Uniform convergence for the incompressible limit of a tumor growth model

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Abstract

We study a model introduced by Perthame and Vauchelet [19] that describes the growth of a tumor governed by Brinkman's Law, which takes into account friction between the tumor cells. We adopt the viscosity solution approach to establish an optimal uniform convergence result of the tumor density as well as the pressure in the incompressible limit. The system lacks standard maximum principle, and thus modification of the usual approach is necessary.
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1. Introduction

We study the following model, which was introduced by Perthame and Vauchelet in [19]. It describes the growth of tumors at the cellular level by providing a law relating the cell density, pressure, and cell multiplication. The tumor cell density $n_k : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ satisfies,

$$\begin{cases} \partial_t n_k - \operatorname{div}(n_k D W_k) = n_k G(p_k), \\ -\nu \Delta W_k + W_k = p_k, \end{cases} \tag{1.1}$$

where the pressure p_k is given by,

$$p_k = \frac{k}{k-1}(n_k)^{k-1}.$$

Here ν is a positive constant and G is a given function that describes the effect that the pressure has on the growth of the tumor. We assume G satisfies,

$$G \in C^1(\mathbb{R}), \ G'(\cdot) \leq -\bar{\alpha} < 0, \ \text{and} \ G(P_M) = 0 \text{ for some } P_M > 0 \text{ and } \bar{\alpha} > 0. \tag{1.2}$$

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The main results of [19] concern the limit as $k \to \infty$, or the so-called incompressible limit, of (1.1). This connects (1.1) to a system that involves a moving front. If the parameter ν were zero (in other words, if the tumor were governed by Darcy's Law), then the system (1.1) would become,

$$\partial_t n_k - \operatorname{div}(n_k D p_k) = n_k G(p_k).$$

This model for tumor growth has been widely studied, and we refer the reader to the introduction of [19] for a variety of references, both about modeling and rigorous mathematical analysis. In particular, in [18], Perthame, Quiròs and Vázquez find that the incompressible limit of the above equation is the Hele–Shaw problem with a forcing term. Kim and Pozar [16] used viscosity solution methods to improve the result in [18]. The model that we study, (1.1) with $\nu > 0$, has been proposed as a better description of tumor growth. Here, the tumor is governed by Brinkman's Law, which takes into account the friction between the tumor cells, and not just of the tumor with its environment. These modeling issues are discussed in, for example, [24,6]. For fixed k, the system (1.1) was also studied by Trivisa and Webber in [23], who established existence of weak solutions and found a convergent numerical method for (1.1).

Of particular interest in the asymptotic limit is the limiting pressure, which represents the incompressibility condition. In the inviscid model ($\nu = 0$), the limiting pressure solves a Hele–Shaw type problem and is continuous as long as the pressure zone is reasonably regular [18]. However, as illustrated in [19], in the viscous model that we study here the limiting pressure is strictly positive on the boundary of its support, and thus is discontinuous. This is an interesting contrast to the inviscid model.

Our goal in this paper is to obtain pointwise convergence results in the framework of viscosity solutions theory, improving the L^1 convergence obtained in [19]. Due to the discontinuity of the limiting pressure, the optimal pointwise convergence result one expects is uniform convergence away from the pressure boundary. This is precisely what we obtain. In addition, knowing that the pressure converges uniformly then allows us to improve the convergence of the W_k as well (see Theorem 1.1 below).

We point out that the system (1.1) does not enjoy the comparison principle – in fact, it is strongly coupled – and thus one needs to modify the existing theory in the analysis. To achieve this we follow the approach in [15], where we rely on the fact that one component of the system can be considered almost fixed due to its strong convergence: in our case that turns out to be the W_k, though their convergence is still weaker than what is available in [15].

Heuristics. Let us briefly recall the formal derivation of the limiting system given in [19] to illustrate additional challenges and main ingredients of our analysis in more detail. We denote the limit of (p_k, n_k, W_k) by $(p_\infty, n_\infty, W_\infty)$. Perhaps the easiest equation to guess is the one for W_∞:

$$-\nu \Delta W_\infty + W_\infty = p_\infty. \tag{1.3}$$

Next we expect that p_∞ is either zero or satisfies $p_\infty - \nu G(p_\infty) = W_\infty$. This is because we can write the n_k equation in terms of p_k as

$$\partial_t p_k - D p_k \cdot D W_k = (k-1)\nu^{-1} p_k (W_k - (Id - \nu G)(p_k)), \tag{1.4}$$

which then translates p_∞ as a singular limit of reaction–diffusion equations. Thus it is reasonable to think that p_∞ will take value either zero or $(Id - \nu G)^{-1}(W_\infty)$. In other words, we expect to have $p_\infty = (Id - \nu G)^{-1}(W_\infty)\chi_{\Omega_t}$ for some region Ω_t. The question now is to characterize Ω_t.

We recall that there is a third component here, namely n_k. Manipulating the equation for n_k and then using the equation that W_k satisfies yields,

$$\partial_t n_k - D n_k \cdot D W_k = n_k (\Delta W_k + G(p_k)) = \frac{n_k}{\nu}(W_k - p_k + \nu G(p_k)). \tag{1.5}$$

The region Ω_t is where the p_k converge to the positive value $(Id - \nu G)^{-1}(W_\infty)$, so by definition we know that the n_k converge to 1 there. When the p_k converge to 0 (in other words, on Ω_t^c) we expect the n_k to converge to zero if initially this is the case (see the discussion in the outline below). Notice that in both situations, the right-hand side of the previous equation is zero. Thus we expect n_∞ to equal χ_{Ω_t} and solve,

$$\partial_t n_\infty - D n_\infty \cdot D W_\infty = 0, \tag{1.6}$$

yielding the normal velocity law for the set Ω_t. Thus, we expect the triple $(p_\infty, n_\infty, W_\infty)$ to solve the system,

$$\begin{cases} -\nu \Delta W_\infty + W_\infty = p_\infty, \\ p_\infty = (Id - \nu G)^{-1}(W_\infty)\chi_{\{n_\infty>0\}}, \quad n_\infty = \chi_{\{n_\infty>0\}}, \\ \partial_t n_\infty - Dn_\infty \cdot DW_\infty = 0. \end{cases} \tag{1.7}$$

The above heuristics are indeed true when the limiting density n_∞ is initially a patch. Then it follows from the transport equation (1.6) that n_∞ is always zero or one at later times. In general the limiting system (1.7) is invalid with the presence of the region $\{0 < n_\infty < 1\}$, due to the interaction of the two convergence regions as $k \to \infty$. In the inviscid model ($\nu = 0$), this was studied by Kim and Pozar [16] and Mellet, Perthame and Quirós [17]. In our situation the normal velocity of the pressure zone in this general setting remains open.

Initial data. Let us now state the conditions on the limiting initial data with the notation

$$F := (Id - \nu G)^{-1}, \tag{1.8}$$

as given in [19] (where H is used instead of F). We assume,

$$n_\infty^0 = \chi_{\Omega_0}, \quad p_\infty^0 = F(W_\infty^0)\chi_{\Omega_0}, \quad -\nu \Delta W_\infty^0 + W_\infty^0 = p_\infty^0, \tag{1.9}$$

where $\Omega_0 \subset \mathbb{R}^n$ is a compact set with measure zero boundary. The last two equations are, as mentioned in [19], to avoid initial layers in the limit system. As for the approximating system, we impose

$$\liminf_{k \to \infty} {}_* p_k(x, 0) > 0 \text{ on } \Omega_0 \text{ and } \liminf_{k \to \infty} dist\left(\{x \,|\, p_k(x, 0) > 0\}, (\bar\Omega_0)^c\right) > 0, \tag{1.10}$$

where $dist$ is the usual distance function. The assumptions (1.10) that we make on the initial data are very similar to those in [19]. They are neither more nor less general, as discussed below.

Main result. Now we are ready to state our main result.

Theorem 1.1. *Let Ω_0 be a compact set in $\mathbb{R}^n$ and let n_k and W_k solve (1.1) with initial data satisfying (1.10). Then, along a subsequence:*

(a) *the W_k converge strongly to W_∞ in $L^\infty((0, T), W_{loc}^{2,p}(\mathbb{R}^n))$,*
(b) *the p_k converge locally uniformly to p_∞ on $(\mathbb{R}^n \times (0, \infty)) \setminus \partial\{n_\infty > 0\}$,*
(c) *the n_k converge locally uniformly to n_∞ on $(\mathbb{R}^n \times (0, \infty)) \setminus \partial\{n_\infty > 0\}$,*

where $(p_\infty, n_\infty, W_\infty)$ solve (1.7) with initial data (1.9). Moreover, $\partial\{n_\infty > 0\}$ has measure zero.

As stated in the theorem, the limiting density has its support evolving by the geometric flow (1.6). Since our goal here is to obtain the convergence of the density and pressure in a strong sense – namely, locally uniformly – we therefore need to employ a sufficiently strong notion of solution for (1.6). For this reason we consider viscosity solutions to (1.6). This allows us to use barrier arguments with smooth test functions, as well as stability properties, to yield the (locally) uniform convergence results that we desire. Since a priori estimates only yield DW_∞ to be integrable in time and log-Lipschitz in space, (1.6) is not covered by standard viscosity solutions theory. Thus, a key part of our work is to define a notion of viscosity solution for (1.6); establish basic results such as stability, existence and a comparison theorem; and describe the unique viscosity solution of (1.6) in terms of the associated flow map. In fact, in the proof of the main result we identify n_∞ with the function given by (2.7) of Theorem 2.7, with $V = -DW_\infty$.

Before we discuss the main ingredients of the proof in more detail, some remarks on the theorem are in order.

Size of $\partial\{n_\infty > 0\}$. Theorem 1.1 tells us the limiting behavior of the pressure and density everywhere except on $\partial\{n_\infty > 0\}$. In Lemma 9.1 we show that $\partial\{n_\infty > 0\}$ has Lebesgue measure zero for all times. If DW_∞ were Lipschitz in space, then the flow generated by DW_∞ would also be Lipschitz, and, for example, the Hausdorff dimension of $\partial\{n_\infty > 0\}$ would be preserved under the flow. This is not quite the case for us: see Section 9.2 for more discussion.

Relationship of our work and [19]. Our results strengthen those of [19] in several ways. First, we obtain locally uniform convergence of p_k and n_k, improving the L^1_{loc} convergence in [19]. Second, we characterize n_∞ as an indicator function. This confirms what was suggested in the numerical examples in [19, Section 3], but was not proven there. Third, as a consequence of the stronger convergence of the p_k, we improve the convergence of the W_k from $L^1((0, T), W^{1,q}_{loc}(\mathbb{R}^n))$ to $L^\infty((0, T), W^{2,p}_{loc}(\mathbb{R}^n))$.

We now discuss in detail the relationship between our assumption on the initial data (1.10) and the analogous assumption in [19]. Indeed, [19] assumes that $p_k(x, 0)$ converge almost everywhere to $H(W_\infty)$ on Ω_0, and are identically zero on Ω_0^c. These assumptions of [19] do not imply that (1.10) holds, because convergence in the almost everywhere sense is weaker than what is needed for (1.10) to hold. On the other hand, our assumptions do not imply that those in [19] hold. We do not need to assume convergence of the $p_k(x, 0)$ to $F(W_\infty)$ on Ω_0; for us it is enough only to assume that the p_k are uniformly positive there. In addition, we do not require all of the $p_k(x, 0)$ to have the same zero set; we simply require a convergence of the zero sets. Thus, our assumptions are neither stronger nor weaker than those in [19].

The initial time. We discuss the behavior of the p_k near the initial time. First, we point out a difference between the assumptions on the initial data for the limiting system (1.9) and for the system at the k-level (1.10) – the second condition in (1.9) states $p^0_\infty = F(W^0_\infty)\chi_{\Omega_0}$; however, at the k-level we assume only $\liminf_* p_k(x, 0) > 0$ on Ω_0. Despite this, we are able to establish that the p_k converge locally uniformly to p_∞ for $t > 0$ (and off of $\partial\{n_\infty > 0\}$).

To see why this should be the case, we look at the equation that p_k satisfies, (1.4), and explain the heuristics. We see that if p_k is even a little bit positive initially, it will approach the stable root of the reaction term for positive times as k approaches infinity. This causes a possible "jump" at time 0: indeed, it is even possible for the initial data $p_k(x, 0)$ to converge to something other than p^0_∞, and yet for the p_k to still converge to p_∞ for $t > 0$.

Main challenges and ingredients. As mentioned above, our goal here is to obtain the convergence of the p_k in a strong sense – namely, locally uniformly – using the viscosity solution approach. We illustrate the main ingredients and challenges of the proof below.

Viscosity solutions for the transport equation. A key part of our work is defining and establishing basic properties for viscosity solutions of (1.6). We prove a comparison result, Theorem 2.4, that is essential to the rest of the paper, and implies that solutions to (1.6) with continuous initial data are unique. In addition, we also establish uniqueness for solutions to (1.6) that have a characteristic function as initial data (see Theorem 2.7). This is an interesting and subtle point – in general, it is possible for a Hamilton Jacobi equation $\partial_t u + H(x, t, Du) = 0$ to enjoy uniqueness for continuous solutions, but not discontinuous solutions (see the counterexample of Barles, Soner and Souganidis in [3, Proposition 4.4] where non-uniqueness occurs for $\partial_t u + (x - t)|Du| = 0$ due to nucleation).

Literature on the transport equation. There is a wide literature on renormalized solutions (in the sense of DiPerna–Lions [11]) and distributional solutions to the transport equation with quite general vector fields. In particular, Ambrosio's [1, Theorem 4.1] establishes uniqueness of distributional solutions to

$$\partial_t u + Du \cdot b(x, t) = 0$$

where $b(\cdot, t) \in BV_{loc}(\mathbb{R}^n)$ for almost all t and satisfies $\text{div}(b) \in L^1((0, T), L^\infty_{loc}(\mathbb{R}^n))$. Our vector field DW_∞ satisfies these hypotheses. However, the aforementioned result concerns solutions in the distributional sense. We do not know whether or not a viscosity solution is a distributional solution, and thus cannot immediately deduce uniqueness or comparison for our situation, so we establish comparison for viscosity solutions of (1.6) directly.

Generalized set evolution. Understanding the evolution of the set $\partial\{n_\infty > 0\}$ is key to finding the asymptotic behavior of our system. The heuristics indicate that this region travels with normal velocity DW_∞. We want a precise and direct way to describe and study such evolution. For this, we extend the definition given by Barles and Souganidis in [2] of generalized flow to velocities that are only integrable in time. Heuristically, the definition involves testing a subset of $\mathbb{R}^n$ from the "inside" or "outside" by smoothly evolving sets. Whether a region Ω_t is a generalized flow with velocity DW_∞ is closely related to whether the indicator of Ω_t is a viscosity solution of (1.6) (this is made precise in Theorem 3.2).

In [2], the authors also introduce a way of studying the development of interfaces in asymptotic limits of reaction–diffusion equations. In [15], such methods were used to study a system with no comparison principle. Although the methods of [2] and [15] provided a lot of inspiration for our work, we do not use the so-called "abstract approach" introduced in [2], and are able to proceed with more basic barrier arguments. This is mostly due to the fact that the equation (1.4) for p_k is first order.

Issues of uniqueness. It is not known whether the solutions to (2.1) or (1.7) are unique. However, if we are given two triplets solving (1.7) but with the same W_∞, then the pressures and densities must agree. In other words, if $(p_\infty, n_\infty, W_\infty)$ and $(p'_\infty, n'_\infty, W_\infty)$ solve (1.7), then we must have $(n_\infty)_* \leq n'_\infty \leq (n_\infty)^*$, and similarly for the pressures (here $u*$ and u_* denote, respectively, the upper and lower semicontinuous envelopes of u). This is a consequence of Theorem 2.7, which establishes uniqueness for discontinuous viscosity solutions of the transport equation (1.6). This also implies, by an argument very similar to that of [3, Section 2] for classical viscosity solutions, that $\partial\{n_\infty > 0\}$ enjoys the empty interior property.

Obtaining the main result. Once we have introduced the notion of generalized flow, we establish:

Proposition 1.2. *For $t \geq 0$, define the sets A^k, Ω_t^1 and Ω_t^2 by,*

$$A_t^k = \{x \,|\, p_k(x, t) > 0\}, \ \Omega_t^1 = \{x \,|\, \liminf_{k \to \infty} {}_* \, p_k(x, t) > 0\} \ and \ \Omega_t^2 = \{x \,|\, \liminf_{k \to \infty} {}_* \, dist(x, A_t^k) > 0\}.$$

Under the hypotheses of Theorem 1.1, we have,

(1) $(\underline{\Omega_t^1})^{int}$ is a generalized superflow with velocity $-DW_\infty$, and,
(2) $(\overline{\Omega_t^2})^c$ is a generalized subflow with velocity $-DW_\infty$.

(Here $\liminf_*$ and $\limsup^*$ are the usual weak limits. We write down the definition in Definition 4.1 for the convenience of the reader.) We view this proposition as the heart of our paper. It captures the basic idea that the limiting behavior of (1.1) can be expressed by saying where the limit of the p_k is zero, where it is positive, and how these two regions evolve in time.

We remark on the definition of Ω_t^2. It says that the p_k are eventually zero, uniformly on compact subsets of Ω_t^2. Knowing $x \in \Omega_t^2$ is strictly stronger than simply $\limsup^* p_k(x, t) = 0$. (As an elementary example, consider the sequence of functions $f_k(x) \equiv 1/k$. This sequence satisfies $\limsup^* f_k = 0$, and yet the f_k are never eventually zero.) In addition, $\limsup^* p_k(x, t) = 0$ does not imply that the limit of the $n_k(x, t)$ is zero, but $x \in \Omega_t^2$ does.

In order to deduce Theorem 1.1 from Proposition 1.2, we also study the sets Ω_t^1 and Ω_t^2 at the initial time, and compare them to the set Ω_0 appearing in the hypotheses on the initial data. Then we establish that in Ω_t^1, not only are the p_k uniformly positive, but that they in fact converge to $(Id - G)^{-1}(W_\infty)$. Finally, use these results, together with estimates on the size of $\partial\{n_\infty > 0\}$, to obtain the improved convergence of the W_k.

Structure of our paper. Viscosity solutions and generalized flows are defined in Section 2 and Section 3, respectively. There we state basic properties of these two notions and of their relationship to each other. We could not find these results elsewhere in our precise setting so we include their proofs. However, since this is not the main focus of our work, these proofs are sketched in the appendices. Section 4 is short and covers some preliminary results about (1.1) that we use in the rest of the paper. Then, Section 5 and Section 6 are devoted to the proofs of items (1) and (2), respectively, of Proposition 1.2. We study the limiting behavior at the initial time in Section 7. Section 8 is devoted to studying the limit of the p_k in the positive region. We put all of the ingredients together in Section 9 and establish our main result.

2. Viscosity solutions

We define a notion of solution for

$$\partial_t u + Du \cdot V(x, t) = 0 \tag{2.1}$$

for V uniformly bounded, log-Lipschitz in x and L^1 in t. Our precise hypotheses are that there exist $M > 0$ and $N > 0$ with:

$$|V(x, t)| \le M \text{ for all } x, t; \tag{2.2}$$

$$|V(x, t) - V(y, t)| \le \sigma(|x - y|) \text{ for all } x, y, t, \text{ where } \sigma(r) = Nr|\ln(r)|; \tag{2.3}$$

$$t \mapsto V(x, t) \text{ is integrable on } (0, T) \text{ for any } x. \tag{2.4}$$

We use USC and LSC to denote, respectively, the classes of real-valued upper-semicontinuous and lower-semicontinuous functions on $\mathbb{R}^n$. We will also employ the upper-semicontinuous and lower-semicontinuous envelopes, which we denote for a given function u by u^* and u_*, respectively.

We follow Ishii [13] in defining viscosity solutions for (2.1). We use H to denote,

$$H(x, t, q) = V(x, t) \cdot q. \tag{2.5}$$

First we define:

Definition 2.1. For any open set $Q \subset \mathbb{R}^n \times [0, T]$, $(x_0, t_0) \in Q$, and $q_0 \in \mathbb{R}^n$ we define H^+ as:

$$H^+(x_0, t_0, q_0) = \{(G, b) \text{ such that } G \in C(Q \times \mathbb{R}^n), b \in L^1(0, T),$$

$$G(x, t, q) + b(t) \ge H(x, t, q) \text{ for all } (x, q) \in B_\delta(x_0, q_0),$$

$$\text{almost all } t \in B_\delta(t_0), \text{ some } \delta > 0\}.$$

The definition $H^-(x_0, t_0, q_0)$ is identical, except $\ge$ is replaced with $\le$.

Definition 2.2. Let Q be an open subset of $\mathbb{R}^n$.

(1) $u \in USC$ is called a *viscosity subsolution* (resp. *supersolution*) in $Q \times (0, T)$ if

$$\partial_t \phi(x_0, t_0) + G(x_0, t_0, D\phi(x_0, t_0)) \le 0 \text{ (resp. } \ge 0)$$

holds for any $\phi \in C^1(Q \times (0, T))$, $(x_0, t_0) \in Q \times (0, T)$, and $(G, b) \in H^-(x_0, t_0, D\phi(x_0, t_0))$ (resp. H^+) such that

$$(x, t) \mapsto u(x, t) + \int_0^t b(s) \, ds - \phi(x, t)$$

has a local maximum (resp. minimum) at (x_0, t_0).

(2) u is called a *viscosity solution* on $Q \times (0, T)$ *with initial data* u_0 if u_* is a supersolution, u^* is a subsolution, $u_*(x, 0) \ge (u_0)_*(x)$ for all $x \in Q$, and $u^*(x, 0) \le (u_0)^*(x)$ for all $x \in Q$.

From now on, we often use "solution" to refer to "viscosity solution", and similarly for sub- and super-solutions. We remark that if u_0 and u are continuous, then u being a viscosity solution with initial data u_0 simply means that u is both a sub- and a super-solution, and equals u_0 at time 0.

2.1. Basic properties

It follows directly from the definitions that if H is continuous in t, then a viscosity solution in this sense is also a viscosity solution in the usual sense. We record this as:

Lemma 2.3. *Suppose H is continuous in t, $u \in USC$ (resp. LSC) is a viscosity subsolution (resp. supersolution), $\phi \in C^1$, and $u - \phi$ has a local maximum (resp. minimum) at (x_0, t_0). Then*

$$\partial_t \phi(x_0, t_0) + H(x_0, t_0, D\phi(x_0, t_0)) \le 0 \text{ (resp. } \ge 0).$$

An important property of classical viscosity solution is stability – if there is a uniformly convergent sequence of viscosity solutions, then the limit is also a viscosity solution. The notion of viscosity solutions we define also enjoys such a property. For a precise statement, we refer the reader to [13, Proposition 7.1]. The proof there carries over to this situation with no modifications.

We establish the following comparison theorem for viscosity solutions:

Theorem 2.4. *Suppose V satisfies hypotheses (2.2), (2.3) and (2.4). Suppose $u \in USC(\mathbb{R}^n \times [0, T])$ and $v \in LSC(\mathbb{R}^n \times [0, T])$ are, respectively, a sub and super solution to (2.1) on $\mathbb{R}^n \times [0, T]$. Then,*

$$\sup_{\mathbb{R}^n \times [0,T]} (u - v) \leq \sup_{\mathbb{R}^n}(u(x, 0) - v(x, 0))^+.$$

This result is essential to our work. The proof is essentially a doubling-variables argument. It has two parts. First, we establish that $u(x, t) - v(x, t)$ is a subsolution to a certain equation. This part of the proof follows [13]. These techniques have also been used by Bourgoing [5,4] to establish comparison and existence for solutions of second order PDEs with L^1 time dependence. However, V is not regular enough in the space variable for any of these results to apply directly. The second part of the proof of Theorem 2.4 is to use that V satisfies the log-Lipschitz hypothesis (2.3) to construct a supersolution to the equation that $u(x, t) - v(y, t)$ satisfies, thus yielding the desired bound from above. The second part of the proof uses ideas from two papers that study Hamilton–Jacobi equations with coefficients that are not necessary Lipshitz. These are [8] of Crandall, Ishii and Lions, as well as Stromberg's [21]. We provide the proof in Appendix A.

Next we prove existence and basic regularity for viscosity solutions of (2.1).

Theorem 2.5. *Let V satisfy (2.2), (2.3) and (2.4). Let $u_0 \in L^\infty(\mathbb{R}^n)$ be uniformly continuous. Then there exists a solution u to (2.1) on $\mathbb{R}^n$ with initial data u_0.*

Moreover, u is uniformly continuous in x and t, with modulus that depends only on the modulus of continuity of u_0, T, and the constants M, N in (2.2) and (2.3).

The main idea of the proof is that, if V were "regular enough," then simply the method of characteristics would provide a solution of (2.1). It turns out that the assumptions (2.2), (2.3) and (2.4) are enough for us to be able to regularize V, obtain classical solutions using the method of characteristics, and then take a limit. In fact, the description of u in terms of characteristics remains valid even after taking the limit, and we have:

Theorem 2.6. *Under the hypotheses of Theorem 2.5, there exists unique $X : (0, T) \times (0, T) \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying,*

$$X(s, t, x) = x + \int_t^s V(X(r, t, x), r) \, dr, \tag{2.6}$$

and we have $u(x, t) = u_0(X(0, t, x))$. Moreover, for every $t > 0$ the map Φ_t defined by, $\Phi_t(x) = X(t, 0, x)$ is Holder continuous with exponent $\exp(-Nt)$, where N is the constant in (2.3).

This characterization of u is useful to us for several reasons. First, it will allow us to deduce information about the size and regularity of the set $\partial\{n_\infty > 0\}$, which, according to Theorem 1.1, is the only region on which we "don't know" what the limit of the p_k is. Second, it connects the notion of generalized flow that we introduce in the next section and use in the rest of the paper with more classical notions. The proofs of Theorem 2.5 and Theorem 2.6 are in Appendix B.

2.2. Discontinuous viscosity solutions to (2.1)

It is clear that Theorem 2.4 implies uniqueness of continuous viscosity solutions to (2.1). The situation for discontinuous solutions is more subtle. In fact, as described in the introduction, there are equations for which uniqueness holds in the class of continuous solutions, but not in the class of discontinuous solutions. We use Theorem 2.4, together with Theorem 2.6, to establish existence and uniqueness of viscosity solutions to (2.1) with initial data a characteristic function:

Theorem 2.7. *Let Ω_0 be an open, bounded domain in $\mathbb{R}^n$ and let V satisfy (2.2), (2.3) and (2.4). Then there exists a viscosity solution u of (2.1) with initial data $u_0(x) := \chi_{\Omega_0}(x)$. Moreover, we have,*

$$u(x, t) = \chi_{\Omega_t}(x), \ \ where \ \Omega_t := \{x : X(0, t, x) \in \Omega_0\}, \tag{2.7}$$

where X is the unique map satisfying (2.6). And, u is unique, in the sense that any other viscosity solution of (2.1) is between u and $u^* = \chi_{\bar{\Omega}_t}$.

In order to establish the uniqueness result stated here, we are taking advantage of the special form of u in terms of the flow X. The proof is at the end of Appendix B.

3. Generalized flows

We introduce a notion of generalized flows with velocity $V(x, t)$, where V satisfies (2.2), (2.4) and (2.3). Throughout, we use Ω^{int} to denote the interior of the set Ω.

Definition 3.1. Let $(\Omega_t)_{t \in (a,b)}$ be a family of open subsets of $\mathbb{R}^n$, and let V satisfy (2.2), (2.3), and (2.4).

- The family $(\Omega_t)_{t \in (a,b)}$ is called a *generalized superflow* with velocity V if for all $\bar{x} \in \mathbb{R}^n$, $t \in (a, b)$, $r > 0$, $\alpha > 0$ and for all smooth functions $\phi : \mathbb{R}^n \to \mathbb{R}$ such that

$$\{x : \phi(x) \geq 0\} \subset \Omega_t \cap B_r(\bar{x}),$$

with $|D\phi| \neq 0$ on $\{x : \phi(x) = 0\}$, there exists $\bar{h} > 0$ such that for all $h \in (0, \bar{h})$,

$$\left\{ x : \phi(x) - \int_t^{t+h} V(x, s) \cdot D\phi(x) \, ds - h\alpha > 0 \right\} \cap \bar{B}_r(\bar{x}) \subset \Omega_{t+h},$$

and $\bar{h}$ depends only on α, $||\phi||_{C^3(B_r(\bar{x}))}$, and the constant M that appears in the hypotheses (2.2), (2.3), and (2.4).
- The family $(\Omega_t)_{t \in (a,b)}$ is called a *generalized subflow* with velocity V if for all $\bar{x} \in \mathbb{R}^n$, $t \in (a, b)$, $r > 0$, $\alpha > 0$ and for all smooth functions $\phi : \mathbb{R}^n \to \mathbb{R}$ such that

$$\{x : \phi(x) \leq 0\} \subset \bar{\Omega}_t^c \cap B_r(\bar{x}),$$

with $|D\phi| \neq 0$ on $\{x : \phi(x) = 0\}$, there exists $\bar{h} > 0$ such that for all $h \in (0, \bar{h})$,

$$\left\{ x : \phi(x) - \int_t^{t+h} V(x, s) \cdot D\phi(x) \, ds - h\alpha < 0 \right\} \cap \bar{B}_r(\bar{x}) \subset \bar{\Omega}_{t+h}^c,$$

and $\bar{h}$ depends only on α, $||\phi||_{C^3(B_r(\bar{x}))}$, and the constant M that appears in the hypotheses (2.2), (2.3), and (2.4).

Whether Ω_t is a generalized flow with velocity V is closely related to whether $\chi_\Omega - \chi_{\bar{\Omega}^c}$ is a solution of (2.1). Precisely:

Theorem 3.2. *Let V satisfy (2.2), (2.3), and (2.4).*

(1) $(\chi_\Omega(x, t) - \chi_{\bar{\Omega}^c}(x, t))_*$ is a supersolution of (2.1) on $\mathbb{R}^n \times [0, T]$ if and only if $(\Omega_t^{int})_{t \in [0,T]}$ is a generalized superflow with velocity V.

(2) $(\Omega_t^{int})_{t \in [0,T]}$ is a generalized subflow with velocity V if and only if $(\chi_\Omega(x, t) - \chi_{\bar{\Omega}^c}(x, t))^*$ is a subsolution of (2.1) on $\mathbb{R}^n \times [0, T]$.

The proof follows along the lines of the proof of Theorem 2.4 of [2]. The key idea is that, for any (smooth enough) function ϕ, the function ψ defined by,

$$\psi(x, r) = \phi(x) - \int_t^r H(x, s, D\phi(x)) \, ds - (r - t)\alpha/2$$

"should be" a subsolution of the equation (2.1). Indeed, if H were differentiable in x and continuous in t, we'd have,

$$D\psi(x, r) = D\phi(x) - \int_t^r H_p(x, s, D\phi(x)) D^2\phi(x) + H_x(x, s, D\phi(x)) \, ds,$$

and hence $D\psi(x, r) = D\phi(x) + O(|r - t|)$. Taking the derivative of ψ in r yields,

$$\psi_r(x, t) = H(x, r, D\phi(x)) - \alpha/2,$$

and since $D\psi(x, r) = D\phi(x) + O(|r - t|)$, we find that the right-hand side of the previous line is bounded from above by $H(x, r, D\psi(x, r))$ for r close enough to t.

To make these ideas precise, we regularize H in the space variable before carrying out this argument. We do so in the following lemma, which will also be useful to us in Section 5. The proofs of the lemma and of Theorem 3.2 are in Appendix C.

Lemma 3.3. *Let $\phi \in C^3(\mathbb{R}^n)$ and let $\alpha > 0$. We take $H(x, t, p) = p \cdot V(x, t)$, where V satisfies (2.2), (2.3), and (2.4). Let ρ be a standard bump function, supported on $B_1(0)$ and with $0 \le \rho \le 1$ everywhere, and let $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right)$. Define H^ε as the convolution in x of H and ρ; namely,*

$$H^\varepsilon(x, t, q) = \int_{y \in \mathbb{R}^n} H(x - y, t, q) \rho_\varepsilon(y) \, dy.$$

Define ψ_ε and $\bar{\psi}_\varepsilon$ by,

$$\psi_\varepsilon(x, r) = \phi(x) - \int_t^r H^\varepsilon(x, s, D\phi(x)) \, ds - (r - t)\alpha/2 \tag{3.1}$$

and

$$\bar{\psi}_\varepsilon(x, r) = \phi(x) - \int_t^r H^\varepsilon(x, s, D\phi(x)) \, ds + (r - t)\alpha/2.$$

There exist constants $\varepsilon_1 > 0$ and $\bar{h} > 0$, both depending only on α, $\|\phi\|_{C^3(\mathbb{R}^n)}$ and the constant N in (2.3), such that ψ_{ε_1} is a subsolution and $\bar{\psi}_{\varepsilon_1}$ is a supersolution of (2.1) on $\mathbb{R}^n \times (t, t + \bar{h})$. Moreover, for this same ε_1 we have, for all $x \in \mathbb{R}^n$ and for all t,

$$|H^{\varepsilon_1}(x, t, D\phi(x)) - H(x, t, D\phi(x))| \le \frac{\alpha}{4}. \tag{3.2}$$

4. Preliminaries

Now that we have introduced the notions of viscosity solution and generalized flow that we will be using, we are almost ready to study the limit of our system. However, we first need to take care of a few preliminaries. We recall that p_k satisfies (1.4). For the remainder of the paper we take $v = 1$. Indeed, if p_k, W_k, G are as in the introduction, then $\tilde{p}_k$, $\tilde{W}_k$, $\tilde{G}$ given by,

$$\tilde{p}_k(x, t) = p_k(\sqrt{v}x, vt); \quad \tilde{W}_k = W_k(\sqrt{v}x, vt); \quad \tilde{G}(u) = vG(u),$$

satisfy

$$\begin{cases} \partial_t p_k - Dp_k \cdot DW_k = kp_k(W_k - p_k + G(p_k)), \\ -v\Delta W_k + W_k = p_k. \end{cases} \tag{4.1}$$

(We have also renumbered, so that $k + 1$ becomes k.) Thus our assumption $\nu = 1$ does not result in any loss of generality. We will focus on (4.1) for the remainder of the paper.

First, we recall for the reader the standard definition of weak limit that we use throughout the paper:

Definition 4.1. Let $\{u_k\}$ be a sequence of functions. We define:

$$\liminf_{k \to \infty} {}_* u_k(x, t) = \lim_{k \to \infty} \inf\{u_j(y, s) : \ j \geq k \text{ and } |(x, t) - (y, s)| \leq k^{-1}\},$$

and,

$$\limsup_{k \to \infty} {}^* u_k(x, t) = \lim_{k \to \infty} \sup\{u_j(y, s) : \ j \geq k \text{ and } |(x, t) - (y, s)| \leq k^{-1}\}.$$

Next, we summarize the results on the system (4.1) that we will use from [19].

Lemma 4.2. *Let p_k and W_k satisfy (4.1) with initial data satisfying (1.10). We have,*

$$0 \leq W_k \leq P_M \text{ and} \tag{4.2}$$
$$0 \leq p_k \leq P_M. \tag{4.3}$$

In addition, there exists $W_\infty \in C(\mathbb{R}^n \times (0, T)) \cap L^\infty((0, T), W^{1,\infty}(\mathbb{R}^n))$ satisfying (1.3) and such that DW_∞ satisfies (2.2), (2.3) and (2.4) and, along a subsequence (still denoted by W_k),

$$DW_k \text{ converge strongly in } L^1((0, T), L^\infty_{loc}(\mathbb{R}^n)) \text{ to } DW_\infty \tag{4.4}$$

and the W_k converge to W_∞ locally uniformly.

Proof. Items (4.3) and (4.4), as well as the fact that DW_∞ satisfies (2.2) and (2.4), follow directly from the statement of [19, Lemma 2.1]. Item (4.2) follows from (4.3) and the maximum principle for the equation that W_k satisfies. That W_∞ satisfies (1.3) is exactly [19, equation (1.14)]. Since p_∞ is, in particular, a function in L^∞, classical results (see, for example, [20]) yield that DW_∞ satisfies (2.3).

The uniform estimates $W_k \in L^\infty((0, T), W^{1,q}(\mathbb{R}^d))$ and $\partial_t W_k \in L^1((0, T), L^q(\mathbb{R}^d))$, for $1 \leq q \leq \infty$ of [19, Lemma 2.1] imply that the W_k are equicontinuous. According to (4.2) the W_k are also uniformly bounded. Hence, by the Arzela Ascoli Theorem, we also have that the W_k converge locally uniformly to W_∞ along a further subsequence. In particular, we conclude W_∞ is continuous. $\square$

Finally we establish some elementary properties of G and of the reaction term in (4.1) that we will use throughout the remainder of the paper.

Lemma 4.3. *Suppose G satisfies (1.2) and that the family W_k satisfies (4.2). Then we have:*

$$G(u) \geq \bar{\alpha}(P_M - u), \tag{4.5}$$
$$u(W_k - u + G(u)) \geq u(\bar{\alpha} P_M - (1 + \bar{\alpha})u), \text{ and} \tag{4.6}$$
$$u(W_k - u + G(u)) \leq u(1 + \bar{\alpha})P_M \tag{4.7}$$

for all $u \in [0, P_M]$. In addition, we have $F'(u) \in (0, 1)$ for all u, and $F(W_\infty(x, t)) \in (0, P_M]$ for any (x, t), where W_∞ is as in Lemma 4.2 and F is given by (1.8).

Proof. First we examine the function G. The properties of G in (1.2) imply, for $u \in [0, P_M]$,

$$-G(u) = G(P_M) - G(u) = \int_u^{P_M} G'(r) \, dr \leq \int_u^{P_M} -\bar{\alpha} \, dr = -\bar{\alpha}(P_M - u),$$

which implies that (4.5) holds. Next we use (4.5) to obtain a lower bound on our reaction term:

$$u(W_k - u + G(u)) \geq u(W_k - u + \bar{\alpha}(P_M - u)) = u(W_k + \bar{\alpha} P_M - (1 + \bar{\alpha})u).$$

Using $W_k \geq 0$, we find,

$$u(W_k - u + G(u)) \geq u(\bar{\alpha} P_M - (1 + \bar{\alpha})u).$$

To obtain the bound from above we use the estimate $W_k \leq P_M$ and obtain,

$$u(W_k - u + G(u)) \leq u(P_M - u + G(u)).$$

Since $-G$ is increasing and $u \geq 0$, we have $u - G(u) \geq -G(0)$. Since, by (4.5), $G(0) \geq \bar{\alpha} P_M$, we have,

$$u(W_k - u + G(u)) \leq u(1 + \bar{\alpha}) P_M. \tag{4.8}$$

The first assertion about F is line (1.7) of [19], where H is used instead of F. We have that $(Id - G)$ maps $[-G(0), P_M]$ onto $[0, P_M]$, and $G(0) > 0$. Hence if $u \in [0, P_M]$, we have $F(u) \leq P_M$. Moreover, $F(-G(0)) = 0$. Together with the fact that F is increasing, this implies that $F(u) > 0$ holds for $u > 0$. According to Lemma 4.2, $0 \leq W_\infty \leq P_M$. Thus the second assertion about F holds as well. $\quad\square$

5. Superflow

Now we will establish item (1) of Proposition 1.2. In this section we use V_k and V to denote,

$$V_k = D W_k \text{ and } V = D W_\infty. \tag{5.1}$$

And, we use $f(u)$ and $\bar{a}$ to denote,

$$f(u) = u(\bar{\alpha} P_M - (1 + \bar{\alpha})u), \quad \bar{a} = \bar{\alpha} P_M (1 + \bar{\alpha})^{-1}.$$

According to (4.6) of Lemma 4.3, p_k is a supersolution of

$$\partial_t u - Du \cdot V_k - k f(u) = 0. \tag{5.2}$$

In the following lemma we construct a barrier that we will use in this proof.

Lemma 5.1. *Assume the hypotheses of Theorem 1.1. Let $x_0 \in \mathbb{R}^n$, $r > 0$, $t_0 > 0$ and let ϕ be a smooth function with $\{\phi \geq 0\} \subset B_r(x_0)$. Let $a \in (0, \bar{a})$, $\beta > 0$. There exists $\bar{h} > 0$, that does not depend on β or a, and a subsolution $\Psi^{k, \beta, a}(x, t)$ of (5.2) in $\mathbb{R}^n \times (t_0, t_0 + \bar{h})$ such that*

$$\Psi^{k, \beta, a}(\cdot, t_0) \leq a \chi_{\{\phi \geq \beta\}} \tag{5.3}$$

for all k large. And, if $(x, h) \in B_r(x_0) \times (0, \bar{h})$ is such that

$$\phi(x) + \int\limits_{t_0}^{t_0 + h} V(x, s) \cdot D\phi(x) \, ds - h\alpha > 2\beta \tag{5.4}$$

holds, then

$$\liminf\nolimits_* \Psi^{k, \beta, a}(x, t_0 + h) = a.$$
$$_{k \to \infty}$$

Proof of Lemma 5.1. Let us assume, without loss of generality, $t_0 = 0$.

Let $(V_k)^\varepsilon$ be the regularization of V_k in space defined in Lemma 3.3. We define ψ and ψ^k by:

$$\psi(x, t) = \phi(x) + \int\limits_0^t D\phi(x) \cdot V(x, s) \, ds - t\alpha - 2\beta,$$

$$\psi^k(x, t) = \phi(x) + \int\limits_0^t D\phi(x) \cdot (V_k)^\varepsilon(x, s) \, ds - t\frac{\alpha}{2} - 2\beta.$$

Notice that $\psi^k = \psi^\varepsilon - 2\beta$, where ψ^ε is as defined in Lemma 3.3. In particular, since, according to Lemma 4.2, we have that the V_k satisfy (2.2) and (2.3) uniformly in k, Lemma 3.3 implies that there exists $\varepsilon > 0$ and $\bar{h}$ such that for all k, ψ^ε, and therefore ψ^k, is a viscosity subsolution of

$$\partial_t \psi^k - D\psi^k \cdot V_k \leq 0 \tag{5.5}$$

on $\mathbb{R}^n \times (0, \bar{h})$. Moreover, since each of the V_k is continuous in t, Lemma 2.3 implies that ψ^k satisfies (5.5) in the classical viscosity sense.

We also recall that, according to Lemma 3.3,

$$|D\phi(x) \cdot V_k(x, s) - D\phi(x) \cdot (V_k)^\varepsilon(x, s)| \leq \frac{\alpha}{4}$$

for all x and for all $s > 0$. We use this to estimate the difference in size between ψ and ψ^k:

$$\psi(x, h) - \psi^k(x, h) = \int_0^h (V(x, s) - (V_k)^\varepsilon(x, s)) \cdot D\phi(x)\, ds - \frac{\alpha}{2} h$$

$$\leq \int_0^h (V(x, s) - V_k(x, s)) \cdot D\phi(x)\, ds - \frac{\alpha}{4} h$$

$$\leq ||D\phi||_\infty \int_0^h |V(x, s) - V_k(x, s)|\, ds - \frac{\alpha}{4} h.$$

Since $V_k \to V$ in $L^1((0, T), L^\infty(\mathbb{R}^n))$, we have that the right-hand side of the previous line is non-positive for all k large enough and for all x, s. Thus we find,

$$\psi(x, s) \leq \psi^k(x, s) \tag{5.6}$$

for all x, for all k large enough, and for all $s > 0$.

Next let us take $q : \mathbb{R} \to [0, a]$ to be a smooth non-decreasing function on $\mathbb{R}$ with $q(-1) = 0$ and $q(1) = a$, and define $\Psi^{k,\beta,a}$ by,

$$\Psi^{k,\beta,a}(x, t) = q(k\psi^k(x, t)).$$

Since β and a are fixed throughout this proof, we will no longer write them in the superscript. We remark that $f(\Psi^k) \geq 0$ holds since $0 \leq q(\zeta) < a < \bar{a}$ for all ζ and $f \geq 0$ on $[0, \bar{a}]$. We have,

$$\partial_t \Psi^k - V_k(x, t) \cdot D\Psi^k = \dot{q}k\psi_t^k - \dot{q}kV_k(x, t) \cdot D\psi^k = \dot{q}k[\psi_t^k - V_k(x, t) \cdot D\psi^k] \leq 0,$$

where the inequality holds for $t \in (0, \bar{h})$, and follows from (5.5) and because $\dot{q} \geq 0$. Finally, since $f(\Psi^k) \geq 0$, we find,

$$\partial_t \Psi^k - V_k(x, t) \cdot D\Psi^k \leq kf(\Psi^k),$$

so that Ψ^k is a subsolution to (5.2) on $(0, \bar{h})$.

Let us now check the behavior of the Ψ^k at time 0. The definitions of Ψ^k and ψ^k yield:

$$\Psi^k(x, 0) = q(k(\phi(x) - 2\beta)). \tag{5.7}$$

Let us suppose x is such that $\phi(x) < \beta$. This implies that, for all $k \geq 1/\beta$,

$$\phi(x) \leq 2\beta - k^{-1}.$$

The previous line holds if and only if

$$k(\phi(x) - 2\beta) \leq -1.$$

Applying q, which is non-decreasing, yields,

$$q(k(\phi(x) - 2\beta)) \leq q(-1) = 0.$$

Together with (5.7), this implies that if x is such that $\phi(x) < \beta$, then $\Psi^k(x, 0) = 0$. In addition, $\Psi^k(x, 0) \leq a$ for all x. Thus, we have shown that (5.3) holds.

Now we study the lim inf of the Ψ^k. Since q is non-decreasing for each k, (5.6) implies,

$$q(k\psi(x, t)) \leq q(k\psi^k(x, t)) = \Psi^k(x, t), \tag{5.8}$$

where the equality is simply the definition of Ψ^k.

Now suppose (x, t) is such that (5.4) holds (with $t_0 = 0$ and t instead of h). This says exactly that $\psi(x, t) > 0$, so that $\psi(y, s) > 0$ for all $(y, s) \in B_{\tilde{r}}(x, t)$ for some $\tilde{r} \geq 0$. Therefore there exists K such that $k\psi(y, s) > 1$ for all $(y, s) \in B_{\tilde{r}}(x, t)$ and for $k \geq K$. Applying q, which is non-decreasing, and then using that $q(1) = a$ yields,

$$q(k\psi(y, s)) \geq q(1) = a \text{ for all } (y, s) \in B_{\tilde{r}}(x, t).$$

We now use (5.8) to estimate the left-hand side of the previous line from above and find,

$$\Psi^k(y, s) \geq a \text{ for all } (y, s) \in B_{\tilde{r}}(x, t).$$

Therefore, taking lim inf$_*$ gives lim inf$_{k \to \infty *} \Psi^k(x, t) \geq a$. □

Proof of item (1) of Proposition 1.2. Let $\bar{x} \in \mathbb{R}^n$, $t \in (0, T)$, $r > 0$, $\alpha > 0$ and let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that

$$\{x : \phi(x) \geq 0\} \subset (\Omega_t^1)^{int} \cap B_r(\bar{x}),$$

with $|D\phi| \neq 0$ on $\{x : \phi(x) = 0\}$. Let us use $A_{\beta,h}$ to denote,

$$A_{\beta,h} = \left\{ x \,\middle|\, \phi(x) + \int_t^{t+h} V(x, s) \cdot D\phi(x) \, ds - h\alpha > 2\beta \right\}.$$

Let $\bar{h}$ be as given by Lemma 5.1. We fix some $\beta > 0$ for the remainder of the proof. We will establish that for $h \in (0, \bar{h})$,

$$A_{\beta,h} \subset (\Omega_{t+h}^1)^{int}.$$

Since β is arbitrary, establishing the previous line will complete the proof.

That $\{x : \phi(x) \geq 0\}$ is contained in $(\Omega_t^1)^{int} \cap B_r(\bar{x})$ implies that there exists $a > 0$ with lim inf$_* p_k \geq 2a$ on $\{x : \phi(x) \geq 0\}$, and so, for all k large enough and x such that $\phi(x) \geq 0$, we have $p_k(x, t) \geq a$. Since $p_k \geq 0$ everywhere, we find, for all k large enough, and for all x,

$$p_k(x, t) \geq a\chi_{\{\phi \geq 0\}}(x, t) \geq a\chi_{\{\phi \geq \beta\}}(x, t). \tag{5.9}$$

Let us use $\Psi^k(x, t)$ to denote $\Psi^{k,\beta,a}(x, t)$ as given in Lemma 5.1. According to (5.9) and (5.3),

$$p_k(x, t) \geq \Psi^k(x, t) \text{ for all } x \in \mathbb{R}^n.$$

In addition, Ψ^k is a subsolution of (5.2) on $(t, t + \bar{h})$, and p_k is a supersolution of (5.2). Therefore, we have, for all $h \in (0, \bar{h})$,

$$p_k(x, t + h) \geq \Psi^k(x, t + h) \text{ for all } x \in \mathbb{R}^n.$$

Now let $y \in A_{\beta,h}$. Since $A_{\beta,h}$ is an open set, there exists $\tilde{r}$ such that $B_{\tilde{r}}(y) \subset A_{\beta,h}$. Thus let take $x \in B_{\tilde{r}}(y)$ and take lim inf$_*$ of the previous line and find,

$$\text{lim inf}_* p_k(x, t + h) \geq \text{lim inf}_* \Psi^k(x, t + h).$$

Since $x \in A_{\beta,h}$, Lemma 5.1 implies that the right-hand side of the previous line equals a. Therefore, lim inf$_* p_k(x, t + h) \geq a > 0$, which means $x \in \Omega_{t+h}^1$, and hence $y \in (\Omega_{t+h}^1)^{int}$, as desired. □

6. Subflow

In this section we prove item (2) of Proposition 1.2. As in the previous section, we will employ the notation (5.1). In addition, we will use that, according to (4.7) and (4.1), p_k is a subsolution of,

$$\partial_t u - Du \cdot V_k - ku(1 + \bar{\alpha})P_M = 0. \tag{6.1}$$

We first construct suitable supersolutions to (6.1) in:

Lemma 6.1. *Assume the hypotheses of Theorem 1.1. Let $x_0 \in \mathbb{R}^n$, $t_0 \geq 0$, $r > 0$, and let ϕ be a smooth function with $\{\phi \leq 0\} \subset B_r(x_0)$. There exists $\bar{h} > 0$ and a supersolution $\Psi_k(x, t)$ of (6.1) in $\mathbb{R}^n \times (t_0, t_0 + \bar{h})$ such that*

$$\Psi_k(\cdot, t_0) \geq P_M \chi_{\{\phi > 0\}} \tag{6.2}$$

for all k. And, if $(x, h) \in B_r(x_0) \times (0, \bar{h})$ is such that

$$\phi(y) + \int_{t_0}^{t_0 + h'} V(y, s) \cdot D\phi(y)\, ds + h'\alpha < 0 \text{ holds for } (y, h') \in \bar{B}_{\bar{r}}(x, h) \tag{6.3}$$

for some $\tilde{r} > 0$, then there exists K such that

$$\Psi_k(y, t_0 + h') = 0 \text{ for } (y, h') \in \bar{B}_{\bar{r}}(x, h) \text{ for all } k \geq K.$$

Proof. The proof of this lemma is similar to that of Lemma 5.1, so we omit some details. Without loss of generality we take $t_0 = 0$, and we write t instead of h. We take $\bar{\psi}^k$ as in Lemma 3.3 and $\bar{\psi}$ to be:

$$\bar{\psi}(x, t) = \phi(x) + \int_0^t D\phi(x) \cdot V(x, s)\, ds + t\alpha,$$

$$\bar{\psi}^k(x, t) = \phi(x) + \int_0^t D\phi(x) \cdot (V_k)^{\varepsilon}(x, s)\, ds + t\frac{\alpha}{2}.$$

As in Lemma 5.1, we find that there exists $\varepsilon > 0$ and $\bar{h}$ such that for all k, $\bar{\psi}^k$ is a viscosity solution (in the classical sense) of

$$\partial_t \bar{\psi}^k - D\bar{\psi}^k \cdot V_k \geq 0 \tag{6.4}$$

on $\mathbb{R}^n \times (0, \bar{h})$. And, again similarly to Lemma 5.1, we find

$$\bar{\psi}(x, s) \geq \bar{\psi}^k(x, s) \tag{6.5}$$

for all x and for all $s > 0$.

Next let us take $q : \mathbb{R} \to [0, a]$ to be a smooth non-decreasing function on $\mathbb{R}$ with $q(-1) = 0$ and $q(0) = P_M$, and define Ψ_k by,

$$\Psi_k(x, t) = q(k\bar{\psi}^k(x, t))e^{k(1 + \bar{\alpha}P_M)t}.$$

We have,

$$\begin{aligned} \partial_t \Psi_k - V_k(x, t) \cdot D\Psi_k &= k(1 + \bar{\alpha}P_M)\Psi_k + \dot{q}k\bar{\psi}_t^k e^{k(1 + \bar{\alpha}P_M)t} - \dot{q}kV_k(x, t) \cdot D\bar{\psi}e^{k(1 + \bar{\alpha}P_M)t} \\ &= k(1 + \bar{\alpha}P_M)\Psi_k + \dot{q}k[\bar{\psi}_t^k - V_k(x, t) \cdot D\bar{\psi}]e^{k(1 + \bar{\alpha}P_M)t} \\ &\geq k(1 + \bar{\alpha}P_M)\Psi_k, \end{aligned}$$

where the last line holds for $t \in (0, \bar{h})$, and follows from (6.4) and because $\dot{q} \geq 0$. Thus we find that Ψ_k is a supersolution to (6.1) on $(0, \bar{h})$.

Let us now check the behavior of the Ψ^k at time 0. The definitions of Ψ_k and $\bar{\psi}^k$ yield:

$$\Psi^k(x,0) = q(k\bar{\psi}(x)). \tag{6.6}$$

Let us suppose x is such that $\bar{\psi}(x) > 0$. Applying q, which is non-decreasing, yields,

$$q(k\bar{\psi}(x)) \geq q(0) = P_M.$$

Together with (6.6), this implies that if x is such that $\bar{\psi}(x) > 0$, then $\Psi^k(x,0) = P_M$. In addition, $\Psi^k(x,0) \geq 0$ for all x. Thus, we have shown that (6.2) holds.

Now suppose (x,t) is such that (6.3) holds on $\bar{B}_{\tilde{r}}(x,t)$ for some $\tilde{r}$. This says exactly that $\bar{\psi}(y,s) < 0$ on $\bar{B}_{\tilde{r}}(x,t)$. Since $\bar{\psi}$ is continuous, there exists $\alpha > 0$ so that

$$\bar{\psi}(y,s) < -\alpha \text{ for } (y,s) \in \bar{B}_{\tilde{r}}(x,t),$$

and hence there exists K so that for $k \geq K$,

$$k\bar{\psi}(y,s) \leq -1 \text{ for } (y,s) \in \bar{B}_{\tilde{r}}(x,t).$$

Applying q yields,

$$q(k\bar{\psi}(y,s)) \leq q(-1) = 0,$$

so that, upon multiplying by $e^{k(1+\bar{\alpha}P_M)s}$ we find,

$$\Psi_k(y,s) = q(k\bar{\psi}(y,s))e^{k(1+\bar{\alpha}P_M)s} \leq 0,$$

and therefore $\Psi_k(y,s) = 0$ for $k \geq K$ and $(y,s) \in \bar{B}_{\tilde{r}}(x,t)$. $\square$

We are now ready to present:

Proof of item (2) of Proposition 1.2. The claim of this proposition is that $((\bar{\Omega}_t^2)^c)_t$ is a generalized subflow. Let us recall that $\bar{\Omega}_t^c$ appears in the definition of Ω being a generalized subflow. Taking $\Omega_t = ((\bar{\Omega}_t^2)^c)_t$ yields

$$\bar{\Omega}_t^c = (\Omega_t^2)^{int}.$$

Thus, let us take $\bar{x} \in \mathbb{R}^n$, $t_0 \in (0,T)$, $r > 0$, $\alpha > 0$, and a smooth function ϕ such that

$$\{x : \phi(x) \leq 0\} \subset (\Omega_t^2)^{int} \cap B_r(\bar{x}),$$

and with $|D\phi| \neq 0$ on $\{\phi = 0\}$. Let us use E_h to denote,

$$E_h = \left\{ x \mid \phi(x) + \int_t^{t+h} V(x,s) \cdot D\phi(x)\,ds + h\alpha < 0 \right\}.$$

Let $\bar{h}$ be as given by Lemma 6.1. We will establish $E_h \subset (\Omega_{t+h}^2)^{int}$ for $h \in (0,\bar{h})$.

That $\{x : \phi(x) \leq 0\}$ is contained in $(\Omega_t^2)^{int} \cap B_r(\bar{x})$ implies that there exists K such that $p_k(x,t) = 0$ on $\{x : \phi(x) \leq 0\}$ for all $k \geq K$. Since $p_k \leq P_M$ everywhere (this is exactly equation (4.3) of Lemma 4.2), we find, for all $k \geq K$, and for all x,

$$p_k(x,t) \leq P_M \chi_{\{\phi>0\}}(x,t). \tag{6.7}$$

Let Ψ_k be as given in Lemma 6.1. According to (6.7) and (6.2), we have,

$$p_k(x,t) \leq \Psi_k(x,t) \text{ for all } x \in \mathbb{R}^n.$$

In addition, we have that Ψ^k is a supersolution of (6.1) on $(t, t+\bar{h})$, and p_k is a subsolution of (6.1). Therefore, we have, for all $h \in (0,\bar{h})$,

$$p_k(x,t+h) \leq \Psi^k(x,t+h) \text{ for all } x \in \mathbb{R}^n.$$

Now let $x \in E_h$. Since

$$(x, h) \mapsto \phi(x) + \int\limits_t^{t+h} V(x, s) \cdot D\phi(x) \, ds + h\alpha$$

is continuous, we find there exists $\tilde{r}$ such that $y \in E_{h'}$ for $(y, h') \in \bar{B}_{\tilde{r}}(x, h)$. Thus, according to Lemma 6.1, there exists K such that for all $k \geq K$ and all $(y, h') \in \bar{B}_{\tilde{r}}(x, h)$, we have $\Psi_k(y, t + h') = 0$. The previous line therefore implies

$$p_k(y, t + h') = 0 \text{ for } (y, h') \in \bar{B}_{\tilde{r}}(x, h), k \geq K.$$

We recall $A_{t+h'}^k = \{x \,|\, p_k(x, t + h') > 0\}$, so from the previous line we find,

$$dist(y, A_{t+h'}^k) \geq \tilde{r}/2 \text{ for } (y, h') \in B_{\tilde{r}/2}(x, h), k \geq K,$$

and hence

$$\liminf{}_* dist(y, A_{t+h}^k) > 0 \text{ for } y \in B_{\tilde{r}/4}(x).$$

This means exactly $y \in (\Omega_{t+h}^2)^{int}$. Thus $E_h \subset (\Omega_{t+h}^2)^{int}$, as desired. $\quad\square$

7. Limiting behavior at the initial time

We established that, for $t > 0$, $(\Omega_t^1)^{int}$ is a superflow, and $(\bar{\Omega}_t^2)^c$ is a subflow, with velocity $-DW_\infty$. In this section we will study these sets at the initial time $t = 0$.

Proposition 7.1. *Under the assumptions of Theorem 1.1,*

(1) $\Omega_0^{int} \subset (\Omega_0^1)^{int}$, *and*
(2) $(\bar{\Omega}_0)^c \subset (\Omega_0^2)^{int}$.

7.1. Positive region

We prove the first part of Proposition 7.1. We use f to denote,

$$f(u) = u(\alpha P_M - (1 + \alpha)u).$$

We will construct a barrier from the solution to the ODE described in the following lemma:

Lemma 7.2. *For each $\xi \in \mathbb{R}$ there exists a unique solution $\omega(\xi, t)$ of the ODE,*

$$\partial_t \omega_t = f(\omega) \text{ for } t \in [0, \infty); \quad \omega(0) = \xi,$$

with $\partial_\xi \omega(\xi, s) > 0$ on $\mathbb{R} \times (0, \infty)$, and $\partial_t \omega(\xi, s) \geq 0$ on $(0, \infty) \times (0, \infty)$.

Proof. This ODE has the solution: $\omega(0, t) \equiv 0$, and for $\xi > 0$ we have,

$$\omega(\xi, t) = \frac{\bar{\alpha} P_M}{1 + \bar{\alpha} + (\bar{\alpha} P_M \xi^{-1} - (1 + \bar{\alpha}))e^{-\bar{\alpha} P_M t}}.$$

From this we can explicitly verify that ω has the desired properties. $\quad\square$

Proof of first part of Proposition 7.1. According to (4.6), we have that p_k is a supersolution of,

$$\partial_t u - V_k \cdot Du = k f(u).$$

Let $x_0 \in (\Omega_0)^{int}$, so that there exists $r > 0$ with $B_{2r}(x_0) \subset \Omega_0$. Assumption (1.10) implies that there exists $a > 0$ such that, for k large enough and $x \in B_{2r}(x_0)$, we have $p_k(x, 0) \geq a$. Let ψ be a smooth function with,

$$0 \le \psi \le a \text{ in } \mathbb{R}^n, \ \psi = 0 \text{ on } \mathbb{R}^n \setminus B_{2r}(x_0), \ \psi = a \text{ on } B_r(x_0).$$

We remark that ψ is chosen so that,

$$\psi(x) \le p_k(x, 0) \text{ for all } x \in \mathbb{R}^n \text{ and all } k \text{ large enough.} \tag{7.1}$$

Let $\omega(\xi, t)$ be as in Lemma 7.2. We define w_k by,

$$w_k(x, t) = \omega((\psi(x) - Kt), kt),$$

where we take K to be $K = 1/(M \sup |D\psi|)$. We have,

$$\partial_t w_k - V_k \cdot D w_k - k f(w_k) = -K \partial_\xi \omega + k \partial_t \omega - V_k \cdot (\partial_\xi \omega D\psi) - k f(\omega).$$

The sum of the second and fourth terms is zero, due to the ODE that ω satisfies. Thus we find,

$$\partial_t w_k + V_k \cdot D w_k - k f(w_k) = \partial_\xi \omega (-K - V_k \cdot D\psi).$$

Since $\partial_\xi \omega > 0$, our choice of K implies that the right-hand side of the previous line is non-positive, and thus w_k is a subsolution to the equation for u_k. In addition, at time 0 we have,

$$w_k(x, 0) = \omega(\psi(x), 0) = \psi(x).$$

Together with (7.1), this implies that, for all k large enough, $w_k(x, 0) \le p_k(x, 0)$ for all $x \in \mathbb{R}^n$. The comparison principle thus implies that for all k large enough,

$$w_k(x, t) \le p_k(x, t) \text{ for all } (x, t) \in \mathbb{R}^n \times (0, \infty). \tag{7.2}$$

We will now establish an appropriate bound from below on w_k, and then use (7.2) to deduce the desired estimate on the limit of the p_k. By definition of ψ, we have $\psi(x) = a$ for $x \in B_r(x_0)$. Together with the definition of w_k this implies, for $x \in B_r(x_0)$,

$$w_k(x, t) = \omega((a - Kt), kt).$$

For $t \le a K^{-1}/2$, we have $a - Kt \ge a/2$. Since ω is non-decreasing in ξ, we find that for $x \in B_r(x_0)$ and for $t \le a K^{-1}/2$,

$$\omega((a - Kt), kt) \ge \omega(a/2, kt).$$

In addition, since ω is non-decreasing in t for $\zeta > 0$, we have, for all t,

$$\omega(a/2, kt) \ge \omega(a/2, 0) = a/2,$$

where the equality follows since ω satisfies $\omega(\xi, 0) = \xi$. Putting the three previous lines together thus yields, for $x \in B_r(x_0)$ and for $t \le a K^{-1}/2$,

$$w_k(x, t) \ge a/2.$$

We use this to bound the left-hand side of (7.2) from below and find,

$$a/2 \le p_k(x, t) \text{ for } x \in B_r(x_0) \text{ and for } t \le a K^{-1}/2.$$

Thus we have for $x \in B_{r/2}(x_0)$,

$$\liminf_* p_k(x_0, 0) \ge a/2 > 0,$$

which means $x_0 \in (\Omega_0^1)^{int}$, as desired. $\quad \square$

7.2. Zero region

In this subsection we study the behavior of $\limsup^* p_k$ at time zero and establish the second part of Proposition 7.1. In the following lemma we construct a barrier.

Lemma 7.3. *Let $M > 0$, $r > 0$ and $x_0 \in \mathbb{R}^n$. The equation*

$$\partial_t v - M|Dv| = 0$$

has a solution $\bar{v}(x, t)$ on $B_{2r}(x_0) \times [0, \frac{r}{2M}]$ with $\bar{v}(x, 0) \equiv 0$ on $B_r(x_0)$, $\bar{v}(x, t) = P_M$ on $\partial B_{2r}(x_0) \times \left[0, \frac{r}{2M}\right]$, and $\bar{v}(x, t) \equiv 0$ on $B_{r/2}(x_0) \times \left(0, \frac{r}{2M}\right)$.

Proof of Lemma 7.3. Let $\phi : \mathbb{R} \to [0, P_M]$ be a smooth and non-decreasing function satisfying,

$$\phi(\xi) = 0 \text{ for } \xi \leq r, \ \phi(\xi) = P_M \text{ for } \xi \geq 2r,$$

and define,

$$\bar{v}(x, t) = \phi(|x - x_0| + Mt).$$

We remark that, although $|x - x_0| + Mt$ is not differentiable at $x = x_0$, we still have that $\bar{v}(x, t)$ is differentiable at x_0 so long as $Mt \leq r$, which is the case in the region we consider. The desired properties thus follow in a straight-forward way from the definitions. $\square$

Proof of second part of Proposition 7.1. Let us take $x_0 \in (\bar{\Omega}_0)^c$. Let $r > 0$ be such that $\bar{B}_{2r}(x_0) \subset (\bar{\Omega}_0)^c$. Our assumption (1.10) on $p_k(x, 0)$ implies that there exists K large so that for $k \geq K$ we have

$$p_k(x, 0) = 0 \text{ for all } x \in \bar{B}_{2r}(x_0). \tag{7.3}$$

Let $\bar{v}$ be as in Lemma 7.3, and define,

$$w_k(x, t) = \bar{v}(x, t)e^{(1+\bar{\alpha}P_M)kt}.$$

We check that w_k is a supersolution to the equation for p_k. To this end, we compute:

$$\partial_t w_k = (\partial_t \bar{v} + \bar{v}(x, t)(1 + \bar{\alpha}P_M)k)e^{(1+\bar{\alpha}P_M)kt} = \partial_t \bar{v}e^{(1+\bar{\alpha}P_M)kt} + (1 + \bar{\alpha}P_M)kw_k,$$

and

$$-V_k \cdot Dw_k = -e^{(1+\bar{\alpha}P_M)kt}V_k \cdot D\bar{v} \geq -e^{(1+\bar{\alpha}P_M)kt}M|D\bar{v}|,$$

where the inequality follows from the uniform bound (2.2) on V_k of Lemma 4.2. Putting this together with $\partial_t w_k$ yields

$$\partial_t w_k - V_k \cdot Dw_k \geq (1 + \bar{\alpha}P_M)kw_k + (\partial_t \bar{v} - M|D\bar{v}|)e^{(1+\bar{\alpha}P_M)kt} \geq (1 + \bar{\alpha}P_M)kw_k,$$

where the last inequality follows because of the equation that $\bar{v}$ satisfies. By the estimate (4.7) on the reaction term of the equation for p_k, we have that p_k is a subsolution of,

$$\partial_t u - V_k \cdot Du = (1 + \bar{\alpha}P_M)ku.$$

Using (7.3), (4.3) and Lemma 7.3, it is easy to check that $p_k(x, 0) \leq w_k(x, 0)$ holds on the parabolic boundary of $B_{2r}(x_0) \times (0, \frac{r}{2M})$ (we omit the details). The comparison principle therefore implies,

$$p_k(x, t) \leq w_k(x, t) \text{ on } B_{2r}(x_0) \times \left(0, \frac{r}{2M}\right) \text{ and for } k \geq K.$$

Let us now take $x \in B_{r/2}(x_0)$ and $t \in (0, r/2M)$. According to Lemma 7.3, we have $\bar{v}(x, t) = 0$. Therefore, the definition of w_k says that we have $w_k(x, t) = 0 \cdot e^{(1+\bar{\alpha}P_M)kt} = 0$. The previous line therefore implies

$$p_k(x, t) \leq 0 \text{ on } B_{r/2}(x_0) \times \left(0, \frac{r}{2M}\right) \text{ and for } k \geq K.$$

Therefore,

$$dist(x, A_t^k) \geq r/4M \text{ for } (x, t) \in B_{r/4}(x_0) \times \left(0, \frac{r}{4M}\right) \text{ and for } k \geq K.$$

This implies,

$$\liminf{}_* dist(x, A_0^k) > 0 \text{ on } B_{r/4}(x_0),$$

so in particular $x_0 \in (\Omega^2)_0^{int}$, as desired. $\square$

8. Convergence in the positive region

So far we have shown that the region

$$\Omega_t^1 = \{x \,|\, \liminf {}_* p_k(x, t) > 0\}$$

is a superflow with velocity $-DW_\infty$. Now we will establish:

Proposition 8.1. *Let Q be a compact subset of $\{(x, t)|t > 0, x \in (\Omega^1)_t^{int}\}$. Then p_k converges uniformly on Q to $F(W_\infty(x, t))$, where F is given by (1.8).*

Proof. Let $(x_0, t_0) \in Q$. We will establish,

$$\liminf {}_* p_k(x_0, t_0) \geq (Id - G)^{-1}(W_\infty(x_0, t_0)) \geq \limsup {}^* p_k(x_0, t_0),$$

which implies the desired result.

There exists $r_0 < 1$ with $B_{r_0}(x_0) \subset \Omega_t^1$ for $|t - t_0| \leq r_0$, and $t_0 - r_0 > 0$. Thus there exists $a > 0$ such that, for k large enough,

$$p_k(x, t) \geq a/2 \text{ for } (x, t) \in Q_{r_0}(x_0, t_0), \tag{8.1}$$

where we use $Q_r(x_0, t_0)$ to denote the cylinder, $Q_r(x_0, t_0) = B_r(x_0, t_0) \times [t_0 - r, t_0 + r]$.

Let us fix $\varepsilon > 0$. Due to the uniform convergence of the W_k (see Lemma 4.2), we have that there exists $r \in (0, r_0)$ such that for all k large enough,

$$|W_k(x, t) - W_\infty(x_0, t_0)| \leq \varepsilon \text{ for } (x, t) \in Q_r(x_0, t_0). \tag{8.2}$$

For the remainder of this proof we use β to denote $\beta = W_\infty(x_0, t_0)$. According to Lemma 4.3,

$$F(\beta) \in (0, P_M]. \tag{8.3}$$

Step one. In this step we will establish,

$$\limsup {}^* p_k(x_0, t_0) \leq F(\beta). \tag{8.4}$$

Using (8.2) and (4.1), together with the uniform bound (4.3), yields that p_k is a subsolution of,

$$\partial_t p_k - Dp_k \cdot DW_k \leq k P_M(W_\infty(x_0, t_0) + \varepsilon - p_k + G(p_k)) \tag{8.5}$$

on $Q_r(x_0, t_0)$, where we assume without loss of generality $P_M \geq 1$.

Let $\psi : \mathbb{R}^+ \to \mathbb{R}$ be a smooth non-decreasing function such that,

$$1 \leq \psi \leq P_M, \quad \psi(z) \equiv 1 \text{ for } z \leq r/4, \quad \psi(z) \equiv \frac{P_M}{F(\beta)} \text{ for } z \geq r/2.$$

(According to (8.3), we have $\frac{P_M}{F(\beta)} \in (1, \infty)$, and hence such a ψ exists.) Define

$$g(t) = F(\beta) + P_M e^{-k(t - (t_0 - r/8M))} + \varepsilon,$$

and

$$v(x, t) = g(t)\psi(|x - x_0| + M(t - (t_0 - r/8M))).$$

First we establish that $v \geq p_k$ holds on the parabolic boundary of $Q := B_r(x_0) \times (t_0 - r/8M, t_0 + r/8M)$. To this end, we first take $x \in \partial B_r(x_0)$ and $t \in (t_0 - r/8M, t_0 + r/8M)$. We have,

$$|x - x_0| + M(t - (t_0 - r/8M)) \geq |x - x_0| = r.$$

Since ψ is non-decreasing, we find

$$\psi(|x - x_0| - M(t - (t_0 - r/8M))) \geq \psi(r) = \frac{P_M}{F(\beta)},$$

where the equality follows from our choice of ψ. Thus we have,

$$v(x,t) \geq g(t)\frac{P_M}{F(\beta)} \geq P_M.$$

Next we take $x \in B_r(x_0)$ and look at the initial time $t_0 - r/8M$, to find,

$$v(x, t_0 - r/8M) \geq g(t_0 - r/8M) = (F(\beta) + P_M + \varepsilon) \geq P_M$$

where the first inequality follows since $\psi \geq 1$ everywhere.

Next we will show that v is a supersolution to (8.5). We have,

$$\begin{aligned}
\partial_t v - Dv \cdot DW_k &= g'(t)\psi + Mg(t)\psi' - g(t)(\psi')(x - x_0) \cdot DW_k \\
&= g'(t)\psi + g(t)(\psi')(M - (x - x_0) \cdot DW_k) \\
&= g'(t)\psi + g(t)|\psi'|(M - (x - x_0) \cdot DW_k),
\end{aligned}$$

where ψ and ψ' are evaluated at $(|x - x_0| + M(t - (t_0 - r/8M)))$ throughout, and the last equality follows since ψ is non-decreasing. According to the uniform supremum bound on DW_k given in (2.2) we have, $|(x - x_0) \cdot DW_k| \leq rM \leq M$ (since $r < 1$), so we find,

$$\partial_t v - Dv \cdot DW_k \geq g'(t)\psi.$$

Thus,

$$\begin{aligned}
\partial_t v - Dv \cdot DW_k - kP_M(\beta + \varepsilon - v + G(v)) &\geq g'(t)\psi - kP_M(\beta + \varepsilon - v + G(v)) \\
&= -kP_M e^{-k(t-(t_0-r/8M))}\psi - kP_M(\beta + \varepsilon - (Id - G)(v)).
\end{aligned}$$

We would like to show that the right-hand side of the previous line is non-negative (and thus v is a supersolution of (8.6)). In order to do this, we need to estimate the difference between $(Id - G)(v)$ and $\beta = (Id - G)(F(\beta))$. First, since $(Id - G)$ is increasing and $\psi \geq 1$, we find,

$$(Id - G)(v) \geq (Id - G)(g) = (Id - G)(F(\beta) + P_M e^{-k(t-(t_0-r/8M))} + \varepsilon).$$

Next, we recall that G is C^1 and, moreover, $G' \leq -\alpha$, so that $(Id - G)' \geq (1 + \alpha)$. We use this to estimate the right-hand side of the previous line from below and find,

$$(Id - G)(v) \geq (Id - G)(F(\beta)) + (1 + \alpha)(P_M e^{-k(t-(t_0-r/8M))} + \varepsilon).$$

Recalling that $(Id - G)$ and F are inverses, we find

$$(Id - G)(v) \geq \beta + (1 + \alpha)(P_M e^{-k(t-(t_0-r/8M))} + \varepsilon).$$

Thus we have,

$$\begin{aligned}
\beta + \varepsilon - (Id - G)(v) &\leq (\beta + \varepsilon) - \beta - (1 + \alpha)(P_M e^{-k(t-(t_0-r/8M))} + \varepsilon) \\
&\leq -(1 + \alpha)(P_M e^{-k(t-(t_0-r/8M))}).
\end{aligned}$$

We use this to estimate from below the last term on the right-hand side of the equation for v and find,

$$\begin{aligned}
\partial_t v - Dv \cdot DW_k - kP_M(\beta + \varepsilon - v + G(v)) &\geq \\
&\geq -kP_M e^{-k(t-(t_0-r/8M))})\psi + kP_M(1 + \alpha)(P_M e^{-k(t-(t_0-r/8M))}) \\
&= kP_M e^{-k(t-(t_0-r/8M))}(-\psi + (1 + \alpha)P_M)).
\end{aligned}$$

We use $\psi \leq P_M \leq (1_\alpha)P_M$ to estimate from below the last term on the right-hand side of the previous line and obtain,

$$\partial_t v - Dv \cdot DW_k - kP_M(\beta + \varepsilon - v + G(v)) \geq 0,$$

and hence v is a supersolution of (8.5). Since $v \geq p_k$ on the parabolic boundary of Q, comparison principle implies $v \geq p_k$ on all of Q.

Consider $(x, t) \in Q_{r/16M}(x_0, t_0) \subset Q$. Then $t \in (t_0 - r/16M, t_0 + r/16M)$, and so, since g is decreasing in t,

$$g(t) \leq F(\beta) + P_M e^{-k(t_0-r/16M-(t_0-r/8M))} + \varepsilon \leq F(\beta) + P_M e^{-k(r/16M)} + \varepsilon.$$

In addition,

$$|x - x_0| + M(t - (t_0 - r/8M)) \le r/16M + M(r/16M + r/8M) \le r/4$$

where we use $M \ge 1$. Thus, since ψ is increasing,

$$\psi(|x - x_0| + M(t - (t_0 - r/8M))) \le \psi(r/4) = 1.$$

Therefore, for $(x, t) \in Q_{r/16M}(x_0, t_0)$, we have,

$$p_k(x, t) \le v(x, t) \le F(\beta) + P_M e^{-k(r/16M)} + \varepsilon.$$

Taking $\limsup^*$ thus yields,

$$\limsup{}^* p_k(x, t) \le F(\beta) + \varepsilon$$

for $(x, t) \in Q_{r/16M}(x_0, t_0)$, where r depends on ε. Since $(x_0, t_0) \in Q_{r/16M}(x_0, t_0)$ for any r, we find that

$$\limsup{}^* p_k(x_0, t_0) \le F(\beta) + \varepsilon$$

holds for any $\varepsilon > 0$. Thus we conclude that the desired result (8.4) holds.

Step two. Next we establish,

$$\liminf{}_* p_k(x_0, t_0) \ge F(\beta).$$

The proof is very similar to that of (8.4), and we provide only a sketch. Using (8.2) to bound from below the right-hand side of the equation (4.1) for p_k implies that p_k is a supersolution of,

$$\partial_t u - Du \cdot DW_k = ku(\beta - \varepsilon - u + G(u)).$$

In addition, $F(\beta - \varepsilon)$ is a solution of the equation in the previous line. Thus,

$$\tilde{p}_k := \min\{p_k, F(\beta - \varepsilon)\}$$

is also a supersolution of that equation (here we mean supersolution in the classical viscosity sense, which suffices for the remainder of this proof). According to (8.1), we have $p_k \ge a/2$ on $Q_r(x_0, t_0)$. This implies

$$\limsup{}^* p_k(x_0, t_0) \ge a/2.$$

Thus, the inequality (8.4) that we established in the first step yields $F(\beta) \ge a/2$. Hence, for ε small enough, $F(\beta - \varepsilon) \ge a/4$. Therefore, $\tilde{p}_k \ge a/4$ on $Q_r(x_0, t_0)$ as well. Therefore, $\tilde{p}_k$ is a supersolution of,

$$\partial_t u - Du DW_k = k(W_\infty(x_0, t_0) - \varepsilon - u + G(u))a/4 \tag{8.6}$$

on $Q_r(x_0, t_0)$. We will now construct a certain subsolution to this equation. Let $\phi : \mathbb{R}^+ \to \mathbb{R}$ be a smooth non-increasing function such that,

$$0 \le \phi \le 1, \quad \phi(z) \equiv 1 \text{ for } z \le r/8, \quad \phi(z) \equiv 0 \text{ for } z \ge r/2.$$

Define $h(t) = F(\beta)(1 - e^{-\frac{a}{4}k(t - (t_0 - r/8M))}) - \varepsilon$ and

$$w(x, t) = h(t)\phi(|x - x_0| + M(t - (t_0 - r/8M))).$$

Just as in the previous step, we find $w \le p_k$ on the parabolic boundary of Q, and that w is a subsolution of (8.6). Since $\tilde{p}_k$ is a supersolution of (8.6) on Q, and $w \le p_k$ on the parabolic boundary of Q, the comparison principle implies,

$$\tilde{p}_k \ge w \text{ on } Q.$$

Similarly to the previous step, we find that for $(x, t) \in Q_{r/16M}(x_0, t_0)$,

$$w(x, t) \ge F(\beta)(1 - e^{-\frac{a}{4}k(9r/16M)}) - \varepsilon.$$

The two previous lines, together with the definition of $\tilde{p}_k$ imply,

$$p_k(x, t) \ge F(\beta)(1 - e^{-\frac{a}{4}k(9r/16M)}) - \varepsilon \text{ on } Q_{r/16M}(x_0, t_0).$$

Taking $\liminf_*$ yields,

$$\liminf{}_* p_k(x, t) \geq F(\beta) - \varepsilon \text{ on } Q_{r/16M}(x_0, t_0),$$

where r depends on ε. However, since $(x_0, t_0) \in Q_{r/16M}(x_0, t_0)$ for all r, we find

$$\liminf{}_* p_k(x_0, t_0) \geq F(\beta) - \varepsilon$$

holds for all ε, and the desired result thus follows. $\quad\square$

9. Proof of the main result

This section puts together the results of the previous ones to establish our main result. Throughout this section we will use the auxiliary function θ, which we define to be the unique solution of

$$\begin{cases} \partial_t \theta - D\theta \cdot DW_\infty = 0 \text{ for } (x, t) \in \mathbb{R}^n \times (0, \infty), \\ \theta(x, 0) = d(x, \partial\Omega_0) \text{ for } x \in \mathbb{R}^n, \end{cases} \tag{9.1}$$

where d is the signed distance function, given by,

$$d(x, \partial\Omega_0) = \begin{cases} dist(x, \partial\Omega) \wedge 1 \text{ for } x \in \Omega_0, \\ -dist(x, \partial\Omega) \vee -1 \text{ for } x \notin \Omega_0. \end{cases} \tag{9.2}$$

(According to Theorems 2.5 and 2.4, as well as the assumption that Ω_0 is compact, θ is well-defined.)

9.1. Proof of parts (b) and (c) of the main result

For the sake of presentation, we prove items (b) and (c) of Theorem 1.1 first. We will deduce item (a) in the next subsection.

Proof of parts (b), (c) of Theorem 1.1. We establish:

- If Q is a compact subset of $\{(x, t) | t > 0, \theta(x, t) > 0\}$, then the p_k converge uniformly on Q to $F(W_\infty)$, and the n_k converge uniformly on Q to 1.
- If Q is a compact subset of $\{(x, t) | t > 0, \theta(x, t) < 0\}$, then there exists K large enough such that $p_k \equiv 0$ on Q and $n_k \equiv 0$ on Q for $k \geq K$.

Together, the two bullet points imply that p_k converge locally uniformly to $p_\infty = F(W_\infty)\chi_{\{p_\infty > 0\}}$, and n_k to $\chi_{\{p_\infty > 0\}}$, on $(\mathbb{R}^n \times (0, \infty)) \setminus \{\theta = 0\}$. Moreover, this identifies $\{\theta > 0\}$ with $\{n_\infty > 0\}$ and $\{\theta = 0\}$ with $\partial\{n_\infty > 0\}$.

We establish the first bullet-point. Let us fix $\varepsilon > 0$. We take ϕ^ε to be a smooth, non-decreasing function such that

$$\phi^\varepsilon(u) = \begin{cases} 1 \text{ for } u > 2\varepsilon, \\ -1 \text{ for } u < \varepsilon, \end{cases}$$

and define v_ε by,

$$v_\varepsilon(x, t) = \phi^\varepsilon(\theta(x, t)).$$

Since θ is a viscosity solution of (9.1), and ϕ^ε is non-decreasing, a direct computation implies that $v_\varepsilon(x, t)$ is also a subsolution of (9.1). (Indeed, here we are using that the equation (9.1) is geometric. See, for example, [22, Lemma 1.3] for further discussion of this property in a more general context.)

According to item (1) of Proposition 1.2, if Ω_t^1 is given by $\Omega_t^1 = \{x \mid \liminf{}_* p_k > 0\}$ then $(\Omega_t^1)^{int}$ is a generalized superflow with velocity $-DW_\infty$. Theorem 3.2 thus implies that w given by $w(x, t) = \chi_{(\Omega_t^1)^{int}}(x) - \chi_{\overline{\Omega_t^1}^c}(x)$ is a viscosity supersolution of (9.1).

We now aim to establish,

$$v_\varepsilon(x, 0) \leq w(x, 0).$$

To this end, let us take x such that $v_\varepsilon(x, 0) > -1$ (for any other x we have that the previous line automatically holds, as $w \geq -1$ everywhere). The definition of v_ε implies that x is such that $\theta(x, 0) \geq \varepsilon$. The definition of $\theta(x, 0)$ therefore implies $x \in \Omega_0^{int}$. Applying Proposition 7.1, we find $x \in (\Omega_0^1)^{int}$. Therefore $w(x, 0) = 1 \geq v_\varepsilon(x, 0)$, as desired.

We may now apply the comparison principle of Theorem 2.4 to find, for all t,

$$v_\varepsilon(x, t) \leq w(x, t). \tag{9.3}$$

Let Q be a compact subset of $\{(x, t)|t > 0, \theta(x, t) > 0\}$. Since (according to Theorem 2.5) θ is continuous, there exists $\delta > 0$ such that $\theta(x, t) \geq \delta$ for $(x, t) \in Q$. Because the ϕ^ε are non-decreasing, we find, for $(x, t) \in Q$,

$$v_\varepsilon(x, t) = \phi^\varepsilon(\theta(x, t)) \geq \phi^\varepsilon(\delta).$$

Let us take $\varepsilon = \delta/2$. Then the right-hand side of the previous line equals 1. Thus $v_\varepsilon(x, t) = 1$ for $(x, t) \in Q$. Now we use (9.3) to find $w = 1$ on Q, which implies $Q \subset (\Omega_1^t)^{int}$. Hence Proposition 8.1 implies that the p_k converge uniformly to $(Id - G)^{-1}(W_\infty)$ on Q.

The proof of the first bullet-point is now complete, and the statement in the second one is proved similarly. In particular, we use the definition of Ω_t^2, the fact that it is a generalized subflow (proved in Proposition 1.2), and the second part of Proposition 7.1. We omit the details. □

9.2. Proof of part (a) of the main result

To establish part (a) of the main result we need to investigate the "size" of the zero set of θ. We do this in the following lemma. Only item (1) is used in the proof of Theorem 1.1; we include item (3) in order to provide a better description of the zero set of θ.

Throughout this section we use $|A|$ to denote the Lebesgue measure of $A \subset \mathbb{R}^n$ and, for $t > 0$, $\Gamma_t := \{x|\theta(x, t) = 0\}$.

Lemma 9.1. *Assume the hypotheses of Theorem 1.1.*

(1) Let Q be a compact subset of $\mathbb{R}^n$ and let $\varepsilon > 0$. There exists an open set A_ε such that $|A_\varepsilon| \leq \varepsilon$ and $Q \cap \Gamma_t \subset A_\varepsilon$.
(2) For any $t > 0$ we have $|\Gamma_t| = 0$.
(3) We have,

$$dim_H(\Gamma_t) \leq \exp(Nt)dim_H(\Gamma_0),$$

where dim_H is Hausdorff dimension and N is the constant from assumption (2.3).

Proof. We provide only a sketch. Due to the characterization of θ provided by Theorem 2.6, Γ_t is exactly the image of Γ_0 under the map Φ_t defined in Theorem 2.6. We recall that Φ_t is Holder continuous. Our assumption that Ω_0 is compact implies that locally $\partial\Omega_0$ is a graph of a uniformly continuous function. Together these two facts imply the claim of item (1) by a standard real analysis argument.

Item 1 implies that $|Q \cap \Gamma_t| = 0$ for any compact set Q. Since there exists a countable cover of $\mathbb{R}^n$ by compact sets, we find that $|\Gamma_t| = 0$, as desired.

The definition and basic properties of Hausdorff dimension may be found in [12]. In particular, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is Holder with exponent α, and $E \subset \mathbb{R}^n$, then we have,

$$dim_H(f(E)) \leq \frac{1}{\alpha}dim_H(E).$$

Item (3) follows from this and the fact that Φ_t is Holder continuous with exponent $\exp(-Nt)$. □

Remark 9.2. The estimate in item (3) is fairly weak – in particular, for times t larger than $N^{-1}\ln(n/dim_H(\Gamma_0))$, the estimate says only $dim_H(\Gamma_t) \leq n$, which holds trivially. However, as far as we can tell, this is the best that we can do, because the map Φ_t is only Holder continuous, but not (necessarily) Lipschitz.

Proof of part (a) of Theorem 1.1. Let $Q \subset\subset Q' \subset\subset \mathbb{R}^n$ and let $p > 0$. Let us use $Z_k(x, t)$ to denote,

$$Z_k(x, t) = W_\infty(x, t) - W_k(x, t).$$

We have that W_∞ and W_k satisfy (1.3) and (4.1), respectively. Subtracting (4.1) from (1.3) we find that Z_k satisfies,

$$-\Delta Z_k + Z_k = p_\infty - p_k.$$

Thus, standard estimates for elliptic equations (see, for example, [14, Theorem 9.11]) yield,

$$||Z_k||_{W^{2,p}(Q)} \le C(||p_\infty - p_k||_{L^p(Q')} + ||Z_k||_{L^p(Q')})$$

where the constant C depends on Q, Q', n and p.

Let $\varepsilon > 0$. According to Lemma 4.2 the W_k converge to W_∞ locally uniformly. Thus we have that for k large enough, the last term on the right-hand side of the previous line is bounded from above by $\varepsilon/2C$.

According to Lemma 9.1, there exists an open set A such that $Q' \cap \Gamma_t \subset A$ and $|A|^p \le \varepsilon/8C P_M$. Let us use Q_1, Q_2, and Q_3 to denote,

$$Q_1 = Q' \cap \{\theta > 0\} \cap A^c, \quad Q_2 = Q' \cap \{\theta < 0\} \cap A^c, \quad Q_3 = Q' \cap A.$$

Thus we have,

$$||p_\infty - p_k||_{L^p(Q')} \le \sum_{i=1,2,3} ||p_\infty - p_k||_{L^p(Q_i)}.$$

According to Theorem 1.1, we have that $p_k \to p_\infty$ locally uniformly on compact subsets of either $\{\theta > 0\}$ or $\{\theta < 0\}$. Since Q_1 and Q_2 are compact subsets of $\{\theta > 0\}$ and $\{\theta < 0\}$, respectively, we find that for k large enough and $i = 1, 2$, $||p_\infty - p_k||_{L^p(Q_i)} \le \varepsilon/8C$. In addition, we have $||p_k||_{L^\infty(\mathbb{R}^n)} \le P_M$, so we find,

$$||p_\infty - p_k||_{L^p(Q_3)} \le |A|^p 2|P_M| \le \frac{\varepsilon}{4C}.$$

Putting everything together yields

$$||Z_k||_{W^{2,p}(Q)} \le \varepsilon$$

for k large enough, as desired. $\quad\square$

Conflict of interest statement

There is no conflict of interest.

Acknowledgements

Appendix A

In this appendix we establish the comparison result Theorem 2.4 using a doubling-variables argument. Thus the strategy has two parts: (1) establish that the difference of a subsolution and a supersolution is a subsolution to the appropriate equation, and (2) construct an appropriate family of test functions (or *penalization functions*) for that equation.

Step (1) is encapsulated in the following lemma, which is the analogy of [13, Lemma 8.2]. The two differences are the regularity of H in x and the fact that now u is USC and v is LSC, while in [13, Lemma 8.2] they are both continuous. The proof is almost identical to that of [13, Lemma 8.2], and we use much of the same notation, and thus is omitted.

Lemma A.1. *Suppose V satisfies hypotheses (2.2), (2.3) and (2.4). Let u, v be, respectively, an USC subsolution and a LSC supersolution of (2.1). Let Q be an open subset of $\mathbb{R}^n \times \mathbb{R}^n \times (0, T)$ and let $\bar{H} : \mathbb{R}^n \times \mathbb{R}^n \times (0, T) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by,*

$$\bar{H}(x, y, t, q, q') = -\sigma(|x - y|)|q| - M|q + q'|. \tag{A.1}$$

Then $w(x, y, t)$ defined by

$$w(x, y, t) = u(x, t) - v(y, t)$$

is a viscosity subsolution (in the classical sense) in Q of,

$$\partial_t w + \bar{H}(x, y, D_x w, D_y w) \le 0. \tag{A.2}$$

The remainder of the proof of the theorem involves constructing the appropriate penalization function for the doubling-variables argument. The one we use here is different from those often appearing in viscosity theory due to the fact H is not Lipschitz in space, but only log-Lipschitz. The presentation of the remainder of the proof closely follows Stromberg [21]. In fact, the ideas we use from there can also be seen in the earlier work [8] of Crandall, Ishii and Lions.

For the remainder of this section we take $r \in [0, \infty)$ and $\alpha \in (0, \infty)$. Let us use θ to denote,

$$\theta(r) = \begin{cases} 0 \text{ if } r = 0, \\ \infty \text{ otherwise.} \end{cases}$$

We will define $\theta_\alpha(r)$ that converges to $\theta(r)$ as $\theta \to \infty$ for all r. We first define G_α and θ_α by,

$$G_\alpha(r) = \frac{1}{\alpha} \exp\left(-\int_r^1 \frac{1}{\sigma(s)} \, ds \right) \quad \text{and} \quad \theta_\alpha(r) = G_\alpha((\alpha^2 + r^2)^{1/2}).$$

For future use we compute, for $r \in [0, 1]$,

$$\int_r^1 \frac{1}{\sigma(s)} \, ds = \int_r^1 \frac{1}{Ns|\ln(s)|} \, ds = -\frac{1}{N} \int_{|\ln(r)|}^0 \frac{1}{u} \, du = \frac{1}{N} [\ln(u)]_0^{|\ln(r)|} = \frac{1}{N} \ln(|\ln(r)|),$$

so that

$$G_\alpha(r) = \frac{1}{\alpha} \exp(-\frac{1}{N} \ln(|\ln(r)|)) = \frac{e^{-\frac{1}{N}}}{\alpha} \frac{1}{|\ln r|}.$$

According to the definition of θ_α we have,

$$\theta_\alpha(r) = G_\alpha((\alpha^2 + r^2)^{1/2}) = \frac{e^{-\frac{1}{N}}}{\alpha} \frac{1}{|\ln(\alpha^2 + r^2)^{1/2}|}. \tag{A.3}$$

We summarize some properties of θ_α in the following lemma.

Lemma A.2. *Let θ_α be defined as above. We have,*

(1) θ_α is non-negative, increasing and smooth on $[0, \infty)$ for each $\alpha > 0$,
(2) $D_x \theta_\alpha(|x - y|) = -D_y \theta_\alpha(|x - y|)$,
(3) $|D_x \theta_\alpha(|x - y|)| \le \dfrac{\theta_\alpha(|x - y|)}{\sigma(|x - y|)}$,
(4) $\theta_\alpha(r) \to \theta(r)$ for all r as $\alpha \to 0$,
(5) if $r_\alpha \to r$, then

$$\theta(r) \le \liminf_{\alpha \to 0} \theta_\alpha(r_\alpha). \tag{A.4}$$

Proof. Item (1) holds because G_α is non-negative, increasing and smooth on $[0, \infty)$. Item (2) follows directly from the definition of θ_α and item (1). To establish item (3), we compute,

$$G'_\alpha(r) = \frac{G_\alpha(r)}{\sigma(r)},$$

and so,

$$\theta_\alpha'(r) = G_\alpha'((\alpha^2 + r^2)^{1/2})\frac{r}{(\alpha^2 + r^2)^{1/2}} = \frac{G_\alpha((\alpha^2 + r^2)^{1/2})}{\sigma((\alpha^2 + r^2)^{1/2})}\frac{r}{(\alpha^2 + r^2)^{1/2}}.$$

Taking absolute value yields,

$$|\theta_\alpha'(r)| \leq \frac{G_\alpha((\alpha^2 + r^2)^{1/2})}{\sigma((\alpha^2 + r^2)^{1/2})} = \frac{\theta_\alpha(r)}{\sigma((\alpha^2 + r^2)^{1/2})},$$

where the equality follows from the definition of θ. Since σ is increasing, we have $\sigma((\alpha^2 + r^2)^{1/2}) \geq \sigma(r)$, and hence we find,

$$|\theta_\alpha'(r)| \leq \frac{\theta_\alpha(r)}{\sigma(r)}. \tag{A.5}$$

Now let us consider $\theta_\alpha(|x - y|)$ for $x, y \in \mathbb{R}^n$. We have,

$$D_x \theta_\alpha(|x - y|) = \theta_\alpha'(|x - y|)\frac{(x - y)}{|x - y|},$$

so taking absolute value and then applying (A.5) with $r = |x - y|$ yields item (3).

To establish item (4), we take $r = 0$ in (A.3) and then take the limit $\alpha \to \infty$, to find,

$$\lim_{\alpha \to 0} \theta_\alpha(0) = \lim_{\alpha \to 0} \frac{1}{\alpha}\frac{1}{|\ln \alpha|} = 0.$$

If $r \neq 0$, then, again using (4), we find,

$$\lim_{\alpha \to 0} \theta_\alpha(r) = \lim_{\alpha \to 0} \frac{e^{-\frac{1}{N}}}{\alpha}\frac{1}{|\ln(\alpha^2 + r^2)^{1/2}|} = \infty.$$

This means that θ_α converges pointwise to θ, as desired.

To establish item (5), we consider the cases $r = 0$ and $r \neq 0$. First, let us suppose $r = 0$. Then (A.4) holds because its left-hand side is zero, and we already know that the θ_α are non-negative. Now let us suppose $r \neq 0$. Then for all α small enough we have $r_\alpha \geq r/2$. Since θ_α is increasing for all α we find,

$$\theta_\alpha(r_\alpha) \geq \theta_\alpha(r/2).$$

Taking $\liminf$ yields,

$$\liminf_{\alpha \to 0} \theta_\alpha(r_\alpha) \geq \liminf_{\alpha \to 0} \theta_\alpha(r/2) = \infty = \theta(r),$$

where the equalities follow from the first part of the lemma. $\quad\square$

We abuse notation and use $\theta(x, y, t)$ as well as $\theta(x, y)$ to mean $\theta(|x - y|)$, and similarly for θ_α.

Lemma A.3. *Let Q be a compact subset of $\mathbb{R}^n$ and let W be upper-semicontinuous and bounded on $Q \times Q \times [0, T]$. Let Ψ_α denote,*

$$\Psi_\alpha(x, y, t) = W(x, y, t) - \theta_\alpha(|x - y|).$$

Let $(x_\alpha, y_\alpha, t_\alpha) \in Q \times Q \times [0, T]$ be where the maximum of Ψ_α is achieved on $Q \times Q \times [0, T]$. Suppose that, along a subsequence, the $(x_\alpha, y_\alpha, t_\alpha)$ converge to some $(\bar{x}, \bar{y}, \bar{t})$ as $\alpha \to 0$. Then:

$$\bar{x} = \bar{y}, \tag{A.6}$$

$$\max_{x \in Q, t \in [0,T]} W(x, x, t) = W(\bar{x}, \bar{x}, \bar{t}), \text{ and} \tag{A.7}$$

$$\lim_{\alpha \to 0} \theta_\alpha(|x_\alpha - y_\alpha|) = 0. \tag{A.8}$$

This lemma is very similar to ones appearing in standard viscosity theory (see for example, [9, Proposition 3.7] or [7, Lemma 4.1]). However, we cannot directly apply any of those results because our penalization function θ_α is not of the form $\alpha\Psi(x, y)$. Nevertheless the proof of this lemma is so similar to those in the literature that we omit it. We remark that items (5) and (A.4) of Lemma A.2 are used in its proof.

Proof of Theorem 2.4. Step 1. In this step we reduce the situation from all of $\mathbb{R}^n$ to compact subsets S_β, which we define for $\beta > 0$ as,

$$S_\beta = \{(x, t) \in \mathbb{R}^n \times [0, T] : f(x, t) \le \beta\}, \text{ for } f(x, t) = t + \frac{1}{M}(1 + |x|^2)^{1/2}.$$

(Note $Df = \frac{1}{M}\frac{x}{(1+|x|^2)^{1/2}}$, so that $|Df| < 1/M$.) We define the functions g and ψ_β by,

$$g(\beta) = \ln(2 + \max_{S_\beta}(u - v)^+) \text{ and } \psi_\beta(x, t) = \exp(g(\beta)(1 + f(x, t) - \beta)).$$

We have that g is positive and non-decreasing and $\lim_{\beta\to\infty}\psi_\beta(x, t) = 0$ holds for all $(x, t) \in \mathbb{R}^n \times [0, T]$. We also find,

$$\partial_t\psi_\beta - M|D\psi_\beta| = \psi_\beta g(\beta)(\partial_t f - M|Df|) > \psi_\beta g(\beta)(1 - M\frac{1}{M}) = 0. \tag{A.9}$$

Step 2. We seek to establish that for all β,

$$(u - v - \psi_\beta)(x, t) \le \sup_{\mathbb{R}^n}(u - v)^+(x, 0) \text{ for } (x, t) \in S_\beta.$$

If this holds, then we may take the limit $\beta \to 0$ of the previous line, use that $\lim_{\beta\to\infty}\psi_\beta(x, t) = 0$, and find that the desired inequality holds, completing the proof.

We proceed to establish the previous line by contradiction. Thus, assume that there exists $\beta > 0$ such that the previous line fails. This implies that there exists $c > 0$ with,

$$\sup_{(x,t)\in S_\beta}(u(x, t) - v(x, t) - \psi_\beta(x, t) - ct) > \sup_{\mathbb{R}^n}(u - v)^+(x, 0).$$

We now stop writing the subscript β, as it is fixed. We use $W(x, y, t)$ to denote,

$$W(x, y, t) = u(x, t) - v(y, t) - \psi_\beta(y, t) - ct,$$

so that the previous line becomes,

$$\sup_{(x,t)\in S}W(x, x, t) > \sup_{\mathbb{R}^n}(u - v)^+(x, 0). \tag{A.10}$$

Let us consider,

$$\Phi_\alpha(x, y, t) = u(x, t) - v(y, t) - \psi_\beta(y, t) - ct - \theta_\alpha(|x - y|),$$

where θ_α is as defined in the previous subsection. Let $(x_\alpha, y_\alpha, t_\alpha)$ be a maximum of Φ_α on $S \times S \times [0, T]$. Notice that this exists as S is compact and Φ_α is upper-semicontinuous.

Step 3. In this step, we establish, for all α small enough:

- $t_\alpha > 0$, and,
- (x_α, t_α) and (y_α, t_α) are contained in the interior of S.

Let us suppose that the first statement is not true. That means that there exists a subsequence $(x_{\alpha_j}, y_{\alpha_j}, t_{\alpha_j})$ converging to $(\bar{x}, \bar{y}, 0)$. We now apply Lemma A.3. We find that $\bar{x} = \bar{y}$ and,

$$\max_{x\in S, t\in[0,T]}W(x, x, t) = W(\bar{x}, \bar{x}, 0) = u(\bar{x}, 0) - v(\bar{x}, 0) - \psi(\bar{x}, 0) \le u(\bar{x}, 0) - v(\bar{x}, 0),$$

where the second equality follows from the definition of W. We now use (A.10) to bound the left-hand side of the previous line from below and find,

$$\sup_{\mathbb{R}^n}(u - v)^+(x, 0) < u(\bar{x}, 0) - v(\bar{x}, 0),$$

which is impossible. Therefore the first item must hold.

Now let us suppose the second item does not hold. This means that there exists a subsequence $(x_{\alpha_j}, y_{\alpha_j}, t_{\alpha_j})$ converging to $(\bar{x}, \bar{x}, \bar{t})$ with $(\bar{x}, \bar{t}) \in \partial S$ (the x_{α_j} and y_{α_j} converge to the same point due to Lemma A.3). The definition of S as the sub level set of f implies, $f(\bar{x}, \bar{t}) = \beta$. The definitions of ψ and g therefore imply,

$$\psi(\bar{x}, \bar{t}) = \exp(g(\beta)) = 2 + \max_S(u - v)^+.$$

Using this in the definition of W yields,

$$W(\bar{x}, \bar{x}, \bar{t}) = u(\bar{x}, \bar{t}) - v(\bar{x}, \bar{t}) - \psi(\bar{x}, \bar{t}) - c\bar{t} = u(\bar{x}, \bar{t}) - v(\bar{x}, \bar{t}) - (2 + \max_S(u - v)^+) - c\bar{t} < 0.$$

We again apply Lemma A.3 and find,

$$\max_{x \in S, t \in [0, T]} W(x, x, t) = W(\bar{x}, \bar{x}, \bar{t}).$$

We had just shown that the right-hand side of the previous line is negative, which contradicts (A.10), as the right hand side of (A.10) is non-negative.

Step 4. Let us recall that, according to Lemma A.1, we have that $w(x, y, t) = u(x, t) - v(y, t)$ is a subsolution (in the standard viscosity sense) to (A.2) with $\bar{H}$ given by (A.1). The claim we established in step 3, together with the fact that $(x_\alpha, y_\alpha, t_\alpha)$ is a maximum of Φ_α on $S \times S \times [0, T]$, therefore yields,

$$\partial_t \psi(y_\alpha, t_\alpha) + c + \bar{H}(x_\alpha, y_\alpha, t_\alpha, D_x \theta_\alpha(|x_\alpha - y_\alpha|), D_y \psi(y_\alpha, t_\alpha) + D_y \theta_\alpha(|x_\alpha - y_\alpha|)) \leq 0.$$

The definition of $\bar{H}$ thus gives,

$$\partial_t \psi(y_\alpha, t_\alpha) + c - \sigma(|x_\alpha - y_\alpha|)|D_x \theta_\alpha(|x_\alpha - y_\alpha|)| - M|D_y \psi(y_\alpha, t_\alpha)| \leq 0,$$

where we have also used that $D_x \theta_\alpha(|x_\alpha - y_\alpha|) = -D_y \theta_\alpha(|x_\alpha - y_\alpha|)$. According to (A.9), the sum of the first and last terms is non-negative, so we find,

$$c - \sigma(|x_\alpha - y_\alpha|)|D_x \theta_\alpha(|x_\alpha - y_\alpha|)| \leq 0.$$

We use item (3) of Lemma A.2 to estimate the left-hand side from below and find,

$$c - \sigma(|x_\alpha - y_\alpha|) \frac{\theta_\alpha(|x_\alpha - y_\alpha|)}{\sigma(|x_\alpha - y_\alpha|)} \leq 0,$$

which becomes,

$$c \leq \theta_\alpha(|x_\alpha - y_\alpha|).$$

However, equation (A.8) of Lemma A.3 says that the limit as α goes to zero of the right hand side of the previous line is zero. This yields the desired contradiction, completing the proof. $\square$

Appendix B

In this section we establish Theorem 2.5, Theorem 2.6, and Theorem 2.7. Classical results (see, for example, [10, Chapter 1]) assert that if $V \in C^1$ satisfies (2.2) and (2.3), then there exists a unique flow X generated by V satisfying (2.6). Moreover, if $u_0 \in C^1(\mathbb{R}^n)$, then u defined on $\mathbb{R}^n \times (0, T)$ by,

$$u(x, t) = u_0(X(0, t, x)) \tag{B.1}$$

is a classical solution of,

$$\begin{cases} \partial_t u + V \cdot Du = 0 & \text{in } \mathbb{R}^n \times (0, T), \\ u(x, 0) = u_0(x). \end{cases} \tag{B.2}$$

We now establish some regularity results for u, which depend only on the constants in (2.2) and (2.3). For this we need two lemmas about the regularity of the flow X. We phrase them in terms of its trajectories, $s \mapsto X(s, t, x)$, which we denote by $\gamma(s)$.

Lemma B.1. *Let V be continuous and satisfy (2.2) and (2.3). Let $t_0 \leq t_1, t_2 \leq T$, $x_1, x_2 \in \mathbb{R}^n$ with $|x_1 - x_2|$, $|t_1 - t_2| \leq 1$, and for $i = 1, 2$, let γ_i solve the ODEs,*

$$\begin{cases} \dot{\gamma}_i(s) = V(\gamma_i(s), s) \text{ on } (t_0, T), \\ \gamma_i(t_i) = x_i. \end{cases}$$

Then for $s \in [t_0, T]$,

$$|\gamma_1(s) - \gamma_2(s)| \leq |x_1 - x_2|^{\exp(-N(T-t_0))} + (M|t_1 - t_2|)^{\exp(-N(T-t_0))}.$$

Proof. We only present the case $t_1 = t_2$ for simplicity. Note that

$$\gamma_1(s) - \gamma_2(s) = x_1 - x_2 + \int_{t_1}^{s} V(\gamma_1(r), r) - V(\gamma_2(r), r) \, dr.$$

Taking absolute value and using (2.3) yields, for all s,

$$|\gamma_1(s) - \gamma_2(s)| \leq |x_1 - x_2| + \int_{t_1}^{s} \sigma(|\gamma_1(r) - \gamma_2(r)|) \, dr. \tag{B.3}$$

Let us use $R(s)$ to denote the quantity on the right-hand side of the previous line. Using a generalized Gronwall's inequality and the explicit expression $\sigma(r) = Nr|\ln(r)|$ we may compute,

$$R(s) \leq |x_1 - x_2|^{\exp(-N(s-t_1))} \leq |x_1 - x_2|^{\exp(-N(T-t_0))}.$$

Together with (B.3), this implies that the desired claim holds. □

We write the bounds in the previous lemma so explicitly to highlight the fact that they depend only on the constants M, N, and not on the C^1 norm of V. The following proposition is a direct consequence of the previous lemma.

Proposition B.2. *Assume $V \in C^1$ satisfies (2.2) and (2.3), let X be the flow associated with V and let u be given by (B.1). Assume that there exists a modulus of continuity ω with,*

$$|u_0(x) - u_0(y)| \leq \omega(|x - y|).$$

We have $\|u\|_{L^\infty} \leq \|u_0\|_{L^\infty}$ and u is uniformly continuous, with modulus that depends only on ω, T, N and M.

We now present the proofs of Theorem 2.5 and Theorem 2.6.

Proof of Theorem 2.5 and Theorem 2.6. Let $\rho^\varepsilon(x, t)$ and $\tilde{\rho}^\varepsilon(x)$ be standard mollifiers with integral 1. We regularize V by convolution in both x and t:

$$V^\varepsilon(x, t) = \int_{\mathbb{R}^n \times \mathbb{R}} V(x - y, t - s)\rho^\varepsilon(y, s) \, dy \, ds,$$

where $V(x, r)$ has been extended to be zero for $r \notin (0, T)$, as in [13, Lemma 8.1]. Since V satisfies (2.2) and (2.3), V^ε does as well. We also have $V^\varepsilon \in C^1$, and hence the classical ODE results summarized above yield the existence of X^ε satisfying,

$$X^\varepsilon(s, t, x) = x + \int_{t}^{s} V^\varepsilon(X^\varepsilon(r, t, x), r) \, dr. \tag{B.4}$$

We also regularize the initial data: we define,

$$u_0^\varepsilon(x) = \int_{\mathbb{R}^n} u_0(x - y)\tilde{\rho}^\varepsilon(y) \, dy,$$

so that u_0^ε is C^1 and uniformly continuous with modulus independent of ε.

Let $u^\varepsilon(x, t) = u_0^\varepsilon(X^\varepsilon(0, t, x))$, so that according to the classical results described above, u^ε satisfies,

$$\begin{cases} \partial_t u^\varepsilon + V^\varepsilon \cdot Du^\varepsilon = 0 \text{ in } \mathbb{R}^n \times (0, T), \\ u^\varepsilon(x, 0) = u_0^\varepsilon(x). \end{cases}$$

According to Proposition B.2, the u^ε are uniformly continuous in x and t with modulus independent of ε and $||u_\varepsilon||_{L^\infty} \leq ||u_0||_{L^\infty} < \infty$. By the Arzela–Ascoli Theorem there exists a subsequence which converges locally uniformly to some u.

Let Q be a compact subset of $\mathbb{R}^n \times (0, T)$. The arguments of [13, Lemma 8.1], together with an estimate similar to (3.2), imply that the V^ε converge to V in $L^1((0, T), C(Q))$ (indeed, Lemma 8.1 of [13] concerns regularizing via convolution in time, while (3.2) is an estimate regarding convolution in space). Thus, by the stability result [13, Proposition 7.1], we conclude that u is a viscosity solution of (2.1). The stated regularity of u is a consequence of the regularity of the u^ε given in Proposition B.2.

We have thus completed the proof of Theorem 2.5, and continue with this setup to establish Theorem 2.6. According to Lemma B.1, for $s, t \in (0, T)$, the map $(s, t, x) \mapsto X^\varepsilon(s, t, x)$ is Holder continuous, uniformly in ε. The L^∞ bound (2.2) on V implies $|X^\varepsilon(s, t, x)| \leq |x| + TM$ for $s, t \in (0, T)$. Thus the Arzela–Ascoli theorem implies that there exists a subsequence of $\varepsilon_j \to 0$ as $j \to \infty$ and an X such that X^{ε_j} converge to X locally uniformly on $(0, T) \times (0, T) \times \mathbb{R}^n$. In addition, X is therefore also Holder continuous.

Since V^{ε_j} converge to V in $L^1((0, T), C(Q))$ for any compact set Q and satisfy (2.3) uniformly in ε_j, we find that for any $s, t \in (0, T)$,

$$\lim_{j \to \infty} \int_t^s V^{\varepsilon_j}(X^{\varepsilon_j}(r, t, x), r)\, dr = \int_t^s V(X(r, t, x), r)\, dr.$$

Thus taking a (pointwise) limit of (B.4) along the subsequence ε_j implies that X satisfies (2.6).

Since X^{ε_j} converge to X locally uniformly, we find $u_0^\varepsilon(X^\varepsilon(0, t, x))$ converges to $u_0(X(0, t, x))$. Thus,

$$u(x, t) = \lim_{j \to \infty} u_0^{\varepsilon_j}(X^{\varepsilon_j}(0, t, x)) = u_0(X(0, t, x)),$$

as desired. The regularity of the map Φ follows from the regularity of X. $\square$

We now have the ingredients needed to establish Theorem 2.7.

Proof. Let us first approximate the initial data u_0 from below by $u_0^{-,k}$ and from above $u_0^{+,k}$, where $u_0^{\pm,k}$ are continuous functions from $\mathbb{R}^n$ to $[0, 1]$. We construct these functions such that $u_0^{\pm,k} = 1$ in $\Omega_0^{-,k} := \{x : d(x, \Omega_0^C) \geq 1/k\}$ and $u_0^{\pm,k} = 0$ outside of $\Omega_0^{+,k} := \{x : d(x, \Omega_0) \leq 1/k\}$.

Next, let $u^{-,k}$ and $u^{+,k}$ be the corresponding unique viscosity solutions of (2.1) with aforementioned initial data $u_0^{-,k}$ and $u_0^{+,k}$. Due to Theorem 2.7 we have representation formulas for these solutions. In particular we have

$$u^{\pm,k} = 1 \text{ in } \Omega_t^{-,k} := \{x : X(0, t, x) \in \Omega_0^{-,k}\}$$

and

$$u^{\pm,k} = 0 \text{ in } \Omega_t^{+,k} := \{x : X(0, t, x) \in \Omega_0^{+,k}\}.$$

The properties of the flow map X summarized in Theorem 2.6, including invertibility and continuity with respect to x and t, imply,

$$\limsup_{k \to \infty} {}^* u^{-,k}(\cdot, t) = \limsup_{k \to \infty} {}^* u^{+,k}(\cdot, t) = \chi_{\bar{\Omega}_t} \tag{B.5}$$

and

$$\liminf_{k \to \infty} {}_* u^{-,k}(\cdot, t) = \liminf_{k \to \infty} {}_* u^{+,k}(\cdot, t) = \chi_{\Omega_t}. \tag{B.6}$$

Due to the standard stability property of viscosity solutions, $\chi_{\bar{\Omega}_t} = u^*$ is a subsolution of (1.6), and $u^* = (u_0)^*$ due to the continuity of $X(0, t, x)$ with respect to t. Similarly we have u is a supersolution of (1.6) with $u_* = (u_0)_*$. So we have at least one solution of (1.6) with initial data u_0.

Now if there is any other solution w of (1.6) with initial data u_0, then by comparison principle $w^* \leq u^{+,k}$ and $w_* \geq u^{-,k}$. Taking lim sup and lim inf, respectively, of these inequalities, and then using (B.5) and (B.6) yields,

$$w^* \leq \chi_{\bar{\Omega}_t} = u^* \quad \text{and} \quad w_* \geq \chi_{\Omega_t} = u.$$

Putting the previous two statements of the previous line together yields the desired result:

$$u \leq w_* \leq w \leq w^* \leq u^*. \quad \square$$

Appendix C

Proof of Lemma 3.3. We fix $\alpha > 0$. Throughout, we us use C_1 to denote,

$$C_1 = ||\phi||_{C^3(\mathbb{R}^n)}.$$

Step one. We first establish (3.2). The definitions of H^ε and H, and that V satisfies (2.3) yield,

$$|H^\varepsilon(x, t, q) - H(x, t, q)| \leq \int \rho_\varepsilon(y) |q| \sigma(y) \, dy = |q| \int \frac{1}{\varepsilon^n} \rho\left(\frac{y}{\varepsilon}\right) \sigma(y) \, dy = |q| \int \rho(z) \sigma(\varepsilon z) \, dz.$$

The right-hand side of the previous line is a continuous decreasing function whose limit as $\varepsilon \to 0$ is 0. Thus, there exists $\varepsilon_1 > 0$ such that for $\varepsilon \leq \varepsilon_1$,

$$C_1 \int \rho(z) \sigma(\varepsilon z) \, dz \leq \frac{\alpha}{4}.$$

Combining the previous two lines yields (3.2). Similar computations yield,

$$|D_x H^\varepsilon(x, t, q)| \leq \varepsilon^{-1} M |q| C, \tag{C.1}$$

where $C = \int D\rho(z) \, dz$. Now that we've fixed ε, we take $\bar{h}$ to be,

$$\bar{h} = \frac{\alpha}{16M(MC_1 + MCC_1\varepsilon^{-1})}. \tag{C.2}$$

Step two. Next we establish that ψ_ε is a subsolution of (2.1) on $\mathbb{R}^n \times (t, t + \bar{h})$ (the proof that $\bar{\psi}_\varepsilon$ is a supersolution is analogous). To this end, we suppose $\gamma \in C^1(\mathbb{R}^n \times [0, T])$, $(x_0, r_0) \in \mathbb{R}^n \times (t, t + \bar{h})$, $G, b \in H^-(x_0, r_0, D\gamma(x_0, r_0))$, and

$$(x, r) \mapsto \psi_\varepsilon(x, r) + \int_0^r b(s) \, ds - \gamma(x, r) \tag{C.3}$$

has a local maximum at (x_0, r_0). We want to show,

$$\gamma_r(x_0, r_0) + G(x_0, r_0, D\gamma(x_0, r_0)) \leq 0.$$

We proceed by contradiction and suppose that the previous line does not hold. Thus there exists some $a > 0$ such that,

$$2a \leq \gamma_r(x_0, r_0) + G(x_0, r_0, D\gamma(x_0, r_0)).$$

The continuity of γ, its derivatives, and G implies,

$$a \leq \gamma_r(x_0, r) + G(x_0, r, D\gamma(x_0, r)),$$

for r near r_0, for instance on (r_1, r_0) for some $r_1 < r_0$. Integrating the previous line thus yields,

$$a(r_0 - r_1) \leq \gamma(x_0, r_0) - \gamma(x_0, r_1) + \int_{r_1}^{r_0} G(x_0, r, D\gamma(x_0, r)) \, dr.$$

Since (x_0, r_0) is a local maximum of the function given in (C.3), we find,

$$\gamma(x_0, r_0) - \gamma(x_0, r_1) \leq \psi_\varepsilon(x_0, r_0) - \psi_\varepsilon(x_0, r_1) + \int_{r_1}^{r_0} b(s)\, ds.$$

The two previous lines thus imply,

$$a(r_0 - r_1) \leq \psi_\varepsilon(x_0, r_0) - \psi_\varepsilon(x_0, r_1) + \int_{r_1}^{r_0} b(r)\, dr + \int_{r_1}^{r_0} G(x_0, r, D\gamma(x_0, r))\, dr.$$

The definition of ψ_ε yields,

$$a(r_0 - r_1) \leq -\int_{r_1}^{r_0} H^\varepsilon(x_0, r, D\phi(x_0))\, dr - (r_0 - r_1)\alpha/2 + \int_{r_1}^{r_0} b(r)\, dr + \int_{r_1}^{r_0} G(x_0, r, D\gamma(x_0, r))\, dr.$$

Next, we use the estimate (3.2) to bound the first term on the right-hand side of the previous line from above, and find,

$$a(r_0 - r_1) \leq -\int_{r_1}^{r_0} H(x_0, r, D\phi(x_0))\, dr - (r_0 - r_1)\alpha/4 + \int_{r_1}^{r_0} b(r)\, dr + \int_{r_1}^{r_0} G(x_0, r, D\gamma(x_0, r))\, dr. \tag{C.4}$$

We now want to estimate the difference between $D\phi$ and $D\gamma$ in order to compare the first and last terms on the right-hand side of the previous line. To this end, we first note that, since (x_0, r_0) is a local maximum of the differentiable function (C.3),

$$D_x\gamma(x_0, r_0) = D_x\psi_\varepsilon(x_0, r_0)$$
$$= D\phi(x_0) - \int_t^{r_0} \partial_p H^\varepsilon(x_0, s, D\phi(x_0)) D^2\phi(x_0) + \partial_x H^\varepsilon(x_0, s, D\phi(x_0))\, ds.$$

Rearranging the previous line and taking absolute value yields,

$$|D_x\gamma(x_0, r_0) - D\phi(x_0)| \leq \int_t^{r_0} |\partial_p H^\varepsilon(x_0, s, D\phi(x_0)) D^2\phi(x_0) + \partial_x H^\varepsilon(x_0, s, D\phi(x_0))|\, ds.$$

The definition of H (2.5), together with the assumed bound (2.2) on V yields

$$|\partial_p H^\varepsilon(x_0, s, D\phi(x_0))| = |V(x_0, s)| \leq M.$$

Together with our estimate (C.1), as well as the estimate $||\phi||_{C^3(\mathbb{R}^n)} \leq C_1$, this yields,

$$|D_x\gamma(x_0, r_0) - D\phi(x_0)| \leq \int_t^{r_0} MC_1 + MCC_1\varepsilon^{-1}\, ds = (r_0 - t)(MC_1 + MCC_1\varepsilon^{-1}).$$

Since $r_0 \leq t + \bar{h}$ we find that the right-hand side of the previous line is bounded from above by $\bar{h}(MC_1 + MCC_1\varepsilon^{-1})$. Since $D\gamma$ is continuous in r, we have, for r near r_0,

$$|D_x\gamma(x_0, r) - D\phi(x_0)| \leq 2\bar{h}(MC_1 + MCC_1\varepsilon^{-1}) \leq \frac{\alpha}{8M},$$

where the second inequality follows by our choice of $\bar{h}$ in (C.2), and since $h \leq \bar{h}$. Thus we find,

$$|H(x, r, D_x\gamma(x_0, r)) - H(x, r, D\phi(x_0))| \leq M|D_x\gamma(x_0, r) - D\phi(x_0)| \leq \frac{\alpha}{8}.$$

We are now ready to complete the proof of the lemma. We may assume that r_1 is close enough to r_0 so that the previous line holds. Thus, the previous line together with (C.4) imply,

$$a(r_0 - r_1) \leq - \int\limits_{r_1}^{r_0} H(x_0, r, D\gamma(x_0, r)) \, dr - (r_0 - r_1)\alpha/8 + \int\limits_{r_1}^{r_0} b(r) \, dr + \int\limits_{r_1}^{r_0} G(x_0, r, D\gamma(x_0, r)) \, dr.$$

The fact that we took $G, b \in H^-(x_0, r_0, D\gamma(x_0, r_0))$ implies that, for r_1 close enough to r_0, the sum of the first, third, and fourth terms on the right-hand side of the previous line is non-positive. Thus we find,

$$a(r_0 - r_1) \leq -(r_0 - r_1)\alpha/8,$$

which yields the desired contradiction (since $r_0 > r_1$) and completes the proof of the lemma. $\quad\square$

Proof of Theorem 3.2 item (1). We provide the proof of item (1); the proof of the other item is analogous. We use W to denote,

$$W(x, t) = (\chi_\Omega(x, t) - \chi_{\bar\Omega^c}(x, t))_*.$$

We establish that if W is a supersolution then Ω_t is a superflow. The proof of the other direction is very similar to that of Theorem 2.4 of [2], and we omit it.

Let us suppose W is a supersolution of (2.1) on $\mathbb{R}^n \times [0, T]$. We aim to show that $(\Omega_t^{int})_{t \in [0,T]}$ is a generalized superflow with velocity $-H$. To this end, let us take $\bar{x} \in \mathbb{R}^n$, $t \in (0, T)$, $r > 0$, $\alpha > 0$ and a smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that

$$\{x : \phi(x) \geq 0\} \subset \Omega_t^{int} \cap B_r(\bar{x}), \tag{C.5}$$

with $|D\phi| \neq 0$ on $\{x : \phi(x) = 0\}$. By modifying ϕ outside $B_{2r(x_0)}$, we may assume, without loss of generality, $||\phi||_{C^3(\mathbb{R}^n)} < \infty$ and $||\phi||_{C^3(\mathbb{R}^n)}$ depends on r and $||\phi||_{C^3(B_{r(x_0)})}$. Let us use C_1 to denote,

$$C_1 = ||\phi||_{C^3(\mathbb{R}^n)}.$$

Let ψ_ε be as defined in Lemma 3.3, and fix $\varepsilon = \varepsilon_1$. Notice that $\psi_\varepsilon(x, t) = \phi(x)$, so according to (C.5) we have $\psi(x, t) \leq W(x, t)$ for all $x \in \mathbb{R}^n$. Since, according to Lemma 3.3, ψ_ε is a subsolution of (2.1), we can apply the comparison theorem and conclude that $\psi_\varepsilon(x, r) \leq W(x, r)$ holds on $\mathbb{R}^n \times (t, t + \bar{h})$. Due to the definition of W this implies, for $h \in (0, \bar{h})$,

$$\{x : \psi_\varepsilon(x, t + h) > 0\} \subset \Omega_{t+h}^{int}. \tag{C.6}$$

Let us take $h \in (0, \bar{h})$ and $x \in \bar{B}_r(\bar{x})$ such that

$$\phi(x) - \int\limits_t^{t+h} H(x, s, D\phi(x)) \, ds - h\alpha > 0.$$

Rearranging and then using the estimate (3.2) yields,

$$\phi(x) > \int\limits_t^{t+h} H(x, s, D\phi(x)) \, ds + h\alpha \geq \int\limits_t^{t+h} H^\varepsilon(x, s, D\phi(x)) - \frac{\alpha}{4} \, ds + h\alpha$$

$$= \int\limits_t^{t+h} H^\varepsilon(x, s, D\phi(x)) \, ds + h\alpha/2 = \phi(x) - \psi_\varepsilon(x, t + h),$$

where last equality follows from the definition of ψ. So, we find $\psi_\varepsilon(x, t + h) > 0$ holds for such x, h.

Thus, we have shown that for $h \in (0, \bar{h})$,

$$\bar{B}_r(\bar{x}) \cap \left\{ x : \phi(x) - \int\limits_t^{t+h} H(x, s, D\phi(x)) \, ds - h\alpha > 0 \right\} \subset \{x : \psi(x, t + h) > 0\}.$$

According to (C.6), the set on the right-hand side of the previous line is contained in Ω_{t+h}^{int}, which completes the proof. $\quad\square$

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