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Orbital stability of semitrivial standing waves for the Klein–Gordon–Schrödinger system

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Abstract

In the present paper, we study the orbital stability and instability of standing waves of the Klein–Gordon–Schrödinger system. Especially, we are interested in a standing wave which is expressed by the unique positive solution w_1 to a certain scalar field equation. By utilizing the property of the positive solution w_1, we can apply the general theory of Grillakis, Shatah and Strauss (1987) [11] and show the stability and instability of the standing wave.
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1. Introduction

We consider the Klein–Gordon–Schrödinger system with Yukawa coupling:

$$\begin{cases} i\partial_t u + \Delta u = -2uv, & (t,x) \in \mathbb{R} \times \mathbb{R}^N, \\ \partial_t^2 v - \Delta v + v = |u|^2, & (t,x) \in \mathbb{R} \times \mathbb{R}^N, \end{cases} \tag{1}$$

where $u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $v : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, and $1 \leqslant N \leqslant 5$. The system (1) describes a classical model of the Yukawa interaction of conserved nucleon field with neutral real meson field. The unknown function u is the complex scalar nucleon field and the unknown function v is the real meson field.

We study the orbital stability and instability of standing waves $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ of the system (1), where $\omega > 0$ and $(\varphi_\omega, \psi_\omega)$ is a nontrivial solution to

$$\begin{cases} -\Delta\varphi + \omega\varphi = 2\varphi\psi, & x \in \mathbb{R}^N, \\ -\Delta\psi + \psi = |\varphi|^2, & x \in \mathbb{R}^N. \end{cases} \tag{2}$$

In our previous papers [14,15], we prove that the standing wave $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ is orbitally stable for sufficiently large $\omega > 0$ and orbitally unstable for sufficiently small $\omega > 0$ in the case where $N = 3$ and $(\varphi_\omega, \psi_\omega)$ is ground state.

In the present paper, we discuss the stability and the instability of standing waves $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ when ω is close to 1. Note that the pair of functions $(\sqrt{2}w_1, w_1)$ satisfies the system (2) with $\omega = 1$, where $w_1 \in H^1(\mathbb{R}^N, \mathbb{R})$ is the unique positive radial solution to the following scalar field equation:

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$$-\Delta w + w - 2w^2 = 0, \quad x \in \mathbb{R}^N \tag{3}$$

(see Berestycki and P.L. Lions [5] for the existence and Kwong [16] for the uniqueness). We will prove that the standing wave $(\sqrt{2}e^{it}w_1, w_1)$ is orbitally stable in the case where $1 \leqslant N \leqslant 3$ and orbitally unstable in the case where $N = 4$ or 5. We remark that Eq. (3) has no nontrivial solution in the case where $N \geqslant 6$.

First, we recall the Cauchy problem for the system (1). The Cauchy problem for the system (1) is locally well-posed in the energy space $X := H^1(\mathbb{R}^N, \mathbb{C}) \times H^1(\mathbb{R}^N, \mathbb{R}) \times L^2(\mathbb{R}^N, \mathbb{R})$ (see [2] and also [3,7,12,13,18]). Namely, for any $(u_0, v_0, v_1) \in X$, there exist $T_{\max} = T_{\max}(u_0, v_0, v_1) > 0$ and a local solution $(u, v, \partial_t v) \in C([0, T_{\max}), X)$ to the system (1) with $(u(0), v(0), \partial_t v(0)) = (u_0, v_0, v_1)$. Moreover, the solution satisfies the conservation laws:

$$E\big(u(t), v(t), \partial_t v(t)\big) = E(u_0, v_0, v_1), \qquad \big\|u(t)\big\|_{L^2} = \|u_0\|_{L^2} \quad \text{for } t \in [0, T_{\max}),$$

where $E(u, v, w) = J(u, v) + \|w\|_{L^2}^2$ and

$$J(u, v) = \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|v\|_{L^2}^2 - 2\int_{\mathbb{R}^N} |u|^2 v\, dx.$$

We note that in the case where $1 \leqslant N \leqslant 3$, the solution to the system (1) is global, that is, $T_{\max} = \infty$.

We now discuss the orbital stability of standing waves. The stability and the instability are formulated as follows:

Definition 1. Let $(\varphi_\omega, \psi_\omega) \in H^1(\mathbb{R}^N, \mathbb{C}) \times H^1(\mathbb{R}^N, \mathbb{R})$ be a solution to the system (2). We say that the standing wave $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ is *orbitally stable* if for any $\epsilon > 0$, there exists $\delta > 0$ such that if $(u_0, v_0, v_1) \in X$ satisfies $\|(u_0, v_0, v_1) - (\varphi_\omega, \psi_\omega, 0)\|_X < \delta$, then the solution $(u(t), v(t))$ to the system (1) with $(u(0), v(0), \partial_t v(0)) = (u_0, v_0, v_1)$ satisfies

$$\inf_{\theta \in \mathbb{R},\, y \in \mathbb{R}^N} \big\| \big(u(t), v(t), \partial_t v(t)\big) - \big(e^{i\theta}\varphi_\omega(\cdot + y), \psi_\omega(\cdot + y), 0\big) \big\|_X < \epsilon$$

for all $t \geqslant 0$. Otherwise, $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ is said to be *orbitally unstable*.

Before stating our results, we recall the results concerning the standing wave $e^{it}w_1$ for the following nonlinear Schrödinger equation:

$$i\partial_t u + \Delta u + 2|u|u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N. \tag{4}$$

It is well known that the standing wave $e^{it}w_1$ is orbitally stable in the case where $1 \leqslant N \leqslant 3$ and orbitally unstable in the case where $N = 4$ or 5 (see Cazenave and P.L. Lions [6] for the stability and Berestycki and Cazenave [4], Weinstein [21] for the instability).

Our main results in this paper are the following:

Theorem 2.

(i) *Let* $1 \leqslant N \leqslant 5$. *There exists a constant* $\epsilon_* > 0$ *such that for* $\omega \in (1 - \epsilon_*, 1 + \epsilon_*)$, *the system* (2) *has a solution* $(\varphi_\omega, \psi_\omega) \in H^1_{\mathrm{rad}}(\mathbb{R}^N, \mathbb{R}) \times H^1_{\mathrm{rad}}(\mathbb{R}^N, \mathbb{R})$. *Moreover, the mapping* $\omega \mapsto (\varphi_\omega, \psi_\omega)$ *is* C^2 *with values in* $H^1_{\mathrm{rad}}(\mathbb{R}^N, \mathbb{R}) \times H^1_{\mathrm{rad}}(\mathbb{R}^N, \mathbb{R})$ *satisfying* $(\varphi_1, \psi_1) = (\sqrt{2}w_1, w_1)$.

(ii) *Let* $(\varphi_\omega, \psi_\omega)$ *be the solution to the system* (2), *which is obtained in Theorem* 2(i). *Then there exists a constant* $\epsilon'_* > 0$ *such that for* $\omega \in (1 - \epsilon'_*, 1 + \epsilon'_*)$, *the standing wave* $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ *is orbitally stable in the case where* $1 \leqslant N \leqslant 3$ *and orbitally unstable in the case where* $N = 4$ *or* 5.

Let us admit Theorem 2(i) for the moment and explain the proof of Theorem 2(ii) briefly. To do this, we fix notation. For each $\omega > 0$, we put

$$S_\omega(u) = J(u) + \omega\|u\|_{L^2}^2.$$

Then we see that $S_\omega \in C^2(H^1(\mathbb{R}^N, \mathbb{C}) \times H^1(\mathbb{R}^N, \mathbb{R}), \mathbb{R})$ and it is known that $(\varphi, \psi) \in H^1(\mathbb{R}^N, \mathbb{C}) \times H^1(\mathbb{R}^N, \mathbb{R})$ is a weak solution to the system (2) if and only if $S'_\omega(\varphi_\omega, \psi_\omega) = 0$. To prove Theorem 2(ii), we use the general theory of Grillakis, Shatah and Strauss [11]. By applying the theory to our system, we have the following proposition.

Proposition 3. *Let $(\varphi_\omega, \psi_\omega) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ be a solution to the system* (2). *Assume that*

(i) *the positive spectrum of the operator $S''_\omega(\varphi_\omega, \psi_\omega)$ is bounded away from zero,*
(ii) *the kernel of the operator $S''_\omega(\varphi_\omega, \psi_\omega)$ is spanned by ${}^t(i\varphi_{\omega,0})$ and ${}^t(\partial_{x_i}\varphi_\omega, \partial_{x_i}\psi_\omega)$ for $i = 1, 2, \dots, N$, that is,*

$$\operatorname{Ker} S''_\omega(\varphi_\omega, \psi_\omega) = \operatorname{Span}\{{}^t(i\varphi_\omega, 0)\} \cup \operatorname{Span}\{{}^t(\partial_{x_i}\varphi_\omega, \partial_{x_i}\psi_\omega) \mid i = 1, 2, \dots, N\},$$

(iii) *the operator $S''_\omega(\varphi_\omega, \psi_\omega)$ has exactly one negative simple eigenvalue.*

Then if $\partial_\omega \|\varphi_\omega\|^2_{L^2} > 0$ (resp. < 0), the standing wave $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ is stable (resp. unstable).

For each $(u, v) \in H^1(\mathbb{R}^N, \mathbb{C}) \times H^1(\mathbb{R}^N, \mathbb{R})$, we have

$$\left\langle S''_\omega(\varphi_\omega, \psi_\omega)\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right\rangle = \left\langle L_{1,\omega}\begin{pmatrix} u_1 \\ v \end{pmatrix}, \begin{pmatrix} u_1 \\ v \end{pmatrix}\right\rangle + \langle L_{2,\omega}u_2, u_2\rangle,$$

where $u_1 = \operatorname{Re} u$, $u_2 = \operatorname{Im} u$ and

$$L_{1,\omega} = \begin{pmatrix} -\Delta + \omega - 2\psi_\omega & -2\varphi_\omega \\ -2\varphi_\omega & -\Delta + 1 \end{pmatrix}, \qquad L_{2,\omega} = -\Delta + \omega - 2\psi_\omega.$$

We first verify the assumptions of Proposition 3 in the case where $\omega = 1$. It is well known that the operator $L_{2,1} = -\Delta + 1 - 2w_1$ is non-negative and $\operatorname{Ker} L_{2,1} = \operatorname{Span}\{w_1\}$ (see e.g. Weinstein [22]). Thus, it is enough to investigate the spectrum of the operator $L_{1,1}$. To do this, diagonalization of the operator $L_{1,1}$, which is already employed by Yew [23] and Angulo and Linares [1], is very useful. More precisely, if we define the unitary operator $A : L^2(\mathbb{R}^N, \mathbb{R}) \times L^2(\mathbb{R}^N, \mathbb{R}) \to L^2(\mathbb{R}^N, \mathbb{R}) \times L^2(\mathbb{R}^N, \mathbb{R})$ by

$$A = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} & 1 \\ 1 & -\sqrt{2} \end{pmatrix},$$

then we see that

$$AL_{1,1}A = \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix}, \tag{5}$$

where $T = -\Delta + 1 - 4w_1$ and $S = -\Delta + 1 + 2w_1$. We note that the operator S is positive and the operator T is the real part of linearized operator of Eq. (3). Since $\operatorname{Ker} T = \operatorname{Span}\{\partial_{x_i} w_1 \mid i = 1, 2, \dots, N\}$ (see Weinstein [22] and Ni and Takagi [17]), we infer that

$$\operatorname{Ker} L_{1,1} = \operatorname{Span}\{{}^t(\sqrt{2}\partial_{x_i} w_1, \partial_{x_i} w_1) \mid i = 1, 2, \dots, N\}. \tag{6}$$

Furthermore, we can show that the operator $L_{1,1}$ has exactly one negative simple eigenvalue from (5).

Remark 4. It follows from (6) that $\operatorname{Ker} L_{1,1}|_{H^1_{\mathrm{rad}} \times H^1_{\mathrm{rad}}} = \{0\}$. Thus, we can use the implicit function theorem and obtain Theorem 2(i).

Next, we calculate $\partial_\omega \|\varphi_\omega\|^2_{L^2}|_{\omega=1}$. In the previous results [11,19,20], the scale invariance of Eq. (4) is used to calculate the derivative. Since the system (1) is not scale invariant, we encounter difficulties when we try to check the sufficient condition. To overcome the difficulties, we use the diagonalization (5) again. Then we find that

$$\frac{1}{2}\partial_\omega \|\varphi_\omega\|^2_{L^2}|_{\omega=1} = \frac{4 - N}{3}\|w_1\|^2_{L^2} - \frac{2}{3}\langle S^{-1}w_1, w_1\rangle$$

(see Section 4 below). This yields that $\partial_\omega \|\varphi_\omega\|^2_{L^2}|_{\omega=1} < 0$ in the case where $N = 4$ or 5. In the case where $1 \leqslant N \leqslant 3$, we need a further investigation. We set $f_1 = S^{-1}w_1$. Then we see that $f_1 \in H^1(\mathbb{R}^N, \mathbb{R})$ is positive radial and decreasing function in $|x|$. These properties play an important role to show $\partial_\omega \|\varphi_\omega\|^2_{L^2}|_{\omega=1} > 0$ in the case where $1 \leqslant N \leqslant 3$.

Since the mapping $\omega \mapsto \varphi_\omega$ is C^2, the sign of $\partial_\omega \|\varphi_\omega\|_{L^2}^2$ does not change for $\omega \in (1-\epsilon'_*, 1+\epsilon'_*)$ if ϵ'_* is sufficiently small. Furthermore, by using a perturbation method, we can verify the spectral assumptions of Proposition 3. Therefore, we can obtain Theorem 2(ii).

The rest of this paper is organized as follows. In Section 2, we investigate the spectrum of the linearized operator at $\omega = 1$ and give the proof of Theorem 2(i). In Section 3, by using a perturbation method, we verify the linearized operator satisfies the assumptions of Proposition 3 when ω is close to 1. In Section 4, we calculate $\partial_\omega \|\varphi_\omega\|_{L^2}^2$ and prove Theorem 2(ii).

2. Spectrum of linearized operator at $\omega = 1$

In this section, we check the linearized operator $S''_1(\sqrt{2}w_1, w_1)$ satisfies the spectral assumptions of Proposition 3 and give the proof of Theorem 2(i). As we mentioned in Section 1, since it is known that the operator $L_{2,1} = -\Delta + 1 - 2w_1$ is non-negative operator and $\operatorname{Ker} L_{2,1} = \operatorname{Span}\{w_1\}$, it is enough to investigate the spectrum of the operator $L_{1,1}$, where

$$L_{1,1} = \begin{pmatrix} -\Delta + 1 - 2w_1 & -2\sqrt{2}w_1 \\ -2\sqrt{2}w_1 & -\Delta + 1 \end{pmatrix}. \tag{7}$$

Let $A : L^2(\mathbb{R}^N, \mathbb{R}) \times L^2(\mathbb{R}^N, \mathbb{R}) \to L^2(\mathbb{R}^N, \mathbb{R}) \times L^2(\mathbb{R}^N, \mathbb{R})$ be defined by

$$A = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} & 1 \\ 1 & -\sqrt{2} \end{pmatrix}.$$

Note that $A = A^* = A^{-1}$. Then by a direct computation, we obtain the following lemma.

Lemma 5. *Let $L_{1,1} : L^2(\mathbb{R}^N, \mathbb{R}) \times L^2(\mathbb{R}^N, \mathbb{R}) \to L^2(\mathbb{R}^N, \mathbb{R}) \times L^2(\mathbb{R}^N, \mathbb{R})$ be defined by* (7). *Then we have*

$$AL_{1,1}A = \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix},$$

where $T = -\Delta + 1 - 4w_1$ and $S = -\Delta + 1 + 2w_1$.

Since the function w_1 is positive, we see that the operator S is positive. Note that the operator T is a real part of linearized operator of Eq. (3). Concerning the operator T, we know that the following lemma holds (see Weinstein [22] and Ni and Takagi [17]).

Lemma 6. *Let $1 \leqslant N \leqslant 5$. The operator T satisfies the following*:

(i) $\sigma_{\mathrm{ess}}(T) = [1, \infty)$,
(ii) $\operatorname{Ker} T = \operatorname{Span}\{\partial_{x_i} w_1 \mid i = 1, 2, \ldots, N\}$,
(iii) *the operator T has exactly one negative eigenvalue $-\lambda_1$ for some $\lambda_1 > 0$.*

Using Lemmas 5 and 6, we can easily get the following proposition.

Proposition 7. *Let $1 \leqslant N \leqslant 5$. The operator $L_{1,1}$ satisfies the following*:

(i) $\sigma_{\mathrm{ess}}(L_{1,1}) = [1, \infty)$,
(ii) $\operatorname{Ker} L_{1,1} = \operatorname{Span}\{{}^t(\sqrt{2}\partial_{x_i} w_1, \partial_{x_i} w_1) \mid i = 1, 2, \ldots, N\}$,
(iii) *the operator $L_{1,1}$ has exactly one negative simple eigenvalue $-\lambda_1$, where $-\lambda_1$ is the first eigenvalue of the operator T.*

We can show Theorem 2(i) from Proposition 7(ii).

Proof of Theorem 2(i). Clearly, we infer that $\operatorname{Ker} L_{1,1}|_{H^1_{\mathrm{rad}} \times H^1_{\mathrm{rad}}} = \{0\}$. Thus, the operator $L_{1,1}$ is injective on $H^1_{\mathrm{rad}}(\mathbb{R}^N, \mathbb{R}) \times H^1_{\mathrm{rad}}(\mathbb{R}^N, \mathbb{R})$. Moreover, since $w_1(x) \to 0$ as $|x| \to \infty$, we see that

$$\begin{pmatrix} 2w_1 & -2\sqrt{2}w_1 \\ -2\sqrt{2}w_1 & 0 \end{pmatrix}$$

is a compact operator. It follows from the Fredholm alternative theorem that the operator $L_{1,1}$ is surjective. Therefore, by using the implicit function theorem, we find that there exist $\epsilon_* > 0$ and the solution $(\varphi_\omega, \psi_\omega) \in H^1_{\mathrm{rad}}(\mathbb{R}^N, \mathbb{R}) \times H^1_{\mathrm{rad}}(\mathbb{R}^N, \mathbb{R})$ to the system (2) for $\omega \in (1-\epsilon_*, 1+\epsilon_*)$ with $(\varphi_1, \psi_1) = (\sqrt{2}w_1, w_1)$. Furthermore, since $S_\omega \in C^2(H^1_{\mathrm{rad}}(\mathbb{R}^N, \mathbb{R}) \times H^1_{\mathrm{rad}}(\mathbb{R}^N, \mathbb{R}), \mathbb{R})$, we see that the map $\omega \mapsto (\varphi_\omega, \psi_\omega)$ is C^2 with values in $H^1_{\mathrm{rad}}(\mathbb{R}^N, \mathbb{R}) \times H^1_{\mathrm{rad}}(\mathbb{R}^N, \mathbb{R})$. This completes the proof. □

3. Verification of spectral assumptions

In this section, we show that the operator $S''_\omega(\varphi_\omega, \psi_\omega)$ satisfies the assumptions of Proposition 3 when ω is close to 1. We first consider the operator $L_{1,\omega}$.

Proposition 8. *There exists $\epsilon_1 > 0$ such that for $\omega \in (1-\epsilon_1, 1+\epsilon_1)$, the operator $L_{1,\omega}$ satisfies the following properties*:

(i) $\sigma_{\mathrm{ess}}(L_{1,\omega}) = [\min\{1, \omega\}, \infty)$,
(ii) *the operator $L_{1,\omega}$ has exactly one negative simple eigenvalue and*

$$\operatorname{Ker} L_{1,\omega} = \operatorname{Span}\left\{ {}^t(\partial_{x_i}\varphi_\omega, \partial_{x_i}\psi_\omega) \;\middle|\; i = 1, 2, \ldots, N \right\}.$$

Proof. (i) immediately follows from the Weyl's essential theorem. Then following Grillakis [10], we show (ii). For each $\omega > 0$, there exists a negative eigenvalue of the operator $L_{1,\omega}$ because $\langle L_{1,\omega}\vec{\varphi}_\omega, \vec{\varphi}_\omega\rangle = -3\int |\varphi_\omega|^2 \psi_\omega\,dx < 0$, where $\vec{\varphi}_\omega = (\varphi_\omega, \psi_\omega)$. Moreover, we can easily find that $\partial_{x_i}\vec{\varphi}_\omega \in \operatorname{Ker} L_{1,\omega}$ for $i = 1, 2, \ldots, N$, where $\partial_{x_i}\vec{\varphi}_\omega = {}^t(\partial_{x_i}\varphi_\omega, \partial_{x_i}\psi_\omega)$.

Thus, it is enough to show that there is no other non-positive eigenvalue of the operator $L_{1,\omega}$. Suppose that there exists $\{\omega_j\} \subset (0, \infty)$ with $\lim_{j\to\infty} \omega_j = 1$ such that the number of the non-positive eigenvalues of the operator L_{1,ω_j} (counting the multiplicity) is at least $N+2$. We denote by N_j the number of the non-positive eigenvalues and by $\vec{\phi}_i(\omega_j) \in H^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N)$ $(i = 1, 2, \ldots, N_j)$ the non-positive eigenfunctions. Let $\vec{\psi}_i \in H^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N)$ $(i = 1, 2, \ldots, N+1)$ be the non-positive eigenfunctions of the operator $L_{1,1}$. Then we can write

$$\vec{\phi}_i(\omega_j) = \sum_{k=1}^{N+1} b_{ik}(\omega_j)\vec{\psi}_k + \vec{r}_i(\omega_j)$$

for some $b_{ik} \in \mathbb{R}$ and $\vec{r}_i(\omega_j) \in H^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N)$ with $\langle \vec{r}_i(\omega_j), \vec{\psi}_k\rangle = 0$ for $k = 1, 2, \ldots, N+1$. Then using the assumption that $N_j > N+1$, there exist $(\alpha_1(\omega_j), \alpha_2(\omega_j), \ldots, \alpha_{N_j}(\omega_j)) \in \mathbb{R}^{N_j} \setminus \{0\}$ such that $\sum_{i=1}^{N_j} \alpha_i(\omega_j) b_{ik}(\omega_j) = 0$ for all $k = 1, 2, \ldots, N+1$. Then we put $\vec{p}(\omega_j) = \sum_{i=1}^{N_j} \alpha_i(\omega_j)\vec{\phi}_i(\omega_j) = \sum_{i=1}^{N_j} \alpha_i(\omega_j)\vec{r}_i(\omega_j)$. Without loss of generality, we may assume $\|\vec{p}(\omega_j)\|_{H^1\times H^1} = 1$. Then we can easily find that

$$\begin{aligned} 0 &\geqslant \liminf_{j\to\infty} \langle L_{1,\omega_j}\vec{p}(\omega_j), \vec{p}(\omega_j)\rangle \\ &= \liminf \left\{ \langle L_{1,1}\vec{p}(\omega_j), \vec{p}(\omega_j)\rangle + \langle (L_{1,\omega_j} - L_{1,1})\vec{p}(\omega_j), \vec{p}(\omega_j)\rangle \right\} \\ &\geqslant \delta \left\| \vec{p}(\omega_j) \right\|^2_{H^1\times H^1} \\ &= \delta \end{aligned}$$

for some $\delta > 0$, which is a contradiction. Thus, we obtain the desired result. □

Concerning the operator $L_{2.\omega}$, we can show the following proposition by an argument similar to that in Proposition 8.

Proposition 9. *There exists $\epsilon_2 > 0$ such that for $\omega \in (1-\epsilon_2, 1+\epsilon_2)$, the operator $L_{2,\omega}$ satisfies the following properties*:

(i) $\sigma_{\mathrm{ess}}(L_{2,\omega}) = [\omega, \infty)$,
(ii) *the operator $L_{2,\omega}$ is non-negative and* $\operatorname{Ker} L_{2,\omega} = \operatorname{Span}\{\varphi_\omega\}$.

4. Proof of Theorem 2(ii)

In this section, we calculate $\partial_\omega \|\varphi_\omega\|_{L^2}^2$ and show Theorem 2(ii). Since the map $\omega \mapsto (\varphi_\omega, \psi_\omega)$ is C^2, it is enough to show the following proposition.

Theorem 10. *Let $(\varphi_\omega, \psi_\omega)$ be the solution to the system* (2)*, which is obtained in Theorem* 2(i)*. Then we have*

$$\text{(i) } \partial_\omega \|\varphi_\omega\|_{L^2}^2|_{\omega=1} < 0 \quad \text{if } N = 4 \text{ or } 5, \qquad \text{(ii) } \partial_\omega \|\varphi_\omega\|_{L^2}^2|_{\omega=1} > 0 \quad \text{if } 1 \leqslant N \leqslant 3.$$

Before proving Theorem 10, we prepare the following lemma.

Lemma 11. *Let w_1 be the unique positive solution to Eq.* (3)*. Then we have $T^{-1}w_1 = -w_1 - x \cdot \nabla w_1/2$, where $T = -\Delta + 1 - 2w_1$.*

Lemma 11 is already known (see e.g. Weinstein [22, Proposition B.1]). However, for the sake of the completeness, we give the proof.

Proof of Lemma 11. We put $w_\lambda(\cdot) = \lambda w_1(\sqrt{\lambda}\cdot)$ for $\lambda > 0$. Then we see that $w_\lambda \in H^1(\mathbb{R}^N)$ satisfies

$$-\Delta w_\lambda + \lambda w_\lambda - 2w_\lambda^2 = 0, \quad x \in \mathbb{R}^N.$$

Differentiating the above equation with respect to $\lambda > 0$ and substituting with $\lambda = 1$, we have $T(d/d\lambda)w_\lambda|_{\lambda=1} = -w_1$. Thus, we obtain $T^{-1}w_1 = -(d/d\lambda)w_\lambda|_{\lambda=1} = -w_1 - x \cdot \nabla w_1/2$. □

We now give the proof of Theorem 10(i).

Proof of Theorem 10(i). Differentiating the system (2) with respect to $\omega > 0$, we have

$$\begin{cases} -\Delta \partial_\omega \varphi_\omega + \omega \partial_\omega \varphi_\omega - 2\partial_\omega \varphi_\omega \psi_\omega - 2\varphi_\omega \partial_\omega \psi_\omega = -\varphi_\omega, & x \in \mathbb{R}^N, \\ -\Delta \partial_\omega \psi_\omega + \partial_\omega \psi_\omega - 2\varphi_\omega \partial_\omega \varphi_\omega = 0, & x \in \mathbb{R}^N. \end{cases}$$

We can rewrite the above system as follows:

$$\begin{pmatrix} -\Delta + \omega - 2\psi_\omega & -2\varphi_\omega \\ -2\varphi_\omega & -\Delta + 1 \end{pmatrix} \begin{pmatrix} \partial_\omega \varphi_\omega \\ \partial_\omega \psi_\omega \end{pmatrix} = \begin{pmatrix} -\varphi_\omega \\ 0 \end{pmatrix}.$$

When $\omega = 1$, we see that

$$\begin{pmatrix} -\Delta + 1 - 2w_1 & -2\sqrt{2}w_1 \\ -2\sqrt{2}w_1 & -\Delta + 1 \end{pmatrix} \begin{pmatrix} \partial_\omega \varphi_1 \\ \partial_\omega \psi_1 \end{pmatrix} = \begin{pmatrix} -\sqrt{2}w_1 \\ 0 \end{pmatrix}.$$

Thus, it follows from Lemma 5 that

$$\begin{pmatrix} \partial_\omega \varphi_1 \\ \partial_\omega \psi_1 \end{pmatrix} = -A \begin{pmatrix} T^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix} A \begin{pmatrix} \sqrt{2}w_1 \\ 0 \end{pmatrix}.$$

Therefore, we obtain

$$\frac{1}{2}\partial_\omega \|\varphi_\omega\|_{L^2}^2|_{\omega=1} = \sqrt{2} \int_{\mathbb{R}^N} \partial_\omega \varphi_1 w_1 \, dx$$

$$= \left\langle \begin{pmatrix} \partial_\omega \varphi_1 \\ \partial_\omega \psi_1 \end{pmatrix}, \begin{pmatrix} \sqrt{2} w_1 \\ 0 \end{pmatrix} \right\rangle$$
$$= -\left\langle A \begin{pmatrix} T^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix} A \begin{pmatrix} \sqrt{2} w_1 \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{2} w_1 \\ 0 \end{pmatrix} \right\rangle$$
$$= -\frac{4}{3}\langle T^{-1} w_1, w_1 \rangle - \frac{2}{3}\langle S^{-1} w_1, w_1 \rangle.$$

From Lemma 11, we obtain

$$-\frac{4}{3}\langle T^{-1} w_1, w_1 \rangle = \frac{4}{3}\left\langle w_1 + \frac{1}{2} x \cdot \nabla w_1, w_1 \right\rangle = \frac{4-N}{3}\|w_1\|_{L^2}^2.$$

This yields that

$$\frac{1}{2}\partial_\omega \|\varphi_\omega\|_{L^2}^2|_{\omega=1} = \frac{4-N}{3}\|w_1\|_{L^2}^2 - \frac{2}{3}\langle S^{-1} w_1, w_1 \rangle. \tag{8}$$

It follows from the positivity of the operator S that $\langle S^{-1} w_1, w_1 \rangle > 0$. Therefore, we infer that $\partial_\omega \|\varphi_\omega\|_{L^2}^2|_{\omega=1} < 0$ if $N = 4$ or 5. □

Next, we prove Theorem 10(ii). We set $f_1 = S^{-1} w_1$. Then the function $f_1 \in H^1(\mathbb{R}^N, \mathbb{R})$ satisfies

$$-\Delta f + f + 2 w_1 f = w_1. \tag{9}$$

Since $S : H^1_{\mathrm{rad}}(\mathbb{R}^N, \mathbb{R}) \to H^1_{\mathrm{rad}}(\mathbb{R}^N, \mathbb{R})$ is bijection and $w_1 \in H^1_{\mathrm{rad}}(\mathbb{R}^N, \mathbb{R})$, we see that the function f_1 is radially symmetric. We can also find that $f_1 \in C^2(\mathbb{R}^N, \mathbb{R})$ by a standard elliptic regularity argument (see e.g. Gilbarg and Trudinger [9, Chapter 8]). Moreover, we have the following lemma.

Lemma 12. *The following properties hold*:

(i) $1/2 \geqslant f_1(x) \geqslant w_1(x)/4w_1(0)$ *for all* $x \in \mathbb{R}^N$,
(ii) *the function* $f_1(x) = f_1(|x|)$ *decreases in* $|x|$.

Proof. (i) Using the fact that $w_1(0) = \max_{x \in \mathbb{R}^N} w_1(x)$ (see Gidas, Ni and Nirenberg [8]), we have

$$\big(-\Delta + 1 + 2w_1(x)\big)\left(f_1(x) - \frac{w_1(x)}{4w_1(0)}\right) = w_1(x) - \frac{w_1^2(x)}{w_1(0)} \geqslant w_1(x) - w_1(x) = 0.$$

It follows from the strong maximum principle that $f_1(x) \geqslant w_1(x)/4w_1(0)$ for all $x \in \mathbb{R}^N$.

Let $x_0 \in \mathbb{R}^N$ be such that $f_1(x_0) = \max_{x \in \mathbb{R}^N} f_1(x)$. Since the function f_1 is positive and $-\Delta f_1(x)|_{x=x_0} \geqslant 0$, we infer that $2w_1(x_0) f_1(x_0) \leqslant w_1(x_0)$, which implies that $f_1(x_0) \leqslant 1/2$.

(ii) We show this by contradiction. Suppose that the function f_1 is not non-increasing in $|x|$. Then there exist local minimum $r_1 \geqslant 0$ and maximum $r_2 > 0$ with $r_2 > r_1$.

We set $x_m = (r_1, 0, \dots, 0)$, $x_M = (r_2, 0, \dots, 0)$ and $d = f_1(x_M) - f_1(x_m) (> 0)$. Since the function f_1 is smooth, there exists $r_0 > 0$ such that

$$f_1(x + x_m) \leqslant f_1(x_m) + d/3, \qquad f_1(x + x_M) \geqslant f_1(x_M) - d/3$$

for all $x \in B(0, r_0)$. Therefore, for $x \in B(0, r_0)$, we have

$$\begin{aligned}
&(-\Delta + 1)\big(f_1(x + x_m) - f_1(x + x_M)\big) \\
&\quad = -2w_1(x + x_m) f_1(x + x_m) + w_1(x + x_m) + 2w_1(x + x_M) f_1(x + x_M) - w_1(x + x_M) \\
&\quad = w_1(x + x_m) - w_1(x + x_M) - 2 f_1(x + x_m)\big(w_1(x + x_m) - w_1(x + x_M)\big) \\
&\qquad + 2w_1(x + x_M)\big(f_1(x + x_M) - f_1(x + x_m)\big).
\end{aligned}$$

We take $r_0 > 0$ sufficiently small so that $|x + x_m| < |x + x_M|$ for all $x \in B(0, r_0)$. Then since the function w_1 decreases in $|x|$, we have

$$\begin{aligned}&(-\Delta + 1)\big(f_1(x + x_m) - f_1(x + x_M)\big)\\&\quad \geqslant w_1(x + x_m) - w_1(x + x_M) - 2 \times \frac{1}{2}\big(w_1(x + x_m) - w_1(x + x_M)\big) + 2w_1(x + x_M) \times \frac{d}{3}\\&\quad = \frac{2d}{3} w_1(x + x_M) > 0.\end{aligned} \tag{10}$$

We have used the fact that $\|f_1\|_{L^\infty} \leqslant 1/2$.

Set $g_1(\cdot) = f_1(\cdot + x_m) - f_1(\cdot + x_M)$. Then it follows from (10) that $\Delta g_1(0) \leqslant g_1(0) = -d$. On the other hand, since the function $g_1(\cdot)$ attains a local minimum at $x = 0$, we have $\Delta g_1(0) \geqslant 0$, which is a contradiction. Thus, we see that the function f_1 is a non-increasing function in $|x|$.

Suppose that there exists an interval $I\,(\subset [0, \infty))$ such that $f_1(r)$ is constant for all $r \in I$. Then from (9), we have $f_1(r) = w_1/(1 + 2w_1)$ for all $r \in I$, which is absurd because the function $w_1/(1 + 2w_1)$ is a non-constant. This completes the proof. □

Lemma 13. *Let* $1 \leqslant N \leqslant 5$. *We have*

$$\text{(i)} \int_{\mathbb{R}^N} w_1^2\,dx = 4 \int_{\mathbb{R}^N} w_1^2 f_1\,dx, \qquad \text{(ii)} \frac{1}{2}\int_{\mathbb{R}^N} w_1^2\,dx > \int_{\mathbb{R}^N} w_1 f_1\,dx.$$

Proof. (i) Multiplying Eq. (3) by f_1 and integrating the resulting equation, we have

$$\int_{\mathbb{R}^N} \nabla w_1 \cdot \nabla f_1\,dx + \int_{\mathbb{R}^N} w_1 f_1\,dx - 2\int_{\mathbb{R}^N} w_1^2 f_1\,dx = 0. \tag{11}$$

Similarly, multiplying Eq. (9) by w_1 and integrating the resulting equation, we obtain

$$\int_{\mathbb{R}^N} \nabla w_1 \cdot \nabla f_1\,dx + \int_{\mathbb{R}^N} w_1 f_1\,dx + 2\int_{\mathbb{R}^N} w_1^2 f_1\,dx = \int_{\mathbb{R}^N} w_1^2\,dx. \tag{12}$$

Subtracting (11) from (12), we have $\int_{\mathbb{R}^N} w_1^2\,dx = 4\int_{\mathbb{R}^N} w_1^2 f_1\,dx$.

(ii) It is known that w_1 is radially symmetric and $\partial_r w_1(r) < 0$ for all $r > 0$ (see Gidas, Ni and Nirenberg [8]). Moreover, it follows from Lemma 12(ii) that $\partial_r f_1 < 0$ for all $r > 0$. Therefore, we see that

$$\int_{\mathbb{R}^N} \nabla w_1 \cdot \nabla f_1\,dx = C_N \int_0^\infty \partial_r w_1(r)\partial_r f_1(r) r^{N-1}\,dr > 0,$$

which yields that $\int_{\mathbb{R}^N} w_1 f_1\,dx < 2\int_{\mathbb{R}^N} w_1^2 f_1\,dx = \int_{\mathbb{R}^N} w_1^2\,dx/2$ from (11) and Lemma 13(i). This completes the proof. □

We are now in position to prove Theorem 10(ii).

Proof of Theorem 10(ii). It follows from Lemma 13(ii) that

$$\frac{2}{3}\langle S^{-1} w_1, w_1\rangle = \frac{2}{3}\int_{\mathbb{R}^N} w_1 f_1\,dx < \frac{2}{3} \times \frac{1}{2}\int_{\mathbb{R}^N} w_1^2\,dx = \frac{1}{3}\|w_1\|_{L^2}^2.$$

Therefore, from (8), we have

$$\frac{1}{2}\partial_\omega \|\varphi_\omega\|_{L^2}^2|_{\omega=1} > \frac{3 - N}{3}\|w_1\|_{L^2}^2.$$

Thus, we obtain the desired result. □

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