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On the Schrödinger–Maxwell equations under the effect of a general nonlinear term ☆

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Abstract

In this paper we prove the existence of a nontrivial solution to the nonlinear Schrödinger–Maxwell equations in $\mathbb{R}^3$, assuming on the nonlinearity the general hypotheses introduced by Berestycki and Lions.

Résumé

Dans cet article on démontre l'existence d'une solution non-banale et positive pour les équations non-linéaires de Schrödinger–Maxwell dans $\mathbb{R}^3$ en supposant que le terme non-linéaire satisfait les hypothèses introduites par Berestycki et Lions.

1. Introduction

In the recent years, the following electrostatic nonlinear Schrödinger–Maxwell equations, also known as nonlinear Schrödinger–Poisson system,

$$\begin{cases} -\Delta u + q\phi u = g(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = q u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{$\mathcal{SM}$}$$

have been object of interest for many authors. Indeed a similar system arises in many mathematical physics contexts, such as in quantum electrodynamics, to describe the interaction between a charge particle interacting with the electromagnetic field, and also in semiconductor theory, in nonlinear optics and in plasma physics. We refer to [4] for more details in the physics aspects.

The greatest part of the literature focuses on the study of the previous system for the very special nonlinearity $g(u) = -u + |u|^{p-1}u$, and existence, nonexistence and multiplicity results are provided in many papers for this particular problem (see [1,2,10,12–14,19–21,24,28]). In [9,11,27], also the linear and the asymptotic linear cases have

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been studied, whereas in [22] the problem has been dealt with in a bounded domain, with Neumann conditions on the boundary.

The aim of this paper is to study the Schrödinger–Maxwell system assuming the same very general hypotheses introduced by Berestycki & Lions, in their celebrated paper [7]. Actually, we assume that the following hold for g:

(g1) $g \in C(\mathbb{R},\mathbb{R})$;
(g2) $-\infty < \liminf_{s\to 0^+} g(s)/s \leqslant \limsup_{s\to 0^+} g(s)/s = -m < 0$;
(g3) $-\infty \leqslant \limsup_{s\to +\infty} g(s)/s^5 \leqslant 0$;
(g4) there exists $\zeta > 0$ such that $G(\zeta) := \int_0^\zeta g(s)\,ds > 0$.

Using similar assumptions on the nonlinearity g, [3,18] and [23] studied, respectively, a nonlinear Schrödinger equation in presence of an external potential and a system of weakly coupled nonlinear Schrödinger equations. We mention also [5] where the Klein–Gordon and in Klein–Gordon–Maxwell equations are considered.

The main result of the paper is

Theorem 1.1. *If g satisfies* **(g1)**–**(g4)**, *then there exists $q_0 > 0$ such that, for any $0 < q < q_0$, problem $(\mathcal{SM})$ admits a nontrivial positive radial solution $(u,\phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$.*

Some remarks on this result are in order:

- the assumptions are trivially satisfied by nonlinearities like $g(u) = -u + |u|^{p-1}u$, for any $p \in]1,5[$;
- hypotheses on g are *almost necessary* in the sense specified in [7, Subsection 2.2];
- the fact that the result is obtained for small q is not surprising for at least two reasons: first, because small q makes, in some sense, less strong the influence of the term ϕu, which constitutes, in the first equation, a perturbation with respect to the classical nonlinear Schrödinger equation treated in [7]; second, there is a nonexistence result for large q and $g(u) = -u + |u|^{p-1}u$ with $p \in]1,2]$ (see [24]).

From the technical point of view, dealing with $(\mathcal{SM})$ under the effect of a general nonlinear term presents several difficulties. Indeed the lack of the following global Ambrosetti–Rabinowitz growth hypothesis on g:

$$\text{there exists} \quad \mu > 2 \quad \text{such that } 0 < \mu G(s) \leqslant g(s)s, \quad \text{for all } s \in \mathbb{R},$$

brings on two obstacles to the standard Mountain Pass arguments both in checking the geometrical assumptions in the functional and in proving the boundedness of its Palais–Smale sequences. To overcome these difficulties, we will use a combined technique consisting in a truncation argument (see [17,21]) and a monotonicity trick *à la* Jeanjean [15] (see also Struwe [26]).

It is natural to ask about multiplicity of solutions of $(\mathcal{SM})$. However, our approach does not seem to work in this direction.

The paper is organized as follows. In Section 2 we introduce the functional framework for solving our problem by a variational approach. In Section 3 we define a sequence of modified functionals on which we can easily apply the Mountain Pass Theorem. Then we study the convergence of the sequence of critical points obtained. Finally Appendix A is devoted to prove a Pohozaev type identity which we use, in Section 3, as a fundamental tool in our arguments.

Notation.

- For any $1 \leqslant s \leqslant +\infty$, we denote by $\|\cdot\|_s$ the usual norm of the Lebesgue space $L^s(\mathbb{R}^3)$;
- $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the norm

$$\|u\|^2 := \int_{\mathbb{R}^3} |\nabla u|^2 + u^2;$$

- $\mathcal{D}^{1,2}(\mathbb{R}^3)$ is completion of $C_0^\infty(\mathbb{R}^3)$ (the compactly supported functions in $C^\infty(\mathbb{R}^3)$) with respect to the norm

$$\|u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^3)} := \int_{\mathbb{R}^3} |\nabla u|^2;$$

- for brevity, we denote $\alpha = 12/5$.

2. Functional setting

We first recall the following well-known facts (see, for instance [4,6,12,24]).

Lemma 2.1. *For every $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ solution of*

$$-\Delta\phi = qu^2, \quad \textit{in } \mathbb{R}^3.$$

Moreover,

(i) $\|\phi_u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^3)} = q\int_{\mathbb{R}^3} \phi_u u^2$;
(ii) $\phi_u \geqslant 0$;
(iii) *for any $\theta > 0$: $\phi_{u_\theta}(x) = \theta^2\phi_u(x/\theta)$, where $u_\theta(x) = u(x/\theta)$;*
(iv) *there exist $C, C' > 0$ independent of $u \in H^1(\mathbb{R}^3)$ such that*

$$\|\phi_u\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} \leqslant Cq\|u\|^2_\alpha,$$

and

$$\int_{\mathbb{R}^3} \phi_u u^2 \leqslant C'q\|u\|^4_\alpha; \tag{1}$$

(v) *if u is a radial function then ϕ_u is radial, too.*

Following [7], define $s_0 := \min\{s \in [\zeta, +\infty[\mid g(s) = 0\}$ ($s_0 = +\infty$ if $g(s) \neq 0$ for any $s \geqslant \zeta$). We set $\tilde{g} : \mathbb{R} \to \mathbb{R}$ the function such that

$$\tilde{g}(s) = \begin{cases} g(s) & \text{on } [0, s_0]; \\ 0 & \text{on } \mathbb{R}_+ \setminus [0, s_0]; \\ (g(-s) - ms)^+ - g(-s) & \text{on } \mathbb{R}_-. \end{cases} \tag{2}$$

By the strong maximum principle and by (ii) of Lemma 2.1, if u is a nontrivial solution of $(\mathcal{SM})$ with $\tilde{g}$ in the place of g, then $0 < u < s_0$ and so it is a positive solution of $(\mathcal{SM})$. Therefore we can suppose that g is defined as in (2), so that (**g1**), (**g2**), (**g4**) and then the following limit

$$\lim_{s\to\pm\infty} \frac{g(s)}{s^5} = 0 \tag{3}$$

hold.

We set

$$g_1(s) := \begin{cases} (g(s) + ms)^+, & \text{if } s \geqslant 0, \\ 0, & \text{if } s < 0, \end{cases}$$

$$g_2(s) := g_1(s) - g(s), \quad \text{for } s \in \mathbb{R}.$$

Since

$$\lim_{s\to 0} \frac{g_1(s)}{s} = 0,$$

$$\lim_{s\to\pm\infty} \frac{g_1(s)}{s^5} = 0, \tag{4}$$

and

$$g_2(s) \geqslant ms, \quad \forall s \geqslant 0, \tag{5}$$

by some computations, we have that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$g_1(s) \leqslant C_\varepsilon s^5 + \varepsilon g_2(s), \quad \forall s \geqslant 0. \tag{6}$$

If we set

$$G_i(t) := \int_0^t g_i(s)\,ds, \quad i = 1, 2,$$

then, by (5) and (6), we have

$$G_2(s) \geqslant \frac{m}{2} s^2, \quad \forall s \in \mathbb{R} \tag{7}$$

and for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$G_1(s) \leqslant \frac{C_\varepsilon}{6} s^6 + \varepsilon G_2(s), \quad \forall s \in \mathbb{R}. \tag{8}$$

The solutions $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ of $(\mathcal{SM})$ are the critical points of the action functional $\mathcal{E}: H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \to \mathbb{R}$, defined as

$$\mathcal{E}_q(u, \phi) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 - \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi|^2 + \frac{q}{2} \int_{\mathbb{R}^3} \phi u^2 - \int_{\mathbb{R}^3} G(u).$$

The action functional $\mathcal{E}_q$ exhibits a strong indefiniteness, namely it is unbounded both from below and from above on infinite dimensional subspaces. This indefiniteness can be removed using the reduction method described in [4,6], by which we are led to study a one variable functional that does not present such a strongly indefinite nature. Hence, it can be proved that $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of $(\mathcal{SM})$ (critical point of functional $\mathcal{E}_q$) if and only if $u \in H^1(\mathbb{R}^3)$ is a critical point of the functional $I_q : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$I_q(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{q}{4} \int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} G(u),$$

and $\phi = \phi_u$.

We will look for critical points of I_q on $H_r^1(\mathbb{R}^3) := \{u \in H^1(\mathbb{R}^3) \mid u \text{ is radial}\}$, which is a natural constraint.

3. The perturbed functional

Kikuchi, in [21], considered $(\mathcal{SM})$, where $g(u) = -u + |u|^{p-1}u$, with $1 < p < 5$. To overcome the difficulty in finding bounded Palais–Smale sequences for the associated functional I_q, following [17], he introduced the cut-off function $\chi \in C^\infty(\mathbb{R}_+, \mathbb{R})$ satisfying

$$\begin{cases} \chi(s) = 1, & \text{for } s \in [0, 1], \\ 0 \leqslant \chi(s) \leqslant 1, & \text{for } s \in]1, 2[, \\ \chi(s) = 0, & \text{for } s \in [2, +\infty[, \\ \|\chi'\|_\infty \leqslant 2, \end{cases} \tag{9}$$

and studied the following modified functional $\widetilde{I_q^T} : H^1(\mathbb{R}^3) \to \mathbb{R}$

$$\widetilde{I_q^T}(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{q}{4} \tilde{k}_T(u) \int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} G(u),$$

where, for every $T > 0$,

$$\tilde{k}_T(u) = \chi\left(\frac{\|u\|^2}{T^2}\right).$$

With this penalization, for T sufficiently large and for q sufficiently small, he is able to find a critical point $\bar{u}$ such that $\|\bar{u}\| \leqslant T$ and so he concludes that $\bar{u}$ is a critical point of I_q.

Let us observe that if $g(u) = f(u) - u$ with f satisfying the Ambrosetti–Rabinowitz growth condition, the arguments of Kikuchi still hold with slide modifications.

On the other hand, in presence of nonlinearities satisfying Berestycki–Lions assumptions, further difficulties arise about the geometry of our functional and compactness. First of all, as in [21], we introduce a similar truncated functional $I_q^T : H_r^1(\mathbb{R}^3) \to \mathbb{R}$

$$I_q^T(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{q}{4}k_T(u)\int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} G(u),$$

where, now,

$$k_T(u) = \chi\left(\frac{\|u\|_\alpha^\alpha}{T^\alpha}\right).$$

The C^1-functional I_q^T satisfies the geometrical assumptions of the Mountain–Pass Theorem but, under our general assumptions on the nonlinearity, we are not able to obtain the boundedness of the Palais–Smale sequences. Therefore we use an indirect approach developed by Jeanjean. We apply the following slight modified version of [15, Theorem 1.1] (see [16]).

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a Banach space and $J \subset \mathbb{R}_+$ an interval. Consider the family of C^1 functionals on X*

$$I_\lambda(u) = A(u) - \lambda B(u), \quad \forall \lambda \in J,$$

with B nonnegative and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$ and such that $I_\lambda(0) = 0$.

For any $\lambda \in J$ we set

$$\Gamma_\lambda := \{\gamma \in C([0,1], X) \mid \gamma(0) = 0,\ I_\lambda(\gamma(1)) < 0\}.$$

If for every $\lambda \in J$ the set Γ_λ is nonempty and

$$c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > 0, \tag{10}$$

then for almost every $\lambda \in J$ there is a sequence $(v_n)_n \subset X$ such that

(i) $(v_n)_n$ *is bounded*;
(ii) $I_\lambda(v_n) \to c_\lambda$;
(iii) $(I_\lambda)'(v_n) \to 0$ *in the dual X^{-1} of X.*

In our case, $X = H_r^1(\mathbb{R}^3)$,

$$A(u) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{q}{4}k_T(u)\int_{\mathbb{R}^3} \phi_u u^2 + \int_{\mathbb{R}^3} G_2(u),$$

$$B(u) := \int_{\mathbb{R}^3} G_1(u),$$

so that the perturbed functional we study is

$$I_{q,\lambda}^T(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{q}{4}k_T(u)\int_{\mathbb{R}^3} \phi_u u^2 + \int_{\mathbb{R}^3} G_2(u) - \lambda\int_{\mathbb{R}^3} G_1(u).$$

Actually, this functional is the restriction to the radial functions of a C^1-functional defined on the whole space $H^1(\mathbb{R}^3)$ and for every $u, v \in H^1(\mathbb{R}^3)$

$$\langle (I_q^T)'(u), v\rangle = \int_{\mathbb{R}^3} (\nabla u \mid \nabla v) + qk_T(u)\int_{\mathbb{R}^3} \phi_u uv + \frac{q\alpha}{4T^\alpha}\chi'\left(\frac{\|u\|_\alpha^\alpha}{T^\alpha}\right)\int_{\mathbb{R}^3}\phi_u u^2 \int_{\mathbb{R}^3}|u|^{\alpha-2}uv + \int_{\mathbb{R}^3} g_2(u)v - \lambda\int_{\mathbb{R}^3} g_1(u)v.$$

In order to apply Theorem 3.1, we have just to define a suitable interval J such that $\Gamma_\lambda \neq \emptyset$, for any $\lambda \in J$, and (10) holds.

Observe that, according to [7], as a consequence of (**g4**), there exists a function $z \in H_r^1(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} G_1(z) - \int_{\mathbb{R}^3} G_2(z) = \int_{\mathbb{R}^3} G(z) > 0. \tag{11}$$

Then there exists $0 < \bar\delta < 1$ such that

$$\bar\delta\int_{\mathbb{R}^3} G_1(z) - \int_{\mathbb{R}^3} G_2(z) > 0. \tag{12}$$

We define J as the interval $[\bar\delta, 1]$.

Lemma 3.2. *$\Gamma_\lambda \neq \emptyset$, for any $\lambda \in J$.*

Proof. Let $\lambda \in J$. Set $\bar\theta > 0$ and $\bar z = z(\cdot/\bar\theta)$.

Define $\gamma : [0,1] \to H_r^1(\mathbb{R}^3)$ in the following way

$$\gamma(t) = \begin{cases} 0, & \text{if } t = 0,\\ \bar z(\cdot/t), & \text{if } 0 < t \leqslant 1.\end{cases}$$

It is easy to see that γ is a continuous path from 0 to $\bar z$. Moreover, we have that

$$I_{q,\lambda}^T(\gamma(1)) \leqslant \frac{\bar\theta}{2}\int_{\mathbb{R}^3}|\nabla z|^2 + \frac{q}{4}\bar\theta^5\chi\left(\frac{\bar\theta^3\|z\|_\alpha^\alpha}{T^\alpha}\right)\int_{\mathbb{R}^3}\phi_z z^2 + \bar\theta^3\left(\int_{\mathbb{R}^3} G_2(z) - \bar\delta\int_{\mathbb{R}^3} G_1(z)\right)$$

and then, if $\bar\theta$ is sufficiently large, by (12) and (9) we get $I_{q,\lambda}^T(\gamma(1)) < 0$. □

Lemma 3.3. *There exists a constant $\tilde c > 0$ such that $c_\lambda \geqslant \tilde c > 0$ for all $\lambda \in J$.*

Proof. Observe that for any $u \in H_r^1(\mathbb{R}^3)$ and $\lambda \in J$, using (7) and (8) for $\varepsilon < 1$, we have

$$I_{q,\lambda}^T(u) \geqslant \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{q}{4}k_T(u)\int_{\mathbb{R}^3}\phi_u u^2 + \int_{\mathbb{R}^3} G_2(u) - \int_{\mathbb{R}^3} G_1(u) \geqslant \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2 + (1-\varepsilon)\frac{m}{2}\int_{\mathbb{R}^3} u^2 - \frac{C_\varepsilon}{6}\int_{\mathbb{R}^3}|u|^6$$

and then, by Sobolev embeddings, we conclude that there exists $\rho > 0$ such that, for any $\lambda \in J$ and $u \in H_r^1(\mathbb{R}^3)$ with $u \neq 0$ and $\|u\| \leqslant \rho$, it results $I_{q,\lambda}^T(u) > 0$. In particular, for any $\|u\| = \rho$, we have $I_{q,\lambda}^T(u) \geqslant \tilde c > 0$. Now fix $\lambda \in J$ and $\gamma \in \Gamma_\lambda$. Since $\gamma(0) = 0$ and $I_{q,\lambda}^T(\gamma(1)) < 0$, certainly $\|\gamma(1)\| > \rho$. By continuity, we deduce that there exists $t_\gamma \in]0,1[$ such that $\|\gamma(t_\gamma)\| = \rho$. Therefore, for any $\lambda \in J$,

$$c_\lambda \geqslant \inf_{\gamma\in\Gamma_\lambda} I_{q,\lambda}^T(\gamma(t_\gamma)) \geqslant \tilde c > 0. \qquad \square$$

We present a variant of the Strauss' compactness result [25] (see also [7, Theorem A.1]). It will be a fundamental tool in our arguments:

Theorem 3.4. *Let P and $Q : \mathbb{R} \to \mathbb{R}$ be two continuous functions satisfying*

$$\lim_{s\to\infty} \frac{P(s)}{Q(s)} = 0,$$

$(v_n)_n$, v and w be measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$, with z bounded, such that

$$\sup_n \int_{\mathbb{R}^N} \big|Q\big(v_n(x)\big)w\big|\,dx < +\infty,$$
$$P\big(v_n(x)\big) \to v(x) \quad \text{a.e. in } \mathbb{R}^N.$$

Then $\|(P(v_n) - v)w\|_{L^1(B)} \to 0$, for any bounded Borel set B.

Moreover, if we have also

$$\lim_{s\to 0} \frac{P(s)}{Q(s)} = 0,$$
$$\lim_{x\to\infty} \sup_n \big|v_n(x)\big| = 0,$$

then $\|(P(v_n) - v)w\|_{L^1(\mathbb{R}^N)} \to 0$.

In analogy with the well-known compactness result in [8], we state the following result

Lemma 3.5. *For any $\lambda \in J$, each bounded Palais–Smale sequence for the functional $I^T_{q,\lambda}$ admits a convergent subsequence.*

Proof. Let $\lambda \in J$ and $(u_n)_n$ be a bounded (PS) sequence for $I^T_{q,\lambda}$, namely

$$\big(I^T_{q,\lambda}(u_n)\big)_n \quad \text{is bounded},$$
$$\big(I^T_{q,\lambda}\big)'(u_n) \to 0 \quad \text{in } \big(H^1_r(\mathbb{R}^3)\big)'. \tag{13}$$

Up to a subsequence, we can suppose that there exists $u \in H^1_r(\mathbb{R}^3)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } H^1_r\big(\mathbb{R}^3\big), \tag{14}$$
$$u_n \to u \quad \text{in } L^p\big(\mathbb{R}^3\big), \quad 2 < p < 6, \tag{15}$$
$$u_n \to u \quad \text{a.e. in } \mathbb{R}^N. \tag{16}$$

If we apply Theorem 3.4 for $P(s) = g_i(s)$, $i = 1, 2$, $Q(s) = |s|^5$, $(v_n)_n = (u_n)_n$, $v = g_i(u)$, $i = 1, 2$ and $w \in C^\infty_0(\mathbb{R}^N)$, by (3), (4) and (16) we deduce that

$$\int_{\mathbb{R}^3} g_i(u_n)w \to \int_{\mathbb{R}^3} g_i(u)w, \quad i = 1, 2.$$

Moreover, by (15) and [24, Lemma 2.1], we have

$$k_T(u_n)\int_{\mathbb{R}^3} \phi_{u_n}u_n w \to k_T(u)\int_{\mathbb{R}^3} \phi_u u w,$$
$$\chi'\bigg(\frac{\|u_n\|^\alpha_\alpha}{T^\alpha}\bigg)\int_{\mathbb{R}^3} \phi_{u_n}u_n^2 \int_{\mathbb{R}^3} |u_n|^{\frac{2}{5}}u_n w \to \chi'\bigg(\frac{\|u\|^\alpha_\alpha}{T^\alpha}\bigg)\int_{\mathbb{R}^3} \phi_u u^2 \int_{\mathbb{R}^3} |u|^{\frac{2}{5}}u w.$$

As a consequence, by (13) and (14) we deduce $(I^T_{q,\lambda})'(u) = 0$ and hence

$$\int_{\mathbb{R}^3} |\nabla u|^2 + qk_T(u)\int_{\mathbb{R}^3} \phi_u u^2 + \frac{q\alpha}{4T^\alpha}\chi'\left(\frac{\|u\|_\alpha^\alpha}{T^\alpha}\right)\|u\|_\alpha^\alpha \int_{\mathbb{R}^3} \phi_u u^2 + \int_{\mathbb{R}^3} g_2(u)u = \lambda \int_{\mathbb{R}^3} g_1(u)u. \tag{17}$$

By weak lower semicontinuity we have:

$$\int_{\mathbb{R}^3} |\nabla u|^2 \leqslant \liminf_n \int_{\mathbb{R}^3} |\nabla u_n|^2. \tag{18}$$

Again, by (15) we have

$$k_T(u_n)\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \to k_T(u)\int_{\mathbb{R}^3} \phi_u u^2, \tag{19}$$

$$\chi'\left(\frac{\|u_n\|_\alpha^\alpha}{T^\alpha}\right)\|u_n\|_\alpha^\alpha \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \to \chi'\left(\frac{\|u\|_\alpha^\alpha}{T^\alpha}\right)\|u\|_\alpha^\alpha \int_{\mathbb{R}^3} \phi_u u^2. \tag{20}$$

If we apply Theorem 3.4 for $P(s) = g_1(s)s$, $Q(s) = s^2 + s^6$, $(v_n)_n = (u_n)_n$, $v = g_1(u)u$, and $w = 1$, by (3), (4) and (16), we deduce that

$$\int_{\mathbb{R}^3} g_1(u_n)u_n \to \int_{\mathbb{R}^3} g_1(u)u. \tag{21}$$

Moreover, by (16) and Fatou's lemma

$$\int_{\mathbb{R}^3} g_2(u)u \leqslant \liminf_n \int_{\mathbb{R}^3} g_2(u_n)u_n. \tag{22}$$

By (17), (19), (20), (21) and (22), and since $\langle (I_\lambda)'(u_n), u_n\rangle \to 0$, we have

$$\begin{aligned}
\limsup_n \int_{\mathbb{R}^3} |\nabla u_n|^2 &= \limsup_n \Bigg[\lambda \int_{\mathbb{R}^3} g_1(u_n)u_n - \int_{\mathbb{R}^3} g_2(u_n)u_n \\
&\qquad - qk_T(u_n)\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 - \frac{q\alpha}{4T^\alpha}\chi'\left(\frac{\|u_n\|_\alpha^\alpha}{T^\alpha}\right)\|u_n\|_\alpha^\alpha \int_{\mathbb{R}^3} \phi_{u_n} u_n^2\Bigg] \\
&\leqslant \lambda \int_{\mathbb{R}^3} g_1(u)u - \int_{\mathbb{R}^3} g_2(u)u - qk_T(u)\int_{\mathbb{R}^3} \phi_u u^2 - \frac{q\alpha}{4T^\alpha}\chi'\left(\frac{\|u\|_\alpha^\alpha}{T^\alpha}\right)\|u\|_\alpha^\alpha \int_{\mathbb{R}^3} \phi_u u^2 \\
&= \int_{\mathbb{R}^3} |\nabla u|^2.
\end{aligned} \tag{23}$$

By (18) and (23), we get

$$\lim_n \int_{\mathbb{R}^3} |\nabla u_n|^2 = \int_{\mathbb{R}^3} |\nabla u|^2, \tag{24}$$

hence

$$\lim_n \int_{\mathbb{R}^3} g_2(u_n)u_n = \int_{\mathbb{R}^3} g_2(u)u. \tag{25}$$

Since $g_2(s)s = ms^2 + h(s)$, with h a positive and continuous function, by Fatou's lemma we have

$$\int_{\mathbb{R}^3} h(u) \leqslant \liminf_n \int_{\mathbb{R}^3} h(u_n),$$

$$\int_{\mathbb{R}^3} u^2 \leqslant \liminf_n \int_{\mathbb{R}^3} u_n^2.$$

These last two inequalities and (25) imply that, up to a subsequence,

$$\int_{\mathbb{R}^3} u^2 = \lim_n \int_{\mathbb{R}^3} u_n^2,$$

which, together with (24), shows that $u_n \to u$ strongly in $H_r^1(\mathbb{R}^3)$. □

Lemma 3.6. *For almost every $\lambda \in J$, there exists $u^\lambda \in H_r^1(\mathbb{R}^3)$, $u^\lambda \neq 0$, such that $(I_{q,\lambda}^T)'(u^\lambda) = 0$ and $I_{q,\lambda}^T(u^\lambda) = c_\lambda$.*

Proof. By Theorem 3.1, for almost every $\lambda \in J$, there exists a bounded sequence $(u_n^\lambda)_n \subset H_r^1(\mathbb{R}^3)$ such that

$$I_{q,\lambda}^T(u_n^\lambda) \to c_\lambda; \tag{26}$$

$$(I_{q,\lambda}^T)'(u_n^\lambda) \to 0 \quad \text{in } (H_r^1(\mathbb{R}^3))'. \tag{27}$$

Up to a subsequence, by Lemma 3.5, we can suppose that there exists $u^\lambda \in H_r^1(\mathbb{R}^3)$ such that $u_n^\lambda \to u^\lambda$ in $H_r^1(\mathbb{R}^3)$. By Lemma 3.3, (26) and (27) we conclude. □

Therefore there exist $(\lambda_n)_n \subset J$ and $(u_n)_n \subset H_r^1(\mathbb{R}^3)$ such that

$$I_{q,\lambda_n}^T(u_n) = c_{\lambda_n}, \qquad (I_{q,\lambda_n}^T)'(u_n) = 0. \tag{28}$$

Lemma 3.7. *Let u_n be a critical point for I_{q,λ_n}^T at level c_{λ_n}. Then, for $T > 0$ sufficiently large, there exists $q_0 = q_0(T)$ such that for any $0 < q < q_0$, up to a subsequence, $\|u_n\|_\alpha \leqslant T$, for any $n \geqslant 1$.*

Proof. We will argue by contradiction.

First of all, since $(I_{q,\lambda_n}^T)'(u_n) = 0$, u_n satisfies the following Pohozaev type identity

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{5q}{4} k_T(u_n) \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 + \frac{3q}{T^\alpha} \chi'\left(\frac{\|u_n\|_\alpha^\alpha}{T^\alpha}\right) \|u_n\|_\alpha^\alpha \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \\ &\quad = 3\lambda_n \int_{\mathbb{R}^3} G_1(u_n) - 3\int_{\mathbb{R}^3} G_2(u_n) \end{aligned} \tag{29}$$

(see Appendix A for the proof).

Moreover, combining (29) with the first of (28) and by (1), we get

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla u_n|^2 &= 3c_{\lambda_n} + \frac{q}{2} k_T(u_n) \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 + \frac{3q}{T^\alpha} \chi'\left(\frac{\|u_n\|_\alpha^\alpha}{T^\alpha}\right) \|u_n\|_\alpha^\alpha \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \\ &\leqslant 3c_{\lambda_n} + C_1 q^2 k_T(u_n) \|u_n\|_\alpha^4 + C_2 \chi'\left(\frac{\|u_n\|_\alpha^\alpha}{T^\alpha}\right) \frac{q^2}{T^\alpha} \|u_n\|_\alpha^{4+\alpha}. \end{aligned} \tag{30}$$

We are going to estimate the right part of the previous inequality. By the min–max definition of the Mountain Pass level, we have

$$\begin{aligned} c_{\lambda_n} &\leqslant \max_\theta I_{q,\lambda_n}^T(z(\cdot/\theta)) \\ &\leqslant \max_\theta \left\{ \frac{\theta}{2} \int_{\mathbb{R}^3} |\nabla z|^2 + \theta^3 \left(\int_{\mathbb{R}^3} G_2(z) - \bar{\delta} \int_{\mathbb{R}^3} G_1(z) \right) \right\} \end{aligned}$$

$$+\max_{\theta}\left\{\frac{q}{4}\theta^5\chi\left(\frac{\theta^3\|z\|_\alpha^\alpha}{T^\alpha}\right)\int_{\mathbb{R}^3}\phi_z z^2\right\}$$
$$= A_1 + A_2(T)$$

where z is the function such that (11) holds.

Now, if $\theta^3 \geqslant 2T^\alpha/\|z\|_\alpha^\alpha$ then $A_2(T)=0$, otherwise we compute

$$A_2(T) \leqslant \frac{q}{4}\left(\frac{2T^\alpha}{\|z\|_\alpha^\alpha}\right)^{\frac{5}{3}}\int_{\mathbb{R}^3}\phi_z z^2 = C_3 q^2 T^4.$$

We also have

$$C_1 q^2 k_T(u_n)\|u_n\|_\alpha^4 \leqslant C_4 q^2 T^4$$
$$C_2\chi'\left(\frac{\|u_n\|_\alpha^\alpha}{T^\alpha}\right)\frac{q^2}{T^\alpha}\|u_n\|_\alpha^{4+\alpha} \leqslant C_5 q^2 T^4.$$

Then, from (30) we deduce that

$$\int_{\mathbb{R}^3}|\nabla u_n|^2 \leqslant 3A_1 + C_6 q^2 T^4. \tag{31}$$

On the other hand, since $\langle (I^T_{q,\lambda_n})'(u_n),(u_n)\rangle = 0$, by (6) we have that

$$\int_{\mathbb{R}^3}|\nabla u_n|^2 + qk_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2 + \frac{q\alpha}{4T^\alpha}\chi'\left(\frac{\|u_n\|_\alpha^\alpha}{T^\alpha}\right)\|u_n\|_\alpha^\alpha\int_{\mathbb{R}^3}\phi_{u_n}u_n^2 + \int_{\mathbb{R}^3}g_2(u_n)u_n$$
$$= \lambda_n\int_{\mathbb{R}^3}g_1(u_n)u_n \leqslant C_\varepsilon\int_{\mathbb{R}^3}|u_n|^6 + \varepsilon\int_{\mathbb{R}^3}g_2(u_n)u_n. \tag{32}$$

Now, by (5) and (32), we obtain

$$m(1-\varepsilon)\int_{\mathbb{R}^3}u_n^2 \leqslant (1-\varepsilon)\int_{\mathbb{R}^3}g_2(u_n)u_n$$
$$\leqslant C_\varepsilon\int_{\mathbb{R}^3}|u_n|^6 - \frac{q\alpha}{4T^\alpha}\chi'\left(\frac{\|u_n\|_\alpha^\alpha}{T^\alpha}\right)\|u_n\|_\alpha^\alpha\int_{\mathbb{R}^3}\phi_n u_n^2$$
$$\leqslant C\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\right)^3 + \bar{C}q^2T^4$$
$$\leqslant C\left(3A_1 + C_6 q^2 T^4\right)^3 + \bar{C}q^2T^4 \tag{33}$$

where in the last inequality we have used (31).

We suppose by contradiction that there exists no subsequence of $(u_n)_n$ which is uniformly bounded by T in the α-norm. As a consequence, for a certain $\bar{n}$ it should result that

$$\|u_n\|_\alpha > T, \quad \forall n \geqslant \bar{n}. \tag{34}$$

Without any loss of generality, we are supposing that (34) is true for any u_n.

Therefore, by (31) and (33), we conclude that

$$T^2 < \|u_n\|_\alpha^2 \leqslant C\|u_n\|^2 \leqslant C_7 + C_8 q^2 T^4 + C_9 q^4 T^8 + C_{10} q^6 T^{12}$$

which is not true for T large and q small enough: indeed we can find $T_0 > 0$ such that $T_0^2 > C_7 + 1$ and $q_0 = q_0(T_0)$ such that $C_8 q^2 T_0^4 + C_9 q^4 T_0^8 + C_{10} q^6 T_0^{12} < 1$, for any $q < q_0$, and we find a contradiction. □

Proof of Theorem 1.1. Let T, q_0 be as in Lemma 3.7 and fix $0 < q < q_0$. Let u_n be a critical point for I^T_{q,λ_n} at level c_{λ_n}. We prove that $(u_n)_n$ is a H^1-bounded Palais–Smale sequence for I_q.

Since by Lemma 3.7

$$\|u_n\|_\alpha \leqslant T, \tag{35}$$

the boundedness in the H^1-norm trivially follows from arguments such as those in (31) and (33). Finally, by (35), certainly we have that

$$I^T_{q,\lambda_n}(u_n) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{q}{4}\int_{\mathbb{R}^3} \phi_u u^2 + \int_{\mathbb{R}^3} G_2(u) - \lambda_n \int_{\mathbb{R}^3} G_1(u),$$

and then, since $\lambda_n \nearrow 1$, we can prove that $(u_n)_n$ is a (PS) sequence for I_q by similar argument as in [3, Theorem 1.1].

Now we conclude arguing as in Lemma 3.5. □

Appendix A. A Pohozaev type identity

In this section we show that if $u, \phi \in H^2_{loc}(\mathbb{R}^3)$ solve

$$\begin{cases} -\Delta u + q k_T(u)\phi u + q\dfrac{\alpha}{T^\alpha}\chi'\Big(\dfrac{\|u\|_\alpha^\alpha}{T^\alpha}\Big)|u|^{2/5}u \displaystyle\int_{\mathbb{R}^3} \phi u^2 = g(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = q u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{36}$$

then the following Pohozaev type identity

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{5q}{4}k_T(u)\int_{\mathbb{R}^3} \phi u^2 + \frac{3q}{T^\alpha}\chi'\Big(\frac{\|u\|_\alpha^\alpha}{T^\alpha}\Big)\|u\|_\alpha^\alpha \int_{\mathbb{R}^3} \phi u^2 = 3\int_{\mathbb{R}^3} G(u) \tag{37}$$

holds.

Indeed, by [13, Lemma 3.1], for every $R > 0$, we have

$$\int_{B_R} -\Delta u(x\cdot\nabla u) = -\frac{1}{2}\int_{B_R} |\nabla u|^2 - \frac{1}{R}\int_{\partial B_R} |x\cdot\nabla u|^2 + \frac{R}{2}\int_{\partial B_R} |\nabla u|^2, \tag{38}$$

$$\int_{B_R} \phi u(x\cdot\nabla u) = -\frac{1}{2}\int_{B_R} u^2(x\cdot\nabla\phi) - \frac{3}{2}\int_{B_R} \phi u^2 + \frac{R}{2}\int_{\partial B_R} \phi u^2, \tag{39}$$

$$\int_{B_R} g(u)(x\cdot\nabla u) = -3\int_{B_R} G(u) + R\int_{\partial B_R} G(u), \tag{40}$$

$$\int_{B_R} |u|^{2/5}u(x\cdot\nabla u) = -\frac{3}{\alpha}\int_{B_R} |u|^\alpha + \frac{R}{\alpha}\int_{\partial B_R} |u|^\alpha, \tag{41}$$

where B_R is the ball of $\mathbb{R}^3$ centered in the origin and with radius R.

Multiplying the first equation of (36) by $x\cdot\nabla u$ and the second equation by $x\cdot\nabla\phi$ and integrating on B_R, by (38), (39), (40) and (41) we get

$$\begin{aligned} &-\frac{1}{2}\int_{B_R} |\nabla u|^2 - \frac{1}{R}\int_{\partial B_R} |x\cdot\nabla u|^2 + \frac{R}{2}\int_{\partial B_R} |\nabla u|^2 \\ &\quad - \frac{q}{2}k_T(u)\int_{B_R} u^2(x\cdot\nabla\phi) - \frac{3q}{2}k_T(u)\int_{B_R} \phi u^2 + \frac{Rq}{2}k_T(u)\int_{\partial B_R} \phi u^2 \end{aligned}$$

$$- \frac{3q}{T^\alpha}\chi'\left(\frac{\|u\|_\alpha^\alpha}{T^\alpha}\right)\int_{\mathbb{R}^3}\phi u^2\int_{B_R}|u|^\alpha + \frac{Rq}{T^\alpha}\chi'\left(\frac{\|u\|_\alpha^\alpha}{T^\alpha}\right)\int_{\mathbb{R}^3}\phi u^2\int_{\partial B_R}|u|^\alpha$$

$$= -3\int_{B_R}G(u) + R\int_{\partial B_R}G(u) \tag{42}$$

and

$$q\int_{B_R}u^2(x\cdot\nabla\phi) = -\frac{1}{2}\int_{B_R}|\nabla\phi|^2 - \frac{1}{R}\int_{\partial B_R}|x\cdot\nabla\phi|^2 + \frac{R}{2}\int_{\partial B_R}|\nabla\phi|^2. \tag{43}$$

Substituting (43) into (42) we obtain

$$\begin{aligned}
&-\frac{1}{2}\int_{B_R}|\nabla u|^2 - \frac{3q}{2}k_T(u)\int_{B_R}\phi u^2 + \frac{1}{4}k_T(u)\int_{B_R}|\nabla\phi|^2 \\
&\quad - \frac{3q}{T^\alpha}\chi'\left(\frac{\|u\|_\alpha^\alpha}{T^\alpha}\right)\int_{\mathbb{R}^3}\phi u^2\int_{B_R}|u|^\alpha + 3\int_{B_R}G(u) \\
&= \frac{1}{R}\int_{\partial B_R}|x\cdot\nabla u|^2 - \frac{R}{2}\int_{\partial B_R}|\nabla u|^2 - \frac{1}{2R}k_T(u)\int_{\partial B_R}|x\cdot\nabla\phi|^2 \\
&\quad + \frac{R}{4}k_T(u)\int_{\partial B_R}|\nabla\phi|^2 - \frac{Rq}{2}k_T(u)\int_{\partial B_R}\phi u^2 \\
&\quad - \frac{Rq}{T^\alpha}\chi'\left(\frac{\|u\|_\alpha^\alpha}{T^\alpha}\right)\int_{\mathbb{R}^3}\phi u^2\int_{\partial B_R}|u|^\alpha + R\int_{\partial B_R}G(u).
\end{aligned}$$

As in [13], the right-hand side goes to zero as $R \to +\infty$ and so we get

$$\begin{aligned}
&-\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2 - \frac{3q}{2}k_T(u)\int_{\mathbb{R}^3}\phi u^2 + \frac{1}{4}k_T(u)\int_{\mathbb{R}^3}|\nabla\phi|^2 \\
&\quad - \frac{3q}{T^\alpha}\chi'\left(\frac{\|u\|_\alpha^\alpha}{T^\alpha}\right)\int_{\mathbb{R}^3}\phi u^2\int_{\mathbb{R}^3}|u|^\alpha + 3\int_{\mathbb{R}^3}G(u) = 0.
\end{aligned}$$

If $(u,\phi_u) \in H^1(\mathbb{R}^3)\times\mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of (36), by standard regularity results, $u,\phi_u \in H^2_{loc}(\mathbb{R}^3)$ and, by (i) of Lemma 2.1, we get (37).

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