$$

It is clear that for all sufficiently large n,

$$\bigl\{t\colon |t-t_k| \leqslant \tau_k\bigr\} \cup \bigl\{t\colon |t-s_k| \leqslant \tau_k\bigr\} \subset \bigl\{t\colon \bigl|t-t^*\bigr| < \beta\bigr\}.$$

Therefore

$$\begin{aligned}\bigl|\overline{\varphi}_{\tau_k}(s_k) - \overline{\varphi}_{\tau_k}(t_k)\bigr| &\leqslant \left|\frac{1}{\tau_k}\int_{\mathbb{R}} \omega\left(\frac{t-s_k}{\tau_k}\right)\bigl(\varphi(t) - \varphi\bigl(t^*\bigr)\bigr)\,dt\right| + \left|\frac{1}{\tau_k}\int_{\mathbb{R}} \omega\left(\frac{t-t_k}{\tau_k}\right)\bigl(\varphi(t) - \varphi\bigl(t^*\bigr)\bigr)\,dt\right| \\ &\leqslant \left|\frac{\delta}{3\tau_k}\int_{\mathbb{R}} \omega\left(\frac{t-s_k}{\tau_k}\right)dt\right| + \left|\frac{\delta}{3\tau_k}\int_{\mathbb{R}} \omega\left(\frac{t-t_k}{\tau_k}\right)dt\right| = \frac{2\delta}{3},\end{aligned}$$

which contradicts (2.12). Hence φ is continuous on all the interval (a_n, a_{n+1}).

Now from (H3), the critical points ϕ_s, $\phi_s^{\pm}$ are non-degenerate. Hence, for $s_0 \in (a_n, a_{n+1})$, $n \in \mathbb{N}$, a unique continuous branch of critical points $\tilde{\varphi}(t)$, for t in a neighbourhood of s_0, is defined by equations $\partial_\phi W(\tilde{\varphi}(t), \vartheta(t)) = 0$, $\tilde{\varphi}(s_0) = \varphi(s_0)$. Therefore φ coincides with $\tilde{\varphi}$ in neighbourhood of each point $s_0 \in (a_n, a_{n+1})$. Moreover the non-degeneracy condition (H3) implies

$$\bigl|\tilde{\varphi}'(s_0)\bigr| = \left|\frac{\partial^2_{\phi\theta} W(\varphi(s_0), \vartheta(s_0)) \cdot \vartheta'(s_0)}{\partial^2_{\phi} W(\varphi(s_0), \vartheta(s_0))}\right| \leqslant c(M)\bigl\|\vartheta'(s_0)\bigr\|.$$

With (H1), this gives (2.10) and the lemma follows. □

Remark 2.7. It is obvious from the proof that (2.10) holds for any piecewise regular minimizer φ.

2.3. Regularized problems

As in (1.10) let $B:\mathbb{R} \to \mathbb{R}$ be a smooth increasing function and, for a set $E \subset (0, 1]$ with a limit point at 0, let $\{\vartheta_\epsilon\colon \epsilon \in E\}$ be bounded in $(H^1_{\text{per}})^d$. Then a solution φ_ϵ to the *regularized* variational problem, for $\epsilon > 0$,

$$\int_0^L \left(\frac{\epsilon}{2}\bigl(B(\varphi_\epsilon)'\bigr)^2 + W(\varphi_\epsilon, \vartheta_\epsilon)\right) ds = \inf_{\varphi \in H^1_{\text{per}}} \int_0^L \left(\frac{\epsilon}{2}\bigl(B(\varphi)'\bigr)^2 + W(\varphi, \vartheta_\epsilon)\right) ds, \tag{2.13}$$

satisfies the Euler–Lagrange equation

$$\epsilon B'\bigl(\varphi_\epsilon(s)\bigr) B(\varphi_\epsilon)''(s) - \partial_\phi W\bigl(\varphi_\epsilon(s), \vartheta_\epsilon(s)\bigr) = 0 \quad \text{on } \mathbb{R}. \tag{2.14}$$

We assume that $\|\vartheta_\epsilon\|_{(H^1_{\text{per}})^d} \leqslant M_E$. Then, by the maximum principle and (H1), solutions to (2.14) satisfy $M^{-1} \leqslant \varphi_\epsilon(s) \leqslant M$ for $s \in \mathbb{R}$, where M depends only on M_E, B and W. Since the adiabatic variable

$$\mathcal{A}_\epsilon(s) := W\bigl(\varphi_\epsilon(s), \vartheta_\epsilon(s)\bigr) - \frac{\epsilon}{2}\bigl(B(\varphi_\epsilon)'(s)\bigr)^2 \tag{2.15}$$

satisfies the identity $\mathcal{A}_\epsilon' = \nabla_\theta W \cdot \vartheta'$, we also have the estimate

$$\|\mathcal{A}_\epsilon\|_{H^1_{\text{per}}} \leqslant c(M_E), \quad \epsilon \in E. \tag{2.16}$$

Hence $\{(\mathcal{A}_\epsilon, \vartheta_\epsilon)\colon \epsilon \in E\}$ is relatively sequentially compact in the weak topology of $(H^1_{\text{per}})^{d+1}$ and $\{\varphi_\epsilon\colon \epsilon \in E\}$ is relatively sequentially compact in the weak* topology of L^∞_{per}. The difficulty is that a weak* limit of a sequence of solutions φ_ϵ to problem (2.14) need not satisfy the limiting equation (2.4).

3. Oscillation defect, energy estimates and jumps

Suppose that as $E \ni \epsilon \to 0$,

$$(\mathcal{A}_\epsilon, \vartheta_\epsilon) \rightharpoonup (\mathcal{A}, \vartheta) \quad \text{in } \left(H^1_{\text{per}}\right)^{d+1} \quad \text{and} \quad \varphi_\epsilon \rightharpoonup^* \varphi \quad \text{in } L^\infty_{\text{per}}, \tag{3.1a}$$

where (2.14) holds and

$$\|\vartheta_\epsilon\|_{(H^1_{\text{per}})^d} + \|\vartheta\|_{(H^1_{\text{per}})^d} \leqslant M_E < \infty, \tag{3.1b}$$

$$M^{-1} \leqslant \varphi_\epsilon(s), \varphi(s) \leqslant M \quad \text{for } s \in \mathbb{R}, \tag{3.1c}$$

$$\|\mathcal{A}_\epsilon\|_{H^1_{\text{per}}} + \|\mathcal{A}\|_{H^1_{\text{per}}} \leqslant M. \tag{3.1d}$$

When solutions $\varphi_\epsilon \in L^\infty_{\text{per}}$ of (2.14) converge weak* as $E \ni \epsilon \to 0$, the functions φ_ϵ can oscillate at points of $\mathcal{F}(\mathcal{A}, \vartheta)$. To characterize the behaviour of φ_ϵ at these points we define an averaging operator as follows. For $\epsilon \in (0, 1/\mathrm{e}]$ and $\varphi \in L^\infty_{\text{per}}$, let

$$\Psi_\epsilon[\varphi](s) = \frac{1}{\sqrt{\epsilon}h(\epsilon)} \int\limits_{s-\sqrt{\epsilon}h(\epsilon)/2}^{s+\sqrt{\epsilon}h(\epsilon)/2} \varphi(t)\,dt, \quad h(\epsilon) = \ln|\ln\epsilon|, \tag{3.2}$$

and note that, if $\varphi, \psi \in L^\infty_{\text{per}}$, then

$$\int\limits_0^L \psi(s)\Psi_\epsilon[\varphi](s)\,ds = \int\limits_0^L \varphi(s)\Psi_\epsilon[\psi](s)\,ds.$$

Oscillation defect. When (3.1) holds and $s_0 \in \mathcal{G}^0_3(\vartheta)$, the *oscillation defect of E at s_0* is defined by

$$\text{osc-def}\, E(s_0) := \min\Big\{ \lim_{\beta\to 0} \liminf_{E\ni\epsilon\to 0} \sup_{|s-s_0|\leqslant\beta} \left|\Psi_\epsilon[\varphi_\epsilon](s) - \phi^-_s(s_0)\right|,$$
$$\lim_{\beta\to 0} \liminf_{E\ni\epsilon\to 0} \sup_{|s-s_0|\leqslant\beta} \left|\Psi_\epsilon[\varphi_\epsilon](s) - \phi^+_s(s_0)\right| \Big\}, \tag{3.3}$$

where $\phi^\pm_s(s_0) = \phi^\pm_s(\vartheta(s_0))$.

The oscillation defect is defined in terms of the family $\{\varphi_\epsilon\colon \epsilon \in E\}$, and not in terms of its weak* limit points. Nevertheless we have the following observation.

Lemma 3.1. *If* (3.1) *holds and* osc-def $E(s_0) = d$*,* $s_0 \in \mathcal{G}^0_3(\vartheta)$*. Then either*

$$\lim_{\beta\to 0} \left\|\varphi - \phi^-_s(s_0)\right\|_{L^\infty(s_0-\beta, s_0+\beta)} \leqslant d \quad \textit{or} \quad \lim_{\beta\to 0} \left\|\varphi - \phi^+_s(s_0)\right\|_{L^\infty(s_0-\beta, s_0+\beta)} \leqslant d.$$

In particular, if osc-def $E(s_0) = 0$*, then φ is essentially continuous at s_0.*

Proof. This depends on the general observation that if $u_k \rightharpoonup^* u$ in any dual space X^*, then $\|u\|_{X^*} \leqslant \liminf_k \|u_k\|_{X^*}$. Suppose that $\varphi_{\epsilon_k} \rightharpoonup^* \varphi$. It is clear that for any continuous L-periodic function f,

$$\int\limits_0^L f(s)\varphi(s)\,ds = \lim_{k\to\infty} \int\limits_0^L \Psi_{\epsilon_k}[f](s)\varphi_{\epsilon_k}(s)\,ds = \lim_{k\to\infty} \int\limits_0^L f(s)\Psi_{\epsilon_k}[\varphi_{\epsilon_k}](s)\,ds.$$

Hence the sequence $\Psi_{\epsilon_k}[\varphi_{\epsilon_k}]$ is weak* convergent in $L^\infty(s_0 - \beta, s_0 + \beta)$ to φ and

$$\left\|\varphi - \phi^\pm_s(s_0)\right\|_{L^\infty(s_0-\beta, s_0+\beta)} \leqslant \liminf_k \left\|\Psi_{\epsilon_k}[\varphi_{\epsilon_k}] - \phi^\pm_s(s_0)\right\|_{L^\infty(s_0-\beta, s_0+\beta)}.$$

Since osc-def $E(s_0) = d$, the result follows. □

Combining this observation with Lemma 2.6 we obtain the following.

Corollary 3.2. *Suppose that φ in* (3.1) *is a relaxed minimizer of the limiting problem and* $\{a_n\colon\ n \in \mathbb{Z}\}$ *is a discrete periodic sequence for which* osc-def $E(s) = 0$ *for all* $s \in \mathcal{G}_3^0(\vartheta) \cap (a_n, a_n + 1)$, $n \in \mathbb{Z}$. *Then φ is a piecewise regular minimizer of* (2.1) *with jump set* $\{S_n\} \subset \{a_n\}$.

We now establish a connection between the oscillation defect of a family of solutions of (2.14) (not necessarily minimizers), the asymptotic behaviour of the energy functional for small ϵ, and, from Lemma 3.1, the possible jumps of a weak* limiting function.

Theorem 3.3. *If* (H1–3) *and* (3.1) *hold,*

$$\liminf_{E\ni\epsilon\to 0} \frac{\sqrt{\epsilon}}{2} \int_0^L \big(B\big(\varphi_\epsilon(s)\big)'\big)^2\, ds \geqslant \sum_{s\in\mathcal{O}(E)} \wp\big(\vartheta(s)\big), \tag{3.4}$$

where the singular set $\mathcal{O}(E)$ is given by

$$\mathcal{O}(E) = [0, L) \cap \big\{s \in \mathcal{F}(\mathcal{A}, \vartheta)\colon\ \text{osc-def}\, E(s) > 0\big\}. \tag{3.5}$$

If $\mathcal{O}(E)$ is infinite, then both sides of (3.4) *are infinite.*

Proof. The proof is divided into a number of steps.

Step 1: Equations with constant coefficients. Fix $(A_0, \theta_0) \in F$. The first two lemmas concern solutions to the equation

$$W\big(u(y), \theta\big) - \frac{1}{2}\big(B(u)'(y)\big)^2 = A \tag{3.6}$$

in which the *constant* parameters $(A, \theta) \in \mathbb{R}^{d+1}$ satisfy

$$\|\theta - \theta_0\| + |A - A_0| < \rho \tag{3.7}$$

for some small ρ. Let

$$0 < \eta < \phi_s^+(\theta_0) - \phi_s^-(\theta_0). \tag{3.8}$$

Lemma 3.4. *Suppose that* (3.7) *and* (3.8) *hold,* $(A_0, \theta_0) \in F$ *and u is a solution of Eq.* (3.6) *on an interval I where*

$$M^{-1} \leqslant u(y) \leqslant M, \tag{3.9}$$

$$\big|u(y) - \phi_s^\pm(\theta_0)\big| \geqslant \eta, \quad u \text{ is monotone}. \tag{3.10}$$

Then there are positive constants ρ_0 and C, depending only on θ_0 and η, such that, for all $\rho < \rho_0$ the length of I is bounded by C.

Proof. If $u = u_c$, a constant, then $A = W(u_c, \theta)$. If $\rho > 0$ is sufficiently small this is impossible because of (3.7) and (3.10). Hence, without loss of generality, we can assume that u is increasing on I and

$$dy = \frac{B'(u)\, du}{\sqrt{2(W(u,\theta) - A)}}.$$

Recall that the function $W(\cdot, \theta_0) - A_0$ is continuous on the interval $[M^{-1}, M]$ and has only two zeros $\phi_s^\pm(\theta_0)$. Therefore, by (3.7) and (3.10), there exist $\rho_0 > 0$ and $c(\eta, \theta_0) > 0$ such that for $\rho < \rho_0$,

$$W(u, \theta) - A > c(\eta, \theta_0)$$

when $u \in [M^{-1}, M]$ and $|u - \phi_s^\pm(\theta_0)| \geqslant \eta$. Thus, for $\rho < \rho_0$,

$$\operatorname{meas} I \leqslant \int_{M^{-1}}^{\phi_s^-(\theta_0)-\eta} \frac{B'(u)\,du}{\sqrt{2(W(u,\theta)-A)}} + \int_{\phi_s^-(\theta_0)+\eta}^{\phi_s^+(\theta_0)-\eta} \frac{B'(u)\,du}{\sqrt{2(W(u,\theta)-A)}} + \int_{\phi_s^+(\theta_0)+\eta}^{M} \frac{B'(u)\,du}{\sqrt{2(W(u,\theta)-A)}}$$
$$\leqslant \int_{M^{-1}}^{M} \frac{B'(u)\,du}{\sqrt{2c(\eta,\theta_0)}},$$

and the lemma follows. □

Next we extend this result to the case of non-monotone u, when θ and A are constants. Suppose $\theta_0 \in G_3^0$. For any function $u: J \mapsto \mathbb{R}$ and $\eta > 0$ let

$$B_\eta[u] = \{ y \in J: \ |u(y) - \phi_s^\pm(\theta_0)| \geqslant \eta \}. \tag{3.11}$$

Lemma 3.5. *For* $(A_0, \theta_0) \in F$*, suppose that u satisfies* (3.6) *on an interval J and* (3.7)–(3.9) *hold. Then, for any* $\delta > 0$ *there exist* $K_2, \varrho_0 > 0$*, depending only on* δ*,* η *and* θ_0*, such that if* $\operatorname{meas} J \geqslant K_2$ *and* $\rho < \rho_0$ *then* $\operatorname{meas} B_\eta[u] < \delta \operatorname{meas} J$.

Proof. Suppose this is false. Then there exists $\delta > 0$ and a sequence of solutions $u_n : J_n \to \mathbb{R}$ of (3.6) corresponding to (A_n, θ_n) which satisfy (3.7)–(3.9), with $(A_n, \theta_n) \to (A_0, \theta_0)$,

$$\operatorname{meas} J_n \to \infty \quad \text{as } n \to \infty \quad \text{and} \quad \frac{\operatorname{meas} B_\eta[u_n]}{\operatorname{meas} J_n} \geqslant \delta. \tag{3.12}$$

To contradict (3.12) we need only prove that

$$(\operatorname{meas} J_n)^{-1} \operatorname{meas} B_\eta[u_n] \to 0 \quad \text{as } \operatorname{meas} J_n \to \infty. \tag{3.13}$$

Let Π denote the set of $(A, \theta) \in \mathbb{R}^{d+1}$ such that the equation $W(\phi, \theta) = A$ has four simple roots $R_1 < R_2 < R_3 < R_4$. After passing to a subsequence, we can assume that either $(A_n, \theta_n) \in \Pi$ or $(A_n, \theta_n) \notin \Pi$.

First consider the case $(A_n, \theta_n) \in \Pi$. There are two possibilities.

The first is that u_n takes its values in interval $(-\infty, R_1(n))$ or in $(R_4(n), \infty)$. In both cases it is monotone and, by virtue of Lemma 3.4,

$$\operatorname{meas} B_\eta[u_n] / \operatorname{meas} J_n \leqslant C / \operatorname{meas} J_n,$$

which yields (3.13).

The second possibility is that u_n is a periodic function which oscillates between $R_2(n)$ and $R_3(n)$ with half-period

$$T_n = \int_{R_2(n)}^{R_3(n)} \frac{B'(\phi)\,d\phi}{\sqrt{2(W(\phi,\theta_n) - A_n)}}.$$

Without loss of generality we can assume that $u_n(0) = R_2(n)$, $u_n(T_n) = R_3(n)$ and u_n is monotone on each interval $[kT_n, (k+1)T_n]$. Let

$$J_n = (\alpha_n, \beta_n) = \bigcup_{i=k}^{m+k} [iT_n, (i+1)T_n] \cup (\alpha_n, kT_n] \cup [(k+m+1)T_n, \beta_n)$$

for integers $k, m \geqslant 0$, where

$$\alpha_n \in [(k-1)T_n, kT_n], \qquad \beta_n \in [(k+m+1)T_n, (k+m+2)T_n].$$

Then u_n is monotone on each of the intervals in this representation of J_n. Hence the intersection of $B_\eta[u_n]$ with these intervals consists of no more that three subintervals on each of which the function u_n is monotone. It follows from Lemma 3.4 that, for all sufficiently large n, the measure of each such subinterval is bounded by a constant depending only on η if $\rho < \rho_0$. Therefore

$$\operatorname{meas} B_\eta[u_n] \cap [\alpha_n, kT_n] + \operatorname{meas} B_\eta[u_n] \cap \big[(k+m+1)T_n, \beta_n\big] + \operatorname{meas} B_\eta[u_n] \cap \big[iT_n, (i+1)T_n\big] \leqslant c(\eta, \theta_0)$$

and

$$(\operatorname{meas} J_n)^{-1} \operatorname{meas} B_\eta[u_n] \leqslant \big(2(\operatorname{meas} J_n)^{-1} + T_n^{-1}\big) c(\eta, \rho_0).$$

To complete the analysis of this case, it suffices to show that $T_n \to \infty$ as $n \to \infty$. To this end note that the function $W(u,\theta)$ is uniformly continuous on the set

$$u \in \big[M^{-1}, M\big], \quad \|\theta - \theta_0\| < \rho,$$

and the equation $W(\phi, \theta_0) = A_0$ has exactly two distinct roots $\phi_s^\pm(\theta_0)$. Hence the limit points of subsequences of $\{R_j(n)\}$ lie in $\{\phi_s^-(\theta_0), \phi_s^+(\theta_0)\}$. We claim that

$$R_3(n) \to \phi_s^+(\theta_0) \quad \text{and} \quad R_2(n) \to \phi_s^-(\theta_0) \quad \text{as } n \to \infty. \tag{3.14}$$

There are only two possibilities: $R_3(n) \to \phi_s^+(\theta_0)$, or, for a subsequence indexed by k, $R_3(k) \to \phi_s^-(\theta_0)$. In the second case $R_j(k) \to \phi_s^-(\theta_0)$ for $j \leqslant 3$, which is impossible, since the roots $\phi_s^\pm(\theta_0)$ of the limiting equation $W(\phi, \theta_0) = A_0$ are non-degenerate, that is, $\partial_\phi^2 W(\phi_s^-(\theta_0), \theta_0) \neq 0$. Therefore $R_3(n) \to \phi_s^+(\theta_0)$ as $n \to \infty$. The same arguments show that $R_2(n) \to \phi_s^-(\theta_0)$. Since

$$\int_{\phi_s^-(\theta_0)}^{\phi_s^+(\theta_0)} \frac{B'(\phi)\, d\phi}{\sqrt{W(\phi, \theta_0) - A_0}} = \infty,$$

it follows that $T_n \to \infty$ as $n \to \infty$ which yields (3.13).

It remains to establish (3.13) when $(A_n, \theta_n) \notin \Pi$. Since the set $G_3 \subset \mathbb{R}^d$ is open, $\theta_n \in G_3$ for all sufficiently large n and we have the following possibilities.

The first is that u_n is a constant and coincides with one of the critical points of $W(\cdot, \theta_n)$. More precisely, $u_n \in \{\phi_\iota(\theta_n){:}\ \iota = s, m, u\}$ when $\theta_n \in G_3 \setminus G_3^0$, and $u_n \in \{\phi_s^\pm(\theta_n), \phi_u(\theta_n)\}$ when $\theta_n \in G_3^0$. Note that $u_n = \phi_u(\theta_n)$ is impossible since it implies that

$$\lim_n A_n = \lim_n W\big(\phi_u(\theta_n), \theta_n\big) = W\big(\phi_u(\theta_0), \theta_0\big) > W\big(\phi_s^\pm(\theta_0), \theta_0\big) = A_0$$

which is false, by (H3). Hence u_n coincides with one of the stable critical points and converges uniformly to $\phi_s^-(\theta_0)$ or $\phi_s^+(\theta_0)$. This means that the set $B_\eta[u_n]$ is empty for all large n which yields (3.13).

The only other possibility is that the functions u_n are not constants. Moreover, they are not periodic since $(A_n, \theta_n) \notin \Pi$. Hence each u_n is either monotone and bounded (a kink solution), or monotone and unbounded, or has exactly one critical point on each side of which it is monotone (a solitary wave). In all these cases, J_n consists of no more than two intervals on each of which u_n is monotone. It follows (from the definition of $B_\eta[u_n]$) that the intersection of $B_\eta[u_n]$ with J_n consists of no more than six subintervals. By Lemma 3.4, the measure of each such subinterval is less than $C(\eta, \theta_0)$, which yields (3.13). □

The last lemma in Step 1 gives a lower bound on the energy of solutions to (3.6).

Lemma 3.6. *If η satisfies* (3.8) *and $(A_n, \theta_n) \to (A_0, \theta_0) \in F$ as $n \to \infty$, let solutions u_n to Eq.* (3.6) *with (A, θ) replaced by (A_n, θ_n) be defined on intervals J_n and suppose that, for each n, there are points $y_n^\pm$ with*

$$u_n\big(y_n^-\big) = \phi_s^-(\theta_0) + \eta, \qquad u_n\big(y_n^+\big) = \phi_s^+(\theta_0) - \eta.$$

Then

$$\liminf_{n\to\infty} \frac{1}{2} \int_{J_n} \big(B(u_n)'\big)^2\, dy \geqslant \frac{1}{\sqrt{2}} \int_{\phi_s^-(\theta_0)+\eta}^{\phi_s^+(\theta_0)-\eta} B'(\phi) \sqrt{W(\phi, \theta_0) - A_0}\, d\phi.$$

Proof. First we show that for all sufficiently large n we can choose $y_n^{\pm}$ as end-points of an interval $I_n \subset J_n$ on which the function u_n is monotone. Recall that there are only two possibilities: $(A_n, \theta_n) \in \Pi$ and u_n is periodic, or u_n has at most one critical point on J_n.

In the first case, u_n oscillates between the roots $R_2(n)$ and $R_3(n)$ of the equation $W(\phi, \theta_n) = A_n$, and is strictly monotone between successive minima and maxima. In the second, u_n has only one critical point which is either an absolute maximum or minimum. The supremum and infimum of u_n are then $R_3(n)$ and $R_2(n)$, respectively. It follows from (3.14) (which holds, by the same argument, in the present situation) that, for all n sufficiently large,

$$\left|R_2(n) - \phi_s^-(\theta_0)\right| < \eta/2 \quad \text{and} \quad \left|R_3(n) - \phi_s^+(\theta_0)\right| < \eta/2,$$

and the existence of the interval I_n is then obvious.

Choosing $y_n^{\pm} \in I_n$ and noting that $W(u_n(y), \theta_n) - A_n \geqslant 0$ for $y \in I_n$, we obtain

$$\begin{aligned}
\frac{1}{2}\int_{J_n} \big(B(u_n)'\big)^2\,dy \geqslant \frac{1}{2}\int_{y_n^-}^{y_n^+} \big(B(u_n)'\big)^2\,dy &= \int_{y_n^-}^{y_n^+} \big(W(u_n, \theta_n) - A_n\big)\,dy \\
&= \frac{1}{\sqrt{2}} \int_{\phi_s^-(\theta_0)+\eta}^{\phi_s^+(\theta_0)-\eta} B'(\phi)\sqrt{W(\phi, \theta_n) - A_n}\,d\phi \\
&\to \frac{1}{\sqrt{2}} \int_{\phi_s^-(\theta_0)+\eta}^{\phi_s^+(\theta_0)-\eta} B'(\phi)\sqrt{W(\phi, \theta_0) - A_0}\,d\phi \quad \text{as } n \to \infty,
\end{aligned}$$

and the lemma follows. □

Step 2: Equations with variable coefficients. For a solution φ_ϵ to (2.14) and $s \in \mathbb{R}$, let

$$v_\epsilon(y, s) = \varphi_\epsilon(s + \sqrt{\epsilon}y), \qquad \widehat{\vartheta}_\epsilon(y, s) = \vartheta_\epsilon(s + \sqrt{\epsilon}y). \tag{3.15}$$

Then

$$B'(v_\epsilon)B(v_\epsilon)'' - \partial_\phi W(v_\epsilon, \widehat{\vartheta}_\epsilon) = 0,$$

in which $'$ denotes differentiation with respect to y. Denote by $u_\epsilon(y, s)$ the solution to the Cauchy problem

$$\begin{aligned}
&B'\big(u_\epsilon(y)\big)B(u_\epsilon)''(y) - \partial_\phi W\big(u_\epsilon(y), \vartheta_\epsilon(s)\big) = 0, \\
&u_\epsilon(0, s) = v_\epsilon(0, s), \qquad \partial_y u_\epsilon(0, s) = \partial_y v_\epsilon(0, s).
\end{aligned} \tag{3.16}$$

From (3.16), u_ϵ satisfies the autonomous equation

$$-\frac{1}{2}\big(B\big(u_\epsilon(y)\big)'\big)^2 + W\big(u_\epsilon(y), \vartheta_\epsilon(s)\big) = \mathcal{A}_\epsilon(s), \tag{3.17}$$

where $\mathcal{A}_\epsilon(s)$ is given by (2.15). Let $h(\epsilon) = \ln|\ln\epsilon| > 0$, $\epsilon \in (0, 1/\mathrm{e}]$.

Lemma 3.7. *For any compact set $K \subset \mathbb{R}$ and $p \in (0, 1/4)$, there is $\epsilon_0 > 0$, depending only on K and p, so that, for all $s \in K$ and $\epsilon \in (0, \epsilon_0)$, the function $u_\epsilon(y, s)$ is defined on the interval $|y| \leqslant h(\epsilon)/2$ where $|u_\epsilon(y, s) - v_\epsilon(y, s)| \leqslant \epsilon^p$.*

Proof. Let

$$V(\phi, \theta) = B'(\phi)^{-1}\partial_\phi W(\phi, \theta), \quad (\phi, \theta) \in (0, \infty) \times \mathbb{R}^d.$$

Since s is fixed, we can suppress it in the notation and write $u_\epsilon(y)$ instead of $u_\epsilon(y, s)$, and similarly for the other variables. Then

$$\begin{aligned}\big(B\big(u_\epsilon(y)\big)-B\big(v_\epsilon(y)\big)\big)'' &= V\big(u_\epsilon(y),\widehat{\vartheta}_\epsilon(0)\big)-V\big(v_\epsilon(y),\widehat{\vartheta}_\epsilon(y)\big)\\ &= \big(u_\epsilon(y)-v_\epsilon(y)\big)\int_0^1 \partial_\phi V\big(tu_\epsilon(y)+(1-t)v_\epsilon(y),\widehat{\vartheta}_\epsilon(0)\big)\,dt\\ &\quad + V\big(v_\epsilon(y),\widehat{\vartheta}_\epsilon(0)\big)-V\big(v_\epsilon(y),\widehat{\vartheta}_\epsilon(y)\big)\\ &= P(y)\big(B\big(u_\epsilon(y)\big)-B\big(v_\epsilon(y)\big)\big)+Q(y),\quad \text{say},\end{aligned}$$

where, for a constant $N_1 = N_1(M_E)$, independent of $s \in K$,

$$\big|P(y)\big| = \frac{u_\epsilon(y)-v_\epsilon(y)}{B(u_\epsilon(y))-B(v_\epsilon(y))}\int_0^1 \partial_\phi V\big(tu_\epsilon(y)+(1-t)v_\epsilon(y),\widehat{\vartheta}_\epsilon(0)\big)\,dt \leqslant N_1,$$

$$\big|Q(y)\big| \leqslant N_1\epsilon^{1/4}\sqrt{|y|},$$

when

$$\big|u_\epsilon(y)-v_\epsilon(y)\big| \leqslant \frac{1}{2}\min\big\{M^{-1},M\big\}. \tag{3.18}$$

Since $Z(y) := B(u_\epsilon(y)) - B(v_\epsilon(y))$ solves the Cauchy problem

$$Z''(y) = P(y)Z(y)+Q(y),\qquad Z(0)=Z'(0)=0,$$

it follows that $|Z(y)| \leqslant z(y)$ where

$$z''(y) = N_1\big(z(y)+\epsilon^{1/4}\sqrt{|y|}\big),\qquad z(0)=z'(0)=0.$$

It is easy to see that

$$z(y) = \epsilon^{1/4}N_1|y|^{5/2}Y(y),$$

where

$$Y(y) = 4\sum_{k=0}^{\infty}A_{2k}y^{2k}\quad\text{and}\quad A_{2k} = \frac{N_1^k 2^{2k}}{(4k+5)(4k+3)\cdots 1} \leqslant \frac{N_1^k}{15(2k)!}.$$

Thus

$$\big|Z(y)\big| \leqslant \frac{4N_1\epsilon^{1/4}}{15}|y|^{5/2}\exp\big(\sqrt{N_1}|y|\big).$$

Now suppose that (3.18) holds on an interval $I \subset (-h(\epsilon), h(\epsilon))$. Then for $y \in I$,

$$\big|Z(y)\big| \leqslant \frac{4N_1\epsilon^{1/4}}{15}\big(\ln|\ln\epsilon|\big)^{5/2}|\ln\epsilon|^{\sqrt{N_1}} \leqslant \epsilon^p,\quad p \in (0,1/4),$$

when $\epsilon < \epsilon_0$, where ϵ_0 depends only on M and p. If, also, $\epsilon_0^p \leqslant \frac{1}{2}\min\{M^{-1}, M\}$, then (3.18) holds on $(-h(\epsilon), h(\epsilon))$. This estimate of Z implies the required estimate of $|v_\epsilon(y) - u_\epsilon(y)|$, and the proof is complete. □

The next result characterizes the behaviour of $u_\epsilon(y,s)$ near a point s_0 with non-zero oscillation defect.

Lemma 3.8. *Under the hypotheses of Theorem* 3.3 *suppose, for some* $s_0 \in \mathcal{F}(\mathcal{A},\vartheta)$*, that* osc-def $E(s_0) > 0$. *Let* $\theta_0 = \vartheta(s_0)$ *and let*

$$\eta_0 = \frac{1}{4}\min\big\{\text{osc-def}\,E(s_0),\phi_s^+(\theta_0)-\phi_s^-(\theta_0)\big\} > 0. \tag{3.19}$$

For any $\eta \in (0,\eta_0)$ *and any sequence* $E \ni \epsilon_n \to 0$, *there is a subsequence* $\{\epsilon_{n_m}\}$ *and there are sequences* $\{s_m\}$, $\{y_m^\pm\}$ *with the following properties*:

$$\begin{aligned}&(\epsilon_{n_m}, s_m) \to (0, s_0)\quad \textit{as } m\to\infty,\qquad y_m^\pm \in \big[-h(\epsilon_{n_m})/2, h(\epsilon_{n_m})/2\big],\\ &u_{\epsilon_{n_m}}\big(y_m^-, s_m\big) = \phi_s^-(\theta_0)+\eta,\qquad u_{\epsilon_{n_m}}\big(y_m^+, s_m\big) = \phi_s^+(\theta_0)-\eta.\end{aligned} \tag{3.20}$$

Proof. Fix an arbitrary sequence $E \ni \epsilon_n \to 0$. Suppose the assertion of the theorem is false. Then there exist $\eta \in (0, \eta_0)$, $\epsilon_0 > 0$, and $b > 0$ so that for every $\epsilon_n \in (0, \epsilon_0)$ and $s \in (s_0 - b, s_0 + b)$, the function $u_{\epsilon_n}(\cdot, s)$ does not take at least one of the values $\phi_s^-(\theta_0) + \eta$ or $\phi_s^+(\theta_0) - \eta$ on $[-h(\epsilon_n)/2, h(\epsilon_n)/2]$. This means that for every $\epsilon_n \in (0, \epsilon_0)$ and $s \in (s_0 - b, s_0 + b)$, either

$$\left|u_{\epsilon_n}(y, s) - \phi_s^-(\theta_0)\right| \geqslant \eta \quad \text{for all } |y| \leqslant h(\epsilon_n)/2, \tag{3.21}$$

or

$$\left|u_{\epsilon_n}(y, s) - \phi_s^+(\theta_0)\right| \geqslant \eta \quad \text{for all } |y| \leqslant h(\epsilon_n)/2. \tag{3.22}$$

Consider first the case when u_{ϵ_n} satisfies (3.21). Now u_{ϵ_n} satisfies (3.17) on $J_n = [-h(\epsilon_n)/2, h(\epsilon_n)/2]$, and

$$J_n = B_\eta\big[u_{\epsilon_n}(\cdot, s)\big] \cup \big\{y \in J_n\colon \left|u_\epsilon(y, s) - \phi_s^+(\theta_0)\right| \leqslant \eta\big\}, \tag{3.23}$$

where the set B_η is defined by (3.11). By hypothesis,

$$\vartheta_{\epsilon_n}(s) \to \vartheta(s_0) = \theta_0, \qquad \mathcal{A}_{\epsilon_n}(s) \to \mathcal{A}(s_0) \quad \text{as } (\epsilon_n, s) \to (0, s_0),$$
$$\text{meas } J_n \to \infty \quad \text{as } \epsilon_n \to 0 \quad \text{and} \quad 1/(2M) \leqslant u_{\epsilon_n} \leqslant 2M \quad \text{by Lemma 3.7.}$$

Since, by hypothesis, $s_0 \in \mathcal{F}(\mathcal{A}, \vartheta)$, we can apply Lemma 3.5 to Eq. (3.17). Let ϵ_0 and b, depending only on η and $\vartheta(s_0)$, be such that, for all $\epsilon_n \in (0, \epsilon_0)$ and $s \in (s_0 - b, s_0 + b)$,

$$\left\|\vartheta_{\epsilon_n}(s) - \vartheta(s_0)\right\| + \left|\mathcal{A}_{\epsilon_n}(s) - \mathcal{A}(s_0)\right| < \rho_0, \qquad \text{meas } J_n = h(\epsilon_n) > K_2,$$

where ρ_0 and K_2 are given by Lemma 3.5 with $\delta = \eta/6M$. Thus

$$\frac{\text{meas } B_\eta[u_{\epsilon_n}(\cdot, s)]}{\text{meas } J_n} < \frac{\eta}{6M}$$

for all $\epsilon_n \in (0, \epsilon_0)$ and $s \in (s_0 - b, s_0 + b)$. Therefore, from (3.23), for all such s and ϵ_n,

$$\left|\left\{\frac{1}{h(\epsilon_n)} \int_{-h(\epsilon_n)/2}^{h(\epsilon_n)/2} u_{\epsilon_n}(y, s)\, dy\right\} - \phi_s^+(\theta_0)\right|$$
$$\leqslant \frac{1}{\text{meas } J_n} \int_{B_\eta[u_{\epsilon_n}(\cdot,s)]} \left|u_{\epsilon_n}(y, s) - \phi_s^+(\theta_0)\right| dy + \frac{1}{\text{meas } J_n} \int_{J_n \setminus B_\eta[u_{\epsilon_n}(\cdot,s)]} \eta\, dy$$
$$\leqslant 3M \frac{\text{meas } B_\eta[u_{\epsilon_n}(\cdot, s)]}{\text{meas } J_n} + \eta \leqslant \frac{\eta}{2} + \eta < 2\eta.$$

Applying Lemma 3.7 and, with $p \in (0, 1/4)$, choosing ϵ_0 sufficiently small we conclude that for all $s \in (s_0 - b, s_0 + b)$ and $\epsilon_n \in (0, \epsilon_0)$,

$$\left|\left\{\frac{1}{h(\epsilon_n)} \int_{-h(\epsilon_n)/2}^{h(\epsilon_n)/2} v_{\epsilon_n}(y, s)\, dy\right\} - \phi_s^+(\theta_0)\right| < 2\eta + \epsilon_n^p.$$

Now recall from (3.2) that

$$\Psi_{\epsilon_n}[\varphi_{\epsilon_n}](s) = \frac{1}{\sqrt{\epsilon_n} h(\epsilon_n)} \int_{s - \sqrt{\epsilon_n} h(\epsilon_n)/2}^{s + \sqrt{\epsilon_n} h(\epsilon_n)/2} \varphi_{\epsilon_n}(t)\, dt = \frac{1}{h(\epsilon_n)} \int_{-h(\epsilon_n)/2}^{h(\epsilon_n)/2} v_{\epsilon_n}(y, s)\, dy.$$

Thus, for all $s \in (s_0 - b, s_0 + b)$ and $\epsilon_n \in (0, \epsilon_0)$ for which (3.21) holds,

$$\left|\Psi_{\epsilon_n}[\varphi_{\epsilon_n}](s) - \phi_s^+(\theta_0)\right| < 2\eta + \epsilon_n^p.$$

Repeating the above arguments we conclude that if u_{ϵ_n} satisfies inequality (3.22), then for all $s \in (s_0 - b, s_0 + b)$ and $\epsilon_n \in (0, \epsilon_0)$,

$$\left|\Psi_{\epsilon_n}[\varphi_{\epsilon_n}](s) - \phi_s^-(\theta_0)\right| < 2\eta + \epsilon_n^p.$$

Recalling the definition (3.3) of the oscillation defect we conclude that

$$0 < \text{osc-def}\, E(s_0) \leqslant \liminf_{n\to\infty} \sup_{|s-s_0|<b} \left|\Psi_{\epsilon_n}[\varphi_{\epsilon_n}](s) - \phi_s^+(\theta_0)\right| \leqslant 2\eta.$$

But this contradicts $\eta < \eta_0$ in (3.19) and the lemma follows. □

Step 3: Proof of Theorem 3.3. To complete the proof, fix an arbitrary $s_0 \in \mathcal{F}(\mathcal{A}, \vartheta)$ with osc-def $E(s_0) > 0$ and let η_0 be defined by (3.19). Now let $\eta \in (0, \eta_0)$ and $\beta > 0$. For a sequence with $E \ni \epsilon_n \to 0$ as $n \to \infty$ and

$$\liminf_{E\ni\epsilon\to 0} \frac{1}{\sqrt{\epsilon}} \int_{s_0-\beta}^{s_0+\beta} \frac{\epsilon}{2}\big(B(\varphi_\epsilon)'\big)^2\, ds = \lim_{n\to\infty} \frac{1}{\sqrt{\epsilon_n}} \int_{s_0-\beta}^{s_0+\beta} \frac{\epsilon_n}{2}\big(B(\varphi_{\epsilon_n})'\big)^2\, ds, \tag{3.24}$$

apply Lemma 3.8 to extract a subsequence $\{\epsilon_{n_m}\} \subset \{\epsilon_n\}$ and sequences $\{s_m\}$, $\{y_m^{\pm}\}$ satisfying (3.20). For convenience let ϵ_{n_m} be denoted by ϵ_m. Then, since $(\epsilon_m, s_m) \to (0, s_0)$ we have, for sufficiently large m,

$$\lim_{m\to\infty} \frac{1}{\sqrt{\epsilon_m}} \int_{s_0-\beta}^{s_0+\beta} \frac{\epsilon_m}{2}\big(B(\varphi_{\epsilon_m})'\big)^2\, ds \geqslant \limsup_{m\to\infty} \frac{1}{\sqrt{\epsilon_m}} \int_{s_m-\sqrt{\epsilon_m}h(\epsilon_m)/2}^{s_m+\sqrt{\epsilon_m}h(\epsilon_m)/2} \frac{\epsilon_m}{2}\big(B(\varphi_{\epsilon_m})'\big)^2\, ds. \tag{3.25}$$

Since $(\epsilon_m/2)(B(\varphi_{\epsilon_m})'(s))^2 = W(\varphi_{\epsilon_m}(s), \vartheta_{\epsilon_m}(s)) - \mathcal{A}_{\epsilon_m}(s)$, we have that

$$\frac{1}{\sqrt{\epsilon_m}} \int_{s_m-\sqrt{\epsilon_m}h(\epsilon_m)/2}^{s_m+\sqrt{\epsilon_m}h(\epsilon_m)/2} \frac{\epsilon_m}{2}\big(B(\varphi_{\epsilon_m})'\big)^2\, ds = \frac{1}{\sqrt{\epsilon_m}} \int_{s_m-\sqrt{\epsilon_m}h(\epsilon_m)/2}^{s_m+\sqrt{\epsilon_m}h(\epsilon_m)/2} \big(W\big(\varphi_{\epsilon_m}(s), \vartheta_{\epsilon_m}(s)\big) - \mathcal{A}_{\epsilon_m}(s)\big)\, ds. \tag{3.26}$$

Moreover, since (3.1) holds, for a constant c depending on M_E and M,

$$\begin{aligned}
&\big|\big(W\big(\varphi_{\epsilon_m}(s), \vartheta_{\epsilon_m}(s)\big) - \mathcal{A}_{\epsilon_m}(s)\big) - \big(W\big(\varphi_{\epsilon_m}(s), \vartheta_{\epsilon_m}(s_m)\big) - \mathcal{A}_{\epsilon_m}(s_m)\big)\big| \\
&\quad \leqslant c\big(\big\|\vartheta_{\epsilon_m}(s) - \vartheta_{\epsilon_m}(s_m)\big\| + \big|\mathcal{A}_{\epsilon_m}(s) - \mathcal{A}_{\epsilon_m}(s_m)\big|\big) \leqslant c|s - s_m|^{1/2}.
\end{aligned}$$

Combining this observation with (3.26) and (3.15) we obtain

$$\begin{aligned}
&\frac{1}{\sqrt{\epsilon_m}} \int_{s_m-\sqrt{\epsilon_m}h(\epsilon_m)/2}^{s_m+\sqrt{\epsilon_m}h(\epsilon_m)/2} \frac{\epsilon_m}{2}\big(B(\varphi_{\epsilon_m})'\big)^2\, ds \\
&\quad = \frac{1}{\sqrt{\epsilon_m}} \int_{s_m-\sqrt{\epsilon_m}h(\epsilon_m)/2}^{s_m+\sqrt{\epsilon_m}h(\epsilon_m)/2} \big(W\big(\varphi_{\epsilon_m}(s), \vartheta_{\epsilon_m}(s_m)\big) - \mathcal{A}_{\epsilon_m}(s_m)\big)\, ds + O\big(\epsilon_m^{1/4} h(\epsilon_m)^{3/2}\big) \\
&\quad = \int_{-h(\epsilon_m)/2}^{h(\epsilon_m)/2} \big(W\big(v_{\epsilon_m}(y, s_m), \vartheta_{\epsilon_m}(s_m)\big) - \mathcal{A}_{\epsilon_m}(s_m)\big)\, dy + o\big(\epsilon_m^{1/8}\big) \quad \text{as } \epsilon_m \to 0.
\end{aligned} \tag{3.27}$$

Recall, from Lemma 3.7, that for all m sufficiently large, and $p \in (0, 1/4)$,

$$\big|v_{\epsilon_m}(y, s_m) - u_{\epsilon_m}(y, s_m)\big| \leqslant \epsilon_m^p \quad \text{when } y \in \big[-h(\epsilon_m)/2, h(\epsilon_m)/2\big],$$

where $u_{\epsilon_m}(y, s)$ satisfies (3.16) with $s = s_m$. With $p = 1/8$ this leads to the estimate

$$\big|W\big(v_{\epsilon_m}(y, s_m), \vartheta_{\epsilon_m}(s_m)\big) - W\big(u_{\epsilon_m}(y, s_m), \vartheta_{\epsilon_m}(s_m)\big)\big| \leqslant c\epsilon_m^{1/8},$$

which in (3.27) yields that

$$\frac{1}{\sqrt{\epsilon_m}} \int_{s_m-\sqrt{\epsilon_m}h(\epsilon_m)/2}^{s_m+\sqrt{\epsilon_m}h(\epsilon_m)/2} \frac{\epsilon_m}{2}\big(B(\varphi_{\epsilon_m})'\big)^2\, ds = \int_{-h(\epsilon_m)/2}^{h(\epsilon_m)/2} \big(W\big(u_{\epsilon_m}(y, s_m), \vartheta_{\epsilon_m}(s_m)\big) - \mathcal{A}_{\epsilon_m}(s_m)\big)\, dy + O\big(\epsilon_m^{1/8}\big). \tag{3.28}$$

Now recall that, in addition to the sequences $\{\epsilon_m\}$ and $\{s_m\}$, we can choose sequences $\{y_m^\pm\}$ satisfying (3.20). Note also that if the function u_{ϵ_m} has different values at $y_m^\pm$, then it is not a constant and there are only three possibilities: it is periodic and monotone between successive points at which it takes absolute minimum and maximum values, or it has only one critical point at which it takes its absolute minimum or maximum, or it is monotone.

It obviously follows that we can choose $y_m^\pm$, satisfying conditions (3.20), so that $u_{\epsilon_m}(y, s_m)$ will be monotone on the interval $I_m \subset [-h(\epsilon_m)/2, h(\epsilon_m)/2]$ which is bounded by y_m^- and y_m^+. Note from Lemma 3.7 that

$$W\big(u_{\epsilon_m}(y, s_m), \vartheta_{\epsilon_m}(s_m)\big) - \mathcal{A}_{\epsilon_m}(s_m) \geqslant 0 \quad \text{for } |y| \leqslant h(\epsilon_m)/2$$

and, on I_m,

$$B'(u_{\epsilon_m})\,du_{\epsilon_m} = \pm\sqrt{2\big(W\big(u_{\epsilon_m}(y, s_m), \vartheta_{\epsilon_m}(s_m)\big) - \mathcal{A}_{\epsilon_m}(s_m)\big)}\,dy,$$

where we take "+" if $y_m^- < y_m^+$ and "−" otherwise. Thus, since $|y_m^\pm| \leqslant h(\epsilon_m)/2$,

$$\int\limits_{-h(\epsilon_m)/2}^{h(\epsilon_m)/2} \big(W\big(u_{\epsilon_m}(y, s_m), \vartheta_{\epsilon_m}(s_m)\big) - \mathcal{A}_{\epsilon_m}(s_m)\big)\,dy \geqslant \int\limits_{I_m} \big(W\big(u_{\epsilon_m}(y, s_m), \vartheta_{\epsilon_m}(s_m)\big) - \mathcal{A}_{\epsilon_m}(s_m)\big)\,dy$$
$$= \frac{1}{\sqrt{2}} \int\limits_{\phi_s^-(\vartheta_0)+\eta}^{\phi_s^+(\vartheta_0)-\eta} \sqrt{W\big(\phi, \vartheta_{\epsilon_m}(s_m)\big) - \mathcal{A}_{\epsilon_m}(s_m)}\,B'(\phi)\,d\phi.$$

Combining this observation with (3.28) we obtain

$$\frac{1}{\sqrt{\epsilon_m}} \int\limits_{s_m-\sqrt{\epsilon_m}h(\epsilon_m)/2}^{s_m+\sqrt{\epsilon_m}h(\epsilon_m)/2} \frac{\epsilon_m}{2}\big(B(\varphi_{\epsilon_m})'\big)^2\,ds \geqslant \frac{1}{\sqrt{2}} \int\limits_{\phi_s^-(\vartheta_0)+\eta}^{\phi_s^+(\vartheta_0)-\eta} \sqrt{W\big(\phi, \vartheta_{\epsilon_m}(s_m)\big) - \mathcal{A}_{\epsilon_m}}\,B'(\phi)\,d\phi + O\big(\epsilon_m^{1/8}\big).$$

Since $(\mathcal{A}_{\epsilon_m}(s_m), \vartheta_{\epsilon_m}(s_m)) \to (\mathcal{A}(s_0), \vartheta(s_0))$ as $m \to \infty$, from (3.24) and (3.25),

$$\liminf_{E\ni\epsilon\to 0} \frac{1}{\sqrt{\epsilon}} \int\limits_{s_0-\beta}^{s_0+\beta} \frac{\epsilon}{2}\big(B(\varphi_\epsilon)'\big)^2\,ds \geqslant \frac{1}{\sqrt{2}} \int\limits_{\phi_s^-(s_0)+\eta}^{\phi_s^+(s_0)-\eta} \sqrt{W\big(\phi, \vartheta(s_0)\big) - \mathcal{A}(s_0)}\,B'(\phi)\,d\phi.$$

Finally, letting $\eta \to 0$, we find that for $s_0 \in \mathcal{F}(\mathcal{A}, \vartheta)$ with osc-def $E(s_0) > 0$ and $\beta > 0$,

$$\liminf_{E\ni\epsilon\to 0} \frac{1}{\sqrt{\epsilon}} \int\limits_{s_0-\beta}^{s_0+\beta} \frac{\epsilon}{2}\big(B(\varphi_\epsilon)'\big)^2\,ds \geqslant \wp\big(\vartheta(s_0)\big)$$

where $\wp$ is defined by (2.3a). If the set $\mathcal{O}(E)$ defined by (3.5), contains n distinct points s_i, $1 \leqslant i \leqslant n$, then, for β sufficiently small, we obtain

$$\liminf_{\epsilon\to 0} \frac{1}{\sqrt{\epsilon}} \int\limits_0^L \frac{\epsilon}{2}\big(B(\varphi_\epsilon)'\big)^2\,ds \geqslant \liminf_{\epsilon\to 0} \frac{1}{\sqrt{\epsilon}} \sum_i \int\limits_{s_i-\beta}^{s_i+\beta} \frac{\epsilon}{2}\big(B(\varphi_\epsilon)'\big)^2\,ds \geqslant \sum_i \wp\big(\vartheta(s_i)\big).$$

If the set $\mathcal{O}(E)$ is finite, this gives (3.4). If the set $\mathcal{O}(E)$ is infinite, then letting $n \to \infty$ we obtain that both sides of (3.4) are equal to ∞ (see Remark 2.2), which completes the proof of Theorem 3.3. □

4. Asymptotic behaviour of energy minimizers

In this section we make the following hypotheses on $\{\vartheta_\epsilon \colon \epsilon \in (0, 1)\}$.

(H4) (i) $\{\vartheta_\epsilon \colon \epsilon \in (0, 1)\}$ is bounded in $(H^1_{\mathrm{per}})^d$.
(ii) $\limsup_{\epsilon\to 0} \epsilon^{-1/2}\|\vartheta_\epsilon - \vartheta\|_{(L^1_{\mathrm{per}})^d} = \Theta_0 < \infty$.

(iii) There exists $\Lambda:(0,1)\to\mathbb{R}$ with $\limsup_{\epsilon\to 0}\sqrt{\epsilon}\Lambda(\epsilon)=\Theta_1<\infty$ and, for almost all $\epsilon\in(0,1)$,

$$\liminf_{\lambda\to 0}\frac{\|\vartheta_{\epsilon-\lambda}-\vartheta_\epsilon\|_{(L^1_{\text{per}})^d}}{\lambda}=\Lambda(\epsilon).$$

With $E=(0,1)$ let M be as in (3.1b), (3.1c) and let C_W be such that

$$\left|W(\phi,\theta_1)-W(\phi,\theta_2)\right|\leqslant C_W\|\theta_1-\theta_2\|,\quad \phi\in\left[M^{-1},M\right],\ \|\theta_1\|,\|\theta_2\|\leqslant M_E.$$

(H5) With ϑ given by (H4), problem (2.1) has a piecewise regular minimizer.

Of course, (H4) is trivial if ϑ_ϵ is independent of ϵ.

Energy minimizers. Let (H4) and (H5) hold and define $\mathfrak{E}:[0,1)\mapsto\mathbb{R}$ by

$$\mathfrak{E}(\epsilon)=\inf_{\varphi\in H^1_{\text{per}}}\int_0^L\left(\frac{\epsilon}{2}\big(B(\varphi)'\big)^2+W(\varphi,\vartheta_\epsilon)\right)ds,\quad \epsilon\in(0,1),$$

$$\mathfrak{E}(0)=\int_0^L\mathcal{A}_\vartheta(s)\,ds\quad\text{where }\mathcal{A}_\vartheta(s)=\inf_\phi W\big(\phi,\vartheta(s)\big).\tag{4.1}$$

Here $\mathcal{A}_\vartheta\in H^1_{\text{per}}$ is as in Lemma 2.5. First we discuss the behaviour of $\mathfrak{E}$ at 0.

Theorem 4.1. *$\mathfrak{E}$ is continuous from the right at* 0.

Proof. Let $\underline{\varphi}$ be a minimizer of (2.1) so that $\underline{\varphi}(s)\in[M^{-1},M]$ and

$$\mathfrak{E}(0)=\int_0^L W(\underline{\varphi},\vartheta)\,ds.$$

For $\delta>0$, let $\underline{\varphi}^\eta$ be a mollification of $\underline{\varphi}$ with $\underline{\varphi}^\eta\in[M^{-1},M]$ and

$$\left|\int_0^L W\big(\underline{\varphi}^\eta,\vartheta\big)\,ds-\int_0^L W(\underline{\varphi},\vartheta)\,ds\right|\leqslant\frac{\delta}{3}.$$

Now let

$$\epsilon_\delta=\frac{2\delta}{3}\left(\int_0^L\big(B\big(\underline{\varphi}^\eta\big)'\big)^2\,ds\right)^{-1}.$$

Then, for $0<\epsilon\leqslant\epsilon_\delta$,

$$\begin{aligned}\mathfrak{E}(0)\leqslant\mathfrak{E}(\epsilon)\leqslant J_\epsilon\big(\underline{\varphi}^\eta\big)&=\int_0^L\left(\frac{\epsilon}{2}\big(B\big(\underline{\varphi}^\eta\big)'\big)^2+W\big(\underline{\varphi}^\eta,\vartheta_\epsilon\big)\right)ds\\&\leqslant\frac{\delta}{3}+\int_0^L\big(W\big(\underline{\varphi}^\eta,\vartheta\big)\big)\,ds+\int_0^L\big(W\big(\underline{\varphi}^\eta,\vartheta_\epsilon\big)-W\big(\underline{\varphi}^\eta,\vartheta\big)\big)\,ds\\&\leqslant\mathfrak{E}(0)+\frac{2\delta}{3}+\int_0^L\big(W\big(\underline{\varphi}^\eta,\vartheta_\epsilon\big)-W\big(\underline{\varphi}^\eta,\vartheta\big)\big)\,ds\\&\leqslant\mathfrak{E}(0)+\delta,\end{aligned}$$

by (H1) and (H4)(ii), if $\epsilon<\epsilon_\delta$ is sufficiently small. This proves the result. □

Theorem 4.2. *Suppose that* (H1–5) *hold and that φ is a piecewise regular minimizer φ of* (2.1). *Then*

$$\limsup_{0<\epsilon\to 0} \frac{\mathfrak{E}(\epsilon)-\mathfrak{E}(0)}{\sqrt{\epsilon}} \leqslant 2\mathcal{W}(\varphi)+C_W\Theta_0, \tag{4.2}$$

where $\mathcal{W}(\varphi)$ is defined in (2.3b) *and C_W and Θ_0 are given by* (H4).

Proof. For *any* $\{\psi_\epsilon \in H^1_{\mathrm{per}}\colon \epsilon\in(0,1)\}$ with $\psi_\epsilon(s)\in[M^{-1},M]$,

$$\begin{aligned}\frac{\mathfrak{E}(\epsilon)-\mathfrak{E}(0)}{\sqrt{\epsilon}} &\leqslant \frac{1}{\sqrt{\epsilon}}\int_0^L \left(\frac{\epsilon}{2}\big(B(\psi_\epsilon)'(s)\big)^2 + W\big(\psi_\epsilon(s),\vartheta(s)\big)-\mathcal{A}_\vartheta(s)\right)ds \\ &\quad + \frac{1}{\sqrt{\epsilon}}\int_0^L \Big(W\big(\psi_\epsilon(s),\vartheta_\epsilon(s)\big)-W\big(\psi_\epsilon(s),\vartheta(s)\big)\Big)\,ds \\ &\leqslant \frac{1}{\sqrt{\epsilon}}\int_0^L \left(\frac{\epsilon}{2}\big(B(\psi_\epsilon)'(s)\big)^2 + W\big(\psi_\epsilon(s),\vartheta(s)\big)-\mathcal{A}_\vartheta(s)\right)ds + C_W\Theta_0 + o(1),\end{aligned} \tag{4.3}$$

by (H4)(ii), as $\epsilon\to 0$. Hence it suffices to make an optimal choice of ψ_ϵ, in order to estimate the first term on the right-hand side of (4.3). We use an auxiliary function Φ defined by the following lemma.

Lemma 4.3. *For constants $(A,\theta)\in F$, the Cauchy problem*

$$W\big(\Phi(y),\theta\big)-\frac{1}{2}\big(B(\Phi)'(y)\big)^2 = A, \qquad \Phi(0)=\frac{1}{2}\big(\phi_s^-(\theta)+\phi_s^+(\theta)\big), \qquad \Phi'(0)>0, \tag{4.4}$$

has a unique solution $\Phi:\mathbb{R}\to\mathbb{R}$. It is monotone, $\lim_{y\to\pm\infty}\Phi(y)=\phi_s^\pm(\theta)$ *and there are positive constants c_1, c_2, depending only on $\|\theta\|$, such that*

$$\big|\Phi(y)-\phi_s^\pm(\theta)\big|\leqslant e^{-c_1|y|} \quad \textit{for } |y|>c_2.$$

Proof. Since the function $W(\cdot,\theta)-A$ is positive on $(\phi_s^-(\theta),\varphi_s^+(\theta))$ and has non-degenerate global minima on $\mathbb{R}$ at $\phi_s^\pm(\theta)$, the statement of the lemma follows from the formula

$$y=\int_{(\phi_s^-(\theta)+\phi_s^+(\theta))/2}^{\Phi(y)} \frac{B'(\phi)\,d\phi}{\sqrt{2(W(\phi,\theta)-A)}}. \qquad \square$$

Turning to the proof of the theorem, for $\epsilon>0$ sufficiently small, let

$$I_n=\big(S_n-2\epsilon^{5/12},\,S_n+2\epsilon^{5/12}\big),$$

where $\{S_n\}$ is the jump set of the piecewise regular minimizer φ. For all small ϵ, these intervals are disjoint and their union covers the set $\{S_n\}$. Set $\theta_n=\vartheta(S_n)$, $A_n=\mathcal{A}_\vartheta(S_n)$, $\phi_n^\pm=\phi_s^\pm(\theta_n)$, and denote by Φ_n the solution to the Cauchy problem (4.4) with $\theta=\theta_n$ and $A=A_n$.

(i) For $s\in\mathbb{R}\setminus\cup I_n$, let $\psi_\epsilon(s)=\varphi(s)$.
(ii) For $|s-S_n|\leqslant\epsilon^{5/12}$, let

$$\psi_\epsilon(s)=\Phi\left(\frac{s-S_n}{\sqrt{\epsilon}}\right) \quad \text{when} \quad \lim_{s\to S_n\pm 0}\varphi(s)=\phi_n^\pm, \tag{4.5}$$

and

$$\psi_\epsilon(s)=\Phi\left(\frac{S_n-s}{\sqrt{\epsilon}}\right) \quad \text{when} \quad \lim_{s\to S_n\pm 0}\varphi(s)=\phi_n^\mp. \tag{4.6}$$

(iii) On $(S_n - 2\epsilon^{5/12}, S_n - \epsilon^{5/12}) \cup (S_n + \epsilon^{5/12}, S_n + 2\epsilon^{5/12})$, let ψ_ϵ be affine so that the resulting function is continuous on $\mathbb{R}$.

The behaviour of ψ_ϵ when it is affine is described by the following lemma.

Lemma 4.4. *Suppose that* (4.5) *holds. Then*

$$\begin{aligned}
&\left|\psi_\epsilon(s) - \phi_n^-\right| \leqslant c\epsilon^{5/24} \quad \text{for } S_n - 2\epsilon^{5/12} < s < S_n - \epsilon^{5/12},\\
&\left|\psi_\epsilon(s) - \phi_n^+\right| \leqslant c\epsilon^{5/24} \quad \text{for } S_n + \epsilon^{5/12} < s < S_n + 2\epsilon^{5/12},\\
&\left|\psi_\epsilon'(s)\right| \leqslant c\epsilon^{-5/24} \quad \text{for } \epsilon^{5/12} < |S_n - s| < 2\epsilon^{5/12}.
\end{aligned} \tag{4.7}$$

If (4.6) *holds, then the roles of* ϕ_n^+ *and* ϕ_n^- *are interchanged.*

Proof. Suppose that (4.5) holds. On $(S_n - 2\epsilon^{5/12}, S_n - \epsilon^{5/12})$ where ψ_ϵ is affine,

$$\begin{aligned}
\left|\psi_\epsilon(s) - \phi_n^-\right| &\leqslant \left|\psi_\epsilon\left(S_n - 2\epsilon^{5/12}\right) - \phi_n^-\right| + \left|\psi_\epsilon\left(S_n - \epsilon^{5/12}\right) - \phi_n^-\right|\\
&= \left|\varphi\left(S_n - 2\epsilon^{5/12}\right) - \phi_n^-\right| + \left|\Phi_n\left(-\epsilon^{-1/12}\right) - \phi_n^-\right|.
\end{aligned}$$

Since the piecewise regular minimizer φ is absolutely continuous and φ' is square-integrable on (S_{n-1}, S_n),

$$\begin{aligned}
\left|\varphi\left(S_n - 2\epsilon^{5/12}\right) - \phi_n^-\right| &= \left|\varphi\left(S_n - 2\epsilon^{5/12}\right) - \lim_{s \to S_n - 0} \varphi(s)\right|\\
&\leqslant \left(2\int_{S_{n-1}}^{S_n} \left(\varphi'\right)^2 ds\right)^{1/2} \epsilon^{5/24} \leqslant c\epsilon^{5/24}.
\end{aligned}$$

Also it follows from Lemma 4.3 that, as $\epsilon \to 0$,

$$\left|\Phi_n\left(-\epsilon^{-1/12}\right) - \phi_n^-\right| \leqslant ce^{-c_1\epsilon^{-1/12}} \leqslant c\epsilon^{5/24}.$$

This gives the first inequality in (4.7). Repeating this arguments on the interval $(S_n + \epsilon^{5/12}, S_n + 2\epsilon^{5/12})$, we obtain the second estimate in (4.7). Finally note that, on the interval $(S_n - 2\epsilon^{5/12}, S_n - \epsilon^{5/12})$,

$$\begin{aligned}
\left|\psi_\epsilon'\right| &= \epsilon^{-5/12}\left|\psi_\epsilon\left(S_n - 2\epsilon^{5/12}\right) - \psi_\epsilon\left(S_n - \epsilon^{5/12}\right)\right|\\
&\leqslant \epsilon^{-5/12}\left(\left|\psi_\epsilon\left(S_n - 2\epsilon^{5/12}\right) - \phi_n^-\right| + \left|\psi_\epsilon\left(S_n - \epsilon^{5/12}\right) - \phi_n^-\right|\right)\\
&= \epsilon^{-5/12}\left(\left|\varphi\left(S_n - 2\epsilon^{5/12}\right) - \phi_n^-\right| + \left|\Phi_n\left(-\epsilon^{-1/12}\right) - \phi_n^-\right|\right) \leqslant c\epsilon^{-5/24},
\end{aligned}$$

which yields the third estimate in (4.7) when (4.5) holds. A similar argument when (4.6) holds completes the proof. □

With this choice of ψ_ϵ we study the integral on the right side of (4.3) in three steps. Choose an arbitrary integer n and without loss of generality assume that $\lim_{s \to S_n \pm 0} \varphi(s) = \phi_n^\pm$.

Step 1. By hypothesis, $\{\|\vartheta(s)\|\colon s \in \mathbb{R}\}$ is bounded and, by construction, $M^{-1} \leqslant \psi_\epsilon(s) \leqslant M$, where M is as in (3.1). Therefore

$$\begin{aligned}
\left|W\left(\psi_\epsilon(s), \vartheta(s)\right) - W\left(\psi_\epsilon(s), \theta_n\right)\right| + \left|A_n - \mathcal{A}_\vartheta(s)\right| &\leqslant c\left(\left\|\vartheta(s) - \vartheta(S_n)\right\| + \left|\mathcal{A}_\vartheta(s) - \mathcal{A}_\vartheta(S_n)\right|\right)\\
&\leqslant c\left(\|\vartheta\|_{(H^1_{\mathrm{per}})^d} + \|\mathcal{A}_\vartheta\|_{H^1_{\mathrm{per}}}\right)|s - S_n|^{1/2} \leqslant c\epsilon^{5/24},
\end{aligned} \tag{4.8}$$

for all $s \in (S_n - 2\epsilon^{5/12}, S_n + 2\epsilon^{5/12})$. Therefore

$$\frac{1}{\sqrt{\epsilon}}\left|\int_{S_n - \epsilon^{5/12}}^{S_n + \epsilon^{5/12}} \left(W\left(\psi_\epsilon(s), \vartheta(s)\right) - W\left(\psi_\epsilon(s), \theta_n\right) + A_n - \mathcal{A}_\vartheta(s)\right) ds\right| \leqslant c\epsilon^{-1/2+5/12+5/24} = c\epsilon^{1/8}.$$

Observe, from (4.4) and the definition of ψ_ϵ, that

$$W\big(\psi_\epsilon(s),\theta_n\big)-A_n=\frac{\epsilon}{2}\big(B(\psi_\epsilon)'(s)\big)^2\quad\text{on }\big[S_n-\epsilon^{5/12},S_n+\epsilon^{5/12}\big].$$

Hence

$$\begin{aligned}
&\frac{1}{\sqrt{\epsilon}}\int_{S_n-\epsilon^{5/12}}^{S_n+\epsilon^{5/12}}\bigg(\frac{\epsilon}{2}\big(B(\psi_\epsilon)'(s)\big)^2+W\big(\psi_\epsilon(s),\vartheta(s)\big)-\mathcal{A}_\vartheta(s)\bigg)\,ds\\
&\quad=\frac{2}{\sqrt{\epsilon}}\int_{S_n-\epsilon^{5/12}}^{S_n+\epsilon^{5/12}}\big(W\big(\psi_\epsilon(s),\theta_n\big)-A_n\big)\,ds+\frac{1}{\sqrt{\epsilon}}\int_{S_n-\epsilon^{5/12}}^{S_n+\epsilon^{5/12}}\big(W\big(\psi_\epsilon(s),\vartheta(s)\big)-W\big(\psi_\epsilon(s),\theta_n\big)+A_n-\mathcal{A}_\vartheta(s)\big)\,ds\\
&\quad=\frac{2}{\sqrt{\epsilon}}\int_{S_n-\epsilon^{5/12}}^{S_n+\epsilon^{5/12}}\big(W\big(\psi_\epsilon(s),\theta_n\big)-A_n\big)\,ds+O\big(\epsilon^{1/8}\big).
\end{aligned}\tag{4.9}$$

Next, from the definition of ψ_ϵ,

$$\begin{aligned}
\frac{2}{\sqrt{\epsilon}}\int_{S_n-\epsilon^{5/12}}^{S_n+\epsilon^{5/12}}\big(W\big(\psi_\epsilon(s),\theta_n\big)-A_n\big)\,ds&=2\int_{-\epsilon^{-1/12}}^{\epsilon^{-1/12}}\big(W\big(\Phi_n(y),\theta_n\big)-A_n\big)\,dy\\
&=\sqrt{2}\int_{\Phi_n(-\epsilon^{-1/12})}^{\Phi_n(\epsilon^{-1/12})}B'(\phi)\sqrt{W(\phi,\theta_n)-A_n}\,d\phi.
\end{aligned}\tag{4.10}$$

Since, by Lemma 4.3,

$$\phi_n^-<\Phi_n<\phi_n^+,\qquad\big|\Phi_n\big(\pm\epsilon^{-1/12}\big)-\phi_n^\pm\big|\leqslant ce^{-c_1\epsilon^{-1/12}},$$

it follows from formula (2.3a) that, as $\epsilon\to 0$,

$$\begin{aligned}
&\left|\sqrt{2}\int_{\Phi_n(-\epsilon^{-1/12})}^{\Phi_n(\epsilon^{-1/12})}B'(\phi)\sqrt{W(\phi,\theta_n)-A_n}\,d\phi-2\wp(\theta_n)\right|\\
&\quad=\sqrt{2}\left\{\int_{\phi_n^-}^{\Phi_n(-\epsilon^{-1/12})}+\int_{\Phi_n(\epsilon^{-1/12})}^{\phi_n^+}\right\}B'(\phi)\sqrt{W(\phi,\theta_n)-A_n}\,d\phi\\
&\quad\leqslant ce^{-c_1\epsilon^{-1/12}}\leqslant c\epsilon^{1/8}.
\end{aligned}$$

Combining this with (4.9) and (4.10) we obtain that, as $\epsilon\to 0$,

$$\frac{1}{\sqrt{\epsilon}}\int_{S_n-\epsilon^{5/12}}^{S_n+\epsilon^{5/12}}\bigg(\frac{\epsilon}{2}\big(B(\psi_\epsilon)'(s)\big)^2+W\big(\psi_\epsilon(s),\vartheta(s)\big)-\mathcal{A}_\vartheta(s)\bigg)\,ds=2\wp(\theta_n)+O\big(\epsilon^{1/8}\big).\tag{4.11}$$

Step 2. Note that

$$\frac{1}{\sqrt{\epsilon}}\int_{S_n-2\epsilon^{5/12}}^{S_n-\epsilon^{5/12}}\big(W\big(\psi_\epsilon(s),\vartheta(s)\big)-\mathcal{A}_\vartheta(s)\big)\,ds$$

$$= \frac{1}{\sqrt{\epsilon}} \int_{S_n-2\epsilon^{5/12}}^{S_n-\epsilon^{5/12}} \big(W\big(\psi_\epsilon(s),\theta_n\big) - A_n\big)\,ds$$
$$+ \frac{1}{\sqrt{\epsilon}} \int_{S_n-2\epsilon^{5/12}}^{S_n-\epsilon^{5/12}} \big(W\big(\psi_\epsilon(s),\vartheta(s)\big) - W\big(\psi_\epsilon(s),\theta_n\big) - \mathcal{A}_\vartheta(s) + A_n\big)\,ds. \tag{4.12}$$

From (4.8) we obtain

$$\left|\frac{1}{\sqrt{\epsilon}} \int_{S_n-2\epsilon^{5/12}}^{S_n-\epsilon^{5/12}} \big(W\big(\psi_\epsilon(s),\vartheta(s)\big) - W\big(\psi_\epsilon(s),\theta_n\big) - \mathcal{A}_\vartheta(s) + A_n\big)\,ds\right| \leqslant c\epsilon^{-1/2+5/12+5/24} = c\epsilon^{1/8} \tag{4.13}$$

and, from Lemma 4.4,

$$\big|W\big(\psi_\epsilon(s),\theta_n\big) - A_n\big| = \big|W\big(\psi_\epsilon(s),\theta_n\big) - W\big(\phi_n^-,\theta_n\big)\big| \leqslant c\big|\psi_\epsilon(s) - \phi_n^-\big| \leqslant c\epsilon^{5/24},$$

which leads to the conclusion that

$$\left|\frac{1}{\sqrt{\epsilon}} \int_{S_n-2\epsilon^{5/12}}^{S_n-\epsilon^{5/12}} \big(W\big(\psi_\epsilon(s),\theta_n\big) - A_n\big)\,ds\right| \leqslant c\epsilon^{-1/2+5/12+5/24} \leqslant c\epsilon^{1/8}. \tag{4.14}$$

Substituting (4.13) and (4.14) into (4.12) we finally obtain

$$\left|\frac{1}{\sqrt{\epsilon}} \int_{S_n-2\epsilon^{5/12}}^{S_n-\epsilon^{5/12}} \big(W\big(\psi_\epsilon(s),\vartheta(s)\big) - \mathcal{A}_\vartheta(s)\big)\,ds\right| \leqslant c\epsilon^{1/8}. \tag{4.15}$$

Furthermore, it follows from Lemma 4.4 that

$$\left|\frac{1}{\sqrt{\epsilon}} \int_{S_n-2\epsilon^{5/12}}^{S_n-\epsilon^{5/12}} \frac{\epsilon}{2}\big(B(\psi_\epsilon)'(s)\big)^2\,ds\right| \leqslant c\epsilon^{1/2}. \tag{4.16}$$

Combining (4.15) and (4.16) gives

$$\left|\frac{1}{\sqrt{\epsilon}} \int_{S_n-2\epsilon^{5/12}}^{S_n-\epsilon^{5/12}} \left(\frac{\epsilon}{2}\big(B(\psi_\epsilon)'(s)\big)^2 + W\big(\psi_\epsilon(s),\vartheta(s)\big) - \mathcal{A}_\vartheta(s)\right)ds\right| \leqslant c\epsilon^{1/8}.$$

Similarly, the same estimate holds on $(S_n + \epsilon^{5/12}, S_n + 2\epsilon^{5/12})$. Hence, by (4.11),

$$\frac{1}{\sqrt{\epsilon}} \int_{I_n} \left(\frac{\epsilon}{2}\big(B(\psi_\epsilon)'(s)\big)^2 + W\big(\psi_\epsilon(s),\vartheta(s)\big) - \mathcal{A}_\vartheta(s)\right)ds = 2\wp(\theta_n) + O\big(\epsilon^{1/8}\big). \tag{4.17}$$

Step 3. Note that $\psi_\epsilon(s)$ coincides with the piecewise regular minimizer $\varphi(s)$ and $\mathcal{A}_\vartheta(s) = W(\varphi(s),\vartheta(s))$ for s outside of the intervals I_n. Hence, for any finite interval (a,b),

$$\frac{1}{\sqrt{\epsilon}} \int_{(a,b)\setminus\cup I_n} \left(\frac{\epsilon}{2}\big(B(\psi_\epsilon)'(s)\big)^2 + W\big(\psi_\epsilon(s),\vartheta(s)\big) - \mathcal{A}_\vartheta(s)\right)ds$$
$$= \frac{\sqrt{\epsilon}}{2} \int_{(a,b)\setminus\cup I_n} \big(B(\varphi)'(s)\big)^2\,ds \leqslant c\sqrt{\epsilon}\int_a^b \big(\varphi'(s)\big)^2\,ds \leqslant c(a,b)\sqrt{\epsilon}.$$

We are now in a position to complete the proof of Theorem 4.2. Since each compact set contains only a finite number of points S_n, there exists $\delta > 0$ so that the intervals $(-\delta, 0)$ and $(L-\delta, L)$ do not contain any S_n, $n \in \mathbb{Z}$. Therefore

$$\begin{aligned}
&\frac{1}{\sqrt{\epsilon}}\int_0^L \left(\frac{\epsilon}{2}\big(B(\psi_\epsilon)'\big)^2 + W(\psi_\epsilon, \vartheta) - \mathcal{A}_\vartheta\right) ds \\
&\quad = \frac{1}{\sqrt{\epsilon}}\int_{-\delta}^{L-\delta} \left(\frac{\epsilon}{2}\big(B(\psi_\epsilon)'\big)^2 + W(\psi_\epsilon, \vartheta) - \mathcal{A}_\vartheta\right) ds \\
&\quad = \sum_{\{n:\ S_n \in (-\delta, L-\delta)\}} \frac{1}{\sqrt{\epsilon}}\int_{I_n} \left(\frac{\epsilon}{2}\big(B(\psi_\epsilon)'\big)^2 + W(\psi_\epsilon, \vartheta) - \mathcal{A}_\vartheta\right) ds \\
&\qquad + \frac{1}{\sqrt{\epsilon}}\int_{(-\delta, L-\delta)\setminus \cup I_n} \left(\frac{\epsilon}{2}\big(B(\psi_\epsilon)'\big)^2 + W(\psi_\epsilon, \vartheta) - \mathcal{A}_\vartheta\right) ds \\
&\quad = \sum_{\{n:\ S_n \in (-\delta, L-\delta)\}} \frac{1}{\sqrt{\epsilon}}\int_{I_n} \left(\frac{\epsilon}{2}\big(B(\psi_\epsilon)'\big)^2 + W(\psi_\epsilon, \vartheta) - \mathcal{A}_\vartheta\right) ds + O\big(\epsilon^{1/2}\big).
\end{aligned}$$

This observation, when combined with (4.17), yields

$$\begin{aligned}
\frac{1}{\sqrt{\epsilon}}\int_0^L \left(\frac{\epsilon}{2}\big(B(\psi_\epsilon)'\big)^2 + W(\psi_\epsilon, \vartheta) - \mathcal{A}_\vartheta\right) ds &= \sum_{\{n:\ S_n \in (-\delta, L-\delta)\}} 2\wp(\theta_n) + O\big(\epsilon^{1/8}\big) \\
&= \sum_{S_n \in [0, L)} 2\wp\big(\vartheta(S_n)\big) + O\big(\epsilon^{1/8}\big).
\end{aligned}$$

Recalling (4.3) we finally arrive at the inequality

$$\frac{\mathfrak{E}(\epsilon) - \mathfrak{E}(0)}{\sqrt{\epsilon}} \leqslant 2\sum_{S_n \in [0, L)} \wp\big(\vartheta(S_n)\big) + O\big(\epsilon^{1/8}\big)$$

as $\epsilon \to 0$, which yields (4.2). □

5. Minimal number of jumps principle

Theorem 5.6, which says that, *almost always,* solutions to the variational problem (2.13) converge weak* to piecewise regular minimizers with a minimal number of jumps, is a corollary of a stronger statement, under more general hypotheses. A set $E \subset (0, 1]$ has *lower Lebesgue density* λ *at* 0 if

$$\liminf_{0<\tau\to 0} \frac{1}{\tau}\big(\operatorname{meas} E \cap [0, \tau]\big) = \lambda. \tag{5.1}$$

We say that E has positive Lebesgue density at 0 if $\lambda > 0$ and E is Lebesgue dense at 0 if $\lambda = 1$. We need a final hypothesis.

(H6) $\mathfrak{E}$ defined in (4.1) is locally absolutely continuous on $(0, 1)$.

This is a rather weak hypothesis which is trivial when ϑ_ϵ is independent of ϵ, because then $\mathfrak{E}$ is concave. More generally it is shown in Corollary A.2 in Appendix A that (H6) holds if there exists a locally absolutely continuous function $h : (0, 1) \to \mathbb{R}$ with the property that, for all $\varphi \in H^1_{\mathrm{per}}$ with range in $[M^{-1}, M]$,

$$\text{the mapping } \epsilon \mapsto h(\epsilon) + \int_0^L W(\varphi, \vartheta_\epsilon)\, ds \text{ is concave on } (0, 1), \tag{5.2}$$

where $M \geqslant 0$ is given by (3.1) with $E = (0, 1)$. Alternately, if $[\delta, 1] \ni \epsilon \mapsto \vartheta_\epsilon \in (L^1_{\text{per}})^d$ is Lipschitz continuous for all $\delta \in (0, 1)$, (H6) follows from Theorem A.3. In the notation of (H4), let

$$\mathcal{L}(\delta) = \left(\frac{\delta}{\delta + C_W \Theta_1 + C_W \Theta_0/2}\right)^2, \quad \delta > 0. \tag{5.3}$$

Theorem 5.1. *Suppose that* (H1–6) *hold. Then, for any $\delta > 0$ there is a set $E_\delta \subset (0, 1]$ with lower Lebesgue density at least $\mathcal{L}(\delta)$ at 0 and the following property. If a sequence $\{\varphi_{\epsilon_n}\}$, $E_\delta \ni \epsilon_n \to 0$, of solutions to problem* (2.13) *converges weak* in L^∞_{per} to some function φ, then φ is a piecewise regular minimizer of* (2.1) *with weighted number of jumps*

$$\mathcal{W}(\varphi) \leqslant \mathcal{W}_{\min} + C_W \Theta_1 + C_W \Theta_0/2 + \delta, \tag{5.4}$$

where $\mathcal{W}_{\min}$ is defined in (2.3c).

Proof. The proof is in a number of steps, but first a remark.

Remark 5.2. To illustrate this observation, consider the example

$$J_\epsilon(\varphi) := \int_0^L \left(\epsilon \varphi'^2 + W(\varphi, \vartheta_\epsilon)\right) ds,$$

in which $d = 1$, $W(\phi, \theta) = (\phi^2 - 1)^2/4 - \theta\phi$ (a Landau potential) and $\vartheta_\epsilon = \sqrt{\epsilon}\gamma\vartheta^*$, where $\gamma \in \mathbb{R}$ and $\vartheta^* \in H^1_{\text{per}}$ are given. Note that (H4) holds with $\Theta_0 \sim |\gamma| \|\vartheta^*\|_{L^1_{\text{per}}}$ and $\Theta_1 \sim |\gamma| \|\vartheta^*\|_{L^1_{\text{per}}}/2$, that the limit of ϑ_ϵ is 0 and that $\wp(0) = \wp_0 = 4/3$, from (1.8). Clearly the minimum possible number of jumps of a minimizer of J_0 is zero, in other words, $\mathcal{W}_{\min} = 0$ in (5.4). A question arises:

how many jumps do the weak limits of minimizers of J_ϵ have?*

To address this question, note that minimizers of J_ϵ are minimizers of a family of scaled Ginzburg–Landau functionals $\mathfrak{J}_\epsilon$ given by (1.5) with $\vartheta = \gamma\vartheta^*$,

$$\mathfrak{J}_\epsilon(\varphi) = \frac{1}{\sqrt{\epsilon}} J_\epsilon(\varphi) := \frac{1}{\sqrt{\epsilon}} \int_0^L \left(\epsilon \varphi'^2 + W(\varphi, \vartheta_\epsilon)\right) ds.$$

Hence weak* limits of minimizers of J_ϵ are minimizers of the Γ-limit $\mathfrak{J}$ given by (1.7), where $\wp_0 \mathcal{N}(\varphi)$ coincides with the weighted number of jumps $\mathcal{W}(\varphi)$ of a minimizer. It follows from (1.7) and (1.8) that weak* limits of minimizers of J_ϵ take the values ± 1 only, and are minimizers of the functional

$$2\mathcal{W}(\varphi) - \gamma \int_0^L \vartheta^* \varphi \, ds, \quad |\varphi| = 1. \tag{5.5}$$

Suppose that $\vartheta^*(t) = \sin(2\pi t/L)$ which has only two simple zeros per period. If φ does not change sign then $\varphi \equiv \pm 1$ and the minimum of (5.5) is zero. If minimizers of (5.5) have jumps, they must occur at $t = 0 \bmod \pi$. However, if φ changes sign twice then (5.5) becomes $16/3 \pm 2\gamma L/\pi$. Hence, the minimum of (5.5) is zero if and only if $|\gamma| \leqslant 8\pi/3L$, and negative otherwise. It follows that weak* limits of minimizers of J_ϵ have no jumps when $|\gamma|$ is small and two jumps when γ is large, notwithstanding the fact that in both cases there exists a minimizer of the limiting problem with no jumps. This discrepancy is allowed for in inequality (5.4).

Step 1. The function $\mathfrak{E}(\epsilon)$. We have seen in Theorem 4.1 that $\mathfrak{E}$ is continuous at zero. Even more is true.

Lemma 5.3. *If* (H1–5) *hold and if a sequence φ_ϵ of minimizers of problem* (2.13) *converges weak* in L^∞_{per} to a function φ, then φ is a minimizer of the relaxed problem* (2.6)*, and is therefore continuous on $\mathbb{R} \setminus \mathcal{G}^0_3(\vartheta)$. Moreover, $\mathcal{A}_\epsilon$ in* (2.15) *converges weakly in H^1_{per} to $\mathcal{A}_\vartheta$, defined in Lemma* 2.5.

Proof. It follows from the definitions that

$$\int_0^L W^{**}(\varphi_\epsilon,\vartheta)\,dt \leqslant \int_0^L W(\varphi_\epsilon,\vartheta)\,dt$$
$$\leqslant \int_0^L \left(\frac{\epsilon}{2}\big(B(\varphi_\epsilon)'\big)^2 + W(\varphi_\epsilon,\vartheta_\epsilon)\right)dt + \int_0^L \big(W(\varphi_\epsilon,\vartheta) - W(\varphi_\epsilon,\vartheta_\epsilon)\big)\,dt$$
$$= \mathfrak{E}(\epsilon) + \int_0^L \big(W(\varphi_\epsilon,\vartheta) - W(\varphi_\epsilon,\vartheta_\epsilon)\big)\,dt.$$

Therefore, if $\varphi_\epsilon \rightharpoonup^* \varphi$ in L^∞_{per}, from the convexity of $W^{**}(\cdot,\vartheta(s))$ which is bounded below, and the characterization of sequentially weak* lower semi-continuous functionals in L^∞_{per} [8, Theorem 6.56], the uniform convergence of ϑ_ϵ to ϑ, and the definition of $\mathfrak{E}(0)$ we find that

$$\mathfrak{E}(0) \leqslant \int_0^L W^{**}(\varphi,\vartheta)\,dt \leqslant \liminf_{\epsilon\to 0}\int_0^L W^{**}(\varphi_\epsilon,\vartheta)\,dt \leqslant \liminf_{\epsilon\to 0}\mathfrak{E}(\epsilon).$$

Moreover, from Theorem 4.1, $\mathfrak{E}$ is continuous at 0. Hence

$$\int_0^L W^{**}(\varphi,\vartheta)\,dt = \mathfrak{E}(0) = \lim_{\epsilon\to 0}\mathfrak{E}(\epsilon),$$

which proves that φ is a relaxed minimizer. Now

$$0 \leqslant \int_0^L \frac{\epsilon}{2}\big(B(\varphi_\epsilon)'\big)^2\,dt + \int_0^L \big(W(\varphi_\epsilon,\vartheta) - W^{**}(\varphi_\epsilon,\vartheta)\big)\,dt$$
$$= \mathfrak{E}(\epsilon) + \int_0^L \big(W(\varphi_\epsilon,\vartheta) - W(\varphi_\epsilon,\vartheta_\epsilon)\big)\,dt - \int_0^L W^{**}(\varphi_\epsilon,\vartheta)\,dt$$
$$\leqslant \mathfrak{E}(\epsilon) - \mathfrak{E}(0) + \int_0^L \big(W(\varphi_\epsilon,\vartheta) - W(\varphi_\epsilon,\vartheta_\epsilon)\big)\,dt \to 0$$

as $\epsilon \to 0$. Since both integrands on the first line are non-negative, it follows that

$$\frac{\epsilon}{2}\int_0^L \big(B(\varphi_\epsilon)'\big)^2\,dt \to 0 \quad \text{as } \epsilon \to 0.$$

By (2.16), $\{\mathcal{A}_\epsilon\colon \epsilon\in(0,1)\}$, defined in (2.15), is bounded in H^1_{per}. Moreover,

$$\int_0^L \mathcal{A}_\epsilon(s)\,ds = \mathfrak{E}(\epsilon) - \epsilon\int_0^L \big(B(\varphi_\epsilon)'\big)^2\,dt \to \mathfrak{E}(0) = \int_0^L W^{**}(\varphi,\vartheta)\,ds \tag{5.6}$$

as $\epsilon \to 0$. However, for $\epsilon > 0$,

$$\mathcal{A}_\epsilon = W(\varphi_\epsilon,\vartheta_\epsilon) - \frac{\epsilon}{2}\big(B(\varphi_\epsilon)'\big)^2$$
$$\geqslant W^{**}(\varphi_\epsilon,\vartheta) + W(\varphi_\epsilon,\vartheta_\epsilon) - W(\varphi_\epsilon,\vartheta) - \frac{\epsilon}{2}\big(B(\varphi_\epsilon)'\big)^2$$
$$\geqslant W^{**}(\varphi,\vartheta) + W(\varphi_\epsilon,\vartheta_\epsilon) - W(\varphi_\epsilon,\vartheta) - \frac{\epsilon}{2}\big(B(\varphi_\epsilon)'\big)^2.$$

Taking limits almost everywhere on the right, for a sequence of $\epsilon \to 0$ we find that

$$\liminf_{\epsilon\to 0} \mathcal{A}_\epsilon(s) \geqslant W^{**}\big(\varphi(s), \vartheta(s)\big) \tag{5.7}$$

almost everywhere. Along with (5.6) and Fatou's lemma, this implies equality almost everywhere in (5.7) for that sequence. Since $\mathcal{A}_\vartheta$ coincides with $W^{**}(\varphi, \vartheta)$ almost everywhere and since $\{\mathcal{A}_\epsilon\}$ is bounded in H^1_{per}, it follows at once that $\mathcal{A}_\epsilon$ converges weakly in H^1_{per} to $\mathcal{A}_\vartheta$. □

Step 2. Monotonicity trick. The following lemma is similar to Struwe's monotonicity argument [14, Chapter II, Section 9]. Denote by $\mathfrak{M}(\epsilon)$, $\epsilon \in (0,1)$, the set of all minimizers φ_ϵ of the variational problem (2.13) and let

$$\mathfrak{B}(\epsilon) = \sup_{\varphi_\epsilon \in \mathfrak{M}(\epsilon)} \frac{1}{2}\int_0^L \big(B(\varphi_\epsilon)'\big)^2\, ds.$$

We note from (H6) that $\mathfrak{E}$ is locally absolutely continuous and hence its derivative $\mathfrak{E}'(\epsilon)$ exists almost everywhere.

Lemma 5.4. *There exists a function $\Lambda_1 : (0,1) \to [0,\infty)$ with*

$$\limsup_{\epsilon\to 0} \sqrt{\epsilon}\,\Lambda_1(\epsilon) \leqslant C_W \Theta_1 \quad \text{and} \quad \mathfrak{B}(\epsilon) \leqslant \mathfrak{E}'(\epsilon) + \Lambda_1(\epsilon)$$

for almost all $\epsilon \in (0,1]$. Hence, $\sqrt{\epsilon}\,\mathfrak{E}'(\epsilon)$ is essentially bounded below as $\epsilon \to 0$.

Proof. Choose an arbitrary $\epsilon \in (0,1]$, $\lambda \in (0,\epsilon)$ and $\varphi_\epsilon \in \mathfrak{M}(\epsilon)$. Then

$$\begin{aligned}
\frac{\lambda}{2}\int_0^L \big(B(\varphi_\epsilon)'\big)^2\, ds &= \int_0^L \bigg(\frac{\epsilon}{2}\big(B(\varphi_\epsilon)'\big)^2 + W\big(\varphi_\epsilon(t), \vartheta_\epsilon(t)\big)\bigg)\, dt - \int_0^L \bigg(\frac{\epsilon-\lambda}{2}\big(B(\varphi_\epsilon)'\big)^2 + W\big(\varphi_\epsilon(t), \vartheta_{\epsilon-\lambda}(t)\big)\bigg)\, dt \\
&\quad + \int_0^L \Big(W\big(\varphi_\epsilon(t), \vartheta_{\epsilon-\lambda}(t)\big) - W\big(\varphi_\epsilon(t), \vartheta_\epsilon(t)\big)\Big)\, dt \\
&\leqslant \mathfrak{E}(\epsilon) - \mathfrak{E}(\epsilon-\lambda) + \int_0^L \Big(W\big(\varphi_\epsilon(t), \vartheta_{\epsilon-\lambda}(t)\big) - W\big(\varphi_\epsilon(t), \vartheta_\epsilon(t)\big)\Big)\, dt.
\end{aligned}$$

Now

$$\frac{1}{\lambda}\left|\int_0^L \Big(W\big(\varphi_\epsilon(t), \vartheta_{\epsilon-\lambda}(t)\big) - W\big(\varphi_\epsilon(t), \vartheta_\epsilon(t)\big)\Big)\, dt\right| \leqslant \frac{C_W}{\lambda}\|\vartheta_{\epsilon-\lambda} - \vartheta_\epsilon\|_{(L^1_{\mathrm{per}})^d}.$$

Hence, by (H4)(iii) and the absolute continuity of $\mathfrak{E}$, for almost all $\epsilon \in (0,1)$,

$$\frac{1}{2}\int_0^L \big(B(\varphi_\epsilon)'\big)^2\, ds \leqslant \mathfrak{E}'(\epsilon) + C_W \Lambda(\epsilon),$$

where $\limsup_{\epsilon\to 0} \sqrt{\epsilon}\,\Lambda(\epsilon) = \Theta_1$. With $\Lambda_1 = C_W \Lambda$, the proof is complete. □

Step 3. Lebesgue density. For any $c > 0$, let

$$\mathcal{E}_c = \big\{\epsilon \in (0,1]:\ \mathfrak{E}'(\epsilon) > c/\sqrt{\epsilon}\big\}. \tag{5.8}$$

Lemma 5.5. *If $c = \mathcal{W}_{\min} + C_W \Theta_0/2 + \delta$, $\delta > 0$, then*

$$\limsup_{\tau\to 0} \frac{1}{\tau}\,\mathrm{meas}\big(\mathcal{E}_c \cap [0,\tau]\big) \leqslant 1 - \mathcal{L}(\delta).$$

In particular, if $\Theta_0 = \Theta_1 = 0$ in (H4)*, the complement of $\mathcal{E}_c$ is Lebesgue dense at 0 for all $c > \mathcal{W}_{\min}$.*

Proof. Suppose there exists a sequence $\{\tau_n\}$ and $\lambda \in (0,1)$ so that

$$\frac{1}{\tau_n}\operatorname{meas}\bigl(\mathcal{E}_c \cap (0,\tau_n]\bigr) > \lambda > 0, \quad \tau_n \to 0 \text{ as } n \to \infty.$$

Let $m_n = 0$ if $(0,\tau_n] \setminus \mathcal{E}_c$ has zero measure and

$$m_n = \operatorname{ess\,inf}\bigl\{\sqrt{\epsilon}\,\mathfrak{E}'(\epsilon)\colon \epsilon \in (0,\tau_n] \setminus \mathcal{E}_c\bigr\} \quad \text{otherwise.}$$

Then $m_n > -\infty$ by Lemma 5.4 and, from Theorem 4.1, (5.8) and (H6),

$$\int_{\mathcal{E}_c\cap(0,\tau_n]} \frac{c}{\sqrt{\epsilon}}\,d\epsilon + \int_{(0,\tau_n]\setminus\mathcal{E}_c} \frac{m_n}{\sqrt{\epsilon}}\,d\epsilon \leqslant \lim_{h\searrow 0}\int_h^{\tau_n} \mathfrak{E}'(\epsilon)\,d\epsilon = \mathfrak{E}(\tau_n) - \mathfrak{E}(0).$$

To estimate the right-hand side we use Theorem 4.2. Choose the minimizer φ_n in the hypotheses of that theorem so that $\mathcal{W}(\varphi_n) \leqslant \mathcal{W}_{\min} + (1/n)$. It then follows from Theorem 4.2 that

$$\mathfrak{E}(\tau_n) - \mathfrak{E}(0) = (2\mathcal{W}_{\min} + C_W\Theta_0)\sqrt{\tau_n} + \zeta_n, \quad \text{where } \limsup_{n\to\infty} \zeta_n/\sqrt{\tau_n} \leqslant 0.$$

Hence

$$\int_{\mathcal{E}_c\cap(0,\tau_n]} \frac{c}{\sqrt{\epsilon}}\,d\epsilon + \int_{(0,\tau_n]\setminus\mathcal{E}_c} \frac{m_n}{\sqrt{\epsilon}}\,d\epsilon \leqslant (2\mathcal{W}_{\min} + C_W\Theta_0)\sqrt{\tau_n} + \zeta_n.$$

Now set $\mu_n = \mathcal{W}_{\min} + C_W\Theta_0/2 - m_n$ and $c = \mathcal{W}_{\min} + C_W\Theta_0/2 + \delta$. Then $\delta > 0$ and

$$\delta \int_{\mathcal{E}_c\cap(0,\tau_n]} \frac{d\epsilon}{\sqrt{\epsilon}} - \zeta_n \leqslant \mu_n \int_{(0,\tau_n]\setminus\mathcal{E}_c} \frac{d\epsilon}{\sqrt{\epsilon}}.$$

Since $\operatorname{meas}(\mathcal{E}_c \cap (0,\tau_n]) > \lambda\tau_n$,

$$\int_{\mathcal{E}_c\cap(0,\tau_n]} \frac{d\epsilon}{\sqrt{\epsilon}} > \int_{(1-\lambda)\tau_n}^{\tau_n} \frac{d\epsilon}{\sqrt{\epsilon}} = 2\sqrt{\tau_n}\bigl(1 - \sqrt{1-\lambda}\bigr),$$

whence

$$2\delta\sqrt{\tau_n}\bigl(1 - \sqrt{1-\lambda}\bigr) - \zeta_n < \mu_n \int_{(0,\tau_n]\setminus\mathcal{E}_c} \frac{d\epsilon}{\sqrt{\epsilon}}.$$

Since $\liminf_{n\to 0}(-\zeta_n/\sqrt{\tau_n}) \geqslant 0$, it follows that μ_n is positive and $(0,\tau_n] \setminus \mathcal{E}_c$ has positive measure for all sufficiently large n. Moreover,

$$\int_{(0,\tau_n]\setminus\mathcal{E}_c} \frac{d\epsilon}{\sqrt{\epsilon}} < \int_0^{(1-\lambda)\tau_n} \frac{d\epsilon}{\sqrt{\epsilon}} = 2\sqrt{\tau_n}\sqrt{1-\lambda}.$$

Hence, for all large n,

$$\frac{1}{\sqrt{1-\lambda}}\left(\delta\bigl(1 - \sqrt{1-\lambda}\bigr) - \frac{\zeta_n}{2\sqrt{\tau_n}}\right) < \mu_n.$$

Since $\liminf_{n\to\infty}\{-\zeta_n/\sqrt{\tau_n}\} \geqslant 0$, we have

$$\liminf_{n\to\infty} \mu_n \geqslant \mu^* := \delta\left(\frac{1}{\sqrt{1-\lambda}} - 1\right) > 0.$$

Since $\operatorname{ess\,inf}_{(0,\tau_n]\setminus\mathcal{E}_c} \sqrt{\epsilon}\,\mathfrak{E}'(\epsilon) = \mathcal{W}_{\min} + C_W\Theta_0/2 - \mu_n$, we have proved that $(0,\tau_n] \setminus \mathcal{E}_c \neq \emptyset$ for all sufficiently large n and

$$\limsup_{n\to\infty}\Bigl\{\operatorname{ess}\inf_{(0,\tau_n]\setminus\mathcal{E}_c} \sqrt{\epsilon}\,\mathfrak{E}'(\epsilon)\Bigr\} \leqslant \mathcal{W}_{\min} + \frac{C_W\Theta_0}{2} - \mu^*.$$

Now recall from Lemma 5.4 that, for all $\epsilon \in (0,1)$, every solution φ_ϵ to the variational problem (2.13) satisfies

$$\frac{\sqrt{\epsilon}}{2}\int_0^L \big(B(\varphi_\epsilon)'\big)^2\,ds \leqslant \sqrt{\epsilon}\mathfrak{B}(\epsilon) \leqslant \sqrt{\epsilon}\mathfrak{E}'(\epsilon) + \sqrt{\epsilon}\Lambda_1(\epsilon),$$

where $\limsup_{\epsilon\to 0}\sqrt{\epsilon}\Lambda_1(\epsilon) \leqslant C_W\Theta_1$. Therefore, there exists $\epsilon_n \in (0,\tau_n)$ and a solution φ_{ϵ_n} to (2.13) such that

$$\lim_{n\to\infty}\frac{\sqrt{\epsilon_n}}{2}\int_0^L \big(B(\varphi_{\epsilon_n})'\big)^2\,ds \leqslant \mathcal{W}_{\min} + \frac{C_W\Theta_0}{2} + C_W\Theta_1 - \mu^* =: \mathcal{W}_{\min} + \nu^*, \quad \text{say.}$$

Without loss of generality we can assume that φ_{ϵ_n} converges weak* in L^∞_{per} to a function φ. Let $E = \{\epsilon_n\colon n \in \mathbb{N}\}$. By Lemma 5.3, φ is a relaxed minimizer and $\mathcal{A}_{\epsilon_n} \rightharpoonup \mathcal{A}_\vartheta$ in H^1_{per}. Hence $\mathcal{F}(\mathcal{A}_\vartheta,\vartheta) = \mathcal{G}^0_3(\vartheta)$ in (3.5). Therefore, by Theorem 3.3,

$$\sum_{s\in\mathcal{O}(E)} \wp\big(\vartheta(s)\big) \leqslant \liminf_{n\to\infty}\frac{\sqrt{\epsilon_n}}{2}\int_0^L \big(B\big(\varphi_{\epsilon_n}(s)\big)'\big)^2\,ds \leqslant \mathcal{W}_{\min} + \nu^* \tag{5.9}$$

where

$$\mathcal{O}(E) = [0,L) \cap \big\{s \in \mathcal{G}^0_3(\vartheta)\colon \text{osc-def}\,E(s) > 0\big\}.$$

Hence $\mathcal{O}(E)$ is a finite set and, by Corollary 3.2, φ is a piecewise regular minimizer with $\mathcal{Q}(\varphi) \subset \mathcal{O}(E)$, where $\mathcal{Q}(\varphi)$ is given by (2.2). Hence

$$\mathcal{W}(\varphi) = \sum_{s\in\mathcal{Q}(\varphi)} \wp\big(\vartheta(s)\big) \leqslant \mathcal{W}_{\min} + \nu^*. \tag{5.10}$$

If $\nu^* < 0$, this contradicts the definition of $\mathcal{W}_{\min}$ and we can infer that

$$\frac{C_W\Theta_0}{2} + C_W\Theta_1 \geqslant \mu^* = \delta\bigg(\frac{1}{\sqrt{1-\lambda}} - 1\bigg).$$

Hence $\lambda \leqslant 1 - \mathcal{L}(\delta)$, where $\mathcal{L}(\delta)$ is defined in (5.3), as required. □

Proof of Theorem 5.1 concluded. By Lemma 5.5, the set $E_\delta := (0,1] \setminus \mathcal{E}_c$ with $c = \mathcal{W}_{\min} + C_W\Theta_0 + \delta$ has lower Lebesgue density at least $\mathcal{L}(\delta)$ at 0. Suppose $E \subset E_\delta$ has a limit point at 0 and φ_ϵ, $\epsilon \in E$, converges weak* in L^∞_{per} to φ as $E \ni \epsilon \to 0$. From Lemma 5.3, φ is a relaxed minimizer. However, by the choice of the set E and Lemma 5.4,

$$\frac{\sqrt{\epsilon}}{2}\int_0^L \big(B\big(\varphi_\epsilon(s)\big)'\big)^2\,ds \leqslant \mathcal{W}_{\min} + C_W\Theta_0 + \delta + \sqrt{\epsilon}\Lambda_1(\epsilon) \quad \text{for all } \epsilon \in E.$$

A repeat of the argument for (5.9) and (5.10) now yields

$$\begin{aligned}\mathcal{W}(\varphi) &= \sum_{s\in\mathcal{Q}(\varphi)} \wp\big(\vartheta(s)\big) \leqslant \sum_{s\in\mathcal{O}(E)} \wp\big(\vartheta(s)\big)\\ &\leqslant \lim_{\epsilon\to 0}\frac{\sqrt{\epsilon}}{2}\int_0^L \big(B\big(\varphi_\epsilon(s)\big)'\big)^2\,ds \leqslant \mathcal{W}_{\min} + C_W\Theta_1 + C_W\Theta_0/2 + \delta,\end{aligned}$$

which completes the proof of Theorem 5.1. □

Theorem 5.6. *Suppose that* (H1–6) *hold, that* $\Theta_0 = \Theta_1 = 0$ *in* (H4) *and that* $\mathcal{N}^*$ *is defined in* (2.3). *Then, for* $\delta > 0$ *sufficiently small,* E_δ *in Theorem* 5.1 *is Lebesgue dense at* 0 (*i.e.,* $\lambda = 1$ *in* (5.1)) *and* $\mathcal{N}(\varphi) \leqslant \mathcal{N}^*$.

Proof. This follows from Lemma 2.3 because $\mathcal{L}(\delta) = 1$ in Theorem 5.1 when $\Theta_0 = \Theta_1 = 0$. □

Appendix A. Simple criteria for absolute continuity of $\mathfrak{E}$

On any non-empty set X consider a variational problem

$$\mathfrak{E}(\epsilon) = \inf_{x\in X}\bigl\{\epsilon f(x) + g(\epsilon, x)\bigr\},$$

where $f : X \to \mathbb{R}$ and $g : (0,1] \times X \to \mathbb{R}$ are arbitrary functions.

Theorem A.1. *Suppose that there is a set $E \subset X$ and a function $h : (0,1] \to \mathbb{R}$ which is locally absolutely continuous with the following properties*:

(A) *for each $\epsilon \in (0,1]$ there exists $x_\epsilon \in E$ with $\mathfrak{E}(\epsilon) = \epsilon f(x_\epsilon) + g(\epsilon, x_\epsilon) > -\infty$;*
(B) *for all $x \in E$, $\epsilon \mapsto g(\epsilon, x) + h(\epsilon)$ is concave on* $(0,1]$.

Then $\mathfrak{E}$ is locally absolutely continuous on $(0,1]$.

Proof. For $\epsilon \in (0,1]$, let

$$\mathfrak{H}(\epsilon) = \inf_{x\in X}\bigl\{\epsilon f(x) + g(\epsilon, x) + h(\epsilon)\bigr\} = \mathfrak{E}(\epsilon) + h(\epsilon). \tag{A.1}$$

By (A), for $\epsilon \in (0,1]$,

$$\mathfrak{H}(\epsilon) = \inf_{x\in E}\bigl\{\epsilon f(x) + g(\epsilon, x) + h(\epsilon)\bigr\} \tag{A.2}$$

and, by hypothesis (B), $\mathfrak{H}$ is concave, and hence locally absolutely continuous, on $(0,1]$. The result follows from (A.1), since h is locally absolutely continuous. □

Corollary A.2. *If* (5.2) *holds, $\mathfrak{E}$ is locally absolutely continuous on* $(0,1]$.

Proof. In the preceding theorem let $X = H^1_{\mathrm{per}}$, for $\varphi \in X$ let

$$f(\varphi) = \frac{1}{2}\int_0^L \bigl(B(\varphi)'\bigr)^2\,ds, \qquad g(\epsilon, \varphi) = \int_0^L W(\varphi, \vartheta_\epsilon)\,ds$$

and let $E = \{\varphi_\epsilon\colon \epsilon \in (0,1]\}$ where φ_ϵ is a solution of (2.13). Because of (5.2), the hypotheses (A) and (B) are satisfied and the result follows. □

Theorem A.3. *Suppose that* (A) *holds and, for all $\delta \in (0,1)$,*

(C) *$\{f(x_\epsilon)\colon \epsilon \in [\delta, 1]\}$ is a bounded set*;
(D) *for $\widehat{\epsilon} \in [\delta, 1]$, $\epsilon \mapsto g(\epsilon, x_{\widehat{\epsilon}})$ is Lipschitz continuous on $[\delta, 1]$ with a Lipschitz constant that is independent of ϵ, $\widehat{\epsilon} \in [\delta, 1]$.*

Then $\mathfrak{E}$ is locally absolutely continuous on $(0,1]$.

Proof. Since

$$\epsilon_1 f(x_{\epsilon_1}) + g(\epsilon_1, x_{\epsilon_1}) = \epsilon_2 f(x_{\epsilon_1}) + g(\epsilon_2, x_{\epsilon_1}) + (\epsilon_1 - \epsilon_2) f(x_{\epsilon_1}) + g(\epsilon_1, x_{\epsilon_1}) - g(\epsilon_2, x_{\epsilon_1})$$

it follows that

$$\mathfrak{E}(\epsilon_2) - \mathfrak{E}(\epsilon_1) \leqslant (\epsilon_2 - \epsilon_1) f(x_{\epsilon_1}) + g(\epsilon_2, x_{\epsilon_1}) - g(\epsilon_1, x_{\epsilon_1}).$$

Similarly,

$$\mathfrak{E}(\epsilon_1) - \mathfrak{E}(\epsilon_2) \leqslant (\epsilon_1 - \epsilon_2) f(x_{\epsilon_2}) + g(\epsilon_1, x_{\epsilon_2}) - g(\epsilon_2, x_{\epsilon_2}).$$

It follows from (C) and (D) that $\mathfrak{E}$ is Lipschitz continuous on $[\delta, 1]$, for all $\delta \in (0,1)$. Hence it is locally absolutely continuous on $(0,1]$, as required. □

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