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A quasi-monotonicity formula and partial regularity for borderline solutions to a parabolic equation

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Abstract

A quasi-monotonicity formula for the solution to a semilinear parabolic equation $u_t = \Delta u + V(x)|u|^{p-1}u$, $p > (N+2)/(N-2)$ in $\Omega \times (0,T)$ with 0-Dirichlet boundary condition is obtained. As an application, it is shown that for some suitable global weak solution u and any compact set $Q \subset \Omega \times (0,T)$ there exists a close subset $Q' \subset Q$ such that u is continuous in Q' and the $(N - \frac{4}{p-1})$-dimensional parabolic Hausdorff measure $\mathcal{H}^{(N-\frac{4}{p-1})}(Q \setminus Q')$ of $Q \setminus Q'$ is finite.

Keywords: Quasi-monotonicity formula; Partial regularity; Borderline solutions; Semilinear parabolic equations; Potential

1. Introduction

In this paper, we are interested in the following semilinear parabolic problem

$$\begin{cases} u_t = \Delta u + V(x)|u|^{p-1}u & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ is a bounded, smooth domain, $p > \frac{N+2}{N-2}$, $u_0 \in L^\infty(\Omega)$, and the potential $V \in C^1(\bar{\Omega})$ satisfies $V(x) \geqslant c$ for some positive constant c and all $x \in \Omega$. It is well known that for any $u_0 \in L^\infty(\Omega)$ problem (1.1) has a unique local in time solution. Specially, if the L^∞-norm of the initial datum is small enough, then (1.1) has a global, classical solution, while the solution to (1.1) ceases to exist after some time $T > 0$ and $\lim_{t \uparrow T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$ provided that the initial datum u_0 is large in some suitable sense. In the latter case we call the solution u to (1.1) blowing up in finite time and T the blow-up time.

When $V \equiv 1$, problem (1.1) is one of the parabolic problems that have been studied extensively in the past. See for example, [1,2,4,8,10–17,19]. Consider (1.1) with initial data of the form $u_0 = \lambda\varphi$ where λ is a positive number and φ is a fixed non-negative function in $L^\infty(\Omega)$ which does not vanish almost everywhere. For large λ, the energy of $\lambda\varphi$ is negative, so the (maximal) solution, u_λ, blows up in finite time. When λ is small, the solution is global and decays to zero at infinity. It is natural to set

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$$\lambda^*(\varphi) \equiv \sup\bigl\{\lambda > 0\text{: The solution } u_\lambda \text{ satisfying } u_\lambda(0) = \lambda\varphi \text{ is global and decays to zero as } t \to \infty\bigr\},$$

and define

$$U_\varphi(x,t) \equiv \lim_{\lambda \uparrow \lambda^*} u_\lambda(x,t). \tag{1.2}$$

Note that by the maximum principle, u_λ is monotone increasing and U_φ coincides with u_{λ^*} on the maximal interval of existence of the latter. In the following we shall not distinguish U_φ and u_{λ^*}, and we call it (positive) borderline solution.

In [24], Ni, Sacks and Tavantzis had examined the properties of this borderline solution for some range of p. Among other things, they have proven the following result under the assumption that Ω is convex:

For $p \geqslant 2^* = (N+2)/(N-2)$, $N \geqslant 3$, u_{λ^*} is a global, L^1-solution to (1.1)*, which must be unbounded.*

The definition of an L^1-solution will be given in Section 1. There was little progress on the critical and supercritical case since then. Considering global, L^1-solutions for radially symmetric and decreasing initial data in a ball, Galaktionov and Vazquez [11] have proven the following results:

(1) *When $p = 2^*$, u_{λ^*} remains bounded for all time and tends to zero uniformly away from the origin as $t \to \infty$ and*
(2) *when $p \in (p^*, p')$, where $p' = (N-4)/(N-10)$, for $N \geqslant 11$ and $p' = \infty$ for $3 \leqslant N \leqslant 10$, u_{λ^*} blows up in finite time.*

Later, Mizoguchi [21] shows that u_{λ^*} blows up in finite time for all supercritical p, that is, the upper bound $(N-4)/(N-10)$ in (2) can be removed. When $2^* < p < \tilde{p}$, where $\tilde{p}$ is the Joseph–Lundgren exponent given by $\tilde{p} = 1 + 4/(N - 4 - 2\sqrt{N-1})$ if $N \geqslant 11$ and $\tilde{p} = \infty$ if $N \leqslant 10$, it is shown in Fila, Matano and Poláčik [9] that the blow-up times of u_{λ^*} form a finite set, which in some cases is a singleton. More information on the corresponding Cauchy problem can be found in [22] and [23].

Recently, we [6] have proven that when Ω is convex the borderline solution u_{λ^*} blows up in finite time and it decays to zero uniformly after some finite time. Moreover, we have established partial regularity theorem for this borderline solution, i.e., there exists a closed set $\mathcal{S}$ in $\Omega \times (0,\infty)$, whose distance to the boundary of $\Omega \times (0,\infty)$ is greater than a positive number and which satisfies $\mathcal{H}^{(N-\frac{4}{p-1})}(\mathcal{S}) = 0$, so that u is continuous in $\Omega \times (0,\infty) \setminus \mathcal{S}$. Here $\mathcal{H}^s(E)$ denotes the s-dimensional Hausdorff measure of the set E with respect to the parabolic metric.

The main purpose of this paper is to improve these results in [6] for more general V, i.e., we will establish the following theorems.

Theorem 1.1. *Consider* (1.1) *where Ω is convex. For any positive, borderline solution u there exists a closed set $\mathcal{S}$ in $\Omega \times (0,\infty)$, whose distance to the boundary of $\Omega \times (0,\infty)$ is greater than a positive number and which satisfies $\mathcal{H}^{(N-\frac{4}{p-1})}(\mathcal{S}) = 0$, so that u is continuous in $\Omega \times (0,\infty) \setminus \mathcal{S}$.*

Theorem 1.2. *Consider* (1.1) *where Ω is convex. Any positive borderline solution must blow up in finite time. After some time, it becomes uniformly bounded and decays to zero as t goes to infinity.*

These results are definitely not a direct consequence of those of [6]. Due to the appearance of the potential V, some extra works should be done. The novelty is to establish a quasi-monotonicity formula for the rescaled local energy and to get the estimates for L^{p+1}-norm of the solution in terms of local energy. When $V = 1$, this quasi-monotonicity formula is almost trivial. When $V \neq 1$, it is not easy. There is a "bad" term

$$\int\limits_{\Omega_s} \left|\frac{\partial \bar{V}}{\partial s}\right| |w|^{p+1} \psi^2 \rho \, dy$$

involved in the derivative of the local energy $\mathcal{E}[w]$ (see e.g. Section 3 for the definition). To eliminate this difficulty, we use a trick similar to what was used in [15] to get the blow-up rate estimate for (1.1). Notice that in [15] the basic assumption is $1 < p < (N+2)/(N-2)$ while in this paper we always assume $p > (N+2)/(N-2)$. Actually, we can get the main estimates for local energy for all $p > 1$ in this paper. To explain more, it is easy to see that

$$\frac{d}{ds}\mathcal{E}[w](s) \leqslant -\frac{1}{2}\int\limits_{\Omega_s} w_s^2\psi^2\rho\,dy + C\int\limits_{\Omega_s}\left|\frac{\partial\bar{V}}{\partial s}\right||w|^{p+1}\psi^2\rho\,dy + C\varphi(s),$$

where φ is an integrable function on $[0,\infty)$ such that $\int_\tau^\infty e^s\varphi(s)\,ds \leqslant Ce^{-\tau}$. Since $\frac{\partial\bar{V}}{\partial s}$ can be written as $\nabla V(x)\cdot ye^{-s/2}$, the integral $\int_{\Omega_s}|\frac{\partial\bar{V}}{\partial s}||w|^{p+1}\psi^2\rho\,dy$ can be controlled by $e^{-s/2}\int_{\Omega_s}|y||w|^{p+1}\psi^2\rho\,dy$. The question is how to estimate the integral $\int_{\Omega_s}|y||w|^{p+1}\psi^2\rho\,dy$. To this end, we introduce

$$\mathcal{E}_{2k}[w](s) = \frac{1}{2}\int\limits_{\Omega_s}\big(|\nabla w|^2+\beta w^2\big)|y|^{2k}\psi^2\rho\,dy - \frac{1}{p+1}\int\limits_{\Omega_s}\bar{V}|w|^{p+1}|y|^{2k}\psi^2\rho\,dy, \quad k\in\mathbb{N}.$$

First, we establish some rough estimates for $\mathcal{E}_{2k}[w]$ using the fact that $\frac{\partial\bar{V}}{\partial s} = \nabla V(x)\cdot ye^{-s/2} = \nabla V(x)\cdot(x-\bar{x})$ is bounded, i.e.,

$$\big|\mathcal{E}_{2k}[w](s)\big| \leqslant M_k e^{2\lambda s}, \qquad \int\limits_0^\infty e^{-2\lambda s}\int\limits_{\Omega_s}|\nabla w|^2|y|^{2k}\psi^2\rho\,dy\,ds \leqslant N_k,$$

for all $k\in\mathbb{N}$ and $s\geqslant 0$. Here M_k, N_k are positive constants depending on k.

Second, by the mathematical induction and $|\frac{\partial\bar{V}}{\partial s}| = |\nabla V(x)\cdot ye^{-s/2}| \leqslant C|y|e^{-s/2}$, we can improve our estimates by at most finite steps to get

$$\big|\mathcal{E}_{2k}[w](s)\big| \leqslant M_k' e^{\alpha s}, \qquad \int\limits_0^\infty e^{-\alpha s}\int\limits_{\Omega_s}|\nabla w|^2|y|^{2k}\psi^2\rho\,dy\,ds \leqslant N_k',$$

for some $\alpha\in(0,1/2)$.

Finally, we get the quasi-monotonicity formula

$$\mathcal{E}[w](s) + \frac{1}{4}\int\limits_\tau^s\int\limits_{\Omega_s} w_s^2\psi^2\rho\,dy\,ds \leqslant \mathcal{E}[w](\tau) + C_3e^{-\delta\tau}, \quad \forall s>\tau\geqslant \underline{s}.$$

Here $\delta\in(0,1/2)$ is a constant. Consequently, we obtain a lower bound estimate

$$\mathcal{E}[w](s) \geqslant -C_4e^{-\delta s}, \quad \forall s\geqslant\underline{s}$$

and

$$\int\limits_\tau^s\int\limits_{\Omega_s}|w|^{p+1}\psi^2\rho\,dy\,ds \leqslant C\big[1+(s-\tau)\big]\eta\big(\mathcal{E}[w](\tau)+C_5e^{-\delta\tau}\big),$$

where $\eta(s) = s + s^{1/2}$. With these estimates in hands, we prove our main theorems as in [6].

The monotonicity formula plays an important role in the partial regularity theories. See for example, Struwe's work [25] on harmonic map heat flow and Caffarelli, Nirenberg amd Kohn's work [3] on Navier–Stokes equations. For more discussion on local monotonicity formulas, please refer to Ecker [7].

Throughout the paper we will denote by C a constant that does not depend on the solution itself. And it may change from line to line. And $K_1, K_2, \dots, L_1, L_2, \dots, M_1, M_2, \dots, N_1, N_2, \dots$ are positive constants depending on p, N, Ω, a lower bound of V, $\|V\|_{C^1(\bar{\Omega})}$ and the initial energy $\mathcal{E}[w_0]$. Here and hereafter $w_0(y) = w(y,\underline{s})$.

2. Preliminaries

Recall that an L^1-solution to (1.1) is a function u in $C([0,T];L^1(\Omega))$ so that $f(x,u)\in L^1(Q_T)$, $Q_T = \Omega\times(0,T)$, and satisfies

$$\int_s^t \int_\Omega \big(u\phi_t + u\Delta\phi + f(x,u)\phi\big)\,dx\,dt - \int_\Omega u\phi|_s^t\,dx = 0,$$

for all $\phi \in C^2(\overline{Q}_T)$, $\phi = 0$ on $\partial\Omega \times (0,T)$ and $0 \leqslant s < t \leqslant T$, here and hereafter $f(x,u) = V(x)|u|^{p-1}u$. We are more interested in a stronger notion of weak solution. A function u in $C([0,T); L^2(\Omega))$ is called an H^1-solution to (1.1) if $\nabla u, u_t \in L^2(Q_T)$, $uf(x,u) \in L^1(Q_T)$ and

$$\int_s^t \int_\Omega \big(u_t\phi + \nabla u \cdot \nabla\phi - f(x,u)\phi\big)\,dx\,dt = 0, \tag{2.1}$$

holds for all $\phi \in C([0,T), H_0^1(\Omega))$ and $0 \leqslant s < t < T$. An L^1- or H^1-solution is called a global L^1- or H^1-solution respectively if it is an L^1- or H^1-solution in $\Omega \times (0,T)$ for every $T > 0$.

For an H^1-solution its energy

$$E(t) = E\big(u(t)\big) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(x,u)\,dx,$$

is well defined for a.e. t. Here $F(x,u) = \int_0^u f(x,t)\,dt$. An H^1-solution is called an energy-decreasing solution if it also satisfies the energy inequality

$$E(t) + \int_s^t \int_\Omega u_t^2\,dx\,dt \leqslant E(s), \tag{2.2}$$

for a.e. $t > s$, including $s = 0$ in $[0,T)$.

The following theorem is established in [6]. See also e.g., [4], [9] and [20].

Theorem 2.1.

(a) *Let u be a global, energy-decreasing solution to* (1.1). *There exists a positive constant C depending on ε_0, C_0, $|\Omega|$ and the initial energy E_0 such that*
  (1) $\operatorname{ess\,inf}_t E(t) \geqslant -C$;
  (2) $\|u_t\|_{L^2(\Omega\times(0,\infty))} \leqslant C$;
  (3) $\|u(t)\|_{L^2} \leqslant C$, $\forall t$;
  (4) $|\|u(t)\|_{L^2} - \|u(s)\|_{L^2}| \leqslant C|t-s|^{1/2}$, $\forall t, s$; *and*
  (5) *the $L^4(0,T; H^1(\Omega))$-norm of u and the $L^2(0,T; L^1(\Omega))$-norm of $uf(\cdot,u)$ are bounded by $C(1+T)$ for every $T > 0$.*

(b) (*Compactness*) *Let $\{u_k\}$ be a sequence of global, energy-decreasing solutions to* (1.1) *where $u_k(0)$ converges to some u_0 in $H_0^1(\Omega)$. Suppose that the initial energies of u_k are uniformly bounded from above. There exists a subsequence $\{u_{k_j}\}$ and a function u such that*

$$\begin{aligned}
&u_{k_j} \to u && \text{in } C\big([0,T); L^2(\Omega)\big),\\
&\nabla u_{k_j} \rightharpoonup \nabla u, \qquad u_{k_j t} \rightharpoonup u_t && \text{in } L^2(Q_T),\\
&F(\cdot,u_{k_j}) \rightharpoonup F(\cdot,u), \qquad u_{k_j} f(\cdot,u_{k_j}) \rightharpoonup uf(\cdot,u) && \text{in } L^1(Q_T),
\end{aligned}$$

for every $T > 0$. Consequently u is a global, H^1-solution to (1.1) *with $u(0) = u_0$. Moreover, if it is known that $u_{k_j} f(\cdot,u_{k_j}) \to uf(\cdot,u)$ in $L^1(\Omega)$ for a.e. t, then*

$$\begin{aligned}
&\nabla u_{k_j} \to \nabla u && \text{in } L^2(Q_T),\\
&F(\cdot,u_{k_j}) \to F(\cdot,u) && \text{in } L^1(Q_T),
\end{aligned}$$

for every $T > 0$ and u is also a global, energy-decreasing solution.

A stationary solution w of (1.1) is called stable if there exists a ball B centered at w in $L^\infty \cap H_0^1(\Omega)$ such that every solution to (1.1) starting inside this ball stays in the ball for all subsequent time. Let

$$\mathcal{U}(w) \equiv \{u_0 \in L^\infty \cap H_0^1(\Omega)\text{: The solution of (1.1) starting at } u_0 \text{ belongs to the ball } B \text{ at some finite time}\}.$$

It is routine to verify that $\mathcal{U} = \mathcal{U}(w)$ is an open, connected subset of $L^\infty \cap H_0^1(\Omega)$. The boundary of $\mathcal{U}$, $\partial\mathcal{U}$, is non-empty. For any boundary point w there exists a sequence $\{u_0^k\}$ in $\mathcal{U}$ converging to w in $L^\infty \cap H_0^1(\Omega)$. Since every u_0^k generates a global H^1-solution, Theorem 2.1 asserts that the maximal solution starting at u_0 can be extended to be a global, H^1-solution. Uniqueness of this global solution is not known generally. We shall call any global, H^1-solution starting at a boundary point of $\mathcal{U}$ a *borderline solution*.

It is easy to see that 0 is a non-negative, stable stationary solution to (1.1). For any non-negative $\varphi \in L^\infty \cap H_0^1(\Omega)$ which does not vanish identically, the solution of (1.1) with $u(0) = \lambda\varphi$, u_λ, belongs to $\mathcal{U}$ for small $\lambda > 0$. Since u_λ blows up in finite time for large λ, we can find some λ^* such that u_λ belongs to $\mathcal{U}$ for all $\lambda < \lambda^*$, and $\lambda^*\varphi$ lies on $\partial\mathcal{U}$. By the comparison principle u_λ converges monotonically to u_{λ^*} as $\lambda \uparrow \lambda^*$.

The monotone convergence theorem implies that

$$\lim_{\lambda\uparrow\lambda^*} \int_\Omega F\big(x, u_\lambda(x)\big)\,dx = \int_\Omega F\big(x, u_{\lambda^*}(x)\big)\,dx.$$

Theorem 2.1(b) is applicable to conclude that this positive borderline solution is also energy-decreasing.

In order to get the lower bound estimates for our energy functionals, we need the following

Lemma 2.2. *Let y, z, g and h be smooth functions on $[0,\infty)$. Suppose y, g and h are non-negative and for some positive constants α, K and L,*

$$\int_t^{t+\tau} g(s)\,ds \leqslant K(1+\tau), \quad \forall t, \tau > 0; \qquad \int_0^\infty e^{-\alpha s} h(s)\,ds \leqslant L.$$

If for some positive constants $a, b, q > 1$, the differential inequalities

$$y'(s) \geqslant -az(s) + by^q(s) - g(s),$$
$$z'(s) \leqslant \alpha z(s) + h(s)$$

hold on $[0,\infty)$, then

$$z(s) \geqslant -2Le^{\alpha s}$$

for all $s \geqslant 0$.

Proof. Suppose the conclusion is not true. Then there exists an $s_1 \geqslant 0$ such that $z(s_1)e^{-\alpha s_1} + 2L < 0$. From the second differential inequality, we see that

$$\frac{d}{ds}\big(e^{-\alpha s} z(s)\big) \leqslant e^{-\alpha s} h(s).$$

So for all $s \geqslant s_1$,

$$e^{-\alpha s} z(s) - e^{-\alpha s_1} z(s_1) \leqslant L, \quad \text{i.e., } e^{-\alpha s} z(s) < -L.$$

Therefore, for $s \geqslant s_1$,

$$y'(s) \geqslant aLe^{\alpha s} - g(s) + by^q(s).$$

Then we deduce that

$$y(s) \geqslant \int_{s_1}^{s} \big[aLe^{\alpha\tau} - g(\tau)\big]\,d\tau + b\int_{s_1}^{s} y^q(\tau)\,d\tau$$
$$\geqslant aL\int_{s_1}^{s} e^{\alpha\tau}\,d\tau - K(s - s_1 + 1) + b\int_{s_1}^{s} y^q(\tau)\,d\tau.$$

It is easy to check that there exists an $s_2 > s_1$, such that $aL\int_{s_1}^{s} e^{\alpha\tau}\,d\tau - K(s - s_1 + 1) > 0$ for all $s > s_2$. Therefore for all $s > s_2$,

$$y(s) \geqslant b\int_{s_1}^{s} y^q(\tau)\,d\tau.$$

And then the quantity $\int_{s_1}^{s} y^q(\tau)\,d\tau$ will blow up in finite time. But this is impossible. So the lemma is proved. □

3. Local energy estimates and quasi-monotonicity formula

Suppose in this section that u is a global classical solution. Let $(\bar{x}, \bar{t}) \in \Omega \times (0, \infty)$ be a fixed point. We introduce the self-similar scaling

$$w(y, s) = (\bar{t} - t)^{\beta} u(\bar{x} + y\sqrt{\bar{t} - t}, t)$$

with $s = -\log(\bar{t} - t)$, $\beta = \frac{1}{p-1}$. If u solves (1.1), then w satisfies

$$w_s - \Delta w + \frac{1}{2} y \cdot \nabla w + \beta w - V\big(\bar{x} + ye^{-s/2}\big)|w|^{p-1}w = 0 \quad \text{in } \Omega_s \times (\underline{s}, \infty)$$

where $\Omega_s = \{y\colon\ \bar{x} + ye^{-s/2} \in \Omega\}$, $\underline{s} = -\log \bar{t}$. We may assume $\bar{t} = 1$ for simplicity as in [15] so that we assume $\underline{s} = 0$. Here and hereafter we will always denote $V(\bar{x} + ye^{-s/2})$ by $\bar{V}(y, s)$.

By introducing a weight function $\rho(y) = \exp(-\frac{|y|^2}{4})$, we can rewrite the equation as the divergence form:

$$\rho w_s = \nabla \cdot (\rho \nabla w) - \beta \rho w + \bar{V}|w|^{p-1}w\rho \quad \text{in } \Omega_s \times (0, \infty). \tag{3.1}$$

Fix a positive number R and let $\psi(y, s) = \phi(e^{-s/2}|y|/R)$ where $\phi(r)$ is the function that is equal to 1 for $r \leqslant 1/2$, to 0 for $r \geqslant 1$ and linear between $r = 1$ and $1/2$. The local energy of w is given by

$$\mathcal{E}[w](s) = \frac{1}{2}\int_{\Omega_s} |\nabla w|^2 \psi^2 \rho\,dy + \frac{\beta}{2}\int_{\Omega_s} w^2 \psi^2 \rho\,dy - \frac{1}{p+1}\int_{\Omega_s} \bar{V}|w|^{p+1}\psi^2 \rho\,dy.$$

Note that the local energy depends on $(\bar{x}, \bar{t})$ and R. Notice that this kind of local energies were firstly introduced by Giga, Matsui and Sasayama in [15,16]. In these papers, $\psi = \psi(y)$ was a cutoff function of a fixed ball. However, in this paper, $\psi = \psi(y, s)$ is a cutoff function of moving balls at time s. In other words, the function ψ is a function of two variables in our case, but one variable in their definition.

Calculating the derivative of $\mathcal{E}[w](s)$ and noting that $w_s|_{\partial\Omega_s} = -\frac{1}{2} y \cdot \nabla w$ we have

$$\frac{d}{ds}\mathcal{E}[w](s) = -\int_{\Omega_s} w_s^2 \psi^2 \rho\,dy - \frac{1}{4}\int_{\partial\Omega_s} |\nabla w|^2 (y \cdot \gamma)\psi^2 \rho\,d\sigma - \frac{1}{p+1}\int_{\Omega_s} \frac{\partial \bar{V}}{\partial s}|w|^{p+1}\psi^2 \rho\,dy$$
$$- 2\int_{\Omega_s} \nabla w \cdot \nabla\psi\,\psi w_s \rho\,dy + \int_{\Omega_s} \bigg(|\nabla w|^2 + \beta w^2 - \frac{2\bar{V}}{p+1}|w|^{p+1}\bigg)\psi\psi_s \rho\,dy$$
$$\leqslant -\frac{1}{2}\int_{\Omega_s} w_s^2 \psi^2 \rho\,dy - \frac{1}{4}\int_{\partial\Omega_s} |\nabla w|^2 (y \cdot \gamma)\psi^2 \rho\,d\sigma - \frac{1}{p+1}\int_{\Omega_s} \frac{\partial \bar{V}}{\partial s}|w|^{p+1}\psi^2 \rho\,dy$$

$$+ 2\int_{\Omega_s} |\nabla w|^2|\nabla\psi|^2\rho\,dy + \int_{\Omega_s}\left(|\nabla w|^2 + \beta w^2 + \frac{2\bar V}{p+1}|w|^{p+1}\right)\psi|\psi_s|\rho\,dy$$

or

$$\frac{1}{2}\int_{\Omega_s} w_s^2\psi^2\rho\,dy \leqslant -\frac{d}{ds}\mathcal{E}[w](s) - \frac{1}{4}\int_{\partial\Omega_s}|\nabla w|^2(y\cdot\gamma)\psi^2\rho\,d\sigma + 2\int_{\Omega_s}|\nabla w|^2|\nabla\psi|^2\rho\,dy$$
$$+ \int_{\Omega_s}\left(|\nabla w|^2 + \beta w^2 + \frac{2\bar V}{p+1}|w|^{p+1}\right)\psi|\psi_s|\rho\,dy$$
$$+ \frac{1}{2(p+1)}\int_{\Omega_s}\nabla\bar V\cdot y|w|^{p+1}\psi^2\rho\,dy.$$

Let us take $R < \operatorname{dist}(\bar x, \partial\Omega)$ so that the boundary integrals above vanish. Using the estimates

$$|\nabla w|^2 = (\bar t - t)^{\frac{p+1}{p-1}}|\nabla u|^2,$$
$$|w| = (\bar t - t)^{\frac{1}{p-1}}|u|,$$
$$|\nabla\psi| = \left|\phi'\frac{e^{-s/2}}{R}\right| \leqslant \frac{2e^{-s/2}}{R}\chi_{A_R}, \quad \text{and}$$
$$|\psi_s| = \left|\phi'\frac{e^{-s/2}}{2R}|y|\right| \leqslant \frac{e^{-s/2}}{R}|y|\chi_{A_R}, \quad A_R = B_R(\bar x)\setminus B_{R/2}(\bar x),$$

we can find a constant C which depends on N, R and $\underline{s}$ such that

$$\left[\exp\left(\frac{N+2}{2} - \frac{2}{p-1}\right)s + \frac{1}{R^2}\exp\left(\frac{N+2}{2} - \frac{p+1}{p-1} - 1\right)s\right]\exp\left(-\frac{R^2}{16}e^s\right) \leqslant Ce^{-2s}.$$

And then by the estimates for u, we get

$$\int_\tau^\infty e^s\int_{\Omega_s}\left[|\nabla w|^2|\nabla\psi|^2 + \left(|\nabla w|^2 + w^2 + |w|^{p+1}\right)\psi|\psi_s|\right]\rho\,dy\,ds$$
$$\leqslant Ce^{-\tau}\int_0^{\bar t}\int_{B_R(\bar x)}\left(|\nabla u|^2 + u^2 + |u|^{p+1}\right)dx\,dt$$
$$\leqslant Ce^{-\tau}.$$

In the last inequality above the constant C also depends on $\bar t$. Denote

$$\varphi(s) = \int_{\Omega_s}\left[|\nabla w|^2|\nabla\psi|^2 + \left(|\nabla w|^2 + w^2 + |w|^{p+1}\right)\psi|\psi_s|\right]\rho\,dy.$$

Then we have

$$\frac{d\mathcal{E}[w]}{ds} \leqslant -\frac{1}{2}\int_{\Omega_s} w_s^2\psi^2\rho\,dy + \frac{1}{2(p+1)}\int_{\Omega_s}\nabla\bar V\cdot y|w|^{p+1}\psi^2\rho\,dy + C\varphi(s), \tag{3.2}$$

where

$$\int_\tau^\infty e^s\varphi(s)\,ds \leqslant Ce^{-\tau}. \tag{3.3}$$

Firstly, we have the following rough estimates for the local energy.

Lemma 3.1. *There exists a constant C depending on N, R, p, $\bar{t}$, the lower bound of V, $\|V\|_{C^1(\bar{\Omega})}$ and $\mathcal{E}[w](\underline{s})$ such that*

$$\left|\mathcal{E}[w](s)\right| \leqslant Ce^{\lambda s}, \quad \text{for all } s \geqslant \underline{s},$$

where $\lambda = \frac{16}{7(p-1)}\frac{d_2}{d_1}$, and d_1, d_2 are constants such that $V(x) \geqslant d_1 > 0$ and $\sup_{x\in\Omega} |\nabla V(x)| \operatorname{diam}(\Omega) \leqslant d_2$.

Proof. We see from (3.1) that

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\int_{\Omega_s} w^2\psi^2\rho\,dy &= \int_{\Omega_s} ww_s\psi^2\rho\,dy + \int_{\Omega_s} w^2\psi\psi_s\rho\,dy \\ &= -\int_{\Omega_s} |\nabla w|^2\psi^2\rho\,dy - \int_{\Omega_s} \beta w^2\psi^2\rho\,dy + \int_{\Omega_s} \bar{V}|w|^{p+1}\psi^2\rho\,dy \\ &\quad + \int_{\Omega_s} w^2\psi\psi_s\rho\,dy - 2\int_{\Omega_s} \nabla w\nabla\psi w\psi\rho\,dy \\ &= -2\mathcal{E}[w] + \frac{p-1}{p+1}\int_{\Omega_s} \bar{V}|w|^{p+1}\psi^2\rho\,dy + \int_{\Omega_s} w^2\psi\psi_s\rho\,dy - 2\int_{\Omega_s} \nabla w\nabla\psi w\psi\rho\,dy. \end{aligned} \tag{3.4}$$

Notice that $\bar{V}$ is bounded below by d_1. By (3.4), using Young's inequality, we have

$$\begin{aligned} -2\mathcal{E}[w] + \frac{p-1}{p+1}d_1\int_{\Omega_s} |w|^{p+1}\psi^2\rho\,dy &\leqslant -2\mathcal{E}[w] + \frac{p-1}{p+1}\int_{\Omega_s} \bar{V}|w|^{p+1}\psi^2\rho\,dy \\ &= \int_{\Omega_s} ww_s\psi^2\rho\,dy + \int_{\Omega_s} \nabla w\nabla\psi w\psi\rho\,dy \\ &\leqslant \int_{\Omega_s} ww_s\psi^2\rho\,dy + \frac{\varepsilon}{2}\int_{\Omega_s} w^2\psi^2\rho\,dy + C(\varepsilon)\int_{\Omega_s} |\nabla w|^2|\nabla\psi|^2\rho\,dy \\ &\leqslant \int_{\Omega_s} ww_s\psi^2\rho\,dy + \frac{\varepsilon}{2}\left(\int_{\Omega_s} |w|^{p+1}\psi^2\rho\,dy + C\right) + C(\varepsilon)\varphi(s) \\ &\leqslant \varepsilon\int_{\Omega_s} w_s^2\psi^2\rho\,dy + \varepsilon\int_{\Omega_s} |w|^{p+1}\psi^2\rho\,dy + C(\varepsilon)\big(1+\varphi(s)\big). \end{aligned}$$

Here we have used the inequality

$$ab \leqslant \varepsilon\big(a^2 + b^{p+1}\big) + C(\varepsilon), \quad p > 1,\ \forall\varepsilon > 0.$$

So we obtain that

$$\int_{\Omega_s} |w|^{p+1}\psi^2\rho\,dy \leqslant 2c(p,d_1)\mathcal{E}[w] + \eta\int_{\Omega_s} |w|^{p+1}\psi^2\rho\,dy + \eta\int_{\Omega_s} w_s^2\psi^2\rho\,dy + C(p,d_1,\eta)\big(1+\varphi(s)\big).$$

Here and hereafter we will denote $\frac{p+1}{(p-1)d_1}$ by $c(p,d_1)$ and $C(p,d_1,\eta)$ denotes a constant depending on $p, d_1, \eta > 0$ and may be different at each occurrence. Take $\eta < 1/8$ and we hence have

$$\begin{aligned}\int_{\Omega_s} |w|^{p+1}\psi^2\rho\,dy &\leqslant \frac{2c(p,d_1)}{1-\eta}\mathcal{E}[w] + \frac{\eta}{1-\eta}\int_{\Omega_s} w_s^2\psi^2\rho\,dy + C(p,d_1,\eta)\big(1+\varphi(s)\big) \\ &\leqslant \frac{16c(p,d_1)}{7}\mathcal{E}[w] + 2\eta\int_{\Omega_s} w_s^2\psi^2\rho\,dy + C(p,d_1,\eta)\big(1+\varphi(s)\big)\end{aligned}$$

$$\leqslant \alpha \frac{16c(p,d_1)}{7}\mathcal{E}[w] + 2\alpha\eta \int_{\Omega_s} w_s^2\psi^2\rho\,dy + \alpha C(p,d_1,\eta)\big(1+\varphi(s)\big), \tag{3.5}$$

for all $\alpha \geqslant 1$. Choosing η small further such that $2\alpha\eta d_2 < 1/4$, we get

$$\int_{\Omega_s} |w|^{p+1}\psi^2\rho\,dy \leqslant \alpha \frac{16c(p,d_1)}{7}\mathcal{E}[w] + \frac{1}{4d_2}\int_{\Omega_s} w_s^2\psi^2\rho\,dy + C(\alpha)\big(1+\varphi(s)\big), \tag{3.6}$$

where $\sup_{y\in\Omega_s}|\nabla\bar{V}||y| = \sup_{x\in\Omega}|\nabla V||x-\bar{x}| \leqslant d_2$. By (3.2), (3.6), we have for any fixed $\alpha \geqslant 1$,

$$\frac{d}{ds}\mathcal{E}[w](s) \leqslant -\frac{1}{4}\int_{\Omega_s} w_s^2\psi^2\rho\,dy + \alpha\lambda\mathcal{E}[w](s) + C(\alpha)\big(1+\varphi(s)\big).$$

Therefore, we obtain that

$$\frac{d}{ds}\mathcal{E}[w](s) \leqslant -\frac{1}{4}\int_{\Omega_s} w_s^2\psi^2\rho\,dy + \mu\mathcal{E}[w](s) + C(\mu)\big(1+\varphi(s)\big), \tag{3.7}$$

for all $\mu \geqslant \lambda$. In particular, we have

$$\frac{d}{ds}\big(e^{-\lambda s}\mathcal{E}[w](s)\big) + \frac{1}{4}e^{-\lambda s}\int_{\Omega_s} w_s^2\psi^2\rho\,dy \leqslant C_1 e^{-\lambda s}\big(1+\varphi(s)\big). \tag{3.8}$$

It follows that $\mathcal{E}[w](s) \leqslant Ce^{\lambda s}$ due to (3.3).

In order to get the lower bound of $\mathcal{E}[w](s)$, we need to estimate the last two terms in (3.4) firstly. For any $\varepsilon > 0$, we have

$$\begin{aligned}
&\left|\int_{\Omega_s} w^2\psi\psi_s\rho\,dy - 2\int_{\Omega_s}\nabla w\nabla\psi w\psi\rho\,dy\right| \\
&\quad\leqslant \int_{\Omega_s} w^2|\psi\psi_s|\rho\,dy + 2\bigg(\int_{\Omega_s}|\nabla w|^2|\nabla\psi|^2\rho\,dy\bigg)^{\frac{1}{2}}\bigg(\int_{\Omega_s} w^2\psi^2\rho\,dy\bigg)^{\frac{1}{2}} \\
&\quad\leqslant \int_{\Omega_s}\big[w^2|\psi\psi_s| + |\nabla w|^2|\nabla\psi|^2\big]\rho\,dy + \int_{\Omega_s} w^2\psi^2\rho\,dy \\
&\quad\leqslant \varphi(s) + \varepsilon\int_{\Omega_s}|w|^{p+1}\psi^2\rho\,dy + C(\varepsilon)\int_{\Omega_s}\psi^2\rho\,dy \\
&\quad\leqslant \varphi(s) + \varepsilon\int_{\Omega_s}|w|^{p+1}\psi^2\rho\,dy + C(\varepsilon).
\end{aligned}$$

Now by (3.4), the above estimate and Jensen's inequality, if we set $y(s) = \int_{\Omega_s} w^2\psi^2\rho\,dy$, then we have

$$y'(s) \geqslant -4\mathcal{E}[w] + Cy^{\frac{p+1}{2}}(s) - C\big(\varphi(s)+1\big). \tag{3.9}$$

Since $C_2 = C_1\int_0^\infty e^{-\lambda s}(1+\varphi(s))\,ds < \infty$, applying Lemma 2.2 for $z(s) = \mathcal{E}[w](s)$, we get $\mathcal{E}[w](s) \geqslant -2C_2e^{\lambda s}$. So the lemma follows. □

To get some refined estimates for $\mathcal{E}[w]$, we introduce

$$\mathcal{E}_{2k}[w] = \frac{1}{2}\int_{\Omega_s}\big(|\nabla w|^2 + \beta w^2\big)|y|^{2k}\psi^2\rho\,dy - \frac{1}{p+1}\int_{\Omega_s}\bar{V}|w|^{p+1}|y|^{2k}\psi^2\rho\,dy, \quad k \in \mathbb{N}.$$

Here $\mathbb{N} = \{0,1,2,3,\dots\}$. For these energy functionals, by straightforward calculation, we can obtain the following identities.

Proposition 3.2.

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\int_{\Omega_s} w^2|y|^{2k}\psi^2\rho\,dy = {} & -2\mathcal{E}_{2k}[w] + \frac{p-1}{p+1}\int_{\Omega_s}\bar{V}|w|^{p+1}|y|^{2k}\psi^2\rho\,dy \\ & + \int_{\Omega_s} k\left(N+2k-2-\frac{1}{2}|y|^2\right)w^2|y|^{2k-2}\psi^2\rho\,dy \\ & + \int_{\Omega_s} w^2|y|^{2k}\psi\psi_s\rho\,dy - \int_{\Omega_s}\nabla w\nabla(\psi^2)w|y|^{2k}\rho\,dy \\ & + k\int_{\Omega_s} y\cdot\nabla(\psi^2)w^2|y|^{2k-2}\rho\,dy. \end{aligned} \tag{3.10}$$

Proposition 3.3.

$$\begin{aligned}\frac{d}{ds}\mathcal{E}_{2k}[w] = {} & -\int_{\Omega_s} w_s^2|y|^{2k}\psi^2\rho\,dy - 2k\int_{\Omega_s}\psi^2\rho(y\cdot\nabla w)w_s|y|^{2k-2}\,dy \\ & - \int_{\Omega_s}\nabla w\nabla(\psi^2)w_s|y|^{2k}\rho\,dy - \frac{1}{p+1}\int_{\Omega_s}\frac{\partial\bar{V}}{\partial s}|w|^{p+1}|y|^{2k}\psi^2\rho\,dy \\ & + \int_{\Omega_s}\left(|\nabla w|^2+\beta w^2-\frac{2}{p+1}\bar{V}|w|^{p+1}\right)\psi\psi_s|y|^{2k}\rho\,dy. \end{aligned} \tag{3.11}$$

Denote

$$\varphi_{2k}(s) = \int_{\Omega_s}\left[\left(|\nabla w|^2+w^2+|w|^{p+1}\right)\psi|\psi_s| + |\nabla w|^2|\nabla\psi|^2\right]|y|^{2k}\rho\,dy + 2k\int_{\Omega_s} w^2\psi|\nabla\psi||y|^{2k-1}\rho\,dy.$$

As before, we can find a constant C depending on N, R, p, k, and $\bar{t}$ such that

$$\int_\tau^\infty e^s\varphi_{2k}(s)\,ds \leqslant Ce^{-\tau}.$$

It is easy to see from (3.10) that

$$\begin{aligned}\int_{\Omega_s} ww_s|y|^{2k}\psi^2\rho\,dy = {} & -2\mathcal{E}_{2k}[w] + \frac{p-1}{p+1}\int_{\Omega_s}\bar{V}|w|^{p+1}|y|^{2k}\psi^2\rho\,dy \\ & + \int_{\Omega_s} k\left(N+2k-2-\frac{1}{2}|y|^2\right)w^2|y|^{2k-2}\psi^2\rho\,dy \\ & - \int_{\Omega_s}\nabla w\nabla(\psi^2)w|y|^{2k}\rho\,dy + k\int_{\Omega_s} y\cdot\nabla(\psi^2)w^2|y|^{2k-2}\rho\,dy. \end{aligned}$$

So

$$\begin{aligned}\frac{p-1}{p+1}\int_{\Omega_s}\bar{V}|w|^{p+1}|y|^{2k}\psi^2\rho\,dy \leqslant {} & \int_{\Omega_s} ww_s|y|^{2k}\psi^2\rho\,dy + 2\mathcal{E}_{2k}[w] \\ & - \int_{\Omega_s} k\left(N+2k-2-\frac{1}{2}|y|^2\right)w^2|y|^{2k-2}\psi^2\rho\,dy \end{aligned}$$

$$
\begin{aligned}
&+\int_{\Omega_s}\nabla w\nabla(\psi^2)w|y|^{2k}\rho\,dy-k\int_{\Omega_s}y\cdot\nabla(\psi^2)w^2|y|^{2k-2}\rho\,dy\\
&\leqslant\int_{\Omega_s}|w||w_s||y|^{2k}\psi^2\rho\,dy+2\mathcal{E}_{2k}[w]+\varphi_{2k}(s)\\
&\quad+\frac{k}{2}\int_{\Omega_s}w^2|y|^{2k}\psi^2\rho\,dy+\int_{\Omega_s}\nabla w\nabla(\psi^2)w|y|^{2k}\rho\,dy\\
&\leqslant\int_{\Omega_s}|w||w_s||y|^{2k}\psi^2\rho\,dy+2\mathcal{E}_{2k}[w]\\
&\quad+\left(1+\frac{k}{2}\right)\int_{\Omega_s}w^2|y|^{2k}\psi^2\rho\,dy+2\varphi_{2k}(s).
\end{aligned}
$$

We have used Cauchy's inequality in the last inequality and the fact that $N+2k-2>0$ in the second inequality. Making use of the inequality

$$ab\leqslant\varepsilon\big(a^2+b^{p+1}\big)+C(\varepsilon),\quad p>1,\ \forall\varepsilon>0,$$

we have

$$\int_{\Omega_s}|w||w_s||y|^{2k}\psi^2\rho\,dy\leqslant\varepsilon\int_{\Omega_s}|w|^{p+1}|y|^{2k}\psi^2\rho\,dy+\varepsilon\int_{\Omega_s}w_s^2|y|^{2k}\psi^2\rho\,dy+C(\varepsilon,k).$$

Applying Young's inequality we obtain that

$$\left(1+\frac{k}{2}\right)\int_{\Omega_s}w^2|y|^{2k}\psi^2\rho\,dy\leqslant\varepsilon\int_{\Omega_s}|w|^{p+1}|y|^{2k}\psi^2\rho\,dy+C(\varepsilon,k).$$

Therefore,

$$
\begin{aligned}
\int_{\Omega_s}|w|^{p+1}|y|^{2k}\psi^2\rho\,dy&\leqslant 2c(p,d_1)\mathcal{E}_{2k}[w]+\eta\int_{\Omega_s}w_s^2|y|^{2k}\psi^2\rho\,dy\\
&\quad+\eta\int_{\Omega_s}|w|^{p+1}|y|^{2k}\psi^2\rho\,dy+C(p,d_1,k)\big(1+\varphi_{2k}(s)\big),
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
\int_{\Omega_s}|w|^{p+1}|y|^{2k}\psi^2\rho\,dy&\leqslant\frac{2c(p,d_1)}{1-\eta}\mathcal{E}_{2k}[w]+\frac{\eta}{1-\eta}\int_{\Omega_s}w_s^2|y|^{2k}\psi^2\rho\,dy+C(p,d_1,k,\eta)\big(1+\varphi_{2k}(s)\big)\\
&\leqslant\alpha\frac{16c(p,d_1)}{7}\mathcal{E}_{2k}[w]+2\alpha\eta\int_{\Omega_s}w_s^2|y|^{2k}\psi^2\rho\,dy+C(\alpha)\big(1+\varphi_{2k}(s)\big)
\end{aligned}
$$

for all $\alpha\geqslant 1$ and $\eta<1/8$.

Choosing η small further such that $2\alpha\eta d_2<1/4$, we get that for all $\alpha\geqslant 1$,

$$\int_{\Omega_s}w^{p+1}|y|^{2k}\psi^2\rho\,dy\leqslant\alpha\frac{16c(p,d_1)}{7}\mathcal{E}_{2k}[w]+\frac{1}{4d_2}\int_{\Omega_s}w_s^2|y|^{2k}\psi^2\rho\,dy+C(\alpha)\big(1+\varphi_{2k}(s)\big).\tag{3.12}$$

Now it is easy to see from Young's inequality that

$$-2k\int_{\Omega_s}\psi^2\rho(y\cdot\nabla w)w_s|y|^{2k-2}\,dy\leqslant\varepsilon\ int_{\Omega_s}w_s^2|y|^{2k}\psi^2\rho\,dy+C(\varepsilon)\int_{\Omega_s}|\nabla w|^2|y|^{2k-2}\psi^2\rho\,dy,$$

and

$$-\int_{\Omega_s} \nabla w \nabla(\psi^2) w_s |y|^{2k} \rho\, dy \leqslant \varepsilon \int_{\Omega_s} w_s^2 |y|^{2k} \psi^2 \rho\, dy + C(\varepsilon) \int_{\Omega_s} |\nabla w|^2 |\nabla \psi|^2 |y|^{2k} \rho\, dy.$$

So by (3.11), the above inequalities, Hölder's inequality and (3.12) we have

$$\begin{aligned}
\frac{d}{ds}\mathcal{E}_{2k}[w] &\leqslant -\frac{1}{2}\int_{\Omega_s} w_s^2 |y|^{2k}\psi^2 \rho\, dy + C\int_{\Omega_s} |\nabla w|^2 |y|^{2k-2}\psi^2\rho\, dy \\
&\quad - \frac{1}{p+1}\int_{\Omega_s} \frac{\partial \bar V}{\partial s}|w|^{p+1}|y|^{2k}\psi^2\rho\, dy + C\varphi_{2k}(s) \\
&\leqslant -\frac{3}{8}\int_{\Omega_s} w_s^2|y|^{2k}\psi^2\rho\, dy + \mu\mathcal{E}_{2k}[w] + C(\mu)\big(1+\varphi_{2k}(s)\big) \\
&\quad + C(\mu)\int_{\Omega_s} |\nabla w|^2|y|^{2k-2}\psi^2\rho\, dy,
\end{aligned} \tag{3.13}$$

for all $\mu \geqslant \lambda$. Here $k \geqslant 1$.

On the other hand, by (3.10), Hölder's inequality, Young's inequality and Jensen's inequality we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}\int_{\Omega_s} w^2|y|^{2k}\psi^2\rho\, dy &\geqslant -2\mathcal{E}_{2k}[w] - C\int_{\Omega_s} w^2|y|^{2k}\psi^2\rho\, dy + C\int_{\Omega_s}|w|^{p+1}|y|^{2k}\psi^2\rho\, dy - C\varphi_{2k}(s) \\
&\geqslant -2\mathcal{E}_{2k}[w] + C\int_{\Omega_s}|w|^{p+1}|y|^{2k}\psi^2\rho\, dy - C\big(1+\varphi_{2k}(s)\big) \\
&\geqslant -2\mathcal{E}_{2k}[w] - C\big(1+\varphi_{2k}(s)\big) + C\bigg(\int_{\Omega_s} w^2|y|^{2k}\psi^2\rho\, dy\bigg)^{\frac{p+1}{2}}.
\end{aligned} \tag{3.14}$$

With these crucial inequalities, (3.13), (3.14), in hands, we can get the following rough estimates.

Lemma 3.4. *For any $k \in \mathbb{N}$, there exist positive constants L_k, M_k, and N_k, such that the following estimates hold*:

$$-L_k e^{2\lambda s} \leqslant \mathcal{E}_{2k}[w](s) \leqslant M_k e^{2\lambda s},$$

$$\int_0^\infty e^{-2\lambda s}\int_{\Omega_s} |\nabla w|^2|y|^{2k}\psi^2\rho\, dy\, ds \leqslant N_k,$$

for all $s \geqslant 0$. Here $\lambda = \frac{16}{7(p-1)}\frac{d_2}{d_1}$, and d_1, d_2 are constants such that $V(x) \geqslant d_1 > 0$ and $\sup_{x\in\Omega}|\nabla V(x)|\,\mathrm{diam}(\Omega) \leqslant d_2$.

Proof. Let $\{\lambda_k\}_{k=0}^\infty \subset (\lambda, 2\lambda)$ be a strictly increasing sequence. It suffices to show the following estimates:

$$-L_k e^{\lambda_k s} \leqslant \mathcal{E}_{2k}[w](s) \leqslant M_k e^{\lambda_k s}, \tag{3.15}$$

$$\int_0^\infty e^{-\lambda_k s}\int_{\Omega_s} |\nabla w|^2|y|^{2k}\psi^2\rho\, dy\, ds \leqslant N_k. \tag{3.16}$$

We prove these estimates by induction.

Step 1. We show that these estimates hold for $k=0$. The inequality (3.15) holds for $k=0$ due to Lemma 3.1. From (3.8) and Lemma 3.1, we deduce that

$$\int_0^\infty e^{-\lambda s}\int_{\Omega_s} w_s^2\psi^2\rho\,dy\,ds \leqslant C.$$

By (3.6) and the definition of $\mathcal{E}[w]$, we have

$$\begin{aligned}\int_{\Omega_s} |\nabla w|^2\psi^2\rho\,dy &\leqslant 2\mathcal{E}[w] + \frac{2}{p+1}\int_{\Omega_s}\bar{V}|w|^{p+1}\psi^2\rho\,dy\\ &\leqslant C\mathcal{E}[w] + C\big(1+\varphi(s)\big) + C\int_{\Omega_s} w_s^2\psi^2\rho\,dy.\end{aligned}$$

Therefore, by Lemma 3.1,

$$\begin{aligned}\int_0^\infty e^{-\lambda_0 s}\int_{\Omega_s}|\nabla w|^2\psi^2\rho\,dy\,ds &\leqslant C\int_0^\infty e^{-\lambda_0 s}\mathcal{E}[w](s)\,ds + C\int_0^\infty e^{-\lambda_0 s}\big(1+\varphi(s)\big)\,ds\\ &\quad + C\int_0^\infty e^{-\lambda_0 s}\int_{\Omega_s} w_s^2\psi^2\rho\,dy\,ds\\ &\leqslant C\end{aligned}$$

since $\lambda_0 > \lambda$. So (3.16) holds for $k=0$.

Step 2. We show that (3.15)–(3.16) holds for all $k\in\mathbb{N}$.

Suppose (3.15)–(3.16) hold for $k\leqslant n$. By (3.13), we have

$$\frac{d}{ds}\big(e^{-\lambda_n s}\mathcal{E}_{2n+2}[w]\big) \leqslant Ce^{-\lambda_n s}\int_{\Omega_s}|\nabla w|^2|y|^{2n}\psi^2\rho\,dy + C(1+\varphi_{2n+2})e^{-\lambda_n s}.$$

Since (3.16) holds for $k=n$, we have

$$e^{-\lambda_n s}\mathcal{E}_{2n+2}[w] \leqslant C(n).$$

Now we need to obtain the lower bound for $\mathcal{E}_{2n+2}[w]$. Denote

$$\begin{aligned} y(s) &= \int_{\Omega_s} w^2|y|^{2n+2}\psi^2\rho\,dy,\\ z(s) &= \mathcal{E}_{2n+2}[w].\end{aligned}$$

Then it follows from (3.13) and (3.14) that

$$y'(s) \geqslant -4z(s) + Cy^{\frac{p+1}{2}}(s) - C\big(1+\varphi_{2n+2}(s)\big), \tag{3.17}$$

$$z'(s) \leqslant \lambda_n z(s) + C\int_{\Omega_s}|\nabla w|^2|y|^{2n}\psi^2\rho\,dy + C\big(1+\varphi_{2n+2}(s)\big). \tag{3.18}$$

By induction hypothesis, we have

$$\int_0^\infty e^{-\lambda_n s}\int_{\Omega_s}|\nabla w|^2|y|^{2n}\psi^2\rho\,dy \leqslant C(n). \tag{3.19}$$

So $N = \int_0^\infty e^{-\lambda_n s}(h(s) + C(1 + \varphi_{2n+2}(s)))\,ds < \infty$, where $h(s) = C\int_{\Omega_s} |\nabla w|^2 |y|^{2n}\psi^2\rho\,dy$.

Then

$$z(s) \geqslant -2Ne^{\lambda_n s}, \quad \forall s \geqslant 0,$$

follows by Lemma 2.2. Therefore $\mathcal{E}_{2n+2}[w] \geqslant -Ce^{\lambda_n s}$ and then $|\mathcal{E}_{2n+2}[w]| \leqslant Ce^{\lambda_n s}$. In particular, (3.15) holds for $k = n+1$.

Finally, by (3.13), we have

$$\frac{d}{ds}\mathcal{E}_{2n+2}[w] \leqslant -\frac{1}{4}\int_{\Omega_s} w_s^2 |y|^{2n+2}\psi^2\rho\,dy + C\int_{\Omega_s} |\nabla w|^2 |y|^{2n}\psi^2\rho\,dy + C(1+\varphi_{2n+2}) + \lambda_n \mathcal{E}_{2n+2}[w].$$

Combining this with the fact that $|\mathcal{E}_{2n+2}[w]| \leqslant Ce^{\lambda_n s}$ and (3.19) we have

$$\int_0^\infty e^{-\lambda_n s}\int_{\Omega_s} w_s^2 |y|^{2n+2}\psi^2\rho\,dy\,ds \leqslant C.$$

By (3.12), it can be shown that

$$\begin{aligned}\int_{\Omega_s} |\nabla w|^2 |y|^{2n+2}\psi^2\rho\,dy &\leqslant 2\mathcal{E}_{2n+2}[w] + \frac{2}{p+1}\int_{\Omega_s} \bar{V}|w|^{p+1}|y|^{2n+2}\psi^2\rho\,dy \\ &\leqslant C\mathcal{E}_{2n+2}[w] + C(1+\varphi_{2n+2}) + C\int_{\Omega_s} w_s^2 |y|^{2n+2}\psi^2\rho\,dy.\end{aligned}$$

Therefore, by $|\mathcal{E}_{2n+2}[w]| \leqslant Ce^{\lambda_n s}$, we get

$$\begin{aligned}&\int_0^\infty e^{-\lambda_{n+1} s}\int_{\Omega_s} |\nabla w|^2 |y|^{2n+2}\psi^2\rho\,dy \\ &\quad\leqslant C\int_0^\infty \big(\mathcal{E}_{2n+2}[w] + 1 + \varphi_{2n+2}\big)e^{-\lambda_{n+1}s}\,ds + C\int_0^\infty e^{-\lambda_n s}\int_{\Omega_s} w_s^2 |y|^{2n+2}\psi^2\rho\,dy\,ds \\ &\quad\leqslant C\int_0^\infty e^{(\lambda_n - \lambda_{n+1})s}\,ds + C \\ &\quad\leqslant C.\end{aligned}$$

Hence (3.16) holds for $k = n+1$. The lemma is proved. □

Next, we need the following

Lemma 3.5. *Suppose $\lambda > \frac{1}{4}$, where $\lambda = \frac{16}{7(p-1)}\frac{d_2}{d_1}$, and d_1, d_2 are constants such that $V(x) \geqslant d_1 > 0$ and $\sup_{x\in\Omega} |\nabla V(x)|\,\mathrm{diam}(\Omega) \leqslant d_2$. If for some $\alpha \in (\frac{1}{2}, 2\lambda]$, there exist positive constants M_k and N_k, such that*

$$\big|\mathcal{E}_{2k}[w](s)\big| \leqslant M_k e^{\alpha s},$$

$$\int_0^\infty e^{-\alpha s}\int_{\Omega_s} |\nabla w|^2 |y|^{2k}\psi^2\rho\,dy\,ds \leqslant N_k,$$

hold for all $k \in \mathbb{N}$ and $s \geqslant 0$, then there exist positive constants M_k' and N_k', such that

$$\left|\mathcal{E}_{2k}[w](s)\right| \leqslant M_k' e^{(\alpha-\frac{1}{4})s},$$

$$\int_0^\infty e^{-(\alpha-\frac{1}{4})s} \int_{\Omega_s} |\nabla w|^2 |y|^{2k} \psi^2 \rho \, dy \, ds \leqslant N_k',$$

hold for all $k \in \mathbb{N}$ and $s \geqslant 0$.

Proof. Let $\{\delta_k\}_{k=0}^\infty \subset [\frac{1}{4}, \frac{1}{3}]$ be a strictly decreasing sequence. It suffices to show the following estimates:

$$\left|\mathcal{E}_{2k}[w](s)\right| \leqslant M_k' e^{(\alpha-\delta_k)s}, \tag{3.20}$$

$$\int_0^\infty e^{-(\alpha-\delta_k)s} \int_{\Omega_s} |\nabla w|^2 |y|^{2k} \psi^2 \rho \, dy \, ds \leqslant N_k'. \tag{3.21}$$

We also prove these estimates by induction.

Step 1. These estimates hold for $k = 0$.

Recalling (3.2) we have

$$\begin{aligned}\frac{d\mathcal{E}[w]}{ds} &\leqslant -\frac{1}{2} \int_{\Omega_s} w_s^2 \psi^2 \rho \, dy + \int_{\Omega_s} \nabla V \cdot y e^{-s/2} |w|^{p+1} \psi^2 \rho \, dy + C\varphi(s) \\ &\leqslant -\frac{1}{2} \int_{\Omega_s} w_s^2 \psi^2 \rho \, dy + Ce^{-s/2} \int_{\Omega_s} |y||w|^{p+1} \psi^2 \rho \, dy + C\varphi(s) \\ &\leqslant -\frac{1}{2} \int_{\Omega_s} w_s^2 \psi^2 \rho \, dy + Ce^{-s/2} \int_{\Omega_s} \left(|y|^2 + 1\right) |w|^{p+1} \psi^2 \rho \, dy + C\varphi(s).\end{aligned} \tag{3.22}$$

Also by the definition of $\mathcal{E}_2[w]$, Hölder inequality and the assumptions we get

$$\begin{aligned}e^{-s/2} \int_{\Omega_s} |y|^2 |w|^{p+1} \psi^2 \rho \, dy &\leqslant Ce^{-s/2} \left(\int_{\Omega_s} |\nabla w|^2 |y|^2 \psi^2 \rho \, dy + \int_{\Omega_s} |w|^{p+1} \psi^2 \rho \, dy + C\mathcal{E}_2[w] + C \right) \\ &\leqslant Ce^{-s/2} \left(\int_{\Omega_s} |\nabla w|^2 |y|^2 \psi^2 \rho \, dy + \int_{\Omega_s} |w|^{p+1} \psi^2 \rho \, dy + Ce^{\alpha s} + C \right).\end{aligned}$$

By (3.6) and assumptions we have

$$\begin{aligned}\frac{d}{ds}\mathcal{E}[w] &\leqslant -\frac{1}{4} \int_{\Omega_s} w_s^2 \psi^2 \rho \, dy + Ce^{-\frac{s}{2}} \int_{\Omega_s} |\nabla w|^2 |y|^2 \psi^2 \rho \, dy + Ce^{(\alpha-\frac{1}{2})s} \\ &\quad + Ce^{-\frac{1}{2}s} \left(\mathcal{E}[w] + 1 + \varphi(s)\right) \\ &\leqslant -\frac{1}{4} \int_{\Omega_s} w_s^2 \psi^2 \rho \, dy + Ce^{-\frac{s}{2}} \int_{\Omega_s} |\nabla w|^2 |y|^2 \psi^2 \rho \, dy + C\varphi(s) + Ce^{(\alpha-\frac{1}{2})s}.\end{aligned} \tag{3.23}$$

So

$$\mathcal{E}[w](s) - \mathcal{E}[w](0) \leqslant C \int_0^s e^{-\frac{\tau}{2}} \int_{\Omega_\tau} |\nabla w|^2 |y|^2 \psi^2 \rho \, dy \, d\tau + Ce^{(\alpha-\frac{1}{2})s}.$$

We claim that

$$\int_0^s e^{-\frac{\tau}{2}} \int_{\Omega_\tau} |\nabla w|^2 |y|^2 \psi^2 \rho \, dy \, d\tau \leqslant C e^{(\alpha - \frac{1}{2})s}. \tag{3.24}$$

Indeed, if we denote the left-hand side of (3.24) by $f(s)$, then $\int_0^\infty e^{-(\alpha-\frac{1}{2})s} f'(s)\,ds \leqslant C$ by the assumption. It follows that

$$C \geqslant \int_0^s e^{-(\alpha-\frac{1}{2})s} f'(s)\,ds \geqslant f(s) e^{-(\alpha-\frac{1}{2})s},$$

by integration by parts. So (3.24) holds and

$$\mathcal{E}[w](s) \leqslant C e^{(\alpha-\frac{1}{2})s}.$$

If we set $y(s) = \int_{\Omega_s} w^2 \psi^2 \rho \, dy$ and $z(s) = \mathcal{E}[w](s)$, then by using (3.7) with $\mu = 2\lambda$ and (3.9) we have

$$y'(s) \geqslant -4z(s) + C y^{\frac{p+1}{2}} - C\varphi(s),$$
$$z'(s) \leqslant 2\lambda z + C\big(1 + \varphi(s)\big) = \left(\alpha - \frac{5}{12}\right) z + h(s),$$

where $h(s) = (2\lambda - \alpha + \frac{5}{12}) z(s) + C(1 + \varphi(s))$. Notice that we have already gotten the upper bound estimates for $z(s)$. Since $\alpha \leqslant 2\lambda$, we have $h(s) \leqslant C e^{(\alpha-\frac{1}{2})s} + C(1 + \varphi(s))$. So

$$z'(s) \leqslant \left(\alpha - \frac{5}{12}\right) z + C e^{(\alpha-\frac{1}{2})s} + C\big(1 + \varphi(s)\big).$$

It is easy to see that $\int_0^\infty e^{-(\alpha-\frac{5}{12})s} (C e^{(\alpha-\frac{1}{2})s} + C(1 + \varphi(s)))\,ds < \infty$. By Lemma 2.2, we have

$$\mathcal{E}[w](s) \geqslant -C e^{(\alpha-\frac{5}{12})s}.$$

Therefore (3.20) holds for $k = 0$.

Furthermore, by (3.23) and (3.24), we deduce that

$$\int_0^s \int_{\Omega_\tau} w_s^2 \psi^2 \rho \, dy \, d\tau \leqslant C e^{(\alpha-\frac{5}{12})s}. \tag{3.25}$$

As usual, we have

$$\int_{\Omega_s} |\nabla w|^2 \psi^2 \rho \, dy \leqslant 2\mathcal{E}[w] + \frac{2}{p+1} \int_{\Omega_s} \bar{V} |w|^{p+1} \psi^2 \rho \, dy$$
$$\leqslant C\mathcal{E}[w] + C \int_{\Omega_s} w_s^2 \psi^2 \rho \, dy + C\big(1 + \varphi(s)\big).$$

Then

$$e^{-(\alpha-\frac{1}{3})s} \int_{\Omega_s} |\nabla w|^2 \psi^2 \rho \, dy \leqslant C\big(\mathcal{E}[w] + 1 + \varphi(s)\big) e^{-(\alpha-\frac{1}{3})s} + C e^{-(\alpha-\frac{1}{3})s} \int_{\Omega_s} w_s^2 \psi^2 \rho \, dy$$
$$\leqslant C e^{-\frac{1}{12}s} + C\varphi(s) + C e^{-(\alpha-\frac{1}{3})s} \int_{\Omega_s} w_s^2 \psi^2 \rho \, dy.$$

Let $f(s) = \int_0^s \int_{\Omega_\tau} w_s^2 \rho \, dy \, d\tau$. Then for any $s > 0$,

$$\int_0^s e^{-(\alpha-\frac{1}{3})\tau}\int_{\Omega_\tau} w_s^2\rho\,dy\,d\tau = \int_0^s f'(\tau)e^{-(\alpha-\frac{1}{3})\tau}\,d\tau$$
$$= f(s)e^{-(\alpha-\frac{1}{3})s} + \left(\alpha-\frac{1}{3}\right)\int_0^s f(\tau)e^{-(\alpha-\frac{1}{3})\tau}\,d\tau$$
$$\leqslant C,$$

due to (3.25). So

$$\int_0^\infty e^{-(\alpha-\frac{1}{3})\tau}\int_{\Omega_\tau} |\nabla w|^2\rho\,dy\,d\tau \leqslant C,$$

i.e., (3.21) holds for $k=0$.

Step 2. (3.20) and (3.21) hold for all $k\in\mathbb{N}$.

Suppose (3.20) and (3.21) hold for all $k=0,1,\dots,n-1$. By the first inequality of (3.13), we have

$$\frac{d\mathcal{E}_{2n}[w]}{ds} \leqslant -\frac{1}{2}\int_{\Omega_s} w_s^2|y|^{2n}\psi^2\rho\,dy + \frac{1}{p+1}\int_{\Omega_s}\left|\frac{\partial\bar V}{\partial s}\right||w|^{p+1}|y|^{2n}\psi^2\rho\,dy$$
$$+ C\int_{\Omega_s}|\nabla w|^2|y|^{2n-2}\psi^2\rho\,dy + C\varphi_{2n}(s).$$

Notice that $|\frac{\partial\bar V}{\partial s}|\leqslant C|y|e^{-\frac{s}{2}}$. By Young's inequality, we obtain for $\varepsilon>0$,

$$\frac{1}{p+1}\int_{\Omega_s}\left|\frac{\partial\bar V}{\partial s}\right||w|^{p+1}|y|^{2n}\psi^2\rho\,dy$$
$$\leqslant Ce^{-\frac{s}{2}}\int_{\Omega_s}|w|^{p+1}|y|^{2n+1}\psi^2\rho\,dy$$
$$\leqslant e^{-\frac{s}{2}}\int_{\Omega_s}|w|^{p+1}\big[\varepsilon|y|^{2n}+C(\varepsilon)|y|^{2n+2}\big]\psi^2\rho\,dy$$
$$= \varepsilon e^{-\frac{s}{2}}\int_{\Omega_s}|w|^{p+1}|y|^{2n}\psi^2\rho\,dy + C(\varepsilon)e^{-\frac{s}{2}}\int_{\Omega_s}|w|^{p+1}|y|^{2n+2}\psi^2\rho\,dy. \tag{3.26}$$

From the definition of $\mathcal{E}_{2n+2}[w]$ and Young's inequality, we get for $\varepsilon>0$,

$$\frac{d_1}{p+1}\int_{\Omega_s}|w|^{p+1}|y|^{2n+2}\psi^2\rho\,dy$$
$$\leqslant \frac{1}{p+1}\int_{\Omega_s}\bar V|w|^{p+1}|y|^{2n+2}\psi^2\rho\,dy$$
$$= \frac{1}{2}\int_{\Omega_s}\big(|\nabla w|^2+\beta w^2\big)|y|^{2n+2}\psi^2\rho\,dy - \mathcal{E}_{2n+2}[w](s)$$
$$\leqslant \int_{\Omega_s}|\nabla w|^2|y|^{2n+2}\psi^2\rho\,dy - \mathcal{E}_{2n+2}[w](s) + \varepsilon\int_{\Omega_s}|w|^{p+1}|y|^{2n+2}\psi^2\rho\,dy + C(\varepsilon).$$

By choosing some small $\varepsilon>0$, we can obtain that

$$\int_{\Omega_s} |w|^{p+1}|y|^{2n+2}\psi^2\rho\,dy \leqslant C\int_{\Omega_s} |\nabla w|^2|y|^{2n+2}\psi^2\rho\,dy + C - C\mathcal{E}_{2n+2}[w](s). \tag{3.27}$$

On the other hand, taking $k = n$ in (3.12), we have

$$\int_{\Omega_s} |w|^{p+1}|y|^{2n}\psi^2\rho\,dy \leqslant C\mathcal{E}_{2n}[w](s) + \frac{1}{4d_2}\int_{\Omega_s} w_s^2|y|^{2n}\psi^2\rho\,dy + C\big(1+\varphi_{2n}(s)\big). \tag{3.28}$$

Combining (3.26)–(3.28) and using the assumptions $|\mathcal{E}_{2k}[w](s)| \leqslant M_k e^{\alpha s}$, we have the following inequality

$$\begin{aligned}
&\frac{1}{p+1}\int_{\Omega_s}\left|\frac{\partial \bar V}{\partial s}\right||w|^{p+1}|y|^{2n}\psi^2\rho\,dy\\
&\quad\leqslant \varepsilon e^{-\frac{s}{2}}\left[C\mathcal{E}_{2n}[w](s) + \frac{1}{4d_2}\int_{\Omega_s} w_s^2|y|^{2n}\psi^2\rho\,dy + C\big(1+\varphi_{2n}(s)\big)\right]\\
&\qquad + C(\varepsilon)e^{-\frac{s}{2}}\left[C\int_{\Omega_s}|\nabla w|^2|y|^{2n+2}\psi^2\rho\,dy + C - C\mathcal{E}_{2n+2}[w](s)\right]\\
&\quad\leqslant \frac14\int_{\Omega_s} w_s^2|y|^{2n}\psi^2\rho\,dy + C\varphi_{2n}(s)\\
&\qquad + Ce^{-\frac{s}{2}}\left[\int_{\Omega_s}|\nabla w|^2|y|^{2n+2}\psi^2\rho\,dy + \big|\mathcal{E}_{2n}[w](s)\big| + 1 + \big|\mathcal{E}_{2n+2}[w](s)\big|\right]\\
&\quad\leqslant \frac14\int_{\Omega_s} w_s^2|y|^{2n}\psi^2\rho\,dy + Ce^{-\frac{s}{2}}\int_{\Omega_s}|\nabla w|^2|y|^{2n+2}\psi^2\rho\,dy + Ce^{(\alpha-\frac12)s} + C\varphi_{2n}(s).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\frac{d\mathcal{E}_{2n}[w]}{ds} &\leqslant -\frac12\int_{\Omega_s} w_s^2|y|^{2n}\psi^2\rho\,dy + \frac{1}{p+1}\int_{\Omega_s}\left|\frac{\partial\bar V}{\partial s}\right||w|^{p+1}|y|^{2n}\psi^2\rho\,dy\\
&\quad + C\int_{\Omega_s}|\nabla w|^2|y|^{2n-2}\psi^2\rho\,dy + C\varphi_{2n}(s)\\
&\leqslant -\frac14\int_{\Omega_s} w_s^2|y|^{2n}\psi^2\rho\,dy + C\int_{\Omega_s}|\nabla w|^2|y|^{2n-2}\psi^2\rho\,dy\\
&\quad + Ce^{-\frac{s}{2}}\int_{\Omega_s}|\nabla w|^2|y|^{2n+2}\psi^2\rho\,dy + Ce^{(\alpha-\frac12)s} + C\varphi_{2n}(s).
\end{aligned}$$

Hence we get

$$\begin{aligned}
\mathcal{E}_{2n}[w](s) - \mathcal{E}_{2n}[w](0) &\leqslant C\int_0^s e^{-\frac{\tau}{2}}\int_{\Omega_\tau}|\nabla w|^2|y|^{2n+2}\psi^2\rho\,dy\,d\tau + Ce^{(\alpha-\frac12)s}\\
&\quad + C\int_0^s\int_{\Omega_\tau}|\nabla w|^2|y|^{2n-2}\psi^2\rho\,dy\,d\tau.
\end{aligned}$$

Since $\int_0^\infty e^{-\alpha s}\int_{\Omega_s}|\nabla w|^2|y|^{2n+2}\psi^2\rho\,dy\,ds \leqslant N_{n+1}$, we get

$$\int_0^s e^{-\frac{\tau}{2}} \int_{\Omega_\tau} |\nabla w|^2 |y|^{2n+2} \psi^2 \rho \, dy \, d\tau \leqslant C e^{(\alpha-\frac{1}{2})s}$$

as before. Let $f(s) = \int_0^s \int_{\Omega_\tau} |\nabla w|^2 |y|^{2n-2} \psi^2 \rho \, dy \, d\tau$. Then by induction hypothesis, we have

$$\int_0^\infty f'(s) e^{-(\alpha-\delta_{n-1})s} \, ds \leqslant N_{n-1}.$$

So

$$\int_0^s f'(\tau) e^{-(\alpha-\delta_{n-1})\tau} \, d\tau = f(s) e^{-(\alpha-\delta_{n-1})s} + (\alpha - \delta_{n-1}) \int_0^s f(\tau) e^{-(\alpha-\delta_{n-1})\tau} \, d\tau \geqslant f(s) e^{-(\alpha-\delta_{n-1})s},$$

i.e., $f(s) \leqslant N_{n-1} e^{(\alpha-\delta_{n-1})s}$.

Therefore

$$\mathcal{E}_{2n}[w] \leqslant N_n e^{(\alpha-\delta_{n-1})s}. \tag{3.29}$$

Now let $y(s) = \int_{\Omega_s} w^2 |y|^{2n} \psi^2 \rho \, dy$, $z(s) = \mathcal{E}_{2n}[w]$. Then by (3.13) and (3.14), we have

$$y'(s) \geqslant -4z(s) + C y^{\frac{p+1}{2}}(s) - C\big(1 + \varphi_{2n}(s)\big),$$
$$z'(s) \leqslant 2\lambda z(s) + C \int_{\Omega_s} |\nabla w|^2 |y|^{2n-2} \psi^2 \rho \, dy + C\big(1 + \varphi_{2n}(s)\big) = 2\lambda z(s) + h(s).$$

We then have $z'(s) \leqslant (\alpha - \delta'_n) z(s) + g(s)$, where $g(s) = (2\lambda - \alpha + \delta'_n) z(s) + h(s)$ and $\delta'_n \in (\delta_n, \delta_{n-1})$. Since $\alpha < 2\lambda$, it follows from (3.29) and induction hypothesis that

$$\int_0^\infty e^{-(\alpha-\delta'_n)s} g(s) \, ds \leqslant C \int_0^\infty e^{(\delta'_n-\delta_{n-1})s} \, ds + C \int_0^\infty e^{-(\alpha-\delta'_n)s} \int_{\Omega_s} |\nabla w|^2 |y|^{2n-2} \psi^2 \rho \, dy \, ds + C \leqslant C.$$

Lemma 2.2 gives us

$$z(s) \geqslant -C e^{(\alpha-\delta'_n)s}. \tag{3.30}$$

From (3.29) and (3.30), we know that (3.20) holds for $k = n$.

From the fact that

$$\frac{d\mathcal{E}_{2n}[w]}{ds} \leqslant -\frac{1}{4} \int_{\Omega_s} w_s^2 |y|^{2n} \psi^2 \rho \, dy + \big(\alpha - \delta'_n\big) \mathcal{E}_{2n}[w] + g(s)$$

and above estimates, we have

$$\int_0^\infty e^{-(\alpha-\delta'_n)s} \int_{\Omega_s} w_s^2 |y|^{2n} \psi^2 \rho \, dy \, ds \leqslant C.$$

As before, we have

$$\int_{\Omega_s} |\nabla w|^2 |y|^{2n} \psi^2 \rho \, dy \leqslant C \mathcal{E}_{2n}[w] + C \int_{\Omega_s} w_s^2 |y|^{2n} \psi^2 \rho \, dy + C\big(1 + \varphi_{2n}(s)\big).$$

Multiplying $e^{-(\alpha-\delta_n)s}$ on both sides and integrating over $(0, \infty)$, we obtain

$$\int_0^\infty e^{-(\alpha-\delta_n)s} \int_{\Omega_s} |\nabla w|^2 |y|^{2n} \psi^2 \rho \, dy \, ds$$
$$\leqslant C \int_0^\infty e^{-(\alpha-\delta_n)s} e^{(\alpha-\delta'_n)s} \, ds + C \int_0^\infty e^{-(\alpha-\delta'_n)s} \int_{\Omega_s} w_s^2 |y|^{2n} \psi^2 \rho \, dy \, ds + C$$
$$\leqslant C,$$

i.e., (3.21) holds for $k = n$. So the proof of this lemma is complete. □

Finally, using the above lemmas, we obtain the following local energy estimates, which include a quasi-monotonicity formula.

Theorem 3.6. *There exist positive constants C_3, C_4 and $\delta < 1/2$ depending on N, R, p, $\bar{t}$, the lower bound of V, $\|V\|_{C^1(\bar{\Omega})}$, $|\Omega|$ and $\mathcal{E}[w](\underline{s})$ such that*

$$\mathcal{E}[w](s) + \frac{1}{4} \int_\tau^s \int_{\Omega_s} w_s^2 \psi^2 \rho \, dy \, ds \leqslant \mathcal{E}[w](\tau) + C_3 e^{-\delta\tau}, \quad \forall s > \tau \geqslant \underline{s}; \tag{3.31}$$

$$\mathcal{E}[w](s) \geqslant -C_4 e^{-\delta s}, \quad \forall s \geqslant \underline{s}. \tag{3.32}$$

Proof. By Lemmas 3.4 and 3.5, there exist two positive constants M, N and some $\alpha \in (0, \frac{1}{2})$ such that

$$\bigl|\mathcal{E}[w](s)\bigr|, \bigl|\mathcal{E}_2[w](s)\bigr| \leqslant M e^{\alpha s},$$
$$\int_0^\infty e^{-\alpha s} \int_{\Omega_s} |\nabla w|^2 |y|^2 \psi^2 \rho \, dy \, ds \leqslant N.$$

Recall from (3.2) that

$$\frac{d\mathcal{E}[w]}{ds} \leqslant -\frac{1}{2} \int_{\Omega_s} w_s^2 \psi^2 \rho \, dy + C e^{-s/2} \int_{\Omega_s} |y|^2 |w|^{p+1} \psi^2 \rho \, dy + C e^{-s/2} \int_{\Omega_s} |w|^{p+1} \psi^2 \rho \, dy.$$

By the lower bound of $\mathcal{E}_2[w]$ and Young's inequality, we get

$$e^{-s/2} \int_{\Omega_s} |y|^2 |w|^{p+1} \psi^2 \rho \, dy \leqslant C e^{-s/2} \biggl(\int_{\Omega_s} |\nabla w|^2 |y|^2 \psi^2 \rho \, dy + \int_{\Omega_s} |w|^{p+1} \psi^2 \rho \, dy + C e^{\alpha s} + C \biggr)$$
$$\leqslant C e^{-s/2} \int_{\Omega_s} |\nabla w|^2 |y|^2 \psi^2 \rho \, dy + C e^{-s/2} \int_{\Omega_s} |w|^{p+1} \psi^2 \rho \, dy$$
$$+ C e^{-s/2} + C e^{(\alpha - \frac{1}{2})s}.$$

Using (3.5) with η small enough, we have

$$\frac{d\mathcal{E}[w]}{ds} \leqslant -\frac{1}{2} \int_{\Omega_s} w_s^2 \psi^2 \rho \, dy + C e^{-s/2} \int_{\Omega_s} |\nabla w|^2 |y|^2 \psi^2 \rho \, dy$$
$$+ C e^{-s/2} \int_{\Omega_s} |w|^{p+1} \psi^2 \rho \, dy + C e^{-s/2} + C e^{(\alpha - \frac{1}{2})s}$$
$$\leqslant -\frac{1}{4} \int_{\Omega_s} w_s^2 \psi^2 \rho \, dy + C e^{-s/2} \int_{\Omega_s} |\nabla w|^2 |y|^2 \psi^2 \rho \, dy$$

$$+ Ce^{-s/2}\big(\mathcal{E}[w] + C\big(1+\varphi(s)\big)\big) + Ce^{(\alpha-\frac{1}{2})s}$$
$$\leqslant -\frac{1}{4}\int_{\Omega_s} w_s^2\psi^2\rho\,dy + Ce^{-s/2}\int_{\Omega_s} |\nabla w|^2|y|^2\psi^2\rho\,dy$$
$$+ Ce^{-s/2}\varphi(s) + Ce^{(\alpha-\frac{1}{2})s}.$$

Therefore, for all $s > \tau$,

$$\mathcal{E}[w](s) + \frac{1}{4}\int_\tau^s\int_{\Omega_s} w_s^2\psi^2\rho\,dy\,ds \leqslant \mathcal{E}[w](\tau) + C_3e^{-\delta\tau},$$

with $\delta = \frac{1}{2} - \alpha > 0$, i.e., (3.31) holds. If we set $y(s) = \int_{\Omega_s} w^2\psi^2\rho\,dy$, then by (3.9), (3.31) and Jensen's inequality we have for all $s > \tau + 1$,

$$y(s) \geqslant y(\tau) - 4\int_\tau^s \mathcal{E}[w](\sigma)\,d\sigma + C\int_\tau^s y^{\frac{p+1}{2}}(\sigma)\,d\sigma - C\int_\tau^s \varphi(\sigma)\,d\sigma$$
$$\geqslant -4\big(\mathcal{E}[w](\tau) + C_3e^{-\delta\tau}\big)(s-\tau) - Ce^{-\tau} + C\int_\tau^s y^{\frac{p+1}{2}}(\sigma)\,d\sigma$$
$$\geqslant -4\big(\mathcal{E}[w](\tau) + C_4e^{-\delta\tau}\big)(s-\tau) + C\int_\tau^s y^{\frac{p+1}{2}}(\sigma)\,d\sigma.$$

So if there is a $\tau \geqslant \underline{s}$ such that $\mathcal{E}[w](\tau) + C_4e^{-\delta\tau} < 0$, then $y(s) \geqslant C\int_\tau^s y^{\frac{p+1}{2}}(\sigma)\,d\sigma$ for all $s > \tau + 1$. Hence $\int_\tau^s y^{\frac{p+1}{2}}(\sigma)\,d\sigma$ will blow up in finite time. This is impossible. The theorem is proved. □

Remark 3.1. We can see from this theorem that the local energy $\mathcal{E}[w]$ is bounded from below and above. When the cutoff function ψ is identically 1, we can simplify the proof and get this property, i.e., the main result in [5], even if the exponent p is critical or supercritical.

The following corollary is crucial to get the ε-regularity of the borderline solution.

Corollary 3.1. *There exists a positive constant C, which depends on N, R, p, $\bar{t}$, the lower bound of V, $\|V\|_{C^1(\bar{\Omega})}$, $|\Omega|$ and $\mathcal{E}[w](\underline{s})$, such that for all $s > \tau \geqslant \underline{s}$,*

$$\int_\tau^s\int_{\Omega_s} |w|^{p+1}\psi^2\rho\,dy\,ds \leqslant C\big[1 + (s-\tau)\big]\eta\big(\mathcal{E}[w](\tau) + C_5e^{-\delta\tau}\big),$$

where $\eta(s) = s + s^{1/2}$ and $C_5 = C_4 + 1$.

Proof. By (3.31) and (3.32), it is easy to see that

$$\int_\tau^s\int_{\Omega_s} w_s^2\psi^2\rho\,dy\,ds \leqslant 4\mathcal{E}[w](\tau) + Ce^{-\delta\tau}. \tag{3.33}$$

It turns out that

$$\int_0^\infty\int_{\Omega_s} w_s^2\psi^2\rho\,dy\,ds \leqslant C.$$

Next,

$$\frac{1}{2}\frac{d}{ds}\int_{\Omega_s} w^2\psi^2\rho\,dy = \int_{\Omega_s} ww_s\psi^2\rho\,dy + \int_{\Omega_s} w^2\psi\psi_s\rho\,dy$$
$$\leqslant \Big(\int_{\Omega_s} w^2\psi^2\rho\,dy\Big)^{1/2}\Big(\int_{\Omega_s} w_s^2\psi^2\rho\,dy + \int_{\Omega_s} w^2\psi_s^2\rho\,dy\Big)^{1/2}$$
$$\leqslant \Big(\int_{\Omega_s} w^2\psi^2\rho\,dy\Big)^{1/2}\Big(\int_{\Omega_s} w_s^2\psi^2\rho\,dy + \varphi(s)\Big)^{1/2}.$$

Let $y(s) = \int_{\Omega_s} w^2\psi^2\rho\,dy$. Then for any $\varepsilon > 0$, $\tau_2 > \tau_1$,

$$\big(y(\tau_2)+\varepsilon\big)^{1/2} - \big(y(\tau_1)+\varepsilon\big)^{1/2} \leqslant \int_{\tau_1}^{\tau_2}\Big(\int_{\Omega_s} w_s^2\psi^2\rho\,dy + \varphi(s)\Big)^{1/2}ds$$
$$\leqslant (\tau_2-\tau_1)^{1/2}\int_{\tau_1}^{\tau_2}\Big(\int_{\Omega_s} w_s^2\psi^2\rho\,dy + \varphi(s)\Big)ds$$
$$\leqslant C(\tau_2-\tau_1)^{1/2}.$$

Letting $\varepsilon \downarrow 0$, we get

$$\big(y(\tau_2)\big)^{1/2} - \big(y(\tau_1)\big)^{1/2} \leqslant C(\tau_2-\tau_1)^{1/2}, \quad \text{for all } \tau_2 > \tau_1. \tag{3.34}$$

On the other hand, by (3.5), for any $\tau > 0$,

$$\int_{\tau}^{\tau+1}\int_{\Omega_s} |w|^{p+1}\psi^2\rho\,dy\,ds \leqslant C\int_{\tau}^{\tau+1}\mathcal{E}[w](s)\,ds + C\int_{\tau}^{\tau+1}\int_{\Omega_s} w_s^2\psi^2\rho\,dy\,ds + C\int_{\tau}^{\tau+1}\big(1+\varphi(s)\big)\,ds$$
$$\leqslant C.$$

It follows from Hölder's inequality that

$$\int_{\tau}^{\tau+1} y(s)^{\frac{p+1}{2}}\,ds \leqslant C\int_{\tau}^{\tau+1}\int_{\Omega_s} |w|^{p+1}\psi^2\rho\,dy\,ds \leqslant C.$$

So for each positive integer j, there exists an $s_j \in [j, j+1]$ such that $y(s_j) \leqslant C$. Combining this with (3.34), we have

$$\int_{\Omega_s} w^2\psi^2\rho\,dy \leqslant C$$

for all $s \geqslant 0$.

From (3.4), we have

$$\frac{p-1}{p+1}\int_{\Omega_s} \bar V|w|^{p+1}\psi^2\rho\,dy = \int_{\Omega_s} ww_s\psi^2\rho\,dy + 2\mathcal{E}[w] + 2\int_{\Omega_s}\nabla w\nabla\psi\, w\psi\rho\,dy$$
$$\leqslant \Big(\int_{\Omega_s} w^2\psi^2\rho\,dy\Big)^{1/2}\Big(\int_{\Omega_s} w_s^2\psi^2\rho\,dy\Big)^{1/2} + 2\mathcal{E}[w] + C\varphi(s)$$
$$\leqslant C\Big(\int_{\Omega_s} w_s^2\psi^2\rho\,dy\Big)^{1/2} + 2\mathcal{E}[w] + C\varphi(s).$$

Finally, by Hölder's inequality and (3.33), we obtain

$$\int_\tau^s \int_{\Omega_s} |w|^{p+1}\psi^2\rho\,dy\,ds \leqslant C\int_\tau^s \biggl(\int_{\Omega_s} w_s^2\psi^2\rho\,dy\biggr)^{1/2} ds + C\int_\tau^s \mathcal{E}[w](s)\,ds + C\int_\tau^s \varphi(s)\,ds$$
$$\leqslant C(s-\tau)^{1/2}\bigl[\bigl(\mathcal{E}[w](\tau)+C_4e^{-\delta\tau}\bigr)+Ce^{-\delta\tau}\bigr]^{1/2}$$
$$+ C\bigl[\mathcal{E}[w](\tau)+C_4e^{-\delta\tau}\bigr](s-\tau)+Ce^{-\delta\tau}$$
$$\leqslant C\bigl[1+(s-\tau)\bigr]\eta\bigl(\mathcal{E}[w](\tau)+C_5e^{-\delta\tau}\bigr),$$

where $\eta(s)=s+s^{1/2}$ and $C_5=C_4+1$. □

4. ε-Regularity and partial regularity

In this section we will establish ε-regularity theorem and partial regularity theorem for a borderline solution to (1.1). To this end, let us rewrite the crucial estimates we have shown in Section 3 back to unscaled form.

For $\bar z=(\bar x,\bar t)\in\Omega\times(0,\infty)$, $R<\operatorname{dist}(\bar x,\partial\Omega)$, and a global classical solution u to (1.1) we define

$$E_{\bar z}(t)=\frac{1}{2}(\bar t-t)^{\frac{p+1}{p-1}}\int_\Omega |\nabla u|^2G_{\bar z}\phi^2\,dx+\frac{1}{2}(\bar t-t)^{\frac{2}{p-1}}\int_\Omega u^2G_{\bar z}\phi^2\,dx$$
$$-\frac{1}{p+1}(\bar t-t)^{\frac{p+1}{p-1}}\int_\Omega \bar V|u|^{p+1}G_{\bar z}\phi^2\,dx,$$

where $\phi=\phi_{\bar x,R}(x)=\phi((x-\bar x)/R)$ and

$$G_{\bar z}(x,t)=\frac{1}{(\bar t-t)^{n/2}}e^{-\frac{|x-\bar x|^2}{4(\bar t-t)}}$$

is a constant multiple of the backward heat kernel at $(\bar x,\bar t)$. Actually, we have $E_{\bar z}(t)=\mathcal{E}[w](\tau)$, under the rescaling described in the previous section. So from (3.31) we have the following quasi-monotonicity formula for the local energy of the solution

$$E_{\bar z}(t)\leqslant E_{\bar z}\bigl(t'\bigr)+C\bigl(\bar t-t'\bigr)^{\delta}, \tag{4.1}$$

where $\tau=-\log(\bar t-t)$, $\tau'=-\log(\bar t-t')$ and $\delta>0$ is the constant described in Theorem 3.6. From Corollary 3.1 we also have for all $0\leqslant t'<t\leqslant\bar t$,

$$\int_{t'}^{t}(\bar t-t)^{\frac{2}{p-1}}\int_\Omega |u|^{p+1}G_{\bar z}\phi^2\,dx\,dt\leqslant C\biggl(1+\log\frac{\bar t-t'}{\bar t-t}\biggr)\eta\bigl(E_{\bar z}\bigl(t'\bigr)+C_5\bigl(\bar t-t'\bigr)^{\delta}\bigr). \tag{4.2}$$

With (4.1) and (4.2) in hands, we can obtain all other results as in [6]. The proofs have little difference from those of [6]. For readers' convenience, we repeat some proofs here.

In order to get the main result, we need the following crucial lemma.

Lemma 4.1. *Let u be a positive borderline solution to* (1.1). *There exist two positive constants ε_0 and ρ_0 depending on N and $p>1$ only, such that if*

$$r^{\frac{4}{p-1}-N}\iint_{P_r(z_0)}|u|^{p+1}\,dx\,dt<\varepsilon_0$$

for all cylinders $P_{2r}(z_0)=B_{2r}(x_0)\times(t_0-4r^2,t_0+4r^2)$, $z_0=(x_0,t_0)$, contained inside the cylinder $P_R(\bar z)$, then

$$\operatorname*{ess\,sup}_{P_{R/4}(\bar z)}|u|\leqslant\rho_0R^{\frac{-2}{p-1}}.$$

Proof. Let u be a classical solution first. Consider

$$M = \sup_{0<r<R'} \Big[\big(R'-r\big)^{\frac{2}{p-1}} \sup_{\overline{P_r(\bar z)}} u \Big],$$

where $R' = R/2$ and let $r_0 \geqslant 0$ and $z^* = (x^*, t^*) \in P_{r_0}(\bar z)$ satisfy

$$M = \big(R' - r_0\big)^{\frac{2}{p-1}} u\big(z^*\big).$$

Let $r_1 = (R' - r_0)/2$. Then $P_{r_1}(z^*) \subseteq P_{R'-r_1}(\bar z)$, so

$$r_1^{\frac{2}{p-1}} \sup_{P_{r_1}(z^*)} |u| \leqslant M.$$

It implies that

$$\sup_{P_{r_1}(z^*)} |u| \leqslant \left(\frac{R' - r_0}{r_1} \right)^{\frac{2}{p-1}} u\big(z^*\big) \leqslant 4^{\frac{1}{p-1}} u\big(z^*\big).$$

We set

$$v(y,s) = \frac{1}{\mu} u\big(x^* + \mu^{\frac{1-p}{2}} y, t^* + \mu^{1-p} s\big), \quad \mu = u\big(z^*\big).$$

Then v satisfies

$$v_s = \Delta v + \widetilde{V} |v|^{p-1} v, \tag{4.3}$$

$$|v| \leqslant 4^{\frac{1}{p-1}}, \qquad \big|v(0,0)\big| = 1, \tag{4.4}$$

in $P_{\mu^{\frac{p-1}{2}} r_1}(0,0)$. Here $\widetilde{V}(y,s) = V(x^* + \mu^{\frac{1-p}{2}} y, t^* + \mu^{1-p} s)$. We claim that $M \leqslant 4^{\frac{2}{p-1}}$. For, if $M > 4^{\frac{2}{p-1}}$, then $\mu^{\frac{p-1}{2}} r_1 \geqslant 2$ and (4.3), (4.4) hold in $P_2(0,0)$. We have

$$\iint\limits_{P_2(0,0)} |v|^{p+1}\, dy\, ds = \mu^{\frac{p-1}{2}N - 2} \iint\limits_{P_{2\mu^{\frac{1-p}{2}}}(z^*)} |u|^{p+1}\, dx\, dt < 2^{N - \frac{4}{p-1}} \varepsilon_0.$$

Notice that $\widetilde{V}$ is bounded. Regarding (4.3) as a linear parabolic equation $v_s = \Delta v + b(y,s)v$ with bounded coefficient b, we infer from interior parabolic estimates, see Ladyzenskaja, Solonnikov and Uralceva [18], that

$$\sup_{P_1(0,0)} |v| \leqslant C \|v\|_{L^{p+1}(P_2(0,0))} \leqslant C' \varepsilon_0^{\frac{1}{p+1}}.$$

By choosing ε_0 so small that $C' \varepsilon_0^{\frac{1}{p+1}} < 1$, a contradiction with (4.4) occurs. Hence we must have $M \leqslant 4^{\frac{2}{p-1}}$. But then the desired result follows by taking $r = R/2$ in the expression of M. Now the general case can be deduced from approximation. □

Using the basic estimates (4.1), (4.2) and the above lemma, we get the following ε-regularity theorem.

Theorem 4.2. *Let u be a classical solution or a positive borderline solution to* (1.1). *For each $(\bar x, \bar t) \in \Omega \times (0, \infty)$ and $R < \operatorname{dist}(\bar x, \partial\Omega)$, there exist constants ε_1, $K > 1$, ρ_0 and $\delta_0 < 1/2$ depending on N, p, R, $\bar t$, and E_0 such that if for some $r \leqslant \delta_0 R$,*

$$r^{\frac{4}{p-1} - N} \int_{\bar t - 9r^2}^{\bar t - 4r^2} \int_{B_{Kr}(\bar x)} \big(|\nabla u|^2 + |u|^{p+1}\big)\, dx\, dt < \varepsilon_1,$$

then

$$\operatorname{ess}\sup_{P_{r/4}(\bar{z})} |u| \leqslant \rho_0 r^{\frac{2}{1-p}}.$$

Proof. Assume that u is classical first. For any ε, $K > 0$ we claim that if

$$r^{\frac{4}{p-1}-n} \int_{\bar{t}-9r^2}^{\bar{t}-4r^2} \int_{B_{Kr}(\bar{x})} \left(|\nabla u|^2 + |u|^{p+1}\right) dx\, dt < \varepsilon, \tag{4.5}$$

for some small r satisfying $Kr < R$, $r < 1$ and $9r^2 \leqslant \bar{t}$, then

$$r^{\frac{4}{p-1}} \int_{\bar{t}-9r^2}^{\bar{t}-4r^2} \int_{\Omega} \left(|\nabla u|^2 + |u|^{p+1}\right) G_{(\bar{x},\bar{t}+20r^2)} \phi^2_{\bar{x},R}\, dx\, dt < \varepsilon + Ce^{-\frac{K^2}{3480}}. \tag{4.6}$$

For, we have

$$\begin{aligned} \frac{G_{(\bar{x},\bar{t}+20r^2)}}{G_{(\bar{x},\bar{t}+21r^2)}}(x,t) &\leqslant \left(\frac{\bar{t}+21r^2-t}{\bar{t}+20r^2-t}\right)^{N/2} \exp\left(-\frac{|x-\bar{x}|^2}{4}\left(\frac{r^2}{(\bar{t}+20r^2-t)(\bar{t}+21r^2-t)}\right)\right) \\ &\leqslant \left(\frac{5}{4}\right)^{N/2} \exp\left(-\frac{K^2}{4 \times 29 \times 30}\right) \end{aligned}$$

for $|x-\bar{x}| \geqslant Kr$ and $t \in [\bar{t}-9r^2, \bar{t}-4r^2]$. It follows from (4.2) (taking $\bar{z}$ to be $(\bar{x}, \bar{t}+21r^2)$) and (4.1) that

$$r^{\frac{4}{p-1}} \int_{\bar{t}-9r^2}^{\bar{t}-4r^2} \int_{\mathbb{R}^N \setminus B_{Kr}(\bar{x})} \left(|\nabla u|^2 + |u|^{p+1}\right) G_{(\bar{x},\bar{t}+20r^2)} \phi^2_{\bar{x},R}\, dx\, dt \leqslant Ce^{-\frac{K^2}{3480}}.$$

Together with (4.5) it gives (4.6).

Next, we claim that by further restricting δ_0 in $r = \delta_0 R$, (4.6) implies

$$(\delta_0 R)^{\frac{4}{p-1}} \int_{\bar{t}-9(\delta_0 R)^2}^{\bar{t}-4(\delta_0 R)^2} \int_{\Omega} \left(|\nabla u|^2 + |u|^{p+1}\right) G_{(x_0,t_0+r^2)} \phi^2_{x_0,R/2}\, dx\, dt \leqslant C_6\left(\varepsilon + e^{-\frac{K^2}{3480}}\right) \tag{4.7}$$

for all $(x_0,t_0) \in B_{\delta_0 R}(\bar{x}) \times [\bar{t}-\delta_0^2 R^2, \bar{t}+\delta_0^2 R^2]$ and $r \leqslant \delta_0 R$. Indeed, for $|x-\bar{x}| > 10\delta_0 R$, $|x-x_0| \geqslant 9\delta_0 R$ and so $|x-x_0|/|x-\bar{x}| \geqslant 9/10$. Hence

$$\begin{aligned} \frac{G_{(x_0,t_0+r^2)}}{G_{(\bar{x},\bar{t}+20(\delta_0 R)^2)}}(x,t) &\leqslant \left(\frac{29}{3}\right)^{N/2} \exp\left(-\frac{|x-x_0|^2}{44\delta_0^2 R^2} + \frac{|x-\bar{x}|^2}{96\delta_0^2 R^2}\right) \\ &\leqslant \left(\frac{29}{3}\right)^{N/2} \end{aligned}$$

for $|x-\bar{x}| \geqslant 10\delta_0 R$ and $t \in [\bar{t}-9(\delta_0 R)^2, \bar{t}-4(\delta_0 R)^2]$. For $|x-\bar{x}| < 10\delta_0 R$, this quotient is bounded by some constant depending only on N. As $\phi_{x_0,R/2} \leqslant \phi_{\bar{x},R}$ for x_0 close to $\bar{x}$, (4.7) holds.

Now, applying the mean value theorem to (4.7) we can find some $\tilde{t} \in (\bar{t}-9\delta_0^2 R^2, \bar{t}-4\delta_0^2 R^2)$ such that

$$(\delta_0 R)^{\frac{2(p+1)}{p-1}} \int_{\Omega} \left(|\nabla u|^2 + |u|^{p+1}\right) G_{(x_0,t_0+r^2)} \phi^2_{x_0,R/2} dx \leqslant C_7\left(\varepsilon + e^{-\frac{K^2}{3480}}\right), \tag{4.8}$$

at $\tilde{t}$. Using (4.1) we have

$$E_{(x_0,t_0+r^2)}\left(\bar{t}-4\delta_0^2 R^2\right) \leqslant E_{(x_0,t_0+r^2)}(\tilde{t}) + C_8 \delta_0^{2\delta}$$

where the cutoff function in $E_{(x_0,t_0+r^2)}$ is given by $\phi_{x_0,R/2}$. The second term in $E_{(x_0,t_0+r^2)}(\tilde{t})$,

$$\frac{1}{2}(\bar{t}-\tilde{t})^{\frac{2}{p-1}}\int_\Omega u^2 G_{(x_0,t_0+r^2)}\phi^2_{x_0,R/2}\,dx,$$

is controlled via Hölder inequality by

$$C(\delta_0 R)^{\frac{4}{p-1}}\left(\int_\Omega |u|^{p+1}G_{(x_0,t_0+r^2)}\phi^2_{x_0,R/2}\,dx\right)^{\frac{2}{p+1}}\left(\int_\Omega G_{(x_0,t_0+r^2)}\,dx\right)^{\frac{p-1}{p+1}}$$
$$\leqslant C\left[(\delta_0 R)^{\frac{2(p+1)}{p-1}}\int_\Omega |u|^{p+1}G_{(x_0,t_0+r^2)}\phi^2_{x_0,R/2}\,dx\right]^{\frac{2}{p+1}}.$$

Therefore, using (4.8) we have

$$E_{(x_0,t_0+r^2)}\big(\bar{t}-4\delta_0^2R^2\big)\leqslant C\big[\big(\varepsilon+e^{-\frac{K^2}{3480}}\big)+\big(\varepsilon+e^{-\frac{K^2}{3480}}\big)^{\frac{2}{p+1}}+\delta_0^{2\delta}\big]. \tag{4.9}$$

Finally, by combining with (4.2), (4.1) and (4.9)

$$\left(\frac{r}{2}\right)^{\frac{4}{p-1}-N}\int_{t_0-\frac{1}{4}r^2}^{t_0+\frac{1}{4}r^2}\int_{B_{r/2}(x_0)}|u|^{p+1}\,dx\,dt\leqslant C\int_{t_0-\frac{1}{4}r^2}^{t_0+\frac{1}{4}r^2}\big(t_0+r^2-t\big)^{\frac{2}{p-1}}\int_\Omega |u|^{p+1}G_{(x_0,t_0+r^2)}\phi^2_{x_0,R/2}\,dx\,dt$$
$$\leqslant C\eta\left(E_{(x_0,t_0+r^2)}\left(t_0-\frac{1}{4}r^2\right)+C'\delta_0^{2\delta}\right)$$
$$\leqslant C\eta\big(E_{(x_0,t_0+r^2)}\big(\bar{t}-4\delta_0^2R_0^2\big)+C'\delta_0^{2\delta}\big)$$
$$\leqslant C\eta\big(C'\big(\varepsilon+e^{\frac{-K^2}{3480}}\big)+\big(\varepsilon+e^{-\frac{K^2}{3480}}\big)^{\frac{2}{p+1}}+C'\delta_0^{2\delta}\big).$$

Now, if we first fix K sufficiently large and then δ_0 sufficiently small, we can make

$$\left(\frac{r}{2}\right)^{\frac{4}{p-1}-N}\iint_{P_{r/2}(z_0)}|u|^{p+1}\,dx\,dt<\varepsilon_0,$$

where ε_0 is specified in Lemma 4.1, for all $P_r(z_0)$ contained inside $B_{\delta_0 R}(\bar{x})\times(\bar{t}-\delta_0^2R^2,\bar{t}+\delta_0^2R^2)$. By Lemma 4.1 the conclusion is drawn. When u is a positive borderline solution the same conclusion holds by an approximation argument. □

We are now in the position to give the partial regularity theorem.

Theorem 4.3. *Let u be a positive borderline solution to* (1.1). *For any subdomain Q' compactly contained in $\Omega\times(0,\infty)$, there exists a compact subset $\mathcal{S}_{Q'}$ in $\overline{Q}'$ with $\mathcal{H}^{N-\frac{4}{p-1}}(\mathcal{S}_{Q'})=0$, so that u is continuous in $\overline{Q}'\setminus\mathcal{S}_{Q'}$.*

Proof. Let

$$\mathcal{S}_{Q'}=\left\{(\bar{x},\bar{t})\in\overline{Q}'\text{: For }(\bar{x},\bar{t}),\text{ there exists }r_0\text{ such that}\right.$$
$$\left.r^{\frac{4}{p-1}-N}\int_{\bar{t}-9r^2}^{\bar{t}-4r^2}\int_{B_{Kr}(\bar{x})}\big(|\nabla u|^2+|u|^{p+1}\big)\,dx\,dt\geqslant\varepsilon_1\text{ for all }r\leqslant r_0\right\},$$

where K and ε_1 are specified in Theorem 4.2 (taking $R=\operatorname{dist}(\overline{Q}',\partial\Omega\times(0,\infty))/2$, say). By the Lebesgue differentiation theorem

$$\lim_{r\to 0} r^{-N-2} \int_{\bar{t}-9r^2}^{\bar{t}+9r^2} \int_{B_{Kr}(\bar{x})} \left(|\nabla u|^2 + |u|^{p+1}\right) dx\, dt$$

exists a.e., so $\mathcal{S}_{Q'}$ is of zero Lebesgue measure in $\Omega \times (0,\infty)$. For any $\varepsilon > 0$, we can find an open set U containing $\mathcal{S}_{Q'}$ such that

$$\iint_U \left(|\nabla u|^2 + |u|^{p+1}\right) dx\, dt < \varepsilon.$$

For each $r_1 \leqslant r_0/K$, consider now the collection $\mathcal{F}$ of the closed cylinders of the form

$$\overline{B_{Kr}(\bar{x})} \times \left[\bar{t} - K^2r^2, \bar{t} + K^2r^2\right], \quad (\bar{x},\bar{t}) \in \mathcal{S},\ r \leqslant r_1,$$

which are contained inside U. Here we assume $K \geqslant 3$. $\mathcal{F}$ forms a cover of $\mathcal{S}_{Q'}$. By a variant of Vitali covering theorem, see Caffarelli, Kohn and Nirenberg [3], we can find a finite collection of these cylinders, $\overline{B_{Kr_j}(\bar{x}_j)} \times [\bar{t}_j - K^2r_j^2, \bar{t}_j + K^2r_j^2]$, $j = 1,\dots,N$, such that they are mutually disjoint, and

$$\mathcal{S}_{Q'} \subseteq \bigcup_{j=1}^{N} \overline{B_{5Kr_j}(\bar{x}_j)} \times \left[\bar{t}_j - 25K^2r_j^2, \bar{t}_j + 25K^2r_j^2\right].$$

We have

$$\begin{aligned}
\varepsilon_1 \sum_j (5Kr_j)^{n-\frac{4}{p-1}} &\leqslant (5K)^{N-\frac{4}{p-1}} \sum_j \int_{\bar{t}_j-9r_j^2}^{\bar{t}_j+9r_j^2} \int_{B_{Kr_j}(\bar{x}_j)} \left(|\nabla u|^2 + |u|^{p+1}\right) dx\, dt \\
&\leqslant \iint_U \left(|\nabla u|^2 + |u|^{p+1}\right) dx\, dt \\
&\leqslant \varepsilon.
\end{aligned}$$

Therefore,

$$\mathcal{H}_{r_1}^{N-4/(p-1)}(\mathcal{S}_{Q'}) \leqslant \frac{\varepsilon}{\varepsilon_1}.$$

Letting $\varepsilon \downarrow 0$ and then $r_1 \downarrow 0$ the theorem holds. □

For a general borderline solution a weaker estimate holds. The proof is similar to that of Theorem 4.3, for details, see e.g. [6].

Theorem 4.4. *Let u be a borderline solution to* (1.1)*. For any subdomain Q' compactly contained in $\Omega \times (0,\infty)$, there exists a compact subset $\mathcal{S}_{Q'}$ in Q' with $\mathcal{H}^{N-\frac{4}{p-1}}(\mathcal{S}_{Q'}) < \infty$, so that u is continuous in $Q' \setminus \mathcal{S}_{Q'}$.*

Proof of Theorems 1.1 and 1.2. When Ω is convex, by the method of moving planes and the L^2-estimates, we can show as in [6] that any positive borderline solution is uniformly bounded near the boundary. Therefore, no singularities can occur in this region. So Theorem 1.1 follows from Theorem 4.3. Furthermore, as an application of ε-regularity theorem, we can show as in [6] Theorem 1.2 holds. □

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