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Gagliardo–Nirenberg inequalities and non-inequalities: The full story

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Abstract

We investigate the validity of the Gagliardo–Nirenberg type inequality

$$\|f\|_{W^{s,p}(\Omega)} \lesssim \|f\|^{\theta}_{W^{s_1,p_1}(\Omega)}\|f\|^{1-\theta}_{W^{s_2,p_2}(\Omega)}, \tag{1}$$

with $\Omega \subset \mathbb{R}^N$. Here, $0 \le s_1 \le s \le s_2$ are non negative numbers (not necessarily integers), $1 \le p_1, p, p_2 \le \infty$, and we assume the standard relations

$$s = \theta s_1 + (1-\theta)s_2,\ 1/p = \theta/p_1 + (1-\theta)/p_2 \text{ for some } \theta \in (0,1).$$

By the seminal contributions of E. Gagliardo and L. Nirenberg, (1) holds when s_1, s_2, s are integers. It turns out that (1) holds for "most" of values of $s_1, \dots, p_2$, but not for all of them. We present an explicit condition on s_1, s_2, p_1, p_2 which allows to decide whether (1) holds or fails.

Keywords: Sobolev spaces; Gagliardo–Nirenberg inequalities; Interpolation inequalities

1. Introduction

In two seminal independent contributions, E. Gagliardo [8] and L. Nirenberg [10] established the interpolation inequality[1]

$$\|f\|_{W^{k,p}} \lesssim \|f\|^{\theta}_{W^{k_1,p_1}}\|f\|^{1-\theta}_{W^{k_2,p_2}},\ \forall f \in W^{k_1,p_1}(\mathbb{R}^N) \cap W^{k_2,p_2}(\mathbb{R}^N), \tag{1.1}$$

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[1] In Eq. (1.1), $A \lesssim B$ means $A \le CB$ for some positive constant C.

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where k_1, k_2, k are non negative integers and $1 \le p_1, p_2, p \le \infty$. These quantities are related by the standard relations

$$k = \theta k_1 + (1-\theta)k_2, \ \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \text{ and } 0 < \theta < 1. \tag{1.2}$$

We investigate the validity of the analogous inequality when the smoothness exponents k_1, k_2, k are not necessarily integers. More specifically, assume that the real numbers $0 \le s_1, s_2, s, \theta \in (0,1)$ and $1 \le p_1, p_2, p \le \infty$ satisfy the relations

$$s = \theta s_1 + (1-\theta)s_2, \ \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \text{ and } 0 < \theta < 1. \tag{1.3}$$

We ask whether the estimate

$$\|f\|_{W^{s,p}(\Omega)} \lesssim \|f\|^{\theta}_{W^{s_1,p_1}(\Omega)} \|f\|^{1-\theta}_{W^{s_2,p_2}(\Omega)}, \ \forall f \in W^{s_1,p_1}(\Omega) \cap W^{s_2,p_2}(\Omega) \tag{1.4}$$

holds. Here, Ω is a *standard domain* in $\mathbb{R}^N$, i.e.,

$$\Omega \text{ is either } \mathbb{R}^N \text{ or a half space or a Lipschitz bounded domain in } \mathbb{R}^N, \tag{1.5}$$

and $\|f\|_{W^{s,p}}$ denotes the usual Sobolev norm (see Section 2).

Let us note that (1.4) holds when $s_1 = s_2$; this is simply Hölder's inequality. In our analysis, we may thus assume that

$$s_1 < s < s_2. \tag{1.6}$$

It has been part of the folklore of the Sobolev spaces theory that (1.4) holds in "most" cases but fails in some "limiting" cases. For example if $0 < s_1 < s_2 < 1$, (1.4) is an immediate consequence of Hölder's inequality. While if $\Omega = (0,1)$, $s_1 = 0$, $s_2 = 1$, $p_1 = \infty$, $p_2 = 1$, $\theta = 1/2$, (1.4) becomes

$$\|f\|_{H^{1/2}((0,1))} \lesssim \|f\|^{1/2}_{W^{1,1}((0,1))} \|f\|^{1/2}_{L^{\infty}((0,1))}, \ \forall f \in W^{1,1}((0,1)), \tag{1.7}$$

which implies

$$\|f\|_{H^{1/2}((0,1))} \lesssim \|f\|^{1/2}_{BV((0,1))} \|f\|^{1/2}_{L^{\infty}((0,1))}, \ \forall f \in BV((0,1)). \tag{1.8}$$

But (1.8) is clearly wrong (take e.g. $f = \mathbb{1}_{(0,1/2)}$), so that (1.7) also fails.

To the best of our knowledge, the precise "dividing line" between the "good" and the "bad" cases in (1.4) was never clarified. It is our goal to fill this gap.

The following condition plays an essential role.[2]

$$s_2 \text{ is an integer } \ge 1, \ p_2 = 1 \text{ and } s_2 - s_1 \le 1 - \frac{1}{p_1}. \tag{1.9}$$

Here is our main result.

Theorem 1. *Inequality (1.4) holds if and only if (1.9) fails.*

More precisely, we have

A) *If (1.9) fails then, for every $\theta \in (0,1)$, there exists a constant C depending on s_1, s_2, p_1, p_2, θ and Ω such that*

$$\|f\|_{W^{s,p}(\Omega)} \le C \|f\|^{\theta}_{W^{s_1,p_1}(\Omega)} \|f\|^{1-\theta}_{W^{s_2,p_2}(\Omega)}, \ \forall f \in W^{s_1,p_1}(\Omega) \cap W^{s_2,p_2}(\Omega). \tag{1.10}$$

B) *If (1.9) holds there exists some $f \in W^{s_1,p_1}(\Omega) \cap W^{s_2,p_2}(\Omega)$ such that $f \notin W^{s,p}(\Omega)$, $\forall \theta \in (0,1)$.*

[2] The latter condition can also be written in the more symmetric form $s_1 - \dfrac{1}{p_1} \ge s_2 - \dfrac{1}{p_2}$.

An amusing consequence is that

$$[(1.10) \text{ holds for all } N, \text{ all standard domains in } \mathbb{R}^N, \text{ and all } \theta \in (0,1)]$$
$$\iff [(1.10) \text{ holds in } (0,1) \text{ with } \theta = 1/2].$$

Remark 1.1. Part A of Theorem 1 covers of course all cases of (1.10) which were known before. We mention in particular:

i) A result of Cohen [5] settling the case $s_1 = 0$, $0 < s_2 < \infty$, $p_1 = 1$ and $1 < p_2 \le \infty$ when s and s_2 are not integers (with a proof involving wavelets).
ii) A result of Oru [11] (unpublished; for a proof, see [3, Section III]) yields in full generality the case $0 \le s_1 < s_2 < \infty$, $1 < p_1 < \infty$ and $1 < p_2 < \infty$ and also implies the validity of (1.10) in some special cases where $p_1 \in \{1, \infty\}$ and/or $p_2 \in \{1, \infty\}$; see Section 5 below.
iii) The case $0 < s_1 < s_2 = 1$, $p_1 < 1/s_1$, $p_2 = 1$ is treated by Cohen, Dahmen, Daubechies and DeVore [6] (with a proof involving again wavelets).
iv) Inequality (1.10) holds when $W^{s,p}$ is obtained by real or complex interpolation from W^{s_1,p_1} and W^{s_2,p_2}. This covers a number of cases, for example $0 \le s_1 < s_2 < \infty$, $p_1 = p_2 = p = 1$ (see e.g. [1, Section 7.32]).

The above results enter as crucial ingredients in the proof of Theorem 1, part A. We should also mention important work by the Soviet school (see Kašin [9] and the references therein), in particular a contribution of Besov which settles the case $s_1 = 0$, $s_2 = 1$, $p_1 = 1$, $p_2 = \infty$. However, this case is not used in our proof (see Case 4 in Section 3.2.2).

We also mention that a special case of Oru's result [11] appears in Runst [12, Section 5.1, Theorem 1], with a proof which can be adapted to the more general situation in [11].

Remark 1.2. Part B of Theorem 1 applied with $s_1 = \alpha \in (0,1)$, $s_2 = 1$, $p_1 = 1$, $p_2 = \infty$, $\theta = 1/2$ asserts that we have the non embedding

$$W^{(1+\alpha)/2,2}((0,1)) \not\subset C^{0,\alpha}((0,1)) \cap W^{1,1}((0,1)), \ \forall \alpha \in (0,1),$$

whose endpoint for $\alpha = 0$ corresponds to the failure of (1.7).

Our paper is organized as follows. In Section 2, we briefly recall the definition of fractional Sobolev spaces and some of their standard properties. Section 3 is devoted to the proof of Theorem 1, part A. Part B is established in Section 4. In some cases, the proof of part A requires an excursion into the world of Triebel–Lizorkin spaces, which is postponed to Section 5. We take advantage of this trip and establish there the analogue of Theorem 1 in these spaces, as well as in the scale of Besov spaces.

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2. Preliminaries on fractional Sobolev spaces

We recall here several equivalent characterizations of fractional Sobolev spaces $W^{s,p}$ (i.e., $W^{s,p}$ with non integer s), involving differences.[3]

[3] In the last section, we will recall the characterization of Sobolev spaces via the Littlewood–Paley theory. Our presentation of such fundamental properties follows mainly [15], [16] and [13]. Another useful reference for fractional Sobolev spaces is [7].

Definition 2.1. A *standard domain* Ω in $\mathbb{R}^N$ is: $\mathbb{R}^N$, or a half space, or a Lipschitz bounded domain.

We start by recalling the definition of the spaces $W^{s,p}(\Omega)$ when s is not an integer.

Let $0 < s < 1$ and $1 \le p < \infty$. Then we set

$$|f|^p_{W^{s,p}} = |f|^p_{W^{s,p}(\Omega)} := \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{N+sp}}\, dxdy, \ \|f\|_{W^{s,p}} := \|f\|_{L^p} + |f|_{W^{s,p}},$$

$$W^{s,p}(\Omega) := \{f : \Omega \to \mathbb{R};\ f \text{ is measurable and } \|f\|_{W^{s,p}} < \infty\}.$$

When $p = \infty$, we let $W^{s,\infty}(\Omega)$ be the Hölder space $C^s(\Omega)$, and set

$$|f|_{W^{s,\infty}} = |f|_{W^{s,\infty}(\Omega)} := |f|_{C^s} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^s}, \ \|f\|_{W^{s,\infty}} := \|f\|_{C^s} = \|f\|_{L^\infty} + |f|_{C^s}.$$

When $s > 1$ is not an integer, we write $s = m + \sigma$, with $m \in \mathbb{N}$ and $\sigma \in (0, 1)$, and then we let

$$W^{s,p}(\Omega) := \{f \in W^{m,p}(\Omega);\ D^m f \in W^{\sigma,p}(\Omega)\},$$

normed with

$$\|f\|_{W^{s,p}} = \|f\|_{W^{s,p}(\Omega)} := \|f\|_{W^{m,p}} + \|D^m f\|_{W^{\sigma,p}}.$$

The above spaces are fractional order Sobolev spaces (also known as Slobodeskii spaces).

Alternatively, it is possible to define the Sobolev spaces inductively [15, Section 2.3.8], [13, Section 2.1.4]:

Proposition 2.2. *Let $s \ge n \ge 1$, with n integer. Let Ω be a standard domain. Let $f \in L^p(\Omega)$. Then $f \in W^{s,p}(\Omega)$ if and only if $D^n f \in W^{s-n,p}(\Omega)$, and the quantity $\|f\|_{L^p} + \|D^n f\|_{W^{s-n,p}}$ is equivalent to $\|f\|_{W^{s,p}}$.*

When $n = 1$, the above theorem asserts the following. For $s \ge 1$, let $\langle\ \rangle_{W^{s,p}}$ denote the following seminorm on $W^{s,p}$:

$$\langle f\rangle_{W^{s,p}} := \|Df\|_{W^{s-1,p}}.$$

Then[4]

$$\|f\|_{W^{s,p}} \approx \|f\|_{L^p} + \langle f\rangle_{W^{s,p}}.$$

The next result (for which we refer to [14, Sections VI.3 and VI.4]) is useful in reducing the analysis of (1.10) to the case where $\Omega = \mathbb{R}^N$.

Proposition 2.3. *Let Ω be a standard domain in $\mathbb{R}^N$. Then there exists a linear extension operator $P : L^1_{loc}(\overline{\Omega}) \to L^1_{loc}(\mathbb{R}^N)$ such that:*

1. *$Pf = f$ in Ω, $\forall f \in L^1_{loc}(\overline{\Omega})$.*
2. *If $s \ge 0$ and $1 \le p \le \infty$, then $P(W^{s,p}(\Omega)) \subset W^{s,p}(\mathbb{R}^N)$ and $\|Pf\|_{W^{s,p}(\mathbb{R}^N)} \approx \|f\|_{W^{s,p}(\Omega)}$, $\forall f \in W^{s,p}(\Omega)$.*
3. *Assume that Ω is bounded, and let U be a bounded open set such that $\overline{\Omega} \subset U$. Then we may construct P such that* supp $Pf \subset U$, $\forall f \in L^1(\Omega)$.

It is possible to characterize the spaces $W^{s,p}$ using differences, as in the case where $s \in (0, 1)$ [16, Section 3.5.3], [13, Section 2.3.1]. More specifically, set $\Delta_h f(x) = f(x+h) - f(x)$, and $\Delta_h^M = \underbrace{\Delta_h \circ \ldots \circ \Delta_h}_{M \text{ times}}$, with $M > 0$ an integer. Define

$$B := \{(x, h) \in \Omega \times (\mathbb{R}^N \setminus \{0\});\ [x, x + Mh] \subset \Omega\}$$

[4] In the equation, $A \approx B$ means $C'A \le B \le CA$ for some positive constants C, C'.

and, for $j = 1, \dots, N$,

$$A_j := \{(x,t) \in \Omega \times (0,\infty);\ [x, x + Mte_j] \subset \Omega\}.$$

Proposition 2.4. *Let $s > 0$ be non integer and let $1 \le p < \infty$. Let $M > s$ be an integer. Let Ω be a standard domain. Then $\|f\|_{W^{s,p}}$ is equivalent to the following quantities*

$$\|f\|_{L^p} + \sum_{j=1}^{N} \left(\iint_{A_j} \frac{|\Delta^M_{te_j} f(x)|^p}{t^{1+sp}}\, dxdt \right)^{1/p}, \tag{2.1}$$

$$\|f\|_{L^p} + \left(\iint_{B} \frac{|\Delta^M_{h} f(x)|^p}{|h|^{N+sp}}\, dxdh \right)^{1/p}. \tag{2.2}$$

When $p = \infty$, the following analogous norm equivalences hold:

$$\|f\|_{W^{s,\infty}} \approx \|f\|_{L^\infty} + \sum_{j=1}^{N} \operatorname*{esssup}_{A_j} \frac{|\Delta^M_{te_j} f(x)|}{t^s}, \tag{2.3}$$

$$\|f\|_{W^{s,\infty}} \approx \|f\|_{L^\infty} + \operatorname*{esssup}_{B} \frac{|\Delta^M_{h} f(x)|}{|h|^s}. \tag{2.4}$$

By (2.1)–(2.4), when $1 \le p \le \infty$ and $\Omega = \mathbb{R}^N$ we have

$$\|f\|_{W^{s,p}(\mathbb{R}^N)} \approx \sum_{j=1}^{N} \left(\int \|f(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_N)\|^p_{W^{s,p}(\mathbb{R})}\, d\widehat{x_j} \right)^{1/p}, \tag{2.5}$$

with the obvious modification when $p = \infty$.

Here, $d\widehat{x_j} = dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_N$.

Lemma 2.5. *Let $s \ge 0$ and $1 \le p \le \infty$. Assume $N = N_1 + N_2$. Let $\psi \in C^\infty_c(\mathbb{R}^{N_2})$, $\psi \not\equiv 0$. Let $f : \mathbb{R}^{N_1} \to \mathbb{R}$. Then*

$$f \otimes \psi \in W^{s,p}(\mathbb{R}^N) \iff f \in W^{s,p}(\mathbb{R}^{N_1})$$

and

$$\|f \otimes \psi\|_{W^{s,p}(\mathbb{R}^N)} \approx 1 + \|f\|_{W^{s,p}(\mathbb{R}^{N_1})}.$$

Proof. When s is not an integer, the conclusion is an immediate consequence of (2.5). We let to the reader the straightforward case where s is an integer. □

We next present another norm equivalence, useful in dimensional reductions. Compared to (2.5), it has the advantage of being valid for both fractional and integer s. If $\omega \in \mathbb{S}^{N-1}$, let $\omega^\perp$ denote the hyperplane $\{x \in \mathbb{R}^N;\ x \cdot \omega = 0\}$. Consider the partial functions

$$\omega^\perp \ni x \mapsto f^x_\omega, \text{ with } \mapsto f^x_\omega(t) := f(x + t\,\omega),\ \forall t \in \mathbb{R}. \tag{2.6}$$

Then we have [2, Proof of Lemma D.2]

Proposition 2.6. *Let $s \ge 0$ and $1 \le p \le \infty$. Then*

$$\|f\|^p_{W^{s,p}(\mathbb{R}^N)} \approx \int_{\mathbb{S}^{N-1}} \left(\int_{\omega^\perp} \|f^x_\omega\|^p_{W^{s,p}(\mathbb{R})}\, dx \right) d\omega, \tag{2.7}$$

with the obvious modification when $p = \infty$.

For further use, we introduce the following distinction between Sobolev spaces.

Definition 2.7. An *ordinary Sobolev space* is a space $W^{s,p}$ such that either s is not an integer or $1 < p < \infty$. The remaining spaces, $W^{k,1}$ and $W^{k,\infty}$ with $k \in \mathbb{N}$, are *exceptional Sobolev spaces*.

3. Proof of Theorem 1, part A

Throughout this section, we assume that $s_1, \dots, p_2$ satisfy (1.3) and (1.6), and that they *do not satisfy* (1.9). We will prove that for such numbers the inequality (1.10) is valid.

3.1. Some standard reductions

The purpose of this subsection is to reduce the study of the general case to the study the validity of (1.10) for some special $N, \Omega, s_1, \dots, p_2$.

3.1.1. Dimensional reduction

We start by recalling a standard argument that reduces the general case to the special case where

$$N = 1, \ \Omega = \mathbb{R}. \tag{3.1}$$

Assume that (1.10) holds in $\mathbb{R}$ (for some $s_1, \dots, p_2$). Assume e.g. that $1 \le p_1, p, p_2 < \infty$ (the remaining cases are similar). Applying (1.10) to the partial functions in (2.6) and then Hölder's inequality to the integral in (2.7), we find that

$$\begin{aligned}
\|f\|^p_{W^{s,p}(\mathbb{R}^N)} &\approx \int_{\mathbb{S}^{N-1}} \left(\int_{\omega^\perp} \|f^x_\omega\|^p_{W^{s,p}(\mathbb{R})}\, dx \right) d\omega \lesssim \int_{\mathbb{S}^{N-1}} \left(\int_{\omega^\perp} \|f^x_\omega\|^{\theta p}_{W^{s_1,p_1}(\mathbb{R})} \|f^x_\omega\|^{(1-\theta)p}_{W^{s_2,p_2}(\mathbb{R})}\, dx \right) d\omega \\
&\le \left(\int_{\mathbb{S}^{N-1}} \left(\int_{\omega^\perp} \|f^x_\omega\|^{p_1}_{W^{s_1,p_1}(\mathbb{R})}\, dx \right) d\omega \right)^{\theta p/p_1} \left(\int_{\mathbb{S}^{N-1}} \left(\int_{\omega^\perp} \|f^x_\omega\|^{p_2}_{W^{s_2,p_2}(\mathbb{R})}\, dx \right) d\omega \right)^{(1-\theta)p/p_2} \\
&\approx \|f\|_{W^{s_1,p_1}(\mathbb{R}^N)}\|^{\theta p} \|f\|_{W^{s_2,p_2}(\mathbb{R}^N)}\|^{(1-\theta)p},
\end{aligned}$$

and thus (1.10) holds in $\mathbb{R}^N$ (for the same $s_1, \dots, p_2$).

Assume next that (1.10) holds in $\mathbb{R}^N$ (for some $s_1, \dots, p_2$). Using Proposition 2.3, we find that (1.10) holds in any standard domain in $\mathbb{R}^N$ (for the same $s_1, \dots, p_2$).

In conclusion, it suffices to establish the validity of (1.10) under the assumption (3.1).

3.1.2. Lowering s_1

We next explain why it suffices to consider the case where

$$0 \le s_1 < 1. \tag{3.2}$$

Assume that (1.10) holds when $N = 1$, $\Omega = \mathbb{R}$ for some $s_1, \dots, p_2$. Let $m \ge 1$ be an integer. Then we claim that (1.10) holds for $\widetilde{s}_1 := s_1 + m, \widetilde{s} := s + m, \widetilde{s}_2 := s_2 + m, \theta, p_1, p, p_2$. To see this, we combine Hölder's inequality applied to f with (1.10) applied to $f^{(m)}$ and find that

$$\|f\|_{L^p} + \|f^{(m)}\|_{W^{s,p}} \lesssim (\|f\|_{L^{p_1}} + \|f^{(m)}\|_{W^{s_1,p_1}})^\theta \times (\|f\|_{L^{p_2}} + \|f^{(m)}\|_{W^{s_2,p_2}})^{1-\theta}. \tag{3.3}$$

We obtain (1.10) for $\widetilde{s}_1, \dots, p_2$ via (3.3) and Proposition 2.2.

By the above discussion, from now on we may assume that (3.2) holds.

3.1.3. Reduction to a semi-norm inequality

Let $\rangle\ \langle_{W^{s,p}}$ be any semi-norm on $W^{s,p}(\mathbb{R})$ such that $\|f\|_{W^{s,p}} \approx \|f\|_{L^p} + \rangle f \langle_{W^{s,p}}$. Assume that, with $s_1, \dots, p_2$ as in (1.3), we have

$$\rangle f \langle_{W^{s,p}} \lesssim \|f\|^{\theta}_{W^{s_1,p_1}} \|f\|^{1-\theta}_{W^{s_2,p_2}}. \tag{3.4}$$

Combining (3.4) with Hölder's inequality $\|f\|_{L^p} \le \|f\|^{\theta}_{L^{p_1}} \|f\|^{1-\theta}_{L^{p_2}}$, we find that (1.10) holds.

3.1.4. Reiteration procedure

This is a very simple technique which allows to generate new cases from a known cases for which (1.10) holds.

The proportionality relation (1.3) is equivalent to the fact that $(s_1, 1/p_1)$, $(s, 1/p)$, $(s_2, 1/p_2)$ are collinear as points in $\mathbb{R}^2$, and that the second point is "between" the first and the third one.

A possible reiteration procedure is the following. Let $s_1 < \sigma_1 < s < \sigma_2 < s_2$. Assume that $(s_1, 1/p_1)$, $(\sigma_1, 1/\rho_1)$, $(s, 1/p)$, $(\sigma_2, 1/\rho_2)$, $(s_2, 1/p_2)$ are collinear. Assume also that (1.10) holds respectively for:

$$(s_1, s_2, \sigma_1, p_1, p_2, \rho_1),\ (s_1, s_2, \sigma_2, p_1, p_2, \rho_2),\ (\sigma_1, \sigma_2, s, \rho_1, \rho_2, p)$$

(and the corresponding θ's, which are uniquely determined by $s_1, s_2, s, \sigma_1, \sigma_2, p_1, p_2$).

Then we claim that (1.10) holds for $(s_1, s_2, s, p_1, p_2, p)$. Indeed, for appropriate $\theta_1, \theta_2, \theta_3$ we have

$$\|f\|_{W^{\sigma_1,\rho_1}} \lesssim \|f\|^{\theta_1}_{W^{s_1,p_1}} \|f\|^{1-\theta_1}_{W^{s_2,p_2}} \tag{3.5}$$

$$\|f\|_{W^{\sigma_2,\rho_2}} \lesssim \|f\|^{\theta_2}_{W^{s_1,p_1}} \|f\|^{1-\theta_2}_{W^{s_2,p_2}} \tag{3.6}$$

$$\|f\|_{W^{s,p}} \lesssim \|f\|^{\theta_3}_{W^{\sigma_1,\rho_1}} \|f\|^{1-\theta_3}_{W^{\sigma_2,\rho_2}}. \tag{3.7}$$

We obtain (1.10) for $(s_1, s_2, s, p_1, p_2, p)$ (with the correct θ) by inserting (3.5)–(3.6) into (3.7).

Here is another illustration of the reiteration procedure. Assume that $s_1 < s < \sigma_2 < s_2$ and that $(s_1, 1/p_1)$, $(s, 1/p)$, $(\sigma_2, 1/\rho_2)$, $(s_2, 1/p_2)$ are collinear. Assume also that (1.10) holds respectively for:

$$(s_1, \sigma_2, s, p_1, \rho_2, p),\ (s, s_2, \sigma_2, p, p_2, \rho_2).$$

Then we claim that (1.10) holds for $(s_1, s_2, s, p_1, p_2, p)$. This time, we rely on

$$\|f\|_{W^{s,p}} \lesssim \|f\|^{\theta_1}_{W^{s_1,p_1}} \|f\|^{1-\theta_1}_{W^{\sigma_2,\rho_2}} \tag{3.8}$$

$$\|f\|_{W^{\sigma_2,\rho_2}} \lesssim \|f\|^{\theta_2}_{W^{s,p}} \|f\|^{1-\theta_2}_{W^{s_2,p_2}} \tag{3.9}$$

and we insert (3.9) into (3.8).

Here is a typical situation where reiteration is useful.

Corollary 3.1. *Assume $0 = s_1 < s_2 \le 1$, $p_1 = 1$ and $1 \le p_2 \le \infty$. Then (1.10) holds.*

Proof. In view of items i) and iv) in Remark 1.1, it suffices to prove that (1.10) holds when

$$s_1 = 0,\ s_2 = 1,\ p_1 = 1 \text{ and } 1 < p_2 \le \infty.$$

We will establish this via reiteration. Fix σ_2, ρ_2 such that $0 < s < \sigma_2 < 1$ and $(0, 1)$, $(s, 1/p)$, $(\sigma_2, 1/\rho_2)$, $(1, 1/p_2)$ are collinear.

Note that (1.10) holds for $(0, \sigma_2, s, 1, \rho_2, p)$ (by i) in Remark 1.1) and for $(s, 1, \sigma_2, p, p_2, \rho_2)$ (by Corollary 5.1 when $1 < p_2 < \infty$ and by Corollary 5.2 when $p_2 = \infty$). [Indeed, we are in position to apply item i) in Remark 1.1 and Corollaries 5.1 and 5.2 since $0 < s < 1$, $1 < p < \infty$, $0 < \sigma_2 < 1$ and $1 < \rho_2 < \infty$.] Therefore, (1.10) holds also for $(0, 1, s, 1, p_2, p)$. □

3.2. Proof of Theorem 1, part A, when $0 \le s_1 < s_2 \le 1$

An easy inspection shows that in this range (1.9) holds exactly when

$$0 \le \frac{1}{p_1} \le s_1 < 1,\ s_2 = 1,\ p_2 = 1. \tag{3.10}$$

Thus we must show that (1.10) holds in *all* cases *except* in the range defined by (3.10). The proof consists of a tedious analysis of all possibilities. We distinguish four subsections:

3.2.1 $0 = s_1 < s_2 < 1$
3.2.2 $s_1 = 0,\ s_2 = 1$
3.2.3 $0 < s_1 < s_2 < 1$
3.2.4 $0 < s_1 < s_2 = 1$

3.2.1. $0 = s_1 < s_2 < 1$

Case 1. $p_1 = 1,\ 1 \le p_2 \le \infty$. Apply Corollary 3.1.

Case 2. $1 < p_1 < \infty,\ 1 \le p_2 \le \infty$. Apply Corollary 5.1.

Case 3. $p_1 = \infty,\ 1 \le p_2 \le \infty$. Apply Corollary 5.2.

3.2.2. $s_1 = 0,\ s_2 = 1$

Case 4. $p_1 = 1,\ 1 \le p_2 \le \infty$. Apply Corollary 3.1.

Case 5. $1 < p_1 < \infty,\ p_2 = 1$. Here we use again reiteration. Let $0 < \theta < 1$ be such that

$$s = 1 - \theta,\ \frac{1}{p} = \frac{\theta}{p_1} + 1 - \theta = \frac{\theta}{p_1} + \frac{1-\theta}{1}.$$

Choose σ_2 and ρ_2 such that $0 < s < \sigma_2 < 1$ and $(0, 1/p_1)$, $(s, 1/p)$, $(\sigma_2, 1/\rho_2)$, $(1, 1)$ are collinear.

Then (1.10) holds for $(0, \sigma_2, s, p_1, \rho_2, p)$ by Case 2 above (since $1 < \rho_2 < \infty$). On the other hand, (1.10) also holds for $(s, 1, \sigma_2, p, 1, \rho_2)$, by item iii) in Remark 1.1 (since $p < 1/s$).

Case 6. $1 < p_1 < \infty,\ 1 < p_2 < \infty$. Apply Corollary 5.1.

Case 7. $1 < p_1 < \infty,\ p_2 = \infty$. Apply Corollary 5.2.

Case 8. $p_1 = \infty,\ p_2 = 1$. Here, (3.10) (and thus (1.9)) holds. There is nothing to prove!

Case 9. $p_1 = \infty,\ 1 < p_2 < \infty$. Apply Corollary 5.2.

Case 10. $p_1 = \infty,\ p_2 = \infty$. This case corresponds to the inequality

$$|f|_{C^s} \lesssim \|f'\|_{L^\infty}^s \|f\|_{L^\infty}^{1-s},\ \forall f \in W^{1,\infty}(\mathbb{R}). \tag{3.11}$$

We have

$$\frac{|f(x) - f(y)|}{|x-y|^s} \le \begin{cases} \|f'\|_{L^\infty} l^{1-s}, & \text{if } |x-y| < l \\ \dfrac{2\|f\|_{L^\infty}}{l^s}, & \text{if } |x-y| \ge l \end{cases}. \tag{3.12}$$

We obtain (3.11) by taking, in (3.12), first $l := \dfrac{\|f\|_{L^\infty}}{\|f'\|_{L^\infty}}$ and then the supremum over x and y.

3.2.3. $0 < s_1 < s_2 < 1$

This case is fully covered by Corollary 5.1.

3.2.4. $0 < s_1 < s_2 = 1$

Case 11. $1 \le p_1 < 1/s_1,\ p_2 = 1$. This follows from item iii) in Remark 1.1.

Case 12. $1 \le p_1 < 1/s_1,\ 1 < p_2 < \infty$. Apply Corollary 5.1.

Case 13. $1 \le p_1 < 1/s_1,\ p_2 = \infty$. Apply Corollary 5.2.

Case 14. $1/s_1 \le p_1 \le \infty,\ p_2 = 1$. Here, (3.10) (and thus (1.9)) holds. There is nothing to prove!

Case 15. $1/s_1 \le p_1 \le \infty,\ 1 < p_2 < \infty$. Apply Corollary 5.1.

Case 16. $1/s_1 \le p_1 \le \infty,\ p_2 = \infty$. Apply Corollary 5.2. □

3.3. Proof of Theorem 1, part A, when $0 \le s_1 < 1$ and $1 < s_2 < \infty$

We must show that (1.10) holds in *all* cases. The proof consists of a tedious analysis of all possibilities. We distinguish two subsections:

3.3.1 $s_1 = 0$, $1 < s_2 < \infty$
3.3.2 $0 < s_1 < 1$, $1 < s_2 < \infty$

3.3.1. $s_1 = 0$, $1 < s_2 < \infty$

Case 17. $p_1 = 1$, $p_2 = 1$. We use item iv) in Remark 1.1.

Case 18. $p_1 = 1$, $1 < p_2 \le \infty$, W^{s_2,p_2} is an ordinary Sobolev space. Apply Corollary 5.1.

Case 19. $p_1 = 1$, $p_2 = \infty$, s_2 is an integer. Apply Corollary 5.2.

Note that Cases 17–19 cover all the possible situations where $s_1 = 0$ and $p_1 = 1$.

Case 20. $1 < p_1 < \infty$, $1 \le p_2 \le \infty$, W^{s_2,p_2} is an ordinary Sobolev space. Apply Corollary 5.1.

Case 21. $1 < p_1 < \infty$, $p_2 = \infty$, s_2 is an integer. Apply Corollary 5.2.

Case 22. $1 < p_1 < \infty$, $p_2 = 1$, s_2 is an integer. In this case, we rely on the lowering s_1 procedure (applied once) and reiteration (applied twice). Choose σ_1, σ_2, ρ_1, ρ_2 such that

$$\max\{s, s_2 - 1\} < \sigma_1 < \sigma_2 < s_2$$

and $(0, 1/p_1)$, $(s, 1/p)$, $(\sigma_1, 1/\rho_1)$, $(\sigma_2, 1/\rho_2)$, $(s_2, 1)$ are collinear.

By item iii) in Remark 1.1, (1.10) holds for $(\sigma_1 - s_2 + 1, 1, \sigma_2 - s_2 + 1, \rho_1, 1, \rho_2)$.

By the lowering s_1 procedure, we find that (1.10) holds for $(\sigma_1, s_2, \sigma_2, \rho_1, 1, \rho_2)$.

On the other hand, (1.10) holds for $(s, \sigma_2, \sigma_1, p, \rho_2, \rho_1)$ (by Corollary 5.1).

By reiteration we find that (1.10) holds for $(s, s_2, \sigma_2, p, 1, \rho_2)$.

We next invoke the fact that (1.10) holds for $(0, \sigma_2, s, p_1, \rho_2, p)$ (by Corollary 5.1). Reiterating again, we obtain (1.10) for $(s_1, s_2, s, p_1, p, 1)$.

Note that Cases 20–22 cover all the possible situations where $s_1 = 0$ and $1 < p_1 < \infty$.

Case 23. $p_1 = \infty$, $1 < p_2 < \infty$. Apply Corollary 5.2.

Case 24. $p_1 = \infty$, $p_2 = \infty$, s_2 is not an integer. Apply Corollary 5.2.

Case 25. $p_1 = \infty$, $p_2 = 1$, s_2 is an integer. Repeat the argument in Case 22, with the only modification that the second reiteration relies on Corollary 5.2 instead of Corollary 5.1.

Case 26. $p_1 = \infty$, $p_2 = \infty$, s is not an integer. This case relies on the lowering s_1 procedure (applied once) and reiteration (applied twice). Choose non integers numbers σ_1, σ_2 such that

$$\max\{s, s_2 - 1\} < \sigma_1 < \sigma_2 < s_2.$$

Let m be the least integer $\ge s_2$. By Case 16 (when s_2 is an integer) and Subsection 3.2.3 (when s_2 is not an integer), (1.10) holds for $(\sigma_1 - m + 1, s_2 - m + 1, \sigma_2 - m + 1, \infty, \infty, \infty)$. By the lowering s_1 procedure, (1.10) also holds for $(\sigma_1, s_2, \sigma_2, \infty, \infty, \infty)$.

On the other hand, Corollary 5.1 implies that (1.10) holds for $(s, \sigma_2, \sigma_1, \infty, \infty, \infty)$ (here, we use the fact that none of s, σ_1, σ_2 is an integer).

By reiteration, (1.10) holds for $(s, s_2, \sigma_2, \infty, \infty, \infty)$.

We next invoke the fact that (1.10) holds for $(0, \sigma_2, s, \infty, \infty, \infty)$ (by Corollary 5.2). Reiterating again, we obtain (1.10) for $(s_1, s_2, s, \infty, \infty, \infty)$.

Case 27. $p_1 = \infty$, $p_2 = \infty$, s is an integer. By the previous case, (1.10) holds for $(0, s_2, s - \varepsilon, \infty, \infty, \infty)$ and for $(0, s_2, s + \delta, \infty, \infty, \infty)$ for sufficiently small $\varepsilon, \delta > 0$. In view of the reiteration procedure, it thus suffices to prove that

$$\text{(1.10) holds for } (s - \varepsilon, s + \delta, s\infty, \infty, \infty). \tag{3.13}$$

By the lowering s_1 procedure, it is enough to establish (3.13) when $s = 1$. Setting $\sigma := 1 - \varepsilon \in (0, 1)$, (3.13) with $s = 1$ amounts to

$$\|f'\|_{L^\infty} \lesssim |f|_{C^\sigma}^{\theta} |f'|_{C^\delta}^{1-\theta}, \ \forall f \in C_c^\infty(\mathbb{R}). \tag{3.14}$$

Here, $\theta \in (0, 1)$ is defined by $1 = \theta\sigma + (1 - \theta)(1 + \delta)$.

In order to obtain (3.14), we start (with $l > 0$ to be defined later) from

$$\begin{aligned} |f'(x)| &\le \left| \frac{f(x+l) - f(x)}{l} - f'(x) \right| + \left| \frac{f(x+l) - f(x)}{l} \right| \\ &\le \sup_{z \in [x, x+l]} |f'(z) - f'(x)| + |f|_{C^\sigma} l^{\sigma - 1} \le |f'|_{C^\delta} l^\tau + |f|_{C^\sigma} l^{\sigma-1}. \end{aligned} \tag{3.15}$$

Taking, in (3.15), first $l := \left(\dfrac{|f|_{C^\sigma}}{|f'|_{C^\delta}} \right)^{1/(1+\delta-\sigma)}$, then the sup over $x \in \mathbb{R}$, we obtain (3.14).

Note that Cases 23–27 cover all the possible situations where $s_1 = 0$ and $p_1 = \infty$.

The analysis of Subsection 3.3.1 is complete.

3.3.2. $0 < s_1 < 1$, $1 < s_2 < \infty$

Case 28. $1 < p < 2 < \infty$. Apply Corollary 5.1.

Case 29. $1 \le p_1 < \infty$, $p_2 = \infty$, s_2 is not an integer. Apply Corollary 5.1.

Case 30. $1 \le p_1 < \infty$, $p_2 = \infty$, s_2 is an integer. Apply Corollary 5.2.

Case 31. $p_1 = \infty$, $p_2 = \infty$. As explained in the analysis of Case 27, if (1.10) holds when s *is not* an integer, then (1.10) holds also when s *is* an integer. We may thus assume that s is not an integer. In this case, the validity of (1.10) follows from Corollary 5.1 (when s is not an integer), respectively from Corollary 5.2 (when s_2 is an integer).

Case 32. $p_1 = 1$, $p_2 = 1$. We rely on item iv) in Remark 1.9.

The proof of Theorem 1, part A, is complete. □

4. Proof of Theorem 1, part B

The main step of the proof consists of establishing the following result, which is a variant of Theorem 1, part B with $s_2 = 1$, $p_2 = 1$ and $\Omega = (0, 1)$.

Lemma 4.1. *Let $0 \le \sigma_1 < 1$ and $1 < r_1 \le \infty$ be such that $\sigma_1 \ge 1/r_1$. For $0 < \theta < 1$ and define $\sigma = \sigma(\theta) \in (\sigma_1, 1)$, $r = r(\theta) \in (1, r_1)$ via the conditions*

$$\sigma = \theta\sigma_1 + 1 - \theta = \theta\sigma_1 + (1 - \theta) \cdot 1, \ \frac{1}{r} = \frac{\theta}{r_1} + 1 - \theta = \frac{\theta}{r_1} + \frac{1 - \theta}{1}. \tag{4.1}$$

Then there exists a sequence (u_j) of Lipschitz functions $u_j : [0, 1] \to [0, 1]$ such that

$$\|u_j\|_{W^{1,1}((0,1))} \to 0, \ \|u_j\|_{W^{\sigma_1, r_1}((0,1))} \to 0, \ \|u_j\|_{W^{\sigma, r}((0,1))} \to \infty, \ \forall \theta \in (0, 1). \tag{4.2}$$

We postpone the proof of Lemma 4.1 and turn to the

Proof of Theorem 1, part B, assuming Lemma 4.1. Let $s_1, \dots, p_2$ be as in (1.9).

Let (u_j) be as in Lemma 4.1, corresponding to $\sigma_1 := s_1 - s_2 + 1$ and $r_1 := p_1$. Thus, if s, p are as in (1.3), then (4.1) is satisfied by $\sigma = s - s_2 + 1$ and $r = p$. If $s_2 = 1$, we set $v_j := u_j$. If $s_2 \geq 2$, we let

$$v_j(x) := \frac{1}{(s_2-2)!}\int_0^x (x-t)^{s_2-2} u_j(t)\, dt, \ \forall x \in [0,1],$$

so that

$$v_j^{(s_2-1)} = u_j \tag{4.3}$$

and (using the fact that $\|u_j\|_{L^\infty((0,1))} \to 0$)

$$\|v_j\|_{L^\infty((0,1))} \to 0. \tag{4.4}$$

Using (4.2)–(4.4) and Proposition 2.2, we find that the sequence $(v_j) \subset W^{s_2,\infty}$ of functions on $[0,1]$ satisfies

$$\|v_j\|_{W^{s_2,1}((0,1))} \lesssim 1,\ \|v_j\|_{W^{s_1,p_1}((0,1))} \lesssim 1,\ \|v_j\|_{W^{s,p}((0,1))} \to \infty,\ \forall s, p \text{ as in (1.3)}. \tag{4.5}$$

By (4.5) and Proposition 2.3, there exists a sequence $(\widetilde{v}_j) \subset W^{s_2,\infty}$ of functions on $\mathbb{R}$ such that

$$\operatorname{supp} \widetilde{v}_j \subset (-1,2),\ \|\widetilde{v}_j\|_{W^{s_2,1}(\mathbb{R})} \lesssim 1,\ \|\widetilde{v}_j\|_{W^{s_1,p_1}(\mathbb{R})} \lesssim 1,\ \|\widetilde{v}_j\|_{W^{s,p}((-1,2))} \to \infty,\ \forall s, p \text{ as in (1.3)}. \tag{4.6}$$

By (4.6) and Lemma 2.5, for every $N \geq 1$ and every ball $B \subset \mathbb{R}^N$ we may construct a sequence $(w_j) \subset W_c^{s_2,\infty}(B)$ satisfying

$$\|w_j\|_{W^{s_2,1}(\mathbb{R}^N)} \lesssim 1,\ \|w_j\|_{W^{s_1,p_1}(\mathbb{R}^N)} \lesssim 1,\ \|w_j\|_{W^{s,p}(B)} \to \infty,\ \forall s, p \text{ as in (1.3)}. \tag{4.7}$$

Let Ω be any standard domain in $\mathbb{R}^N$. Define the numbers s^ℓ, p^ℓ, θ^ℓ, $\ell \geq 2$, by

$$\theta^\ell := 1 - 1/\ell,\ s^\ell := \theta^\ell s_1 + (1-\theta^\ell) s_2,\ \frac{1}{p^\ell} := \frac{\theta^\ell}{p_1} + \frac{1-\theta^\ell}{1}. \tag{4.8}$$

Consider also a sequence of mutually disjoint balls $B^k \subset \Omega$, $k \geq 1$. By (4.7), there exist functions $w^k \in W_c^{s_2,\infty}(B^k)$ such that

$$\|w^k\|_{W^{s_2,1}(\mathbb{R}^N)} \leq \frac{1}{k^2},\ \|w^k\|_{W^{s_1,p_1}(\mathbb{R}^N)} \leq \frac{1}{k^2},\ \|w^k\|_{W^{s^\ell,p^\ell}(B^k)} \geq k,\ \forall k, \forall \ell \leq k. \tag{4.9}$$

Set $f := \sum_k w^k$. By (4.9), we have

$$f \in W^{s_1,p_1}(\Omega) \cap W^{s_2,1}(\Omega), \tag{4.10}$$

while

$$\|f\|_{W^{s_\ell,p_\ell}(\Omega)} \geq \liminf_{k\to\infty} \|f\|_{W^{s^\ell,p^\ell}(B^k)} = \infty,$$

and thus

$$f \notin W^{s^\ell,p^\ell}(\Omega),\ \forall \ell. \tag{4.11}$$

Using (4.11), we find that

$$f \notin W^{s,p}(\Omega),\ \forall s, p \text{ such that (1.3) holds.} \tag{4.12}$$

Indeed, argue by contradiction and assume that

$$f \in W^{s,p}(\Omega) \text{ for some } s \text{ and } p \text{ as in (1.3)}. \tag{4.13}$$

For ℓ sufficiently large, we have $s > s^\ell$. Since $(s_1, s, s^\ell, p_1, p, p^\ell)$ *does not* satisfy (1.9), we find that Theorem 1, part A applies for this sextuple, and thus (using (4.10) and (4.13)) we find that $f \in W^{s^\ell,p^\ell}(\Omega)$. This contradicts (4.12), and completes the proof of Theorem 1, part B, granted Lemma 4.1. □

Proof of Lemma 4.1. As explained in the proof of Theorem 1, part B, it suffices to establish the seemingly weaker form of (4.2). There exists a sequence (u_j) such that, with $\sigma^\ell := \sigma(\theta^\ell)$, $r^\ell := r(\theta^\ell)$, we have

$$\|u_j\|_{W^{1,1}((0,1))} \to 0,\ \|u_j\|_{W^{\sigma_1,r_1}((0,1))} \to 0,\ \|u_j\|_{W^{\sigma^\ell,r^\ell}((0,1))} \to \infty,\ \forall\,\ell \ge 2. \tag{4.14}$$

Step 1. Construction of (u_j) *when* $\sigma_1 = 1/r_1$. In this case, we have $\sigma = 1/r$.

For $k \ge 3$, let $w^k(x) := \begin{cases} 0, & \text{if } x \le 1/2 \\ 1, & \text{if } x \ge 1/2 + 1/k \\ k(x-1/2), & \text{if } 1/2 \le x \le 1/2 + 1/k \end{cases}$. A direct calculation shows that

$$\|w^k\|_{W^{1,1}} \approx 1,\ \|w^k\|_{W^{1/q,q}} \approx (\ln k)^{1/q} \text{ as } k \to \infty,\ \forall\, 1 < q \le \infty. \tag{4.15}$$

We set, for a sequence (k_j) tending to ∞ sufficiently fast, $u_j := \dfrac{1}{(\ln k_j)^{1/r_1} \ln\ln k_j} w^{k_j}$. Then clearly u_j satisfies (4.14) (since $r^\ell < r_1, \forall\,\ell$).

Step 2. Construction of (u_j) *when* $\sigma_1 > 1/r_1$.

Step 2.1. Outline of the construction. In view of the relation (4.1), the points $(\sigma_1, 1/r_1)$, $(\sigma, 1/r)$, $(1,1) \in \mathbb{R}^2$ are collinear. The line they determine intersects the x-axis at the point $(\alpha, 0)$, where $\alpha := \dfrac{\sigma_1 - 1/r_1}{1 - 1/r_1} \in (0,1)$.

Consider the line segment L and the arc of hyperbola H given respectively by

$$L := \{\theta(\alpha,0) + (1-\theta)(1,1);\ \theta \in (0,1]\} \text{ and } H := \{(s,p);\ (s,1/p) \in L\}. \tag{4.16}$$

We note that, in particular, we have $(\sigma_1, r_1) \in H$ and $(\sigma, r) \in H$.

We will construct, by induction on $j \in \mathbb{N}^*$, sequences $\{w_j^k\}_{k\ge 2}$ such that:

$$w_j^k : [0,1] \to [0,1],\ \forall\, j,\ \forall\, k, \tag{4.17}$$

$$w_j^k \text{ is Lipschitz},\ \forall\, j,\ \forall\, k, \tag{4.18}$$

$$w_j^k \text{ is non decreasing (and thus } \|w_j^k\|_{W^{1,1}} \le 2),\ \forall\, j,\ \forall\, k, \tag{4.19}$$

$$\liminf_{k\to\infty} |w_j^k|_{W^{s,p}} \approx j^{1/p},\ \limsup_{k\to\infty} |w_j^k|_{W^{s,p}} \approx j^{1/p},\ \forall\, j,\ \forall\,(s,p) \in H. \tag{4.20}$$

Note that in particular estimate (4.20) holds for $s = \sigma_1$ and $p = r_1$, resp. for $s = \sigma^\ell$ and $p = r^\ell$.

Granted the existence of w_j^k, we set, for a sequence (k_j) tending to ∞ sufficiently fast, $u_j := \dfrac{1}{j^{1/r_1} \ln j} w_j^{k_j}$. Then clearly u_j satisfies (4.2) (since $r^\ell < r_1, \forall\,\ell$).

Step 2.2. Construction of w_1^k. Let $\varepsilon = \varepsilon_k := k^{-1/\alpha}$, so that $0 < \varepsilon < 1$ and $k\varepsilon^\alpha = 1$. Consider the following $2k$ intervals

$$I_1^k := [0,\varepsilon],\ I_2^k := [\varepsilon^\alpha, \varepsilon^\alpha + \varepsilon],\ I_3^k := [2\varepsilon^\alpha, 2\varepsilon^\alpha + \varepsilon] \ldots,\ I_k^k := [(k-1)\varepsilon^\alpha, (k-1)\varepsilon^\alpha + \varepsilon] \tag{4.21}$$

$$J_1^k := [\varepsilon, \varepsilon^\alpha],\ J_2^k := [\varepsilon^\alpha + \varepsilon, 2\varepsilon^\alpha],\ J_3^k := [2\varepsilon^\alpha + \varepsilon, 3\varepsilon^\alpha] \ldots,\ J_k^k := [(k-1)\varepsilon^\alpha + \varepsilon, k\varepsilon^\alpha]. \tag{4.22}$$

Clearly, these intervals have disjoint interiors and cover $[0,1]$ (since, by the definition of ε, we have $k\varepsilon^\alpha = 1$).

We uniquely define $w_1^k : [0,1] \to [0,1]$ via its following properties.

1. w_1^k is continuous.
2. w_1^k is constant on each J_ℓ^k.
3. $w_1^k(0) = 0$.
4. On each I_ℓ^k, w_1^k is affine and increases by the value $\varepsilon^\alpha = 1/k$.

Analytically, for each $\ell \in \{1, \ldots, k\}$ we have

$$w_1^k(x) = \begin{cases} (\ell-1)\varepsilon^\alpha + \varepsilon^{\alpha-1}(x - (\ell-1)\varepsilon^\alpha), & \text{if } (\ell-1)\varepsilon^\alpha \le x \le (\ell-1)\varepsilon^\alpha + \varepsilon \\ \ell\varepsilon^\alpha, & \text{if } (\ell-1)\varepsilon^\alpha + \varepsilon \le x \le \ell\varepsilon^\alpha \end{cases}. \tag{4.23}$$

The above formula defines w_1^k on [0, 1].

Note that the graph of w_1^k consists of k oblique line segments and of k horizontal line segments.

Step 2.3. Construction of w_2^k. The idea consists of modifying w_1^k only on the set where it is not locally constant, i.e., on each of the intervals I_ℓ^k. More specifically, w_2^k is obtained by replacing, on I_ℓ^k, the function w_1^k by an appropriate rescaled copy of w_1^k; this copy is uniquely determined by the requirement that w_2^k is continuous (we will give below the analytical formula of w_2^k). Thus, while on I_ℓ^k the graph of w_1^k is an oblique line segment, the one of w_2^k consists of k oblique line segments and of k horizontal line segments.

Analytically, w_2^k is defined as follows:

$$w_2^k(x) = \begin{cases} w_1^k(x), & \text{if } x \in J_\ell^k \text{ for some } \ell \\ (\ell-1)\varepsilon^\alpha + \varepsilon^\alpha w_1^k((x-(\ell-1)\varepsilon^\alpha)/\varepsilon), & \text{if } x \in I_\ell^k \text{ for some } \ell \end{cases}. \tag{4.24}$$

Note that the graph of w_2^k consists of k^2 oblique line segments and of k^2 horizontal segments, but unlike in the case of w_1^k the horizontal segments are not of equal length.

Step 2.4. Construction of w_j^k for $j \geq 3$. We iterate the above construction. There are two possible ways to iterate, and both lead to the same functions. The first one consists of replacing, on each maximal interval on which w_{j-1}^k is not locally constant, w_{j-1}^k by an adapted rescaled copy of w_1^k. The other one consists of replacing, on each maximal interval on which w_1^k is not locally constant, w_1^k by an adapted rescaled copy of w_{j-1}^k. We adopt the latter point of view and define, by induction on j,

$$w_j^k(x) = \begin{cases} w_1^k(x), & \text{if } x \in J_\ell^k \text{ for some } \ell \\ (\ell-1)\varepsilon^\alpha + \varepsilon^\alpha w_{j-1}^k((x-(\ell-1)\varepsilon^\alpha)/\varepsilon), & \text{if } x \in I_\ell^k \text{ for some } \ell \end{cases}. \tag{4.25}$$

Step 3. Proof of (4.17)–(4.19) and of (4.20) when $p = \infty$.

Step 3.1. First properties of w_2^k. Clearly, w_2^k satisfies (4.17)–(4.19). In addition,

$$w_j^k \text{ is constant on each } J_\ell^k,\ \ell = 1, \dots, k,\ \forall k,\ \forall j. \tag{4.26}$$

Step 3.2. Property (4.20) holds for $s = \alpha$ and $p = \infty$. More specifically, we will prove, by induction on j, that

$$\lim_{k\to\infty} |w_j^k|_{C^\alpha} = 1,\ \forall j \geq 1. \tag{4.27}$$

We start by noting that property (4.26) has the following consequence. Let $0 \leq x < y \leq 1$ and assume e.g. that $y \in J_\ell^k$. Let z be the right endpoint of I_ℓ^k, so that $w_j^k(z) = w_j^k(y)$. We thus have

$$\frac{|w_j^k(y) - w_j^k(x)|}{|y-x|^\alpha} = \frac{w_j^k(y) - w_j^k(x)}{(y-x)^\alpha} \leq \frac{w_j^k(z) - w_j^k(x)}{(z-x)^\alpha} = \frac{|w_j^k(z) - w_j^k(x)|}{|z-x|^\alpha}.$$

Thus we may "project" y on I_ℓ^k without decreasing the quotient $\dfrac{|w_j^k(y) - w_j^k(x)|}{|y-x|^\alpha}$. A similar observation holds for x. We find that

$$|w_j^k|_{C^\alpha} = \sup\left\{\frac{w_j^k(y) - w_j^k(x)}{(y-x)^\alpha};\ 0 \leq x < y \leq 1,\ x \in I_\ell^k,\ y \in I_m^k \text{ for some } \ell, m\right\}. \tag{4.28}$$

Inequality $|w_j^k|_{C^\alpha} \geq 1$ follows from $w_j^k(0) = 0$ and $w_j^k(1) = 1$, $\forall k$, $\forall j$. It thus suffices to prove that

$$\limsup_{k\to\infty} |w_j^k|_{C^\alpha} \leq 1,\ \forall j \geq 1. \tag{4.29}$$

Step 3.2.1. Proof of (4.29) when $j = 1$. If $x, y \in I_\ell^k$ (same ℓ), then

$$w_1^k(y) - w_2^k(x) = \varepsilon^{\alpha-1}(y-x) \leq (y-x)^\alpha. \tag{4.30}$$

On the other hand, if $x \in I_\ell^k$ and $y \in I_m^k$ for some $\ell < m$, write $x = (\ell - 1)\varepsilon^\alpha + \lambda\varepsilon$, $y = (m-1)\varepsilon^\alpha + \delta\varepsilon$, with $0 \le \lambda, \delta \le 1$. Set $t := \lambda - \delta \in [-1, 1]$ and $n := m - \ell \in \{1, \dots, k-1\}$. Then

$$w_1^k(y) - w_1^k(x) = n\varepsilon^\alpha + t\varepsilon^\alpha, \; y - x = n\varepsilon^\alpha + t\varepsilon,$$

and thus

$$\begin{aligned} |w_1^k|_{C^\alpha} &= \max\left\{1, \sup\left\{\frac{n\varepsilon^\alpha + t\varepsilon^\alpha}{(n\varepsilon^\alpha + t\varepsilon)^\alpha}; \; 1 \le n \le k-1, -1 \le t \le 1\right\}\right\} \\ &\le \max\left\{1, \sup\left\{\frac{(n+1)\varepsilon^\alpha}{(n\varepsilon^\alpha - \varepsilon)^\alpha}; \; 1 \le n \le k-1\right\}\right\}. \end{aligned} \tag{4.31}$$

Let us next note that, in the expression $\dfrac{(n+1)\varepsilon^\alpha}{(n\varepsilon^\alpha - \varepsilon)^\alpha}$, the numerator is affine (thus convex) in n and the denominator is concave in n. We find that the maximal value of this expression is achieved either for $n = 1$ or for $n = k-1$. Going back to (4.31), we find that

$$|w_1^k|_{C^\alpha} \le \max\left\{1, \frac{2\varepsilon^\alpha}{(\varepsilon^\alpha - \varepsilon)^\alpha}, \frac{k\varepsilon^\alpha}{((k-1)\varepsilon^\alpha - \varepsilon)^\alpha}\right\} \to 1 \text{ as } k \to \infty;$$

here, we took into account the fact that $k\varepsilon^\alpha = 1$. This proves (4.29) for $j = 1$.

Step 3.2.2. Proof of (4.29) *when* $j \ge 2$. Assume that (4.29) holds for $j - 1$. Let $x, y \in I_\ell^k$ (same ℓ). Taking into account the definition (4.25) of w_j^k, we find that

$$\frac{w_j^k(y) - w_j^k(x)}{(y-x)^\alpha} = \frac{\varepsilon^\alpha w_{j-1}^k((y - (\ell-1)\varepsilon^\alpha)/\varepsilon) - \varepsilon^\alpha w_{j-1}^k((x - (\ell-1)\varepsilon^\alpha)/\varepsilon)}{(y-x)^\alpha} \le |w_{j-1}^k|_{C^\alpha}. \tag{4.32}$$

If $x \in I_\ell^k$ and $y \in I_m^k$ for some $\ell < m$, we estimate, as for $j = 1$,

$$\frac{w_j^k(y) - w_j^k(x)}{(y-x)^\alpha} \le \frac{(n+1)\varepsilon^\alpha}{(n\varepsilon^\alpha - \varepsilon)^\alpha} \text{ with } n := m - \ell.$$

We find that

$$|w_j^k|_{C^\alpha} \le \max\left\{|w_{j-1}^k|_{C^\alpha}, \frac{2\varepsilon^\alpha}{(\varepsilon^\alpha - \varepsilon)^\alpha}, \frac{k\varepsilon^\alpha}{((k-1)\varepsilon^\alpha - \varepsilon)^\alpha}\right\} \to 1 \text{ as } k \to \infty,$$

i.e., (4.27) holds for j.

Our final task is to prove that (4.20) holds when $(s, p) \in H \setminus \{(\alpha, \infty)\}$, i.e., if

$$\alpha < s < 1, \; 1 < p < \infty \text{ and } (s, p) \in H. \tag{4.33}$$

This will be done in the next two steps of the proof.

Let us note that

$$[\alpha < s < 1, \; 1 < p < \infty \text{ and } (s, p) \in H] \Longleftrightarrow [\alpha < s < 1, \; 1 < p < \infty \text{ and } \alpha(p-1) = sp - 1]. \tag{4.34}$$

Step 4. Proof of the lower bound in (4.20) *when* $p < \infty$. More specifically, we will prove the following. Let (s, p) satisfy (4.33). Then

$$\liminf_{k \to \infty} |w_j^k|_{W^{s,p}}^p \ge Cj, \; \forall j \ge 1, \text{ for some } C > 0. \tag{4.35}$$

The proof is by induction on $j \ge 1$.

Step 4.1. Proof of (4.35) *when* $j = 1$. The starting point is the inequality

$$\begin{aligned} |w_1^k|_{W^{s,p}}^p \ge S^k &:= \sum_{1 \le \ell < m \le k} \iint_{J_\ell^k \times J_m^k} \frac{(w_1^k(y) - w_1^k(x))^p}{(y-x)^{1+sp}} \, dxdy \\ &= \sum_{1 \le \ell < m \le k} (m-\ell)^p \varepsilon^{\alpha p} \iint_{J_\ell^k \times J_m^k} \frac{1}{(y-x)^{1+sp}} \, dxdy. \end{aligned} \tag{4.36}$$

Noting that

$$y - x \le (m - \ell + 1)\varepsilon^\alpha \le 2(m-\ell)\varepsilon^\alpha,\ \forall\, 1 \le \ell < m \le k,\ \forall\, x \in J^k_\ell,\ \forall\, y \in J^k_m,$$

and that, for large k (and thus for small ε) we have $|J^k_\ell| \ge \varepsilon^\alpha/2$, we find that

$$\begin{aligned}\liminf_{k\to\infty} S^k &\ge \liminf_{k\to\infty} \underbrace{2^{-sp-3}}_{C_1} \varepsilon^{\alpha(p-sp+1)} \sum_{1\le\ell<m\le k} (\underbrace{m-\ell}_{n})^{p-sp-1}\\ &= C_1 \liminf_{k\to\infty} \varepsilon^{\alpha(p-sp+1)} \sum_{1\le n\le k-1} (k-n)n^{p-sp-1}\\ &\ge \frac{C_1}{2} \liminf_{k\to\infty} k\varepsilon^{\alpha(p-sp+1)} \sum_{1\le n\le k/2} n^{p-sp-1}\\ &= C \liminf_{k\to\infty} \varepsilon^{\alpha(p-sp+1)} k^{p-sp+1} = C \liminf_{k\to\infty} [k\varepsilon^\alpha]^{p-sp+1} = C.\end{aligned} \tag{4.37}$$

In the last line, we use successively the fact that $p - sp - 1 > -1$ and thus

$$\sum_{1\le n\le k/2} n^{p-sp-1} \sim ck^{p-sp} \text{ as } k \to \infty \text{ for some constant } c > 0,$$

resp. the equality $k\varepsilon^\alpha = 1$.

This completes the proof of (4.35) when $j = 1$.

Let us note that this first induction step does not use the fact that $(s, p) \in H$ (but the next one does).

Step 4.2. Proof of (4.35) *when* $j \ge 2$. Assume that (4.35) holds for $j - 1$, with C the constant in (4.37). Then we estimate, using the analytical definition (4.25) of w^k_j in terms of w^k_{j-1} and the scaling of the semi-norm $|\ |_{W^{s,p}}$,

$$\begin{aligned}\liminf_{k\to\infty} |w^k_j|^p_{W^{s,p}} &\ge \liminf_{k\to\infty} S^k + \liminf_{k\to\infty} \sum_{1\le\ell\le k} |w^k_j|^p_{W^{s,p}(I^k_\ell)}\\ &\ge C + \liminf_{k\to\infty} k\varepsilon^{\alpha p-sp+1} |w^k_{j-1}|^p_{W^{s,p}}\\ &\ge C + C(j-1) \liminf_{k\to\infty} k\varepsilon^{\alpha p-sp+1} \ge C + C(j-1) = Cj;\end{aligned} \tag{4.38}$$

here, we rely on the fact that $k = \varepsilon^{-\alpha}$ and thus

$$k\varepsilon^{\alpha p-sp+1} = \varepsilon^{-\alpha+\alpha p-sp+1} = \varepsilon^{\alpha(p-1)-(sp-1)} = 1 \text{ (using (4.34))}. \tag{4.39}$$

This completes the proof of (4.35).

Step 5. Proof of the upper bound in (4.20) *when* $p < \infty$. Let (s, p) satisfy (4.33). We will prove that

$$\limsup_{k\to\infty} |w^k_j|^p_{W^{s,p}} \le C' j,\ \forall\, j \ge 1,\ \text{for some } C' > 0. \tag{4.40}$$

As in Step 4, the proof is by induction on j. It will be convenient to prove a slightly stronger assertion. We extend w^k_j to $(-1, 2)$ by setting $w^k_j(x) = \begin{cases} 0, & \text{if } x \le 0\\ 1, & \text{if } x \ge 1\end{cases}$.

We will prove by induction on j that for every (s, p) satisfying (4.33) we have

$$\limsup_{k\to\infty} |w^k_j|^p_{W^{s,p}((-1,2))} \le C'' j,\ \forall\, j \ge 1,\ \text{for some } C'' > 0. \tag{4.41}$$

Step 5.1. Proof of (4.41) *when* $j = 1$. Set $J^k_0 := (-\varepsilon^\alpha + \varepsilon, 0]$, $J^k_{k+1} := [1, 1 + \varepsilon^\alpha - \varepsilon)$, $I^k_0 := \emptyset$, $I^k_{k+1} := \emptyset$, $I^k_{k+2} := \emptyset$, and

$$A^k := (-\varepsilon^\alpha + \varepsilon, 1 + \varepsilon^\alpha - \varepsilon) = I^k_0 \cup J^k_0 \cup I^k_1 \cup J^k_1 \cup \ldots \cup J^k_{k-1} \cup I^k_k \cup J^k_k \cup I^k_{k+1} \cup J^k_{k+1} \cup I^k_{k+2}.$$

We have

$$|w_1^k|^p_{W^{s,p}((-1,2))} \le 2\left(\sum_{\ell=1}^k T_\ell^k + \sum_{\ell=1}^k U_\ell^k + \sum_{1\le \ell<m\le k+1} V_{\ell,m}^k + \sum_{0\le m<\ell\le k+1} Z_{m,\ell}^k + P^k + Q^k + R^k\right), \tag{4.42}$$

where

$$T_\ell^k := \iint\limits_{I_\ell^k\times I_\ell^k} \frac{|w_1^k(y)-w_1^k(x)|^p}{|y-x|^{1+sp}}\,dxdy,\ U_\ell^k := \iint\limits_{I_\ell^k\times(J_{\ell-1}^k\cup J_\ell^k\cup I_{\ell+1}^k)} \dots dxdy,$$

$$V_{\ell,m}^k := \iint\limits_{(J_{\ell-1}^k\cup I_\ell^k\cup J_\ell^k)\times(J_m^k\cup I_{m+1}^k)} \dots dxdy,\ Z_{m,\ell}^k := \iint\limits_{(I_m^k\cup J_m^k)\times(J_{\ell-1}^k\cup I_\ell^k\cup J_\ell^k)} \dots dxdy,$$

$$P^k := \iint\limits_{(-1,-\varepsilon^\alpha+\varepsilon)\times A^k} \dots dxdy,\ Q^k := \int\limits_{A^k\times(1+\varepsilon^\alpha-\varepsilon,2)} \dots dxdy,\ R^k := \iint\limits_{(-1,-\varepsilon^\alpha+\varepsilon)\times(1+\varepsilon^\alpha-\varepsilon,2)} \dots dxdy.$$

By scaling and the relation $k\varepsilon^{\alpha p-sp+1}=1$ (see (4.39)), we have

$$T_\ell^k = c_1\varepsilon^{\alpha p-sp+1},\ \text{and thus}\ \sum_{\ell=1}^k T_\ell^k = c_1 k\varepsilon^{\alpha p-sp+1} = c_1 \text{ for some } c_1>0. \tag{4.43}$$

By symmetry and scaling, we have

$$\begin{aligned}
U_\ell^k \le& 2\iint\limits_{(0,\varepsilon)\times(\varepsilon,\varepsilon^\alpha)} \varepsilon^{\alpha p}\frac{(1-x/\varepsilon)^p}{(y-x)^{1+sp}}\,dxdy + 2^p\varepsilon^{\alpha p}\iint\limits_{(0,\varepsilon)\times(\varepsilon^\alpha,\varepsilon^\alpha+\varepsilon)} \frac{1}{(y-x)^{1+sp}}\,dxdy\\
=&2\varepsilon^{\alpha p-sp+1}\iint\limits_{(0,1)\times(1,\varepsilon^{\alpha-1})} \frac{(1-X)^p}{(Y-X)^{1+sp}}\,dXdY\\
&+2^p\varepsilon^{\alpha p-sp+1}\iint\limits_{(0,1)\times(\varepsilon^{\alpha-1},\varepsilon^{\alpha-1}+1)} \frac{1}{(Y-X)^{1+sp}}\,dXdY\\
\le& c_2\varepsilon^{\alpha p-sp+1}\left(\int_0^1 (1-X)^{p-sp}\,dX+1\right) = c_3\varepsilon^{\alpha p-sp+1} \text{ for some } c_2,\ c_3>0.
\end{aligned} \tag{4.44}$$

As in (4.43), using (4.44) we obtain

$$\sum_{\ell=1}^k U_\ell^k \le c_3. \tag{4.45}$$

We next estimate $V_{\ell,m}^k$. The estimate of $Z_{m,\ell}^k$ is similar and will not be detailed.
Assume first that $\ell<m-1$. If $x\in J_{\ell-1}^k\cup I_\ell^k\cup J_\ell^k$ and $y\in J_m^k\cup I_{m+1}^k$ with $\ell<m-1$, then

$$|w_1^k(y)-w_1^k(x)| = w_1^k(y)-w_1^k(x) \le (m-\ell+2)\varepsilon^\alpha \le 3(m-\ell)\varepsilon^\alpha,$$
$$|y-x| = y-x \ge (m-\ell-1)\varepsilon^\alpha \ge (m-\ell)\varepsilon^\alpha/2.$$

For such ℓ, m, we find that

$$V_{\ell,m}^k \le c_4\frac{(m-\ell)^p\varepsilon^{\alpha p+2\alpha}}{(m-\ell)^{1+sp}\varepsilon^{\alpha(1+sp)}} = c_4(m-\ell)^{p-sp-1}\varepsilon^{\alpha(p-sp+1)}. \tag{4.46}$$

Arguing as in the proof of (4.37), we find that

$$\begin{aligned}\limsup_{k\to\infty}\sum_{1\le\ell<m-1\le k} V^k_{\ell,m} &\le c_4\varepsilon^{\alpha(p-sp+1)}\sum_{n=1}^{k}(k-n)n^{p-sp-1}\\ &\le c_4k\varepsilon^{\alpha(p-sp+1)}\sum_{n=1}^{k}n^{p-sp-1}\le c_5k^{p-sp+1}\varepsilon^{\alpha(p-sp+1)}=c_5.\end{aligned}\tag{4.47}$$

We now estimate $V^k_{\ell,\ell+1}$ with $1\le\ell\le k$. If $x\in J^k_{\ell-1}\cup I^k_\ell\cup J^k_\ell$ and $y\in J^k_{\ell+1}\cup I^k_{\ell+1}$, then

$$|w^k_1(y)-w^k_1(x)|=w^k_1(y)-w^k_1(x)\le 3\varepsilon^\alpha$$

and thus

$$\begin{aligned}V^k_{\ell,\ell+1}&\le c_6\varepsilon^{\alpha p}\iint\limits_{(-2\varepsilon^\alpha,0)\times(\varepsilon,\varepsilon^\alpha+\varepsilon)}\frac{1}{(y-x)^{1+sp}}\,dxdy\\ &=c_6\varepsilon^{\alpha p-sp+1}\iint\limits_{(-2\varepsilon^{\alpha-1},0)\times(1,\varepsilon^{\alpha-1}+1)}\frac{1}{(Y-X)^{1+sp}}\,dXdY\\ &\le c_7\varepsilon^{\alpha p-sp+1}\int_1^{\varepsilon^{\alpha-1}+1}\frac{1}{Y^{sp}}\,dX\le c_8\varepsilon^{\alpha p-sp+1}\ (\text{since } sp>1).\end{aligned}\tag{4.48}$$

This implies, as above, that

$$\sum_{\ell=1}^{k}V^k_{\ell,\ell+1}\le c_8.\tag{4.49}$$

The estimates of P^k, Q^k and R^k are straightforward, using the fact that

$$(\ell-1)\varepsilon^\alpha\le w^k_j(x)\le\ell\varepsilon^\alpha,\ \forall k,\ \forall j,\ \forall\ell\in\{1,\dots,k\},\ \forall x\in[(\ell-1)\varepsilon^\alpha,\ell\varepsilon^\alpha].\tag{4.50}$$

We find for example that, for large k (such that $\varepsilon^{1-\alpha}<1/2$),

$$\begin{aligned}P^k&\le\sum_{\ell=1}^{k+1}\ell^p\varepsilon^{\alpha p}\iint\limits_{(-1,-\varepsilon^\alpha+\varepsilon)\times((\ell-1)\varepsilon^\alpha,\ell\varepsilon^\alpha)}\frac{1}{(y-x)^{1+sp}}\,dxdy\\ &=\varepsilon^{\alpha(p-sp+1)}\sum_{\ell=1}^{k+1}\ell^p\iint\limits_{(-\varepsilon^{-\alpha},-1+\varepsilon^{1-\alpha})\times(\ell-1,\ell)}\frac{1}{(Y-X)^{1+sp}}\,dXdY\\ &\le\varepsilon^{\alpha(p-sp+1)}\sum_{\ell=1}^{k+1}\ell^p\iint\limits_{(-\infty,-1/2)\times(\ell-1,\ell)}\frac{1}{(Y-X)^{1+sp}}\,dXdY\\ &\le c_9\varepsilon^{\alpha(p-sp+1)}\sum_{\ell=1}^{k+1}\frac{\ell^p}{(\ell-1/2)^{sp}}\le 2^{sp}c_9\varepsilon^{\alpha(p-sp+1)}\sum_{\ell=1}^{k+1}\ell^{p-sp}\\ &\le c_{10}\varepsilon^{\alpha(p-sp+1)}k^{p-sp+1}\ (\text{since } p-sp>-1)\ =c_{10}[k\varepsilon^\alpha]^{p-sp+1}=c_{10}.\end{aligned}\tag{4.51}$$

Similarly we have

$$Q^k\le c_{10}.\tag{4.52}$$

On the other hand,

$$R^k \le \iint\limits_{(-1,0)\times(1,2)} \frac{1}{(y-x)^{1+sp}}\,dxdy = c_{11} < \infty. \tag{4.53}$$

Combining (4.42), (4.43), (4.45), (4.47), (4.49) and the analogues of (4.47) and (4.49) for $Z^k_{m,\ell}$ and (4.51)–(4.53), we find that

$$\limsup_{k\to\infty} |w^k_1|^p_{W^{s,p}((-1,2))} := c_{12} < \infty. \tag{4.54}$$

Step 5.2. Proof of (4.41) *when* $j \ge 2$. Consider the segments

$$L^k_\ell := [(\ell-1)\varepsilon^\alpha - \varepsilon, (\ell-1)\varepsilon^\alpha + 2\varepsilon] = \{(\ell-1)\varepsilon^\alpha\} + \varepsilon(-1,2),\ \ell = 1,\dots,k. \tag{4.55}$$

Using (4.55) and the definition (4.25) of w^k_j in terms of w^k_{j-1}, we find that

$$|w^k_j|^p_{W^{s,p}(L^k_\ell)} = \varepsilon^{\alpha p - sp + 1}|w^k_{j-1}|^p_{W^{s,p}((-1,2))},\ \forall j \ge 2,\ \forall k, \tag{4.56}$$

and thus (by (4.39))

$$\sum_{\ell=1}^k |w^k_j|^p_{W^{s,p}(L^k_\ell)} = k\varepsilon^{\alpha p - sp+1}|w^k_{j-1}|^p_{W^{s,p}((-1,2))} = |w^k_{j-1}|^p_{W^{s,p}((-1,2))}. \tag{4.57}$$

Clearly, by (4.34),

$$|w^k_j|^p_{W^{s,p}((-1,2))} \le \sum_{\ell=1}^k |w^k_j|^p_{W^{s,p}(L^k_\ell)} + 2\underbrace{\iint\limits_{((-1,2)\setminus\cup_{\ell=1}^k L^k_\ell)\times(\cup_{\ell=1}^k L^k_\ell)} \frac{|w^k_j(y)-w^k_j(x)|^p}{|y-x|^{1+sp}}\,dxdy}_{Y^k_j}. \tag{4.58}$$

Assume for the moment that

$$\limsup_{k\to\infty} Y^k_j \le c_{13} \text{ for some } c_{13} < \infty \text{ independent of } j \ge 2. \tag{4.59}$$

Combining (4.54) with (4.57)–(4.59), we obtain by induction on j that (4.41) holds with $C'' := \max\{c_{12}, 2c_{13}\}$.

It remains to prove (4.59). The proof is very similar to the one of (4.54), and we do not provide all the details.

Set, for each $\ell \in \{1,\dots,k\}$, $M^k_\ell := J^k_\ell \setminus \cup_{m=1}^k L^k_m$. For large k, we have $M^k_\ell = (2\varepsilon + (\ell-1)\varepsilon^\alpha, \ell\varepsilon^\alpha - \varepsilon)$. We split Y^k_j as follows:

$$\begin{aligned} Y^k_j = &\sum_{\ell=1}^k\sum_{m=1}^k \underbrace{\iint\limits_{M^k_m\times L^k_\ell} \frac{|w^k_j(y)-w^k_j(x)|^p}{|y-x|^{1+sp}}\,dxdy}_{B^k_{\ell,m}} + \underbrace{\iint\limits_{(-1,-\varepsilon^\alpha+\varepsilon)\times(\cup_{\ell=1}^k L^k_\ell)} \dots\,dxdy}_{P^k} \\ &+ \underbrace{\iint\limits_{(\cup_{\ell=1}^k L^k_\ell)\times(1+\varepsilon^\alpha-\varepsilon,2)} \dots\,dxdy}_{Q^k} + \underbrace{\iint\limits_{(-1,-\varepsilon^\alpha+\varepsilon)\times(1+\varepsilon^\alpha-\varepsilon,2)} \dots\,dxdy}_{R^k}. \end{aligned} \tag{4.60}$$

One of the crucial ingredients in the proof of (4.59) is the fact that the sets L^k_ℓ and M^k_ℓ do not depend on j. We will combine this fact with j-independent estimates of the quantity $|w^k_j(y) - w^k_j(x)|^p$; this will lead to j-independent estimates of the integrals in (4.60) and to the desired conclusion (4.59).

When $m > \ell$ or $m < \ell - 1$, we estimate $B^k_{\ell,m}$ as we did for $V^k_{\ell,m}$ when $\ell < m-1$. As in (4.46), we find that

$$B^k_{\ell,m} \le c_4|m-\ell|^{p-sp-1}\varepsilon^{\alpha(p-sp+1)},\ \forall j \ge 2,\ \forall \ell, m \in \{1,\dots,k\} \text{ such that } m > \ell \text{ or } m < \ell - 1. \tag{4.61}$$

As in (4.47), this leads to

$$\sum_{m>\ell \text{ or } m<\ell-1} B^k_{\ell,m} \le 2c_5. \tag{4.62}$$

The estimates of $B^k_{\ell,\ell}$ and $B^k_{\ell,\ell-1}$ are similar to the one of $V^k_{\ell,\ell+1}$. For example, in order to estimate $B^k_{\ell,\ell}$ we take advantage of the fact that w^k_j is constant on J^k_ℓ and find, as in (4.48), that

$$\begin{aligned} B^k_{\ell,\ell} &= \iint\limits_{((\ell-1)\varepsilon^\alpha-\varepsilon,(\ell-1)\varepsilon^\alpha)\times((\ell-1)\varepsilon^\alpha+2\varepsilon,\ell\varepsilon^\alpha)} \frac{|w^k_j(y)-w^k_j(x)|^p}{|y-x|^{1+sp}}\,dxdy \\ &\le \varepsilon^{\alpha p} \iint\limits_{(-\varepsilon,0)\times(2\varepsilon,\varepsilon^\alpha)} \frac{1}{(y-x)^{1+sp}}\,dxdy \le c_8\varepsilon^{\alpha p-sp+1}. \end{aligned}$$

We are led to

$$\sum_{m=\ell-1 \text{ or } \ell} B^k_{\ell,m} \le 2c_9. \tag{4.63}$$

Finally, exactly as in (4.51)–(4.53) we have

$$P^k \le c_{10}, \;\; Q^k \le c_{10} \tag{4.64}$$

and

$$R^k \le c_{11}. \tag{4.65}$$

Combining (4.62)–(4.65), we obtain (4.59).

The proof of Lemma 4.1 is complete. □

5. Gagliardo–Nirenberg inequalities in Triebel–Lizorkin and Besov spaces

In the first part of this section, we recall the definition of these spaces. We next investigate the validity of the Gagliardo–Nirenberg inequalities in such functional settings. As we have already seen in the proof of Theorem 1, part A, part of the corresponding analysis is relevant for Sobolev spaces.

We start by recalling the (most commonly used) Littlewood–Paley decomposition of a temperate distribution.

Definition 5.1. Let $\psi \in C^\infty_c(\mathbb{R}^N)$ be such that $\psi = 1$ in $B_{4/3}(0)$ and $\operatorname{supp}\psi \subset B_{3/2}(0)$. Define

$$\psi_0 = \psi \text{ and, for } j \ge 1,\, \psi_j(x) := \psi(x/2^j) - \psi(x/2^{j-1}). \tag{5.1}$$

Set $\varphi_j := \mathscr{F}^{-1}\psi_j \in \mathscr{S}$.[5] Then for each temperate distribution f we have

$$f = \sum_{j=0}^{\infty} f_j \text{ in } \mathscr{S}',\; \text{ with } f_j := f * \varphi_j. \tag{5.2}$$

$f = \sum_{j=0}^{\infty} f_j$ is *"the" Littlewood–Paley decomposition* of $f \in \mathscr{S}'$.

Note that $\mathscr{F} f_j = \psi_j \mathscr{F} f$ is compactly supported, and therefore $f_j \in C^\infty$ for each j.

Definition 5.2. Starting from the Littlewood–Paley decomposition, we define the *Triebel–Lizorkin spaces* $F^s_{p,q} = F^s_{p,q}(\mathbb{R}^N)$ as follows: for $s \ge 0$, $1 \le p, q \le \infty$, we let

[5] Equivalently, we have $\varphi_0 = \mathscr{F}^{-1}\psi$ and, for $j \ge 1$, $\varphi_j(x) = 2^{Nj}\varphi_0(2^j x) - 2^{N(j-1)}\varphi_0(2^{j-1}x)$.

$$\|f\|_{F^s_{p,q}} := \left\| \left\| \left(2^{sj} f_j(x)\right)_{j\ge 0} \right\|_{l^q(\mathbb{N})} \right\|_{L^p(\mathbb{R}^N)}, \quad F^s_{p,q} := \{f \in \mathscr{S}';\ \|f\|_{F^s_{p,q}} < \infty\}. \tag{5.3}$$

This definition has to be changed when $p = \infty$ and $1 < q < \infty$ [13, p. 9], but this case will not be considered in what follows.

Most of the Sobolev spaces can be identified with Triebel–Lizorkin spaces [15, Section 2.3.5], [13, Section 2.1.2].

Proposition 5.3. *The following equalities of spaces hold, with equivalence of norms:*

1. *If $s > 0$ is not an integer and $1 \le p \le \infty$, then $W^{s,p}(\mathbb{R}^N) = F^s_{p,p}(\mathbb{R}^N)$.*
2. *If $s \ge 0$ is an integer and $1 < p < \infty$, then $W^{s,p}(\mathbb{R}^N) = F^s_{p,2}(\mathbb{R}^N)$.*

When $s \ge 0$ is an integer and either $p = 1$ or $p = \infty$, the Sobolev space $W^{s,p}$ cannot be identified with a Triebel–Lizorkin space.

Remark 5.4. By Definition 2.7 and Proposition 5.3, ordinary Sobolev spaces in the sense of Definition 2.7 are precisely the Sobolev spaces $W^{s,p}$ which can be identified with a Triebel–Lizorkin space.

Reversing the roles of ℓ^q and L^p in (5.3), we obtain the Besov spaces.

Definition 5.5. We define the *Besov spaces* $B^s_{p,q} = B^s_{p,q}(\mathbb{R}^N)$ as follows: for $s \ge 0$, $1 \le p, q \le \infty$, we let

$$\|f\|_{B^s_{p,q}} := \left\| \left(\left\| 2^{sj} f_j \right\|_{L^p(\mathbb{R}^N)} \right)_{j\ge 0} \right\|_{l^q(\mathbb{N})}, \quad B^s_{p,q} := \{f \in \mathscr{S}';\ \|f\|_{B^s_{p,q}} < \infty\}. \tag{5.4}$$

By Proposition 5.3 item 1, when $s > 0$ is not an integer and $1 \le p \le \infty$ we have $W^{s,p} = B^s_{p,p}$.

Given $s_1, \dots p_2$ satisfying (1.3) and (1.6), we discuss the validity of the following analogues of (1.4):

$$\|f\|_{F^s_{p,q}} \lesssim \|f\|^{\theta}_{F^{s_1}_{p_1,q_1}} \|f\|^{1-\theta}_{F^{s_2}_{p_2,q_2}}, \forall f \in F^{s_1}_{p_1,q_1} \cap F^{s_2}_{p_2,q_2}, \tag{5.5}$$

respectively

$$\|f\|_{B^s_{p,q}} \lesssim \|f\|^{\theta}_{B^{s_1}_{p_1,q_1}} \|f\|^{1-\theta}_{B^{s_2}_{p_2,q_2}}, \forall f \in B^{s_1}_{p_1,q_1} \cap B^{s_2}_{p_2,q_2}. \tag{5.6}$$

It turns out that the analysis of (5.5) and (5.6) is much easier than the one of (1.4).

In the scale of Triebel–Lizorkin spaces, we have the following remarkable result due to Oru [11] (unpublished); for a proof, see [3, Lemma 3.1 and Section III].

Proposition 5.6. *Let $s_1, \dots, p_2$ satisfy (1.3) and (1.6). Then for every $q_1, q_2, q \in [1, \infty]$ we have*

$$\|f\|_{F^s_{p,q}} \lesssim \|f\|^{\theta}_{F^{s_1}_{p_1,q_1}} \|f\|^{1-\theta}_{F^{s_2}_{p_2,q_2}}, \forall f \in F^{s_1}_{p_1,q_1} \cap F^{s_2}_{p_2,q_2}. \tag{5.7}$$

[If one of the p_1, p_2, p equals ∞, then the corresponding q has to be > 1.]

We emphasize the fact that the values of q_1, q_2, q are irrelevant for the validity of (5.7).

Combining Propositions 2.3, 5.3 and 5.6, we obtain the following

Corollary 5.1. *Let $s_1, \dots, p_2$ satisfy (1.3) and (1.6). Let Ω be a standard domain in $\mathbb{R}^N$. If W^{s_1,p_1}, $W^{s,p}$ and W^{s_2,p_2} are ordinary Sobolev spaces, then*

$$\|f\|_{W^{s,p}(\Omega)} \lesssim \|f\|^{\theta}_{W^{s_1,p_1}(\Omega)} \|f\|^{1-\theta}_{W^{s_2,p_2}(\Omega)}, \forall f \in W^{s_1,p_1}(\Omega) \cap W^{s_2,p_2}(\Omega). \tag{5.8}$$

Corollary 5.2. *Inequality (1.10) holds when $p_1 = \infty$ and $W^{s,p}$, W^{s_2,p_2} are ordinary, resp. $p_2 = \infty$ and W^{s_1,p_1}, $W^{s,p}$ are ordinary.*

Proof of Corollary 5.2. Assume e.g. that we are in the first case; the other one is similar. If s_1 in not an integer, then all the three spaces are ordinary, and we are done, by the previous corollary. When s_1 is an integer, we rely on the (well-known) embedding

$$W^{k,\infty}(\mathbb{R}^N) \hookrightarrow F^k_{\infty,\infty}(\mathbb{R}^N),\ \forall k \in \mathbb{N},\ \forall N, \tag{5.9}$$

whose proof we sketch here. Let $f \in W^{k,\infty}(\mathbb{R}^N)$ and let f_j be as in (5.2). Then

$$\|f\|_{F^k_{\infty,\infty}} = \sup_{j\ge 0} 2^{kj}\|f_j\|_{L^\infty} = \max\left\{\|f_0\|_{L^\infty}, \sup_{j\ge 1} 2^{kj}\|f_j\|_{L^\infty}\right\} \lesssim \max\left\{\|f\|_{L^\infty}, \|f^{(k)}\|_{L^\infty}\right\} \approx \|f\|_{W^{k,\infty}};$$

for the justification of the last inequality (via "direct" and "reverse" Nikolski'i inequalities) see e.g. [4, Lemma 2.1.1].

By Proposition 5.6 and (5.9), we have

$$\|u\|_{W^{s,p}} \lesssim \|u\|^{\theta}_{F^{s_1}_{\infty,\infty}} \|u\|^{1-\theta}_{W^{s_2,p_2}} \lesssim \|u\|^{\theta}_{W^{s_1,\infty}} \|u\|^{1-\theta}_{W^{s_2,p_2}},$$

and thus (1.10) holds. □

In the scale of Besov spaces, we have the following result.

Proposition 5.7. *Let $s_1, \ldots, p_2$ satisfy (1.3) and (1.6). Then we have*

$$\|f\|_{B^s_{p,q}} \lesssim \|f\|^{\theta}_{B^{s_1}_{p_1,q_1}} \|f\|^{1-\theta}_{B^{s_2}_{p_2,q_2}},\ \forall f \in B^{s_1}_{p_1,q_1} \cap B^{s_2}_{p_2,q_2} \tag{5.10}$$

if and only if

$$\frac{1}{q} \le \frac{\theta}{q_1} + \frac{(1-\theta)}{q_2}. \tag{5.11}$$

Proof. Assume first that (5.11) holds. Let $\widetilde{q}$ satisfy $\dfrac{1}{\widetilde{q}} = \dfrac{\theta}{q_1} + \dfrac{1-\theta}{q_2}$. By (5.11), we have $q \ge \widetilde{q}$, and thus $\ell^{\widetilde{q}} \hookrightarrow \ell^q$. Using twice Hölder's inequality, we find that

$$\begin{aligned}\|f\|_{B^s_{p,q}} \le \|f\|_{B^s_{p,\widetilde{q}}} &\le \left\|\left(\left\|2^{s_1 j} f_j\right\|^{\theta}_{L^{p_1}(\mathbb{R}^N)} \left\|2^{s_2 j} f_j\right\|^{1-\theta}_{L^{p_2}(\mathbb{R}^N)}\right)_{j\ge 0}\right\|_{l^{\widetilde{q}}(\mathbb{N})} \\ &\le \left\|\left(\left\|2^{s_1 j} f_j\right\|_{L^{p_1}(\mathbb{R}^N)}\right)_{j\ge 0}\right\|^{\theta}_{l^{q_1}(\mathbb{N})} \left\|\left(\left\|2^{s_2 j} f_j\right\|_{L^{p_2}(\mathbb{R}^N)}\right)_{j\ge 0}\right\|^{1-\theta}_{l^{q_2}(\mathbb{N})} = \|f\|^{\theta}_{B^{s_1}_{p_1,q_1}} \|f\|^{1-\theta}_{B^{s_2}_{p_2,q_2}}.\end{aligned}$$

Conversely, assume that (5.10) holds. Let ψ_j be as in (5.1). Then $\psi_j \equiv 0$ in $B(0,3/2) \setminus \overline{B}(0,4/3)$, $\forall j \ge 2$, while $\psi_1 \not\equiv 0$ in $B(0,3/2) \setminus \overline{B}(0,4/3)$. Consider some $g \in C^\infty_c(B(0,3/2) \setminus \overline{B}(0,4/3))$ such that $g\psi_1 \not\equiv 0$, and let $h := \mathscr{F}^{-1} g$. By our choice of g, we have

$$h * \varphi_1 = \mathscr{F}^{-1}(g\psi_1) \not\equiv 0. \tag{5.12}$$

Define $f^m := \sum_{j=m}^{2m} \alpha_j h(2^j \cdot)$, $\forall m \ge 2$. The numbers $\alpha_j > 0$ will be chosen later. It follows from the definition of h that for every $m \ge 2$ and $j \ge 0$ we have

$$f^m * \varphi_j = \begin{cases} \alpha_j h(2^j \cdot) * \varphi_j = \alpha_j (h * \varphi_1)(2^j \cdot), & \text{if } m \le j \le 2m \\ 0, & \text{otherwise} \end{cases}. \tag{5.13}$$

For $\widetilde{s} \ge 0$ and $1 \le \widetilde{p}, \widetilde{q} \le \infty$, we find using (5.12) and (5.13) that

$$\|f^m * \varphi_j\|_{L^{\widetilde{p}}(\mathbb{R}^N)} \approx 2^{-Nj/\widetilde{p}} \alpha_j,\ \forall m \le j \le 2m, \text{ and } \|f^m\|_{B^{\widetilde{s}}_{\widetilde{p},\widetilde{q}}} \approx \left\|\left(2^{(\widetilde{s}-N/\widetilde{p})j}\alpha_j\right)_{j=m}^{2m}\right\|_{l^{\widetilde{q}}}. \tag{5.14}$$

We now let b be such that

$$b < \min\left\{-s + \frac{N}{p}, -s_1 + \frac{N}{p_1}, -s_2 + \frac{N}{p_2}\right\} \tag{5.15}$$

and set $\alpha_j := j2^{bj}$.

It follows from (5.14) and (5.15) that

$$\|f^m\|_{B^s_{p,q}} \approx m^{1/q} 2^{m(s-N/p+b)}, \ \|f^m\|_{B^{s_j}_{p_j,q_j}} \approx m^{1/q_j} 2^{m(s_j-N/p_j+b)}, \ j = 1, 2. \tag{5.16}$$

Combining (5.10) and (5.16) and letting $m \to \infty$, we find that (5.11) holds. □

Remark 5.8. Triebel–Lizorkin $F^s_{p,q}$ and Besov spaces $B^s_{p,q}$ are defined when $s \in \mathbb{R}$ and $0 < p, q \le \infty$.[6] It is easy to see that Propositions 5.6 and 5.7 hold when $-\infty < s_1 < s < s_2 < \infty$, $0 < p_1, p, p_2 \le \infty$ and $0 < q, q_1, q_2 \le \infty$.

Conflict of interest statement

There is no conflict of interest.

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[6] With the exception of $F^s_{\infty,q}$, where one has to take $1 < q \le \infty$.