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# Positive solutions for the $p$-Laplacian involving critical and supercritical nonlinearities with zeros

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**Abstract**

In this paper we show the existence of multiple solutions to a class of quasilinear elliptic equations when the continuous nonlinearity has a positive zero and it satisfies a $p$-linear condition only at zero. In particular, our approach allows us to consider superlinear, critical and supercritical nonlinearities.


## 1. Introduction

In this paper, we look for positive $\mathcal{C}^1(\overline{\Omega})$ weak solutions of the problem

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$P_\lambda$}$$

where $\Omega$ is a convex bounded domain in $\mathbb{R}^N$ with smooth boundary, $N > p > 1$, $\lambda$ is a positive parameter and $f$ satisfies $f(0) = f(1) = 0$, $f(x) > 0$ for any $x \notin \{0; 1\}$; we will show the existence of at least two positive solutions for $\lambda$ large, without restrictions on the growth of the nonlinearity at infinity.

It is known from [17,18] that if the domain $\Omega$ is star-shaped and the nonlinearity is $|u|^{r-2}u$ with $r$ greater or equal to the critical exponent $p^* = pN/(N - p)$, then no nontrivial solution exists. A solution could be recovered either by considering more topologically complex domains, or by perturbing the nonlinearity. In this second direction several authors considered nonlinearities with any growth at infinity but which behave like $|u|^{q-2}u$ with $q \in (p, p^*)$ near zero; for instance, [3,16,9] assume this type of condition, then they truncate the nonlinearity and look for estimates on the possible solutions. These estimates allow to prove that the solutions are below the truncation point for suitable values of $\lambda$, and then a solution of the original problem is obtained.

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Problems with a nonlinearity which is nonnegative but has a zero at a positive value were first considered in [14] for the Laplacian operator, and two solutions were obtained, through topological degree arguments, in the subcritical case. The existence and behavior of a solution below the zero of $f$ are studied in many works (see for instance [7] and references therein), and it can be proved that this solution converges to 1 (the positive zero of the nonlinearity), when $\lambda \to \infty$. The existence of a solution whose maximum is above 1 is more delicate and usually requires some hypotheses on the growth of $f$ at infinity. In [11], we considered the $p$-Laplacian operator and we allowed $f$ to depend also on the variable $x \in \Omega$, but only in the subcritical case: two positive solutions were obtained for $\lambda$ above the first eigenvalue of the asymptotical problem at the origin, and it was proved that both solutions converge at least pointwise to 1 when $\lambda \to \infty$.

This behavior suggests that also for this problem, truncation procedures like those in [3,16,9] could be used to prove the existence of two solutions when considering critical or supercritical nonlinearities. However, the pointwise convergence is not enough to guarantee a suitable control on the $L^\infty$ norm of the solutions.

In this paper we suppose that $\Omega$ is convex and that $f$ is independent of $x \in \Omega$, in order to use suitable monotonicity results (such as [2,6]) which imply a better knowledge of the geometry of the solutions, and then allow to estimate the $L^\infty$ norm when $\lambda \to \infty$ and finally obtain the existence result for critical or supercritical nonlinearities. Even in the subcritical case, our result gives new information with respect to [11] in the sense that we may substitute global hypotheses on $f$ with (much weaker) local ones (see Remark 1.1).

We remand to [11] for a further discussion on the literature related to $(P_\lambda)$.

### 1.1. Statement of the results

We will consider the following hypotheses on $f$.

$(F_1)$ $f : [0, +\infty) \to [0, +\infty)$ is a continuous function which is locally Lipschitz continuous in $(0, \infty)$, $f(0) = f(1) = 0$ and $f(x) > 0$ for $x \notin \{0; 1\}$.

$(F_2)$ $\liminf_{s\to 0^+} \frac{f(s)}{s^{p-1}} \geqslant 1$.

$(F_3)$ There exist $\gamma > 0$ and $\sigma \in (p-1, p_* - 1)$ such that

$$\lim_{t\to 1} \frac{f(t)}{|t-1|^\sigma} = \gamma,$$

where $p_*$ denotes the Serrin's exponent given by $p_* = \frac{(N-1)p}{N-p}$.

$(F_4)$ There exist $k > 0$ and $T > 1$ such that the map $t \mapsto f(t) + kt^{p-1}$ is increasing for $t \in [0, T]$.

Our result is the following

**Theorem 1.1.** *Assume that $\Omega$ is a convex smooth domain. Then, under the hypotheses $(F_1)$ through $(F_4)$, there exists $\lambda^* > 0$ such that the problem $(P_\lambda)$ has at least two $\mathcal{C}^1$ weak positive solutions $u_{1,\lambda}$, $u_{2,\lambda}$, for $\lambda > \lambda^*$.*
*Moreover, these solutions satisfy $\|u_{1,\lambda}\|_\infty \to 1^-$ and $\|u_{2,\lambda}\|_\infty \to 1^+$, when $\lambda \to \infty$.*

A simple example of a function $f$ satisfying the four assumptions of the preceding theorem is $f(u) = u^{p-1}e^u|1-u|^\sigma$ where $\sigma \in (p-1, p_* - 1)$.

### 1.2. Some comments on the problem

In [11], two solutions for the problem $(P_\lambda)$ were encountered by mainly variational techniques, the first solution being a local minimum (which could also be obtained via sub- and supersolutions), while the second one was obtained via mountain pass. It was also proved, using a combination of a Liouville-type theorem, a priori estimates and the blow-up technique, that both solutions tend pointwise to 1.

For the above result of pointwise convergence, a blow-up argument centered at an arbitrary fixed point in $\Omega$ and a new Liouville-type theorem in $\mathbb{R}^N$ (see Lemma 2.2) were combined. A stronger result could be achieved if one centers the blow-up at the maximum point of the solution; however, in that case we did not know if the maximum of the solutions stayed far from the boundary or not, so that the limiting problem could be in a half-space instead of $\mathbb{R}^N$, and Liouville-type theorems in the half-space are not available for the kind of nonlinearity that we are considering.

Since we are aiming to treat also supercritical nonlinearities, variational techniques cannot be used directly here. For this reason we perform a truncation of the nonlinearity and we look for solutions below the truncation point. As observed above, in order to obtain an estimate on the $L^\infty$ norm, when applying the blow-up argument, we need to be sure that the maximum point of the solutions stays far from the boundary, so that the limiting problem is defined in the whole of $\mathbb{R}^N$. This will be obtained by assuming the convexity of $\Omega$ and using [6].

However, the results in [6] hold for locally Lipschitz and strictly positive nonlinearities. The first condition imposes a restriction on $p$ and $N$ (actually, assumption $(F_3)$ is not possible for a Lipschitz function if $p < 2$ and $N$ is large, see Remark 1.1) while, due to the second condition, we will need to solve first an auxiliary problem, where a perturbation is added which makes the nonlinearity strictly positive for $u > 0$. A first solution for the perturbed problem is obtained via sub- and supersolutions, and a second one by using topological degree (see Propositions 3.3–3.4).

Finally, the first solution in Theorem 1.1 is just the same as in [11], while the second one is obtained as the limit of the solutions of the perturbed problem; since we need to distinguish these two solutions, it is crucial to know that one lies below 1 and the other does not: for this reason, instead of the mountain pass theorem used in [11], we obtain the second solution for the perturbed problem by a degree approach, which has the advantage to furnish the information that its maximum is greater than the supersolution available.

We conclude this Introduction with some remarks on the hypotheses.

**Remark 1.1.**

- Hypothesis $(F_2)$ is classical in order to have a subsolution for $\lambda$ above the first eigenvalue of the operator. On the other hand, the constant function 1 is always a supersolution for $(P_\lambda)$, but not for the perturbed problem: hypothesis $(F_3)$ will be used to obtain a family of supersolutions, in particular a supersolution strictly below 1 and one strictly above: this will help to distinguish the two solutions when taking limit.
- Hypothesis $(F_3)$ is required also to obtain inequality (2.1) for the truncated nonlinearity, which is necessary for applying the Liouville-type theorem from [11] (Lemma 2.2). We also remark that, in fact, $(F_3)$ avoids the formation of the so-called flat core (a solution which coincides with 1 in a whole open set, see [12,20] for instance); actually, if this phenomenon could occur, then it would become difficult to separate the two solutions and obtain the multiplicity result.
- Hypothesis $(F_4)$ is a standard condition required in order to apply the sub- and supersolution method and comparison principles. We remark that in [11] we had to impose hypothesis $(F_4)$ with $T = \infty$ and condition (2.1) had to be imposed directly; here these hypotheses are replaced by the local conditions $(F_3)$ and $(F_4)$, since it will be possible to verify the global conditions when performing the truncation of the nonlinearity.
- We observe that in fact Theorem 1.1 is meaningful only for $p > 4/3$, moreover, if $p \in (4/3, 2)$ we have an upper bound for the dimension $N$: actually, hypothesis $(F_3)$ is possible for a Lipschitz function only if the Serrin's exponent $p_* > 2$, which implies, for $p < 2$, that $p < N < p/(2 - p)$, and this cannot be satisfied if $p \leqslant 4/3$.

## 2. Preliminaries

We will denote by $\lambda_1$ the first eigenvalue of $(-\Delta_p)$ in $\Omega$ and by $\phi_1$ the first eigenfunction, which can be chosen positive in $\Omega$.

By hypothesis $(F_3)$ there exist $R > 1$ and $\gamma' > 0$ such that $f(t) \geqslant \gamma'|t-1|^\sigma$ for $t \in [1, R]$; without loss of generality we may suppose that $R \leqslant T$ from hypothesis $(F_4)$. Then we truncate $f$ as follows

$$f_R(t) = \begin{cases} f(t^+), & t \leqslant R, \\ \frac{f(R)}{R^\sigma} t^\sigma, & t \geqslant R, \end{cases}$$

where $t^+ = \max\{0, t\}$. With this definition, $f_R$ has a power growth at infinity with exponent below the Serrin's exponent and satisfies the following properties:

$$f_R(t) \geqslant \gamma''|t-1|^\sigma \quad \text{for } t \geqslant 1, \tag{2.1}$$

if $\gamma'' = \min\{\gamma', \frac{f(R)}{R^\sigma}\} > 0$ and

$$\text{the map } t \mapsto f_R(t) + kt^{p-1} \text{ is increasing for } t \in [0, \infty], \tag{2.2}$$

where $k$ is as in hypothesis $(F_4)$.

We consider then the auxiliary problem

$$\begin{cases} -\Delta_p u = \lambda f_R(u) + \tau (u^+)^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$Q_{\lambda,\tau}$}$$

where $\tau$ is a nonnegative parameter.

We remark that, by the strong maximum principle (see [23]), the nontrivial solutions of the problem $(Q_{\lambda,\tau})$ are positive and, by hypothesis $(F_1)$ and since $\sigma < p_* - 1$, they are in $\mathcal{C}^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ (see [8]); moreover, since $f_R \geqslant 0$, $(Q_{\lambda,\tau})$ has no positive solution if $\tau > \lambda_1$.

The following lemma is a consequence of the results in [6], and will be used in our argument.

**Lemma 2.1.** *Under the hypotheses $(F_1)$ and $(F_3)$, if $\Omega$ is convex, there exists $\delta_\Omega > 0$ which depends only on $\Omega$ (but not on $f$, $R$, $\tau$ and $\lambda$) with the following property: for any $\mathcal{C}^1(\overline{\Omega})$ weak solution $u$ of $(Q_{\lambda,\tau})$ with $\tau > 0$, there exists a point $x \in \Omega$ such that* $\operatorname{dist}(x, \partial\Omega) > \delta_\Omega$ *and $u(x) = \|u\|_\infty$.*

Our purpose will be to obtain a solution for $(Q_{\lambda,0})$ as the limit of solutions of $(Q_{\lambda,\tau})$ with $\tau > 0$, so that the conclusion of Lemma 2.1 holds also for this solution. Then it will be possible to prove that, for large $\lambda$, this is also a solution for $(P_\lambda)$; for this we will make use of the following result, which (using the fact that $f_R$ was constructed satisfying (2.1) and with power growth with exponent below $p_* - 1$) is a consequence of the Liouville-type theorem proved in [11].

**Lemma 2.2.** *Under hypotheses $(F_1)$, $(F_2)$ and $(F_3)$, any $\mathcal{C}^1$ weak solution of the problem*

$$\begin{cases} -\Delta_p w = f_R(w) \\ w \geqslant 0 \end{cases} \quad \text{in } \mathbb{R}^N, \tag{2.3}$$

*is either $w \equiv 0$ or $w \equiv 1$.*

**Remark 2.1.** We observe that hypothesis $(F_2)$ is weaker than the one appearing in [11], but the extension to include this case is straightforward.

## 3. Proofs

Our first step will be to derive some a priori estimates for the solutions of $(Q_{\lambda,\tau})$; we remark that this result holds also for $\tau = 0$.

**Lemma 3.1.** *Under hypotheses $(F_1)$ and $(F_2)$, we have*

(1) *given $\tilde{\lambda} > 0$, there exists a constant $D_{\tilde{\lambda}}$ such that, if $u \in \mathcal{C}^1(\overline{\Omega})$ is a weak solution of the problem $(Q_{\lambda,\tau})$ with $\lambda > \tilde{\lambda}$ and $\tau \geqslant 0$ then*

$$\|u\|_\infty \leqslant D_{\tilde{\lambda}};$$

(2) *given $\lambda > 0$, there exist constants $C_\lambda > 0$ and $\alpha \in (0, 1)$ such that one has also the estimate*

$$\|u\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})} \leqslant C_\lambda. \tag{3.4}$$

**Proof.** Suppose, for sake of contradiction, that there exists a sequence $\{(u_n, \lambda_n, \tau_n)\}_{n\in\mathbb{N}}$ with $u_n$ being a positive $\mathcal{C}^1$-solution of $(Q_{\lambda_n,\tau_n})$, such that $S_n := \max_{\overline{\Omega}} u_n = u_n(x_n) \xrightarrow[n\to\infty]{} \infty$, where $\{x_n\} \subset \Omega$ is a sequence of points where the maximum is attained. We remark that since we are not supposing $\tau > 0$ at this point, this sequence may not be bounded away from the boundary.

Let now $\delta_n = \operatorname{dist}(x_n, \partial\Omega)$ and define $w_n(y) = S_n^{-1} u_n(A_n y + x_n)$, where $A_n$ will be fixed later; then $w_n$ satisfies

$$-\Delta_p w_n(y) = \lambda_n \frac{A_n^p}{S_n^{p-1}} f_R\big(S_n w_n(y)\big) + \tau_n A_n^p w_n(y) \quad \text{in } B\big(0, \delta_n A_n^{-1}\big) \tag{3.5}$$

and $w_n(0) = \max w_n = 1$.

We choose $A_n^p = \lambda_n^{-1} S_n^{p-1-\sigma} f(R)^{-1} R^\sigma$: since $S_n \to \infty$, $\lambda_n > \tilde{\lambda}$ and $\tau_n \leqslant \lambda_1$ (since no positive solution of $(Q_{\lambda,\tau})$ exists for $\tau > \lambda_1$), we conclude that $A_n \to 0$ and $\tau_n A_n^p \to 0$. Then, the right hand side of (3.5) becomes $\frac{R^\sigma f_R(S_n w_n)}{f(R) S_n^\sigma} + o(1)$ and then by the continuity of $f$ and the definition of $f_R$ it is bounded. This allows us to apply the regularity theorem in [22] for the $p$-Laplacian operator, indeed, if $\Omega_n$ is the rescaled domain then, according to whether the limit of $\delta_n/A_n$ is infinity or not, $\Omega_n$ tends to $\mathbb{R}^N$ or to a half-space; fixed an open subset $\widetilde{\Omega}$ such that $\widetilde{\Omega} \subseteq \Omega_n$ for $n$ large, for any compact set $\Omega' \subseteq \widetilde{\Omega}$ one obtains $\alpha \in (0,1)$ and $C > 0$ such that, since $w_n$ is also uniformly bounded in $L^\infty$, the estimate $\|w_n\|_{\mathcal{C}^{1,\alpha}(\Omega')} \leqslant C$ holds.

Then, up to a subsequence, $w_n \to w$ in the $\mathcal{C}^1$ norm in compact sets, where $w$ is a $\mathcal{C}^1$-function defined on $\mathbb{R}^N$ or on a half-space.

Finally, taking the limit in (3.5), we have that $w$ satisfies, in the weak sense, the following:

$$\begin{cases} -\Delta_p w = w^\sigma, \\ w > 0, \\ w(0) = \max w = 1; \end{cases}$$

this contradicts the Liouville-type theorem in [19, Corollary II] in the case of $\mathbb{R}^N$ and in [15] for the half-space.

This contradiction proves that $\|u\|_\infty \leqslant C$ for any solution of the problem $(Q_{\lambda,\tau})$ with $\lambda > \tilde{\lambda}$ and $\tau \geqslant 0$, that is, the item (1) of Lemma 3.1, and also for any solution with a given $\lambda$ and $\tau \geqslant 0$. In this second case, using the regularity theorem in [13], one obtains the uniform bound also for the $\mathcal{C}^{1,\alpha}$ norm, as claimed in the item (2). □

Now we look for a family of supersolutions: for this purpose, let $e \in W_0^{1,p}(\Omega)$ be the solution of

$$\begin{cases} -\Delta_p e = 1 & \text{in } \Omega, \\ e = 0 & \text{on } \partial\Omega \end{cases}$$

and $n := \|e\|_\infty$.

**Lemma 3.2.** *Under hypothesis $(F_3)$, for any $\lambda > 0$ there exist $\tau_\lambda^*, \delta_\lambda > 0$ such that $v_\xi = 1 + \xi + \frac{\delta_\lambda}{4n} e$ is a supersolution for $(Q_{\lambda,\tau})$ for any $\xi \in [-\delta_\lambda, \delta_\lambda/2]$ and $\tau \in [0, \tau_\lambda^*)$. Moreover, we may choose $\delta_\lambda$ as a nonincreasing function of $\lambda$.*

**Proof.** Fixed $\lambda > 0$, by the hypothesis $(F_3)$ we have that

$$\lim_{t \to 1} \frac{\lambda f_R(t)}{|t-1|^{p-1}} = 0,$$

and then there exists $\delta > 0$ such that $\lambda f_R(t) < (\frac{|t-1|}{8n})^{p-1} < (\frac{\delta}{8n})^{p-1}$ for $|t-1| \leqslant \delta$. Since this estimate still holds for lower values of $\lambda$ we deduce that $\delta$ may be chosen as a nonincreasing function of $\lambda$.

If $\tau^* > 0$ is such that $\tau u^{p-1} < (\frac{\delta}{8n})^{p-1}$ for $\tau \in [0, \tau^*)$, $u \in (0, 1+\delta]$, then

$$\lambda f_R(u) + \tau u^{p-1} < \left(\frac{\delta}{4n}\right)^{p-1} \quad \text{for } \tau \in \big[0, \tau^*\big),\ u \in [1-\delta, 1+\delta].$$

If we define $v_\xi = 1 + \xi + \frac{\delta}{4n} e$, we have that $v_\xi \in [1-\delta, 1+\delta]$ provided $\tau \in [0, \tau^*)$, $\xi \in [-\delta, \delta/2]$ and then

$$-\Delta_p v_\xi = \left(\frac{\delta}{4n}\right)^{p-1} > \lambda f_R(v_\xi) + \tau v_\xi^{p-1},$$

which proves that $v_\xi$ is a supersolution. □

Now we prove the existence of a first solution for $(Q_{\lambda,\tau})$ via the sub- and supersolution method: for this we need hypothesis $(F_4)$.

**Proposition 3.3.** *If hypotheses* $(F_1)$–$(F_4)$ *hold, then the problem* $(Q_{\lambda,\tau})$ *has a positive solution* $u_{1,\lambda,\tau} < 1$ *for* $\lambda > \lambda_1$ *and* $0 \leqslant \tau < \tau^*_\lambda$.

*Moreover, the following property holds: given* $\bar\lambda > \lambda_1$ *there exists* $\varepsilon > 0$ *such that* $\varepsilon\phi_1 \leqslant u_{1,\lambda,\tau} < 1$ *for any* $\lambda > \bar\lambda$ *and* $\tau \in [0, \tau^*_\lambda)$.

**Proof.** In a standard way, using $(F_2)$, we may find a $\varepsilon > 0$ (as small as desired) such that $\lambda f_R(t) > \lambda_1 t^{p-1}$ for any $t \in (0, \max\{\varepsilon\phi_1\})$ and any $\lambda > \bar\lambda > \lambda_1$; then $\varepsilon\phi_1$ is a subsolution for the problem $(Q_{\lambda,\tau})$ for any $\tau \geqslant 0$ and $\lambda > \bar\lambda$.

For $\tau \in [0, \tau^*_\lambda)$, we have the supersolution $v_{-\delta_\lambda} < 1$ from Lemma 3.2; since $\delta_\lambda$ is not increasing in $\lambda$, we may choose $\varepsilon$ such that $\varepsilon\phi_1 < v_{-\delta_\lambda/2}$ for any $\lambda > \bar\lambda$. Then the sub- and supersolutions method gives a solution $u_{1,\lambda,\tau}$ with the claimed properties. □

Now, we work with $\tau > 0$ and we show that a second solution exists: we will apply a topological degree argument, adapting a result obtained, for $p = 2$, by de Figueiredo and Lions in [4], see also [10] for the general case.

**Proposition 3.4.** *In the same hypotheses as Proposition* 3.3*, if* $\lambda > \lambda_1$ *and* $\tau_0 \in (0, \tau^*_\lambda)$*, then* $(P_{\lambda,\tau_0})$ *has a second positive solution* $u_{2,\lambda,\tau_0}$. *Moreover* $\|u_{2,\lambda,\tau_0}\|_\infty > 1$.

**Proof.** Let us fix $\lambda > \lambda_1$ and denote by $X$ the Banach space of $\mathcal{C}^1$-functions on $\overline{\Omega}$ which are 0 on $\partial\Omega$, endowed with the usual $\mathcal{C}^1$-norm. Also, we will write $u \ll v$ to say that $u < v$ in $\Omega$ and $\frac{\partial u}{\partial \nu} > \frac{\partial v}{\partial \nu}$ on $\partial\Omega$, where $\nu$ denotes the unitary outward normal to $\partial\Omega$. Let $k$ be as in (2.2) and $K_\tau : X \to X$ be defined as follows: $K_\tau v = u$, where $u$ is the unique solution of the Dirichlet problem

$$\begin{cases} -\Delta_p u + \lambda k u^{p-1} = \lambda f_R(v) + (\lambda k + \tau) v^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega; \end{cases} \tag{3.6}$$

the mapping $K_\tau$ so defined is compact.

We consider the bounded open set

$$\mathcal{O} = \{u \in X\colon\ \|u\|_X < C_\lambda + B_\lambda + 1,\ u \gg \varepsilon\phi_1\},$$

where $C_\lambda, B_\lambda > 0$ will be chosen below (see in (3.7) and (3.9), respectively) and $\varepsilon > 0$ is as in the proof of Proposition 3.3, so that $\varepsilon\phi_1 < 1$ and it is a strict subsolution for all problems $(Q_{\lambda,\tau})$, $\tau \geqslant 0$ (in particular $\lambda_1(\varepsilon\phi_1)^{p-1} < \lambda f_R(\varepsilon\phi_1)$).

We need that $0 \notin (I - K_\tau)(\partial\mathcal{O})$ (i.e., no solution of $(Q_{\lambda,\tau})$ lies on $\partial\mathcal{O}$), so that the degree $\deg(I - K_\tau, \mathcal{O}, 0)$ will be well defined and independent of $\tau$. To obtain this we get $C_\lambda$ from Lemma 3.1 part (2), so that

$$\|u\|_X \leqslant C_\lambda \tag{3.7}$$

for all possible solutions of $(Q_{\lambda,\tau})$ with $\tau \geqslant 0$.

Then, we claim that any solution $u$ of $(Q_{\lambda,\tau})$ such that $u \geqslant \varepsilon\phi_1$ in $\Omega$ satisfies $u \gg \varepsilon\phi_1$ (and then it is not on $\partial\mathcal{O}$).

Actually, we have

$$\begin{cases} -\Delta_p u + \lambda k u^{p-1} = \lambda f_R(u) + (\lambda k + \tau) u^{p-1}, \\ -\Delta_p(\varepsilon\phi_1) + \lambda k(\varepsilon\phi_1)^{p-1} = \lambda_1(\varepsilon\phi_1)^{p-1} + (\lambda k + \tau)(\varepsilon\phi_1)^{p-1}, \end{cases} \tag{3.8}$$

by hypothesis $(F_4)$ and since $u \geqslant \varepsilon\phi_1$, we have $\lambda f_R(u) + (\lambda k + \tau)u^{p-1} \geqslant \lambda f_R(\varepsilon\phi_1) + (\lambda k + \tau)(\varepsilon\phi_1)^{p-1}$, and then a strict inequality holds between the (continuous) right hand sides of (3.8). Thus, using the comparison result in [5], the claim is proved.

By the above computations, we obtain that

$$\deg(I - K_\tau, \mathcal{O}, 0) = 0 \quad \text{for any } \tau > 0,$$

since $(Q_{\lambda,\tau})$ has no solutions for $\tau > \lambda_1$.

At this point we fix $\tau = \tau_0$, we consider the supersolution $a := v_{\xi=0} > 1$ from Lemma 3.2, and we assume that no solution of $(Q_{\lambda,\tau_0})$ touches it, otherwise such a solution would satisfy the claim and the proposition would be true. Using the $L^\infty$ estimate in [1] and then [13] we obtain that we may choose the constant $B_\lambda > 0$ such that

$$\|K_{\tau_0} v\|_X \leqslant B_\lambda, \quad \forall v \in X\colon 0 \leqslant v \leqslant a; \tag{3.9}$$

we consider the open subset of $\mathcal{O}$

$$\mathcal{O}' = \{u \in \mathcal{O}\colon\ u < a \text{ in } \Omega\}$$

and we claim that $\deg(I - K_{\tau_0}, \mathcal{O}', 0) = 1$.

Observe that $K_{\tau_0}$ maps $\overline{\mathcal{O}'}$ into $\overline{\mathcal{O}'}$. Indeed, if $v \in \overline{\mathcal{O}'}$, then $\|K_{\tau_0} v\|_X \leqslant B_\lambda$ by (3.9), and if we consider $u = K_{\tau_0} v$ we have

$$\begin{cases} -\Delta_p a + \lambda k a^{p-1} \geqslant \lambda f_R(a) + (\lambda k + \tau_0) a^{p-1}, \\ -\Delta_p u + \lambda k u^{p-1} = \lambda f_R(v) + (\lambda k + \tau_0) v^{p-1}, \\ -\Delta_p(\varepsilon\phi_1) + \lambda k(\varepsilon\phi_1)^{p-1} = \lambda_1(\varepsilon\phi_1)^{p-1} + (\lambda k + \tau_0)(\varepsilon\phi_1)^{p-1}, \end{cases} \tag{3.10}$$

then, since $\varepsilon\phi_1 \leqslant v \leqslant a$, the comparison principle in [21] implies that $\varepsilon\phi_1 \leqslant K_{\tau_0} v \leqslant a$.

Now, let $u_0 \in \mathcal{O}'$ and consider the constant mapping $C : \overline{\mathcal{O}'} \to \overline{\mathcal{O}'}$ defined by $C(u) = u_0$: one obtains that $I - \mu K_{\tau_0}(v) - (1-\mu)u_0$, $\mu \in [0,1]$, is a homotopy between $I - K_{\tau_0}$ and $I - C$ in $\overline{\mathcal{O}'}$ without zeros on $\partial\mathcal{O}'$: in fact, if $v \in \partial\mathcal{O}'$ then (since $\mathcal{O}'$ is convex) $\mu K_{\tau_0}(v) + (1-\mu)u_0 \in \mathcal{O}'$ for $\mu \neq 1$, and then it is different from $v$, while for $\mu = 1$ we have $v \neq K_{\tau_0}(v)$ since we are assuming that no solution touches $a$.

Hence $\deg(I - K_{\tau_0}, \mathcal{O}', 0) = \deg(I - C, \mathcal{O}', 0) = 1$, as we claimed.

Then, applying the excision property, it follows that $\deg(I - K_{\tau_0}, \mathcal{O} \setminus \overline{\mathcal{O}'}, 0) = -1$, so $(Q_{\lambda,\tau_0})$ has a solution $u_2 \in \mathcal{O} \setminus \overline{\mathcal{O}'}$; in particular, $u_2(x_0) > a(x_0) > 1$ in some point $x_0 \in \Omega$, since otherwise it would be on $\partial\mathcal{O}'$, and then $u_2$ is distinct from $u_{1,\lambda,\tau_0}$ from Proposition 3.3. □

Now, we will obtain a solution for $(Q_{\lambda,0})$ as the limit of the solutions obtained in the previous proposition; as a result, such solution inherits the property in Lemma 2.1.

**Lemma 3.5.** *In the same hypotheses as Propositions* 3.3–3.4*, if moreover $\Omega$ is convex, then given $\lambda > \lambda_1$, there exists a solution $u_{2,\lambda,0}$ for the problem $(Q_{\lambda,0})$, which satisfies $\|u_{2,\lambda,0}\|_\infty \geqslant 1$.*

*Moreover, there exists $x \in \Omega$ such that $d := \operatorname{dist}(x, \partial\Omega) \geqslant \delta_\Omega$ and $u_{2,\lambda,0}(x) = \|u_{2,\lambda,0}\|_\infty$.*

**Proof.** Given $\lambda > \lambda_1$ we will consider a sequence $\tau_n \to 0$ and we will focus on the solution $u_n := u_{2,\lambda,\tau_n}$ from Proposition 3.4, so that we know that $\|u_n\|_\infty > 1$, and that, by Lemma 2.1, there exists $x_n \in \Omega$ such that $d_n := \operatorname{dist}(x_n, \partial\Omega) > \delta_\Omega$ and $u_n(x_n) = \|u_n\|_\infty$.

By Lemma 3.1 point (2), we have a uniform bound for $\|u_n\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})}$ for some $\alpha \in (0,1)$. Then, up to a subsequence, $u_n \to u$ in $\mathcal{C}^1(\overline{\Omega})$, where $u$ is a nonnegative weak solution of $(Q_{\lambda,0})$.

From $\|u_n\|_\infty > 1$ we obtain $\|u\|_\infty \geqslant 1$, thus $u$ is nontrivial and then positive. Finally, up to a subsequence, $x_n \to x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) \geqslant \delta_\Omega$ and taking limit $u(x) = \|u\|_\infty$. □

The following lemma will show that, for $\lambda$ large, the solution from Lemma 3.5 is a solution also for the supercritical problem $(P_\lambda)$.

**Lemma 3.6.** *The solutions $u_{2,\lambda,0}$ from Lemma* 3.5 *satisfy $\|u_{2,\lambda,0}\|_\infty \to 1$ when $\lambda \to \infty$.*

*In particular, there exists $\lambda^*$ such that if $\lambda > \lambda^*$ then $\|u_{2,\lambda,0}\|_\infty \leqslant R$.*

**Proof.** Given $\eta > 1$, suppose by contradiction that there exists a sequence $\lambda_n \to \infty$ such that the corresponding solutions $u_n := u_{2,\lambda_n,0}$ satisfy $\|u_n\| > \eta$, in particular there exists a sequence $x_n \in \Omega$ such that $d_n := \operatorname{dist}(x_n, \partial\Omega) \geqslant \delta_\Omega$ and $u_n(x_n) = \|u_n\|_\infty > \eta$.

Letting $w_n(x) = u_n(x_n + \lambda_n^{-\frac{1}{p}} x)$ we see that $w_n$ satisfies

$$-\Delta_p w_n(x) = f_R(w_n) \quad \text{in } B\big(0, d_n \lambda_n^{1/p}\big) \tag{3.11}$$

and $w_n(0) = u_n(x_n)$.

As in the proof of point (2) in Lemma 3.1, we obtain (since $w_n$ is bounded in $L^\infty$ by the point (1) in the same lemma) also a uniform bound in the $\mathcal{C}^{1,\alpha}$ norm in compact sets, for some $\alpha \in (0,1)$; then, up to a subsequence,

$w_n \to w$ in the $\mathcal{C}^1$ norm in compact sets, where now $w$ is a $\mathcal{C}^1$ function defined in $\mathbb{R}^N$, since $d_n \lambda_n^{1/p} \to \infty$. Thus, $w$ is a weak solution of the problem

$$\begin{cases} -\Delta_p w = f_R(w) & \text{in } \mathbb{R}^N, \\ w \geqslant 0. \end{cases} \tag{3.12}$$

According to Lemma 2.2 we conclude that either $w \equiv 0$ or $w \equiv 1$.

This contradicts the fact that $w_n(0) = u_n(x_n) > \eta > 1$, and then the lemma is proved. □

We are now in a position to prove our main result.

**Proof of Theorem 1.1.** The first solution is $u_{1,\lambda,0}$ from Proposition 3.3, and satisfies $\|u_{1,\lambda,0}\|_\infty < 1$. By Lemma 3.6 we see that, for $\lambda$ large, the solutions $u_{2,\lambda,0}$ from Lemma 3.5 satisfy $1 \leqslant \|u_{2,\lambda,0}\|_\infty < R$, and then are solutions of the supercritical problem $(P_\lambda)$. Therefore we have obtained the existence of a second solution.

We have already proved that $\|u_{2,\lambda,0}\|_\infty \to 1$ when $\lambda \to \infty$.

By hypotheses $(F_1)$ and $(F_2)$, if $t_\lambda$ is the largest real such that $\lambda f(t) > \lambda_1 t^{p-1}$ for $t \in (0, t_\lambda)$, then $t_\lambda \to 1$ when $\lambda \to \infty$. Since no positive solution of $(P_\lambda)$ may exist below $t_\lambda$, we deduce that also $\|u_{1,\lambda,0}\|_\infty \to 1$ when $\lambda \to \infty$. □

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