

The mean curvature at the first singular time of the mean curvature flow

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Abstract

Consider a family of smooth immersions $F(\cdot, t) : M^n \rightarrow \mathbb{R}^{n+1}$ of closed hypersurfaces in \mathbb{R}^{n+1} moving by the mean curvature flow $\frac{\partial F(p,t)}{\partial t} = -H(p,t) \cdot \nu(p,t)$, for $t \in [0, T)$. We prove that the mean curvature blows up at the first singular time T if all singularities are of type I. In the case $n = 2$, regardless of the type of a possibly forming singularity, we show that at the first singular time the mean curvature necessarily blows up provided that either the Multiplicity One Conjecture holds or the Gaussian density is less than two. We also establish and give several applications of a local regularity theorem which is a parabolic analogue of Choi–Schoen estimate for minimal submanifolds.

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1. Introduction

Let M^n be a compact n -dimensional hypersurface without boundary, and let $F_0 : M^n \rightarrow \mathbb{R}^{n+1}$ be a smooth immersion of M^n into \mathbb{R}^{n+1} . Consider a smooth one-parameter family of immersions

$$F(\cdot, t) : M^n \rightarrow \mathbb{R}^{n+1}$$

satisfying $F(\cdot, 0) = F_0(\cdot)$ and

$$\frac{\partial F(p,t)}{\partial t} = -H(p,t)\nu(p,t), \quad \forall (p,t) \in M \times [0, T). \quad (1.1)$$

Here $H(p,t)$ and $\nu(p,t)$ denote the mean curvature and a choice of unit normal for the hypersurface $M_t = F(M^n, t)$ at $F(p,t)$, respectively. We will sometimes also write $x(p,t) = F(p,t)$ and refer to (1.1) as to the mean curvature flow equation. Furthermore, for any compact n -dimensional hypersurface M^n which is smoothly embedded in \mathbb{R}^{n+1} by $F : M^n \rightarrow \mathbb{R}^{n+1}$, let us denote by $g = (g_{ij})$ the induced metric, $A = (h_{ij})$ the second fundamental form, $d\mu = \sqrt{\det(g_{ij})} dx$ the volume form, ∇ the induced Levi-Civita connection. Then the mean curvature of M^n is given by $H = g^{ij} h_{ij}$.

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Without any special assumptions on M_0 , the mean curvature flow (1.1) will in general develop singularities in finite time, characterized by a blow up of the second fundamental form $A(\cdot, t)$.

Theorem 1.1 (Huisken [11]). *Suppose $T < \infty$ is the first singularity time for a compact mean curvature flow. Then $\sup_{M_t} |A|(\cdot, t) \rightarrow \infty$ as $t \rightarrow T$.*

By the work of Huisken and Sinestrari [14] the blow up of H near a singularity is known for mean convex hypersurfaces. They show that when $H \geq 0$ one has a pinching curvature estimate stating that $|A|^2 \leq C_1 H^2 + C_2$, for uniform constants C_1, C_2 . In [22] a similar pinching estimate has been proven for star shaped hypersurfaces. The present article establishes the blow up of the mean curvature in the case of type-I singularities.

Definition 1.1. We say that the mean curvature flow (1.1) develops a singularity of type I at $T < \infty$ if the blow-up rate of the curvature satisfies an upper bound of the form

$$\max_{M_t} |A|^2(\cdot, t) \leq \frac{C_0}{T-t}, \quad 0 \leq t < T. \quad (1.2)$$

In this paper, we prove the following

Theorem 1.2. *Assume (1.2) for the mean curvature flow (1.1). If*

$$\max_{M_t} |H|^2(\cdot, t) \leq C_0 \quad (1.3)$$

then the flow can be extended past time T .

In fact, the above theorem is a consequence of the following result.

Theorem 1.3. *Assume (1.2) for the mean curvature flow (1.1). If for some $\alpha \geq n + 2$*

$$\|H\|_{L^\alpha(M \times [0, T])} \leq C_0 \quad (1.4)$$

then the flow can be extended past time T .

The proofs of Theorems 1.2 and 1.3 are based on blow-up arguments using Huisken's monotonicity formula, the classification of self-shrinkers and White's local regularity theorem for mean curvature flow.

Remark 1.1. To some extent, the condition $\alpha \geq n + 2$ appearing in Theorem 1.3 is optimal as illustrated by the mean curvature flow of the standard sphere S^n .

Our Theorems 1.2 and 1.3 left open the question on the possible blow up of the mean curvature at the first singular time T for mean curvature flows with singularities other than type I. This seems to be a difficult question. However, assuming the validity of Multiplicity One Conjecture (see p. 7 of [15] and the precise statement in Conjecture 3.1 of the present article), we prove the following

Theorem 1.4. *Let M^2 be a compact, smooth and embedded 2-dimensional manifold in \mathbb{R}^3 . If the Multiplicity One Conjecture holds and if*

$$\max_{M_t} |H|^2(\cdot, t) \leq C_0 \quad (1.5)$$

then the flow can be extended past time T .

The next result is independent of the Multiplicity One Conjecture. It is in some sense a refinement of White's local regularity theorem [26]. White gives uniform curvature bounds in regions of spacetime where the Gaussian density is close to one. We prove the following

Theorem 1.5. Let M^2 be a compact, smooth and embedded 2-dimensional manifold in \mathbb{R}^3 . Suppose that (1.5) holds. Let $y_0 \in \mathbb{R}^3$ be a point reached by the mean curvature flow (1.1) at time T . If

$$\lim_{t \nearrow T} \int \rho_{y_0, T} d\mu_t := \lim_{t \nearrow T} \int \frac{1}{[4\pi(T-t)]^{n/2}} \exp\left(-\frac{|y-y_0|^2}{4(T-t)}\right) d\mu_t < 2 \tag{1.6}$$

then (y_0, T) is a regular point of the mean curvature flow (1.1).

Remark 1.2. Our theorem says that for mean curvature flow of surfaces with Gaussian density $\lim_{t \nearrow T} \int \rho_{y_0, T} d\mu_t$ below 2, for every y_0 reached by the flow at time T , the mean curvature must blow up at the first singular time. In [23], Stone calculated the Gaussian density on spheres and cylinders. On spheres, the density is $4/e \approx 1.47$ and on cylinders it is $\sqrt{2\pi}/e \approx 1.52$.

Remark 1.3. In a recent paper, Cooper [7] gave a new characterization of the singular time of the mean curvature flow, without assuming type-I singularities. Roughly speaking, his result says that at the first singular time of the mean curvature flow, the product of the mean curvature and the norm of the second fundamental form blows up. His method also gave a new proof of our Theorem 1.2.

We also give the following characterization of a finite time singularity of (1.1) that works in all dimensions $n \geq 2$.

Theorem 1.6. Assume that for the mean curvature flow (1.1), we have the following integral bound on the second fundamental form

$$\|A\|_{L^{p,q}(M \times [0, T])} := \left(\int_0^T \left(\int_{M_t} |A|^q d\mu \right)^{p/q} dt \right)^{1/p} < \infty \tag{1.7}$$

where $p, q \in (0, \infty)$ satisfy

$$\frac{n}{q} + \frac{2}{p} = 1.$$

Then the flow can be extended past time T .

The previously mentioned results were all global characterizations ensuring that the flow cannot develop any singularities as long as some global quantities are bounded uniformly in time. We also give a result regarding the local regularity theory.

Theorem 1.7. Suppose $\mathcal{M} = (M_t)$ is a smooth, properly embedded solution of the mean curvature flow in $B(x_0, \rho) \times (t_0 - \rho^2, t_0)$ which reaches x_0 at time t_0 . There exists $\varepsilon_0 = \varepsilon_0(M_0) > 0$ such that if $0 < \sigma \leq \rho$ and

$$\int_{t_0 - \sigma^2}^{t_0} \int_{M_t \cap B(x_0, \sigma)} |A|^{n+2} d\mu dt < \varepsilon_0 \tag{1.8}$$

then

$$\max_{0 \leq \delta \leq \sigma/2} \sup_{t \in [t_0 - (\sigma - \delta)^2, t_0]} \sup_{x \in B(x_0, \sigma - \delta) \cap M_t} \delta^2 |A|^2(x, t) < \varepsilon_0^{\frac{-2}{n+2}} \left(\int_{t_0 - \sigma^2}^{t_0} \int_{M_t \cap B(x_0, \sigma)} |A|^{n+2} d\mu dt \right)^{\frac{2}{n+2}}. \tag{1.9}$$

Our theorem is a parabolic version of Choi–Schoen estimate [3] for minimal surfaces. Related results can be found in Ecker [8] and Ilmanen [15]. The precise estimate of the form (1.9) for the case of minimal submanifolds can be found in Shen–Zhu [20], Proposition 2.2 (see also [4]). Moreover, in [18,27,28], the authors showed that if the L^{n+2} norm in space–time of the second fundamental form (or the mean curvature but under various convexity assumptions) is finite then it is possible to extend the mean curvature flow beyond the time interval under consideration. Our theorem

can be viewed as a local version of these results without imposing any convexity assumptions. It turns out that the conclusion of Theorem 1.7 also holds in the case when the ambient space is a complete Riemannian manifold with bounded geometry. We show that in Corollary 5.1. We would like to mention that, as far as regularity is concerned, our ε -regularity result in Theorem 1.7 is a direct corollary of Theorem 14 in [15], which works in any codimension. However, the essence of our theorem is the precise estimate (1.9). Note that the constant in Theorem 1.7 does not depend on the initial hypersurface, but only on a dimension. Finally, up to a constant depending on the dimension n and the initial hypersurface M_0 , our estimate (1.9) is implied as well by the estimate (27) in Theorem 14 of [15]. This can be seen using Hölder’s inequality and volume bound during the mean curvature flow as in Lemma 1.4 in Ecker [8].

The organization of the paper is as follows. In Section 2 we give the proofs of Theorems 1.2 and 1.3. In Section 3 we prove Theorems 1.4 and 1.5. The proof of Theorem 1.6 will be given in Section 4. We conclude the paper with Section 5 in which we prove Theorem 1.7 and give some applications to it.

2. Characterization of type-I singularities

This section is concerned with the proofs of Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Without loss of generality, assume that $M^n \subset B_1(0) \subset \mathbb{R}^{n+1}$. Let $y_0 \in \mathbb{R}^{n+1}$ be a point reached by the mean curvature flow (1.1) at time T , that is, there exists a sequence (y_j, t_j) with $t_j \nearrow T$ so that $y_j \in M_{t_j}$ and $y_j \rightarrow y_0$. We show that (y_0, T) is a regular point of (1.1).

Note that the distance estimate [9, Corollary 3.6] gives

$$\text{dist}(M_t, y_0) \leq \sqrt{2n(T - t)}, \quad \text{for } t < T. \tag{2.1}$$

Consider the parabolic dilation $D_\lambda : \mathbb{R}^{n+1} \times [0, T) \rightarrow \mathbb{R}^{n+1} \times [-\lambda^2 T, 0)$ of scale $\lambda > 0$ at (y_0, T) defined by

$$D_\lambda(y, t) = (\lambda(y - y_0), \lambda^2(t - T)). \tag{2.2}$$

Denote the new time parameter by s . Then $t = T + \frac{s}{\lambda^2}$. Let

$$M_s^\lambda \equiv M_s^{(y_0, T), \lambda} = D_\lambda(M_t) = \lambda(M_{T + \frac{s}{\lambda^2}} - y_0).$$

Then (M_s^λ) is a solution of the mean curvature flow in $B_\lambda(0)$ for $s \in [-\lambda^2 T, 0)$. Denote by $d\mu_s^\lambda$ the induced volume form on M_s^λ . Let $\rho_{y_0, T} : \mathbb{R}^{n+1} \times (-\infty, T) \rightarrow \mathbb{R}$ be the backward heat kernel at (y_0, T) , i.e.,

$$\rho_{y_0, T}(y, t) = \frac{1}{[4\pi(T - t)]^{n/2}} \exp\left(-\frac{|y - y_0|^2}{4(T - t)}\right). \tag{2.3}$$

The monotonicity formula of Huisken [12] says that

$$\frac{d}{dt} \int_{M_t} \rho_{y_0, T} d\mu_t = - \int_{M_t} \rho_{y_0, T} \left| H + \frac{F^\perp}{2(T - t)} \right|^2 d\mu_t, \tag{2.4}$$

from which it follows that the limit $\lim_{t \rightarrow T} \int_{M_t} \rho_{y_0, T} d\mu_t$ exists. Here $F^\perp(\cdot, t)$ is the normal component of the position vector $F(\cdot, t) \in \mathbb{R}^{n+1}$ in the normal space of M_t in \mathbb{R}^{n+1} . Via the parabolic dilation, (2.4) becomes

$$\frac{d}{ds} \int_{M_s^\lambda} \rho_{0,0} d\mu_s^\lambda = - \int_{M_s^\lambda} \rho_{0,0} \left| H_s^\lambda - \frac{(F_s^\lambda)^\perp}{2s} \right|^2 d\mu_s^\lambda. \tag{2.5}$$

Fix $s_0 < 0$. Integrating both sides of (2.5) from $s_0 - \tau$ to s_0 for $\tau > 0$, we get

$$\int_{s_0 - \tau}^{s_0} \int_{M_s^\lambda} \rho_{0,0} \left| H_s^\lambda - \frac{(F_s^\lambda)^\perp}{2s} \right|^2 d\mu_s^\lambda ds = \int_{M_{s_0 - \tau}^\lambda} \rho_{0,0} d\mu_{s_0 - \tau}^\lambda - \int_{M_{s_0}^\lambda} \rho_{0,0} d\mu_{s_0}^\lambda. \tag{2.6}$$

Let $t_1 = T + \frac{s_0}{\lambda^2}$. Then, by the invariance of $\int_{M_t} \rho_{y_0, T} d\mu_t$ under the parabolic scaling,

$$\int_{M_{t_1}} \rho_{y_0, T} d\mu_{t_1} = \int_{M_{s_0}^\lambda} \rho_{0,0} d\mu_{s_0}^\lambda.$$

Letting $\lambda \rightarrow \infty$, one has $t_1 \rightarrow T$ and

$$\lim_{\lambda \rightarrow \infty} \int_{M_{s_0}^\lambda} \rho_{0,0} d\mu_{s_0}^\lambda = \lim_{t \rightarrow T} \int_{M_t} \rho_{y_0, T} d\mu_t.$$

Similarly,

$$\lim_{\lambda \rightarrow \infty} \int_{M_{s_0-\tau}^\lambda} \rho_{0,0} d\mu_{s_0-\tau}^\lambda = \lim_{t \rightarrow T} \int_{M_t} \rho_{y_0, T} d\mu_t.$$

Therefore, by (2.6),

$$\lim_{\lambda \rightarrow \infty} \int_{s_0-\tau}^{s_0} \int_{M_s^\lambda} \rho_{0,0} \left| H_s^\lambda - \frac{(F_s^\lambda)^\perp}{2s} \right|^2 d\mu_s^\lambda ds = 0. \tag{2.7}$$

On the other hand, the second fundamental form of M_s^λ satisfies

$$\max |A|^2(\cdot, s)(M_s^\lambda) = \frac{1}{\lambda^2} \max |A|^2(\cdot, t)(M_t) = -\frac{1}{s}(T - t) \max |A|^2(\cdot, t)(M_t)$$

and thus, by (1.2),

$$\max |A|^2(\cdot, s)(M_s^\lambda) \leq \frac{-C_0}{s}, \quad \forall s \in [-\lambda^2 T, 0).$$

In particular, for fixed $\delta \in (0, 1/2)$, the inequality

$$|A(y)|^2 \leq \frac{C_0}{\delta^2} \tag{2.8}$$

holds for $y \in M_s^\lambda \cap B_\lambda$ and $s \in [-\lambda^2 T, -\delta^2]$ and therefore for $y \in M_s^\lambda \cap B_{1/\delta}$ and $s \in [-1/\delta^2, -\delta^2]$ for λ sufficiently large depending on δ and T , say $\lambda \geq \lambda_{\delta, T}$. By the interior estimate [10], one has for all $m \geq 0$

$$|\nabla^m A(y)|^2 \leq \frac{C(C_0, m, n)}{\delta^{2(m+1)}} \tag{2.9}$$

for $y \in M_s^\lambda \cap B_{1/2\delta}$ and $s \in [-1/4\delta^2, -\delta^2]$. Moreover, by (2.1),

$$\text{dist}(0, M_s^\lambda) = \lambda \text{dist}(y_0, M_{T+\frac{s}{\lambda^2}}) \leq \lambda \sqrt{2n \left(\frac{-s}{\lambda^2} \right)} = \sqrt{-2ns}$$

for the above times s and $\lambda \geq \lambda_{\delta, T}$. By Arzela–Ascoli theorem combined with a diagonal sequence argument when letting $\delta \searrow 0$ for local graph representations of (M_s^λ) , we can find a subsequence $\lambda_i \rightarrow \infty$ such that $(M_s^{\lambda_i})$ converges smoothly on compact subsets of $\mathbb{R}^{n+1} \times (-\infty, 0)$ to a smooth solution $(M_s^\infty)_{s < 0}$ of mean curvature flow. From (2.7), one sees that $H = \frac{1}{2s} F^\perp$ on M_s^∞ for $s \in (s_0 - \tau, s_0)$.

Take $s_0 \rightarrow 0$ and $\tau \rightarrow \infty$ to see that $H = \frac{1}{2s} F^\perp$ on M_s^∞ for $-\infty < s < 0$. In other words, $(M^\infty)_s$ is a self-shrinking mean curvature flow. Moreover, one deduces from (1.3) and $|H_s^\lambda| = |\frac{H_t}{\lambda}|$ that $H = 0$ on M_s^∞ . Thus M_s^∞ is a minimal cone for each $s < 0$; see Corollary 2.8 in [6]. Because M_s^∞ is smooth, it is a hyperplane. Now, fix $s_0 < 0$. One has, as $i \rightarrow \infty$, $M_{s_0}^{\lambda_i} \rightarrow M_{s_0}^\infty \cong \mathbb{R}^n$ and $d\mu_{s_0}^{\lambda_i} \rightarrow dx^n$. Thus

$$\lim_{i \rightarrow \infty} \int_{M_{s_0}^{\lambda_i}} \rho_{0,0} d\mu_{s_0}^{\lambda_i} = \int_{M_{s_0}^\infty} \rho_{0,0} dx^n = 1.$$

This implies that, for $t_i = T + \frac{s_0}{\lambda_i^2}$

$$\lim_{t_i \rightarrow T} \int_{M_{t_i}} \rho_{y_0, T} d\mu_{t_i} = 1, \tag{2.10}$$

and therefore

$$\lim_{t \rightarrow T} \int_{M_t} \rho_{y_0, T} d\mu_t = 1.$$

By White’s regularity theorem [26], the second fundamental form $|A|(\cdot, t)$ of M_t is bounded as $t \rightarrow T$ and (y_0, T) is a regular point. Thus, the flow can be extended past time T . \square

Proof of Theorem 1.3. We will split the proof of Theorem 1.3 in the following two lemmas. \square

Lemma 2.1. *Theorem 1.3 holds for $\alpha > n + 2$.*

Proof. We use the same notations as in the proof of Theorem 1.2. Note that under the parabolic dilations D_{λ_i} , the inequality (1.4) becomes

$$C_0^\alpha \geq \int_0^T \int_{M_t} |H|^\alpha d\mu_t dt = \lambda_i^{\alpha-(n+2)} \int_{-\lambda_i^2 T}^0 \int_{M_s^{\lambda_i}} |H_s^{\lambda_i}|^\alpha d\mu_s^{\lambda_i} ds.$$

Thus

$$\int_{-\lambda_i^2 T}^0 \int_{M_s^{\lambda_i}} |H_s^{\lambda_i}|^\alpha d\mu_s^{\lambda_i} ds \leq \frac{C_0^\alpha}{\lambda_i^{\alpha-(n+2)}}. \tag{2.11}$$

Now, letting $i \rightarrow \infty$ as in the proof of Theorem 1.2, we get a self-shrinking mean curvature flow $(M^\infty)_s$ with the property that

$$\int_{-\infty}^0 \int_{M_s^\infty} |H|^\alpha du_s^\infty ds = 0 \tag{2.12}$$

because $\alpha > n + 2$. Therefore $H = 0$ on M_s^∞ . Now we can argue similarly as in the proof of Theorem 1.2. \square

Lemma 2.2. *Theorem 1.3 holds for $\alpha = n + 2$.*

Proof. We use the same notation as in the proof of Theorem 1.2. Under the parabolic dilations D_{λ_i} , the inequality (1.4) becomes

$$C_0 \geq \int_0^T \int_{M_t} |H|^{n+2} d\mu_t dt = \int_{-\lambda_i^2 T}^0 \int_{M_s^{\lambda_i}} |H_s^{\lambda_i}|^{n+2} d\mu_s^{\lambda_i} ds.$$

Letting $i \rightarrow \infty$ as before we get a complete and smooth self-shrinker M_s^∞ in the limit with the property that

$$\int_{-\infty}^0 \int_{M_s^\infty} |H|^{n+2} d\mu_s^\infty ds \leq C_0 < \infty. \tag{2.13}$$

Our self-shrinker satisfies

$$H = \frac{\langle x, \nu \rangle}{(-2s)},$$

which is equivalent to saying that $M_s^\infty = \sqrt{-s}M_{-1}^\infty$, where M_s^∞ satisfies the mean curvature flow. Notice that

$$\int_{M_s^\infty} |H|^{n+2}(\cdot, s) d\mu_s = \int_{M_{-1}^\infty} \left(\frac{|H|(\cdot, -1)}{\sqrt{-s}} \right)^{n+2} \cdot (-s)^{\frac{n}{2}} d\mu_{-1} = \frac{1}{(-s)} \cdot \int_{M_{-1}^\infty} |H|^{n+2}(\cdot, -1) d\mu_{-1} = \frac{a}{(-s)},$$

where $a := \int_{M_{-1}^\infty} |H|^{n+2}(\cdot, -1) d\mu_{-1}$. If $a > 0$ then

$$\int_{-\infty}^0 \int_{M_s^\infty} |H|^{n+2} d\mu_s^\infty ds = a \cdot \int_{-\infty}^0 \frac{ds}{(-s)} = \infty,$$

which contradicts (2.13). Therefore $a = 0$, which implies $H(\cdot, -1) = 0$ on M_{-1}^∞ . Similar argument shows that $H(\cdot, s) = 0$ on M_s^∞ for every $s < 0$. To prove that (y_0, T) is a regular point of the flow we argue as in the proof of Theorem 1.2. \square

3. Extension results for surfaces

In this section, we prove Theorems 1.4 and 1.5. First, we recall the Multiplicity One Conjecture. In [15] Ilmanen proposed the following conjecture.

Conjecture 3.1 (Multiplicity One Conjecture). *If M_0^2 is embedded in \mathbb{R}^3 , then for any family of rescalings $\lambda_j(M_{\lambda_j^{-2}s+T} - y_0)$ with $\lambda_j \rightarrow \infty$, there is a subsequence smoothly converging and with multiplicity one to the blowup N_τ , that is, there are no concentration points or multiple layers in the limit.*

Theorem 1.4 assumes that the conjecture above holds and its proof is given below.

Proof of Theorem 1.4. In this proof, $n = 2$. Without loss of generality, assume that $M^n \subset B_1(0) \subset \mathbb{R}^{n+1}$. Let $y_0 \in \mathbb{R}^{n+1}$ be a point reached by the mean curvature flow (1.1) at time T , that is, there exists a sequence (y_j, t_j) with $t_j \nearrow T$ so that $y_j \in M_{t_j}$ and $y_j \rightarrow y_0$. We show that (y_0, T) is a regular point of (1.1).

As in the proof of Theorem 1.2, let $t = T + \frac{s}{\lambda^2}$ and

$$M_s^\lambda \equiv M_s^{(y_0, T), \lambda} = \lambda(M_{T+\frac{s}{\lambda^2}} - y_0).$$

Then (M_s^λ) is a solution of the mean curvature flow in $B_\lambda(0)$ for $s \in [-\lambda^2 T, 0)$. For any set $A \subset \mathbb{R}^{n+1}$, let us define the parabolically rescaled measures at (y_0, T) :

$$\mu_s^\lambda(A) = \lambda^{-n} \mathcal{H}^n \lfloor M_s^\lambda(\lambda \cdot A).$$

Let $\rho_{y_0, T} : \mathbb{R}^{n+1} \times (-\infty, T) \rightarrow \mathbb{R}$ be the backward heat kernel at (y_0, T) as defined in (2.3). Then, a result on weak existence of blow ups of Ilmanen and White (see Lemma 8, p. 14 of [15] and also [24]) says that: there exists a subsequence λ_j and a limiting Brakke flow [1] $\{\nu_s\}_{s < 0}$ (also known as a *tangent flow*) such that $\mu_s^{\lambda_j} \rightharpoonup \nu_s$ in the sense of Radon measures for all $s < 0$ and the following statements hold:

- (a) (self-similarity) $\nu_s(A) = \nu_s^\lambda(A) \equiv \lambda^{-n} \nu_{\lambda^2 s}(\lambda \cdot A)$, for all $s < 0$ and for all $\lambda > 0$;
- (b) (tangent flow is a self-shrinker) ν_{-1} satisfies

$$\vec{H}(x) + \frac{S(x)^\perp \cdot x}{2} = 0, \quad \nu_{-1} \text{ a.e. } x; \tag{3.1}$$

- (c) furthermore, Huisken’s integral converges

$$\int \rho_{0,0}(x, -1) d\nu_s(x) = \lim_{t \nearrow T} \int \rho_{y_0, T} d\mu_t, \quad s < 0. \tag{3.2}$$

Equivalently, a subsequence of rescaled solutions M_s^λ converges weakly to a limiting flow X_s that is called a *tangent flow* at (y_0, T) . We know X_s is a self shrinker. Ilmanen showed in [16] that it has to be smooth. Our proof will rely on this fact and the validity of the Multiplicity One Conjecture. Let us briefly explain the notations used in (b).

For a locally n -rectifiable Radon measure μ , we define its n -dimensional approximate tangent plane $T_x\mu$ (which exists μ -a.e. x) by

$$T_x\mu(A) = \lim_{\lambda \rightarrow 0} \lambda^{-n} \mu(x + \lambda \cdot A).$$

The tangent plane $T_x\mu$ is a positive multiple of $\mathcal{H}^n \llcorner P$ for some n -dimensional plane P . Let $S : \mathbb{R}^{n+1} \rightarrow G(n+1, n)$ denotes the μ -measurable function that maps x to the geometric tangent plane, denoted by P above. An important quantity is the first variation of μ , defined by $\delta V_\mu(X) := \int \operatorname{div}_S X(x) d\mu(x)$ for $X \in C_c^\infty(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$. Here $\operatorname{div}_S X = \sum_{i=1}^n D_{e_i} X \cdot e_i$ where e_1, \dots, e_n is any orthonormal basis of S . We also denote by S the orthogonal projection onto S and thus $\operatorname{div}_S X$ can be written as $S : DX$. Now, if the total first variation $\|\delta V_\mu\|$ is a Radon measure and is absolutely continuous with respect to μ , then we can define the generalized mean curvature vector $\vec{H} = \vec{H}_\mu \in L^1_{\text{loc}}(\mu)$ of μ as follows

$$\int \operatorname{div}_S X d\mu = \int -\vec{H} \cdot X d\mu \tag{3.3}$$

for all $X \in C_c^\infty(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$. For further information on geometric measure theory, we refer the reader to Simon’s lecture notes [21]. Note that when μ is the surface measure of a smooth n -dimensional manifold M , the generalized mean curvature vector \vec{H}_μ of μ exists and is also the classical mean curvature vector of M ; see Corollary 4.3 in [19]. Therefore, we can apply (3.3) to μ_s^λ , which is the rescaled surface measure of the smooth manifold M_s^λ . From (3.3) and the definition of μ_s^λ , one sees that the mean curvature vector \vec{H}_s^λ of μ_s^λ is $\frac{\vec{H}_t}{\lambda}$ where \vec{H}_t is the mean curvature vector of M_t where $t = T + \frac{s}{\lambda^2}$. The lower semicontinuity of $\int |H| d\mu$ asserts that

$$\int |\vec{H}_s| d\nu_s \leq \liminf_{\lambda \rightarrow \infty} \int |\vec{H}_s^\lambda| d\mu_s^\lambda \leq \limsup_{\lambda \rightarrow \infty} \int \frac{C_0}{\lambda} d\mu_s^\lambda = 0.$$

Thus $\vec{H}_s = 0$ for all $s < 0$. Now, because X_s is smooth for all $s < 0$, the weak mean curvature vector \vec{H}_s coincides with the mean curvature vector in classical sense. Thus we have a smooth solution X_s that is a self-shrinker with $H = 0$ and therefore by the result in [6] it has to be a hyperplane. Furthermore ν_s represents the surface measure of the plane X_s with multiplicity one by the validity of the Multiplicity One Conjecture. Using the convergence of Huisken’s integral (3.2), we see that

$$\lim_{t \nearrow T} \int \rho_{y_0, T} d\mu_t = 1.$$

By White’s regularity theorem [26], the second fundamental form $|A|(\cdot, t)$ of M_t is bounded as $t \rightarrow T$ and (y_0, T) is a regular point. Thus, the flow can be extended past time T . \square

We conclude this section by the proof of Theorem 1.5, which can be viewed as a local regularity result without a smallness condition. Note that the Multiplicity One Conjecture is not assumed in this theorem.

Proof of Theorem 1.5. We will use the same notation as in the proof of Theorem 1.4. Note that, when $n = 2$, by the fact that $\int H_s^2$ is bounded (follows from the Gauss–Bonnet theorem for surfaces) and Allard’s Compactness Theorem [21], each Radon measure ν_s is integer 2-rectifiable, that is

$$d\nu_s = \theta_s(x) d\mathcal{H}^2 \llcorner X_s$$

where X_s is an \mathcal{H}^2 -measurable, 2-rectifiable set and θ_s is an $\mathcal{H}^2 \llcorner X_s$ -integrable, integer valued “multiplicity function”.

Furthermore the mean curvature vector \vec{H}_s of ν_s satisfies $\vec{H}_s \in L^\infty(\nu_s)$. Here is the only place we wish to use (1.5). The same argument as in the proof of Theorem 1.4 implies that $\vec{H}_s = \vec{0}$ and X_s is a plane.

The key point of our proof is the following Constancy Theorem.

Theorem 3.1 (Constancy Theorem). *Let $\mu = \theta \mathcal{H}^n \llcorner M$ be an integral n -varifold in the open set $\Omega \subset \mathbb{R}^{n+m}$, $M \subset \Omega$ a connected C^1 - n -manifold, $\theta : M \rightarrow N_0$ be \mathcal{H}^n -measurable with weak mean curvature $\vec{H}_\mu \in L^1_{\text{loc}}(\mu)$, that is*

$$\int \operatorname{div}_\mu \eta \, d\mu = \int_M \operatorname{div}_M \eta \theta \, d\mathcal{H}^n = - \int \langle \vec{H}_\mu, \eta \rangle \, d\mu, \quad \forall \eta \in C_0^1(\Omega, \mathbb{R}^{n+m}). \tag{3.4}$$

Then θ is a constant: $\theta \equiv \theta_0 \in N_0$. Here N_0 is the set of all nonnegative integers and $\langle \cdot \rangle$ is the standard Euclidean inner product on \mathbb{R}^{n+m} .

Now θ_s is a constant and X_s is a plane. Thus by the convergence of Huisken’s integral (3.2), we see that

$$\lim_{t \nearrow T} \int \rho_{y_0, T} \, d\mu_t = \int \rho_{0,0}(x, -1) \, dv_s(x) = \int \rho_{0,0}(x, -1) \theta_s \, d\mathcal{H}^2 \llcorner X_s = \theta_s.$$

By (1.6) and Proposition 2.10 in [26], $1 \leq \theta_s < 2$. It follows from the integrality of θ_s that $\theta_s \equiv 1$. Now, our result follows from White’s local regularity theorem [26]. \square

Proof of Theorem 3.1. For the case of stationary varifold μ , i.e., $\vec{H}_\mu = \vec{0}$, the proof of Theorem 3.1 can be found in [21], Theorem 41.1. For the general case stated above, due to Schätzle, the proof can be found in [17], Theorem 4.3. However, in our application, when M is a plane and $\vec{H}_\mu = \vec{0}$, Theorem 3.1 has a very simple proof which we include it here. Choose $\eta \in C_0^1(\Omega, \mathbb{R}^{n+m})$ such that $\eta \in TM$ -the tangent bundle of M . Then (3.4) gives

$$\int_M (\operatorname{div}_M \eta) \theta \, d\mathcal{H}^n = 0.$$

Calculating in local coordinates, this yields $\nabla_M \theta = 0$ weakly. Hence $\theta \equiv \theta_0$ is constant, as M is connected. \square

We conclude this section with the following result, which has been suggested to us by one of the referees. It serves to somewhat remove the Multiplicity One Conjecture in Theorem 1.4.

Corollary 3.1. *Let M^2 be a compact, smooth and embedded 2-dimensional manifold in \mathbb{R}^3 . If*

$$\max_{M_t} |H|^2(\cdot, t) \leq C_0$$

and if no quasi-static higher multiplicity planes occur as tangent flows at any point (y_0, T) where $y_0 \in \mathbb{R}^3$ is a point reached by the mean curvature flow (1.1) at time T , then the flow can be extended past time T .

Remark 3.1. For a precise definition of quasi-static limit flow, we refer the reader to White [25]. In the course of the proof of Theorem 1.4, we showed (without the Multiplicity One Conjecture) that, under the boundedness of the mean curvature, any tangent flow at (y_0, T) is a quasi-static plane, maybe with higher multiplicity. The Multiplicity One Conjecture corresponds to the case that all tangent flows (even non-flat ones) are of multiplicity one. In Corollary 3.1, the only requirement is that, if a tangent flow happens to be a quasi-static plane, it must have a point of multiplicity one. Thus, we do not require any information on the multiplicity of non-flat tangent flows.

Proof of Corollary 3.1. As in the proof of Theorem 1.5, the bound on the mean curvature implies that at any (y_0, T) , the tangent flows are planes. By Theorem 3.1, each of these planes has constant multiplicity. But at one point, the plane has multiplicity one. Thus the whole plane must have multiplicity one. Now, our result follows from White’s local regularity theorem [26]. \square

4. Some global results on the extension of the mean curvature flow

In this section we give global conditions for extending a smooth solution to (1.1), which has been a subject of study in [18].

Proof of Theorem 1.6. We argue by contradiction. Suppose that T is the extinction time of the flow. Then, by Theorem 1.1, $|A|$ is unbounded. Therefore, there exists a sequence of points (x_i, t_i) with $x_i \in M_{t_i}$ such that

$$Q_i := |A|(x_i, t_i) = \max_{0 \leq t \leq t_i} \max_{x \in M_t} |A|(x, t) \rightarrow +\infty. \tag{4.1}$$

Consider the sequence \tilde{M}_t^i of rescaled solutions for $t \in [0, 1]$ defined by

$$\tilde{F}_i(\cdot, t) = Q_i \left(F \left(\cdot, t_i + \frac{t-1}{Q_i^2} \right) - x_i \right).$$

The sequence of rescaled solutions \tilde{M}_t^i converges (see [2]) to a complete smooth solution to the mean curvature flow, call it \tilde{M}_t for $t \in [0, 1]$ with the property that

$$|\tilde{A}|(0, 1) = 1. \tag{4.2}$$

If g and $A := \{h_{jk}\}$ are the induced metric, and the second fundamental form of M_t , respectively, then the corresponding rescaled quantities are given by

$$\tilde{g}_i = Q_i^2 g; \quad |\tilde{A}_i|^2 = \frac{|A|^2}{Q_i^2}.$$

We calculate

$$\begin{aligned} & \lim_{i \rightarrow \infty} \left\{ \int_0^1 \left(\int_{(\tilde{M}_t^i)_t \cap B(0,1)} |\tilde{A}_i|^q d\mu \right)^{\frac{p}{q}} dt \right\}^{\frac{1}{p}} \\ &= \lim_{i \rightarrow \infty} \left\{ \int_{t_i - \frac{1}{Q_i^2}}^{t_i} \left(\int_{M_t \cap B(0, \frac{1}{Q_i})} |A|^q d\mu \right)^{\frac{p}{q}} dt \right\}^{\frac{1}{p}} \leq \lim_{i \rightarrow \infty} \left\{ \int_{t_i - \frac{1}{Q_i^2}}^{t_i} \left(\int_{M_t} |A|^q d\mu \right)^{\frac{p}{q}} dt \right\}^{\frac{1}{p}} = 0. \end{aligned} \tag{4.3}$$

The last step follows from the facts that

$$\left(\int_0^T \left(\int_{M_t} |A|^q d\mu \right)^{p/q} dt \right)^{1/p} < \infty; \quad \lim_{i \rightarrow \infty} \frac{1}{Q_i^2} = 0.$$

By Fatou’s lemma and (4.3) it follows that

$$\int_0^1 \left(\int_{\tilde{M}_t \cap B(0,1)} |\tilde{A}|^q \right)^{\frac{p}{q}} = 0.$$

By the smoothness of \tilde{M}_t , this implies $|\tilde{A}|(x, t) \equiv 0$ for all $x \in \tilde{M}_t \cap B(0, 1)$ and all $t \in [0, 1]$. This contradicts (4.2). \square

5. Some local regularity results and applications

5.1. ε -regularity theorem for the mean curvature flow

In this section we prove Theorem 1.7, which is parabolic version of the epsilon regularity theorem for minimal surfaces proven by Choi and Schoen [3].

Proof of Theorem 1.7. We may assume without loss of generality that the flow $\mathcal{M} = (M_t)_{t < t_0}$ is smooth up to and including time t_0 , because we can first prove the theorem with t_0 replaced by $t_0 - \alpha$ for fixed $\alpha > 0$ and then let $\alpha \searrow 0$ afterwards.

Let

$$F(\delta) = \sup_{t \in [t_0 - (\sigma - \delta)^2, t_0]} \sup_{x \in B(x_0, \sigma - \delta) \cap M_t} \delta^2 |A|^2(x, t).$$

Since our flow is smooth up to time t_0 , $F(0) = 0$. Thus, there exists $\delta_* \in (0, \sigma/2]$ such that $F(\delta_*) = \max_{0 \leq \delta \leq \sigma/2} F(\delta)$. It suffices to show that

$$F(\delta_*) < \varepsilon_0^{-\frac{2}{n+2}} \left(\int_{t_0 - \sigma^2}^{t_0} \int_{M_t \cap B(x_0, \sigma)} |A|^{n+2} d\mu dt \right)^{\frac{2}{n+2}} \equiv (\varepsilon_0^{-1} \eta)^{\frac{2}{n+2}}. \tag{5.1}$$

Suppose not, then

$$F(\delta_*) \geq (\varepsilon_0^{-1} \eta)^{\frac{2}{n+2}}. \tag{5.2}$$

Because the flow is defined up to and including time t_0 , we can find $t_* \in [t_0 - (\sigma - \delta_*)^2, t_0]$ and $x_* \in \bar{B}(x_0, \sigma - \delta_*) \cap M_{t_0}$ such that

$$\delta_*^2 |A|^2(x_*, t_*) = F(\delta_*). \tag{5.3}$$

It follows from $\delta_* \in (0, \sigma/2]$ that

$$B\left(x_*, \frac{\delta_*}{2}\right) \subset B\left(x_0, \sigma - \frac{\delta_*}{2}\right); \quad \left[t_* - \frac{\delta_*^2}{4}, t_*\right] \subset \left[t_0 - \left(\sigma - \frac{\delta_*}{2}\right)^2, t_0\right]. \tag{5.4}$$

By the choice of δ_* , t_* and x_* ,

$$\left(\frac{\delta_*}{2}\right)^2 \sup_{t \in [t_0 - (\sigma - \frac{\delta_*}{2})^2, t_0]} \sup_{x \in B(x_0, \sigma - \frac{\delta_*}{2}) \cap M_t} |A|^2(x, t) \leq F(\delta_*) = \delta_*^2 |A|^2(x_*, t_*).$$

Hence

$$\sup_{t \in [t_0 - (\sigma - \frac{\delta_*}{2})^2, t_0]} \sup_{x \in B(x_0, \sigma - \frac{\delta_*}{2}) \cap M_t} |A|^2(x, t) \leq 4|A|^2(x_*, t_*)$$

and thus, it follows from (5.4) that

$$\sup_{t \in [t_* - \frac{\delta_*^2}{4}, t_*]} \sup_{x \in B(x_*, \frac{\delta_*}{2}) \cap M_t} |A|^2(x, t) \leq 4|A|^2(x_*, t_*). \tag{5.5}$$

We now rescale our mean curvature flow by setting

$$\tilde{F}(\cdot, t) = QF\left(\cdot, t_* + \frac{t-1}{Q^2}\right), \quad \tilde{M}_t = \tilde{F}(M^n, t)$$

where

$$Q = 2(\varepsilon_0 \eta^{-1})^{\frac{1}{n+2}} |A|(x_*, t_*).$$

Then we have a mean curvature flow \tilde{M}_t on $B(x_*, Q\delta_*/2)$ for $t \in [0, 1]$. Let $\tilde{g} = Q^2g$ be the induced metric on \tilde{M}_t and let $\tilde{B}(x_*, r)$ be the geodesic ball w.r.t. the metric \tilde{g} and centered at x_* with radius r . By (5.2) and (5.3), we have

$$\frac{Q\delta_*}{2} \geq 1. \tag{5.6}$$

This combined with (5.5) gives

$$\sup_{t \in [0, 1]} \sup_{x \in \tilde{B}(x_*, 1) \cap \tilde{M}_t} |\tilde{A}|^2(x, t) \leq 4|\tilde{A}|^2(x_*, 1) \leq 1. \tag{5.7}$$

Note that the last inequality follows from the facts that

$$4|\tilde{A}|^2(x_*, 1) = \frac{4|A|^2(x_*, t_*)}{Q^2} = (\varepsilon_0^{-1}\eta)^{\frac{2}{n+2}} \leq 1.$$

To obtain a contradiction, we will use the inequality

$$(\partial_t - \Delta)|\tilde{A}|^2 \leq 2|\tilde{A}|^4. \tag{5.8}$$

This is a differential inequality of the form $(\partial_t - \Delta)v \leq fv$ where $v = |\tilde{A}|^2$ and $f = 2|\tilde{A}|^2$. Furthermore f satisfies a smallness condition:

$$\int_0^1 \int_{\tilde{M}_t \cap \tilde{B}(x_*, 1)} f^{\frac{n+2}{2}} d\tilde{\mu} dt \leq 2^{n+2}\eta \leq 2^{n+2}\varepsilon_0.$$

Thus, we can localize our estimates in Lemmas 5.1 and 6.1 in [18] to obtain the following inequality

$$|\tilde{A}|^2(x_*, 1) \leq \sup_{t \in [1/12, 1]} \sup_{x \in \tilde{B}(x_*, 1/2)} |\tilde{A}|^2(x, t) \leq C(n) \left(\int_0^1 \int_{\tilde{M}_t \cap \tilde{B}(x_*, 1)} |\tilde{A}|^{n+2} d\tilde{\mu} dt \right)^{\frac{2}{n+2}} \leq C(n)\eta^{\frac{2}{n+2}}. \tag{5.9}$$

There is a simple proof of this inequality. It goes as follows. Note that for $t \in [0, 1]$ and $x \in \tilde{B}(x_*, 1)$, (5.7) and (5.8) give

$$(\partial_t - \Delta)|\tilde{A}|^2 \leq 2|\tilde{A}|^2$$

or equivalently $(\partial_t - \Delta)(e^{-2t}|\tilde{A}|^2) \leq 0$. Now, we can apply Moser’s mean value inequality [8, Proposition 1.6], for $e^{-2t}|\tilde{A}|^2$ to obtain a constant $C_1(n)$ depending only on n such that

$$|\tilde{A}|^2(x_*, 1) \leq C_1(n) \int_0^1 \int_{\tilde{M}_t \cap \tilde{B}(x_*, 1)} |\tilde{A}|^2 d\tilde{\mu} dt. \tag{5.10}$$

By Hölder inequality

$$|\tilde{A}|^2(x_*, 1) \leq C_1(n) \left(\int_0^1 \int_{\tilde{M}_t \cap \tilde{B}(x_*, 1)} d\tilde{\mu} dt \right)^{\frac{n}{n+2}} \left(\int_0^1 \int_{\tilde{M}_t \cap \tilde{B}(x_*, 1)} |\tilde{A}|^{n+2} d\tilde{\mu} dt \right)^{\frac{2}{n+2}}. \tag{5.11}$$

By (5.7) and the Gauss equation

$$\tilde{R}_{ik} = \tilde{H}\tilde{h}_{ik} - \tilde{h}_{il}\tilde{g}^{lj}\tilde{h}_{jk}$$

one easily sees that the Ricci tensor satisfies $\tilde{R}_{ik} \geq -(n - 1)$. By the Bishop–Gromov volume comparison theorem, for each time $t \in [0, 1]$, one has $\int_{\tilde{M}_t \cap \tilde{B}(x_*, 1)} d\tilde{\mu} \leq V(n)$ where $V(n)$ denotes the volume of a unit geodesic ball in an n -dimensional space form of constant curvature -1 . Thus

$$\begin{aligned} |\tilde{A}|^2(x_*, 1) &\leq C_1(n) V(n)^{\frac{n}{n+2}} \left(\int_0^1 \int_{\tilde{M}_t \cap \tilde{B}(x_*, 1)} |\tilde{A}|^{n+2} d\tilde{\mu} dt \right)^{\frac{2}{n+2}} \\ &= C_1(n) V(n)^{\frac{n}{n+2}} \left(\int_{t_* - \frac{1}{Q^2}}^{t_*} \int_{M_t \cap B(x_*, \frac{1}{Q})} |A|^{n+2} d\mu dt \right)^{\frac{2}{n+2}} \\ &\leq C_1(n) V(n)^{\frac{n}{n+2}} \left(\int_{t_* - \delta_*^2/2}^{t_*} \int_{M_t \cap B(x_*, \delta_*/2)} |A|^{n+2} d\mu dt \right)^{\frac{2}{n+2}} \end{aligned}$$

$$\leq C_1(n)V(n)^{\frac{n}{n+2}} \left(\int_{t_0-\sigma^2}^{t_0} \int_{M_t \cap B(x_0, \sigma)} |A|^{n+2} d\mu dt \right)^{\frac{2}{n+2}} = C_1(n)V(n)^{\frac{n}{n+2}} \eta^{\frac{2}{n+2}}. \tag{5.12}$$

Consequently,

$$C_1(n)V(n)^{\frac{n}{n+2}} \eta^{\frac{2}{n+2}} \geq |\tilde{A}|^2(x_*, 1) = \frac{1}{4}(\varepsilon_0^{-1} \eta)^{\frac{2}{n+2}}.$$

This is a contradiction if ε_0 is small. \square

Remark 5.1. In general, we can modify the proof of Theorem 1.7 to obtain the following result. Suppose $\mathcal{M} = (M_t)$ is a smooth, properly embedded solution of the mean curvature flow in $B(x_0, \rho) \times (t_0 - \rho^2, t_0)$ which reaches x_0 at time t_0 . Let p and q be positive numbers satisfying

$$\frac{n}{q} + \frac{2}{p} = 1.$$

Then, there exists $\varepsilon_0 = \varepsilon_0(M_0, p, q) > 0$ such that if $0 < \sigma \leq \rho$ and

$$\int_{t_0-\sigma^2}^{t_0} \left(\int_{M_t \cap B(x_0, \sigma)} |A|^q d\mu \right)^{p/q} dt < \varepsilon_0 \tag{5.13}$$

then

$$\max_{0 \leq \delta \leq \sigma/2} \sup_{t \in [t_0 - (\sigma - \delta)^2, t_0)} \sup_{x \in B(x_0, \sigma - \delta) \cap M_t} \delta^2 |A|^2(x, t) < \varepsilon_0^{\frac{-2}{p}} \left(\int_{t_0-\sigma^2}^{t_0} \left(\int_{M_t \cap B(x_0, \sigma)} |A|^q d\mu \right)^{p/q} dt \right)^{\frac{2}{p}}. \tag{5.14}$$

Theorem 1.7 can be extended to the case when an ambient manifold is an arbitrary Riemannian manifold.

Corollary 5.1. Let $n \geq 2$ and N^{n+1} be a smooth complete, locally symmetric Riemannian manifold with bounded geometry. Let M_0 be a compact connected hypersurface without boundary which is smoothly immersed in $B(x_0, \rho) \subset N$. Suppose that $\mathcal{M} = (M_t)$ is a smooth, properly embedded solution of the mean curvature flow in $B(x_0, \rho) \times (t_0 - \rho^2, t_0)$ which reaches x_0 at time t_0 . There exists $\varepsilon_0 = \varepsilon_0(M_0, N)$ such that if $0 < \sigma \leq \rho$ and

$$\int_{t_0-\sigma^2}^{t_0} \int_{M_t \cap B(x_0, \sigma)} |A|^{n+2} d\mu dt < \varepsilon_0 \tag{5.15}$$

then

$$\max_{0 \leq \delta \leq \sigma/2} \sup_{t \in [t_0 - (\sigma - \delta)^2, t_0)} \sup_{x \in B(x_0, \sigma - \delta) \cap M_t} \delta^2 |A|^2(x, t) < \varepsilon_0^{\frac{-2}{n+2}} \left(\int_{t_0-\sigma^2}^{t_0} \int_{M_t \cap B(x_0, \sigma)} |A|^{n+2} d\mu dt \right)^{\frac{2}{n+2}}. \tag{5.16}$$

Proof. In the formulas that follow, if we mean the metric or the connection on N , this will be indicated by a bar, for example $\bar{g}_{\alpha\beta}$, etc. The Riemann curvature tensors on M and N will be denoted by $\text{Rm} = \{R_{ijkl}\}$ and $\bar{\text{Rm}} = \{\bar{R}_{\alpha\beta\gamma\delta}\}$. Let ν be the outer unit normal to M_t . For a fixed time t , we choose a local field of frame e_0, e_1, \dots, e_n in N such that when restricted to M_t , we have $e_0 = \nu, e_i = \frac{\partial F}{\partial x_i}$. The relations between $A = (h_{ij}), \text{Rm}$ and $\bar{\text{Rm}}$ are given by the equations of Gauss and Codazzi:

$$R_{ijkl} = \bar{R}_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk}, \tag{5.17}$$

$$\nabla_k h_{ij} - \nabla_j h_{ik} = \bar{R}_{0ijk}. \tag{5.18}$$

Observe that we have the following evolution equation:

$$\begin{aligned} \frac{\partial}{\partial t} |A|^2 &= \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^2(|A|^2 + \overline{\text{Ric}}(\nu, \nu)) \\ &\quad - 4(h^{ij}h_j^m \bar{R}_{mli}^l - h_{ij}h^{lm} \bar{R}_{milj}) - 2h^{ij}(\bar{\nabla}_j \bar{R}_{0li}^l + \bar{\nabla}_l \bar{R}_{0ij}^l), \end{aligned}$$

whose derivation can be found in [13]. Since N is, by our assumption, locally symmetric, we have $\bar{\nabla} \overline{\text{Rm}} = 0$ and therefore the previous equation just reads as

$$\frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^2(|A|^2 + \overline{\text{Ric}}(\nu, \nu)) - 4(h^{ij}h_j^m \bar{R}_{mli}^l - h_{ij}h^{lm} \bar{R}_{milj}). \tag{5.19}$$

Using the evolution equation (5.19) and the bounds on the geometry of N we obtain

$$\frac{\partial}{\partial t} |A|^2 \leq \Delta |A|^2 + 2|A|^4 + C|A|^2.$$

After rescaling our solution and using (5.7), we obtain

$$\left(\frac{\partial}{\partial t} - \Delta\right) |\tilde{A}|^2 \leq C|\tilde{A}|^2.$$

If $f := e^{-C \cdot t} |\tilde{A}|^2$ then we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) f \leq 0.$$

If we take a trace of (5.17) in jl we obtain

$$R_{ik} = g^{jl} \bar{R}_{ijkl} + h_{ik} H - h_{il} h_{jk} g^{jl} \geq -C.$$

Applying the Moser mean value inequality to f will lead to a contradiction in the same way as in the proof of Theorem 1.7. \square

5.2. Some applications of Theorem 1.7

In this section, we give three applications of the local regularity results obtained in Section 5.1.

The first application is a simple consequence of the Remark 5.1. It gives a sufficient integral condition for (1.1) to have a type-I singularity. This will be achieved by showing that any type-I control on the L^s -norm ($s > n$) of the second fundamental form for all time slice t gives a type-I control on the second fundamental form. Precisely, we prove the following

Corollary 5.2. *Let $s \in (n, \infty)$. Suppose there is a constant $C_s > 0$ such that for any $T/2 \leq t < T$, we have*

$$\|A\|_{L^s(M_t)} \leq \frac{C_s}{(T-t)^{\frac{s-n}{2s}}}. \tag{5.20}$$

Then (1.2) holds.

Remark 5.2. We say that the L^s -control on the second fundamental form given by (5.20) is of type I. Notice that for the shrinking spheres S^n we have the equality in (5.20).

Proof of Corollary 5.2. For $q = s > n$ there exists a positive number p such that

$$\frac{n}{s} + \frac{2}{p} = 1.$$

Let (x_0, t_0) be arbitrary, where $0 < t_0 < T$. Let $\sigma \in (0, \sqrt{t_0})$. Then, for any $t \in (t_0 - \sigma^2, t_0)$ we have

$$\int_{t_0-\sigma^2}^{t_0} \left(\int_{B(x_0,\sigma)\cap M_t} |A|^s \right)^{\frac{2}{s-n}} \leq \int_{t_0-\sigma^2}^{t_0} \frac{dt}{T-t} \leq \frac{C\sigma^2}{T-t_0} =: C\alpha, \tag{5.21}$$

where $\alpha(T - t_0) = \sigma^2$. Fix α sufficiently small so that $C\alpha \leq \varepsilon_0(M_0, p, q)$ where $\varepsilon_0(M_0, p, q)$ is the small constant in Remark 5.1. For this choice of α , the estimate (5.14), taking $\delta = \frac{\sigma}{2}$ gives

$$|A|^2(x_0, t_0) \leq \frac{4}{\sigma^2} = \frac{C}{T - t_0},$$

and this completes the proof of our corollary. \square

The second application is a lower bound on the L^s -norm ($s > n$) of the second fundamental form at each time slice. This lower bound can be viewed as a slight generalization of Huisken’s estimate [12] where the case $s = \infty$ was considered. Let $s \in (n, \infty)$. Suppose that T is the first singular time of the mean curvature flow. We are interested in the following question:

Does there exist a constant $C'_s > 0$ such that for any $t < T$, we have

$$\|A\|_{L^s(M_t)} \geq \frac{C'_s}{(T - t)^{\frac{s-n}{2s}}}? \tag{5.22}$$

We prove a weaker version of the above inequality as follows

Corollary 5.3. *For $t < T$, let $f(t) = \sup_{t_1 \leq t} \|A\|_{L^s(M_{t_1})}$. Then there exists a constant $C'_s > 0$ such that*

$$f(t) \geq \frac{C'_s}{(T - t)^{\frac{s-n}{2s}}}. \tag{5.23}$$

Proof of Corollary 5.3. For $q = s > n$ there exists a positive number p such that

$$\frac{n}{s} + \frac{2}{p} = 1.$$

Let (x_0, t_0) be arbitrary, where $0 < t_0 < T$. Let $\sigma \in (0, \sqrt{t_0})$. Then, for any $t \in (t_0 - \sigma^2, t_0)$ we have

$$\int_{t_0-\sigma^2}^{t_0} \left(\int_{B(x_0,\sigma)\cap M_t} |A|^s \right)^{\frac{2}{s-n}} \leq \int_{t_0-\sigma^2}^{t_0} (f(t_0))^{\frac{2s}{s-n}} = \sigma^2 (f(t_0))^{\frac{2s}{s-n}}. \tag{5.24}$$

Fix σ so that $\sigma^2 (f(t_0))^{\frac{2s}{s-n}} = \varepsilon_0(M_0, p, q)$ where $\varepsilon_0(M_0, p, q)$ is the small constant in Remark 5.1. For this choice of σ , the estimate (5.14), taking $\delta = \frac{\sigma}{2}$ gives

$$|A|^2(x_0, t_0) \leq \frac{4}{\sigma^2} = \frac{4(f(t_0))^{\frac{2s}{s-n}}}{\sigma^2 (f(t_0))^{\frac{2s}{s-n}}} = \frac{4(f(t_0))^{\frac{2s}{s-n}}}{\varepsilon_0(p, q)}.$$

Taking the supremum of the left hand side with respect to x_0 and in view of Huisken’s estimate [12] on the lower bound of $\sup_{M_{t_0}} |A|^2$, we get

$$\frac{2}{T - t_0} \leq \sup_{M_{t_0}} |A|^2 \leq \frac{4(f(t_0))^{\frac{2s}{s-n}}}{\varepsilon_0(p, q)}.$$

This gives the desired inequality. \square

The third application is a regularity result without a smallness condition. We will use the curvature estimate in Theorem 1.7 to obtain other curvature estimates without any smallness condition for mean curvature flow of surfaces. Our result in this direction states

Corollary 5.4. *Suppose $\mathcal{M} = (M_t)$ is a smooth, properly embedded solution of the mean curvature flow in $B(x_0, 4\sigma) \times (t_0 - (4\sigma)^2, t_0) \subset \mathbb{R}^3 \times (t_0 - (4\sigma)^2, t_0)$ which reaches x_0 and time t_0 . Given a constant $C_I > 0$, there is a constant $C_P > 0$ so that if*

$$\int_{t_0 - (4\sigma)^2}^{t_0} \int_{B(x_0, 4\sigma) \cap M_t} |A|^4 d\mu dt \leq C_I \tag{5.25}$$

then

$$\sup_{t \in [t_0 - \sigma^2, t_0)} \sup_{x \in B(x_0, \sigma) \cap M_t} |A|^2(x, t) \leq C_P \sigma^{-2}. \tag{5.26}$$

Proof of Corollary 5.4. Our strategy is to use annuli as in the proof of Lemma 1.10 in Colding–Minicozzi [5]. The idea is that the L^∞ -bounds on the annulus and the L^2 -bounds of the second fundamental form on each time slice will give the L^∞ -bounds for the fundamental form in the interior of the annulus. See (5.32) and (5.33) for the precise statements. Moreover, by (5.25) and Theorem 1.6 for $n = 2$ and $p = q = 4$, we have the regularity of the mean curvature flow up to and including time t_0 . Thus the second fundamental form is uniformly bounded. The essence of Corollary 5.4 is the explicit bound (5.26). Because the second fundamental form is uniformly bounded, (M_t) is locally a graph over some hyperplane. This will replace the use of Rado’s theorem for minimal surfaces as in [5].

We start with the following claim.

Claim 5.1. *Given (5.25) there is a constant C so that*

$$\sup_{t \in [t_0 - (2\sigma)^2, t_0)} \int_{B(x_0, 2\sigma) \cap M_t} |A|^2 dx \leq C.$$

Proof. Let $\eta(x, t)$ be a cut off function compactly supported in $B(x_0, 4\sigma) \cap M_t \times [t_0 - (4\sigma)^2, t_0)$, identically equal to one on $B(x_0, 2\sigma) \times [t_0 - (3\sigma)^2, t_0)$ (the same one that Ecker used in [8]). Multiply the evolution equation of $|A|^2$

$$\frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4,$$

by η^2 and integrate it over M_t . Using the evolution equation of the volume form $\frac{d}{dt} \mu = -H^2 d\mu$, we see that

$$\begin{aligned} \frac{d}{dt} \int_{M_t} |A|^2 \eta^2 &\leq \int_{M_t} \frac{d}{dt} |A|^2 \eta^2 + |A|^2 \frac{d}{dt} \eta^2 = \int_{M_t} (\Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4) \eta^2 + |A|^2 \frac{d}{dt} \eta^2 \\ &= \int_{M_t} |A|^2 2\eta \left(\frac{d}{dt} - \Delta \right) \eta + 2|A|^4 \eta^2 + 2|A|^2 \eta \Delta \eta + \Delta |A|^2 \eta^2 - 2|\nabla A|^2 \eta^2. \end{aligned} \tag{5.27}$$

Integrating by parts gives

$$\begin{aligned} \int_{M_t} 2|A|^2 \eta \Delta \eta + \Delta |A|^2 \eta^2 - 2|\nabla A|^2 \eta^2 &= \int_{M_t} -2\nabla(|A|^2 \eta) \nabla \eta - \nabla |A|^2 \nabla \eta^2 - 2|\nabla A|^2 \eta^2 \\ &= \int_{M_t} -8|A| \nabla |A| \eta \nabla \eta - 2|A|^2 |\nabla \eta|^2 - 2|\nabla A|^2 \eta^2. \end{aligned} \tag{5.28}$$

Using Kato’s inequality $|\nabla |A|| \leq |\nabla A|$ and Cauchy–Schwarz’s inequality, one deduces from (5.28) that

$$\int_{M_t} 2|A|^2 \eta \Delta \eta + \Delta |A|^2 \eta^2 - 2|\nabla A|^2 \eta^2 \leq \int_{M_t} 8|A|^2 |\nabla \eta|^2. \tag{5.29}$$

Combining (5.27) and (5.29), we get

$$\frac{d}{dt} \int_{M_t} |A|^2 \eta^2 \leq \int_{M_t} |A|^2 2\eta \left(\frac{d}{dt} - \Delta \right) \eta + 8|A|^2 |\nabla \eta|^2 + 2|A|^4 \eta^2.$$

Using the estimate in Lemma 1.1 in [9] that

$$\sup_{M \times [t_0 - 1, t_0]} \left(|\nabla \eta|^2 + 2\eta \left| \left(\frac{d}{dt} - \Delta \right) \eta \right| \right) \leq \frac{c}{\sigma^2},$$

and that $\text{vol}(B(x_0, 4\sigma) \cap M_t) \leq C\sigma^2$ (this can be proved using Huisken’s monotonicity formula [12]; see for example Lemma 1.4 in [8]) we have

$$\frac{d}{dt} \int_{B(x_0, 4\sigma)} |A|^2 \eta^2 d\mu \leq \frac{C}{\sigma^2} \int_{B(x_0, 4\sigma) \cap M_t} |A|^2 d\mu + C \int_{B(x_0, 4\sigma) \cap M_t} |A|^4 d\mu.$$

Choose a cut off function $\psi(t)$ in time so that $\psi = 0$ for $t \in [0, t_0 - (4\sigma)^2]$, $\psi(t) = 1$ for $t \geq t_0 - (2\sigma)^2$ and in between grows linearly. Multiply the previous inequality by $\psi(t)$ and integrate it over $[t_0 - (4\sigma)^2, t]$, where $t \geq t_0 - (2\sigma)^2$. Then,

$$\int_{B(x_0, 2\sigma) \cap M_t} |A|^2 d\mu \leq \frac{C}{\sigma^2} \int_{t_0 - (4\sigma)^2}^{t_0} \int_{B(x_0, 4\sigma) \cap M_t} |A|^2 d\mu + \tilde{C}.$$

By Hölder inequality and the Euclidean volume growth we have

$$\begin{aligned} \int_{B(x_0, 2\sigma) \cap M_t} |A|^2 d\mu &\leq \frac{C}{\sigma^2} \left(\int_{t_0 - (4\sigma)^2}^{t_0} \int_{B(x_0, 4\sigma) \cap M_t} |A|^4 d\mu \right)^{\frac{1}{2}} \cdot \left(\int_{t_0 - (4\sigma)^2}^{t_0} \int_{B(x_0, 4\sigma) \cap M_t} d\mu \right)^{\frac{1}{2}} + \tilde{C} \\ &\leq \frac{CC_I}{\sigma^2} \cdot (16\sigma^2 \cdot c\sigma^2)^{\frac{1}{2}} + \tilde{C} = \tilde{C}, \end{aligned}$$

where \tilde{C} is a uniform constant, independent of σ . \square

Having (5.25) and Claim 5.1 we can continue as follows. Let $\varepsilon \in (0, \varepsilon_0)$ be a small number to be determined. Here ε_0 is as in Theorem 1.7. Let N be an integer greater than C_I/ε . Given $x \in B(x_0, \sigma) \cap M_t$ where $t \in [t_0 - \sigma^2, t_0)$, there exists $1 \leq j \leq N$ with

$$\int_{t_0 - (2\sigma)^2}^{t_0} \int_{B(x, 9^{1-j}\sigma) \setminus B(x, 9^{-j}\sigma) \cap M_t} |A|^4 d\mu dt \leq C_I/N \leq \varepsilon \leq \varepsilon_0.$$

Note that, if $s = 9^{-j}\sigma$ then $B(x, 9s) \subset B(x_0, 2\sigma)$. Therefore

$$\int_{t_0 - (2\sigma)^2}^{t_0} \int_{B(x, 9s) \cap M_t} |A|^4 d\mu dt \leq \int_{t_0 - (2\sigma)^2}^{t_0} \int_{B(x_0, 2\sigma) \cap M_t} |A|^4 d\mu dt \leq C_I. \tag{5.30}$$

From the estimate

$$\int_{t_0 - (2\sigma)^2}^{t_0} \int_{B(x, 9s) \setminus B(x, s) \cap M_t} |A|^4 d\mu dt \leq \varepsilon \leq \varepsilon_0$$

we have, by the Choi–Schoen type estimate in Theorem 1.7

$$\begin{aligned} & \sup_{t \in [t_0 - (2\sigma - s)^2, t_0]} \sup_{y \in B(s, 8s) \setminus B(x, 2s) \cap M_t} |A|^2(y, t) \\ & \leq \varepsilon_0^{-1/2} s^{-2} \left(\int_{t_0 - (2\sigma)^2}^{t_0} \int_{B(x, 9s) \setminus B(x, s) \cap M_t} |A|^4 d\mu dt \right)^{1/2} \leq \varepsilon_0^{-1/2} \varepsilon^{1/2} s^{-2}. \end{aligned} \tag{5.31}$$

The proof of the above inequality is similar to the proof of (1.9) in Theorem 1.7. To obtain (5.31), we need to verify that (5.9) goes through for the annuli used here. Indeed, instead of using (5.10) to deduce (5.9), we use the more general estimate given in the proof of Proposition 1.6 in Ecker [8], namely

$$\sup_{t \in [0, 1]} \sup_{x \in \tilde{B}(x_*, 1)} |\tilde{A}|^2(x, t) \eta^{2(n+1)}(x, t) \leq C(n) \iint_{\text{spt } \eta} |\tilde{A}|^2 d\tilde{\mu} dt.$$

Now, for the case of annuli considered here, (5.9) follows by choosing a suitable cut off function η , compactly supported in the annulus.

Moreover, inspecting the proof, we can replace extrinsic balls by intrinsic balls $B(x, s)$. Thus, for each time slice $t \in [t_0 - (2\sigma - s)^2, t_0]$, we have the following two estimates

$$\sup_{y \in B(x, 8s) \setminus B(x, 2s) \cap M_t} |A|^2(y, t) \leq \varepsilon_0^{-1/2} \varepsilon^{1/2} s^{-2} \tag{5.32}$$

and

$$\int_{B(x, 9s)} |A|^2(t) d\mu \leq \int_{B(x_0, 2\sigma) \cap M_t} |A|^2 d\mu \leq C_I. \tag{5.33}$$

Now, arguing as in the proof of Colding–Minicozzi [5], Lemma 1.10, one can find a small number ε depending only on C_I and ε_0 such that (5.32) and (5.33) imply the following curvature estimate

$$\sup_{B(x, s) \subset M_t} |A|^2 \leq s^{-2} = (9^{-j} \sigma)^2 \leq 9^{2N} \sigma^{-2}.$$

Hence, for $x \in B(x_0, \sigma) \cap M_t$ where $t \in [t_0 - \sigma^2, t_0]$, the following estimate holds

$$|A|^2(x, t) \leq 9^{2N} \sigma^{-2}.$$

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