Regularity for solutions of nonlocal, nonsymmetric equations

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Abstract

We study the regularity for solutions of fully nonlinear integro differential equations with respect to nonsymmetric kernels. More precisely, we assume that our operator is elliptic with respect to a family of integro differential linear operators where the symmetric parts of the kernels have a fixed homogeneity σ and the skew symmetric parts have strictly smaller homogeneity τ. We prove a weak ABP estimate and $C^{1,\alpha}$ regularity. Our estimates remain uniform as we take $\sigma \to 2$ and $\tau \to 1$ so that this extends the regularity theory for elliptic differential equations with dependence on the gradient.
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1. Introduction

We are interested in studying integro differential equations that arise when studying discontinuous stochastic processes. By the Lèvy–Khintchine formula, the generator of an n-dimensional Lèvy process is given by

$$Lu(x) = \sum_{i,j} a_{i,j} u_{i,j} + \sum_i b_i u_i + \int_{\mathbb{R}^n} \left(u(x+y) - u(x) - \nabla u(x) \cdot y \chi_{B_1}(y) \right) d\mu_x(y),$$

where μ is a positive measure such that $\int |y|^2/(|y|^2 + 1) \, d\mu(y) < \infty$. The first and second terms correspond to the diffusion and drift parts, and the third one corresponds to the jump. The effect of first term is already well understood as it regularizes the solution. The type of equations that we will study comes from processes with only the jump part,

$$Lu(x) = \int_{\mathbb{R}^n} \left(u(x+y) - u(x) - \nabla u(x) \cdot y \chi_{B_1}(y) \right) d\mu_x(y). \tag{1.1}$$

More general than the linear operator are the fully nonlinear ones, which are also important in stochastic control as seen in [9]. For example, a convex type of equation takes the form

$$Iu(x) = \sup_\alpha L_\alpha u(x). \tag{1.2}$$

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Eq. (1.2) can be seen as a one player game, for which he can choose different strategies at each step to maximize the expected value of some function at the first exit point of the domain. A natural extension for (1.2), when there are two players competing is

$$Iu(x) = \inf_\beta \sup_\alpha L_{\alpha\beta} u(x).$$

We are mainly interested in studying interior regularity for solutions of

$$Iu(x) = f(x), \quad \text{in } \Omega, \tag{1.3}$$

for f continuous, Ω a given domain and I a fully nonlinear operator of fractional order to be defined in the next section. In [8] the regularity for this type of problem was already established by using analytic techniques. However those estimates blow up as the order of the equation goes to the classical one, so it was expected that better estimates could be possible. Those results are more elaborated and presented in [4–6] in the case that the kernels are symmetric. We remove this symmetry hypothesis of the kernel and are able to obtain C^α regularity and $C^{1,\alpha}$ regularity for translation invariant equations.

The paper is divided as follows. In Section 2 we give most of the relevant definitions and point out some important examples to keep in mind. Specifically we will introduce the notions of fully nonlinear, nonlocal operators, ellipticity and viscosity solution. In Section 3 we state the main results of this work, which is C^α and $C^{1,\alpha}$ regularity for solutions of equations of the form (1.3) under different hypotheses on the kernels. In Section 4 we study the basic stability properties of the elliptic integro differential operators, the comparison principle and prove existence of solution of the Dirichlet problem by using Perron's method. Sections 5 and 6 are the core of this paper. In Section 5 we prove a weak ABP estimate which combined with a rescaling argument will allow us to prove, in Section 6, a point estimate lemma. In Section 7 we deal with the Hölder regularity by applying the previous point estimate to show a geometric decay of the oscillation of the solution. Finally in Section 8 we use that, for translation invariant equations, the incremental quotients are also solutions of equations in the same ellipticity class in order to show Hölder regularity for the first derivatives.

2. Preliminaries and viscosity solutions

In this work we restrict ourselves to measures $d\mu_x = K(x, y)\, dy$. From Eq. (1.1) we formally can write

$$Lu(x) = \int \delta_e(u, x; y) K_e(x; y)\, dy + \int \delta_o(u, x; y) K_o(x; y)\, dy + b(x) \cdot \nabla u(x), \tag{2.4}$$

where

$$\delta_e(u, x; y) = u(x + y) + u(x - y) - 2u(x),$$
$$\delta_o(u, x; y) = u(x + y) - u(x - y).$$

$K_{e,o}$ are the even and odd parts of K with respect to y and b is a vector valued function given by

$$b(x) = \int_{B_1} K_o(x; y) y\, dy.$$

Notice that if the total kernel K is even the last two terms in (2.4) disappear. This was convenient in [4] as these bring additional difficulties with the scaling as can be noticed in [7].

The second term can be considered as a drift term, in the sense that it has a "direction", K_o being odd. If the singularity of K_o at the origin is of order $n + \tau$, with $\tau \to 1^-$, then this integral becomes a gradient term. For this reason, one can consider studying the regularizing effect of the first two terms. The linear operators we are interested are always of the form

$$Lu(x) = P.V. \int_{\mathbb{R}^n} \big(u(x + y) - u(x)\big) K(x; y)\, dy$$

$$:= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon} \big(u(x + y) - u(x)\big) K(x; y)\, dy. \tag{2.5}$$

2.1. Integrability conditions

We want here to make sense of the decomposition (2.4). All we need for that is that the kernels are not too singular whenever $u \in C^{1,1}(x_0)$. The following definition is the same as in [4].

Definition 2.1. We say that a function u is $C^{1,1}$ at the point x_0 and write $u \in C^{1,1}(x_0)$ if and only if there exist a vector $v \in \mathbb{R}^n$ and a number $M > 0$ such that

$$\left| u(x+y) - u(x) - v \cdot y \right| < M|y|^2 \quad \text{for } |y| \text{ small enough.}$$

This implies in particular that $|\delta_e(u, x_0; y)| = O(|y|^2)$ and $|\delta_o(u, x_0; y)| = O(|y|)$ as $|y|$ goes to zero.

With this notion at hand we ask for the kernel K, when decomposed in its symmetric and skew symmetric parts, $K = K_e + K_o$ respectively, to satisfy the following integrability conditions,

$$\int \frac{|y|^2}{|y|^2 + 1} \left| K_e(y) \right| dy < \infty, \tag{2.6}$$

$$\int \frac{|y|}{|y| + 1} \left| K_o(y) \right| dy < \infty. \tag{2.7}$$

These conditions allow us to write rigorously

$$Lu(x) = \int \delta_e(u, x; y) K_e(y) \, dy + \int \delta_o(u, x; y) K_o(y) \, dy,$$

for $u \in C^{1,1}(x) \cap L^\infty(\mathbb{R}^n)$.

We say that a family $\mathcal{L}$ of linear operators satisfies the integrability conditions uniformly when the upper bounds in (2.6) and (2.7) can be taken independent of $L \in \mathcal{L}$.

2.2. Nonlinear, nonlocal operators

Before defining what will be for us a fully nonlinear nonlocal operator we present some examples to keep in mind. They are constructed from the linear operators in (2.5):

$$\text{(Inf-sup type) } Iu(x) = \inf_{\beta} \sup_{\alpha} L_{\alpha,\beta} u(x), \tag{2.8}$$

$$\text{(Maximal) } \mathcal{M}_{\mathcal{L}}^+ u(x) = \sup_{L \in \mathcal{L}} Lu(x), \tag{2.9}$$

$$\text{(Minimal) } \mathcal{M}_{\mathcal{L}}^- u(x) = \inf_{L \in \mathcal{L}} Lu(x). \tag{2.10}$$

Definition 2.2. We say that I is a nonlocal fully nonlinear operator if it satisfies the following:

 (i) If u is any bounded $C^{1,1}(x)$ function then $Iu(x)$ is well defined.
(ii) If $u \in C^{1,1}(\Omega)$ for some open set $\Omega \subseteq \mathbb{R}^n$, then Iu is a continuous function in Ω.

Our examples satisfy immediately (i) in the definition above. In order to have the continuity stated in (ii) we need to check a uniform integrability condition in the kernels.

Lemma 2.1. Let I be of the form (2.8) where $\mathcal{L} = \{L_{\alpha,\beta}\}$ satisfy the integrability conditions (2.6) and (2.7) uniformly. Then $Iu \in C(\Omega)$ for every $u \in C^{1,1}(\Omega)$.

Proof. We need to prove that $L_{\alpha,\beta} u$ are equicontinuous over compact sets of Ω in order to conclude by Arzela–Ascoli's Theorem. Fix $\delta > 0$ and let's work over the points $x \in \Omega$ that are at least δ away from $\mathbb{R}^n \setminus \Omega$.

Let $L_{\alpha,\beta}$ have associated the kernels $K_{\alpha,\beta}(y) = (K_{\alpha,\beta})_e(y) + (K_{\alpha,\beta})_o(y)$, decomposed in its symmetric and skew symmetric parts. Because $u \in C^{1,1}(\Omega)$ we can write for $x \in \Omega$,

$$L_{\alpha,\beta}u(x) = \int \delta_e(u, x; y)(K_{\alpha,\beta})_e(y)\,dy + \int \delta_o(u, x; y)(K_{\alpha,\beta})_o(y)\,dy$$

$$= \int_{B_r} \delta_e(u, x; y)(K_{\alpha,\beta})_e(y)\,dy + \int_{\mathbb{R}^n \setminus B_r} \delta_e(u, x; y)(K_{\alpha,\beta})_e(y)\,dy$$

$$+ \int_{B_r} \delta_o(u, x; y)(K_{\alpha,\beta})_o(y)\,dy + \int_{\mathbb{R}^n \setminus B_r} \delta_o(u, x; y)(K_{\alpha,\beta})_o(y)\,dy.$$

The first and third integrals can be smaller than any $\varepsilon > 0$ if r is small enough. Use that u is $C^{1,1}$ to get that $|\delta_e(u, x; y)| \leqslant C|y|^2$ and $|\delta_o(u, x; y)| \leqslant C|y|$ if $r < \delta$ and for some constant C independent of x. By the integrability condition and the absolute continuity of the integral we get that, for even smaller radius r, the aforementioned terms are smaller than ε, independently of x and $L_{\alpha,\beta}$.

Now if we fix a radius r, we get that the second and fourth terms are equicontinuous in x. For this we just need to apply Lemma 4.1 in [4].

As a consequence of the previous two paragraphs, we obtain that the difference $|L_{\alpha,\beta}u(x) - L_{\alpha,\beta}u(x')|$ is arbitrarily small when $|x - x'|$ is sufficiently small, independently of x, x' (both at least δ away from $\mathbb{R}^n \setminus \Omega$) and $L_{\alpha,\beta}$. $\quad \square$

2.3. Extremal operators comparable to the fractional Laplacians

An important family, that will be used for the study of regularity, is given by $\mathcal{L}_0 = \mathcal{L}_0(\sigma, \tau, \lambda, \Lambda, b)$ with all the linear operators L such that the kernels $K_{e,o}$ are comparable to those of the σ fractional Laplacian and some derivation of order τ:

$$(2 - \sigma)\frac{\lambda}{|y|^{n+\sigma}} \leqslant K_e \leqslant (2 - \sigma)\frac{\Lambda}{|y|^{n+\sigma}}, \tag{2.11}$$

$$|K_o| \leqslant (1 - \tau)\frac{b}{|y|^{n+\tau}}. \tag{2.12}$$

In order to satisfy the integrability conditions all we need is $\sigma \in (0, 2)$ and $\tau \in (0, 1)$.

In this family the operators (2.9), (2.10) take the explicit form

$$\mathcal{M}^+_{\mathcal{L}_0}v(x) = \mathcal{M}^+_\sigma v(x) + b(1 - \tau)\int_{\mathbb{R}^n} \frac{|\delta_o(v, x, y)|}{|y|^{n+\tau}}, \tag{2.13}$$

$$\mathcal{M}^-_{\mathcal{L}_0}v(x) = \mathcal{M}^-_\sigma v(x) - b(1 - \tau)\int_{\mathbb{R}^n} \frac{|\delta_o(v, x, y)|}{|y|^{n+\tau}}, \tag{2.14}$$

where $\mathcal{M}^\pm_\sigma$ are the extremal operators found in [4], i.e.

$$\mathcal{M}^+_\sigma v(x) = (2 - \sigma)\int_{\mathbb{R}^n} \frac{\Lambda\delta^+_e(v, x; y) - \lambda\delta^-_e(v, x; y)}{|y|^{n+\sigma}},$$

$$\mathcal{M}^-_\sigma v(x) = (2 - \sigma)\int_{\mathbb{R}^n} \frac{\lambda\delta^+_e(v, x; y) - \Lambda\delta^-_e(v, x; y)}{|y|^{n+\sigma}}.$$

$\delta^\pm_{e,o}$ denote the positive and negative parts of $\delta_{e,o}$ respectively ($\delta_{e,o} = \delta^+_{e,o} - \delta^-_{e,o}$).

For ease of notation we also introduce what we call the maximal τ derivative $|D_\tau|$, given by

$$|D_\tau|v(x) = (1 - \tau)\int_{\mathbb{R}^n} \frac{|\delta_o(v, x; y)|}{|y|^{n+\tau}}\,dy,$$

so that we can rewrite the operators as

$$\mathcal{M}^\pm_{\mathcal{L}_0}v(x) = \mathcal{M}^\pm_\sigma v(x) \pm b|D_\tau|v(x).$$

The factors $(2 - \sigma)$ and $(1 - \tau)$ become important as $\sigma \to 2$, and $\tau \to 1$, as they will allow us to recover second order differential equations with gradient terms as limits of integro differential equations.

Notice that this family admits kernels that could be positive and negative. The natural assumption, due the positivity of the measure in the Lèvy–Khintchine formula, is to consider operators which are elliptic with respect to a family $\mathcal{L}$ with nonnegative kernels. Because of this reason we consider also the family $\tilde{\mathcal{L}}_0 \subseteq \mathcal{L}_0$, given by all possible operators L with total kernel $K = K_e + K_o \geqslant 0$ satisfying the conditions (2.11) and (2.12). We point out that given v smooth, we have the following natural inequalities,

$$\mathcal{M}^+_{\mathcal{L}_0} v(x) \geqslant \mathcal{M}^+_{\tilde{\mathcal{L}}_0} v(x) \geqslant \mathcal{M}^-_{\tilde{\mathcal{L}}_0} v(x) \geqslant \mathcal{M}^-_{\mathcal{L}_0} v(x).$$

This control will be useful, since we have explicit formulas for the maximal operators in the larger class $\mathcal{L}_0$.

2.4. Ellipticity

The reason why we introduce extremal operators is because they are the ones that control *elliptic* nonlinear operators. Here is the definition of ellipticity for a general family $\mathcal{L}$ of linear operators.

Definition 2.3. Let $\mathcal{L}$ be a class of linear integro differential operators satisfying (2.6) and (2.7). We say that a fully nonlinear operator I is elliptic with respect to the class $\mathcal{L}$ if

$$\mathcal{M}^-_{\mathcal{L}}(u - v)(x) \leqslant Iu(x) - Iv(x) \leqslant \mathcal{M}^+_{\mathcal{L}}(u - v)(x). \tag{2.15}$$

2.5. Scaling

A tool we will be using frequently is the scaling. Consider a smooth bounded function u and an operator I, elliptic with respect to $\mathcal{L} \subseteq \mathcal{L}_0(\sigma, \tau, \lambda, \Lambda, b)$, such that

$$Iu = f \quad \text{in } \Omega.$$

If we rescale u by $u_{\alpha,\beta}(x) = \alpha u(\beta x)$ then the equation gets rescaled in the following way,

$$I_{\alpha,\beta} u_{\alpha,\beta} = f_{\alpha,\beta} \quad \text{in } \beta^{-1}\Omega,$$

where

$$(I_{\alpha,\beta} v)(x) = \alpha I\big(\alpha^{-1} v\big(\beta^{-1}\cdot\big)\big)(\beta x),$$

$$f_{\alpha,\beta}(x) = \alpha f(\beta x).$$

In particular, if $I = L$ is linear with kernel K then the kernel $K_{\alpha,\beta}$ for $L_{\alpha,\beta}$ gets transformed according to the change of variables formula,

$$K_{\alpha,\beta}(x, y) = \beta^n K(\beta x, \beta y).$$

The extremal operators $\mathcal{M}^\pm_\sigma$ and $|D_\tau|$ scale with order σ and τ respectively, because by the change of variables formula,

$$\mathcal{M}^\pm_\sigma u_{\alpha,\beta}(x) = \alpha \beta^{-\sigma} \big(\mathcal{M}^\pm_\sigma u\big)(\beta x),$$

$$|D_\tau| u_{\alpha,\beta}(x) = \alpha \beta^{-\tau} \big(|D_\tau| u\big)(\beta x).$$

This implies that, going back to I nonlinear, the operator $I_{\alpha,\beta}$ belongs to some rescaled family of linear operators $\mathcal{L}_{\alpha,\beta} \subseteq \mathcal{L}_0(\sigma, \tau, \beta^{-\sigma}\lambda, \beta^{-\sigma}\Lambda, \beta^{-\tau}b)$.

At many points we will use that when $\sigma > \tau$ and β is small then the rescaled equation is dominated by the derivatives of order σ.

2.6. Viscosity solutions

Viscosity solutions provide the right framework to study fully nonlinear equations, as seen in the local case in [3], and also in the nonlocal case in [1].

Definition 2.4. A bounded function $u : \mathbb{R}^n \to \mathbb{R}$, upper (lower) semicontinuous in $\bar{\Omega}$, is said to be a sub solution (super solution) to $Iu = f$, and we write $Iu \geqslant f$ ($Iu \leqslant f$), if every time φ is a second order polynomial touching u by above (below) at x in a neighborhood N, i.e.

(i) $\varphi(x) = u(x)$,
(ii) $\varphi(y) > u(y)$ ($\varphi(y) < u(y)$) for every $x \in N \setminus \{x\}$,

then $Iv(x) \geqslant f(x)$ ($Iv(x) \leqslant f(x)$), for v defined as

$$v = \begin{cases} \varphi & \text{in } N, \\ u & \text{in } \mathbb{R}^n \setminus N. \end{cases}$$

Later in Section 4 we will see that in many cases this definition is equivalent to one which includes many more test functions.

3. Statement of results

In this section we state the main results obtained in this paper. An important tool used to prove the following theorems is a point estimate, also known as L^ε Lemma. This comes from a partial ABP inequality similar to the one in [4] and a scaling argument which decreases the effect of the lower order term.

In order to prove our regularity results we will need to impose some assumptions on σ and τ. Given $\sigma_0, \tau_0, m, A_0 > 0$, considered as universal constants, we will assume that the following holds.

(H1) $2 > \sigma \geqslant \sigma_0 > 0$, $\min(1, \sigma) > \tau \geqslant \tau_0 > 0$,
(H2) $\sigma - \tau \geqslant m > 0$,
(H3) $\lambda A_0 (2 - \sigma) \geqslant b(1 - \tau)$.

Theorem 3.1. *Let $\sigma_0, \tau_0, m, A_0 > 0$ and assume that* (H1), (H2) *and* (H3) *hold. Let u be a bounded function in $\mathbb{R}^n$ such that in B_1,*

$$\mathcal{M}^+_{\tilde{\mathcal{L}}_0} u \geqslant -C_0 \quad and \quad \mathcal{M}^-_{\tilde{\mathcal{L}}_0} u \leqslant C_0,$$

in the viscosity sense. Then there exists a universal exponent $\alpha > 0$ such that $u \in C^\alpha(B_{1/2})$ and

$$\|u\|_{C^\alpha(B_{1/2})} \leqslant C(\|u\|_\infty + C_0)$$

for some universal constant $C > 0$.

An immediate corollary is the following.

Corollary 3.2. *Let $\sigma_0, \tau_0, m, A_0 > 0$ and assume that* (H1), (H2) *and* (H3) *hold. Let I be an elliptic operator of the inf-sup type as in* (2.8) *with all the linear operators in $\tilde{\mathcal{L}}_0$ and let $f \in C(\bar{B}_1)$. Let u be a bounded function in $\mathbb{R}^n$ such that in B_1,*

$$Iu = f,$$

in the viscosity sense. Then there exists a universal exponent $\alpha > 0$ such that $u \in C^\alpha(B_{1/2})$ and

$$\|u\|_{C^\alpha(B_{1/2})} \leqslant C(\|u\|_\infty + \|f\|_\infty)$$

for some universal constant $C > 0$.

Coming back to Theorem 3.1, we would like to point out that our bounds remain uniform as $\sigma \to 2$ and $\tau \to 1$, which allows us to recover Hölder regularity for equations with bounded measurable coefficients including gradient terms. For fixed σ and τ these results were proven in [8] and [2] by using analytic techniques. These estimates are not uniform in σ and blow up as the order goes to the classical one.

The order α of our Hölder estimates deteriorates as $\tau \to \sigma$. In this critical case $\sigma = \tau$, both terms in the equation are of the same order and rescaling the equation doesn't have any effect on the τ derivative, hence our argument doesn't work. It is known from the previous work in [8] and [2] that the same result holds even when $\sigma = \tau$. By combining both results, we can get regularity uniformly in σ and τ, disregarding the separation between σ and τ (hypothesis (H2)).

To get higher regularity we will need to add an extra assumption to the kernels, which is a modulus of continuity of K_e and K_o in measure. More precisely, given $\rho_0 > 0$, we define the class $\mathcal{L}_1 = \mathcal{L}_1(\sigma, \tau, \lambda, \Lambda, b, \rho_0, C) \subseteq \tilde{\mathcal{L}}_0(\sigma, \tau, \lambda, \Lambda, b)$ such that it contains all the linear operators L with kernels $K = K_e + K_o \geqslant 0$ such that K_e and K_o satisfy (2.11) and (2.12) respectively and

$$\int_{\mathbb{R}^n \setminus B_{\rho_0}} \frac{|K(y) - K(y - h)|}{|h|} \, dy \leqslant C \tag{3.16}$$

for every $|h| \leqslant \rho_0/2$.

A sufficient condition for (3.16) is for example that $|\nabla K(y)| \leqslant \Lambda/|y|^{n+1+\sigma}$.

In this smaller class we are able to get $C^{1,\alpha}$ by studying the incremental quotients of solutions and using the a priori C^α estimates given by Theorem 3.1. The proof follows the ideas of [3] and [4].

Theorem 3.3. *Let $\sigma_0, \tau_0, m, A_0 > 0$ and assume that* (H1), (H2) *and* (H3) *hold. Let I be an elliptic operator of the inf-sup type as in* (2.8) *with all the linear operators in $\mathcal{L}_1$. There is $\rho_0 > 0$ small enough so that if u is a bounded function in $\mathbb{R}^n$ such that in B_1,*

$$Iu = 0,$$

in the viscosity sense, then there is a universal $\alpha > 0$ such that $u \in C^{1,\alpha}(B_{1/2})$ and

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leqslant C\|u\|_\infty$$

for some universal $C > 0$.

In the proofs of our regularity results the odd part doesn't have to be of a fixed order. We could ask for example

$$|K_o| \leqslant b \max\left(\frac{1 - \tau_1}{|y|^{n+\tau_1}}, \frac{1 - \tau_2}{|y|^{n+\tau_2}}\right)$$

with $0 < \tau_1 \leqslant \tau_2 < \min(1, \sigma)$. The reason is that the proofs will treat the lower order term as a perturbation term that can be made small enough after a dilation large enough. For the sake of keeping the exposition simpler we decided to restrict to the case of $\tau_1 = \tau_2 = \tau$.

4. Qualitative properties

This section is devoted to prove basic results that concern the definition of viscosity solution. First we take a look to the monotonicity properties which are inherited from assuming that the operator I is elliptic with respect to a family $\mathcal{L}$ with nonnegative kernels. Second we see how the set of test functions can be enlarged in the definition of viscosity solutions. We use these tools to prove the stability, comparison and maximum principle and existence of solutions for the Dirichlet problem.

4.1. Monotonicity

Lemma 4.1 *(Monotonicity). Let I be elliptic with respect to a family $\mathcal{L}$ of linear operators with nonnegative kernels. Let u and v be two bounded functions in $C^{1,1}(x)$ such that $v \geqslant u$ and $v(x_0) = u(x_0)$, then*

$$Iv(x_0) \geqslant Iu(x_0).$$

Proof. By the ellipticity,

$$Iv(x_0) - Iu(x_0) \geqslant \mathcal{M}_{\mathcal{L}}^-(v - u)(x_0).$$

Let $w(y) = (v - u)(x_0 + y)$ such that $w(y) \geqslant 0$ with equality for $y = 0$. Then for any $L \in \mathcal{L}$ with kernel $K \geqslant 0$ we have

$$Lw(0) = P.V. \int w(y)K(y) \geqslant 0.$$

By taking the infimum we get that $\mathcal{M}_{\mathcal{L}}^- w(x_0) \geqslant 0$ which concludes the proof.   □

Lemma 4.2. *Let I be an elliptic operator with respect to a class $\mathcal{L}$ of nonnegative kernels. Let u, v be viscosity solutions of $Iu \leqslant f$, then $w = \min(u, v)$ is also a super solution.*

Proof. Let φ be a function touching w by below at x in N and assume without loss of generality that $w(x_0) = u(x_0)$. Then φ also touches u by below at x_0 in N and we use its equation. For

$$v = \begin{cases} \varphi & \text{in } N, \\ u & \text{in } \mathbb{R}^n \setminus N, \end{cases}$$

we have $Iv(x_0) \leqslant f(x_0)$.

Let $\tilde{v}$ be defined by

$$\tilde{v} = \begin{cases} \varphi & \text{in } N, \\ w & \text{in } \mathbb{R}^n \setminus N. \end{cases}$$

Then by the monotonicity Lemma 4.1 applied to v and $\tilde{v}$ at x_0 we get $I\tilde{v}(x_0) \leqslant Iv(x_0) \leqslant f(x_0)$ which concludes that $Iw \leqslant f$.   □

4.2. A larger class of test functions

Lemma 4.3. *Let I be elliptic with respect to a class $\mathcal{L}$ of nonnegative kernels satisfying* (2.6) *and* (2.7) *uniformly. Let $u : \mathbb{R}^n \to \mathbb{R}$ such that $Iu \geqslant f$ in the viscosity sense and φ touching u by above at x in a neighborhood N. Then $Iv(x) \geqslant f(x)$ for v defined as*

$$v = \begin{cases} \varphi & \text{in } N, \\ u & \text{in } \mathbb{R}^n \setminus N, \end{cases}$$

given that $\varphi \in C^{1,1}(x)$.

Proof. Fix p and q second order polynomials that touch φ, by below and above respectively, at x in $B_r(x) \subseteq N$. Let

$$w = \begin{cases} q & \text{in } B_r(x), \\ u & \text{in } \mathbb{R}^n \setminus B_r(x), \end{cases} \qquad v_r = \begin{cases} q & \text{in } B_r(x), \\ v & \text{in } \mathbb{R}^n \setminus B_r(x). \end{cases}$$

By the ellipticity

$$Iv(x) \geqslant Iv_r(x) + \mathcal{M}_{\mathcal{L}}^-(v - v_r)(x),$$

and thanks to the monotonicity Lemma 4.1 applied to $v_r \geqslant w$ we have $Iv_r(x) \geqslant Iw(x)$, so that

$$Iv(x) \geqslant Iw(x) + \mathcal{M}_{\mathcal{L}}^-(v - v_r)(x).$$

Note that $Iw(x) \geqslant f(x)$, so we only need to estimate the second term. Now, $v - v_r$ is supported in $B_r(x)$ and it is equal to $\varphi - q$ which is bounded by $-(q - p)$ and zero. For $L \in \mathcal{L}$ with kernel K,

$$\begin{aligned} L(v - v_r)(x) &= \int_{B_r} \delta_e\big((v - v_r), x; y\big) K_e(y)\,dy + \int_{B_r} \delta_o\big((v - v_r), x; y\big) K_o(y)\,dy \\ &\geqslant -C \left\{ \int_{B_r} |y|^2 K_e(y)\,dy + \int_{B_r} |y| K_o(y)\,dy \right\} \\ &\geqslant -C\varepsilon, \end{aligned}$$

for $\varepsilon > 0$ arbitrarily small if $r = r(\varepsilon)$ is small enough (independent of $L \in \mathcal{L}$). By taking the infimum above among every $L \in \mathcal{L}$ we get that $Iv(x) \geqslant f(x) - C\varepsilon$ and we just need to take $\varepsilon \to 0$ to conclude. $\square$

Next we have an even stronger result, that tells us that we can compute I classically every time we have a $\varphi \in C^{1,1}(x)$ touching by below.

Lemma 4.4. *Let I be an elliptic operator of the inf-sup (sup-inf) type as in* (2.8) *with all the linear operators in $\tilde{\mathcal{L}}_0$ satisfying* (H1). *Let $u : \mathbb{R}^n \to \mathbb{R}$ such that $Iu \leqslant f$ in the viscosity sense and φ touching u by below at x in a neighborhood N. Then $Iu(x)$ is defined in the classical sense and we have $Iu(x) \leqslant f(x)$ given that $\varphi \in C^{1,1}(x)$.*

To prove Lemma 4.4 we need an interpolation result that will allow us to replace the τ derivative by the σ derivative and a residue term evaluated at the test function φ. This result is also useful when the function touching by below is the convex envelope as $\delta_e^-(\varphi) = 0$.

Lemma 4.5. *Let $u : \mathbb{R}^n \to \mathbb{R}$, $x \in \mathbb{R}^n$, $2 > \sigma > \tau > 0$ and $r_0 > 0$ such that the following integrals are finite,*

$$\int_{B_{r_0}} \frac{\delta_e^+(u, x; y)}{|y|^{n+\sigma}} \, dy \quad and \quad \int_{B_{r_0}} \frac{|\delta_o(u, x; y)|}{|y|^{n+\tau}} \, dy.$$

Let φ be a function defined in $B_{r_0}(x)$ and touching u by below at x. Then

$$\int_{B_{r_0}} \lambda(2-\sigma) \frac{\delta_e^+(u, x; y)}{|y|^{n+\sigma}} - b(1-\tau) \frac{|\delta_o(u, x; y)|}{|y|^{n+\tau}} \, dy$$

$$\geqslant \int_{B_{r_0}} \alpha\lambda(2-\sigma) \frac{\delta_e^+(u, x; y)}{|y|^{n+\sigma}} - b(1-\tau) \frac{\delta_e^-(\varphi, x; y) + |\delta_o(\varphi, x; y)|}{|y|^{n+\tau}} \, dy,$$

for $\alpha \in (0, 1)$ given that

$$r_0 \leqslant \left(\frac{(1-\alpha)\lambda(2-\sigma)}{b(1-\tau)} \right)^{1/(\sigma-\tau)}. \tag{4.17}$$

Proof. Since φ touches u by below, we have that for every $y \in B_r$,

$$\delta_e^+(u - \varphi, x; y) = (u - \varphi)(x + y) + (u - \varphi)(x - y)$$
$$\geqslant \left| (u - \varphi)(x + y) - (u - \varphi)(x - y) \right|$$
$$= \left| \delta_o(u - \varphi, x; y) \right|,$$

and also,

$$\delta_e^+(u, x; y) \geqslant \delta_e^+(u - \varphi, x; y) - \delta_e^-(\varphi, x; y),$$
$$\left| \delta_o(u - \varphi, x; y) \right| \geqslant \left| \delta_o(u, x; y) \right| - \left| \delta_o(\varphi, x; y) \right|,$$

so that

$$\delta_e^+(u, x; y) - \left| \delta_o(u, x; y) \right| \geqslant -\delta_e^-(\varphi, x; y) - \left| \delta_o(\varphi, x; y) \right|.$$

Now we can replace $|\delta_o|$ by δ_e^+ in the integral,

$$\int_{B_{r_0}} \lambda(2-\sigma) \frac{\delta_e^+(u)}{|y|^{n+\sigma}} - b(1-\tau) \frac{|\delta_o(u)|}{|y|^{n+\tau}} \, dy$$

$$\geqslant \int_{B_{r_0}} \delta_e^+(u) \left\{ \frac{\lambda(2-\sigma)}{|y|^{n+\sigma}} - \frac{b(1-\tau)}{|y|^{n+\tau}} \right\} - b(1-\tau) \frac{\delta_e^-(\varphi) + |\delta_o(\varphi)|}{|y|^{n+\tau}} \, dy.$$

By using that

$$r_0 \leqslant \left(\frac{(1-\alpha)\lambda(2-\sigma)}{b(1-\tau)} \right)^{1/(\sigma-\tau)},$$

and that $\sigma > \tau$ we can substitute the difference of the fractions by α times $|y|^{-(n+\sigma)}$,

$$\int_{B_{r_0}} \delta_e^+(u) \left\{ \frac{\lambda(2-\sigma)}{|y|^{n+\sigma}} - \frac{b(1-\tau)}{|y|^{n+\tau}} \right\} dy \geqslant \alpha\lambda(2-\sigma) \int_{B_{r_0}} \frac{\delta_e^+(u)}{|y|^{n+\sigma}}. \qquad \square$$

Proof of Lemma 4.4. We check first that Lu can be computed in the classical sense at x. Because u is bounded we only care about the convergence of the integrals around the origin.

Let φ be defined in $B_{r_0}(x)$ and for $r \leqslant r_0$

$$v_r(y) = \begin{cases} u & \text{in } B_r(x), \\ \varphi & \text{in } \mathbb{R}^n \setminus B_r. \end{cases}$$

The differences $\delta_e^-(v_r, x; y)$, parametrized by r, decrease to $\delta_e^-(u, x; y)$ as r goes to zero. Since

$$\int_{B_{r_0}} \frac{\delta_e^-(v_{r_0}, x; y)}{|y|^{n+\sigma}} dy < \infty,$$

we have by monotone convergence that

$$\int_{B_{r_0}} \frac{\delta_e^-(u, x; y)}{|y|^{n+\sigma}} dy < \infty.$$

By using $f(x) \geqslant \mathcal{M}_{\tilde{\mathcal{L}}_0}^- v_r(x)$, which implies $f(x) \geqslant \mathcal{M}_{\mathcal{L}_0}^- v_r(x)$, and the boundedness of u,

$$M \geqslant \int_{B_{r_0}} \lambda(2-\sigma) \frac{\delta_e^+(v_r, x; y)}{|y|^{n+\sigma}} - b(1-\tau) \frac{|\delta_o(v_r, x; y)|}{|y|^{n+\tau}} dy$$

for some M independent of r. Use now Lemma 4.5 to keep only the term with $\delta_e^+(v_r)$, this requires r_0 sufficiently small,

$$M + b(1-\tau) \int_{B_{r_0}} \frac{|\delta_o(\varphi, x; y)| + \delta_e^-(\varphi, x; y)}{|y|^{n+\tau}} dy \geqslant \frac{\lambda(2-\sigma)}{2} \int_{B_{r_0}} \frac{\delta_e^+(v_r, x; y)}{|y|^{n+\sigma}} dy.$$

The left hand side above is finite and independent of r. By Fatou's Lemma,

$$\int_{B_{r_0}} \frac{\delta_e^+(u, x; y)}{|y|^{n+\sigma}} dy < \infty.$$

Now recall from the proof of Lemma 4.5 the identity

$$\left| \delta_o(v_r, x; y) \right| \leqslant \left| \delta_o(\varphi, x; y) \right| + \delta_e^+(v_r, x; y) + \delta_e^-(\varphi, x; y).$$

The last two terms are integrable against $|y|^{-(n+\sigma)}$ around the origin and therefore they are also integrable against $|y|^{-(n+\tau)}$ around the origin as well as the whole right hand side. Moreover the integral can be bounded by above independently of r. By Fatou's Lemma we then get that $|\delta_o(u, x; y)|$ is integrable against $|y|^{-(n+\tau)}$ in B_{r_0}.

We have shown that each term $\delta_e(u, x; y)/|y|^{n+\sigma}$ and $|\delta_o(u, x; y)|/|y|^{n+\tau}$ is integrable, then for every linear operator $L_{\alpha,\beta}u(x)$ is well defined. Therefore $Iu(x)$ can be computed by being an inf-sup combination of $L_{\alpha,\beta}$. To see that $Iu(x) \leqslant f(x)$ we use the ellipticity,

$$Iu(x) \leqslant Iv_r(x) + \mathcal{M}_{\mathcal{L}}^+(u - v_r)(x) \leqslant f(x) + \mathcal{M}_\sigma^+(u - v_r)(x) + b|D_\tau|(u - v_r)(x).$$

Both integrals go to zero by absolute continuity. $\square$

4.3. Stability

We are interested in studying limit of sub or super solutions. To state the result we need first to recall the definition of Γ convergence.

Definition 4.1. We say that a sequence of lower semicontinuous functions u_k Γ-converge to u in a set Ω if the two following conditions hold:

(i) For every sequence $x_k \to x$ in Ω, $\liminf_{\kappa \to \infty} u_k(x_k) \geqslant u(x)$.
(ii) For every $x \in \Omega$, there is a sequence $x_k \to x$ in Ω such that
$$\limsup_{k \to \infty} u_k(x_k) = u(x).$$

Lemma 4.6. *Let I be an elliptic operator with respect to a class $\mathcal{L}$ with nonnegative kernels and satisfying the integrability conditions* (2.6) *and* (2.7) *uniformly. Let u_k be a sequence of functions that are uniformly bounded in $\mathbb{R}^n$ and lower semicontinuous in $\Omega \subseteq \mathbb{R}^n$ such that*

(i) *$Iu_k \leqslant f_k$ in Ω,*
(ii) *$u_k \to u$ in the Γ-sense in Ω,*
(iii) *$u_k \to u$ a.e. in $\mathbb{R}^n$, and*
(iv) *$f_k \to f$ locally uniformly in Ω for some continuous function f.*

Then $Iu \leqslant f$ in Ω.

Proof. Let φ be a test function touching u by below at x in N. Because $u_k - \varphi$ Γ-converges to $(u - \varphi)$ there exists a sequence $x_k \to x$ such that
$$(u_k - \varphi)(x_k) = \inf_N (u_k - \varphi) = d_k.$$

Therefore $\varphi + d_k$ touches u_k at x_k in N, starting at some k sufficiently large.

Let
$$v_k = \begin{cases} \varphi + d_k & \text{in } B_r(x), \\ u_k & \text{in } \mathbb{R}^n \setminus B_r(x), \end{cases} \qquad v = \begin{cases} \varphi & \text{in } B_r(x), \\ u & \text{in } \mathbb{R}^n \setminus B_r(x). \end{cases}$$

By using the equation we know that $Iv_k(x_k) \leqslant f_k(x_k)$.

For $z \in B_{r/2}(x)$ we have by ellipticity,
$$\big|Iv_k(z) - Iv(z)\big| \leqslant \max\big(\big|\mathcal{M}_{\mathcal{L}}^+(v_k - v)\big|, \big|\mathcal{M}_{\mathcal{L}}^+(v_k - v)\big|\big)$$
$$\leqslant \sup_{L \in \mathcal{L}} \big|L(v_k - v)(z)\big|.$$

For a given $L \in \mathcal{L}$ with kernel K we have
$$\big|L(v_k - v)(z)\big| \leqslant \int\limits_{\mathbb{R}^n \setminus B_{r/2}} \big|(v_k - v)(x + y)\big| K(y)\, dy.$$

The integrand goes to zero a.e. when $k \to \infty$ and it is dominated by
$$\big(\|v_k\|_\infty + \|v\|_\infty\big) K(y) \chi_{\mathbb{R}^n \setminus B_{r/2}} \in L^1.$$

Then by dominated convergence $|L(v_k - v)(z)| \to 0$ as $k \to \infty$ uniformly in $z \in B_{r/2}(x)$ and $L \in \mathcal{L}$. This implies that $|Iv_k(z) - Iv(z)|$ also goes to zero uniformly in $z \in B_{r/2}(x)$.

Finally, using that Iv is continuous in $B_{r/2}(x)$,
$$\big|Iv_k(x_k) - Iv(x)\big| \leqslant \big|Iv_k(x_k) - Iv(x_k)\big| + \big|Iv(x_k) - Iv(x)\big| \to 0.$$

Finally we have

$$
\begin{aligned}
Iv(x) &\leqslant Iv_k(x_k) + \big|Iv_k(x_k) - Iv(x)\big| \\
&\leqslant f(x_k) + \big|Iv_k(x_k) - Iv(x)\big| \\
&\leqslant f(x) + \big|Iv_k(x_k) - Iv(x)\big| + \big|f_k(x_k) - f(x)\big|.
\end{aligned}
$$

Take $k \to \infty$ and use also that $f_k \to f$ locally uniformly to conclude. □

4.4. Comparison and maximum principle for viscosity solutions

Lemma 4.7 says that the difference of two viscosity solutions is the solution of an equation in the same ellipticity class. Theorem 4.10 is the comparison principle which implies in particular the maximum principle for sub solution. Instead of having to prove an ABP type result, as it is used in Chapter 5 of [3], we take advantage of Lemma 4.4 in order to evaluate the operators in the classical sense whenever is needed.

Lemma 4.7. *Let I be an elliptic operator of the inf-sup type as in* (2.8) *with all the linear operators in $\tilde{\mathcal{L}}_0$ satisfying* (H1). *Let u and v be two bounded functions such that $Iu \geqslant f$ and $Iv \leqslant g$ in the viscosity sense in Ω. Then $\mathcal{M}^+_{\tilde{\mathcal{L}}_0}(u-v) \geqslant f-g$ in the viscosity sense in Ω.*

The proof is straightforward when either u or v is smooth because of the nonnegativity of the kernels. In the general case we proceed by regularizing the functions by their inf or sup convolutions.

Definition 4.2. *Given a lower (upper) semicontinuous function u and a parameter $\varepsilon > 0$ the inf (sup) convolution u_ε (u^ε) is given by*

$$
u_\varepsilon(x) = \inf_y u(x+y) + \frac{|y|^2}{\varepsilon} \quad \left(u^\varepsilon(x) = \sup_y u(x+y) - \frac{|y|^2}{\varepsilon} \right).
$$

The proof of the following property con be found for instance in the beginning of Chapter 5 in [3].

Lemma 4.8. *If u is bounded and lower semicontinuous in $\mathbb{R}^n$ then u_ε Γ-converges to u.*

Lemma 4.9. *If f is a continuous function and I is elliptic with respect to a class $\mathcal{L}$ with nonnegative kernels, then if $Iu \leqslant f$ in the viscosity sense, $Iu_\varepsilon \leqslant f - d_\varepsilon$ also in the viscosity sense, where $d_\varepsilon \to 0$ as $\varepsilon \to 0$ depending only on the modulus of continuity ρ of f.*

Proof. Let φ be a test function that touches u_ε by below at x in N.

For ε sufficiently small, there is some $(x+h) \in N$ such that $u_\varepsilon(y) \leqslant u(y+h) + |h|^2/\varepsilon$ with equality at $y = x$. (See the beginning of Chapter 5 in [3].)

Then $\varphi - |h|^2/\varepsilon$ touches u at $x+h$ in N and $Iv(x+h) \leqslant f(x+h) \leqslant f(x) + \rho(|h|)$ for

$$
v = \begin{cases} \varphi - \dfrac{|h|^2}{\varepsilon} & \text{in } B_{r/2}(x+h), \\ u & \text{in } \mathbb{R}^n \setminus B_{r/2}(x+h). \end{cases}
$$

By ellipticity the value of Iv does not change by adding a constant, $I(v + |h|^2/\varepsilon)(x) \leqslant f(x) + \rho(|h|)$. Then by the monotonicity Lemma 4.1 we also have that $Iw(x) \leqslant f(x) + \rho(|h|)$ for

$$
w = \begin{cases} \varphi & \text{in } N, \\ u_\varepsilon & \text{in } \mathbb{R}^n \setminus N, \end{cases}
$$

because as we already noticed $u_\varepsilon(y) \leqslant u(y+h) + |h|^2/\varepsilon$. This concludes the proof. □

Proof of Lemma 4.7. Assume first that u is upper semicontinuous in $\mathbb{R}^n$ and v is lower semicontinuous in $\mathbb{R}^n$.

Thanks to Lemma 4.9, we have that $Iu^\varepsilon \geqslant f - d_\varepsilon$ and $Iv_\varepsilon \leqslant f + d_\varepsilon$ with $-u^\varepsilon \to -u$ and $v_\varepsilon \to v$ in the Γ-sense and $d_\varepsilon \to 0$. By the stability of viscosity solutions, Lemma 4.6, we just need to prove that $\mathcal{M}^+_{\tilde{\mathcal{L}}_0}(u^\varepsilon - v_\varepsilon) \geqslant f - g - 2d_\varepsilon$ in Ω in the viscosity sense.

Let φ be a test function touching $u^\varepsilon - v_\varepsilon$ from above at a point x. For any $\varepsilon > 0$, u^ε, $-v_\varepsilon$ and $u^\varepsilon - v_\varepsilon$ are semiconvex functions, hence there is a paraboloid for each of them touching then from below at x. If $\varphi \in C^{1,1}(x)$ then both u^ε and v_ε are also $C^{1,1}(x)$. By Lemma 4.4 we can evaluate $Iu^\varepsilon(x)$, $Iv_\varepsilon(x)$ and $\mathcal{M}^+_{\tilde{\mathcal{L}}_0}(u^\varepsilon - v_\varepsilon)(x)$ in the classical sense and they satisfy

$$\mathcal{M}^+_{\tilde{\mathcal{L}}_0}(u^\varepsilon - v_\varepsilon)(x) \geqslant Iu^\varepsilon(x) - Iv_\varepsilon(x) \geqslant f(x) - g(x) - 2d_\varepsilon.$$

Since φ touches $u^\varepsilon - v_\varepsilon$ from above at x

$$\mathcal{M}^+_{\tilde{\mathcal{L}}_0}\varphi(x) \geqslant f(x) - g(x) - 2d_\varepsilon.$$

This says that $\mathcal{M}^+_{\tilde{\mathcal{L}}_0}(u^\varepsilon - v_\varepsilon) \geqslant f - g - 2d_\varepsilon$ in the viscosity sense and completes the proof under the semicontinuity assumptions in $\mathbb{R}^n$.

Now we will not assume the lower and upper semicontinuity outside of $\bar{\Omega}$. There are sequences u_k and v_k, upper and lower semicontinuous respectively such that

(i) $u_k = u$ and $v_k = v$ in $\bar{\Omega}$ for every k,
(ii) $u_k \to u$ and $v_k \to v$ a.e. in $\mathbb{R}^n \setminus \bar{\Omega}$,
(iii) $Iu_k \geqslant f_k$ and $Iv_k \leqslant g_k$, with $f_k \to f$, $g_k \to g$ locally uniformly in Ω.

By having such sequences we just have to apply the first part of this proof and the stability, Lemma 4.6, to conclude the proof.

We can construct the sequences satisfying the first two items above by doing a standard mollification of u and v away from Ω and then filling the gap in a semicontinuous way. The function $u_k - u$ vanishes in Ω, hence $\mathcal{M}^-_{\tilde{\mathcal{L}}_0}(u_k - u)$ is defined in the classical sense in Ω and

$$\mathcal{M}^-_{\tilde{\mathcal{L}}_0}(u_k - u)(x) \geqslant - \int_{\mathbb{R}^n \setminus B_{\mathrm{dist}(x,\partial\Omega)}} |u_k(x+y) - u(x)| K(y)\, dy = h_k(x),$$

where $K = \Lambda \frac{2-\sigma}{|y|^{n+\sigma}} + b\frac{1-\tau}{|y|^{n+\tau}}$.

The functions $h_k(x)$ are continuous in Ω and by dominated convergence $h_k \to 0$ locally uniformly in Ω as $k \to 0$. Let $\varphi \in C^{1,1}(x)$ touching u_k from above at x in N and v_k defined by

$$v_k = \begin{cases} \varphi & \text{in } N, \\ u_k & \text{in } \mathbb{R}^n \setminus N. \end{cases}$$

The functions $v_k + u - u_k$ are also in $C^{1,1}(x)$ and touch u by above at x. By Lemma 4.3 we have that $I(v_k + u - u_k)(x) \geqslant f(x)$. By ellipticity

$$\begin{aligned} Iv_k(x) &\geqslant I(v_k + u - u_k)(x) + \mathcal{M}^-_{\tilde{\mathcal{L}}_0}(u - u_k)(x) \\ &\geqslant f(x) + h_k(x). \end{aligned}$$

So we have that (iii) above is also satisfied. $\quad\square$

Theorem 4.10 (Comparison principle). *Let I be an elliptic operator of the inf-sup type as in* (2.8) *with all the linear operators in $\tilde{\mathcal{L}}_0$ satisfying* (H1). *Let Ω be a bounded open set and u, v two bounded functions such that*

(i) *$Iu \geqslant f$ and $Iv \leqslant f$ in Ω in the viscosity sense for some $f \in C(\Omega)$,*
(ii) *$u \leqslant v$ in $\mathbb{R}^n \setminus \Omega$.*

Then $u \leqslant v$ in Ω.

Here, as in [4], the proof is also based on using a barrier function as

$$\varphi(x) = \min\left(1, |x|^2/4\right).$$

Lemma 4.11. *Let $s \in (0,1)$ and $\varphi_s = \varphi(sx)$ for φ defined above. There exist $\delta > 0$ and some s small enough such that*

$$\mathcal{M}^-_{\mathcal{L}_0} \varphi_s \geqslant \delta \quad \text{in } B_1.$$

Proof. First take $s = 1$ to get that

$$\mathcal{M}^-_{\sigma} \varphi \geqslant \delta_1 \quad \text{in } B_1.$$

This inequality comes from the fact that $\delta_e(\varphi, x; y) \geqslant \delta_2$ for $x \in B_1$ and every y. In fact, if $x \pm y$ are both in B_2 or both outside B_2 it is immediate. If only $x + y$ is in B_2 then we use that, $\varphi(x) \leqslant 1/4$ for $x \in B_1$,

$$\delta_e(\varphi, x; y) = 1 + \varphi(x + y) - 2\varphi(x) \geqslant 1/2.$$

On the other hand, since φ is smooth we have that $|D_\tau|\varphi \leqslant \delta_3$ in B_1, for some finite $\delta_3 > 0$. Now recall scaling properties from Section 2. We have that

$$\mathcal{M}^-_{\sigma} \varphi_s(x) = s^{-\sigma}\left(\mathcal{M}^-_{\sigma} \varphi\right)(sx) \geqslant s^{-\sigma}\delta_1,$$
$$|D_\tau|\varphi_s(x) = s^{-\tau}\left(|D_\tau|\varphi\right)(sx) \leqslant s^{-\tau}\delta_3,$$

which implies

$$\mathcal{M}^-_{\mathcal{L}_0} \varphi_s \geqslant s^{-\sigma}\left(\delta_1 - s^{\tau-\sigma}\delta_3\right)$$
$$\geqslant \delta$$

for s small enough. $\square$

Proof of Theorem 4.10. By Lemma 4.7 we know that for $w = u - v$, $\mathcal{M}^+_{\tilde{\mathcal{L}}_0} w \geqslant 0$ in the viscosity sense in Ω. We will prove from here that $\sup_\Omega w \leqslant \sup_{\mathbb{R}^n \setminus \Omega} w := M$.

Let $\Omega \subseteq B_R$ ($R \geqslant 1$) and take $\psi(x) = \varphi_s(x/R)$ for φ_s as in the previous lemma. Since $\tilde{\mathcal{L}}_0 \subseteq \mathcal{L}_0$,

$$\mathcal{M}^-_{\tilde{\mathcal{L}}_0} \psi \geqslant \mathcal{M}^-_{\mathcal{L}_0} \psi$$
$$\geqslant R^{-\sigma}\left(\mathcal{M}^-_{\mathcal{L}_0} \varphi_s\right)(x/R)$$
$$\geqslant R^{-\sigma}\delta,$$

in Ω. Fix $\varepsilon > 0$ and consider

$$\psi_\varepsilon(x) = M + \varepsilon\left(1 - \psi(x)\right),$$

which satisfies $\mathcal{M}^+_{\tilde{\mathcal{L}}_0} \psi_\varepsilon \leqslant -\varepsilon R^{-\sigma}\delta < 0$ in Ω.

If $\inf(\psi_\varepsilon - w) < 0$ then there is some translation $\psi_\varepsilon + d$ such that $\psi_\varepsilon + d$ touches w by above in $x \in \Omega$. This cannot happen because of Lemma 4.3 which says that in that case $\mathcal{M}^+_{\tilde{\mathcal{L}}_0}(\psi_\varepsilon + d)(x) = \mathcal{M}^+_{\tilde{\mathcal{L}}_0} \psi_\varepsilon(x) \geqslant 0$. Therefore $\psi_\varepsilon \geqslant w$ and by letting $\varepsilon \to 0$ we get to the conclusion of the theorem. $\square$

4.5. Existence of solutions for the Dirichlet problem

Theorem 4.12. *Let I be an elliptic operator of the inf-sup type as in* (2.8) *with all the linear operators in $\tilde{\mathcal{L}}_0$ satisfying* (H1). *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set satisfying the exterior ball condition. Let $g : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$ be a function which is globally bounded and continuous on $\partial\Omega$. Then there exists a viscosity solution $u \in C(\bar{\Omega})$ of*

$$Iu(x) = 0, \quad \text{in } \Omega,$$
$$u = g, \quad \text{in } \mathbb{R}^n \setminus \Omega.$$

The proof is based on the Perron's method. The first two lemmas account to the construction of a solution and the third one regards with achieving the boundary data.

Lemma 4.13. *Let I be an elliptic operator with respect to a class $\mathcal{L}$ with nonnegative kernels and satisfying the integrability conditions* (2.6) *and* (2.7) *uniformly. Let S be a set of viscosity solutions of $Iv \geqslant 0$ in Ω. Then $\bar{u}$, the upper semicontinuous envelope in Ω of the function u defined by*

$$u(x) = \sup_{v \in S} v(x),$$

also satisfies $I\bar{u} \geqslant 0$ in Ω in the viscosity sense.

Proof. Let φ be a test function touching $\bar{u}$ by above at $x \in \Omega$ in a neighborhood N.

The fact that $\bar{u}$ is defined as the upper semicontinuous envelope of $u(y) = \sup_{v \in S} v(y)$ implies that for our given x there exists a sequence $\{(x_k, v_k)\} \subseteq (N \cap \Omega) \times S$ such that

(i) $(x_k, v(x_k)) \to (x, \bar{u}(x))$ as $k \to \infty$,
(ii) for any $y_k \to x$ we have that $\liminf_{k \to \infty} v_k(y_k) \geqslant \bar{u}(x)$.

These are the two sufficient conditions to prove the stability Lemma 4.6. The same proof applies here to show that $I\varphi(x) \geqslant 0$ and then conclude that $Iu \geqslant 0$ in the viscosity sense. $\quad\square$

Lemma 4.14. *Let I be an elliptic operator with respect to a class $\mathcal{L}$ with nonnegative kernels and satisfying the integrability conditions* (2.6) *and* (2.7) *uniformly. Let u be a viscosity sub solution of $Iu \geqslant 0$ in Ω such that $\underline{u}$, its lower semicontinuous envelope, is not a viscosity super solution of $Iu \leqslant 0$. Then there is function U such that*

(i) *U is a viscosity sub solution of $IU \geqslant 0$ in Ω,*
(ii) *$U = u$ in $\mathbb{R}^n \setminus \Omega$,*
(iii) *$\sup_{x \in \Omega}(U - u)(x) > 0$.*

Proof. Let φ be a test function touching $\underline{u}$ by below in $x_0 \in B_{r_0}(x_0) \subseteq \Omega$ such that $Iv(x_0) > 0$ for

$$v = \begin{cases} \varphi & \text{in } B_{r_0}(x_0), \\ \underline{u} & \text{in } \mathbb{R}^n \setminus B_{r_0}(x_0). \end{cases}$$

By continuity we also have that $Iv > \delta > 0$ in $B_{r_1}(x_0) \subseteq B_{r_0/2}(x_0)$.

Let $\varepsilon_1, \varepsilon_2 > 0$ be fixed and

$$\psi(y) = \varphi(y) - \varepsilon_1 |y - x_0|^2 + \varepsilon_2.$$

We have that $\psi \leqslant u$ in $\mathbb{R}^n \setminus B_{r_1}(x_0) \subseteq \mathbb{R}^n \setminus \Omega$ if $\varepsilon_2 < r_1^2$. We want to choose ε_1 and ε_2 such that $U = \min(\psi, u)$ satisfies $IU \geqslant 0$.

Let η be a test function touching U by above at $x_1 \in \Omega$ in a neighborhood N. If $U(x_1) = u(x_1)$ then the inequality follows from Lemma 4.3. If $U(x_1) = \psi(x_1) > u(x_1)$ then necessarily $x_1 \in B_{r_1}(x_0)$. By the lower semicontinuity, $\psi > u$ in some open neighborhood around x_1 and contained in $B_{r_1}(x_0)$. Because ψ is smooth, $IU(x_0)$ is classically defined and we just have to check that it is nonnegative.

Let

$$w = \begin{cases} \psi & \text{in } B_{r_0}(x_0), \\ \underline{u} & \text{in } \mathbb{R}^n \setminus B_{r_0}(x_0). \end{cases}$$

By monotonicity and ellipticity,

$$\begin{aligned} IU(x_1) &\geqslant Iw(x_1) \\ &\geqslant Iv(x_1) + \mathcal{M}_{\mathcal{L}}^-(w - v)(x_1) \\ &\geqslant \delta + \inf_{L \in \mathcal{L}} L(w - v)(x_1). \end{aligned}$$

Notice that $w - v = \varepsilon_2 - \varepsilon_1 |y - x_0|^2$ in $B_{r_0}(x_0)$ and it is zero outside. Recall that $x_1 \in B_{r_0/2}(x_0)$, so for any $L \in \mathcal{L}$ with kernel K,

$$
\begin{aligned}
L(w - v)(x_1) &= \int \delta(w - v, x_1; y) K(y) \, dy \\
&\geqslant -\varepsilon_1 \int_{B_{2r_0}} \delta(|\cdot - x_0|^2, x_1; y) K(y) \, dy - \min(\varepsilon_1 r_0^2, \varepsilon_2) \int_{\mathbb{R}^n \setminus B_{r_0/2}} K(y) \, dy.
\end{aligned}
$$

The second term in the inequality appears since

$$
\delta(w - v, x_1, y) \geqslant -\min(\varepsilon_1 r_0^2, \varepsilon_2)
$$

for any x_1 in $B_{r_0/2}(x_0)$ and $|y| \geqslant B_{r_0/2}$. Therefore,

$$
L(w - v)(x_1) \geqslant -C(\varepsilon_1 + \min(\varepsilon_1 r_0^2, \varepsilon_2)).
$$

Then we choose $\varepsilon_2 = r_1^2/2$ and ε_1 sufficiently small to make $L(w - v)(x) \geqslant -\delta/2$ uniformly in $L \in \mathcal{L}$ and $x \in B_{r_1}(x_0)$. This finally implies that $IU \geqslant 0$ and concludes the proof of the lemma. $\quad\square$

Lemma 4.15. *Let* $\varphi(x) = \min(1, C(|x| - 1)_+^\alpha)$, *where* C *and* α *have been chosen as in* [5]. *Then for any pair* σ, τ *satisfying* (H1) *we have*

$$
\mathcal{M}_{\tilde{\mathcal{L}}_0}^+ \varphi(x) \leqslant 0, \quad x \in \mathbb{R}^n \setminus B_1.
$$

Moreover,

$$
\mathcal{M}_{\tilde{\mathcal{L}}_0}^+ \varphi(x) \leqslant -\delta < 0, \quad x \in B_2 \setminus B_1.
$$

Proof. Let r_0 and α be the radius and exponent from Lemma 3.1 in [5]. We know that $\mathcal{M}_\sigma^+ v(x_0) = -d$ $(d > 0)$ and $|D_\tau| v(x_0) = e < \infty$ for every $x_0 \in \partial B_{1+r_0}$.

Let $s \in (0, 1)$ and rescale v by

$$
v_s(x) = s^{-\alpha} v(sx) = (|x| - s^{-1})_+^\alpha.
$$

Recall the scaling remarks in Section 2. For $L \in \tilde{\mathcal{L}}_0(\sigma, \tau, \lambda, \Lambda, b)$ with kernel K we consider $L_s \in \tilde{\mathcal{L}}_0(\sigma, \tau, s^{-\sigma}\lambda, s^{-\sigma}\Lambda, s^{-\tau}b)$ with kernel $K_s(y) = s^n K(sy)$ such that

$$
L_s v_s(x) = s^{-\alpha}(Lv)(sx).
$$

Let x be such that $(1 + r_0)x = (1 + sr_0)x_0$ and translate v_s such that it remains below v but touches it in a whole ray passing through x and x_0. We still denote the translation v_s. By the scaling,

$$
Lv(x) = s^\alpha(L_s v_s)(x_0) \leqslant (L_s v_s)(x_0).
$$

By the monotonicity Lemma 4.1 applied to v and v_s at x_0,

$$
L_s v_s(x_0) \leqslant L_s v(x_0).
$$

Since $L_s \in \tilde{\mathcal{L}}_0(\sigma, \tau, s^{-\sigma}\lambda, s^{-\sigma}\Lambda, s^{-\tau}b) \subseteq \mathcal{L}_0(\sigma, \tau, s^{-\sigma}\lambda, s^{-\sigma}\Lambda, s^{-\tau}b)$,

$$
\begin{aligned}
L_s v(x_0) &\leqslant s^{-\sigma} \mathcal{M}_\sigma^+ v(x_0) + s^{-\tau} b |D_t| v(x_0) \\
&\leqslant s^{-\sigma} \{-d + s^{\sigma-\tau} b e\} \\
&\leqslant -d/2,
\end{aligned}
$$

if s is small enough. By transitivity $Lv(x) \leqslant -d/2$ for any x with $|x| = 1 + sr_0$ with $s \in (0, s_0)$. Finally we just multiply v by a constant C big enough such that $Cv(s_0 r_0) \geqslant 1$ and use Lemma 4.2 to conclude the lemma for the truncation of v. $\quad\square$

Proof of Theorem 4.12. Let S be the set of all viscosity sub solutions of $Iv \geqslant 0$ with boundary data smaller than g,

$$S = \{v \in USC(\Omega) \cap L^\infty(\mathbb{R}^n): Iv \geqslant 0 \text{ in viscosity in } \Omega \text{ and } v \leqslant g \text{ in } \mathbb{R}^n \setminus \Omega\}.$$

The set S is nonempty because the constant function $u = -\|g\|_\infty$ satisfies $Iu = 0$ given that I is of the inf-sup type.

The first lemma assures us that $\bar{u}$, defined as the upper semicontinuous envelope in Ω of $u(x) = \sup_{v \in S} v(x)$, is a viscosity sub solution of $I\bar{u} \geqslant 0$. Then $\bar{u} \in S$ and $\bar{u} = u$ is a sub solution too. By the second lemma the lower semicontinuous envelope $\underline{u}$, is a super solution. If not that would contradict the fact that u is the biggest sub solution. We conclude, by the comparison principle, that $\underline{u} \geqslant u$ and therefore both have to be equal and u is a viscosity solution of $Iu = 0$ in Ω.

The next step is to prove that we actually attain the boundary values in a continuous way. We have to show that for any $x \in \mathbb{R}^n \setminus \Omega$ and any $\varepsilon > 0$ we can find continuous barriers v and w such that

(i) $Iw \leqslant 0$ and $Iv \geqslant 0$ in Ω in the viscosity sense,
(ii) $w \geqslant g$ and $v \leqslant g$ in $\mathbb{R}^n \setminus \Omega$,
(iii) $w(x) \leqslant g(x) + \varepsilon$ and $v(x) \geqslant g(x) - \varepsilon$.

We just prove it for w.

If x belongs to the interior of $\mathbb{R}^n \setminus \Omega$ then a function w which is equal to $\|g\|_\infty$ for every $y \neq x$ and equal to $g(x)$ for $y = x$ is in $USC(\Omega)$ and is a super solution. If $x \in \partial\Omega$ then there is a ball $B_{r_0}(x + r_0\eta)$ such that $\bar{B}_{r_0}(x + r_0\eta) \cap \partial\Omega = \{x\}$, where η is a unitary vector and r_0 less than one. Let

$$w(y) = 2\|g\|_\infty \varphi\left(\frac{y - (x + r\eta)}{r}\right) + g(x) + \varepsilon$$

with φ from Lemma 4.15 and some $r < r_0$. By the construction of φ we already have that (i) and (iii) are satisfied.

To check (ii) let $\delta > 0$ such that $|g(y) - g(x)| \leqslant \varepsilon$ whenever $|x - y| \leqslant \delta$. Take r such that $B_{2r}(x + r\eta) \subseteq B_\delta(x)$. If $y \in B_\delta(x) \cap (\mathbb{R}^n \setminus \Omega)$ then $w(y) = g(x) + \varepsilon \geqslant g(y)$. If $y \in \mathbb{R}^n \setminus B_\delta(x) \subseteq \mathbb{R}^n \setminus B_{2r}(x + r\eta)$ then $\varphi \geqslant 1$ and $w(y) \geqslant \|g\|_\infty \geqslant g(y)$. $\quad\square$

5. Partial ABP estimates

The classical ABP theorem states that for a super solution, positive in ∂B_3, the supremum of u^- is controlled by the L^n norm of the right hand side, integrated only over the contact set for the convex envelope. These estimates are useful to get lower bounds in the measure of the contact set which are then needed to get point estimates.

We denote by Γ the convex envelope supported in B_3. For a lower semicontinuous function $u \geqslant 0$ in $\mathbb{R}^n \setminus B_1$,

$$\Gamma(x) = \sup\{v(x): v: B_3 \to \mathbb{R} \text{ is convex and } v \leqslant u^-\}.$$

We get the same definition if v is only affine. Every time we refer to $\nabla\Gamma(x)$ we are actually referring to a sub differential of Γ at x which always exists.

In the next lemma we see that we can almost put a paraboloid above Γ, with the opening controlled by $f(x)$, the supremum of u outside B_1 and the τ derivative of Γ at x.

Lemma 5.1. *Let $u \geqslant 0$ in $\mathbb{R}^n \setminus B_1$ be a globally bounded viscosity solution of*

$$\mathcal{M}^-_{\tilde{\mathcal{L}}_0} u \leqslant f \quad \text{in } B_1,$$

and $x \in \{u = \Gamma\}$. Assume (H1) *holds ($2 > \sigma > \sigma_0$ and $\min(1, \sigma) > \tau > \tau_0$) and $2b \leqslant \lambda(2-\sigma)/(1-\tau)$. Let $\rho_0 = 1/(128\sqrt{n})$, $r_k = \rho_0 2^{-1/(2-\sigma)-k}$ and $R_k = B_{r_k} \setminus B_{r_{k+1}}$. Then there is a constant C_0 such that for any $M > 0$ there is a k such that*

$$\left|\left\{y \in R_k: u(y + x) > u(x) + y \cdot \nabla\Gamma(x) + Mr_k^2\right\}\right| \leqslant C_0 \frac{F(x)}{M}|R_k|,$$

where

$$F(x) = f(x) + (1 - \tau)b \int_{B_2} \frac{|\delta_0(\Gamma, x; y)|}{|y|^{n+\tau}}\, dy + \frac{1 - \tau}{\tau} b \|u^+\|_{L^\infty(\mathbb{R}^n \setminus B_1)}.$$

Proof. Notice that $\delta_e^-(u, x; y) = 0$. If $x \pm y \in B_3$ then we use that there is a plane touching u by below in B_3. If $x + y \notin B_3$ then $x - y \notin B_1$ and the boundary value gives that $u(x \pm y) \geqslant 0$ and then $\delta_e(u, x; y) \geqslant 0$ because $u(x) \leqslant 0$.

By Lemma 4.4 the following quantities can be computed and satisfy

$$\int\limits_{B_{r_0}} (2 - \sigma)\lambda \frac{\delta_e^+(u, x; y)}{|y|^{n+\sigma}} - (1 - \tau)b \frac{|\delta_o(u, x; y)|}{|y|^{n+\tau}} \, dy$$

$$\leqslant f(x) + (1 - \tau)b \int\limits_{\mathbb{R}^n \setminus B_{r_0}} \frac{|\delta_o(u, x; y)|}{|y|^{n+\tau}} \, dy$$

$$\leqslant f(x) + C(n)b \frac{1 - \tau}{\tau} \|u^+\|_{L^\infty(\mathbb{R}^n \setminus B_1)}.$$

We want to use Lemma 4.5 with Γ as the test function. The assumption $2b \leqslant \lambda(2 - \sigma)/(1 - \tau)$ guarantees (4.17) with $\alpha = 1/2$,

$$\int\limits_{B_{r_0}} (2 - \sigma)\lambda \frac{\delta_e^+(u, x; y)}{|y|^{n+\sigma}} - (1 - \tau)b \frac{|\delta_o(u, x; y)|}{|y|^{n+\tau}} \, dy$$

$$\geqslant \int\limits_{B_{r_0}} \frac{(2 - \sigma)\lambda}{2} \frac{\delta_e^+(u, x; y)}{|y|^{n+\sigma}} \, dy - (1 - \tau)b \int\limits_{B_{r_0}} \frac{|\delta_0(\Gamma, x; y)|}{|y|^{n+\tau}} \, dy.$$

Adding what we have so far

$$(2 - \sigma) \int\limits_{B_{r_0}} \frac{\delta_e^+(u, x; y)}{|y|^{n+\sigma}} \, dy \leqslant C F(x). \tag{5.18}$$

The rest of the proof goes as in [4]. Fix M and assume that none of the dyadic rings satisfies the conclusion of the lemma for C_0 still to be fixed. For every $y \in R_k$ where

$$u(y + x) > u(x) + y \cdot \nabla\Gamma(x) + Mr_k^2,$$

we have that

$$\delta_e(u, x; y) = u(x + y) + u(x - y) - 2u(x)$$
$$> y \cdot \nabla\Gamma(x) + Mr_k^2 + u(x - y) - u(x)$$
$$\geqslant Mr_k^2,$$

because by the convexity of Γ,

$$-y \cdot \nabla\Gamma(x) + u(x) = -y \cdot \nabla\Gamma(x) + \Gamma(x) \leqslant \Gamma(x - y) \leqslant u(x - y).$$

Adding all the contributions into the estimate (5.18) we get

$$(2 - \sigma) \int\limits_{B_{r_0}} \frac{\delta_e^+(u, x; y)}{|y|^{n+\sigma}} \geqslant (2 - \sigma)C_0 M \frac{F(x)}{M} \sum_{k=0}^{\infty} r_k^{2-\sigma} \geqslant C_0 F(x) r_0^{2-\sigma} \frac{2 - \sigma}{1 - 2^{-(2-\sigma)}}.$$

Now it just a matter to take C_0 large enough to get a contradiction. Notice that the quotient $(2 - \sigma)/(1 - 2^{-(2-\sigma)})$ is uniformly bounded by above and away from zero when σ varies in $(0, 2)$. $\quad\square$

The following is just a modification of the previous lemma. The aim is to replace the second term in $F(x)$ by $\|u\|_\infty$.

Remark 5.2. By the intermediate value theorem, for each $x \in B_1$ and $y \in B_2$, $|\delta_0(\Gamma, x; y)|$ is equal to $2|\nabla\Gamma(x')||y|$ for x' an intermediate point in the segment between $x + y$ and $x - y$. So that

$$\int_{B_2} \frac{|\delta_0(\Gamma, x, y)|}{|y|^{n+\tau}} \, dy \leqslant \frac{C(n)}{1 - \tau} \|\nabla\Gamma\|_\infty.$$

By the geometry of the convex envelope $\|\nabla\Gamma\|_\infty \leqslant \|u^-\|_\infty/2$. We can also consider that $\tau > \tau_0 > 0$ for τ_0 universal, so that $F(x)$ can be simplified to

$$F(x) = f(x) + b\|u\|_\infty.$$

Notice that we haven't absorb the constant b into the universal constants of the estimate. The importance of this choice will be seen in the results of the next sections.

Corollary 5.3. *Under the assumptions of Lemma* 5.1 *there exist a small fraction* $\varepsilon_0 > 0$ *and a constant* $M_0 > 0$ *such that for some radius* r *and* $R = B_r \setminus B_{r/2}$:

(i) $|\{y \in R: u(y + x) > u(x) + y \cdot \nabla\Gamma(x) + M_0 F(x)r^2\}| \leqslant \varepsilon_0|R|.$
(ii) $\Gamma(y + x) \leqslant u(x) + y \cdot \nabla\Gamma(x) + M_0 F(x)r^2$ *for* $y \in B_{r/2}.$
(iii) $|\nabla\Gamma(B_{r/4}(x))| \leqslant C|B_{r/4}|F(x)^n.$

Proof. Let A be the following set

$$A = (B_1 \setminus B_{1/2}) \cap \{x_1 > 1/2\}.$$

Take

$$\varepsilon_0 = \frac{|A|}{2|B_1 \setminus B_{1/2}|} \quad \text{and} \quad M_0 = \frac{C_0}{\varepsilon_0},$$

with C_0 from Lemma 5.1. Apply Lemma 5.1 with $M = F(x)M_0$ to get a radius $r(= r_k)$ such that

$$\left|\{y \in R: u(y + x) > u(x) + y \cdot \nabla\Gamma(x) + M_0 F(x)r^2\}\right| \leqslant \varepsilon_0|R|. \tag{5.19}$$

By convexity we can assume without loss of generality that Γ attains its maximum N on $B_{r/2}(x)$ at the point $(r/2)e_1 + x$ and

$$\Gamma(y + x) \geqslant \Gamma\big((r/2)e_1 + x\big),$$

for every $y \in R$ with $y \cdot e_1 \geqslant r/2$. Therefore,

$$2\varepsilon_0|R| \leqslant \left|\{y \in R: \Gamma(y + x) \geqslant N\}\right| \leqslant \left|\{y \in R: u(y + x) \geqslant N\}\right|.$$

Then N has to be smaller than or equal to $u(x) + y \cdot \nabla\Gamma(x) + M_0 F(x)r^2$ because otherwise we get a contradiction with (5.19). This implies (ii).

Finally, by Γ being trapped between two planes in $B_{r/2}$, separated by a distance $M_0 F(x)r^2$, we get by the geometry of convex functions a control in the oscillation of $\nabla\Gamma$ in $B_{r/4}$. Namely $\nabla\Gamma(B_{r/4})$ is contained in the ball of radius $4M_0 F(x)r$ with center at $\nabla\Gamma(x)$. This concludes the proof. $\quad\square$

Now we are able to state and prove an ABP type estimate.

Theorem 5.4. *Let* $u \geqslant 0$ *in* $\mathbb{R}^n \setminus B_1$ *be a globally bounded viscosity solution of*

$$\mathcal{M}^-_{\tilde{\mathcal{L}}_0} u \leqslant f \quad \text{in } B_1.$$

Assume (H1) *holds and* $2b \leqslant \lambda(2 - \sigma)/(1 - \tau)$. *There is a disjoint family of cubes* Q_j *with diameters* $d_j \leqslant \rho_0 2^{-1/(2-\sigma)}$ *($\rho_0 = 1/(32\sqrt{n})$) which covers the contact set* $\{\Gamma = u\}$ *such that the following holds*

(i) $\{u = \Gamma\} \cap \bar{Q}_j \neq \emptyset$ for any Q_j,

(ii) $|\{y \in 8\sqrt{n}Q_j : u(y) < \Gamma(y) + C(\max_{x \in Q_j \cap \{\Gamma = u\}} F(x))d_j^2\}| \geqslant \mu|Q_j|$,

(iii) $|\nabla\Gamma(\bar{Q}_j)| \leqslant C(\max_{x \in Q_j \cap \{\Gamma = u\}} F(x))^n|Q_j|$,

where μ $(= (1 - \varepsilon_0)$ from Corollary 5.3) and C above are universal (independent of σ and τ) and

$$F(x) = f(x) + (1 - \tau)b \int_{B_2} \frac{|\delta_0(\Gamma, x; y)|}{|y|^{n+\tau}} \, dy + \frac{1 - \tau}{\tau} b\|u^+\|_{L^\infty(\mathbb{R}^n \setminus B_1)}.$$

Proof. Let's proceed as in [4] and cover B_1 with a tiling of cubes of diameter $\rho_0 2^{-1/(2-\sigma)}$. We discard all those that do not intersect the contact set $\{u = \Gamma\}$. Whenever a cube does not satisfy (ii) and (iii), we split it into 2^n congruent cubes of half diameter and discard those whose closure does not intersect $\{u = \Gamma\}$. We want to prove that eventually this procedure finishes.

Let's assume that the covering process does not stop. We end up getting a sequence of nested cubes intersecting at a point $x_0 \in \{u = \Gamma\}$. We will prove that there is a cube in the family that did not split, reaching then a contradiction.

Due to Corollary 5.3 there is a radius $0 < r < \rho_0 2^{-1/(2-\sigma)}$ such that for $R = B_r \setminus B_{r/2}$,

$$\left|\{y \in R : u(y + x) > u(x) + y \cdot \nabla\Gamma(x) + CF(x_0)r^2\}\right| \leqslant \varepsilon_0|R|,$$

and

$$\left|\nabla\Gamma(B_{r/4}(x_0))\right| \leqslant CF(x_0)^n|B_{r/4}|.$$

There is a cube Q_j with diameter $r/8 \leqslant d_j < r/4$ such that $B_{r/4}(x_0) \supset \bar{Q}_j$ and $B_r(x_0) \subset 32\sqrt{n}Q_j$.

Using the fact that the diameter of the cube and the radius are comparable and that, by the convexity of Γ, $\Gamma(y) \geqslant u(x_0) + (y - x_0) \cdot \nabla\Gamma(x_0)$, we get

$$\left|\left\{y \in 32\sqrt{n}Q_j : u(y) \leqslant \Gamma(y) + C\left(\max_{Q_j \cap \{\Gamma = u\}} F\right)d_j^2\right\}\right|$$

$$\geqslant \left|\left\{y \in 32\sqrt{n}Q_j : u(y) \leqslant u(x_0) + (y - x_0) \cdot \nabla\Gamma(x_0) + C\left(\max_{Q_j \cap \{\Gamma = u\}} F\right)d_j^2\right\}\right|$$

$$\geqslant (1 - \varepsilon_0)|R| \geqslant \mu|Q_j|.$$

This is (ii) in the statement of the theorem. Since $\bar{Q}_j$ is contained in B_r we conclude also that (iii) holds and Q_j did not split. □

As τ and σ go to one and two respectively in a controlled way (recall the hypothesis $2b \leqslant \lambda(2 - \sigma)/(1 - \tau)$), this theorem recovers a sufficient step to complete the proof of the classical ABP estimate. However, to prove regularity for u it will be sufficient to use a weaker version where $F(x) = f(x) + b\|u\|_\infty$ (see Remark 5.2).

We also point out that the condition on b is not too restrictive. By the scaling discussion in Section 2 we can always consider a dilation of u to make the assumption valid.

6. Point estimate

The point estimate for nonlinear operators works in someway like the mean value theorem for super harmonic functions. If a nonnegative super harmonic function is bigger than or equal to 1 in half of the points in B_1 (in measure) then it gets automatically separated from zero at $B_{1/4}$ a fixed quantity. This is the key step to prove a decay of oscillation and then Hölder regularity for the solutions of our equations. For nonlocal operators, point estimates were already given in [8]. Those estimates are easier to obtain than in the local case because the definition of the nonlocal operators already involve some sort of averaging. However, the estimates in [8] blow up when the order of the equation goes to the classical one. Our goal here is to see that the same estimates still hold with constant that remains uniform when $\sigma \to 2$ and $\tau \to 1$ in a controlled way.

From this point on we will always assume that, for $\sigma_0, \tau_0, m, A_0 > 0$ given, the set of hypotheses (H1), (H2) and (H3) holds.

(H1) $2 > \sigma \geqslant \sigma_0 > 0$, $\min(1, \sigma) > \tau \geqslant \tau_0 > 0$,
(H2) $\sigma - \tau \geqslant m > 0$,
(H3) $\lambda A_0(2 - \sigma) \geqslant b(1 - \tau)$.

We recall the special function constructed in [4].

Lemma 6.1. *Let $2 > \sigma_0 > 0$; there is a function Φ such that*

   (i) *Φ is continuous in $\mathbb{R}^n$,*
  (ii) *$\Phi(x) = 0$ for x outside $B_{2\sqrt{n}}$,*
 (iii) *$\Phi(x) < -2$ for x in Q_3, and*
  (iv) *$\mathcal{M}_\sigma^+ \Phi \leqslant \psi(x)$ in $\mathbb{R}^n$ for some nonnegative function $\psi(x)$ supported in $\bar{B}_{1/4}$*

for every $\sigma > \sigma_0$.

The following lemma provides the first and also the inductive step towards an inductive proof of the point estimate.

Lemma 6.2. *Let $\sigma_0, \tau_0, m, A_0 > 0$ and assume* (H1), (H2) *and* (H3). *There exist constants $\mu \in (0, 1)$, $\varepsilon_0 > 0$ and $M > 1$, such that if*

   (i) *$u \geqslant 0$ in $\mathbb{R}^n$,*
  (ii) *$\inf_{Q_{3\kappa}} u \leqslant 1$,*
 (iii) *$\mathcal{M}_{\tilde{\mathcal{L}}_0}^- u \leqslant 1$ in $Q_{4\sqrt{n}\kappa}$,*

then

$$\left|\{u \leqslant M\} \cap Q_\kappa\right| > \mu|Q_\kappa|,$$

for

$$\kappa = \frac{\varepsilon_0}{(1 + \|u\|_\infty)^{1/(\sigma - \tau)}}.$$

Proof. Consider $\tilde{u}(x) = u(\kappa x)$ and note that by the scaling of the equation $\tilde{u}$ satisfies (i), (ii) in the cube of side 3 and

$$\mathcal{M}_{\tilde{\mathcal{L}}_0(\tilde{b})} u\tilde{u} \leqslant \kappa^\sigma \quad \text{in } Q_{4\sqrt{n}},$$

where $\tilde{\mathcal{L}}_0(\tilde{b}) = \tilde{\mathcal{L}}_0(\sigma, \tau, \lambda \Lambda, \tilde{b})$, for $\tilde{b} = \kappa^{\sigma - \tau} b$. We will prove that the lemma holds for $\tilde{u}$ in Q_1, which implies the desired result. The proof follows as in [4] but we point out that the ABP type results that we have are different.

First thing we require from ε_0^m is to be small enough, with respect to A_0, such that the condition of smallness on $\tilde{b} \leqslant \varepsilon_0^m b$ from Lemma 5.1 and Theorem 5.4 holds. Namely, $2\varepsilon_0^m \leqslant A_0^{-1}$.

Consider $v = \tilde{u} + \Phi$, where Φ is the special function given in [4]. We have that v satisfies in $Q_{4\sqrt{n}}$ (since $\tilde{\mathcal{L}}_0(\tilde{b}) \subseteq \mathcal{L}_0(\tilde{b})$)

$$\mathcal{M}_\sigma^- v - \tilde{b}|D_\tau|v \leqslant \kappa^\sigma + \mathcal{M}_\sigma^+ \Phi + \kappa^{\sigma - \tau} b|D_\tau|\Phi$$
$$\leqslant \kappa^\sigma + \psi + \kappa^{\sigma - \tau} bC,$$

for a universal constant C.

Let Γ be the concave envelope of v supported in the ball $B_{6\sqrt{n}}$. Let Q_j be the cubes from a rescaled version of our partial ABP estimate. We have

$$\max_{B_{2\sqrt{n}}} v^- \leqslant C\left|\nabla \Gamma(B_{2\sqrt{n}})\right|^{1/n} \leqslant C\left(\sum_j |\nabla \Gamma(\bar{Q}_j)|\right)^{1/n}$$

$$\leqslant C\left(\sum_j \left(\max_{Q_j} \psi + \kappa^\sigma + \kappa^{\sigma - \tau} b(1 + \|v\|_\infty)\right)^n\right)|Q_j|^{1/n}.$$

We can make the terms $\kappa^\sigma + \kappa^{\sigma-\tau} b(1 + \|v\|_\infty)$ small enough by choosing ε_0 small enough,

$$\kappa^\sigma + \kappa^{\sigma-\tau} b\big(1 + \|v\|_\infty\big) \leqslant C\big(\varepsilon_0^\sigma + \varepsilon_0^{\sigma-\tau}\big) \leqslant C\varepsilon_0^m.$$

Using that $\Phi \leqslant -2$ in Q_3

$$1 \leqslant C\varepsilon_0^m + C\bigg(\sum_j \Big(\max_{Q_j} \psi^+ \Big)^n |Q_j| \bigg)^{1/n},$$

which implies then, for ε_0 small enough, the following inequality,

$$\frac{1}{2} \leqslant C\bigg(\sum_j \Big(\max_{Q_j} \psi^+ \Big)^n |Q_j| \bigg)^{1/n}.$$

Since ψ is supported in $\bar{B}_{1/4}$ and is bounded, we get

$$\sum_{Q_j \cap \bar{B}_{1/4} \neq \emptyset} |Q_j| \geqslant c, \tag{6.20}$$

where c is universal. Now, the diameters of all cubes Q_j are bounded by $\rho_0 2^{-1/(2-\sigma)}$, which is smaller than $\rho_0 = 1/(128\sqrt{n})$. So, every time we have that Q_j intersects $B_{1/4}$ the cube $32\sqrt{n} Q_j$ will be contained in $B_{1/2}$.

Since ε_0 is universal, the partial ABP estimates translate into

$$\big|\big\{ x \in 32\sqrt{n} Q_j : \ v(x) \leqslant \Gamma(x) + C d_j^2 \big\}\big| \geqslant c|Q_j|, \tag{6.21}$$

for C universal and $C d_j^2 < C\rho_0^2$. Let us consider now the cubes $32\sqrt{n} Q_j$ for every Q_j that intersects $B_{1/4}$. This provides an open cover of the union of the corresponding cubes $\bar{Q}_j$ and it is contained in $B_{1/2}$. Taking a subcover with finite overlapping and using (6.20) and (6.21) we get

$$\big|\big\{ x \in B_{1/2} : v(x) \leqslant \Gamma(x) + C\rho_0^2 \big\}\big| \geqslant c.$$

Hence, if we let $-M_0 = \min_{B_{1/2}} \Phi$ we get

$$\big|\big\{ x \in B_{1/2} : \ \tilde{u}(x) \leqslant M_0 + C\rho_0^2 \big\}\big| \geqslant c.$$

Finally let $M = M_0 + C\rho_0^2$ and $\mu = c$. Since $B_{1/2} \subset Q_1$,

$$\big|\big\{ x \in Q_1 : \tilde{u}(x) \leqslant M \big\}\big| \geqslant c,$$

which concludes the result for $\tilde{u}$. $\quad\square$

Remark 6.3. In the previous proof the scaling is necessary to have:

(i) $\tilde{b}(1 + \|u\|_\infty) \leqslant \varepsilon_0^m$,
(ii) right hand side smaller than ε_0^m.

In future references we will use that if these identities hold, then the conclusion also holds without any further scaling.

Remark 6.4. Consider for u and $\tilde{u}$ as before and $0 < r \leqslant 1$, $v(x) = \tilde{u}(rx)$. Then v satisfies

$$\mathcal{M}_{\tilde{\mathcal{L}}_0(b(r\kappa)^{\sigma-\tau})} v \leqslant (r\kappa)^\sigma,$$

and in particular

$$\mathcal{M}_\sigma^- v(x) - b(r\kappa)^{\sigma-\tau} |D_\tau v| \leqslant (r\kappa)^\sigma.$$

From the previous remark we check that $b(r\kappa)^{\sigma-\tau}(1 + \|v\|_\infty) \leqslant \varepsilon_0^m$ and that the right hand side is also smaller than or equal to ε_0^m if $r \leqslant 1$.

In particular, the transformations required to prove the full L^ε Lemma are of the form

$$v(y) = \frac{u(x_0 + 2^{-i} y)}{M^k}.$$

The previous lemma can still be applied and we can iterate by means of a Calderón–Zygmund decomposition as in [3].

Lemma 6.5. *Let $\sigma_0, \tau_0, m, A_0 > 0$ and $u \geqslant 0$, $\tilde{u}$ as in Lemma 6.2. Then we have*

$$\big|\{\tilde{u} > M^k\} \cap Q_1\big| \leqslant (1 - \mu)^k,$$

for $k = 1, 2, \ldots$ where M and μ are as in Lemma 6.2. As a consequence we have the following inequality,

$$\big|\{\tilde{u} > t\} \cap Q_1\big| \leqslant dt^{-\varepsilon}, \quad \forall t > 0,$$

where d and ε are positive universal constants.

By standard covering arguments one can pass from cubes to balls.

Corollary 6.6. *Let $\sigma_0, \tau_0, m, A_0 > 0$ and $u \geqslant 0$, $\tilde{u}$ as in Lemma 6.2 with u super solution of $\mathcal{M}^-_{\tilde{\mathcal{L}}_0} u \leqslant 1$ in $B_{2\kappa}$ and $u(0) \leqslant 1$. Then we have*

$$\big|\{\tilde{u} > t\} \cap B_1\big| \leqslant ct^{-\varepsilon}, \quad \forall t > 0,$$

where c and ε are positive universal constants.

By using Remark 6.4 one more time we can prove a rescaled version of Corollary 6.6.

Corollary 6.7. *Let $\sigma_0, \tau_0, m, A_0 > 0$, $u \geqslant 0$, $\tilde{u}$ as in Lemma 6.2 with u super solution of $\mathcal{M}^-_{\tilde{\mathcal{L}}_0} u \leqslant C_0$ in $B_{2\kappa r}$, for $r \leqslant 1$. Then we have*

$$\big|\{\tilde{u} > t\} \cap B_r\big| \leqslant Cr^n \big(u(0) + C_0 r^\sigma\big)^\varepsilon t^{-\varepsilon}, \quad \forall t > 0, \tag{6.22}$$

where C and ε are positive universal constants.

7. Hölder regularity

The first lemma in this section is the decay of oscillation, that follows from the point estimate already proved, applied at every scale. It is well known that when the oscillation of a function decays geometrically in geometrically decaying balls it implies a Hölder modulus of continuity at the center of such ball. By applying it at every point of a ball strictly contained in the domain we get Hölder regularity.

We still assume the general hypotheses, (H1), (H2) and (H3), of the previous section.

Lemma 7.1. *Let $\sigma_0, \tau_0, m, A_0 > 0$ and assume* (H1), (H2) *and* (H3). *Let u be a function such that*:

(i) $\frac{-1}{2} \leqslant u \leqslant \frac{1}{2}$ *in $\mathbb{R}^n$,*
(ii) $\mathcal{M}^+_{\tilde{\mathcal{L}}_0} u \geqslant -1$ *and $\mathcal{M}^-_{\tilde{\mathcal{L}}_0} u \leqslant 1$ in B_κ,*

in the viscosity sense. Let $\tilde{u}(x) = u(\kappa x)$ for

$$\kappa = \frac{\varepsilon_1}{(1 + \|u\|_\infty)^{1/(\sigma - \tau)}}.$$

Then there are universal $\alpha, C > 0$ such that

$$\big|\tilde{u}(x) - \tilde{u}(0)\big| \leqslant C|x|^\alpha.$$

Our proof relies in noticing that a dilation powerful enough puts us in the same hypothesis as in the proof of [4]. The detail is that the rescaling considered in such proof consists of a dilation of the domain, which as we already saw are good for our situation, times some constants that grow geometrically which compete against the smallness condition on the coefficient b. We want to check that by making α small enough we can control the effect of this second multiplication.

Proof of Lemma 7.1. Let $\varepsilon_1^m \leqslant \varepsilon_0^m/2$ to start such that the estimates from the previous section are valid with the same constants.

We will show that there exist sequences m_k and M_k such that $m_k \leqslant \tilde{u} \leqslant M_k$ in $B_{4^{-k}}$ and

$$M_k - m_k = 4^{-\alpha k}$$

so that result holds for $C = 4^\alpha$.

For $k = 0$ we choose $m_0 = -1/2$ and $M_0 = 1/2$ and by (i) we have $m_0 \leqslant \tilde{u} \leqslant M_0$ in $\mathbb{R}^n$. We construct the sequence by induction. Assume then that we have the sequences up to some k and then we want to find m_{k+1} and M_{k+1}. In the ball $B_{4^{-(k+1)}}$, either $\tilde{u} \geqslant (M_k + m_k)/2$ in at least half the points (in measure) or we have the other inequality. Let's assume, without loss of generality, that

$$\left| \left\{ \tilde{u} \geqslant \frac{M_k + m_k}{2} \right\} \cap B_{4^{-(k+1)}} \right| \geqslant \frac{|B_{4^{-(k+1)}}|}{2}.$$

Consider now

$$v(x) = \frac{\tilde{u}(4^{-k}x) - m_k}{(M_k - m_k)/2},$$

so that $v \geqslant 0$ in B_1 and $|\{v \geqslant 1\} \cap B_{1/4}| \geqslant |B_{1/4}|/2$.

From the inductive hypothesis, we have that for any index j between 1 and k,

$$v \geqslant \frac{m_{k-j} - m_k}{(M_k - m_k)/2} \geqslant \frac{m_{k-j} - M_{k-j} + M_k - m_k}{(M_k - m_k)/2}$$
$$\geqslant 2(1 - 4^{\alpha j}),$$

in B_{4^j}. Therefore $v(x) \geqslant -2(|4x|^\alpha - 1)$ outside B_1. Let $w = v^+$, it satisfies

$$\mathcal{M}^-_{\bar{\mathcal{L}}_0(4^{-k(\sigma-\tau)}\kappa^{\sigma-\tau}b)} w \leqslant 4^{-k\alpha}\kappa^\sigma + \mathcal{M}^+_\sigma v^- + 4^{-k(\sigma-\tau)}\kappa^{\sigma-\tau}b|D_\tau|v^-$$
$$\leqslant \varepsilon_1^m + \mathcal{M}^+_\sigma v^- + \varepsilon_1^m b|D_\tau|v^-. \tag{7.23}$$

We still have $|\{w \geqslant 1\} \cap B_{1/4}| \geqslant |B_{1/4}|/2$. Use the other bound $v^- \leqslant 2(|4x|^\alpha - 1)$ outside B_1, also proved by induction, and $v^- = 0$ in B_1 to get that the right hand side can be made smaller than ε_0^m in $B_{3/4}$ by choosing a small exponent α.

We recall the conditions in Remark 6.3. So far we have shown the second one which is satisfied with a right hand side ε_0^m. For the first condition note that

$$4^{-k(\sigma-\tau)}\kappa^{\sigma-\tau}b(1 + \|w\|_\infty) \leqslant 4^{-k(\sigma-\tau)}\varepsilon_0^m(1 + 4^{\alpha k}) \leqslant 4^{-k(m-\alpha)}\varepsilon_0^m$$

so we have to choose $\alpha \leqslant m$.

Now, given any $x \in B_{1/4}$ we can apply Corollary 6.7 in $B_{1/2}(x)$ to get

$$C(w(x) + \varepsilon_1)^\varepsilon \geqslant |\{w > 1\} \cap B_{1/2}(x)| \geqslant \frac{|B_{1/4}|}{2},$$

hence, since ε_1 can be made even smaller, we conclude $w \geqslant \theta > 0$ in $B_{1/4}$ for some $\theta > 0$. If we let $M_{k+1} = M_k$ and $m_{k+1} = m_k + \theta(M_k - m_k)/2$ we have the inductive step

$$m_{k+1} \leqslant \tilde{u} \leqslant M_{k+1}, \quad \text{in } B_{4^{-(k+1)}}.$$

Moreover $M_{k+1} - m_{k+1} = (1 - \theta/2)4^{-\alpha k}$, so choosing α and θ such that $1 - \theta/2 = 4^{-\alpha}$ we conclude $M_{k+1} - m_{k+1} = 4^{-\alpha(k+1)}$. $\quad\square$

As a result we get that $\tilde{u}$ is C^α. Here is the proof of Theorem 3.1.

Proof of Theorem 3.1. Let u be as in Theorem 3.1 and consider now

$$v(x) = \frac{u(x)}{2(\|u\|_\infty + C_0)}.$$

Since the equation is homogeneous of degree 1 we are now under the hypothesis of the previous lemma. Hence we conclude that the dilation $\tilde{v}$ of v satisfies

$$\left|\tilde{v}(x) - \tilde{v}(0)\right| \leqslant C|x|^\alpha,$$

where C is a universal constant. Coming back to v, this translates to

$$\left|v(x) - v(0)\right| \leqslant C|x|^\alpha \kappa^{-\alpha} = \frac{C}{\varepsilon_0^\alpha}\left(\|v\|_\infty + 1\right)^{\alpha/(\sigma-\tau)}|x|^\alpha,$$

since $\|v\|_\infty \leqslant 1/2$, $\sigma - \tau \geqslant m$ and ε_0 is universal we get

$$\left|v(x) - v(0)\right| \leqslant C|x|^\alpha,$$

for a different universal C. In terms of u we recover the estimate

$$\left|u(x) - u(0)\right| \leqslant C\left(\|u\|_\infty + C_0\right)|x|^\alpha.$$

Hence u is C^α at 0 and its C^α seminorm is controlled as desired. This concludes the proof. $\quad\square$

8. $C^{1,\alpha}$ regularity

For translation invariant equations, $C^{1,\alpha}$ regularity comes by proving C^α regularity for the incremental quotients of a given solution. This procedure allows to improve the regularity from C^α to $C^{2\alpha}$ and so forth all the way up to $C^{0,1}$ and then to $C^{1,\alpha}$, see [3]. We need to use the comparison principle to see that these incremental quotients satisfy a uniformly elliptic equation with bounded measurable coefficients and zero right hand side, for which we already have C^α estimates. The difficulty in this case is that we need, in each step, these incremental quotients to be uniformly bounded in $\mathbb{R}^n$. The previous regularity only guarantees this on $B_{r-\delta}$, given that the equation is satisfied in B_r.

Recall the class $\mathcal{L}_1 = \mathcal{L}_1(\sigma, \tau, \lambda, \Lambda, b, \rho_0) \subseteq \tilde{\mathcal{L}}_0(\sigma, \tau, \lambda, \Lambda, b)$ of all possible linear operators L with nonnegative kernels K such that they satisfy (2.11) and (2.12), and the following integrability assumption for some radius ρ_0,

$$\int_{\mathbb{R}^n \setminus B_{\rho_0}} \frac{|K(y) - K(y-h)|}{|h|}\, dy \leqslant C \quad \text{every time } |h| < \frac{\rho_0}{2}.$$

Theorem 8.1. *Let $\sigma_0, \tau_0, m, A_0 > 0$ and assume that* (H1), (H2) *and* (H3) *hold. There is $\rho_0 > 0$ small enough so that if I is an elliptic operator of the inf-sup type as in* (2.8) *with all the linear operators in $\mathcal{L}_1$ and u a bounded viscosity solution of $Iu = 0$ in B_1, then there is a universal $\alpha > 0$ such that $u \in C^{1,\alpha}(B_{1/2})$ and*

$$\|u\|_{C^{1,\alpha}(B_{1/4})} \leqslant C\|u\|_\infty$$

for some universal $C > 0$.

Proof. Let $\bar{\alpha}$ be the Hölder exponent obtained by Theorem 3.1 and assume that it is not the reciprocal of an integer by making it smaller if necessary. Let $\delta = 1/(4[1/\bar{\alpha}])$. We want to see that, for $k = 0, 1, \ldots, [1/\bar{\alpha}] - 1$, the estimate

$$\|u\|_{C^{0,k\bar{\alpha}}(B_{3/4-k\delta})} \leqslant C(k)\|u\|_\infty \tag{8.24}$$

implies the next estimate,

$$\|u\|_{C^{0,(k+1)\bar{\alpha}}(B_{3/4-(k+1)\delta})} \leqslant C(k+1)\|u\|_\infty. \tag{8.25}$$

Fix a unit vector $e \in \mathbb{R}^n$ and η a smooth cut-off function supported in $B_{(3/4-k\delta)-\delta/4}$ and equal to one in $B_{(3/4-k\delta)-\delta/2}$. For given $h \in (-\delta/8, \delta/8)$ we define the following incremental quotients

$$w_h(x) = \frac{u(x + he) - u(x)}{|h|^{\bar{\alpha}k}},$$

$$w_1^h(x) = \frac{(\eta u)(x + he) - (\eta u)(x)}{|h|^{\bar{\alpha}k}},$$

$$w_2^h(x) = \frac{((1 - \eta)u)(x + he) - ((1 - \eta)u)(x)}{|h|^{\bar{\alpha}k}}.$$

When $x \in B_{(3/4-k\delta)-\delta/8}$, $|w_1^h(x)|$ is bounded above by $C(k, \eta)\|u\|_\infty$. By interpolation and (8.24),

$$\begin{aligned}
\left|w_1^h(x)\right| &\leqslant \|\eta u\|_{C^{0,k\bar{\alpha}}(B_{3/4-k\delta})} \\
&\leqslant \|u\|_{C^{0,k\bar{\alpha}}(B_{3/4-k\delta})} + \|u\|_\infty \|\eta\|_{C^{0,k\bar{\alpha}}(B_{3/4-k\delta})} \\
&\leqslant C(k, \eta)\|u\|_\infty.
\end{aligned}$$

If $x \in \mathbb{R}^n \setminus B_{(3/4-k\delta)-\delta/8}$ then $w_1^h(x)$ just cancels.

By using that the equation is translation invariant we have that u and $u(\cdot + he)$ satisfy equations in the same ellipticity family, *with positive kernels*. Then w_h also satisfies an equation in the same ellipticity family by Lemma 4.7. The function w_1^h satisfies a similar equation as w_h, the difference is on the right hand side introduced by the cut-off,

$$\mathcal{M}_{\mathcal{L}_1}^+ w_1^h \geqslant -\mathcal{M}_{\mathcal{L}_1}^+ w_2^h \quad \text{and} \quad \mathcal{M}_{\mathcal{L}_1}^- w_1^h \leqslant -\mathcal{M}_{\mathcal{L}_1}^- w_2^h.$$

For $x \in B_{(3/4-k\delta)-3\delta/4}$ the terms $|\mathcal{M}_{\mathcal{L}_1}^\pm w_2^h|$ are controlled by $\|u\|_\infty$ by using that

$$\int_{\mathbb{R}^n \setminus B_{\rho_0}} \frac{|K(y) - K(y - h)|}{|h|}\, dy \leqslant C \quad \text{every time } |h| < \frac{\rho_0}{2}$$

with $\rho_0 = \delta/8$. Indeed, for $L \in \mathcal{L}_1$ with kernel K and $x \in B_{(3/4-k\delta)-3\delta/4}$ and $|y| \leqslant \delta/8$, $w_2^h(x + y) = 0$ and

$$\begin{aligned}
\left|Lw_2^h(x)\right| &= \left|\int w_2^h(x + y)K(y)\, dy\right| \\
&= \left|\int_{\mathbb{R}^n \setminus B_{\delta/8}} \frac{(1 - \eta)u(x + y + h) - (1 - \eta)u(x + y)}{|h|^{\bar{\alpha}k}} K(y)\, dy\right| \\
&= \left|\int_{\mathbb{R}^n \setminus B_{\delta/8}} (1 - \eta)u(x + y)|h|^{1-\bar{\alpha}k} \frac{K(y) - K(y - h)}{|h|}\, dy\right| \\
&\leqslant C\|u\|_\infty.
\end{aligned}$$

We get then the equations for w_h^1 in $B_{(3/4-k\delta)-3\delta/4}$

$$\mathcal{M}_{\mathcal{L}_1}^+ w_1^h \geqslant C\|u\|_\infty \quad \text{and} \quad \mathcal{M}_{\mathcal{L}_1}^- w_1^h \leqslant -C\|u\|_\infty.$$

By applying Theorem 3.1 to w_1^h from $B_{(3/4-k\delta)-3\delta/4}$ to $B_{3/4-(k+1)\delta}$ we conclude that for a constant $C(k + 1)$ independent of h,

$$\left\|w_h^1\right\|_{C^{0,\bar{\alpha}}(B_{3/4-(k+1)\delta})} \leqslant C(k + 1)\|u\|_\infty.$$

This implies the estimate (8.25) by using Lemma 5.6 in [3].

From $k = [1/\bar{\alpha}] - 1$ to $k + 1 = [1/\bar{\alpha}]$ we get that u is Lipschitz in $B_{3/4}$ with the estimate

$$\|u\|_{C^{0,1}(B_{3/4})} \leqslant C\|u\|_\infty.$$

By applying the previous step one more time to the Lipschitz quotient we conclude the theorem. $\quad\square$

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