

# Some new results in competing systems with many species

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Received 3 August 2009; received in revised form 1 November 2009; accepted 1 November 2009

Available online 10 November 2009

## Abstract

In this paper, we prove some uniqueness and convergence results for a competing system and its singular limit, and an interior measure estimate of the free boundary for the singular limit.

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MSC: 35B40; 35R35; 92D25; 35K57; 58E20

Keywords: Competing species; Free boundary problem; Harmonic map into singular space

## 1. Introduction

The Lotka–Volterra model of competing species describes the competition of a number of species in a fixed domain. Its general form is as following:

$$\begin{cases} \frac{\partial u_i}{\partial t} - \Delta u_i = f_i(u_i) - u_i \sum_{j \neq i} b_{ij} u_j, & \text{in } \Omega \times (0, +\infty), \\ u_i = \phi_i, & \text{on } \Omega \times \{0\}, \end{cases}$$

where  $b_{ij} \geq 0$  are constants and  $1 \leq i, j \leq M$ , and  $M$  is the number of the species and  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  ( $n \geq 1$ ) with smooth boundary. Usually we consider homogeneous Dirichlet or Neumann boundary condition. We only study nonnegative solutions, that is,  $u_i \geq 0$  for all  $i$ .

The study of this reaction–diffusion systems has a long history and there exist a great amount of works. However, most of these works are concerned with the case of two species. As far as we know, the study in case of many competing species is not so much, in 1990s Dancer and Du studied three species competition systems and got very interesting existence results. In fact, it's believed that generally this system has complicated dynamics (see [8,9]), even in the ordinary differential equation cases (see [15]).

In recent years, people show a lot of interests in strongly competing systems with many species, that is, the system (or its elliptic case)

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<sup>1</sup> Supported by National Natural Science Foundation of China (10671195, 10831005, 10971046).

$$\frac{\partial u_i}{\partial t} - \Delta u_i = -\kappa u_i \sum_{j \neq i} b_{ij} u_j,$$

where  $\kappa$  is sufficiently large (or its limit at  $\kappa = +\infty$ ). Conti, Terracini and Verzini [5,6], Caffarelli, Karakhanyan and Lin [1,2], etc., established the regularity of the singular limit (and the partial regularity of its free boundary) as  $\kappa \rightarrow +\infty$  and the uniform regularity for all  $\kappa > 0$ . Conti, Terracini and Verzini find that in the singular limit species are spatially segregated and they satisfy a remarkable system of differential inequalities, and these two conditions are also satisfied by the solution of a variational problem. Although it's not fully established, it's very possible that this singular limit has a variational structure. That is, the solution of corresponding elliptic problem is the harmonic map from the domain  $\Omega$  into a metric space  $\Sigma$  with nonpositive curvature, which has been studied by many authors since the work of [11]. Here the metric space  $\Sigma$  is defined as follows:

$$\Sigma := \{(u_1, u_2, \dots, u_M) \in \mathbb{R}^M : u_i \geq 0, u_i u_j = 0 \text{ for } i \neq j\}.$$

Under the intrinsic metric structure, it is a metric space of nonpositive curvature (for the definition, please see [11]). The harmonic map is the critical point (in weak sense) of the following functional

$$\sum_i \int_{\Omega} |\nabla u_i|^2,$$

defined in the class of functions  $u = (u_1, u_2, \dots, u_M) \in (H^1(\Omega))^M$  satisfying  $u_i \geq 0$  and  $u_i u_j = 0$ , a.e., see [5].

In this paper, we present some results concerning this problem. First we prove the uniqueness result of the following Dirichlet boundary value problem of elliptic systems in a smooth domain  $\Omega$  in  $\mathbb{R}^n$  for  $\forall n \geq 1$ :

$$\begin{cases} \Delta u_i = \kappa u_i \sum_{j \neq i} b_{ij} u_j, & \text{in } \Omega, \\ u_i = \varphi_i, & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

Here  $b_{ij} > 0$  are constants and satisfy  $b_{ij} = b_{ji}$ ,  $\varphi_i$  are given Lipschitz continuous functions on  $\partial\Omega$ , which satisfy  $\varphi_i \geq 0$ . In the paper we will simply take  $b_{ij} = 1$ , without loss of generality.

In the paper [6], they prove the existence of the positive solution of (1.1), using Leray–Schauder degree theory. However, the uniqueness of the solution was not known. Here, we will use the sub- and sup-solution method to show that the uniqueness is indeed right. That is

**Theorem 1.1.**  $\forall \kappa \geq 0$ , there exists a unique positive solution  $(u_1, \dots, u_M)$  of (1.1).

The application of the sub- and sup-solution method in nonlinear elliptic systems was known for a long time, see for example [13]. Our main contribution here is a simple observation which leads to the uniqueness in our current situation.

This method can also be applied to the parabolic case. We consider the parabolic analogue of Eq. (1.1), that is, the following initial–boundary value problem:

$$\begin{cases} \frac{\partial u_i}{\partial t} - \Delta u_i = -\kappa u_i \sum_{j \neq i} b_{ij} u_j, & \text{in } \Omega \times (0, +\infty), \\ u_i = \varphi_i, & \text{on } \partial\Omega \times (0, +\infty), \\ u_i = \phi_i, & \text{on } \Omega \times \{0\}. \end{cases} \quad (1.2)$$

Here  $\varphi_i$  are given Lipschitz continuous functions on  $\partial\Omega$ , which satisfy  $\varphi_i \geq 0$ ; and  $\phi_i$  are given Lipschitz continuous functions on  $\Omega$ , which satisfy  $\phi_i \geq 0$  and  $\phi_i = \varphi_i$  on  $\partial\Omega$ . We have the following theorem:

**Theorem 1.2.**  $\forall \kappa \geq 0$ , the solution  $(u_1(x, t), \dots, u_M(x, t))$  of (1.2) exists globally, and it converges to the (unique) solution of the stationary equation (1.1) as  $t \rightarrow +\infty$  (in  $C(\overline{\Omega})$ ).

Then we give a uniform Lipschitz estimate, using Kato inequality (this inequality was also used in [10]) and our observation from the symmetric assumption  $b_{ij} = b_{ji}$ :

**Theorem 1.3.** *There exists a constant  $C$  independent of  $\kappa$ , such that for any solution  $u_{i,\kappa}$  of (1.2) we have*

$$\sup_{\Omega \times [0, +\infty)} \text{Lip}(u_{i,\kappa}) \leq C.$$

The elliptic case can be treated similarly.

**Theorem 1.4.** *There exists a constant  $C$  independent of  $\kappa$ , such that for solution  $u_{i,\kappa}$  of (1.1) we have*

$$\sup_{\Omega} |\nabla u_{i,\kappa}| \leq C.$$

Next we consider the uniqueness of the singular limit of (1.1) as  $\kappa \rightarrow +\infty$ . We know that, as  $\kappa \rightarrow +\infty$ , solutions of (1.1) converge to some  $(u_1, \dots, u_M)$  which satisfy the following conditions (see [6]):

$$\begin{cases} \Delta u_i \geq 0, & \text{in } \Omega, \\ \Delta \left( u_i - \sum_{j \neq i} u_j \right) \leq 0, & \text{in } \Omega, \\ u_i = \varphi_i, & \text{on } \partial\Omega, \\ u_i u_j = 0, & \text{in } \Omega. \end{cases} \tag{1.3}$$

First we establish some results concerning the estimate of the  $(n - 1)$ -dimensional Hausdorff measure of the free boundary. From the regularity theory in [1], we know that  $\partial\{u_i > 0\}$  and  $\partial\{v_i > 0\}$  are smooth hypersurface except a closed set of dimension  $n - 2$ . What we show is that they have finite  $n - 1$  dimension Hausdorff measure in the interior of  $\Omega$ .

**Theorem 1.5.** *For any compact set  $\Omega' \Subset \Omega$ , we have*

$$H^{n-1}(\Omega' \cap \partial\{u_i > 0\}) < +\infty.$$

This result is valid for locally energy minimizing maps too, because it also satisfy the same conditions such as monotonicity of the frequency function (see [3]). We also establish a uniform interior estimate of the level surface.

Then we consider the uniqueness problem of (1.3). In the paper [7], the authors prove the uniqueness and least energy property of  $(u_1, \dots, u_M)$  which satisfies (1.3) in the case of  $M = 3$  and in dimension 2. Here we will generalize their result to arbitrary dimension and arbitrary number of species.

**Theorem 1.6.** *Given a solution  $(u_1, \dots, u_M)$  of (1.3), it must be the harmonic map into the space  $\Sigma$ .*

By definition the harmonic map is the critical point of the energy functional  $\int_{\Omega} \sum_i |\nabla u_i|^2 dx$  (under the same boundary condition, see [11] or [3]). Because this functional is convex with respect to the geodesic homotopy, then it must be the (unique) energy minimizing map. The uniqueness of energy minimizer has been proved by M. Conti, S. Terracini and G. Verzini in [5], see their Theorem 4.2. We also use the construction of the test functions in their proof in our Section 6.

Our method is to compute the derivative of the energy functional with respect to the geodesic homotopy between  $u$  and a comparison (an energy minimizing map  $v$  with same boundary values). This involves some procedures of integration by parts. In order to make this procedure rigorous, we first calculate in an approximate setting where we can avoid the free boundary which may contain singularity, at this stage we can also cancel (or place a good control on) the terms which involve the integration on the free boundary of  $v$ . This control is necessary, even with our knowledge on the interior measure estimate of the free boundary of  $v$ , because we do not have the corresponding estimate near

the boundary (this may be true, if we can choose the boundary value good enough). At last, for those integration on the free boundary of  $u$ , we can control them well enough and after take the limit, they all cancel.

At last, we consider the uniqueness of the initial–boundary value problems and the asymptotic of following singular limit of (1.2):

$$\begin{cases} \frac{\partial u_i}{\partial t} - \Delta u_i \leq 0, & \text{in } \Omega \times (0, +\infty), \\ \left(\frac{\partial}{\partial t} - \Delta\right)\left(u_i - \sum_{j \neq i} u_j\right) \geq 0, & \text{in } \Omega \times (0, +\infty), \\ u_i u_j = 0, & \text{in } \Omega \times (0, +\infty), \\ u_i = \varphi_i, & \text{on } \partial\Omega \times (0, +\infty), \\ u_i = \phi_i, & \text{on } \Omega \times \{0\}. \end{cases} \quad (1.4)$$

Various regularity results concerning this system are proved in [1]. We give a simple proof of the following result:

**Theorem 1.7.** *There exists a unique solution of (1.4), and it converges to the unique solution of (1.3) as  $t \rightarrow +\infty$ .*

There is another simple result which we would like to mention. A simple blow up argument shows that  $\frac{\partial u_{i,\kappa}}{\partial t}$  are uniformly bounded as  $\kappa \rightarrow +\infty$ , so for solution  $u$  of (1.4),  $\frac{\partial u_i}{\partial t}$  are bounded. Although we don't need this result in our paper, we hope it will be useful in other settings.

At last, we would like to add a remark on the symmetric assumptions on  $b_{ij}$  in the above equation. This assumption is essential for our proof. This can also be seen from the regularity results in [1], which, according to [6], may be wrong if  $b_{ij}$  is not symmetric.

The organization of this paper is as follows. In Section 2, we prove Theorem 1.1. In Section 3, we prove Theorem 1.2. In Section 3, we prove Theorem 1.3 and Theorem 1.4. The methods in these three sections are very easy. In Section 5, we prove Theorem 1.5. In Section 6, we prove Theorem 1.6. These two sections are the main part of this paper. In Section 7, we prove Theorem 1.7. This again, is a simple treatment.

## 2. Uniqueness in the elliptic case

We use the following iteration scheme to prove the uniqueness of solutions for (1.1). First, we know the following harmonic extension is possible:

$$\begin{cases} \Delta u_{i,0} = 0, & \text{in } \Omega, \\ u_{i,0} = \varphi_i, & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

that is, this equation has a unique positive solution  $u_{i,0}$ .

Then the iteration can be defined as:

$$\begin{cases} \Delta u_{i,m+1} = \kappa u_{i,m+1} \sum_{j \neq i} u_{j,m}, & \text{in } \Omega, \\ u_{i,m+1} = \varphi_i, & \text{on } \partial\Omega, \end{cases} \quad (2.2)$$

this is a linear equation, and it satisfies the Maximum Principle, so the existence and uniqueness of the solution is clear.

Now concerning these  $u_{i,m}$  we have the following result:

**Proposition 2.1.** *In  $\Omega$*

$$u_{i,0}(x) > u_{i,2}(x) > \cdots > u_{i,2m}(x) > \cdots > u_{i,2m+1}(x) > \cdots > u_{i,3}(x) > u_{i,1}(x).$$

**Proof.** We divide the proof into several claims.

**Claim 1.**  $\forall i, m, u_{i,m} > 0$  in  $\Omega$ .

Because  $\sum_{j \neq i} u_{j,0} > 0$  in  $\Omega$ , Eq. (2.2) satisfies the maximum principle. From the boundary condition  $\varphi_i \geq 0$ , then we have  $u_{i,1} > 0$  in  $\Omega$ . By induction, we see the claim is right for all  $u_{i,m}$ .

**Claim 2.**  $u_{i,1} < u_{i,0}$  in  $\Omega$ .

From the equation, now we have

$$\begin{cases} \Delta u_{i,1} \geq 0, & \text{in } \Omega, \\ u_{i,1} = u_{i,0}, & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

so we have  $u_{i,1} < u_{i,0}$  from the comparison principle.

In the following we assume the conclusion of the proposition is valid until  $2m + 1$ , that is in  $\Omega$

$$u_{i,0} > \dots > u_{i,2m} > u_{i,2m+1} > u_{i,2m-1} > \dots > u_{i,1}.$$

Then we have:

**Claim 3.**  $u_{i,2m+1} \leq u_{i,2m+2}$ .

By (2.2) we have

$$\Delta u_{i,2m+2} \leq \kappa u_{i,2m+2} \sum_{j \neq i} u_{j,2m}, \tag{2.4}$$

$$\Delta u_{i,2m+1} = \kappa u_{i,2m+1} \sum_{j \neq i} u_{j,2m}. \tag{2.5}$$

Because  $u_{i,2m+1}$  and  $u_{i,2m+2}$  have the same boundary value, comparing (2.4) and (2.5), by the comparison principle again we obtain that  $u_{i,2m+1} \leq u_{i,2m+2}$ .

**Claim 4.**  $u_{i,2m+2} \leq u_{i,2m}$ .

This can be seen by comparing the equations they satisfy:

$$\begin{cases} \Delta u_{i,2m+2} = \kappa u_{i,2m+2} \sum_{j \neq i} u_{j,2m+1}, \\ \Delta u_{i,2m} = \kappa u_{i,2m} \sum_{j \neq i} u_{j,2m-1}. \end{cases} \tag{2.6}$$

By assumption we have  $u_{j,2m+1} \geq u_{j,2m-1}$ , so the claim follows from the comparison principle again.

**Claim 5.**  $u_{i,2m+3} \geq u_{i,2m+1}$ .

This can be seen by comparing the equations they satisfy:

$$\begin{cases} \Delta u_{i,2m+3} = \kappa u_{i,2m+2} \sum_{j \neq i} u_{j,2m+2}, \\ \Delta u_{i,2m+1} = \kappa u_{i,2m+1} \sum_{j \neq i} u_{j,2m}. \end{cases} \tag{2.7}$$

By Claim 4 we have  $u_{j,2m} \geq u_{j,2m+2}$ , so the claim follows from the comparison principle again.  $\square$

Now we know that there exist two family of functions  $u_i$  and  $v_i$ , such that  $\lim_{m \rightarrow \infty} u_{j,2m}(x) = u_j(x)$  and  $\lim_{m \rightarrow \infty} u_{j,2m+1}(x) = v_j(x)$ ,  $\forall x \in \Omega$ . Moreover, from the elliptic estimate, we know this convergence is smooth in  $\Omega$  and uniformly on  $\bar{\Omega}$ . So by taking the limit in (2.2) we obtain the following equations:

$$\begin{cases} \Delta u_i = \kappa u_i \sum_{j \neq i} v_j, \\ \Delta v_i = \kappa v_i \sum_{j \neq i} u_j. \end{cases} \quad (2.8)$$

Because  $u_{i,2m+1} \leq u_{j,2m}$ , by taking limit we also have

$$v_i \leq u_i. \quad (2.9)$$

Now summing (2.8) we have

$$\begin{cases} \Delta \left( \sum_i u_i \right) = \kappa \sum_i \left( u_i \sum_{j \neq i} v_j \right), \\ \Delta \left( \sum_i v_i \right) = \kappa \sum_i \left( v_i \sum_{j \neq i} u_j \right). \end{cases} \quad (2.10)$$

It is easily seen that

$$\sum_i \left( u_i \sum_{j \neq i} v_j \right) = \sum_i v_i \left( \sum_{j \neq i} u_j \right),$$

so we must have  $\sum_i u_i \equiv \sum_i v_i$  because they have the same boundary value. This means, by (2.9),  $u_i \equiv v_i$ . In particular, they satisfy Eq. (1.1).

**Proposition 2.2.** *If there exists another positive solution  $w_i$  of (1.1), we must have  $u_i \equiv w_i$ .*

**Proof.** We will prove  $u_{i,2m} \geq w_i \geq u_{j,2m+1}$ ,  $\forall m$ , then the proposition follows immediately. We divide the proof into several claims.

**Claim 1.**  $w_i \leq u_{i,0}$ .

This is because

$$\begin{cases} \Delta w_i \geq 0, & \text{in } \Omega, \\ w_i = u_{i,0}, & \text{on } \partial\Omega. \end{cases} \quad (2.11)$$

**Claim 2.**  $w_i \geq u_{i,1}$ .

This is because

$$\begin{cases} \Delta w_i = \kappa w_i \sum_{j \neq i} w_j, \\ \Delta u_{i,1} = \kappa u_{i,1} \sum_{j \neq i} u_{j,0}. \end{cases} \quad (2.12)$$

Noting that we have  $w_j < u_{j,0}$ , so the comparison principle applies.

In the following we assume that our claim is valid until  $2m + 1$ , that is

$$u_{i,2m} \geq w_i \geq u_{i,2m+1}.$$

Then we have

**Claim 3.**  $u_{i,2m+2} \geq w_i$ .

This can be seen by comparing the equations they satisfy:

$$\begin{cases} \Delta w_i = \kappa w_i \sum_{j \neq i} w_j, \\ \Delta u_{i,2m+2} = \kappa u_{i,2m+3} \sum_{j \neq i} u_{j,2m+1}. \end{cases} \tag{2.13}$$

By assumption we have  $u_{j,2m+1} \leq w_j$ , so the claim follows from the comparison principle again.

**Claim 4.**  $u_{i,2m+3} \leq w_i$ .

This can be seen by comparing the equations they satisfy:

$$\begin{cases} \Delta w_i = \kappa w_i \sum_{j \neq i} w_j, \\ \Delta u_{i,2m+3} = \kappa u_{i,2m+3} \sum_{j \neq i} u_{j,2m+2}. \end{cases} \tag{2.14}$$

By Claim 3 we have  $u_{j,2m+2} \geq w_j$ , so the claim follows from the comparison principle again.  $\square$

**Remark 2.3.** From our proof, we know that the uniqueness result still holds for equations of more general form:

$$\begin{cases} \Delta u_i = u_i \sum_{j \neq i} b_{ij}(x) u_j, & \text{in } \Omega, \\ u_i = \varphi_i, & \text{on } \partial\Omega, \end{cases} \tag{2.15}$$

where  $b_{ij}(x)$  are positive (and smooth enough) functions defined in  $\overline{\Omega}$ , which satisfy  $b_{ij} \equiv b_{ji}$ .

### 3. Asymptotics in the parabolic case

In this section we prove Theorem 1.2. We can use the method of Section 2 to prove there exists a globally unique solution  $u_i(x, t)$ . Moreover we can get a result about the asymptotic behavior of this solution from this method.

**Proof.** Let's consider the iteration scheme analogous to (2.2). First we consider

$$\begin{cases} \frac{\partial u_{i,0}}{\partial t} - \Delta u_{i,0} = 0, & \text{in } \Omega \times (0, +\infty), \\ u_{i,0} = \varphi_i, & \text{on } \partial\Omega \times (0, +\infty), \\ u_{i,0} = \phi_i, & \text{on } \Omega \times \{0\}. \end{cases} \tag{3.1}$$

We know this equation has a unique positive solution  $u_{i,0}(x, t)$ . We also have

$$\lim_{t \rightarrow +\infty} u_{i,0}(x, t) = u_{i,0}(x), \tag{3.2}$$

where the convergence is (for example), in the space of  $C^0(\overline{\Omega})$  and  $u_{i,0}(x)$  is the solution of (2.1). In fact, we can prove that

$$\int_{\Omega} \left| \frac{\partial u_{i,0}}{\partial t} \right|^2 dx \leq C_1 e^{-C_2 t} \tag{3.3}$$

for some positive constants  $C_1$  and  $C_2$ .

Now the iteration can be defined as:

$$\begin{cases} \frac{\partial u_{i,m+1}}{\partial t} - \Delta u_{i,m+1} = -\kappa u_{i,m+1} \sum_{j \neq i} u_{j,m}, & \text{in } \Omega \times (0, +\infty), \\ u_{i,m+1} = \varphi_i, & \text{on } \partial\Omega \times (0, +\infty), \\ u_{i,m+1} = \phi_i, & \text{on } \Omega \times \{0\}. \end{cases} \quad (3.4)$$

This is just a linear parabolic equation, and there exists a unique global solution  $u_{i,m+1}(x, t)$ . Differentiating (3.4) in time  $t$  we get

$$\frac{\partial}{\partial t} \frac{\partial u_{i,m+1}}{\partial t} - \Delta \frac{\partial u_{i,m+1}}{\partial t} = -\kappa \frac{\partial u_{i,m+1}}{\partial t} \sum_{j \neq i} u_{j,m}, -\kappa u_{i,m+1} \sum_{j \neq i} \frac{\partial u_{j,m}}{\partial t}. \quad (3.5)$$

By the induction assumption and maximum principle we know for  $t > 1$  we have for some constant  $C'_m$

$$\sum_{j \neq i} u_{j,m+1} \leq C'_m, \quad (3.6)$$

and

$$\int_{\Omega} \left| \frac{\partial u_{i,m}}{\partial t} \right|^2 dx \leq C_{m,1} e^{-C_{m,2}t} \quad (3.7)$$

for some positive constants  $C_{m,1}$  and  $C_{m,2}$ .

Multiplying (3.5) by  $\frac{\partial u_{i,m+1}}{\partial t}$ , with the help of (3.6), we get (note that we have the boundary condition  $\frac{\partial u_{i,m+1}}{\partial t} = 0$  on  $\partial\Omega$ )

$$\frac{d}{dt} \int_{\Omega} \frac{1}{2} \left| \frac{\partial u_{i,m+1}}{\partial t} \right|^2 + \int_{\Omega} \left| \nabla \frac{\partial u_{i,m+1}}{\partial t} \right|^2 \leq \kappa C'_m \int_{\Omega} \sum_{j \neq i} \left| \frac{\partial u_{j,m}}{\partial t} \right| \left| \frac{\partial u_{i,m+1}}{\partial t} \right|. \quad (3.8)$$

Using Cauchy inequality, we get

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \frac{1}{2} \left| \frac{\partial u_{i,m+1}}{\partial t} \right|^2 + \int_{\Omega} \left| \nabla \frac{\partial u_{i,m+1}}{\partial t} \right|^2 \\ & \leq \kappa C'_m \left( \int_{\Omega} \sum_{j \neq i} \left| \frac{\partial u_{j,m}}{\partial t} \right|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} \left| \frac{\partial u_{i,m+1}}{\partial t} \right|^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (3.9)$$

From (3.7) and Poincaré inequality, we get

$$\int_{\Omega} \left| \frac{\partial u_{i,m+1}}{\partial t} \right|^2 dx \leq C_{m+1,1} e^{-C_{m+1,2}t}, \quad (3.10)$$

for some positive constants  $C_{m+1,1}$  and  $C_{m+1,2}$ .

By standard parabolic estimate this also implies

$$\sup_{\Omega} \left| \frac{\partial u_{i,m+1}}{\partial t} \right| \leq C'_{m+1} e^{-C_{m+1}t}, \quad (3.11)$$

for another two constants  $C'_{m+1}$  and  $C_{m+1}$ . At last we get

$$\lim_{t \rightarrow +\infty} u_{i,m+1}(x, t) = u_{i,m+1}(x). \quad (3.12)$$



Furthermore, the convergence can be taken (for example), in the space of  $C^0(\overline{\Omega})$  and  $u_{i,m+1}(x)$  is the solution of (2.2).

The same method of Section 6.1 gives, in  $\Omega \times (0, +\infty)$

$$u_{i,0} > \dots > u_{i,2m} > u_{i,2m+2} > \dots > u_i > \dots > u_{i,2m+1} > u_{i,2m-1} > \dots > u_{i,1}.$$

Now our Theorem 1.2 can be easily seen. In fact,  $\forall \epsilon > 0$ , there exists an  $m$ , such that

$$\max_{\overline{\Omega}} |u_{i,2m}(x) - u_i(x)| < \epsilon$$

and

$$\max_{\overline{\Omega}} |u_{i,2m+1}(x) - u_i(x)| < \epsilon.$$

We also have that there exists a  $T > 0$ , depending on  $m$  only, such that,  $\forall t > T$ ,

$$\max_{\overline{\Omega}} |u_{i,2m}(x, t) - u_{i,2m}(x)| < \epsilon,$$

and

$$\max_{\overline{\Omega}} |u_{i,2m+1}(x, t) - u_{i,2m+1}(x)| < \epsilon.$$

Combing these together, we get  $\forall t > T$ ,

$$\max_{\overline{\Omega}} |u_i(x, t) - u_i(x)| < 4\epsilon. \tag{3.13}$$

This implies,  $u_i(x, t)$  converges to the solution  $u_i(x)$  of (1.1) as  $t \rightarrow +\infty$ , uniformly on  $\overline{\Omega}$ . (If the boundary values are sufficiently smooth, we know that in fact the convergence in Theorem 1.2 is smooth enough.)  $\square$

#### 4. A uniform Lipschitz estimate

Here we use the Kato inequality to establish the uniform Lipschitz bound of solutions for (1.1) and (1.2). We will only treat the parabolic case, the elliptic case is similar.

First differentiating Eq. (1.2) in a space direction  $e$  we obtain an equation for  $D_e u := e \cdot \nabla u$ :

$$\left(\frac{\partial}{\partial t} - \Delta\right) D_e u_{i,\kappa} = -\kappa D_e u_{i,\kappa} \sum_{j \neq i} u_{j,\kappa} - \kappa u_{i,\kappa} \sum_{j \neq i} D_e u_{j,\kappa}. \tag{4.1}$$

Now using the Kato inequality we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) |D_e u_{i,\kappa}| \leq -\kappa |D_e u_{i,\kappa}| \sum_{j \neq i} u_{j,\kappa} + \kappa u_{i,\kappa} \sum_{j \neq i} |D_e u_{j,\kappa}|. \tag{4.2}$$

Sum these in  $i$  we get

$$\left(\frac{\partial}{\partial t} - \Delta\right) \sum_i |D_e u_{i,\kappa}| \leq 0. \tag{4.3}$$

On the other hand, for  $\Phi_i$  to be the solution of

$$\begin{cases} \frac{\partial \Phi_i}{\partial t} - \Delta \Phi_i = 0, & \text{in } \Omega \times (0, +\infty), \\ \Phi_i = \varphi_i, & \text{on } \partial\Omega \times (0, +\infty), \\ \Phi_i = \phi_i, & \text{on } \Omega \times \{0\}, \end{cases} \tag{4.4}$$

we have (see [6] for the elliptic case)

$$\begin{cases} \Phi_i \geq u_{i,\kappa}, \\ \Phi_i - \sum_{j \neq i} \Phi_j \leq u_{i,\kappa} - \sum_{j \neq i} u_{j,\kappa}. \end{cases} \quad (4.5)$$

This implies

$$\sup_{\partial\Omega \times (0, +\infty)} \left| \frac{\partial u_{i,\kappa}}{\partial \nu} \right| \leq C,$$

for all  $i$ , where  $\nu$  is the outward unit normal vector and  $C$  is independent of  $\kappa$ . With the assumption of Lipschitz continuity of the initial–boundary values, we in fact have

$$\sup_{\partial\Omega \times (0, +\infty)} |\nabla u_{i,\kappa}| \leq C,$$

with a constant  $C$  independent of  $\kappa$  again. Next we also have at  $t = 0$ ,  $u_{i,\kappa} = \phi_i$ , so

$$\sup_{\Omega \times \{0\}} |\nabla u_{i,\kappa}| = \sup_{\Omega} |\nabla \phi_i|.$$

Now Maximum Principle implies a global uniform bound:

$$\sup_{\Omega \times [0, +\infty)} |\nabla u_{i,\kappa}| \leq C.$$

Then standard method means we also have a uniform Lipschitz bound with respect to the parabolic distance.

**Remark 4.1.** Without the boundary regularity, we can still get an interior uniform bound. Multiplying the equation by  $u_{i,\kappa}$  and integrating by parts, we can get an  $L^2$  bound for any  $T > 0$

$$\sum_i \int_T^{T+1} \int_{\Omega} |\nabla u_{i,\kappa}|^2 \leq C, \quad (4.6)$$

with  $C$  independent of  $\kappa$ . Then we can use the mean value property for sub-caloric (or subharmonic function) to give a uniform upper bound for  $|\nabla u_{i,\kappa}|$ .

**Remark 4.2.** If we consider the original Lotka–Volterra system

$$\frac{\partial u_i}{\partial t} - \Delta u_i = a_i u_i - u_i^2 - \kappa u_i \sum_{j \neq i} u_j, \quad (4.7)$$

with homogeneous Dirichlet boundary condition, the above results still hold. In fact, we only need to prove a boundary gradient estimate, which can be guaranteed by the following argument: if we define  $v_i$  to be the solution of

$$\frac{\partial v_i}{\partial t} - \Delta v_i = a_i v_i - v_i^2, \quad (4.8)$$

with the same initial value, then by maximum principle we have for each  $\kappa$

$$u_{i,\kappa} \leq v_i,$$

which, together with the boundary condition, implies

$$\left| \frac{\partial u_{i,\kappa}}{\partial \nu} \right| \leq \left| \frac{\partial v_i}{\partial \nu} \right|,$$

where  $\nu$  is the unit outward normal vector to  $\partial\Omega$ ; using the boundary condition once again we get on the boundary

$$|\nabla u_{i,\kappa}| \leq |\nabla v_i|,$$

where the right-hand side is independent of  $\kappa$ .

### 5. Interior measure estimate

In this section we prove Theorem 1.5.

In order to prove this theorem, we need some lemmas (following the same ideas in Section 5.2 of [12]). The first is a compactness result, so given an  $N \geq 1$ , let's define

$$H_N^1 := \left\{ u: B_1(0) \rightarrow \Sigma, \text{ satisfies (1.3) except the boundary condition, } \right. \\ \left. \frac{\int_{B_1(0)} \sum_i |\nabla u_i^2|}{\int_{\partial B_1(0)} \sum_i u_i^2} \leq N, \int_{\partial B_{\frac{1}{2}}(0)} \sum_i u_i^2 = 1 \right\}.$$

Then we have

**Lemma 5.1.**  $H_N^1$  is compact in  $L^2(B_1(0))$ .

**Proof.** First from the monotonicity of the frequency we have a well-known doubling property, which implies

$$\int_{\partial B_1(0)} \sum_i u_i^2 \leq C(N), \tag{5.1}$$

where  $C(N)$  depends only on  $N$ . By the definition and Poincaré inequality (noting here we have a boundary constraint), we have

$$\int_{B_1(0)} \sum_i |\nabla u_i^2| + \sum_i u_i^2 \leq C(N), \tag{5.2}$$

for another constant  $C(N)$ . So for any sequence  $u_m \in H_N^1$ , there exists a subsequence converging to  $u$ , weakly in  $H^1(B_1)$  and strongly in  $L^2(B_1)$ .

We claim that the limit  $u$  must be in  $H_N^1$ , too. First, we know those properties in (1.3) are preserved under weak convergence in  $H^1(B_1)$  and strong convergence in  $L^2(B_1)$ . Next, we claim for any  $r < 1$ ,  $u_m$  converges to  $u$  strongly in  $H^1(B_r)$ . This is because, if we take a smooth cut-off function  $\zeta$ , from the continuity of  $u_m$  and the fact that  $\Delta u_{i,m}$  is a Radon measure supported on  $\partial\{u_m > 0\}$ , we have

$$0 = \int \Delta u_{i,m} \cdot u_{i,m} \zeta^2 \\ = - \int |\nabla u_{i,m}|^2 \zeta^2 + 2\zeta u_{i,m} \nabla u_{i,m} \nabla \zeta.$$

So from the weak convergence of  $u_{i,m}$  in  $H^1(B_1)$  and the fact that  $u_{i,m}$  converges to  $u_i$  uniformly, we get

$$\lim_{m \rightarrow +\infty} \int |\nabla u_{i,m}|^2 \zeta^2 = \int |\nabla u_i|^2 \zeta^2.$$

From Trace Theorem, we also have

$$\int_{\partial B_{\frac{1}{2}}(0)} \sum_i u_i^2 = \lim_{m \rightarrow +\infty} \int_{\partial B_{\frac{1}{2}}(0)} \sum_i u_{i,m}^2 = 1.$$

Thus for any  $r < 1$

$$\frac{\int_{B_r(0)} \sum_i |\nabla u_i^2|}{\int_{\partial B_r(0)} \sum_i u_i^2} = \lim_{m \rightarrow +\infty} \frac{\int_{B_r(0)} \sum_i |\nabla u_{i,m}^2|}{\int_{\partial B_r(0)} \sum_i u_{i,m}^2} \leq N.$$

By the monotonicity and continuity of the frequency this implies

$$\frac{\int_{B_1(0)} \sum_i |\nabla u_i^2|}{\int_{\partial B_1(0)} \sum_i u_i^2} \leq N. \quad \square$$

The next step is to divide the free boundary into two parts: the good parts are those which are uniformly smooth (the gradient has a uniform lower bound there), while for the bad parts we have a control on its size. In the following we shall denote the free boundary of  $u$  as  $\mathcal{F}(u)$ .

**Lemma 5.2.** *For any  $u \in H_N^1$ , there exist finite balls  $B_{r_k}(x_k)$  with  $r_k \leq \frac{1}{2}$  such that*

$$\left\{ x \in B_{\frac{1}{2}}, \sum_i |\nabla u_i^2| \leq \gamma(N) \right\} \cap \mathcal{F}(u) \subset \bigcup_k B_{r_k}(x_k), \tag{5.3}$$

and

$$\sum_k r_k^{n-1} \leq \frac{1}{2}, \tag{5.4}$$

where  $\gamma(N)$  is a constant depending only on the dimension  $n$  and  $N$ .

**Proof.** For any  $u_0 \in H_N^1$ , the singular set of the free boundary  $\text{sing}(\mathcal{F}(u_0))$  has vanishing  $(n - 1)$ -dimensional Hausdorff measure:

$$H^{n-1}(\text{sing}(\mathcal{F}(u_0))) = 0.$$

So there exist finite balls  $B_{r_k}(x_k)$  with  $r_k \leq \frac{1}{2}$  such that

$$\text{sing}(\mathcal{F}(u_0)) \subset \bigcup_k B_{\frac{r_k}{2}}(x_k), \tag{5.5}$$

and

$$\sum_k r_k^{n-1} \leq \frac{1}{2^n}. \tag{5.6}$$

Of course, there exists a constant  $\gamma(u_0) > 0$ , such that, on the set  $B_{\frac{1}{2}} \cap \mathcal{F}(u_0) \setminus (\bigcup_k B_{\frac{r_k}{2}}(x_k))$ ,

$$\sum_i |\nabla u_{i,0}|^2 \geq 3\gamma(u_0). \tag{5.7}$$

Now we claim there exists an  $\epsilon(u_0) > 0$  such that for any  $u \in H_N^1$  with  $\|u - u_0\|_{L^2(B_1)} \leq \epsilon(u_0)$ , on  $B_{\frac{1}{2}} \cap \mathcal{F}(u) \setminus (\bigcup_k B_{r_k}(x_k))$ ,

$$\sum_i |\nabla u_i|^2 \geq \gamma(u_0).$$

With the compactness of  $H_N^1$  in  $L^2(B_1)$ , our conclusion is easily seen.

Assume this claim is not true, then there exists a sequence of  $u_m \in H_N^1$  with  $\|u_m - u_0\|_{L^2(B_1)} \leq \frac{1}{m}$ , but  $\exists x_m \in B_{\frac{1}{2}} \cap \mathcal{F}(u_m) \setminus (\bigcup_k B_{r_k}(x_k))$ ,

$$\sum_i |\nabla u_{i,m}|^2(x_m) \leq \gamma(u_0). \tag{5.8}$$

Then from the uniform Lipschitz estimate, we have  $u_m$  converge to  $u$  uniformly on any compact subset of  $B_1(0)$ . This implies for any  $\delta > 0$ , for  $m$  large enough depending only on  $\delta$ ,  $\mathcal{F}(u_m)$  is in the  $\delta$  neighborhood of  $\mathcal{F}(u)$ . However, near  $B_{\frac{1}{2}} \cap \mathcal{F}(u_0) \setminus (\bigcup_k B_{r_k}(x_k))$ , locally, there exist exact two components of  $u_0$  which are non-vanishing here, without loss of generality, assuming to be  $u_{1,0}$  and  $u_{2,0}$ , which satisfies

$$\Delta(u_{1,0} - u_{2,0}) = 0.$$

Using the same method of the Clean Up Lemma in [1], we can show that locally, for  $m$  large, we also have, only  $u_{1,m}$  and  $u_{2,m}$  are non-vanishing. (This can also be proven by the upper semicontinuity of the frequency function under the convergence of  $u_m \rightarrow u$  and  $x_m \rightarrow x$ .) Then we also have locally

$$\Delta(u_{1,m} - u_{2,m}) = 0.$$

In view of their convergence in  $L^2_{loc}$ , we have here locally

$$u_{1,m} - u_{2,m} \rightarrow u_{1,0} - u_{2,0} \quad \text{smoothly.}$$

Now coming back to (5.8), without loss of generality, assuming  $x_m \rightarrow x_0$ , which lies in  $B_{\frac{1}{2}} \cap \mathcal{F}(u_0) \setminus (\bigcup_k B_{r_k}(x_k))$ , we can take the limit in (5.8) to get

$$\sum_i |\nabla u_{i,0}|^2(x_0) \leq \gamma(u_0), \tag{5.9}$$

which contradicts (5.7).  $\square$

First we need to control the measure of the good part. This needs a comparison with some standard models, for example, in [12], they use the comparison with harmonic functions. But here we have no such smooth model to compare, instead, we will compare it with the homogeneous elements in  $H^1_N$ , which has the property

$$u(rx) = r^d u(x), \quad \text{for some } d > 0.$$

It can be represented by  $u(r\theta) = r^d \varphi(\theta)$ , for  $\varphi$  defined on  $\mathbb{S}^{n-1}$ , satisfying

$$\begin{cases} (\Delta_\theta + \lambda)\varphi_i \geq 0, \\ (\Delta_\theta + \lambda)\left(\varphi_i - \sum_{j \neq i} \varphi_j\right) \leq 0, \\ \varphi_i \geq 0, \\ \varphi_i \varphi_j = 0, \end{cases} \tag{5.10}$$

where  $\lambda$  satisfies  $d(d + n - 2) = \lambda$  and  $\lambda \leq N$ , and  $\Delta_\theta$  is the Laplacian on  $\mathbb{S}^{n-1}$ . By induction on the dimension, we can assume

$$H^{n-2}(\mathcal{F}(\varphi) \cap \mathbb{S}^{n-1}) \leq C(N, n). \tag{5.11}$$

Note here in dimension  $n = 2$ , each  $\varphi$  can be computed explicitly.

**Lemma 5.3.** *With the assumptions of the preceding lemma, if moreover*

$$\frac{\int_{B_1(0)} \sum_i |\nabla u_i^2|}{\int_{\partial B_1(0)} \sum_i u_i^2} - N(u, 0) \leq \sigma, \tag{5.12}$$

where  $\sigma$  is a constant depending only on the dimension  $n$  and  $N$ , and

$$N(u, 0) = \lim_{r \rightarrow 0} \frac{r \int_{B_r(0)} \sum_i |\nabla u_i^2|}{\int_{\partial B_r(0)} \sum_i u_i^2}. \tag{5.13}$$

Then

$$H^{n-1}\left(\mathcal{F}(u) \cap B_{\frac{1}{2}} \setminus \left(\bigcup_k B_{r_k}(x_k)\right)\right) \leq C(N),$$

where  $C(N)$  is a constant depending only on the dimension  $n$  and  $N$ .

**Proof.** Take a  $\delta > 0$  small enough. If we choose  $\sigma$  small enough too, then by compactness there exists a homogeneous  $w \in H^1_N$  such that

$$\|u - w\|_{L^2(B_1)} \leq \delta,$$

and  $\mathcal{F}(u)$  is in the  $\delta$  neighborhood of  $\mathcal{F}(w)$ . Define

$$S_1 := \left\{ x \in B_{\frac{1}{2}}, \sum_i |\nabla u_i^2| \geq \gamma(N) \right\} \cap \mathcal{F}(u),$$

$$S_2 := \left\{ x \in B_{\frac{1}{2}}, \sum_i |\nabla w_i^2| \geq \gamma(N) \right\} \cap \mathcal{F}(w).$$

If  $\sigma$  is small,  $S_1$  is in the  $\delta$  neighborhood of  $S_2$ , too. Take an  $\varepsilon \gg \delta$ , and take a maximal  $\varepsilon$  separated sets  $\{y_k\}$  of  $S_2$ . Then we have  $\text{dist}(y_k, y_l) \geq \frac{\varepsilon}{2}$  and  $S_2 \subset \bigcup_k B_\varepsilon(y_k)$ .

In each  $B_\varepsilon(y_k)$ ,  $w$  has exactly two components which are non-vanishing. Moreover, the free boundary  $\mathcal{F}(w) \cap B_\varepsilon(y_k)$  can be represented by the graph of a  $C^1$  function defined on the tangent plane to  $\mathcal{F}(w)$  at  $y_k$ . Now if  $\delta$  is small enough, this property is also valid for  $u$ . The same method of Lemma 5.25 in [12] gives our conclusion.  $\square$

Now the proof of Theorem 1.5 can be easily done by an iteration procedure exactly as in [12]. Here, we just need to note that in Lemma 5.2, those radius  $r_i$  can be chosen arbitrarily small so that the assumptions in Lemma 5.3 are satisfied.

At last, we give a theorem on the uniform estimate of the measure of the level surface  $\{u_i = \delta\}$ .

**Theorem 5.4.** For  $u \in H^1_N$ ,  $\forall \delta > 0$  and  $1 \leq i \leq M$ , we have

$$H^{n-1}(B_{\frac{1}{2}} \cap \{u_i = \delta\}) \leq C(N).$$

This is also valid if we consider the local energy minimizing map.

**Proof.** First, because each  $u_i$  is subharmonic, from the  $L^2(B_1)$  bound we have

$$\sup_{B_{\frac{1}{2}}} u_i \leq C(N). \tag{5.14}$$

We claim that,  $\forall \delta > 0, \exists C(\delta, N)$ , such that,  $\forall t > \delta$

$$H^{n-1}(B_{\frac{1}{2}} \cap \{u_i = t\}) \leq C(\delta, N). \tag{5.15}$$

If this is not true, then  $\exists t_k \geq \delta$  and  $u_k \in H^1_N$  such that

$$H^{n-1}(B_{\frac{1}{2}} \cap \{u_{i,k} = t_k\}) \geq k. \tag{5.16}$$

By (5.14), we can assume  $t_k \rightarrow t \geq \delta$ . By the compactness of  $H^1_N$ , we can assume  $u_k \rightarrow u$  in  $L^2(B_1)$ . By the uniform Hölder continuity [1], we can also assume  $u_{i,k} \rightarrow u_i$  in  $C(B_{\frac{2}{3}})$ .

If  $u_i \equiv 0$ , then for  $k$  large

$$\sup_{B_{\frac{2}{3}}} u_{i,k} < \frac{t}{2}.$$

This is impossible, so  $u_i$  is not 0. In fact,  $\{u_i = t\} \cap B_{\frac{1}{2}} \neq \emptyset$ . Because in  $\{u_i > 0\}$ ,  $u_i$  is harmonic, we have

$$H^{n-1}(B_{\frac{1}{2}} \cap \{u_i = t\}) < +\infty. \tag{5.17}$$

We also have for  $k$  large,  $B_{\frac{1}{2}} \cap \{u_{i,k} = t_k\}$  lies in a small neighborhood of  $B_{\frac{1}{2}} \cap \{u_i = t\}$ . In this neighborhood,  $u_{i,k}$  converge to  $u_i$  smoothly, so for  $k$  large,  $\exists C > 0$  such that

$$H^{n-1}(B_{\frac{1}{2}} \cap \{u_{i,k} = t_k\}) \leq C. \tag{5.18}$$

This is a contradiction, so our claim follows.

Now we can use an iteration to prove our theorem. For any  $u \in H_N^1$ , take a covering of the singular set of the free boundary as in Lemma 5.2:

$$\text{sing}(\mathcal{F}(u)) \subset \bigcup_k B_{\frac{r_k}{2}}(x_k) \tag{5.19}$$

with

$$\sum_k r_k^{n-1} \leq \frac{1}{2^n}. \tag{5.20}$$

By the uniform interior Lipschitz estimate,  $\exists \delta(N) > 0$ , such that

$$\sup_{\bigcup_k B_{r_k}(x_k)} \sum_i u_i \leq \delta(N). \tag{5.21}$$

While for  $x \in B_{\frac{1}{2}} \cap \mathcal{F}(u) \setminus \bigcup_k B_{\frac{r_k}{2}}(x_k)$

$$\sum_i |\nabla u_i|^2 \geq \gamma(N), \tag{5.22}$$

so  $\exists C(N)$ , such that  $\forall \delta > 0$

$$H^{n-1}\left(B_{\frac{1}{2}} \cap \{u_i = \delta\} \setminus \bigcup_k B_{\frac{r_k}{2}}(x_k)\right) \leq C(N).$$

Now we can rescale  $u$  in  $B_{r_k}$ :

$$\hat{u} = L_k u(x_k + r_k x).$$

If we choose  $L_k$  appropriately,  $\hat{u}$  is still in  $H_N^1$ , and we can iterate the above procedure. This iteration will stop in finite times and at last we get our original estimate.  $\square$

### 6. Uniqueness of the singular limit

In this section we prove Theorem 1.6.

In order to prove the energy minimizing property, we need to prove that for given Lipschitz map  $w : \Omega \rightarrow \Sigma$  such that  $w \equiv u$  outside a compact set  $\Omega' \Subset \Omega$ , we have

$$\int_{\Omega'} \sum_i |\nabla u_i|^2 dx \leq \int_{\Omega'} \sum_i |\nabla w_i|^2 dx.$$

In fact, we will prove that, if  $v$  minimizes  $\{\int_{\Omega'} \sum_i |\nabla w_i|^2 dx : w \equiv u \text{ outside } \Omega'\}$ , then  $u \equiv v$  in  $\Omega'$ .

First,  $\forall i$ , in  $\{u_i > 0\}$  (or  $\{v_i > 0\}$ ),  $u_i$  (or  $v_i$ ) is harmonic, thus real analytic. If we enlarge  $\Omega'$ , we can assume  $\partial\Omega'$  is real analytic and smooth (without singularity). Choose two constants  $\delta > \sigma > 0$ , such that the level surfaces  $\{u_i = \delta\}$  and  $\{v_i = \delta\}$ ,  $\forall i$ , are regular real analytic hypersurface up to the boundary (that is,  $\{u_i = \delta\} \cap \partial\Omega'$  and  $\{v_i = \delta\} \cap \partial\Omega'$  are regular real analytic hypersurface in  $\partial\Omega'$ ). With this setting, we know that the divergence theorem is valid for domains separated by these hypersurfaces.

Define  $u_i^\delta := \max\{u_i - \delta, 0\}$  and  $v_i^\sigma := \max\{v_i - \sigma, 0\}$ . We consider the geodesic homotopy  $u^t : \Omega \rightarrow \Sigma$  between  $u^\delta$  and  $v^\sigma$  for  $t \in [0, 1]$ , that is,  $u^t(x)$  is the point on the unique geodesic between  $u^\delta(x)$  and  $v^\sigma(x)$  which is characterized uniquely by  $d(u^t(x), u^\delta(x)) = td(u^\delta(x), v^\sigma(x))$ . (Here  $d$  denotes the intrinsic distance of  $\Sigma$ .)

We can write down the expression of  $u^t(x)$  explicitly from the concrete form of  $\Sigma$ , following the construction of the test functions used in [5]. In the set  $A_i := \{x : u_i^\delta(x) > 0, v_i^\sigma(x) > 0\}$ :

$$u_i^t(x) = (1 - t)u_i^\delta(x) + tv_i^\sigma(x); \tag{6.1}$$

in the set  $B_{ij} := \{x: u_i^\delta(x) > 0, v_j^\sigma(x) > 0 \text{ and } u_i^\delta(x) - t(u_i^\delta(x) + v_j^\sigma(x)) > 0\}$  for some  $j \neq i$ :

$$u_i^t(x) = u_i^\delta(x) - t(u_i^\delta(x) + v_j^\sigma(x)); \tag{6.2}$$

in the set  $C_{ij} := \{x: u_j^\delta(x) > 0, v_i^\sigma(x) > 0 \text{ and } t(u_j^\delta(x) + v_i^\sigma(x)) - u_j^\delta(x) > 0\}$  for some  $j \neq i$ :

$$u_i^t(x) = t(u_j^\delta(x) + v_i^\sigma(x)) - u_j^\delta(x); \tag{6.3}$$

in the set  $D_i := \{x: u_i^\delta(x) > 0, v_j^\sigma(x) = 0, \forall j\}$ :

$$u_i^t(x) = (1 - t)u_i^\delta(x); \tag{6.4}$$

in the set  $E_i := \{x: v_i^\sigma(x) > 0, u_j^\delta(x) = 0, \forall j\}$ :

$$u_i^t(x) = tv_i^\sigma(x); \tag{6.5}$$

on the remaining part  $u_i^t(x) = 0$ .

Now we have (note that in  $\{u_i^\delta(x) > 0\}$ ,  $\nabla u_i^\delta(x) = \nabla u_i(x)$  a.e.)

$$\begin{aligned} \int_{\Omega} |\nabla u_i^t(x)|^2 dx &= \int_{A_i} |(1-t)\nabla u_i(x) + t\nabla v_i(x)|^2 dx + \sum_{j \neq i} \int_{B_{ij}} |(1-t)\nabla u_i(x) - t\nabla v_j(x)|^2 dx \\ &\quad + \sum_{j \neq i} \int_{C_{ij}} |-(1-t)\nabla u_j(x) + t\nabla v_i(x)|^2 dx + \int_{D_i} |(1-t)\nabla u_i(x)|^2 dx \\ &\quad + \int_{E_i} |t\nabla v_i(x)|^2 dx. \end{aligned} \tag{6.6}$$

We need to compute  $\frac{dE}{dt}|_{t=0}$ . Noticing that in the first term of (6.6) the domain is fixed, and  $\Delta u_i = 0$  in the open set  $\{u_i > \delta\}$ , we can integrate by parts to get

$$2 \int_{A_i} \nabla u_i(x)(\nabla v_i(x) - \nabla u_i(x)) dx = 2 \int_{\partial A_i} \frac{\partial u_i}{\partial \nu} [v_i(x) - u_i(x)]. \tag{6.7}$$

Now we have

$$\partial A_i = A_{i,1} \cup A_{i,2} \cup A_{i,3},$$

where

$$\begin{cases} A_{i,1} = \{u_i = \delta\} \cap \{v_i > \sigma\}, \\ A_{i,2} = \{v_i = \sigma\} \cap \{u_i > \delta\}, \\ A_{i,3} = \partial\Omega' \cap \{v_i > \sigma\} \cap \{u_i > \delta\}, \\ A_{i,4} = \{u_i = \delta, v_i = \sigma\}. \end{cases} \tag{6.8}$$

On  $A_{i,3}$  we have  $u_i = v_i$ , while the  $n - 1$  dimension Hausdorff measure of  $A_{i,4}$  is 0 (it lies in the interior of  $\Omega'$ ), so they do not appear in (6.7). On  $A_{i,1}$  we have  $u_i = \delta$  and on  $A_{i,2}$  we have  $v_i = \sigma$ . Thus we get

$$2 \int_{A_{i,1}} \frac{\partial u_i}{\partial v_{i,1}^-} v_i - 2 \int_{A_{i,2}} \frac{\partial u_i}{\partial v_{i,2}^-} u_i - 2\delta \int_{A_{i,1}} \frac{\partial u_i}{\partial v_{i,1}^-} + 2\sigma \int_{A_{i,2}} \frac{\partial u_i}{\partial v_{i,2}^-}. \tag{6.9}$$

Here  $v_{i,1}^-$  and  $v_{i,2}^-$  are the outward unit normal vector to  $\partial\{u_i > \delta\}$  and  $\partial\{v_i > \delta\}$  respectively.

Next let's consider the second term in (6.6). Here we must be careful because the domain changes as  $t$  changes. That is

$$B_{ij} := \{x: u_i(x) > \delta, v_j(x) > \sigma\} \setminus \left\{ u_i^\delta(x) < \frac{t}{1-t} v_j^\sigma(x) \right\}.$$



So the second term of (6.6) can be written as

$$\begin{aligned} & \sum_{j \neq i} \int_{\{x: u_i(x) > \delta, v_j(x) > \sigma\}} |(1-t)\nabla u_i(x) - t\nabla v_j(x)|^2 dx \\ & - \sum_{j \neq i} \int_{\{x: u_i^\delta(x) \leq \frac{t}{1-t} v_j^\sigma(x)\}} |(1-t)\nabla u_i(x) - t\nabla v_j(x)|^2 dx. \end{aligned} \tag{6.10}$$

In the first term, the domain is fixed, and the derivative can be calculated directly. The second term can be written in another form using the Co-Area formula (see, for example, [14]):

$$\sum_{j \neq i} \int_0^{\frac{t}{1-t}} \left[ \int_{\{x: \frac{u_i^\delta}{v_j^\sigma} = \tau\}} \frac{|(1-t)\nabla u_i(x) - t\nabla v_j(x)|^2}{|\nabla \frac{u_i^\delta}{v_j^\sigma}|} \right] d\tau. \tag{6.11}$$

Its derivative at  $t = 0$  is

$$\sum_{j \neq i} \int_{\{x: \frac{u_i^\delta}{v_j^\sigma} = 0\}} \frac{|\nabla u_i(x)|^2}{|\nabla \frac{u_i^\delta}{v_j^\sigma}|}. \tag{6.12}$$

After calculation we get

$$\sum_{j \neq i} \int_{\{x: u_i = \delta, v_j > \sigma\}} |\nabla u_i(x)| v_j^\sigma(x). \tag{6.13}$$

At last we get that the derivative of the second term of (6.6) at  $t = 0$  is

$$-2 \int_{\{u_i > \delta, v_j > \sigma\}} \nabla u_i(x) (\nabla v_j(x) + \nabla u_i(x)) dx - \int_{\{u_i = \delta\} \cap \{v_j > \sigma\}} |\nabla u_i(x)| v_j^\sigma. \tag{6.14}$$

Through an integration by parts (and the same remark as before concerning this procedure of integration by parts), the first term of (6.14) can be transformed into the boundary term, and notice that on  $\{u_i = \delta\}$  we have  $u_i = \delta$  and on  $\{v_j = \sigma\}$  we have  $v_j = \sigma$ , so the first term of (6.14) is

$$-2 \int_{\{u_i = \delta\} \cap \{v_j > \sigma\}} \frac{\partial u_i}{\partial v_{i,1}^-} v_j - 2 \int_{\{v_j = \sigma\} \cap \{u_i > \delta\}} \frac{\partial u_i}{\partial v_{i,2}^-} u_i - 2\delta \int_{\{u_i = \delta\} \cap \{v_j > \sigma\}} \frac{\partial u_i}{\partial v_{i,1}^-} - 2\sigma \int_{\{v_j = \sigma\} \cap \{u_i > \delta\}} \frac{\partial u_i}{\partial v_{i,2}^-}. \tag{6.15}$$

Here  $v_{i,1}^-$  and  $v_{i,2}^-$  are the outward normal vectors to  $\partial\{u_i > \delta\}$  and  $\partial\{v_j > \delta\}$  respectively.

The third term can be calculated similarly (notice that the domain is enlarged so here the positive sign comes out):

$$\int_{\{u_i = \delta\} \cap \{v_i > \sigma\}} |\nabla u_j(x)| v_i^\sigma. \tag{6.16}$$

The result from the fourth term is

$$\begin{aligned} \frac{d}{dt} \int_{E_i} (1-t)^2 |\nabla u_i|^2 &= -2 \int_{\{u_i > \delta\} \cap (\cup_j \{v_j < \sigma\})} |\nabla u_i|^2 \\ &= -2 \int_{\{u_i = \delta\} \cap (\cup_j \{v_j < \sigma\})} \frac{\partial u_i}{\partial v_{i,1}^-} u_i - 2 \sum_j \int_{\{u_i > \delta\} \cap \{v_j = \sigma\}} \frac{\partial u_i}{\partial v_{i,2}^+} u_i, \end{aligned} \tag{6.17}$$

where  $v_{i,1}^-$  and  $v_{i,2}^+$  are the outward normal vectors to  $\partial\{u_i > \delta\}$  and  $\partial\{v_j < \delta\}$  respectively.

The fifth term is of the order  $t^2$ , so there is no contribution. Now put all of these terms for all  $i$  together:

$$\begin{aligned}
 & 2 \sum_i \int_{\{u_i=\delta\} \cap \{v_i>\sigma\}} \frac{\partial u_i}{\partial v_{i,1}^-} v_i - 2 \sum_i \int_{\{v_i=\sigma\} \cap \{u_i>\delta\}} \frac{\partial u_i}{\partial v_{i,2}^-} u_i - 2 \sum_i \delta \int_{\{u_i=\delta\} \cap \{v_i>\sigma\}} \frac{\partial u_i}{\partial v_{i,1}^-} \\
 & + 2 \sum_i \sigma \int_{\{v_i=\sigma\} \cap \{u_i>\delta\}} \frac{\partial u_i}{\partial v_{i,2}^-} - \sum_{i \neq j} \int_{\{u_i=\delta\} \cap \{v_j>\sigma\}} |\nabla u_i(x)| v_j^\sigma - 2 \sum_{i \neq j} \int_{\{u_i=\delta\} \cap \{v_j>\sigma\}} \frac{\partial u_i}{\partial v_{i,1}^-} v_j \\
 & - 2 \sum_{i \neq j} \int_{\{v_j=\sigma\} \cap \{u_i>\delta\}} \frac{\partial u_i}{\partial v_{i,2}^-} u_i - 2 \sum_{i \neq j} \delta \int_{\{u_i=\delta\} \cap \{v_j>\sigma\}} \frac{\partial u_i}{\partial v_{i,1}^-} - 2 \sum_{i \neq j} \sigma \int_{\{v_j=\sigma\} \cap \{u_i>\delta\}} \frac{\partial u_i}{\partial v_{i,2}^-} \\
 & + \sum_{i \neq j} \int_{\{u_j=\delta\} \cap \{v_i>\sigma\}} |\nabla u_j(x)| v_i^\sigma - 2 \sum_{i,j} \delta \int_{\{u_i=\delta\} \cap \{v_j<\sigma\}} \frac{\partial u_i}{\partial v_{i,1}^-} - 2 \sum_{i,j} \int_{\{u_i>\delta\} \cap \{v_j=\sigma\}} \frac{\partial u_i}{\partial v_{i,2}^+} u_i. \tag{6.18}
 \end{aligned}$$

In these twelve terms, let's see the second, the seventh and the twelfth terms (modulo the constant  $-2$ ):

$$\begin{aligned}
 & \sum_i \int_{\{v_i=\sigma\} \cap \{u_i>\delta\}} \frac{\partial u_i}{\partial v_{i,2}^-} u_i + \sum_{i \neq j} \int_{\{v_j=\sigma\} \cap \{u_i>\delta\}} \frac{\partial u_i}{\partial v_{i,2}^-} u_i + \sum_{i,j} \int_{\{u_i>\delta\} \cap \{v_j=\sigma\}} \frac{\partial u_i}{\partial v_{i,2}^+} u_i \\
 & = \sum_i \sum_j \int_{\{v_i=\sigma\} \cap \{u_j>\delta\}} \frac{\partial u_i}{\partial v_{i,2}^-} u_i + \sum_{i,j} \int_{\{v_i=\sigma\} \cap \{u_j>\delta\}} \frac{\partial u_i}{\partial v_{i,2}^+} u_i \\
 & = 0,
 \end{aligned}$$

because the normal vector field  $v_{i,2}^+$  and  $v_{i,2}^-$  on  $\{v_i = \sigma\}$  have opposite directions.

The fourth and the ninth terms are

$$2 \sum_i \sigma \int_{\{v_i=\sigma\} \cap \{u_i>\delta\}} \frac{\partial u_i}{\partial v_{i,2}^-}$$

and

$$-2 \sum_{i \neq j} \sigma \int_{\{v_j=\sigma\} \cap \{u_i>\delta\}} \frac{\partial u_i}{\partial v_{i,2}^-}.$$

We show that the integration in these terms are uniformly bounded in  $\sigma$ , thus as  $\sigma \rightarrow 0$ , these two terms converge to 0. We only calculate the first, the second is similar. First

$$\int_{\{v_i=\sigma\} \cap \{u_i>\delta\}} \frac{\partial u_i}{\partial v_{i,2}^-} = \int_{\{v_i>\sigma\} \cap \{u_i>\delta\}} \Delta u_i - \int_{\{v_i>\sigma\} \cap \{u_i=\delta\}} \frac{\partial u_i}{\partial v_{i,1}^-} - \int_{\{v_i>\sigma\} \cap \{u_i>\delta\} \cap \partial \Omega'} \frac{\partial u_i}{\partial v},$$

where  $v$  is the outward unit normal vector field to  $\partial \Omega'$ . In the right-hand side, the first term is less than the total mass of the measure  $\Delta u_i$  on  $\Omega$ ; the second term can be controlled by

$$\int_{\{u_i=\delta\}} |\nabla u_i|,$$

at the end of this subsection we will show this is uniformly bounded in  $\delta$ ; the third term is also uniformly bounded by the area of  $\partial \Omega'$  times the sup norm of  $\nabla u_i$ , and we conclude.

The eleventh term in Eq. (6.18) converges to 0 as  $\sigma \rightarrow 0$ . Now we can take the limit in the remaining terms as  $\sigma \rightarrow 0$  to get:

$$\begin{aligned}
 & 2 \sum_i \int_{\{u_i=\delta\} \cap \{v_i>0\}} \frac{\partial u_i}{\partial v_{i,1}^-} v_i - 2\delta \sum_i \int_{\{u_i=\delta\} \cap \{v_i>0\}} \frac{\partial u_i}{\partial v_{i,1}^-} - \sum_{i \neq j} \int_{\{u_i=\delta\} \cap \{v_j>0\}} |\nabla u_i(x)| v_j \\
 & - 2 \sum_{i \neq j} \int_{\{u_i=\delta\} \cap \{v_j>0\}} \frac{\partial u_i}{\partial v_{i,1}^-} v_j - 2\delta \sum_{i \neq j} \int_{\{u_i=\delta\} \cap \{v_j>0\}} \frac{\partial u_i}{\partial v_{i,1}^-} + \sum_{i \neq j} \int_{\{u_j=\delta\} \cap \{v_i>0\}} |\nabla u_j(x)| v_i.
 \end{aligned} \tag{6.19}$$

Noting that on  $\{u_i = \delta\}$ ,  $\frac{\partial u_i}{\partial v_{i,1}^-} = -|\nabla u_i|$ , so we have

$$\begin{aligned}
 & -2 \sum_i \int_{\{u_i=\delta\} \cap \{v_i>0\}} |\nabla u_i| v_i - \sum_{i \neq j} \int_{\{u_i=\delta\} \cap \{v_j>0\}} |\nabla u_i(x)| v_j + 2 \sum_{i \neq j} \int_{\{u_i=\delta\} \cap \{v_j>0\}} |\nabla u_i| v_j \\
 & + \sum_{i \neq j} \int_{\{u_j=\delta\} \cap \{v_i>0\}} |\nabla u_j(x)| v_i + 2\delta \sum_{i \neq j} \int_{\{u_i=\delta\} \cap \{v_j>0\}} |\nabla u_i| + 2\delta \sum_i \int_{\{u_i=\delta\} \cap \{v_i>0\}} |\nabla u_i|.
 \end{aligned} \tag{6.20}$$

The integration in the last two terms will be shown to be uniformly bounded in  $\delta$  at the end of this subsection, thus as  $\delta \rightarrow 0$ , they converge to 0.

As  $\delta \rightarrow 0$ , the remaining terms converge to (see the end of this subsection, too)

$$\begin{aligned}
 & -2 \sum_i \int_{\partial\{u_i>0\} \cap \{v_i>0\}} |\nabla u_i| v_i - \sum_{i \neq j} \int_{\partial\{u_i>0\} \cap \{v_j>0\}} |\nabla u_i(x)| v_j \\
 & + 2 \sum_{i \neq j} \int_{\partial\{u_i>0\} \cap \{v_j>0\}} |\nabla u_i| v_j + \sum_{i \neq j} \int_{\partial\{u_j>0\} \cap \{v_i>0\}} |\nabla u_j(x)| v_i.
 \end{aligned} \tag{6.21}$$

In  $\{v_i > 0\}$ , if  $\partial\{u_i > 0\} \cap \partial\{u_j > 0\} \neq \emptyset$ , then  $\frac{\partial u_i}{\partial v_{i,1}^-} = \frac{\partial u_j}{\partial v_{j,1}^-}$ , so the first term cancels some terms in the third term, with

$$\sum_i \sum_{j, k \neq i} \int_{\partial\{u_j>0\} \cap \partial\{u_k>0\} \cap \{v_i>0\}} |\nabla u_j(x)| v_i$$

left. The integral  $\int_{\partial\{u_j>0\} \cap \{v_i>0\}} |\nabla u_j(x)| v_i$  appears twice in the second term and the fourth term with different signs, so these terms cancel each other, too.

So we have

$$\frac{dE}{dt} \Big|_{t=0} \geq 0.$$

However,  $E(t)$  is a convex function of  $t$ , so 0 is its minimal point. But from our choice of  $v$  we also have 1 is its minimal point. Therefore we must have  $E(t) \equiv const.$ , this implies  $u$  is the energy minimizer and  $u \equiv v$ .

### 6.1. Verification of the convergence of the integration

Here we will show the uniform boundedness and the convergence of the various integrations appearing before.

We only consider the integration

$$\int_{\{u_i=\delta\} \cap \{v_i>0\}} |\nabla u_i| v_i,$$

others can be treated similarly.

We know that the singular set of  $\partial\{u_i > 0\}$  is of Hausdorff dimension  $n - 2$ , so its  $(n - 1)$ -dimensional Hausdorff measure is 0. In particular,  $\forall \varepsilon > 0$ , there exist some balls  $B(x_k, r_k)$ , such that

$$\text{Sing}(\partial\{u_i > 0\}) \subset \bigcup B(x_k, r_k),$$

with

$$\sum_k r_k^{n-1} \leq \varepsilon.$$

Outside  $\bigcup B(x_k, r_k)$ ,  $\partial\{u_i > 0\}$  is a regular smooth hypersurface where  $\inf |\nabla u_i| > 0$ , then it is easily seen that (noting that the integrand are continuous up to  $\partial\{u_i > 0\}$  outside  $\bigcup B(x_k, r_k)$ ), there exists a  $\delta_0 > 0$ , such that for any  $\delta < \delta_0$  (if the level surface  $\{u_i = \delta\}$  is regular)

$$\left| \int_{\{u_i=\delta\} \setminus (\bigcup B(x_k, r_k))} |\nabla u_i| v_i(x) - \int_{\partial\{u_i>0\} \setminus (\bigcup B(x_k, r_k))} |\nabla u_i| v_i(x) \right| \leq \varepsilon. \tag{6.22}$$

On the other hand, in  $B(x_k, r_k)$ , we have

$$\begin{aligned} 0 &= \int_{\{u_i>\varepsilon\} \cap B(x_k, r_k)} \Delta u_i \\ &= - \int_{\{u_i=\varepsilon\} \cap B(x_k, r_k)} |\nabla u_i(x)| + \int_{\{u_i>\varepsilon\} \cap \partial B(x_k, r_k)} \frac{\partial u_i}{\partial \nu}, \end{aligned}$$

where  $\nu$  is the unit outward normal vector of  $\partial B(x_k, r_k)$ . We also have on  $\{u_i > \varepsilon\} \cap \partial B(x_k, r_k)$

$$\left| \frac{\partial u_i}{\partial \nu} \right| \leq |\nabla u_i| \leq C.$$

Combing these two facts we get

$$\int_{\{u_i=\delta\} \cap B(x_k, r_k)} |\nabla u_i(x)| \leq C r_k^{n-1}. \tag{6.23}$$

Sum these to get

$$\sum_k \int_{\{u_i=\delta\} \cap B(x_k, r_k)} |\nabla u_i(x)| \leq C\varepsilon.$$

From  $H^{n-1}(\partial\{u_i > 0\} \cap \Omega') < \infty$ , we can select balls  $B(x_k, r_k)$  small enough so that (here the integration, as usual, is understood as on the regular part)

$$\sum_k \int_{\partial\{u_i>0\} \cap B(x_k, r_k)} |\nabla u_i(x)| \leq \varepsilon.$$

In view of the arbitrary choice of  $\varepsilon$ , now it is clear that as  $\delta \rightarrow 0$ ,

$$\int_{\{u_i=\delta\} \cap \{v_i>0\}} |\nabla u_i| v_i(x) \rightarrow \int_{\partial\{u_i>0\} \cap \{v_i>0\}} |\nabla u_i| v_i(x).$$

Then the left-hand side is also uniformly bounded in  $\delta$ .

### 7. Uniqueness and asymptotics of the singular parabolic system

In this section we prove Theorem 1.7.

First, we prove the uniqueness of the solution. If there exist two solutions of (1.4),  $u$  and  $v$ , define the distance

$$d(u, v)(x, t) := \sum_i |u_i(x, t) - v_i(x, t)|.$$

Because  $\forall i$ ,  $u_i$  and  $v_i$  are Lipschitz continuous with respect to the parabolic distance,  $d$  is Lipschitz too. Now we claim that in  $\{d > 0\}$

$$\left(\Delta - \frac{\partial}{\partial t}\right)d \geq 0. \tag{7.1}$$

We prove this case by case:

1. Where  $u_i > 0$  and  $v_i > 0$  with  $u_i - v_i > 0$ , we have

$$d = u_i - v_i,$$

$$\text{so } \left(\Delta - \frac{\partial}{\partial t}\right)d = 0.$$

2. Where  $u_i > 0$  and  $v_i > 0$  with  $v_i - u_i > 0$ , we have

$$d = v_i - u_i,$$

$$\text{so } \left(\Delta - \frac{\partial}{\partial t}\right)d = 0.$$

3. Where  $u_i > 0$  and  $v_j > 0$  for some  $j \neq i$ , we have

$$d = v_j + u_i,$$

$$\text{so } \left(\Delta - \frac{\partial}{\partial t}\right)d = 0.$$

4. Where  $u_i(X_0) > 0$  and  $v_j(X_0) = 0, \forall j$ , then in a neighborhood of  $X_0$ , we have

$$d = u_i - v_i + \sum_{j \neq i} v_j,$$

so

$$\left(\Delta - \frac{\partial}{\partial t}\right)d = \left(\Delta - \frac{\partial}{\partial t}\right)\left(-v_i + \sum_{j \neq i} v_j\right) \geq 0.$$

5. Where  $v_i(X_0) > 0$  and  $u_j(X_0) = 0, \forall j$ , then in a neighborhood of  $X_0$ , we have

$$d = v_i - u_i + \sum_{j \neq i} u_j,$$

so

$$\left(\Delta - \frac{\partial}{\partial t}\right)d = \left(\Delta - \frac{\partial}{\partial t}\right)\left(-u_i + \sum_{j \neq i} u_j\right) \geq 0.$$

Take an  $\epsilon > 0$ , and define

$$\hat{d} = e^{-\epsilon t} d,$$

then

$$\left(\Delta - \frac{\partial}{\partial t}\right)\hat{d} > 0, \tag{7.2}$$

strictly on the open set  $\{d > 0\}$ . Now from the boundary condition we have

$$\hat{d} = d = 0, \quad \text{on } \partial_p(\Omega \times (0, +\infty)).$$

By the maximum principle we get  $\hat{d} \equiv 0$ , or in other words

$$u_i \equiv v_i.$$

That is, the solution is unique.

Next, let's consider the singular limit of the following system (this was considered by Caffarelli and Lin in [4]):

$$\begin{cases} \frac{\partial v_{i,\kappa}}{\partial t} - \Delta v_{i,\kappa} = -\kappa v_{i,\kappa} \sum_{j \neq i} v_{j,\kappa}^2, & \text{in } \Omega \times (0, +\infty), \\ v_{i,\kappa} = \varphi_i, & \text{on } \partial\Omega \times (0, +\infty), \\ v_{i,\kappa} = \phi_i, & \text{on } \Omega \times \{0\}. \end{cases} \quad (7.3)$$

This system is the decreasing gradient flow of the functional

$$\int_{\Omega} 2 \sum_i |\nabla v_i|^2 + \kappa \sum_{i \neq j} v_i^2 v_j^2.$$

We claim that its singular limit  $v_i$  as  $\kappa \rightarrow +\infty$  satisfy the inequalities in (1.4). We know that the singular limit satisfy

$$\frac{\partial v_i}{\partial t} - \Delta v_i = - \sum_{j \neq i} \mu_{ij}, \quad (7.4)$$

where  $\mu_{ij}$  are positive Radon measure supported on  $\partial\{v_i > 0\} \cap \partial\{v_j > 0\}$ . We just need to show

$$\mu_{ij} = \mu_{ji}, \quad \forall j \neq i.$$

This comes from the regularity theory of the free boundary, which shows that

$$\mu_{ij} = |\nabla v_i| H^{n-1} \llcorner_{\partial\{v_i > 0\}} \wedge dt.$$

But on  $\partial\{v_i > 0\} \cap \partial\{v_j > 0\}$ , we have

$$|\nabla v_i| = |\nabla v_j|, \quad H^{n-1} \text{ a.e.}$$

Then our claim is proven.

From the above proof of the uniqueness, we know this singular limit  $v$  coincide with  $u$ , the solution of (1.4). But  $v$  has an energy identity induced from (7.3):

$$\frac{d}{dt} \int_{\Omega} \sum_i |\nabla v_i|^2 = - \int_{\Omega} \sum_i \left| \frac{\partial v_i}{\partial t} \right|^2. \quad (7.5)$$

Of course, this is also valid for  $u$ . Now it is easy to conclude that as  $t \rightarrow +\infty$ ,  $u$  converge to the unique stationary solution. This is because, for any sequence  $t_i \rightarrow +\infty$ , the translation  $u(t_i + t)$  has a subsequence converges to a solution  $w$  of (1.4) defined on  $(-\infty, +\infty)$ . However, from the energy decreasing property, we know

$$\frac{d}{dt} \int_{\Omega} \sum_i |\nabla w_i|^2 = - \int_{\Omega} \sum_i \left| \frac{\partial w_i}{\partial t} \right|^2 = 0. \quad (7.6)$$

So

$$\frac{\partial w_i}{\partial t} = 0, \quad \text{a.e.}$$

that is,  $w$  is a stationary solution of (1.4), or solution of (1.3). From Theorem 1.6, we know such  $w$  is unique, thus we proved that for any sequence  $t_i \rightarrow +\infty$

$$u(t_i) \rightarrow w,$$

with  $w$  the unique solution of (1.3).

**Remark 7.1.** The above method can be easily generalized to systems with the form

$$\left\{ \begin{array}{ll} \frac{\partial u_i}{\partial t} - \Delta u_i \leq f_i(u_i), & \text{in } \Omega \times (0, +\infty), \\ \left( \frac{\partial}{\partial t} - \Delta \right) \left( u_i - \sum_{j \neq i} u_j \right) \geq f_i(u_i) - \sum_{j \neq i} f_j(u_j), & \text{in } \Omega \times (0, +\infty), \\ u_i u_j = 0, & \text{in } \Omega \times (0, +\infty), \\ u_i = \varphi_i, & \text{on } \partial\Omega \times (0, +\infty), \\ u_i = \phi_i, & \text{on } \Omega \times \{0\}, \end{array} \right. \quad (7.7)$$

where  $f_i(u_i)$  are Lipschitz continuous function on  $\mathbb{R}$ .

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