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Some nonlinear differential inequalities and an application to Hölder continuous almost complex structures

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Abstract

We consider some second order quasilinear partial differential inequalities for real-valued functions on the unit ball and find conditions under which there is a lower bound for the supremum of nonnegative solutions that do not vanish at the origin. As a consequence, for complex-valued functions $f(z)$ satisfying $\partial f/\partial \bar{z} = |f|^\alpha$, $0 < \alpha < 1$, and $f(0) \neq 0$, there is also a lower bound for $\sup |f|$ on the unit disk. For each α, we construct a manifold with an α-Hölder continuous almost complex structure where the Kobayashi–Royden pseudonorm is not upper semicontinuous.

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1. Introduction

We begin with an analysis of a second order quasilinear partial differential inequality for real-valued functions of n real variables,

$$\Delta u - B|u|^\varepsilon \geqslant 0, \tag{1}$$

where $B > 0$ and $\varepsilon \in [0, 1)$ are constants. In Section 2, we use a Comparison Principle argument to show that (1) has "no small solutions," in the sense that there is a uniform lower bound $M > 0$ for the supremum of solutions u which are nonnegative on the unit ball and nonzero at the origin.

We also consider a generalization of (1):

$$u\Delta u - B|u|^{1+\varepsilon} - C|\vec{\nabla} u|^2 \geqslant 0, \tag{2}$$

and find conditions under which there is a similar property of no small solutions, in Theorem 2.4.

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As an application of the results on the inequality (1), we show failure of upper semicontinuity of the Kobayashi–Royden pseudonorm for a family of 4-dimensional manifolds with almost complex structures of regularity $\mathcal{C}^{0,\alpha}$, $0 < \alpha < 1$. This generalizes the $\alpha = \frac{1}{2}$ example of [5]; it is known [6] that the Kobayashi–Royden pseudonorm is upper semicontinuous for almost complex structures with regularity $\mathcal{C}^{1,\alpha}$.

Our construction of the almost complex manifolds in Section 4 is very similar to that of [5]; we give the details for the convenience of the reader, and to show how the argument breaks down as $\alpha \to 1^-$, due to a shrinking radius of the domain. We also take the opportunity in Section 3 to state some lemmas which allow for a more quantitative description than that of [5].

One of the steps in [5] is a Maximum Principle argument applied to a complex-valued function $h(z)$ satisfying the equation $\partial h/\partial \bar{z} = |h|^{1/2}$, to get the property of no small solutions. The main difference between our paper and [5] is the use of a Comparison Principle in Section 2 instead of the Maximum Principle, and we arrive at this result:

Theorem 1.1. *For any $\alpha \in (0, 1)$, suppose $h(z)$ is a continuous complex-valued function on the closed unit disk, and on the set $\{z\colon |z| < 1,\ h(z) \neq 0\}$, h has continuous partial derivatives and satisfies*

$$\frac{\partial h}{\partial \bar{z}} = |h|^{\alpha}. \tag{3}$$

If $h(0) \neq 0$ then $\sup|h| > S_\alpha$, where the constant $S_\alpha > 0$ is defined by:

$$S_\alpha = \left(\frac{2(1-\alpha)}{2-\alpha}\right)^{1/(1-\alpha)}. \tag{4}$$

2. Some differential inequalities

Let D_R denote the open ball in $\mathbb{R}^n$ centered at $\vec{0}$ with radius $R > 0$, and let $\overline{D}_R$ denote the closed ball.

Lemma 2.1. *Given constants $B > 0$ and $0 \leqslant \varepsilon < 1$, let*

$$M = \left(\frac{B(1-\varepsilon)^2}{2(2\varepsilon + n(1-\varepsilon))}\right)^{\frac{1}{1-\varepsilon}} > 0.$$

Suppose the function $u : \overline{D}_1 \to \mathbb{R}$ satisfies:

- *u is continuous on $\overline{D}_1$,*
- *$u(\vec{x}) \geqslant 0$ for $\vec{x} \in D_1$,*
- *on the open set $\omega = \{\vec{x} \in D_1\colon u(\vec{x}) \neq 0\}$, $u \in \mathcal{C}^2(\omega)$,*
- *for $\vec{x} \in \omega$:*

$$\Delta u(\vec{x}) - B\big(u(\vec{x})\big)^{\varepsilon} \geqslant 0. \tag{5}$$

If $u(\vec{0}) \neq 0$, then $\sup_{\vec{x} \in D_1} u(\vec{x}) > M$.

Proof. Define a comparison function

$$v(\vec{x}) = M|\vec{x}|^{\frac{2}{1-\varepsilon}},$$

so $v \in \mathcal{C}^2(\mathbb{R}^n)$ since $0 \leqslant \varepsilon < 1$. By construction of M, it can be checked that v is a solution of this nonlinear Poisson equation on the domain $\mathbb{R}^n$:

$$\Delta v(\vec{x}) - B\big|v(\vec{x})\big|^{\varepsilon} \equiv 0.$$

Suppose, toward a contradiction, that $u(\vec{x}) \leqslant M$ for all $\vec{x} \in D_1$. For a point $\vec{x}_0$ on the boundary of $\omega \subseteq \mathbb{R}^n$, either $|\vec{x}_0| = 1$, in which case by continuity, $u(\vec{x}_0) \leqslant M = v(\vec{x}_0)$, or $0 < |\vec{x}_0| < 1$ and $u(\vec{x}_0) = 0$, so $u(\vec{x}_0) \leqslant v(\vec{x}_0)$. Since $u \leqslant v$ on the boundary of ω, the Comparison Principle [4, Theorem 10.1] applies to the subsolution u and the solution v on the domain ω. The relevant hypothesis for the Comparison Principle in this case is that the second term expression of (5), $-BX^\varepsilon$, is weakly decreasing, which uses $B>0$ and $\varepsilon \geqslant 0$. (To satisfy this technical condition for all $X \in \mathbb{R}$, we define a function $c : \mathbb{R} \to \mathbb{R}$ by $c(X) = -BX^\varepsilon$ for $X \geqslant 0$, and $c(X) = 0$ for $X \leqslant 0$. Then c is weakly decreasing in X, v satisfies $\Delta v(\vec{x}) + c(v(\vec{x})) \equiv 0$ and u satisfies $\Delta u(\vec{x}) + c(u(\vec{x})) \geqslant 0$.)

The conclusion of the Comparison Principle is that $u \leqslant v$ on ω, however $\vec{0} \in \omega$ and $u(\vec{0}) > v(\vec{0})$, a contradiction. □

Of course, the constant function $u \equiv 0$ satisfies the inequality (5), and so does the radial comparison function v, so the initial condition $u(\vec{0}) \neq 0$ is necessary.

Example 2.2. In the $n = 1$ case, $M = (\frac{B(1-\varepsilon)^2}{2(1+\varepsilon)})^{\frac{1}{1-\varepsilon}}$. For points $c_1, c_2 \in \mathbb{R}$, $c_1 < c_2$, define a function

$$u(x) = \begin{cases} M(x-c_2)^{\frac{2}{1-\varepsilon}} & \text{if } x \geqslant c_2, \\ 0 & \text{if } c_1 \leqslant x \leqslant c_2, \\ M(c_1 - x)^{\frac{2}{1-\varepsilon}} & \text{if } x \leqslant c_1. \end{cases}$$

Then $u \in \mathcal{C}^2(\mathbb{R})$, and it is nonnegative and satisfies $u'' = B|u|^\varepsilon$ (the case of equality in the $n = 1$ version of (5)). For $c_1 < 0 < c_2$, this gives an infinite collection of solutions of the ODE $u'' = B|u|^\varepsilon$ which are identically zero in a neighborhood of 0, so the ODE does not have a unique continuation property. For $c_1 > 0$ or $c_2 < 0$, the function u satisfies $u(0) \neq 0$ and the other hypotheses of Lemma 2.1, and its supremum on $(-1, 1)$ exceeds M even though it can be identically zero on an interval not containing 0.

Example 2.3. In the case $n = 2$, $B = 1$, $\varepsilon = 0$, (5) becomes the linear inequality $\Delta u \geqslant 1$ and the number $M = \frac{1}{4}$ agrees with Lemma 2 of [5], which was proved there using a Maximum Principle argument.

By applying Lemma 2.1 to the Laplacian of a power of u, we get the following generalization.

Theorem 2.4. *Given constants $B > 0$, $C \in \mathbb{R}$, and $\varepsilon < 1$, let*

$$M = \begin{cases} (\frac{B(1-\varepsilon)^2}{2(2(\varepsilon - C) + n(1-\varepsilon))})^{\frac{1}{1-\varepsilon}} & \text{if } C \leqslant \varepsilon, \\ (\frac{B(1-\varepsilon)}{2n})^{\frac{1}{1-\varepsilon}} & \text{if } C \geqslant \varepsilon. \end{cases}$$

Suppose the function $u : \overline{D}_1 \to \mathbb{R}$ satisfies:

- *u is continuous on $\overline{D}_1$,*
- *$u(\vec{x}) \geqslant 0$ for $\vec{x} \in D_1$,*
- *on the open set $\omega = \{\vec{x} \in D_1\colon u(\vec{x}) \neq 0\}$, $u \in \mathcal{C}^2(\omega)$,*
- *for $\vec{x} \in \omega$:*

$$u(\vec{x})\Delta u(\vec{x}) \geqslant B\left|u(\vec{x})\right|^{1+\varepsilon} + C\left|\vec{\nabla} u(\vec{x})\right|^2.$$

If $u(\vec{0}) \neq 0$, then $\sup_{\vec{x} \in D_1} u(\vec{x}) > M$.

Proof. Let $\mu = \min\{\varepsilon, C\}$, so $\mu \leqslant \varepsilon < 1$, and on the set ω,

$$u(\vec{x})\Delta u(\vec{x}) \geqslant B\left|u(\vec{x})\right|^{1+\varepsilon} + \mu\left|\vec{\nabla} u(\vec{x})\right|^2.$$

Consider the function $u^{1-\mu}$ on $\overline{D}_1$, so $u^{1-\mu} \in \mathcal{C}^0(\overline{D}_1) \cap \mathcal{C}^2(\omega)$, and on the set ω,

$$\begin{aligned} \Delta\left(u^{1-\mu}\right) &= (1-\mu)u^{-\mu-1}\left(u\Delta u - \mu|\vec{\nabla} u|^2\right) \\ &\geqslant (1-\mu)u^{-\mu-1}Bu^{1+\varepsilon} \\ &= (1-\mu)B\left(u^{1-\mu}\right)^{(\varepsilon-\mu)/(1-\mu)}. \end{aligned}$$

Since $(1-\mu)B > 0$, and $\mu \leqslant \varepsilon < 1 \Rightarrow 0 \leqslant \frac{\varepsilon-\mu}{1-\mu} < 1$, Lemma 2.1 applies to $u^{1-\mu}$. If $(u(\vec{0}))^{1-\mu} \neq 0$, then

$$\sup u^{1-\mu} > \left(\frac{(1-\mu)B(1-\frac{\varepsilon-\mu}{1-\mu})^2}{2(2\frac{\varepsilon-\mu}{1-\mu}+n(1-\frac{\varepsilon-\mu}{1-\mu}))}\right)^{\frac{1}{1-\frac{\varepsilon-\mu}{1-\mu}}} \quad \Rightarrow \quad \sup u > \left(\frac{B(1-\varepsilon)^2}{2(2(\varepsilon-\mu)+n(1-\varepsilon))}\right)^{\frac{1}{1-\varepsilon}}. \qquad \square$$

Functions satisfying a differential inequality of the form (1) or (2) also satisfy a Strong Maximum Principle; the only condition is $B > 0$.

Theorem 2.5. *Given any open set $\Omega \subseteq \mathbb{R}^n$, and any constants $B > 0$, $C, \varepsilon \in \mathbb{R}$, suppose the function $u : \Omega \to \mathbb{R}$ satisfies*:

- *u is continuous on Ω,*
- *on the set $\omega = \{\vec{x} \in \Omega\colon u(\vec{x}) > 0\}$, $u \in \mathcal{C}^2(\omega)$,*
- *on the set ω, u satisfies*

$$u\Delta u - B|u|^{1+\varepsilon} - C|\vec{\nabla} u|^2 \geqslant 0.$$

If $u(\vec{x}_0) > 0$ for some $\vec{x}_0 \in \Omega$, then u does not attain a maximum value on Ω.

Proof. Note that the constant function $u \equiv 0$ is the only locally constant solution of the inequality for $B > 0$. If $B = 0$ then obviously any constant function would be a solution.

Given a function u satisfying the hypotheses, ω is a nonempty open subset of Ω. Suppose, toward a contradiction, that there is some $\vec{x}_1 \in \Omega$ with $u(\vec{x}) \leqslant u(\vec{x}_1)$ for all $x \in \Omega$. In particular, $u(\vec{x}_1) \geqslant u(\vec{x}_0) > 0$, so $\vec{x}_1 \in \omega$. Let ω_1 be the connected component of ω containing $\vec{x}_1$.

For $\vec{x} \in \omega_1$, u satisfies the linear, uniformly elliptic inequality

$$\Delta u(\vec{x}) + \left(-B\big(u(\vec{x})\big)^{\varepsilon-1}\right)u(\vec{x}) + \left(-C\frac{\vec{\nabla} u(\vec{x})}{u(\vec{x})}\right) \cdot \vec{\nabla} u(\vec{x}) \geqslant 0,$$

where the coefficients (defined in terms of the given u) are locally bounded functions of $\vec{x}$, and $(-B(u(\vec{x}))^{\varepsilon-1})$ is negative for all $\vec{x} \in \omega$. It follows from the Strong Maximum Principle [4, Theorem 3.5] that since u attains a maximum value at $\vec{x}_1$, then u is constant on ω_1. Since the only constant solution is 0, it follows that $u(\vec{x}_1) = 0$, a contradiction. $\square$

The next lemma shows how an inequality like (5) with $n = 2$ can arise from a first order PDE for a complex-valued function.

Lemma 2.6. *Consider constants $\alpha, \gamma \in \mathbb{R}$ with $0 < \alpha < 1$. Let $\omega \subseteq \mathbb{C}$ be an open set, and suppose $h : \omega \to \mathbb{C}$ satisfies*:

- *$h \in \mathcal{C}^1(\omega)$,*
- *$h(z) \neq 0$ for all $z \in \omega$,*
- *$\frac{\partial h}{\partial \bar{z}} = |h|^\alpha$ on ω.*

Then, the following inequality is satisfied on ω:

$$\Delta\big(|h|^{(1-\alpha)\gamma}\big) \geqslant \big(4(1-\alpha)\gamma - (2-\alpha)^2\big)|h|^{(1-\alpha)(\gamma-2)}. \tag{6}$$

Remark. The special case $\alpha = \frac{1}{2}$, $\gamma = \frac{3}{2}$ is Lemma 1 of [5]; its proof there is a long calculation in polar coordinates, which can be generalized to some other values of α by an analogous argument. However, using z, $\bar{z}$ coordinates allows for a shorter calculation.

Proof of Lemma 2.6. We first want to show that h is smooth on ω, applying the regularity and bootstrapping technique of PDE to the equation $\partial h/\partial \bar{z} = |h|^\alpha$. We recall the following fact (for a more general statement, see

Theorem 15.6.2 of [1]): for a nonnegative integer ℓ, and $0 < \beta < 1$, if $\varphi \in \mathcal{C}^{\ell,\beta}_{loc}(\omega)$ and $g : \omega \to \mathbb{C}$ has first derivatives in $L^2_{loc}(\omega)$ and is a solution of $\partial g/\partial\bar{z} = \varphi$, then $g \in \mathcal{C}^{\ell+1,\beta}_{loc}(\omega)$. In our case, $\varphi = |h|^\alpha \in \mathcal{C}^1(\omega) \subseteq \mathcal{C}^{0,\beta}_{loc}(\omega)$ (since $h \in \mathcal{C}^1(\omega)$ and is nonvanishing), and $g = h$ has continuous first derivatives, so we can conclude that $g = h \in \mathcal{C}^{1,\beta}_{loc}(\omega)$. Repeating gives that $h \in \mathcal{C}^{2,\beta}_{loc}(\omega)$, etc.

Since the conclusion is a local statement, it is enough to express ω as a union of open subsets ω_k and establish the conclusion on each subset. For each $z_k \in \omega$, there is a sufficiently small disk ω_k containing z_k, where real exponentiation of $h(z)$ is well defined on ω_k, by choosing a single-valued branch of log to define $h^r = \exp(r\log(h))$.

The condition $\frac{\partial h}{\partial \bar{z}} = |h|^\alpha$ can be re-written

$$h_{\bar{z}} = (\bar{h})_z = |h|^\alpha = h^{\alpha/2}\bar{h}^{\alpha/2}.$$

This leads to

$$\begin{aligned} h_{z\bar{z}} &= (h_{\bar{z}})_z = \left(h^{\alpha/2}\bar{h}^{\alpha/2}\right)_z \\ &= \frac{\alpha}{2}\left(h^{(\alpha/2)-1}\bar{h}^{\alpha/2}h_z + h^\alpha\bar{h}^{\alpha-1}\right) \\ &= \overline{\left((\bar{h})_{z\bar{z}}\right)}, \end{aligned}$$

which is used in a line of the next step. For an arbitrary exponent $m \in \mathbb{R}$,

$$\begin{aligned} \left(|h|^m\right)_{z\bar{z}} &= \left(h^{m/2}\bar{h}^{m/2}\right)_{z\bar{z}} \\ &= \frac{\partial}{\partial z}\left(\frac{m}{2}h^{\frac{m}{2}-1}h_{\bar{z}}\bar{h}^{\frac{m}{2}} + h^{\frac{m}{2}}\frac{m}{2}\bar{h}^{\frac{m}{2}-1}(\bar{h})_{\bar{z}}\right) \\ &= \frac{m}{2}\frac{\partial}{\partial z}\left(h^{\frac{m}{2}-1+\frac{\alpha}{2}}\bar{h}^{\frac{m}{2}+\frac{\alpha}{2}} + h^{\frac{m}{2}}\bar{h}^{\frac{m}{2}-1}(\bar{h})_{\bar{z}}\right) \\ &= \frac{m}{2}\Bigg[\left(\frac{m}{2}+\frac{\alpha}{2}-1\right)h^{\frac{m}{2}+\frac{\alpha}{2}-2}h_z\bar{h}^{\frac{m}{2}+\frac{\alpha}{2}} + h^{\frac{m}{2}+\frac{\alpha}{2}-1}\left(\frac{m}{2}+\frac{\alpha}{2}\right)\bar{h}^{\frac{m}{2}+\frac{\alpha}{2}-1}(\bar{h})_z \\ &\quad + \frac{m}{2}h^{\frac{m}{2}-1}h_z\bar{h}^{\frac{m}{2}-1}(\bar{h})_{\bar{z}} + h^{\frac{m}{2}}\left(\frac{m}{2}-1\right)\bar{h}^{\frac{m}{2}-2}(\bar{h})_z(\bar{h})_{\bar{z}} + h^{\frac{m}{2}}\bar{h}^{\frac{m}{2}-1}(\bar{h})_{z\bar{z}}\Bigg]. \\ &= \frac{m}{2}\left[\operatorname{Re}\left((m+\alpha-2)|h|^{m+\alpha-4}\bar{h}^2h_z\right) + \left(\frac{m}{2}+\alpha\right)|h|^{m+2\alpha-2} + \frac{m}{2}|h|^{m-2}|h_z|^2\right]. \end{aligned}$$

With the aim of applying Lemma 2.1 to the function $|h|^m$, we consider the expression (8), with real constants B, ε, and $m \neq 0$. In line (9), we assign

$$\varepsilon = \frac{1}{m}(m + 2\alpha - 2) \tag{7}$$

to be able to combine like terms, and in line (10), we choose $B = 4m - (2-\alpha)^2$ to complete the square.

$$\begin{aligned} &\Delta\left(|h|^m\right) - B\left(|h|^m\right)^\varepsilon && (8) \\ &= 4\left(|h|^m\right)_{z\bar{z}} - B|h|^{m\varepsilon} \\ &= 2m\left[\operatorname{Re}\left((m+\alpha-2)|h|^{m+\alpha-4}\bar{h}^2h_z\right) + \left(\frac{m}{2}+\alpha\right)|h|^{m+2\alpha-2} + \frac{m}{2}|h|^{m-2}|h_z|^2\right] - B|h|^{m\varepsilon} \\ &= \left(m(m+2\alpha) - B\right)|h|^{m+2\alpha-2} && (9) \\ &\quad + \operatorname{Re}\left(2m(m+\alpha-2)|h|^{m+\alpha-4}\bar{h}^2h_z\right) + m^2|h|^{m-2}|h_z|^2 \\ &\geqslant |h|^{m-2}\left(\left(m^2+2\alpha m - B\right)|h|^{2\alpha} - 2|m||m+\alpha-2||h|^\alpha|h_z| + m^2|h_z|^2\right) \\ &= |h|^{m-2}\left(|m+\alpha-2||h|^\alpha - |m||h_z|\right)^2 \geqslant 0. && (10) \end{aligned}$$

Considering the form of (7), it is convenient to choose $m = (1-\alpha)\gamma$ for some constant $\gamma \neq 0$. The claim of the lemma follows; the $\gamma = 0$ case can be checked separately. □

The parameter γ can be chosen arbitrarily large; to apply Lemma 2.1 to get the "no small solutions" result of Theorem 1.1, we need the RHS coefficient in (6) to be positive, so $\gamma > \frac{(2-\alpha)^2}{4(1-\alpha)}$, and also the RHS exponent $(1-\alpha)(\gamma-2)$ to be nonnegative, so $\gamma \geqslant 2$. In contrast, the $\alpha = \frac{1}{2}$, $\gamma = \frac{3}{2}$ case appearing in Lemma 1 of [5] has RHS exponent $-\frac{1}{4}$. The approach of Theorem 2 of [5] is to use the negative exponent together with the result of Example 2.3 to show that assuming h has a small solution leads to a contradiction. As claimed, their method can be generalized to apply to other nonpositive exponents, but $\frac{(2-\alpha)^2}{4(1-\alpha)} < \gamma \leqslant 2$ holds only for $\alpha < 2(\sqrt{2}-1) \approx 0.8284$.

Proof of Theorem 1.1. Given a continuous $h : \overline{D}_1 \to \mathbb{C}$ satisfying the hypotheses of Theorem 1.1, on the set $\omega = \{z \in D_1\colon\ h(z) \neq 0\}$, $h \in \mathcal{C}^1(\omega)$, and the conclusion of Lemma 2.6 can be re-written:

$$\Delta\big(|h|^{(1-\alpha)\gamma}\big) \geqslant \big(4(1-\alpha)\gamma - (2-\alpha)^2\big)\big(|h|^{(1-\alpha)\gamma}\big)^{1-\frac{2}{\gamma}}. \tag{11}$$

The hypotheses of Lemma 2.1 are satisfied with $n = 2$, $u(x, y) = |h(x+iy)|^{(1-\alpha)\gamma}$, and $u(\vec{0}) \neq 0$, when the RHS of (11) has a positive coefficient (so $\gamma > \frac{(2-\alpha)^2}{4(1-\alpha)}$) and the quantity $\varepsilon = 1 - \frac{2}{\gamma}$ is in $[0, 1)$ (for $\gamma \geqslant 2$). The conclusion of Lemma 2.1 is:

$$\sup_{z\in D_1} \big|h(z)\big|^{(1-\alpha)\gamma} > M = \left(\frac{1}{4}\cdot\big(4(1-\alpha)\gamma - (2-\alpha)^2\big)\cdot\left(\frac{2}{\gamma}\right)^2\right)^{\gamma/2}$$

$$\Rightarrow\quad \sup_{z\in D_1}\big|h(z)\big| > \left(\frac{4(1-\alpha)\gamma - (2-\alpha)^2}{\gamma^2}\right)^{\frac{1}{2(1-\alpha)}}.$$

We can optimize this lower bound, using elementary calculus to show that the maximum value of $\frac{4(1-\alpha)\gamma-(2-\alpha)^2}{\gamma^2}$ is achieved at the critical point $\gamma = \frac{(2-\alpha)^2}{2(1-\alpha)} > \max\{2, \frac{(2-\alpha)^2}{4(1-\alpha)}\}$, and the lower bound for the sup is S_α as appearing in (4). □

Note that S_α is decreasing for $0 < \alpha < 1$, with $S_{1/2} = \frac{4}{9}$, $S_{2/3} = \frac{1}{8}$, and $S_\alpha \to 0$ as $\alpha \to 1^-$. This theorem is used in the proof of Theorem 4.3.

Example 2.7. As noted by [5], a 1-dimensional analogue of Eq. (3) in Theorem 1.1 is the well-known (for example, [2, §I.9]) ODE $u'(x) = B|u(x)|^\alpha$ for $0 < \alpha < 1$ and $B > 0$, which can be solved explicitly. By an elementary separation of variables calculation, the solution on an interval where $u \neq 0$ is $|u(x)| = (\pm(1-\alpha)(Bx+C))^{\frac{1}{1-\alpha}}$. The general solution on the domain $\mathbb{R}$ is, for $c_1 < c_2$,

$$u(x) = \begin{cases} (1-\alpha)^{\frac{1}{1-\alpha}}(B(x-c_2))^{\frac{1}{1-\alpha}} & \text{if } x \geqslant c_2,\\ 0 & \text{if } c_1 \leqslant x \leqslant c_2,\\ -(1-\alpha)^{\frac{1}{1-\alpha}}(B(c_1-x))^{\frac{1}{1-\alpha}} & \text{if } x \leqslant c_1. \end{cases}$$

So $u \in \mathcal{C}^1(\mathbb{R})$, and if $u(0) \neq 0$, then $\sup_{-1<x<1}|u(x)| > ((1-\alpha)B)^{\frac{1}{1-\alpha}}$.

3. Lemmas for holomorphic maps

We continue with the D_R notation for the open disk in the complex plane centered at the origin. The following quantitative lemmas on inverses of holomorphic functions $D_R \to \mathbb{C}$ are used in a step of the proof of Theorem 4.3 where we put a map $D_r \to \mathbb{C}^2$ into a normal form, (14).

Lemma 3.1. *(See [3, Exercise I.1].) Suppose $f : D_1 \to D_1$ is holomorphic, with $f(0) = 0$, $|f'(0)| = \delta > 0$. For any $\eta \in (0, \delta)$, let $s = (\frac{\delta-\eta}{1-\eta\delta})\eta$; then the restricted function $f : D_\eta \to D_1$ takes on each value $w \in D_s$ exactly once.* □

The hypotheses imply $\delta \leqslant 1$ by the Schwarz Lemma.

Lemma 3.2. *For a holomorphic map $Z_1 : D_r \to D_2$ with $Z_1(0) = 0$, $Z_1'(0) = 1$, if $r > \frac{4\sqrt{2}}{3}$ then there exists a continuous function $\phi : \overline{D}_1 \to D_r$ which is holomorphic on D_1 and which satisfies $(Z_1 \circ \phi)(z) = z$ for all $z \in \overline{D}_1$.*

Remark. It follows from the Schwarz Lemma that $r \leqslant 2$, and it follows from the fact that ϕ is an inverse of Z_1 that $\phi(0) = 0$ and $\phi'(0) = 1$.

Proof of Lemma 3.2. Define a new holomorphic function $f : D_1 \to D_1$ by

$$f(z) = \frac{1}{2} \cdot Z_1(r \cdot z),$$

so $f(0) = 0$, $f'(0) = \frac{r}{2}$, and Lemma 3.1 applies with $\delta = \frac{r}{2}$. If we choose $\eta = \frac{3r}{8}$, then $s = \frac{3r^2}{64-12r^2}$, and the assumption $r > \frac{4\sqrt{2}}{3}$ implies $s > \frac{1}{2}$. It follows from Lemma 3.1 that there exists a function $\psi : D_s \to D_\eta$ such that $(f \circ \psi)(z) = z$ for all $z \in \overline{D}_{1/2} \subseteq D_s$; this inverse function ψ is holomorphic on $D_{1/2}$. The claimed function $\phi : \overline{D}_1 \to D_{r\eta} \subseteq D_r$ is defined by $\phi(z) = r \cdot \psi(\frac{1}{2} \cdot z)$, so for $z \in \overline{D}_1$,

$$Z_1\big(\phi(z)\big) = Z_1\left(r \cdot \psi\left(\frac{1}{2} \cdot z\right)\right) = 2 \cdot f\left(\psi\left(\frac{1}{2} \cdot z\right)\right) = 2 \cdot \frac{1}{2} \cdot z = z. \qquad \square$$

4. *J*-holomorphic disks

For $S > 0$, consider the bidisk $\Omega_S = D_2 \times D_S \subseteq \mathbb{C}^2$, as an open subset of $\mathbb{R}^4$, with coordinates $\vec{x} = (x_1, y_1, x_2, y_2) = (z_1, z_2)$ and the trivial tangent bundle $T\Omega_S \subseteq T\mathbb{R}^4$. Consider an almost complex structure J on Ω_S given by a complex structure operator on $T_{\vec{x}}\Omega_S$ of the following form:

$$J(\vec{x}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -1 \\ \lambda & 0 & 1 & 0 \end{pmatrix}, \tag{12}$$

where $\lambda : \Omega_S \to \mathbb{R}$ is any function.

A differentiable map $Z : D_r \to \Omega_S$ is a J-holomorphic disk if $dZ \circ J_{std} = J \circ dZ$, where J_{std} is the standard complex structure on $D_r \subseteq \mathbb{C}$. Let $z = x + iy$ be the coordinate on D_r. For J of the form (12), if $Z(z)$ is defined by complex-valued component functions,

$$Z : D_r \to \Omega_S : Z(z) = \big(Z_1(z), Z_2(z)\big), \tag{13}$$

then the J-holomorphic property implies that $Z_1 : D_r \to D_2$ is holomorphic in the standard way.

Example 4.1. If the function $\lambda(z_1, z_2)$ satisfies $\lambda(z_1, 0) = 0$ for all $z_1 \in D_2$, then the map $Z : D_2 \to \Omega_S : Z(z) = (z, 0)$ is a J-holomorphic disk.

Definition 4.2. The Kobayashi–Royden pseudonorm on Ω_S is a function $T\Omega_S \to \mathbb{R} : (\vec{x}, \vec{v}) \mapsto \|(\vec{x}, \vec{v})\|_K$, defined on tangent vectors $\vec{v} \in T_{\vec{x}}\Omega_S$ to be the number

$$\operatorname{glb}\left\{\frac{1}{r} \colon \exists \text{ a } J\text{-holomorphic } Z : D_r \to \Omega_S,\ Z(0) = \vec{x},\ dZ(0)\left(\frac{\partial}{\partial x}\right) = \vec{v}\right\}.$$

Under the assumption that $\lambda \in \mathcal{C}^{0,\alpha}(\Omega_S)$, $0 < \alpha < 1$, it is shown by [6] and [7] that there is a nonempty set of J-holomorphic disks through $\vec{x}$ with tangent vector $\vec{v}$ as in the definition, so the pseudonorm is a well-defined function. Further, each such disk satisfies $Z \in \mathcal{C}^1(D_r)$.

At this point we pick $\alpha \in (0, 1)$ and set $\lambda(z_1, z_2) = -2|z_2|^\alpha$. Let $S = S_\alpha > 0$ be the constant defined by formula (4) from Theorem 1.1. Then, (Ω_S, J) is an almost complex manifold with the following property:

Theorem 4.3. *If* $0 \neq b \in D_S$ *then* $\|(0,b),(1,0)\|_K \geqslant \frac{3}{4\sqrt{2}}$.

Remark. Since $\frac{3}{4\sqrt{2}} \approx 0.53$, and $\|(0,0),(1,0)\|_K \leqslant \frac{1}{2}$ by Example 4.1, the theorem shows that the Kobayashi–Royden pseudonorm is not upper semicontinuous on $T\Omega_S$.

Proof. Consider a J-holomorphic map $Z : D_r \to \Omega_S$ of the form (13), and suppose $Z(0) = (0,b) \in \Omega_S$ and $dZ(0)(\frac{\partial}{\partial x}) = (1,0)$. Then the holomorphic function $Z_1 : D_r \to D_2$ satisfies $Z_1(0) = 0$, $Z_1'(0) = 1$, and $Z_2 \in \mathcal{C}^1(D_r)$ satisfies $Z_2(0) = b$.

Suppose, toward a contradiction, that there exists such a map Z with $b \neq 0$ and $r > \frac{4\sqrt{2}}{3}$. Then Lemma 3.2 applies to Z_1: there is a re-parametrization ϕ which puts Z into the following normal form:

$$(Z \circ \phi) : \overline{D}_1 \to \Omega_S,$$
$$z \mapsto \big(Z_1(\phi(z)), Z_2(\phi(z))\big) = \big(z, f(z)\big), \tag{14}$$

where $f = Z_2 \circ \phi : \overline{D}_1 \to D_S$ satisfies $f \in \mathcal{C}^0(\overline{D}_1) \cap \mathcal{C}^1(D_1)$. From the fact that $Z \circ \phi$ is J-holomorphic on D_1, it follows from the form (12) of J that if $f(z) = u(x,y) + iv(x,y)$, then f satisfies this system of nonlinear Cauchy–Riemann equations on D_1:

$$\frac{du}{dy} = -\frac{dv}{dx} \quad \text{and} \quad \frac{du}{dx} + \lambda\big(z, f(z)\big) = \frac{dv}{dy} \tag{15}$$

with the initial conditions $f(0) = b$, $u_x(0) = u_y(0) = v_x(0) = 0$ and $v_y(0) = \lambda(0,b) = -2|b|^\alpha$. The system of equations implies

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x}(u+iv) + i\frac{\partial}{\partial y}(u+iv)\right)\\ &= \frac{1}{2}\big(u_x - v_y + i(v_x + u_y)\big)\\ &= -\frac{1}{2}\lambda\big(z, f(z)\big) = |f|^\alpha.\end{aligned} \tag{16}$$

So, Theorem 1.1 applies, with $f = h$. The conclusion is that

$$\sup_{z \in D_1} \big|f(z)\big| > S_\alpha,$$

but this contradicts $|f(z)| < S = S_\alpha$. □

The previously mentioned existence theory for J-holomorphic disks shows there are interesting solutions of Eq. (16), and therefore also the inequality (11).

Example 4.4. For $0 < \alpha < 1$, (Ω_S, J), $\lambda(z_1, z_2) = -2|z_2|^\alpha$ as above, a map $Z : D_r \to \Omega_S$ of the form $Z(z) = (z, f(z))$ is J-holomorphic if $f(x,y) = u(x,y) + iv(x,y)$ is a solution of (15). Again generalizing the $\alpha = \frac{1}{2}$ case of [5], examples of such solutions can be constructed (for small r) by assuming $v \equiv 0$ and u depends only on x, so (15) becomes the ODE $u'(x) - 2|u(x)|^\alpha = 0$. This is the equation from Example 2.7; we can conclude that J-holomorphic disks in Ω_S do not have a unique continuation property.

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