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Pulsating fronts for nonlocal dispersion and KPP nonlinearity

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Abstract

In this paper we are interested in propagation phenomena for nonlocal reaction–diffusion equations of the type:

$$\frac{\partial u}{\partial t} = J * u - u + f(x,u) \quad t \in \mathbb{R},\ x \in \mathbb{R}^N,$$

where J is a probability density and f is a KPP nonlinearity periodic in the x variables. Under suitable assumptions we establish the existence of pulsating fronts describing the invasion of the 0 state by a heterogeneous state. We also give a variational characterization of the minimal speed of such pulsating fronts and exponential bounds on the asymptotic behavior of the solution.

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Keywords: Periodic front; Nonlocal dispersal; KPP nonlinearity

1. Introduction

In this paper we are interested in propagation phenomena for nonlocal reaction–diffusion equations of the type:

$$\frac{\partial u}{\partial t} = J * u - u + f(x,u) \quad t \in \mathbb{R},\ x \in \mathbb{R}^N, \tag{1.1}$$

where J is a probability density and f is a nonlinearity which is KPP in u and periodic in the x variables, that is,

$$f(x,u) = f(x+k,u) \quad \forall x \in \mathbb{R}^N,\ k \in \mathbb{Z}^N,\ u \in \mathbb{R}.$$

More precisely, we are interested in the existence/nonexistence and the characterization of front type solutions called pulsating fronts. A pulsating front connecting 2 stationary periodic solutions p_0, p_1 of (1.1) is an entire solution that has the form $u(x,t) := \psi(x \cdot e + ct, x)$ where e is a unit vector in $\mathbb{R}^N$, $c \in \mathbb{R}$, and $\psi(s,x)$ is periodic in the x variable, and such that

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$$\lim_{s\to-\infty}\psi(s,x)=p_0(x)\quad\text{uniformly in } x,$$
$$\lim_{s\to+\infty}\psi(s,x)=p_1(x)\quad\text{uniformly in } x.$$

The real number c is called the effective speed of the pulsating front.

Using an equivalent definition, pulsating fronts were first defined and used by Shigesada, Kawasaki and Teramoto [58,59] in their study of biological invasions in a heterogeneous environment modeled by the following reaction–diffusion equation

$$\frac{\partial u}{\partial t}=\nabla\cdot\big(A(x)\nabla u\big)+f(x,u)\quad\text{in } \mathbb{R}^+\times\mathbb{R}^N, \tag{1.2}$$

where $A(x)$ and $f(x,u)$ are respectively a periodic smooth elliptic matrix and a smooth periodic function. Using heuristics and numerical simulations, in a one-dimensional situation and for the particular nonlinearity $f(x,u):=u(\eta(x)-\mu u)$, Shigesada, Kawasaki and Teramoto were able to recover earlier results on the minimal speed of spreading obtained by probabilistic methods by Gärtner and Freidlin [34,35].

The above definition of pulsating front has been introduced by Xin [62,63] in his study of flame propagation. This definition is a natural extension of the definition of the sheared traveling fronts studied for example in [10,11]. Within this framework, Xin [62,63] has proved existence and uniqueness up to translation of pulsating fronts for Eq. (1.2) with a homogeneous bistable or ignition nonlinearity. Since then, much attention has been drawn to the study of periodic reaction–diffusion equations and the existence and the uniqueness of pulsating front have been proved in various situations, see for example [5,8,9,38–41,47,61–64]. In particular, Berestycki, Hamel and Roques [8,9] have showed that when $f(x,u)$ is of KPP type, then the existence of a unique nontrivial stationary solution $p(x)$ to (1.2) is governed by the sign of the periodic principal eigenvalue of the following spectral problem

$$\nabla\cdot\big(A(x)\nabla\phi\big)+f_u(x,0)\phi+\lambda_p\phi=0.$$

Furthermore, they have showed that there exists a critical speed c^* so that a pulsating front with speed $c\geqslant c^*$ in the direction e connecting the two equilibria 0 and $p(x)$ exists and no pulsating front with speed $c<c^*$ exists. They also gave a precise characterization of c^* in terms of some periodic principal eigenvalue. Versions of (1.2) with periodicity in time, or more general media are studied in [5–7,48,50–53,55,66]. It is worth noticing that when the matrix A and f are homogeneous, then Eq. (1.2) reduces to a classical reaction–diffusion equation with constant coefficients and the pulsating front (ψ,c) is indeed a traveling front which have been well studied since the pioneering works of Kolmogorov, Petrovsky and Piskunov [44].

Here we are concerned with a nonlocal version of (1.2) where the classical local diffusion operator $\nabla\cdot(A(x)\nabla u)$ is replaced by the integral operator $J*u-u$. The introduction of such type of long range interaction finds its justification in many problems ranging from micro-magnetism [26–28], neural network [31] to ecology [16,19,29,45,49,60]. For example, in some population dynamic models, such long range interaction is used to model the dispersal of individuals through their environment, [32,33,42]. Regarding Eq. (1.1) we quote [1,2,18,20,21,23,25] for the existence and characterization of traveling fronts for this equation with homogeneous nonlinearity and [3,22,24,36,42] for the study of the stationary problem.

In what follows, we assume that $J:\mathbb{R}^N\to\mathbb{R}$ satisfies

$$\begin{cases} J\geqslant 0, \quad \displaystyle\int_{\mathbb{R}^N} J=1, \quad J(0)>0,\\ J \text{ is smooth, symmetric with support contained in the unit ball,}\end{cases} \tag{1.3}$$

and that $f:\mathbb{R}^N\times[0,\infty)\to\mathbb{R}$ is $[0,1]^N$-periodic in x and satisfies

$$\begin{cases} f\in C^3\big(\mathbb{R}^N\times[0,\infty)\big),\\ f(\cdot,0)\equiv 0,\\ f(x,u)/u \text{ is decreasing with respect to } u \text{ on } (0,+\infty),\\ \text{there exists } M>0 \text{ such that } f(x,u)\leqslant 0 \text{ for all } u\geqslant M \text{ and all } x.\end{cases} \tag{1.4}$$

The model example is

$$f(x,u)=u\big(a(x)-u\big)$$

where $a(x)$ is a periodic, C^3 function.

Before constructing pulsating fronts, we discuss the existence of solutions of the stationary equation

$$J*u-u+f(x,u)=0 \quad x\in\mathbb{R}^N. \tag{1.5}$$

Under the assumption (1.4), 0 is a solution of (1.5) and, as shown in [22], the existence of a positive periodic stationary solution $p(x)$ is characterized by the sign of a *generalized principal eigenvalue* of the linearization of (1.5) around 0, defined by

$$\mu_0=\sup\big\{\mu\in\mathbb{R}\;\big|\;\exists\phi\in C_{per}\big(\mathbb{R}^N\big),\ \phi>0,\ \text{such that } J*\phi-\phi+f_u(x,0)\phi+\mu\phi\leqslant 0\big\} \tag{1.6}$$

where $C_{per}(\mathbb{R}^N)$ is the space of continuous periodic functions in $\mathbb{R}^N$.

More precisely, we have

Theorem 1.1. *The stationary equation* (1.5) *has a positive continuous periodic solution $p(x)$ if and only if $\mu_0<0$. Moreover the positive solution is Lipschitz and unique in the class of positive bounded periodic functions.*

This result is analogous to the characterization of stationary positive solutions of the differential equation (1.2) with f of type KPP in u. The main difference is that μ_0 is not always an eigenvalue, that is, the supremum in (1.6) is not always achieved. Similar results for (1.5), but assuming that μ_0 is an eigenvalue and for the one-dimensional case (i.e. $N=1$), have been obtained in [3,24]. In this particular situation, the uniqueness of the positive solution of (1.5) in the class of bounded measurable functions has been proved in [24]. For the multidimensional case, the existence and uniqueness of a stationary solution in the class of periodic functions has been obtained by Shen and Zhang [56] assuming that μ_0 is eigenvalue and by Coville [22] without this assumption. The difference of Theorem 1.1 and [22] is that we obtain a Lipschitz continuous solution.

The question whether μ_0 is really a principal eigenvalue, that is, if there exists $\phi\in C_{per}(\mathbb{R}^N)$, $\phi>0$ such that

$$J*\phi-\phi+f_u(x,0)\phi+\mu_0\phi=0 \quad \text{in } \mathbb{R}^N \tag{1.7}$$

has been studied in [22,56] where simple criteria on $f_u(x,0)$ have been derived to ensure the existence of a principal eigenfunction ϕ. For instance, the following criterion proposed in [22]

$$\int\limits_{[0,1]^N}\frac{1}{A-f_u(x,0)}\,dx=+\infty, \qquad \text{where } A=\max_{x\in\mathbb{R}^N} f_u(x,0),$$

guarantees that μ_0 is a principal eigenvalue. Some properties of μ_0 and the existence criteria will be discussed in Section 3.

Our main result on pulsating fronts is the following:

Theorem 1.2. *Assume $\mu_0<0$ and that there exists $\phi\in C_{per}(\mathbb{R}^N)$, $\phi>0$ satisfying* (1.7)*. Then, given any unit vector $e\in\mathbb{R}^N$ there is a number $c_e^*>0$ such that for $c\geqslant c_e^*$* (1.1) *has a pulsating front solution $u(x,t)=\psi(x\cdot e+ct,x)$ with effective speed c, and for $c<c_e^*$ there is no such solution.*

The minimal speed c_e^* is given by

$$c_e^*:=\inf_{\lambda>0}\left(\frac{-\mu_\lambda}{\lambda}\right) \tag{1.8}$$

where μ_λ is the periodic principal eigenvalue of the following problem

$$J_\lambda*\phi-\phi+f_u(x,0)\phi+\mu\phi=0 \quad \text{in } \mathbb{R}^N \tag{1.9}$$

with $J_\lambda(x):=J(x)e^{\lambda x\cdot e}$. We will see in Section 3 that this eigenvalue problem is solvable under the assumptions of Theorem 1.2.

Shen and Zhang showed in [56] that c_e^* corresponds to the speed of spreading for this equation in the following sense. For reasonable initial conditions, the solution of (1.1) satisfies

$$\limsup_{t\to+\infty} \sup_{x\cdot e+ct\leqslant 0} u(x,t)=0 \quad \text{if } c>c_e^*,$$

while

$$\liminf_{t\to+\infty} \inf_{x\cdot e+ct\geqslant 0} \big(u(x,t)-p(x)\big)=0 \quad \text{if } c<c_e^*.$$

The nonexistence statement in Theorem 1.2 is a consequence of the these spreading speed results. Along our analysis, we also obtain some asymptotic behavior of $\psi(s,x)$ as $s\to\pm\infty$ where ψ is the pulsating front constructed in Theorem 1.2. More precisely, let $\lambda(c)$ denote the smallest positive λ such that $c=\frac{-\mu_\lambda}{\lambda}$.

Theorem 1.3. *Assume $\mu_0<0$ and that there exists $\phi\in C_{per}(\mathbb{R}^N)$, $\phi>0$ satisfying (1.7). Then, given any unit vector $e\in\mathbb{R}^N$ and $c\geqslant c_e^*$ we have*:

a) *For any positive λ so that $\lambda<\lambda(c)$ there exists $C>0$ such that*

$$\psi(s,x)\leqslant Ce^{\lambda s} \quad \forall x\in\mathbb{R}^N,\ \forall s\in\mathbb{R}.$$

b) *There are $\sigma, C>0$ such that*

$$0\leqslant p(x)-\psi(s,x)\leqslant Ce^{-\sigma s} \quad \forall x\in\mathbb{R}^N,\ \forall s\geqslant 0.$$

Eq. (1.1) can be related to a class of problems studied by Weinberger in [61]. However, as observed in [23,56], one of the main difficulties in dealing with the nonlocal equation (1.1) comes from the lack of regularizing effect of (1.1), which makes the framework developed by Weinberger not applicable, since the compactness assumption required in [61] does not hold.

Another difficulty in the construction of pulsating fronts is that the equation satisfied by the function ψ (see (2.1) below) involves an integral operator in time and space, which is in some sense degenerate. This difficulty also appears in the classical reaction–diffusion case, and it becomes delicate to proceed using the standard approaches used in [10, 11,44].

Finally, we comment on some of the hypotheses made in the construction. Regarding smoothness of the data, one can deal with less regularity of J and f, but some arguments would have to be modified. The hypothesis on the support of J in (1.3) can be weakened. For example, we believe that the same results are true assuming that J satisfies the so-called Mollison condition:

$$\forall\lambda>0, \quad \int_{\mathbb{R}^N} J(z)e^{\lambda|z|}\,dz<+\infty.$$

Finally, the hypothesis that μ_0 is an eigenvalue seems crucial in our approach. It is an interesting open problem to understand whether some type of pulsating front exists in the case where μ_0 is not an eigenvalue. We believe that if such solutions exist, they will be qualitatively different from the ones constructed in Theorem 1.2. See also Remark 3.11 for other observations on this hypothesis.

In the preparation of this work, we were informed of a very recent work of Shen and Zhang [57] done independently dealing with the existence and properties of pulsating front for a nonlocal equation like (1.1). The construction of pulsating front proposed by Shen and Zhang relies on a completely different method and another definition of pulsating front. With their method, they are able to construct bounded measurable pulsating fronts for any speed $c>c_e^*$ but fail to construct pulsating front for the critical speeds c_e^* due to the lack of good Lipschitz regularity estimates on the fronts. Some additional properties, such as exact exponential behavior as $t\to-\infty$, uniqueness of the profile in an appropriate class and some kind of stability of the front are also studied in this work. The main differences between the results obtained by Shen and Zhang and ours concern essentially the regularity of the fronts. Whereas they obtained bounded measurable front, we obtained uniform Lipschitz front which is a significant part of our work. We also have the feeling that our approach is more robust, in the sense that it does not strongly rely on the KPP structure and can be adapted to other situations such as a monostable or ignition nonlinearity which seems not be the case for the method used in [57]. We have in mind a problem like

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} J\left(\frac{x-y}{g(x)g(y)}\right)\big[u(y)-u(x)\big]\,dy + f(u) \quad t \in \mathbb{R},\ x \in \mathbb{R}^N,$$

where f is monostable nonlinearity, J a smooth probability density and g a continuous positive periodic function. It is worth noticing that in [57], the existence of a principal eigenvalue for (1.7) is also a crucial hypothesis.

2. Scheme of the construction

The proof of Theorem 1.1 is contained in Section 5, and follows by now standard arguments.

To construct a pulsating front solution u of (1.1) in the direction $-e$ with effective speed c connecting 0 and a positive periodic stationary solution p, we let $\psi(s,x) = u(\frac{s-x\cdot e}{c}, x)$. Then we need to find ψ satisfying

$$\begin{cases} c\psi_s = M[\psi] - \psi + f(x,\psi) \quad \forall s \in \mathbb{R},\ x \in \mathbb{R}^N, \\ \psi(s, x+k) = \psi(s,x) \quad \forall s \in \mathbb{R},\ x \in \mathbb{R}^N,\ k \in \mathbb{Z}^N, \\ \lim\limits_{s\to-\infty} \psi(s,x) = 0 \quad \text{uniformly in } x, \\ \lim\limits_{s\to\infty} \psi(s,x) = p(x) \quad \text{uniformly in } x, \end{cases} \tag{2.1}$$

where M is the integral operator

$$M[\psi](s,x) = \int_{\mathbb{R}^N} J(x-y)\psi\big(s+(y-x)\cdot e, y\big)\,dy.$$

To analyze (2.1) we introduce a regularized problem, namely, we consider for $\varepsilon > 0$

$$c\psi_s = M[\psi] - \psi + f(x,\psi) + \varepsilon\Delta\psi \quad \forall s \in \mathbb{R},\ x \in \mathbb{R}^N \tag{2.2}$$

where Δ is the Laplacian with respect to the x variables. The stationary version of this equation is a perturbation of (1.5):

$$0 = J * u - u + f(x,u) + \varepsilon\Delta u, \quad x \in \mathbb{R}^N. \tag{2.3}$$

We will see in Section 5 that under the assumption that (1.5) has a positive periodic continuous solution p, for small $\varepsilon > 0$ Eq. (2.3) also has a stationary positive solution p_ε and $p_\varepsilon \to p$ uniformly as $\varepsilon \to 0$.

As a step to prove Theorem 1.2, for small $\varepsilon > 0$ we will find $c_e^*(\varepsilon)$ such that for $c \geqslant c_e^*(\varepsilon)$ there exists a solution ψ_ε to (2.2) satisfying

$$\begin{cases} \lim\limits_{s\to-\infty} \psi(s,x) = 0, \\ \lim\limits_{s\to+\infty} \psi(s,x) = p_\varepsilon(x), \\ \psi(s,x) \text{ is increasing in } s \text{ and periodic in } x. \end{cases} \tag{2.4}$$

This is done in Section 6, following in part the methods developed in [9].

A substantial part of this article is devoted to obtain estimates for ψ_ε that will allow us to prove that $\psi = \lim_{\varepsilon\to0}\psi_\varepsilon$ exists and solves (2.1). These estimates are based on the expected exponential decay of ψ as $s \to -\infty$, which we discuss next. Suppose ψ is a solution of (2.1). One may expect that for some $\lambda > 0$

$$\psi(s,x) = e^{\lambda s}w(x) + o\big(e^{\lambda s}\big) \quad \text{as } s \to -\infty,\ x \in \mathbb{R}^N$$

where w is a positive periodic function, at least when $c > c_e^*$. Then at main order the equation in (2.1) yields

$$c\lambda w = \int_{\mathbb{R}^N} J(x-y)e^{\lambda(y-x)\cdot e}w(y)\,dy - w + f_u(x,0)w \quad \text{in } \mathbb{R}^N. \tag{2.5}$$

Define

$$J_\lambda(x) = J(x)e^{-\lambda x\cdot e},$$

then (2.5) can be written as the periodic eigenvalue problem

$$\begin{cases} J_\lambda * w - w + f_u(x,0)w + \mu_\lambda w = 0 & \text{in } \mathbb{R}^N, \\ w > 0 \text{ is continuous and periodic,} \end{cases} \tag{2.6}$$

which will be studied in Section 3. In particular, under the assumptions of Theorem 1.2, we will see that it has a principal eigenvalue μ_λ in the space of continuous periodic functions. Then the speed of the traveling front should be given by $c = -\frac{\mu_\lambda}{\lambda}$, and this leads to the formula for the minimal speed (1.8).

For the solutions of (2.2) and (2.4) one can guess a similar asymptotic behavior as $s \to -\infty$ and a formula for the minimal speed

$$c_e^*(\varepsilon) = \min_{\lambda > 0}\left(-\frac{\mu_{\varepsilon,\lambda}}{\lambda}\right) \tag{2.7}$$

where $\mu_{\varepsilon,\lambda}$ is the principal eigenvalue of $-L_{\varepsilon,\lambda}$ where

$$L_{\varepsilon,\lambda} w = \varepsilon \Delta w + J_\lambda w - w + f_u(x,0)w$$

in the space of C^2 periodic functions.

Based on the estimates developed in Section 7 for the operator $L_{\varepsilon,\lambda} u$, we prove in Section 8 exponential bounds of the form: for $0 < \lambda < \lambda_\varepsilon(c)$

$$\psi_\varepsilon(s,x) \leqslant C e^{\lambda s} \quad \forall x \in \mathbb{R}^N,\ \forall s \in \mathbb{R} \tag{2.8}$$

where $\lambda_\varepsilon(c)$ is the smallest positive λ such that $c = -\frac{\mu_{\varepsilon,\lambda}}{\lambda}$, and C does not depend on $\varepsilon > 0$. This exponential bound is obtained by studying the two sided Laplace transform of ψ_ε, an idea present in [17].

The exponential estimate (2.8) allows us in Section 9 to obtain uniform control of local Sobolev norms $\|\psi_\varepsilon\|_{W^{1,p}}$ with $p > N$, which in turn implies that we obtain a locally uniform limit $\psi = \lim_{\varepsilon \to 0} \psi_\varepsilon$ for some subsequence. The final step is to verify that ψ satisfies all the requirements in (2.1).

3. Principal eigenvalue for nonlocal operators

Let us recall the notation

$$C_{per}(\mathbb{R}^N) = \{\phi \in C(\mathbb{R}^N) \mid \phi \text{ is } [0,1]^N\text{-periodic}\}.$$

For the rest of the article it is crucial to understand the eigenvalue problem (2.6), and the purpose of this section is to study its properties. We will write (2.6) in the form

$$\begin{cases} L_\lambda \phi + \mu \phi = 0 & \text{in } \mathbb{R}^N, \\ \phi \in C_{per}(\mathbb{R}^N), & \phi > 0 \end{cases} \tag{3.1}$$

where

$$L_\lambda w = J_\lambda * w + a(x)w$$

and $a(x) = f_u(x,0) - 1 \in C_{per}(\mathbb{R}^N)$.

We say that L_λ has a principal eigenfunction if for some $\mu \in \mathbb{R}$ there is a solution in $C_{per}(\mathbb{R}^N)$ of (3.1).

As we will see later, it is not true in general that L_λ has a principal eigenfunction, but it is convenient to define in all cases

$$\mu_\lambda = \sup\{\mu \in \mathbb{R} \mid \exists \phi \in C_{per}(\mathbb{R}^N),\ \phi > 0,\ \text{such that } L_\lambda \phi + \mu \phi \leqslant 0\} \tag{3.2}$$

and call it the generalized principal eigenvalue of $-L_\lambda$. The name is motivated by the following result.

Proposition 3.1. *Let $\lambda \in \mathbb{R}$. If there is $\mu \in \mathbb{R}$, $\phi \in C_{per}(\mathbb{R}^N)$, $\phi \geqslant 0$ and nontrivial satisfying $L_\lambda \phi + \mu \phi = 0$, then μ is given by (3.2) and it is simple eigenvalue of L_λ.*

The proof of this is a direct adaptation of Lemma 3.2 in [22].

The next proposition characterizes the existence of a principal eigenfunction.

Proposition 3.2. *If $a \in C_{per}(\mathbb{R}^N)$, then $\max a(x) + \mu_\lambda \leqslant 0$. Moreover, $\max a(x) + \mu_\lambda < 0$ if and only if L_λ admits a principal eigenfunction.*

For the proof of the above result and the following two (Proposition 3.3 and Corollary 3.4) see later in this section.

Proposition 3.3. *The function $-\mu_\lambda$ is convex in $\mathbb{R}$ and even. In particular, $-\mu_\lambda$ is nondecreasing in $[0,\infty)$ and nonincreasing in $(-\infty, 0]$.*

Corollary 3.4. *If L_0 has a principal eigenfunction then for all $\lambda \in \mathbb{R}$, L_λ has a principal eigenfunction.*

In general it is difficult to describe precisely in terms of J and a whether L_λ has a principal eigenfunction, but we have sufficient and necessary conditions.

Proposition 3.5. *Assume $a \in C_{per}(\mathbb{R}^N)$ and let $A := \max_{\mathbb{R}^N} a(x)$. There are constants $C_1, C_2, m > 0$ that depend on J_λ such that:*

a) *if*

$$\int_{[0,1]^N} \frac{1}{A - a(x)}\,dx \geqslant C_1 \|a\|^m_{L^\infty}, \tag{3.3}$$

then L_λ admits a principal eigenfunction,

b) *if*

$$\int_{[0,1]^N} \frac{1}{A - a(x)}\,dx \leqslant C_2,$$

then L_λ has no principal eigenfunction.

We give the proof of this proposition later on inside this section.

Finally, we need the next proposition to show that the formula (1.8) is well defined and gives a positive number.

Proposition 3.6. *The function $\lambda \mapsto \mu_\lambda$ is continuous and for all $\varepsilon > 0$ there exists $\sigma > 0$ such that*

$$-\mu_\lambda \geqslant -\mu_0 - \varepsilon + \sigma e^{\sigma|\lambda|} \quad \forall \lambda \in \mathbb{R}.$$

The above proposition is proved later on inside this section.

Remark 3.7. Many of the previous results have appeared in similar contexts, or have been proved under slightly different conditions. Existence of a principal eigenfunction was obtained for symmetric nonlocal operators in [42], and later also in [3,22,24,56]. A condition like (3.3) is always explicitly or implicitly assumed in these works. The motivation for definition (3.2) is taken from [12]. It has been adapted to many elliptic operators, and was first introduced for nonlocal operators in [22]. In this work the author obtained many of the results described here for an integral operator on a domain in $\mathbb{R}^N$. A characterization like Proposition 3.2 for μ_λ was first obtained in [22]. The convexity of $-\mu_\lambda$, Proposition 3.3, is proved in [56] under the assumption that a principal eigenfunction exists. Examples of nonlocal operators with no principal eigenvalue are also presented in [22,56].

The rest of this section is devoted to prove Propositions 3.2, 3.3, Corollary 3.4, and Propositions 3.5 and 3.6. We start with some basic facts about the definition (3.2). The following results are simple adaptations from results found in [22].

Proposition 3.8. *(Proposition 1.1 [22].) Given $a \in C_{per}(\mathbb{R}^N)$, and $J : \mathbb{R}^N \to \mathbb{R}$, $J \geqslant 0$ in $L^1(\mathbb{R}^N)$ define*

$$\mu_p(J,a) = \sup\{\mu \in \mathbb{R} \mid \exists \phi \in C_{per}(\mathbb{R}^N),\ \phi > 0,\ \text{such that } J * \phi + a\phi + \mu\phi \leqslant 0\}.$$

Then the following hold:

(i) *If $a_1 \geqslant a_2$, then*
$$\mu_p(J,a_2) \geqslant \mu_p(J,a_1).$$
(ii) *If $J_1 \geqslant J_2$ then*
$$\mu_p(J_2,a) \geqslant \mu_p(J_1,a).$$
(iii) *$\mu_p(J,a)$ is Lipschitz in a, more precisely*
$$\left|\mu_p(J,a_1) - \mu_p(J,a_2)\right| \leqslant \|a_1 - a_2\|_\infty.$$

To prove Proposition 3.5 we will need a generalization of the Krein–Rutman theorem [46] for positive not necessarily compact operators due to Edmunds, Potter and Stuart [30]. For this we recall some definitions. A cone in a real Banach space X is a nonempty closed set K such that for all $x, y \in K$ and all $\alpha \geqslant 0$ one has $x + \alpha y \in K$, and if $x \in K$, $-x \in K$ then $x = 0$. A cone K is called reproducing if $X = K - K$. A cone K induces a partial ordering in X by the relation $x \leqslant y$ if and only if $x - y \in K$. A linear map or operator $T : X \to X$ is called positive if $T(K) \subseteq K$.

If $T : X \to X$ is a bounded linear map on a complex Banach space X, its essential spectrum (according to Browder [15]) consists of those λ in the spectrum of T such that at least one of the following conditions holds: (1) the range of $\lambda I - T$ is not closed, (2) λ is a limit point of the spectrum of T, (3) $\bigcup_{n=1}^{\infty} ker(\lambda I - T)^n$ is infinite dimensional. The radius of the essential spectrum of T, denoted by $r_e(T)$, is the largest value of $|\lambda|$ with λ in the essential spectrum of T. For more properties of $r_e(T)$ see [54].

Theorem 3.9. *(See Edmunds, Potter, Stuart [30].) Let K be a reproducing cone in a real Banach space X, and let $T \in \mathcal{L}(X)$ be a positive operator such that $T^m(u) \geqslant cu$ for some $u \in K$ with $\|u\| = 1$, some positive integer m and some positive number c. If $c^{1/m} > r_e(T)$, then T has an eigenvector $v \in K$ with associated eigenvalue $\rho \geqslant c^{1/m}$ and T^* has an eigenvector $v^* \in K^*$ corresponding to the eigenvalue ρ.*

If the cone K has nonempty interior and T is strongly positive, i.e. $u \geqslant 0$, $u \neq 0$ implies $Tu \in int(K)$, then ρ is the unique $\lambda \in \mathbb{R}$ for which there exists nontrivial $v \in K$ such that $Tv = \lambda v$ and ρ is simple, see [65].

Proof of Proposition 3.5. a) Write the eigenvalue problem (3.1) in the form

$$J_\lambda * u + b(x)u = \nu u$$

where

$$b(x) = a(x) + k, \qquad \nu = -\mu + k$$

and $k > 0$ is a constant such that $\inf b > 0$. Sometimes we will use the operator notation $J_\lambda[\phi] = J_\lambda * \phi$. We study this eigenvalue problem in the space $C_{per}(\mathbb{R}^N)$ with uniform norm, where the operator J_λ is compact. Let $u \in C_{per}(\mathbb{R}^N)$, $u \geqslant 0$ and $m \in \mathbb{N}$. Since u and b are nonnegative and J_λ is a positive operator, we see that

$$\left(J_\lambda + b(x)\right)^m[u] \geqslant J_\lambda^m[u] + b(x)^m u. \tag{3.4}$$

We observe that there are m and $d > 0$ depending on J such that for $u \in C_{per}(\mathbb{R}^N)$, $u \geqslant 0$,

$$J_\lambda^m[u] \geqslant d \int_{[0,1]^N} u.$$

Indeed,

$$J_\lambda^m[u] = J_\lambda^{(m)} * u,$$

where $J_\lambda^{(m)}$ denotes the m-fold convolution $J_\lambda * \cdots * J_\lambda$. Let $B_R(x_0)$ with $R > 0$ be such that $J_\lambda(x) > 0$ for points $x \in B_R(x_0)$. Then $J_\lambda * J_\lambda(x) > 0$ for $x \in B_{2R}(2x_0)$. Iterating this argument we get $J_\lambda^{(m)}(x) > 0$ for $x \in B_{mR}(mx_0)$. We choose now m large so that $B_{mR}(mx_0)$ contains some closed cube Q with vertices in $\mathbb{Z}^N$. Let $d = \inf_{x \in Q} J_\lambda^{(m)}(x) > 0$. Then, for $u \in C_{per}(\mathbb{R}^N)$, $u \geqslant 0$,

$$J_\lambda^m[u](x) = \int_{\mathbb{R}^n} J_\lambda^{(m)}(x-y)u(y)\,dy \geqslant \int_Q J_\lambda^{(m)}(z)u(x-z)\,dz$$
$$\geqslant d \int_Q u(x-z)\,dz = \int_{[0,1]^N} u,$$

since u is $[0,1]^N$-periodic.

Let $\varepsilon > 0$ and define the continuous periodic positive function

$$u_\varepsilon(x) = \frac{1}{\max b^m - b(x)^m + \varepsilon}.$$

We claim that choosing ε and C_1 in (3.3) appropriately there is $\delta > 0$ such that

$$J_\lambda^m u_\varepsilon + b(x)^m u_\varepsilon \geqslant (\max b + \delta)^m u_\varepsilon \quad \text{in } \mathbb{R}^N. \tag{3.5}$$

Indeed, taking C_1 large in (3.3) and then $\varepsilon > 0$ small, we have

$$d \int_{[0,1]^N} \frac{1}{\max b^m - b(x)^m + \varepsilon}\,dx > 1.$$

Then to prove (3.5) it is sufficient to have

$$1 > \frac{(\max b + \delta)^m - b(x)^m}{\max b^m - b(x)^m + \varepsilon} \quad \text{in } \mathbb{R}^N.$$

This last condition holds provided we take δ sufficiently small. Therefore, by (3.4) and (3.5) we have

$$\big(J_\lambda + b(x)\big)^m[u_\varepsilon] \geqslant (\max b + \delta)^m u_\varepsilon.$$

Using the compactness of the operator J_λ, we have $r_e(J_\lambda + b(x)) = \max_{x \in \mathbb{R}^N} b(x)$, and by Theorem 3.9 we obtain the desired conclusion. We observe that the principal eigenvalue is simple since the cone of positive periodic functions has nonempty interior and, for a sufficiently large p, the operator $(J_\lambda + b)^p$ is strongly positive. Any point ν in the spectrum of $(J_\lambda + b)$ with $|\nu| > r_e(J_\lambda + b)$ is isolated, see [15]. In particular the principal eigenvalue is an isolated point in the spectrum.

b) As before, without loss of generality we can assume $a > 0$. Suppose there exists a principal periodic eigenfunction ϕ with eigenvalue μ. Then $\max a(x) + \mu < 0$. Let $\mathcal{C} = [0,1]^N$ and note that

$$J_\lambda * \phi(x) = \int_{\mathbb{R}^N} J(x-y)e^{\lambda(x-y)\cdot e}\phi(y)\,dy = \int_{\mathcal{C}} \sum_{k \in \mathbb{Z}^N} J(x-z-k)e^{\lambda(x-z-k)\cdot e}\phi(z)\,dz$$
$$\leqslant \left(\int_{\mathcal{C}} \phi\right) \sup_{x,z \in \mathcal{C}} \sum_{k \in \mathbb{Z}^N} J(x-z-k)e^{\lambda(x-z-k)\cdot e}.$$

But then

$$\phi(x) \leqslant \frac{1}{-(a(x)+\mu)} \left(\int_{\mathcal{C}} \phi\right) \sup_{x,z \in \mathcal{C}} \sum_{k \in \mathbb{Z}^N} J(x-z-k)e^{\lambda(x-z-k)\cdot e}.$$

Integrating the above inequality we obtain

$$\int_{\mathcal{C}} \phi \leqslant \int_{\mathcal{C}} \frac{1}{-(a(x)+\mu)}\,dx \cdot \int_{\mathcal{C}} \phi \cdot \sup_{x,z \in \mathcal{C}} \sum_{k \in \mathbb{Z}^N} J(x-z-k)e^{\lambda(x-z-k)\cdot e},$$

and hence

$$1 \leqslant \int\limits_{\mathcal{C}} \frac{1}{-(a(x)+\mu)}\,dx \cdot \sup_{x,z\in\mathcal{C}} \sum_{k\in\mathbb{Z}^N} J(x-z-k)e^{\lambda(x-z-k)\cdot e}.$$

Since $\mu \leqslant -\max a(\cdot)$

$$1 \leqslant \int\limits_{\mathcal{C}} \frac{1}{\max a(\cdot)-a(x)}\,dx \cdot \sup_{x,z\in\mathcal{C}} \sum_{k\in\mathbb{Z}^N} J(x-z-k)e^{\lambda(x-z-k)\cdot e}.$$

Let

$$M = \sup_{x,z\in\mathcal{C}} \sum_{k\in\mathbb{Z}^N} J(x-z-k)e^{\lambda(x-z-k)\cdot e}.$$

If

$$M\int\limits_{\mathcal{C}} \frac{1}{\max a(\cdot)-a(x)}\,dx < 1$$

there cannot exist a principal eigenfunction. □

Proof of Proposition 3.2. From the definition we obtain directly $\max a(x) + \mu_\lambda \leqslant 0$ for all $\lambda \in \mathbb{R}$. If there exists a principal eigenfunction $\phi \in C_{per}(\mathbb{R}^N)$, then clearly $\max a(x) + \mu_\lambda < 0$.

Now suppose that $\max a(x) + \mu_\lambda < 0$. We approximate a by functions $a_\varepsilon \in C_{per}(\mathbb{R}^N)$ such that $\max a = \max a_\varepsilon$, $\|a - a_\varepsilon\|_\infty \to 0$ as $\varepsilon \to 0$, and

$$\int\limits_{[0,1]^N} \frac{1}{\max a_\varepsilon - a_\varepsilon(x)}\,dx = +\infty. \tag{3.6}$$

Then, by Proposition 3.5 there exists a positive, periodic ϕ_ε, with $\|\phi_\varepsilon\|_\infty = 1$, such that

$$J_\lambda * \phi_\varepsilon + \big(a_\varepsilon(x) + \mu_\lambda^\varepsilon\big)\phi_\varepsilon = 0 \quad \text{in } \mathbb{R}^N.$$

Since by Proposition 3.8, $\mu_\lambda^\varepsilon \to \mu_\lambda$, there exists $\delta > 0$ such that $a_\varepsilon(x) + \mu_\lambda^\varepsilon < -\delta$ for all x and ε. Therefore, by a simple compactness argument, we have that $\phi_\varepsilon \to \phi$ uniformly as $\varepsilon \to 0$, with ϕ positive satisfying (4.1), which concludes the proof. □

Remark 3.10. If L_λ has a principal eigenfunction $\phi \in C_{per}(\mathbb{R}^N)$, and additionally $a \in C^k$, $k \geqslant 1$ and J is C^k, then ϕ is also C^k, which follows from

$$J_\lambda \phi = (-\mu_\lambda - a)\phi$$

and $-\mu_\lambda - a \geqslant \delta$ for some $\delta > 0$.

Proof of Proposition 3.3. To prove this result, we will first suppose that a satisfies (3.6), and then we proceed by an approximation argument. We will prove the convexity using an idea from [56]. Let $\lambda_1, \lambda_2 \in \mathbb{R}$, and $t \in (0,1)$. If a satisfies (3.6) then by Proposition 3.5 there exist ϕ_1, ϕ_2 positive solutions of (3.1), with corresponding eigenvalues μ_1, μ_2, for λ_1, λ_2 respectively. Consider $\phi = \phi_1^t \phi_2^{1-t}$. Then by Hölder's inequality we have that

$$J_\lambda * \phi \leqslant (J_{\lambda_1} * \phi_1)^t (J_{\lambda_2} * \phi_2)^{1-t}.$$

Using the inequality above and that ϕ_1 and ϕ_2 are solutions of (3.1) we obtain that

$$J_\lambda * \phi \leqslant \big(\big(-a(x)-\mu_1\big)\phi_1\big)^t \big(\big(-a(x)-\mu_2\big)\phi_2\big)^{1-t} = \big(-a(x)-\mu_1\big)^t \big(-a(x)-\mu_2\big)^{1-t}\phi$$

and then using Young's inequality we obtain that

$$J_\lambda * \phi \leqslant \big(t\big(-a(x)-\mu_1\big) + (1-t)\big(-a(x)-\mu_2\big)\big)\phi = \big(-a(x) + t\mu_1 + (1-t)\mu_2\big)\phi,$$

from where

$$\mu_{t\lambda_1+(1-t)\lambda_2} \geqslant t\mu_1 + (1-t)\mu_2,$$

which gives the convexity.

To conclude when (3.6) does not hold, we just approximate a by a_ε satisfying (3.6) and $a_\varepsilon \to a$ uniformly in $\mathbb{R}^N$. Then the result follows by Proposition 3.8 (iii).

Finally, we claim that the function μ_λ is even. Indeed, suppose first μ_λ is the principal eigenvalue of L_λ, so $\mu_\lambda + \max a(x) < 0$. Considering L_λ in the space of $L^2_{loc}(\mathbb{R}^N)$ periodic functions, we have that $L_{-\lambda}$ is its adjoint, and therefore μ_λ is in the spectrum of $L_{-\lambda}$. Using $\mu_\lambda + \max a(x) < 0$ it is easy to see that μ_λ is the principal eigenvalue of $L_{-\lambda}$. In the case L_λ has no principal eigenfunction, we directly deduce $\mu_\lambda = \mu_{-\lambda}$.

Since $-\mu_\lambda$ is even and convex, we obtain, that μ is nondecreasing in $(0,\infty)$ and nonincreasing in $(-\infty,0)$. □

Proof of Proposition 3.6. For the continuity of $\lambda \mapsto \mu_\lambda$ we argue as follows. Suppose first that a satisfies (3.6) and $\lambda_j \to \lambda_\infty$. It is easy to see that μ_{λ_j} is bounded, so up to a subsequence $\mu_{\lambda_j} \to \mu$. Let $\phi_j \in C_{per}(\mathbb{R}^N)$ be the principal eigenfunction associated with μ_{λ_j} $(j = 1,2,\ldots)$ normalized so that $\|\phi_j\|_{L^\infty} = 1$. Since $\mu + \max a < 0$, we have $\mu_{\lambda_j} + \max a \leqslant -\delta < 0$ for some $\delta > 0$ and all j large. Then from

$$J_{\lambda_j} * \phi_j = (-\mu_{\lambda_j} - a)\phi_j$$

we obtain compactness to say that for a subsequence ϕ_j converges uniformly to a nontrivial, nonnegative function $\phi \in C_{per}(\mathbb{R}^N)$ satisfying the eigenvalue problem

$$J_{\lambda_\infty} * \phi = (-\mu - a)\phi.$$

Because of the uniqueness of the principal eigenvalue, Proposition 3.1, $\mu = \mu_{\lambda_\infty}$.

If a does not satisfy (3.6) we argue approximating a by a_ε that satisfy (3.6). Let μ_λ^ε denote the principal eigenvalue of $-J_\lambda - a_\varepsilon$. We note that the convergence $\mu_\lambda^\varepsilon \to \mu_\lambda$ as $\varepsilon \to 0$ is uniform by Proposition 3.8 (iii), so continuity of μ_λ^ε with respect to λ for all ε yields continuity of $\lambda \mapsto u_\lambda$.

Next we show the exponential growth of $-\mu_\lambda$. Observe that if $\phi \in C_{per}(\mathbb{R}^N)$ then

$$J_\lambda * \phi = \int_{[0,1]^N} k_\lambda(x,y) e^{-\lambda(x-y)\cdot e} \phi(y)\,dy,$$

where

$$k_\lambda(x,y) = \sum_{k\in\mathbb{Z}^N} e^{\lambda k\cdot e} J(x-y-k).$$

The function $k_\lambda(\cdot,y)$ is $[0,1]^N$-periodic. We consider the following eigenvalue problem

$$\hat{L}_\lambda \phi + (\mu + \varepsilon)\phi = 0 \quad \text{with } \phi \in C\big([0,1]^N\big),$$

where $\varepsilon > 0$ and

$$\hat{L}_\lambda \phi = \int_{[0,1]^N} k_\lambda(x,y) e^{-\lambda(x-y)\cdot e} \phi(y)\,dy + a(x)\phi + \mu_0\phi.$$

We will assume first that the support of J is large, so that for some constants $b, d > 0$:

$$k_\lambda(x,y) \geqslant d e^{b\lambda} \quad \forall x, y \in [0,1]^N.$$

Let $w(y) = e^{-\lambda y\cdot e}$. Then

$$\hat{L}_\lambda w \geqslant \big(d e^{b\lambda} + a(x) + \mu_0 + \varepsilon\big) w \geqslant \delta e^{b\lambda} w$$

where $\delta > 0$ and where we take λ large. If $\lambda > 0$ is large enough, by Theorem 3.9 we obtain a principal eigenfunction $\hat{\phi} \in C([0,1]^N)$ of $\hat{L}_\lambda$, with principal eigenvalue $-\hat{\mu}_\lambda \geqslant \delta e^{b\lambda}$. Since $k_\lambda(x,y)e^{\lambda(x-y)\cdot e}$ is periodic in x, we see that $\hat{\phi}$ is periodic. Therefore, extending it periodically to $\mathbb{R}^N$, we find that it is the principal eigenfunction of L_λ and

$-\mu_\lambda + \mu_0 + \varepsilon = -\hat{\mu}_\lambda \geqslant \delta e^{b\lambda}$. Now since $-\mu_\lambda$ is nondecreasing in λ we have $-\mu_\lambda + \mu_0 + \varepsilon \geqslant \varepsilon$ and by taking δ smaller if necessary we achieve for all λ

$$-\mu_\lambda \geqslant -\mu_0 - \varepsilon + \delta e^{b\lambda}.$$

Without the assumption that the support of J is large, we can assume that $a(x) \geqslant 0$ and work with m large so that the support of J^m is large. Then

$$\left(J_\lambda + a(x)\right)^m \geqslant J_\lambda^m + a(x)^m.$$

Notice that

$$J_\lambda^m(x) = e^{\lambda x \cdot e} J^m(x)$$

so the previous argument applies and we deduce that the principal eigenvalue of $J_\lambda^m + a(x)^m$ grows exponentially as $\lambda \to +\infty$. Then the same holds for $(J_\lambda + a(x))^m$ and therefore for $J_\lambda + a(x)$. □

Remark 3.11. We would like to comment here on the hypothesis in Theorem 1.2 that there is a principal eigenvalue for problem (1.7). In fact, the proof of Theorem 1.2 reveals that we actually need only that (2.6) has a principal eigenvalue for all $\lambda > 0$, which holds under the stated hypotheses that (1.7) has a principal eigenvalue (this is a consequence of Propositions 3.2 and 3.3). Then it is natural to ask whether it is always true that (2.6) has a principal eigenfunction, even if (1.7) does not. Thanks to Proposition 3.5 one can construct examples where (2.6) has no principal eigenvalue for λ in some interval around 0.

4. Convergence of the principal eigenvalue and eigenfunction

Given $\varepsilon \geqslant 0$ we study here the eigenvalue problem:

$$\begin{cases} \varepsilon \Delta w + J_\lambda * w - w + f_u(x,0)w + \mu w = 0 & \text{in } \mathbb{R}^N, \\ w > 0 \text{ periodic and } C^2. \end{cases} \tag{4.1}$$

We will write

$$L_{\varepsilon,\lambda} w = \varepsilon \Delta w + J_\lambda * w - w + f_u(x,0)w \tag{4.2}$$

and $L_\lambda = L_{0,\lambda}$.

In this section we will assume that μ_0 is a principal eigenvalue for $-L_0$. Observe that by Corollary 3.4 μ_λ is a principal eigenvalue of $-L_\lambda$. By the Krein–Rutman theorem, we know that for $\varepsilon > 0$, $L_{\varepsilon,\lambda}$ has a principal eigenvalue $\mu_{\varepsilon,\lambda}$ and there are principal C^2 periodic eigenfunctions $\phi_{\varepsilon,\lambda} > 0$ of $L_{\varepsilon,\lambda}$ and $\phi^*_{\varepsilon,\lambda} > 0$ of $L^*_{\varepsilon,\lambda}$, that is,

$$L_{\varepsilon,\lambda}\phi_{\varepsilon,\lambda} + \mu_{\varepsilon,\lambda}\phi_{\varepsilon,\lambda} = 0 \quad \text{and} \quad L^*_{\varepsilon,\lambda}\phi^*_{\varepsilon,\lambda} + \mu_{\varepsilon,\lambda}\phi^*_{\varepsilon,\lambda} = 0.$$

Lemma 4.1. *Assume that μ_0 is a principal eigenvalue for $-L_0$. For $\varepsilon \geqslant 0$*

$$\mu_{\varepsilon,\lambda} = \sup\{\mu \in \mathbb{R}\colon \exists \phi > 0 \ L_{\varepsilon,\lambda}\phi + \mu\phi \leqslant 0\} \tag{4.3}$$

$$= \inf\{\mu \in \mathbb{R}\colon \exists \phi > 0 \ L_{\varepsilon,\lambda}\phi + \mu\phi \geqslant 0\}, \tag{4.4}$$

where the sup *and* inf *are taken over C^2 periodic functions if $\varepsilon > 0$ and over continuous periodic functions if $\varepsilon = 0$.*

Proof. Let us write

$$\mu^+_{\varepsilon,\lambda} = \sup\{\mu\colon \exists \phi > 0 \ L_{\varepsilon,\lambda}\phi + \mu\phi \leqslant 0\},$$
$$\mu^-_{\varepsilon,\lambda} = \inf\{\mu\colon \exists \phi > 0 \ L_{\varepsilon,\lambda}\phi + \mu\phi \geqslant 0\}.$$

Using ϕ_ε in the definitions we see that

$$\mu^-_{\varepsilon,\lambda} \leqslant \mu_{\varepsilon,\lambda} \leqslant \mu^+_{\varepsilon,\lambda}.$$

Let us prove $\mu_{\varepsilon,\lambda} = \mu^-_{\varepsilon,\lambda}$. Let $\mu \in \mathbb{R}$ be such that there exists $\psi > 0$ C^2 periodic such that $L_{\varepsilon,\lambda}\psi + \mu\psi \geqslant 0$. Then

$$\mu_{\varepsilon,\lambda}\langle\psi, \phi^*_{\varepsilon,\lambda}\rangle = -\langle\psi, L^*_{\varepsilon,\lambda}\phi^*_{\varepsilon,\lambda}\rangle = -\langle L_{\varepsilon,\lambda}\psi, \phi^*_{\varepsilon,\lambda}\rangle \leqslant \mu\langle\psi, \phi^*_{\varepsilon,\lambda}\rangle$$

where $\langle\,,\rangle$ denotes L^2 inner product on $[0,1]^N$. Since $\langle\psi, \phi^*\rangle > 0$ we deduce that $\mu_{\varepsilon,\lambda} \leqslant \mu$. Hence $\mu_{\varepsilon,\lambda} \leqslant \mu^-_{\varepsilon,\lambda}$.

The proof of $\mu^+_{\varepsilon,\lambda} \leqslant \mu_{\varepsilon,\lambda}$ is similar. □

Lemma 4.2. *Assume that μ_0 is a principal eigenvalue for $-L_0$. Let $\mu_{\varepsilon,\lambda}$ be the principal eigenvalue of (4.1) in the space of C^2 periodic functions. Then*

$$\mu_{\varepsilon,\lambda} \to \mu_\lambda \quad \text{as } \varepsilon \to 0,$$

and the convergence is uniform for λ in bounded intervals.

Let $\phi_{\varepsilon,\lambda}$ be the principal periodic eigenfunction of $L_{\varepsilon,\lambda}$ normalized so that

$$\|\phi_{\varepsilon,\lambda}\|_{L^2([0,1]^N)} = 1.$$

Then

$$\phi_{\varepsilon,\lambda} \to \phi_\lambda \quad \text{in } C\big(\mathbb{R}^N\big) \text{ as } \varepsilon \to 0$$

where ϕ_λ is the principal periodic eigenfunction of L_λ.

Proof. Under the stated hypotheses (1.3), (1.4) on J and f, ϕ_λ is C^2 by Proposition 3.5. Let $\mu > \mu_\lambda$. Then

$$L_{\varepsilon,\lambda}\phi_\lambda + \mu\phi_\lambda = \varepsilon\Delta\phi_\lambda + (\mu - \mu_\lambda)\phi_\lambda \geqslant 0$$

if ε is small. Using formula (4.4) we see that for small ε, $\mu_{\varepsilon,\lambda} \leqslant \mu$. Thus

$$\limsup_{\varepsilon\to 0} \mu_{\varepsilon,\lambda} \leqslant \mu_\lambda.$$

Using (4.3) we can prove

$$\liminf_{\varepsilon\to 0} \mu_{\varepsilon,\lambda} \geqslant \mu_\lambda.$$

Next we prove the uniform convergence of $\phi_{\varepsilon,\lambda}$ and for this we derive *a priori* estimates. Since $\phi_{\varepsilon,\lambda}$ satisfies (4.1) and $f_u(x,0)$ is C^2 we see that $\phi_{\varepsilon,\lambda}$ is in $C^{3,\alpha}(\mathbb{R}^N)$ for any $\alpha \in (0,1)$. Fix $i \in \{1,\dots,N\}$ and differentiate (4.1) with respect to x_i. Let us write $w_i = \partial_{x_i}\phi_{\varepsilon,\lambda}$. Then

$$\varepsilon\Delta w_i + g_i - w_i + f_u(x,0)w_i + \mu_{\varepsilon,\lambda}w_i = 0 \quad \text{in } \mathbb{R}^N, \tag{4.5}$$

where

$$g_i(x) = \int_{\mathbb{R}^N} \big(\partial_{x_i}J(x-y) - \lambda e_i\big)e^{\lambda(y-x)\cdot e}\phi_{\varepsilon,\lambda}(y)\,dy + \partial^2_{x_i u}f(x,0)\phi_{\varepsilon,\lambda}.$$

Let $p \geqslant 1$. Multiplying (4.5) by $|w_i|^{p-2}w_i$ and integrating on the period $[0,1]^N$ we get

$$\varepsilon\int_{[0,1]^N} \Delta w_i|w_i|^{p-2}w_i\,dx + \int_{[0,1]^N} g_i|w_i|^{p-2}w_i\,dx + \int_{[0,1]^N} \big(-1 + f_u(x,0) + \mu_{\varepsilon,\lambda}\big)|w_i|^p\,dx = 0.$$

Integrating by parts

$$\varepsilon(p-1)\int_{[0,1]^N} |w_i|^{p-2}|\nabla w_i|^2 + \int_{[0,1]^N} \big(1 - f_u(x,0) - \mu_{\varepsilon,\lambda}\big)|w_i|^p\,dx = \int_{[0,1]^N} g_i|w_i|^{p-2}w_i\,dx$$

and therefore

$$\int_{[0,1]^N} \big(1 - f_u(x,0) - \mu_{\varepsilon,\lambda}\big)|w_i|^p\,dx \leqslant \int_{[0,1]^N} g_i|w_i|^{p-1}\,dx.$$

By Hölder's inequality

$$\int_{[0,1]^N} \big(1 - f_u(x,0) - \mu_{\varepsilon,\lambda}\big)|w_i|^p\,dx \leqslant \Big(\int_{[0,1]^N} |w_i|^p\Big)^{1-1/p}\Big(\int_{[0,1]^N} |g_i|^p\Big)^{1/p}. \tag{4.6}$$

Since the operator L_λ has a principal eigenfunction $\phi_\lambda > 0$ from the relation

$$J_\lambda * \phi_\lambda = \big(1 - f_u(x,0) - \mu_\lambda\big)\phi_\lambda$$

we see that

$$\inf_{x\in\mathbb{R}^N} \big(1 - f_u(x,0) - \mu_\lambda\big) > 0.$$

Since $\mu_{\varepsilon,\lambda} \to \mu_\lambda$ as $\varepsilon \to 0$, for sufficiently small $\varepsilon > 0$ we have

$$\big(1 - f_u(x,0) - \mu_{\varepsilon,\lambda}\big) \geqslant c > 0 \quad \text{for all } x \in \mathbb{R}^N.$$

We deduce from this and (4.6) that

$$\|w_i\|_{L^p([0,1]^N)} \leqslant C\|g_i\|_{L^p([0,1]^N)}$$

with C independent of ε. But

$$\|g_i\|_{L^p([0,1]^N)} \leqslant C\|\phi_{\varepsilon,\lambda}\|_{L^p([0,1]^N)}$$

and therefore, recalling the definition of w_i, we obtain

$$\|\nabla\phi_{\varepsilon,\lambda}\|_{L^p([0,1]^N)} \leqslant C\|\phi_{\varepsilon,\lambda}\|_{L^p([0,1]^N)} \tag{4.7}$$

with C independent of ε. Since we have normalized $\|\phi_{\varepsilon,\lambda}\|_{L^2([0,1]^N)} = 1$, using (4.7) repeatedly and Sobolev's inequality we deduce that for any $p > 1$

$$\|\nabla\phi_{\varepsilon,\lambda}\|_{L^p([0,1]^N)} \leqslant C$$

for some constant C. By Morrey's inequality we deduce that $\phi_{\varepsilon,\lambda}$ is bounded in $C^\alpha([0,1]^N)$ for any $0 < \alpha < 1$. Therefore, for a subsequence we have that $\phi_{\varepsilon,\lambda} \to \phi$ uniformly on $[0,1]^N$ to some continuous function ϕ. Then, multiplying (4.1) by a periodic smooth function and integrating by parts twice we deduce that $\phi \geqslant 0$ is a periodic eigenfunction of L_λ with eigenvalue μ_λ. Then ϕ is a multiple of ϕ_λ and since both have L^2 norm equal to 1, we conclude that $\phi = \phi_\lambda$. We also deduce that the whole family $\phi_{\varepsilon,\lambda}$ converges to ϕ_λ as $\varepsilon \to 0$. □

5. The stationary problem

In this section we give the proof of Theorem 1.1. The same result for Dirichlet boundary condition appears in [22]. First we state a result analogous to Theorem 1.1 for the perturbed problem.

Proposition 5.1. *Assume* (1.4). *Let μ_ε denote the principal periodic eigenvalue of $-L_\varepsilon$ where for $\varepsilon > 0$*

$$L_\varepsilon\phi = \varepsilon\Delta\phi + J * \phi - \phi + f_u(x,0)\phi.$$

The perturbed stationary equation (2.3) *has a positive periodic solution if and only if $\mu_\varepsilon < 0$ and this solution is unique.*

We will omit the proof, since it is very similar to [8,24].

Lemma 5.2. *Assume $\mu_0 < 0$, so for $\varepsilon > 0$ small $\mu_\varepsilon < 0$ and there exists a positive solution p_ε of* (2.3). *Then there is a constant $C > 0$ such that for $\varepsilon > 0$ small*

$$\frac{1}{C} \leqslant p_\varepsilon(x) \leqslant C \quad \forall x \in \mathbb{R}^N.$$

Also, p_ε is uniformly Lipschitz for $\varepsilon > 0$ small, i.e., there is C such that

$$\big|p_\varepsilon(x) - p_\varepsilon(x')\big| \leqslant C\big|x - x'\big| \quad \text{for all } x, x' \in \mathbb{R}^N$$

and for all $\varepsilon > 0$ small.

Proof. For the proof of upper and lower bounds, it suffices to exhibit super and subsolutions which are bounded and bounded away from zero, uniformly for $\varepsilon > 0$ small. As a supersolution we just take a large fixed constant.

Let us proceed with the construction of a subsolution. We follow an argument developed in [22]. Let $a(x) := f_u(x,0) - 1$ and $\sigma := \sup_{\mathbb{R}^N} a(x)$. Since $a(x)$ is smooth and periodic there exists a point x_0 such that $\sigma = a(x_0)$. By continuity of $a(x)$, for each n there exists η_n such that for all $x \in B_{\eta_n}(x_0)$ we have $|\sigma - a(x)| \leqslant \frac{2}{n}$.

Now let us consider a sequence of real numbers $(\varepsilon_n)_{n\in\mathbb{N}}$ which converges to zero such that $\varepsilon_n \leqslant \frac{\eta_n}{2}$. Next, let $(\chi_n)_{n\in\mathbb{N}}$ be the following sequence of cut-off functions: $\tilde{\chi}_n(x) := \chi(\frac{\|x-x_0\|}{\varepsilon_n})$ where χ is a smooth function such that $0 \leqslant \chi \leqslant 1$, $\chi(x) = 0$ for $|x| \geqslant 2$ and $\chi(x) = 1$ for $|x| \leqslant 1$. Next, we let

$$\chi_n(x) = \sum_{k\in\mathbb{Z}^N} \tilde{\chi}_n(x-k)$$

so that for n large, χ_n is well defined, smooth, and $[0,1]^N$-periodic.

Let us consider the following sequence of continuous periodic functions $(a_n)_{n\in\mathbb{N}}$, defined by

$$a_n(x) := \max\{a(x), \sigma\chi_n\}.$$

Then $\|a_n - a\|_\infty \to 0$ as $n \to \infty$. Now consider a C^∞ regularization $b_n(x) := \rho_n * a_n(x)$ where ρ_n is an adequate sequence of mollifiers with support in $B_{\frac{\varepsilon_n}{4}}(0)$, such that $\|b_n - a_n\|_\infty \leqslant \|a_n - a\|_\infty$. Let $\phi_{\varepsilon,n} > 0$ be the principal eigenfunction of the following eigenvalue problem

$$\varepsilon\Delta\phi_{\varepsilon,n} + J * \phi_{\varepsilon,n} + b_n(x)\phi_{\varepsilon,n} + \mu_{\varepsilon,n}\phi_{\varepsilon,n} = 0 \quad \text{in } \mathbb{R}^N.$$

Since b_n is constant in a small neighborhood of x_0, which is a point where it attains its maximum, by Proposition 3.5, there is a principal eigenvalue μ_n and eigenfunction $\phi_n > 0$ for the problem

$$J * \phi_n + b_n(x)\phi_n + \mu_n\phi_n = 0 \quad \text{in } \mathbb{R}^N.$$

We normalize $\|\phi_n\|_{L^\infty([0,1]^N)} = 1$.

Using that $\|b_n(x) - a(x)\|_\infty \to 0$ as $n \to \infty$, from the Lipschitz continuity with respect to $a(x)$ (Proposition 3.8) it follows that for n big enough, say $n \geqslant n_0$, we have

$$\mu_n \leqslant \frac{\mu_0}{2} < 0.$$

We fix n_0 large so that

$$\|b_{n_0} - a\|_\infty \leqslant \frac{|\mu_0|}{8}.$$

Having fixed n_0, we work with $\varepsilon_0 > 0$ small so that

$$\mu_{\varepsilon,n_0} \leqslant \frac{\mu_0}{4} < 0 \quad \text{for all } 0 < \varepsilon \leqslant \varepsilon_0,$$

which is possible since $\mu_{\varepsilon,n_0} \to \mu_{n_0}$ as $\varepsilon \to 0$ by Lemma 4.2.

Now for $\sigma > 0$ we have

$$\begin{aligned} \varepsilon\sigma\Delta\phi_{\varepsilon,n_0} + J * \sigma\phi_{\varepsilon,n_0} - \sigma\phi_{\varepsilon,n_0} + f(x,\sigma\phi_{\varepsilon,n_0}) &\geqslant -\left(\left\|a(x) - b_{n_0}(x)\right\|_\infty + \mu_{\varepsilon,n_0}\right)\sigma\phi_{\varepsilon,n_0} + o(\sigma\phi_{\varepsilon,n_0}) \\ &\geqslant -\frac{\mu_0}{8}\sigma\phi_{\varepsilon,n_0} + o(\sigma\phi_{\varepsilon,n_0}) > 0. \end{aligned}$$

Therefore, for $\sigma > 0$ sufficiently small, $\sigma\phi_{\varepsilon,n_0}$ is a subsolution of (1.5). By Lemma 4.2, $\phi_{\varepsilon,n_0} \to \phi_{n_0}$ uniformly in $\mathbb{R}^N$ as $\varepsilon \to 0$. Since $\phi_{n_0} > 0$ we find the lower bound $p_\varepsilon \geqslant 1/C$ for some $C > 0$ and all $\varepsilon > 0$ small.

Let us prove now that p_ε is uniformly Lipschitz. Let $v = \frac{\partial p_\varepsilon}{\partial x_j}$ for some $j \in \{1,\dots,N\}$. Then v satisfies

$$J * v - v + \varepsilon\Delta v + f_u(x,p_\varepsilon)v + f_{x_j}(x,p_\varepsilon) = 0 \quad x \in \mathbb{R}^N.$$

We use that $f(x,u)/u$ is a decreasing function for $u > 0$. This implies that $f(x,u) - f_u(x,u)u > 0$ for all $x \in \mathbb{R}^N$ and all $u > 0$. Since there is a fixed lower bound for $p_\varepsilon \geqslant \frac{1}{C}$ ($\varepsilon > 0$ small) we find a fixed lower bound for the quantity

$$f(x,p_\varepsilon) - f_u(x,p_\varepsilon)p_\varepsilon \geqslant \delta_0 > 0 \quad \forall x \in \mathbb{R}^N$$

and all $\varepsilon > 0$ small. Then p_ε satisfies

$$\varepsilon\Delta p_\varepsilon + J * p_\varepsilon - p_\varepsilon + f_u(x, p_\varepsilon)p_\varepsilon = f_u(x, p_\varepsilon)p_\varepsilon - f(x, p_\varepsilon) \leqslant -\delta_0.$$

By the maximum principle we conclude that

$$|v| \leqslant \frac{\|f_{x_j}\|_\infty}{\delta_0} p_\varepsilon \leqslant C \quad \text{in } \mathbb{R}^N.$$

Thus p_ε is uniformly Lipschitz. □

Proof of Theorem 1.1. Uniqueness is proved as in [24,22]. Also the proof that $\mu_0 < 0$ is necessary for existence is very similar to [24,22], so we omit the details.

Assume now $\mu_0 < 0$ and let us prove that there exists a continuous solution. Let p_ε be the positive solution of (2.3), which exists since $\mu_\varepsilon < 0$ for $\varepsilon > 0$ small. By Lemma 5.2, p_ε is uniformly Lipschitz and therefore, up to subsequence p_ε, converges uniformly in $[0, 1]^N$ as $\varepsilon \to 0$ to a continuous function $p > 0$ which is periodic and solves (1.5). By the uniqueness of the positive periodic solution of (1.5), we have convergence of the whole family p_ε. □

Directly from the previous proof we get the following result.

Corollary 5.3. *Assume $\mu_0 < 0$, so $\mu_\varepsilon < 0$ for $\varepsilon > 0$ small. Let p be the positive continuous periodic solution of* (1.5) *and p_ε be the positive periodic solution of* (2.3) *for $\varepsilon > 0$ small. Then*

$$p_\varepsilon \to p \quad \text{uniformly as } \varepsilon \to 0.$$

6. Construction of approximate pulsating fronts

Let $\varepsilon > 0$ be small enough so that

$$0 = J * p - p + \varepsilon\Delta p + f(x, p), \quad x \in \mathbb{R}^N$$

has a positive periodic solution p_ε, which is unique.

Here the main result is the following.

Proposition 6.1. *Let $c_e^*(\varepsilon)$ be defined by* (2.7). *For $c \geqslant c_e^*(\varepsilon)$ there is a solution to*

$$c\partial_s\psi = M\psi - \psi + \varepsilon\Delta\psi + f(x, \psi) \quad \text{in } \mathbb{R} \times \mathbb{R}^N \tag{6.1}$$

such that

$$\begin{cases} \lim\limits_{s\to-\infty} \psi(s, x) = 0, \\ \lim\limits_{s\to+\infty} \psi(s, x) = p_\varepsilon(x), \\ \psi(s, x) \text{ is increasing in } s \text{ and periodic in } x. \end{cases} \tag{6.2}$$

To prove this result, we first work with an elliptic regularization $\mathcal{L}_\kappa$ of the operator $M - Id + \varepsilon\Delta_x - c\partial_s$ as it is done in [5,21,25] and introduce a truncated problem as follows. Given $\kappa, r, R > 0$, $\sigma \geqslant 0$ and $c \in \mathbb{R}$ consider the problem

$$\begin{cases} \mathcal{L}_\kappa\psi + f(x, \psi) + H(s, x) = 0 \quad \text{in } (-r, R) \times \mathbb{R}^N, \\ \psi(s, \cdot) = \sigma\phi \quad \text{for } s \leqslant -r, \\ \psi(s, \cdot) = p_\varepsilon \quad \text{for } s \geqslant R, \\ \psi(s, \cdot) \text{ is } [0, 1]^N\text{-periodic for all } s \end{cases} \tag{6.3}$$

where

$$\mathcal{L}_\kappa\psi := \int\limits_{[-r\leqslant s+(y-x)\cdot e\leqslant R]} J(x - y)\psi\big(s + (y - x)\cdot e, y\big)\,dy - \psi + \varepsilon\Delta_x\psi + \kappa\partial_{ss}\psi - c\partial_s\psi,$$

ϕ_ε is the principal periodic eigenfunction associated with the principal eigenvalue μ_ε of the following problem

$$\varepsilon\Delta\phi + J*\phi - \phi + f_u(x,0)\phi + \mu_\varepsilon\phi = 0,$$

and

$$H(s,x) = \sigma\int_{[s+(y-x)\cdot e\leqslant -r]} J(x-y)\phi_\varepsilon(y)\,dy + \int_{[s+(y-x)\cdot e\geqslant R]} J(x-y)p_\varepsilon(y)\,dy.$$

Proposition 6.2. *There exists σ_0 such that for all $0\leqslant\sigma\leqslant\sigma_0$ and for any $c\in\mathbb{R}$ there exists a unique solution of (6.3). Moreover, the corresponding solution is increasing in s, and continuous with respect to σ with values in $C^2([-r,R]\times\mathbb{R}^N)$.*

Proof. Note that by construction, since J is smooth then $H(s,x)$ is also smooth and the problem (6.3) can be solved by super and subsolutions techniques. We call a function $\psi\in C^2(\mathbb{R}^N\times[-r,R])$ a supersolution of (6.3) if

$$\begin{aligned}
&\mathcal{L}_\kappa\psi + f(x,\psi) + H(s,x)\leqslant 0 \quad -r<s<R,\\
&\psi(-r,x)\geqslant\sigma\phi_\varepsilon, \qquad \psi(R,x)\geqslant p_\varepsilon(x) \quad \forall x\in\mathbb{R}^N,\\
&\psi \text{ is periodic in } x.
\end{aligned}$$

Subsolutions are defined similarly reversing the inequalities. If there exist a subsolution $\Psi_1\in C^2([-r,R]\times\mathbb{R}^N)$ and a supersolution $\Psi_2\in C^2([-r,R]\times\mathbb{R}^N)$ such that $\Psi_1\leqslant\Psi_2$, then using monotone iterations one can construct a minimal solution $\underline{\psi}$ and a maximal solution $\bar\psi$ of (6.3) such that $\Psi_1\leqslant\underline{\psi}\leqslant\bar\psi\leqslant\Psi_2$. The monotone iterations can be taken for instance of the form

$$\psi_0 = \Psi_1$$

and ψ_n defined recursively as

$$\begin{cases}
-\varepsilon\Delta_x\psi_{n+1} - \kappa\partial_{ss}\psi_{n+1} + c\partial_s\psi_{n+1} + (A+1)\psi_{n+1}\\
\quad = \tilde M\psi_n + f(x,\psi_n) + A\psi_n + H(x,s) \quad \text{in } (-r,R)\times\mathbb{R}^N,\\
\psi_{n+1}(-r,x) = \sigma\phi_\varepsilon, \quad \psi_{n+1}(R,x) = p_\varepsilon(x) \quad \forall x\in\mathbb{R}^N,\\
\psi_{n+1} \text{ is periodic in } x,
\end{cases} \tag{6.4}$$

where $\tilde M$ denotes the operator

$$\tilde M\psi(s,x) = \int_{[-r\leqslant s+(y-x)\cdot e\leqslant R]} J(x-y)\psi\big(s+(y-x)\cdot e, y\big)\,dy.$$

Here $A>0$ is a large constant such that $u\mapsto f(x,u)+Au$ is increasing for all $u\in[0,\max p_\varepsilon]$ and all x. Then the right hand side of (6.4) is a monotone operator.

Now since, p_ε and w are bounded and strictly positive functions, the following quantity σ^* is well defined

$$\sigma^* := \sup\{\sigma>0 \mid \sigma\phi_\varepsilon\leqslant p_\varepsilon\}.$$

Take now $0\leqslant\sigma\leqslant\sigma^*$. Then from the definition of $H(s,x)$ we see that p_ε is a supersolution of (6.3). Indeed, a short computation shows that

$$\mathcal{L}_\kappa[p_\varepsilon] + f(x,p_\varepsilon) + H(x,s) \leqslant (J*p_\varepsilon - p_\varepsilon) + f(x,p_\varepsilon) + \varepsilon\Delta_x p_\varepsilon = 0.$$

Working with $\varepsilon>0$ sufficiently small we have that $\mu_\varepsilon<0$. Let us now observe that when $0\leqslant\sigma\leqslant\sigma^*$ and σ is small enough the function $\sigma\phi_\varepsilon$ is a subsolution of (6.3). Indeed, as above using that $\sigma\phi_\varepsilon\leqslant p_\varepsilon$ a short computation shows that

$$\begin{aligned}
\mathcal{L}_\kappa[\sigma\phi_\varepsilon] + f(x,\sigma\phi_\varepsilon) + H(x,s) &\geqslant \sigma(J*\phi_\varepsilon - \phi_\varepsilon) + f(x,\sigma\phi_\varepsilon) + \varepsilon\sigma\Delta_x\phi_\varepsilon\\
&\geqslant \sigma\phi_\varepsilon\left(-\mu_\varepsilon + \frac{f(x,\sigma\phi_\varepsilon)}{\sigma\phi_\varepsilon} - f_u(x,0)\right).
\end{aligned}$$

Since ϕ_ε is uniformly bounded, using the regularity of $f(x,s)$ we have for $\sigma \geqslant 0$ small enough say $\sigma \leqslant \sigma_1$

$$\left(-\mu_\varepsilon + \frac{f(x,\sigma\phi_\varepsilon)}{\sigma\phi_\varepsilon} - f_u(x,0)\right) \geqslant -\frac{\mu_\varepsilon}{2} \geqslant 0.$$

Thus for $\sigma \leqslant \sigma_0 := \inf\{\sigma_1, \sigma^*\}$, $\sigma\phi_\varepsilon$ is a subsolution to (6.3) with $\sigma\phi_\varepsilon \leqslant p_\varepsilon$.

We prove now that for all $\sigma \leqslant \sigma_0$ the corresponding problem (6.3) has a unique positive solution denoted ψ_σ. To this end we use a standard sliding method. First observe that for any $0 \leqslant \sigma \leqslant \sigma_0$, then any bounded solution ψ of the corresponding problem (6.3) satisfies

$$\sigma\phi_\varepsilon < \psi < p_\varepsilon.$$

Indeed, let us start with the proof of the inequality $\psi \leqslant p_\varepsilon$. Since p_ε is bounded away from 0 the following quantity is well defined

$$\gamma^* := \inf\{\gamma > 0 \mid \psi \leqslant \gamma p_\varepsilon\}.$$

To prove the inequality, we are reduced to show that $\gamma^* \leqslant 1$. Assume by contradiction that $\gamma^* > 1$. From the definition of γ^*, using the periodicity of the functions ψ, p_ε and a standard argument we see that there exists a point $(s_0, x_0) \in (-r, R) \times \mathbb{R}^N$ such that $\gamma^* p_\varepsilon(s_0, x_0) = \psi(s_0, x_0)$.

Observe that since $\frac{f(x,s)}{s}$ is a decreasing function of s, the function $\gamma^* p_\varepsilon$ is a supersolution of (6.3). Moreover, for some positive constant A big enough, the function $\gamma^* p_\varepsilon - \psi$ satisfies

$$\mathcal{L}_\kappa\big(\gamma^* p_\varepsilon - \psi\big) - A\big(\gamma^* p_\varepsilon - \psi\big) \leqslant 0 \quad \text{in } (-r, R) \times \mathbb{R}^N,$$
$$\big(\gamma^* p_\varepsilon - \psi\big)(-r, x) \geqslant 0, \qquad \big(\gamma^* p_\varepsilon - \psi\big)(R, x) \geqslant 0 \quad \forall x \in \mathbb{R}^N.$$

Since $\mathcal{L}_\kappa$ is elliptic in $(-r, R) \times \mathbb{R}^N$ and $\gamma^* p_\varepsilon(s_0, x_0) = \psi(s_0, x_0)$, from the strong maximum principle it follows that

$$\gamma^* p_\varepsilon \equiv \psi \quad \text{in } (-r, R) \times \mathbb{R}^N,$$

which is impossible since $\gamma^* p_\varepsilon(x) > p_\varepsilon(x) \geqslant \sigma\phi_\varepsilon(x) = \psi(-r, x)$. Therefore we have $\gamma^* \leqslant 1$ and $\psi \leqslant p_\varepsilon$. The strict inequality comes from the strong maximum principle. Now observe that to obtain the other inequality $\sigma\phi_\varepsilon < \psi$ we can just reproduce the above argumentation with $\sigma\phi_\varepsilon$ in the role of ψ and ψ in the role of p_ε.

We are now in position to prove the uniqueness of the solution of (6.3). Suppose ψ_1, ψ_2 are 2 solutions of (6.3). Define the following continuous functions

$$\bar\psi_1(s,x) := \begin{cases} \sigma\phi_\varepsilon(x) & \text{if } s < -r \text{ and } x \in \mathbb{R}^N, \\ \psi_1(s,x) & \text{if } -r \leqslant s \leqslant R \text{ and } x \in \mathbb{R}^N, \\ p_\varepsilon(x) & \text{if } s > R \text{ and } x \in \mathbb{R}^N \end{cases}$$

and

$$\bar\psi_2(s,x) := \begin{cases} \sigma\phi_\varepsilon(x) & \text{if } s < -r \text{ and } x \in \mathbb{R}^N, \\ \psi_2(s,x) & \text{if } -r \leqslant s \leqslant R \text{ and } x \in \mathbb{R}^N, \\ p_\varepsilon(x) & \text{if } s > R \text{ and } x \in \mathbb{R}^N. \end{cases}$$

Note that with this notation Eq. (6.3) satisfied by ψ_1 and ψ_2 can be rewritten

$$\varepsilon\Delta\psi_i + \kappa\partial_{ss}\psi_i - c\partial_s\psi_i - \psi_i + f(x,\psi_i) = -M\bar\psi_i \quad \text{in } (-r, R) \times \mathbb{R}^N \tag{6.5}$$

with $i \in \{1, 2\}$.

Let us define

$$\bar\psi_1^\tau(s,x) := \bar\psi_1(s+\tau, x)$$

with $\tau \in \mathbb{R}$. Obviously, we have

$$\bar\psi_1^\tau(s,x) := \psi_1(s+\tau, x) \quad \text{in } (-r, R-\tau) \times \mathbb{R}^N.$$

We claim that for all $\tau \in [0, R+r]$

$$\bar\psi_1^\tau(s,x) > \bar\psi_2(s,x) \quad \text{for } (s,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{6.6}$$

By construction we easily see that $\bar\psi_1^{R+r} \geqslant \bar\psi_2$ in $\mathbb{R}\times\mathbb{R}^N$ since we know that

$$\sigma\phi_\varepsilon \leqslant \psi_i \leqslant p_\varepsilon \quad \text{for } (s,x)\in\mathbb{R}\times\mathbb{R}^N.$$

Moreover, using that we have a strict inequality in $(-r,R)$, that is to say

$$\sigma\phi_\varepsilon < \psi_i < p_\varepsilon \quad \text{for } (s,x)\in(-r,R)\times\mathbb{R}^N,$$

we can find a positive ε such that for any $\tau\in[R+r-\varepsilon, R+r]$ we have

$$\bar\psi_1^\tau(s,x) > \bar\psi_2(s,x) \quad \text{for } (s,x)\in\mathbb{R}\times\mathbb{R}^N.$$

Note also that by construction for all $\tau\geqslant 0$ we have

$$\bar\psi_1^\tau \geqslant \bar\psi_2 \quad \text{in } \big((-\infty,-r]\cup[R-\tau,+\infty)\big)\times\mathbb{R}^N. \tag{6.7}$$

Now let us define

$$\tau^* = \inf\big\{\tau\in[0,R]:\ \bar\psi_1^{\tau'} \geqslant \bar\psi_2 \text{ for } \tau'\in[\tau,R+r]\big\}$$

then $0\leqslant\tau^* < R+r$. Assume that $\tau^*>0$. In this case

$$\bar\psi_1^{\tau^*} \geqslant \bar\psi_2 \quad \text{in } \mathbb{R}\times\mathbb{R}^N$$

and since $J\geqslant 0$ we have

$$M\big(\bar\psi_1^{\tau^*} - \bar\psi_2\big) \geqslant 0.$$

Now, fix $A>0$ large so that $f(x,u)+Au$ is monotone increasing in $[0,\max p_\varepsilon]$. Let us denote $z := \bar\psi_1^{\tau^*} - \bar\psi_2$. Then using the definition of $\bar\psi_1^\tau$ and $\bar\psi_2$ in $(-r,R-\tau^*)\times\mathbb{R}^N$, we have

$$\begin{aligned}
&\varepsilon\Delta z + \kappa\partial_{ss} z - c\partial_s z - (A+1)z \leqslant -M\big(\bar\psi_1^{\tau^*} - \bar\psi_2\big) \leqslant 0,\\
&z(-r,x) > 0 \quad \text{for all } x\in\mathbb{R}^N,\\
&z\big(R-\tau^*,x\big) > 0 \quad \text{for all } x\in\mathbb{R}^N.
\end{aligned}$$

By the strong maximum principle, it follows that $z>0$ in $(-r,R-\tau^*)\times\mathbb{R}^N$. Therefore, we have $\bar\psi_1^{\tau^*} - \bar\psi_2 > 0$ in $[-r,R-\tau^*]\times\mathbb{R}^N$ and by continuity for δ small we have for any τ in $(\tau^*-\delta,\tau^*)$

$$\bar\psi_1^\tau - \bar\psi_2 \geqslant 0 \quad \text{in } [-r,R-\tau]\times\mathbb{R}^N. \tag{6.8}$$

Combining the later with (6.7) it follows that for any positive τ in $(\tau^*-\delta,\tau^*)$ we have

$$\bar\psi_1^\tau - \bar\psi_2 \geqslant 0 \quad \text{in } \mathbb{R}\times\mathbb{R}^N,$$

which contradicts the definition of τ^*. Therefore, $\tau^*=0$ and $\bar\psi_1\geqslant\bar\psi_2$. By interchanging the role of ψ_1 and ψ_2 in the above argument we end up with $\bar\psi_1\geqslant\bar\psi_2\geqslant\bar\psi_1$, which prove the uniqueness of the solution of (6.3).

Taking $\psi_2=\psi$ in (6.6) shows that ψ is increasing in s. Finally, denoting ψ_σ the unique solution of the corresponding problem (6.3) one can see that the map $\sigma\mapsto\psi_\sigma$ is continuous, thanks to the uniqueness of the solution to (6.3) and standard elliptic estimates. □

Proposition 6.3. *Suppose $c > c_e^*(\varepsilon)$. Then there exists $r_0>0$, $\kappa(c)>0$ and $k>0$ such that for $r\geqslant r_0$, $R\geqslant r_0$, $\kappa\leqslant\kappa(c)$ there is $\sigma\in(0,\sigma_0)$ for which the unique increasing solution ψ of (6.3) satisfies*

$$\max_{x\in[0,1]^N}\psi(0,x) = \frac{1}{k}\min_{\mathbb{R}^N} p_\varepsilon.$$

Proof. Let ψ_σ denote the unique solution of (6.3) constructed in Proposition 6.2.

Choose $k>0$, so that

$$\sigma_0\max_{\mathbb{R}^N}\phi_\varepsilon > \frac{1}{k}\min_{\mathbb{R}^N} p_\varepsilon,$$

where ϕ_ε denotes the positive periodic principal eigenfunction associated with the eigenvalue problem

$$J * \phi - \phi + \varepsilon \Delta \phi + f_u(x,0)\phi + \mu_\varepsilon \phi = 0.$$

Observe that since ψ_σ is increasing in s, we have $\max_{\mathbb{R}^N} \psi_{\sigma_0}(0,x) > \frac{1}{k} \min_{\mathbb{R}^N} p_\varepsilon$. Next we prove that for $\sigma = 0$, we have $\max_{x\in\mathbb{R}^N} \psi_0(0,x) < \frac{1}{k} \min_{\mathbb{R}^N} p_\varepsilon$.

Recall that

$$c_e^*(\varepsilon) := \inf_{\lambda>0}\left(-\frac{\mu_{\varepsilon,\lambda}}{\lambda}\right),$$

where $\mu_{\varepsilon,\lambda}$ is the principal periodic eigenvalue of the problem

$$J_\lambda * \phi - \phi + \varepsilon \Delta \phi + f_u(x,0)\phi + \mu_{\varepsilon,\lambda}\phi = 0.$$

Since $c > c_e^*(\varepsilon)$ there is $\bar\lambda > 0$ such that $c\bar\lambda + \mu_{\varepsilon,\bar\lambda} > 0$. Let us denote $\phi_{\varepsilon,\bar\lambda}$ the principal periodic eigenfunction associated with $\mu_{\varepsilon,\bar\lambda}$ and consider the function

$$w := e^{\bar\lambda(s-s_0)}\phi_{\varepsilon,\bar\lambda},$$

where $s_0 \in \mathbb{R}$ is chosen so that

$$e^{-\bar\lambda s_0} \max_{\mathbb{R}^N} \phi_{\varepsilon,\bar\lambda} < \frac{1}{k} \min_{\mathbb{R}^N} p_\varepsilon,$$

and take $R > 0$ large so that

$$e^{\bar\lambda(R-s_0)} \min_{\mathbb{R}^N} \phi_{\varepsilon,\bar\lambda} \geqslant p_\varepsilon(x).$$

Since w is monotone increasing in s we have

$$w(s,x) \geqslant p_\varepsilon(x) \quad \text{for any } (s,x) \in [R,+\infty) \times \mathbb{R}^N.$$

Finally, observe that

$$e^{\bar\lambda(-r-s_0)}\phi_{\varepsilon,\bar\lambda}(x) \geqslant 0 \quad \text{for any } (s,x) \in \mathbb{R} \times \mathbb{R}^N.$$

We claim that the function w is a supersolution of (6.3) with $\sigma = 0$ for κ small enough. Indeed, in $(-r,R)$ we have

$$\begin{aligned}\mathcal{L}_\varepsilon w + f(x,w) + H(s,x) &\leqslant \big(J_{\bar\lambda} * \phi_{\varepsilon,\bar\lambda} - \phi_{\varepsilon,\bar\lambda} + \varepsilon\Delta\phi_{\varepsilon,\bar\lambda} + f_u(x,0)\phi_{\varepsilon,\bar\lambda} - c\bar\lambda\phi_{\varepsilon,\bar\lambda} + \kappa\bar\lambda^2\phi_{\varepsilon,\bar\lambda}\big)e^{\bar\lambda s} \\ &\leqslant -\big(\mu_{\varepsilon,\bar\lambda} + c\bar\lambda - \kappa\bar\lambda^2\big)w.\end{aligned}$$

Therefore, for $\kappa \leqslant \frac{c+\mu_{\bar\lambda}}{\bar\lambda^2} =: \kappa(c)$ we have

$$\begin{aligned}&\mathcal{L}_\varepsilon w + f(x,w) + H(s,x) \leqslant 0 \quad \text{for all } (s,x) \in (-r,R) \times \mathbb{R}^N,\\ &w(-r,x) > 0 \quad \text{for all } x \in \mathbb{R}^N,\\ &w(R,x) > p_\varepsilon \quad \text{for all } x \in \mathbb{R}^N.\end{aligned}$$

Since 0 is a subsolution of (6.3) with $\sigma = 0$ and $w \geqslant 0$ using the uniqueness of the solution of (6.3) we must have $\psi_0(s,x) \leqslant w(s,x)$. Therefore

$$\max_{\mathbb{R}^N} \psi_0(0,x) \leqslant \max_{\mathbb{R}^N} w(0,x) < \min_{\mathbb{R}^N} \frac{p_\varepsilon}{k}.$$

With $R > 0$ fixed, we see that the map $\sigma \in [0,\sigma_0] \mapsto \psi_\sigma$ is continuous, and at σ_0 satisfies $\max \psi_{\sigma_0}(0,x) > \min \frac{p_\varepsilon}{k}$ and $\max \psi_0(0,x) < \min \frac{p_\varepsilon}{k}$. By continuity there is $\sigma \in [0,\sigma_0]$ such that $\max \psi_\sigma(0,x) = \min \frac{p_\varepsilon}{k}$. □

Proposition 6.4. *For $c > c_e^*(\varepsilon)$ and $\kappa \leqslant \kappa(c)$ there is a solution to*

$$c\partial_s\psi = M\psi - \psi + \varepsilon\Delta\psi + \kappa\partial_{ss}\psi + f(x,\psi) \quad \text{in } \mathbb{R} \times \mathbb{R}^N \tag{6.9}$$

such that

$$\lim_{s\to-\infty}\psi(s,x)=0,$$
$$\lim_{s\to+\infty}\psi(s,x)=p_\varepsilon(x),$$
$\psi(s,x)$ *is increasing in* s *and periodic in* x.

Proof. For $r>0$ large, let ψ_r be the solution of (6.3) with $R=r$ obtained in Proposition 6.3 where $\sigma=\sigma(r)\in(0,\sigma_0)$ is such that

$$\max_{x\in\mathbb{R}^N}\psi_r(0,x)=\min_{x\in\mathbb{R}^N}\frac{p_\varepsilon(x)}{k}. \tag{6.10}$$

We let $r\to\infty$. Since ψ_r is locally bounded in $C^{1,\alpha}$, there is a subsequence such that ψ_r converges locally in $C^{1,\alpha}$ to a function $\psi:\mathbb{R}\times\mathbb{R}^N$ which satisfies (6.9) with the speed c, is increasing in s and periodic in x.

The limit $w(x)=\lim_{s\to-\infty}\psi(s,x)$ exists and is a solution of the stationary problem. By Proposition 5.1 this solution is either 0 or the unique positive stationary solution p_ε. By (6.10) we conclude that $w\equiv 0$. Similarly $\lim_{s\to+\infty}\psi(s,x)=p_\varepsilon(x)$. □

In the next proposition we establish some *a priori* estimates satisfied by the solutions of (6.9). Namely, we have

Proposition 6.5. *Let* $c>c_e^*(\varepsilon)$ *and* $\kappa\leqslant\kappa(c)$ *then the solution* $(\psi_{\kappa,\varepsilon},c)$ *of* (6.9) *satisfies*:

(i)
$$c\int_{\mathbb{R}\times\mathcal{C}}|\partial_s\psi_{\kappa,\varepsilon}|^2=-\frac{\varepsilon}{2}\int_{\mathcal{C}}|\nabla_x p_\varepsilon|^2-\frac{1}{4}\int_{\mathcal{C}^2}\tilde{\mathcal{J}}(x,y)\big(p_\varepsilon(x)-p_\varepsilon(y)\big)^2+\int_{\mathcal{C}}F(x,p_\varepsilon)$$

where $\mathcal{C}=[0,1]^N$ *and* $\tilde{\mathcal{J}}=\sum_{k\in\mathbb{Z}^N}J(x-y-k)$ *is a symmetric positive kernel.*

(ii) *For all compact set* $\mathcal{K}\subset\mathbb{R}\times\mathbb{R}^N$, *there exists* $R>0$, *a constant* $\gamma(R)$ *and* $n\in\mathbb{N}$ *so that*

$$\int_{\mathcal{K}}|\nabla_x\psi_{\kappa,\varepsilon}|^2\leqslant\gamma(R)(2n)^N.$$

(iii) *Given* $R>0$, *let*

$$Q_R=\big\{(s,x)\in\mathbb{R}\times\mathbb{R}^N:\ |x|<R,\ |s|<R\big\}.$$

Then there exist positive constants M, M' *independent of* ε *such that*

$$\sup_{Q_{R/4}}|\nabla_x\psi_{\kappa,\varepsilon}|\leqslant M\bigg(|c|+\frac{1}{R}+R\Big(2\sup_{Q_R}\big|p_\varepsilon(x)\big|+\sup_{Q_R}\big|f_u(x,0)\big|\Big)\bigg)\sup_{Q_R}|\psi_{\kappa,\varepsilon}|,$$
$$\sup_{Q_{R/4}}\frac{|\psi_{\kappa,\varepsilon}(t_1,x)-\psi_{\kappa,\varepsilon}(t_2,x)|}{|t_1-t_2|^{\frac{1}{2N}}}\leqslant M'\sup_{Q_R}|\nabla_x\psi_{\kappa,\varepsilon}|.$$

We give the proof of this proposition in Appendix A. We are now in a position to prove Proposition 6.1.

Proof of Proposition 6.1. Let us first assume that $c>c_e^*(\varepsilon)$. Then from the above construction, for any $\kappa\leqslant\kappa(c)$, there exists a function $\psi_{\kappa,\varepsilon}(s,x)$ increasing in s and periodic in $x\in\mathbb{R}^N$ that is solution of (6.9). Without loss of generality, we can assume that $\psi_{\kappa,\varepsilon}$ is normalized as follows

$$\max_{\mathbb{R}^N}\psi_{\kappa,\varepsilon}(0,x)=\min_{\mathbb{R}^N}\frac{p_\varepsilon}{k}.$$

We let $\kappa\to0$ along a sequence. Thanks to the a priori estimates of Proposition 6.5, we can extract a subsequence of $(\psi_{\kappa_n,\varepsilon})_{n\in\mathbb{N}}$ which converges locally uniformly in $\mathbb{R}\times\mathbb{R}^N$ to a function $\psi_\varepsilon\in H^1_{loc}(\mathbb{R}^N)\cap C^\alpha(\mathbb{R}\times\mathbb{R}^N)$ for some $\alpha\in(0,1)$, that satisfies (6.1) in the sense of distributions. Since $\psi_{\kappa_n,\varepsilon}$ is periodic in x, monotone increasing in s, and

$0 \leqslant \psi_{\kappa_n,\varepsilon} \leqslant p_\varepsilon$, we also have that ψ_ε is periodic in x, monotone nondecreasing in s, and $0 \leqslant \psi_\varepsilon \leqslant p_\varepsilon$. Note also that from the normalization condition, since $\psi_{\kappa_n,\varepsilon} \to \psi_\varepsilon$ locally uniformly, we also deduce that

$$\max_{\mathbb{R}^N} \psi_\varepsilon(0,x) = \min_{\mathbb{R}^N} \frac{p_\varepsilon}{k}. \tag{6.11}$$

Furthermore, using standard parabolic estimate, one can show that ψ_ε is a classical solution of (6.1). Thus ψ_ε satisfies

$$\begin{cases} \varepsilon\Delta\psi_\varepsilon - c\partial_s\psi_\varepsilon + M[\psi_\varepsilon] - \psi_\varepsilon + f(x,\psi_\varepsilon) = 0 & \text{in } \mathbb{R}\times\mathbb{R}^N, \\ 0 \leqslant \psi \leqslant p_\varepsilon, \quad \partial_s\psi \geqslant 0 & \text{in } \mathbb{R}\times\mathbb{R}^N, \\ \psi_\varepsilon(s,\cdot) \text{ is } [0,1]^N\text{-periodic for all } s. \end{cases}$$

By standard estimates the limit $w(x) = \lim_{s\to-\infty}\psi_\varepsilon(s,x)$ exists and is a solution of the stationary problem. By Proposition 5.1 this solution is either 0 or the unique positive stationary solution p_ε. By (6.11) we conclude that $w \equiv 0$. Similarly $\lim_{s\to+\infty}\psi_\varepsilon(s,x) = p_\varepsilon(x)$. $\square$

7. Estimates for $L_{\varepsilon,\lambda}$

Recall the notation from (4.2):

$$L_{\varepsilon,\lambda}u = \varepsilon\Delta u + J_\lambda * u - u + f_u(x,0)u.$$

Lemma 7.1. *Let λ be such that $0 < \lambda c < -\mu_{\varepsilon,\lambda}$, where $\mu_{\varepsilon,\lambda}$ is the principal periodic eigenvalue of the operator $-L_{\varepsilon,\lambda}$ defined in Section 4. If $u \in C^2(\mathbb{R}^N)$, $u \geqslant 0$ is a periodic solution to*

$$L_{\varepsilon,\lambda}u - \lambda c u = h \quad \text{in } \mathbb{R}^N$$

then

$$\|u\|_{L^\infty([0,1]^N)} \leqslant C_{\varepsilon,\lambda}\|h\|_{L^\infty([0,1]^N)}.$$

Note that for any $\varepsilon > 0$ and $0 < \lambda_0 < \lambda_1 < -\mu_{\varepsilon,\lambda}/c$ we have

$$\sup_{\lambda_0\leqslant\lambda\leqslant\lambda_1} C_{\varepsilon,\lambda} < \infty,$$

but the constant depends on ε.

Proof of Lemma 7.1. Let $\phi^*_{\varepsilon,\lambda}$ be the principal eigenfunction of the adjoint operator $L^*_{\varepsilon,\lambda}$. Then multiplying the equation by $\phi^*_{\varepsilon,\lambda}$ and integrating we find

$$(-\mu_{\varepsilon,\lambda} - \lambda c)\int_{[0,1]^N} u\phi^*_{\varepsilon,\lambda} = \int_{[0,1]^N} h\phi^*_{\varepsilon,\lambda}.$$

Since $\lambda c < -\mu_{\varepsilon,\lambda}$, $u \geqslant 0$ and $\phi^*_{\varepsilon,\lambda}$ is strictly positive and bounded, we obtain

$$\|u\|_{L^1([0,1]^N)} \leqslant C_{\varepsilon,\lambda}\|h\|_{L^1([0,1]^N)}.$$

The uniform norm follows because of standard elliptic estimates for the operator $L_{\varepsilon,\lambda}$. $\square$

Proposition 7.2. *There is $\rho > 0$, such that for any $0 < \rho' < \rho$ there is $\varepsilon_0 > 0$ and $\overline{C}$ such that for any $0 < \varepsilon \leqslant \varepsilon_0$, any λ that satisfies $(-\mu_{\varepsilon,\lambda} - \rho)/c \leqslant \lambda \leqslant (-\mu_{\varepsilon,\lambda} - \rho')/c$ and any $u \geqslant 0$ that is a periodic solution to*

$$L_{\varepsilon,\lambda}u - \lambda c u = h \quad \text{in } \mathbb{R}^N \tag{7.1}$$

for some $h \in L^\infty$ we have

$$\|u\|_{L^\infty([0,1]^N)} \leqslant \overline{C}\|h\|_{L^\infty([0,1]^N)}.$$

The constant $\rho > 0$ does not depend on ε or λ.

Proof. Let μ_λ be the principal eigenvalue of $-L_\lambda$. Recall that $\inf_{x\in[0,1]^N}(1-f_u(x,0)-\mu_0)>0$, so we can fix $\rho>0$ such that $\inf_x(1-f_u(x,0)-\mu_0-\rho>0)$. Since $\mu_\lambda\leqslant\mu_0$, see Proposition 3.3, also $\inf_x(1-f_u(x,0)-\mu_\lambda-\rho>0)$. Let $0<\rho'<\rho$ and let us proceed by contradiction. Assume that there exist sequences $\varepsilon_n\to 0$, $\lambda_n\in\mathbb{R}$, periodic functions (h_n) in L^∞, (u_n) in C^2, such that: λ_n satisfies $(-\mu_n-\rho)/c\leqslant\lambda_n\leqslant(-\mu_n-\rho')/c$, where $\mu_n=\mu_{\varepsilon_n,\lambda_n}$, u_n solves (7.1) and

$$\|h_n\|_{L^\infty}\to 0 \quad \text{and} \quad \|u_n\|_{L^\infty}=1.$$

We write Eq. (7.1) as

$$\varepsilon_n\Delta u_n-a_n(x)u_n=-g_n \tag{7.2}$$

where

$$a_n(x)=1-f_u(x,0)+\lambda_n c \quad \text{and} \quad g_n=J_{\lambda_n}u_n-h_n.$$

After extracting a subsequence we may assume that $\lambda_n\to\lambda$, $u_n\to u$ weakly-* in $L^\infty([0,1]^N)$ and then $J_{\lambda_n}u_n\to J_\lambda u$ uniformly. Hence $g_n\to g=J_\lambda u$ uniformly, and g is continuous. By Lemma 4.2 we have $\mu_n=\mu_{\varepsilon_n,\lambda_n}\to\mu_\lambda$ as $n\to\infty$. Since

$$a_n(x)=1-f_u(x,0)+\lambda_n c\geqslant 1-f_u(x,0)-\mu_n-\rho$$

and $1-f_u(x,0)-\mu_\lambda-\rho>0$, by working with n large we may assume that

$$\inf_x a_n(x)\geqslant a_0>0 \quad \text{for all } n.$$

Note that $a_n\to a=1-f_u(x,0)+\lambda c$, which is a continuous positive function, and the convergence is uniform. We claim that $u_n\to g/a$ uniformly. For the next argument we will assume that $g_n>0$, which we can achieve by replacing u_n by u_n+M and g_n by g_n+a_nM where $M>0$ is large. Note that (7.2) and $g_n\to g$ uniformly still hold. Let $0<\sigma<1/2$ and $x_0\in\mathbb{R}^N$. By uniform convergence $g_n\to g$, $a_n\to a$ and the continuity of g and a, we have

$$\inf_{x\in B_r(x_0)}\frac{g_n(x)}{\beta+a_n(x)}\geqslant(1-\sigma)\frac{g(x_0)}{a(x_0)} \quad \text{in } B_r(x_0)$$

provided we choose $r>0$, $\beta>0$ small and $n\geqslant n_0$ with n_0 large, and this is uniform in x_0. Let z be the principal eigenfunction for $-\Delta$ in $B_r(x_0)$ such that $\max_{B_r(x_0)}z=1$ and let $\nu_r=C/r^2$ be the corresponding principal eigenvalue, that is,

$$\begin{cases}\Delta z+\nu_r z=0, & z>0 \text{ in } B_r(x_0),\\ z=0 & \text{on } \partial B_r(x_0).\end{cases}$$

Define

$$v_n=u_n-zd_n \quad \text{where } d_n=\inf_{B_r(x_0)}\frac{g_n(x)}{\nu_r\varepsilon_n+a_n(x)}.$$

Then

$$\varepsilon_n\Delta v_n-a_nv_n=-g_n+d_n(\varepsilon_n\nu_r+a_n)z\leqslant 0$$

by the choice of d_n and $z\leqslant 1$. Since $v_n=u_n\geqslant 0$ on $\partial B_r(x_0)$ by the maximum principle we deduce that

$$u_n\geqslant\left(\inf_{B_r(x_0)}\frac{g_n(x)}{\nu_r\varepsilon_n+a_n(x)}\right)z \quad \text{in } B_r(x_0).$$

In particular, if $n\geqslant n_0$ is large enough so that $\nu_r\varepsilon_n\leqslant\beta$ we obtain

$$u_n(x_0)\geqslant(1-\sigma)\frac{g(x_0)}{a(x_0)}.$$

This proves that

$$\liminf_{n\to\infty}\inf_x(u_n-g/a)\geqslant 0.$$

A similar argument shows that

$$\limsup_{n\to\infty}\sup_x (u_n - g/a) \leqslant 0$$

which proves the uniform convergence $u_n \to g/a$. We deduce that $u = g/a$, and therefore u solves the equation

$$J_\lambda u - u + f_u(x,0)u - \lambda c u = 0.$$

But since $\|u_n\|_{L^\infty} = 1$ and u_n converges uniformly we also deduce that $\|u\|_{L^\infty} = 1$. Moreover $u \geqslant 0$. Then necessarily λc is the principal eigenvalue $-\mu_\lambda$ of L_λ. This not possible because we assumed $\lambda_n c \leqslant -\mu_n - \rho'$, so $\lambda c \leqslant -\mu_\lambda - \rho'$, a contradiction. □

8. Exponential bounds

Suppose we have a solution of

$$\begin{cases} c\psi_s = \varepsilon\Delta\psi + M[\psi] - \psi + f(x,\psi) & \forall s \in \mathbb{R},\ x \in \mathbb{R}^N, \\ \psi(\cdot,x) \text{ is nondecreasing for all } x, \\ \psi(s,\cdot) \text{ is } [0,1]^N \text{ periodic for all } s, \\ \psi(s,x) \to 0 \quad \text{as } s \to -\infty, \\ \psi(s,x) \to p_\varepsilon(x) \quad \text{as } s \to \infty. \end{cases} \tag{8.1}$$

Let $\delta > 0$ be fixed. We assume the following normalization on ψ:

$$\max_{x\in[0,1]^N} \psi(0,x) = \delta. \tag{8.2}$$

Let $\lambda_\varepsilon(c)$ be the smallest positive λ such that $c = -\frac{\mu_{\varepsilon,\lambda}}{\lambda}$. The main result in this section is the following.

Proposition 8.1. *For any $0 < \lambda < \lambda_\varepsilon(c)$ there are $\delta > 0$, $C > 0$ such that if ψ satisfies (8.1) and (8.2), then*

$$\psi(s,x) \leqslant Ce^{\lambda s} \quad \forall x \in \mathbb{R}^N,\ \forall s \leqslant 0, \tag{8.3}$$

where C does not depend on $\varepsilon > 0$.

As a corollary we have:

Proposition 8.2. *For all $\varepsilon > 0$ small and any fixed λ such that $0 < \lambda < \lambda_\varepsilon(c)$ there exists C_λ independent of ε such that if ψ satisfies (8.1) and (8.2), then*

$$\left|\psi_s(s,x)\right| \leqslant C_\lambda e^{\lambda s} \quad \forall s \leqslant 0,\ \forall x \in \mathbb{R}^N, \tag{8.4}$$

$$\varepsilon^{1/2}\left|\nabla_x\psi(s,x)\right| \leqslant C_\lambda e^{\lambda s} \quad \forall s \leqslant 0,\ \forall x \in \mathbb{R}^N, \tag{8.5}$$

$$\varepsilon\left|\nabla_x^2\psi(s,x)\right| \leqslant C_\lambda e^{\lambda s} \quad \forall s \leqslant 0,\ \forall x \in \mathbb{R}^N. \tag{8.6}$$

The proof of this proposition is based on scaling in the x variable and applying Schauder estimates for parabolic equations. We omit the proof.

The proof has several steps.

Lemma 8.3. *There exists $\lambda_0 > 0$ and $C > 0$ such that if $\delta > 0$ is sufficiently small and ψ satisfies (8.1) and (8.2), then*

$$\int_{[0,1]^N}\int_{-\infty}^{\infty} \psi(s,x)e^{-\lambda s}\,ds\,dx \leqslant C \quad \forall 0 < \lambda \leqslant \lambda_0 \tag{8.7}$$

where the constants do not depend on $\varepsilon > 0$. Moreover,

$$\int_{-\infty}^{\infty} \psi(s,x)e^{-\lambda s}\,ds \leqslant C_\varepsilon \quad \forall 0<\lambda\leqslant\lambda_0$$

where C_ε depends on ε.

Proof. Let $\eta_n:\mathbb{R}\to\mathbb{R}$ be a smooth function such that $\eta_n(s)=1$ for all $s\geqslant -n$, $\eta_n(s)=0$ for all $s\leqslant -2n$, $\eta_n'\geqslant 0$. Let $\lambda>0$ and define

$$U_n(x,\lambda)=\int_{-\infty}^{\infty}\psi(s,x)e^{-\lambda s}\eta_n(s)\,ds.$$

We multiply (8.1) by $\eta_n(s)e^{-\lambda s}$ and integrate on $(-\infty,\infty)$. The term involving $M\psi$ yields

$$\begin{aligned}\int_{-\infty}^{\infty} M\psi(s,x)\eta_n(s)e^{-\lambda s}\,ds &= \int_{-\infty}^{\infty}\int_{\mathbb{R}^N} J(x-y)\psi\big(s+(y-x)\cdot e,y\big)\eta_n(s)e^{-\lambda s}\,dy\,ds\\ &= \int_{\mathbb{R}^N} J(x-y)e^{-\lambda(x-y)\cdot e}\int_{-\infty}^{\infty}\psi\big(s+(y-x)\cdot e,y\big)\eta_n(s)e^{-\lambda(s+(y-x)\cdot e)}\,ds\,dy\\ &= \int_{\mathbb{R}^N} J(x-y)e^{-\lambda(x-y)\cdot e}\int_{-\infty}^{\infty}\psi(\tau,y)e^{-\lambda\tau}\eta_n\big(\tau-(y-x)\cdot e\big)\,d\tau\,dy\end{aligned}$$

and we write this term as

$$J_\lambda U_n(\cdot,\lambda)+\int_{\mathbb{R}^N} J(x-y)e^{-\lambda(x-y)\cdot e}\int_{-\infty}^{\infty}\psi(\tau,y)e^{-\lambda\tau}\big[\eta_n\big(\tau-(y-x)\cdot e\big)-\eta_n(\tau)\big]\,d\tau\,dy.$$

Hence

$$\varepsilon\Delta U_n+J_\lambda U_n-U_n+f_u(x,0)U_n-c\lambda U_n=D_n+E_n+F_n \tag{8.8}$$

where

$$D_n=\int_{\mathbb{R}^N} J(x-y)e^{-\lambda(x-y)\cdot e}\int_{-\infty}^{\infty}\psi(\tau,y)e^{-\lambda\tau}\big[\eta_n(\tau)-\eta_n\big(\tau-(y-x)\cdot e\big)\big]\,d\tau\,dy,$$

$$E_n=\int_{-\infty}^{\infty}\big(f\big(x,\psi(s,x)\big)-f_u(x,0)\psi(s,x)\big)e^{-\lambda s}\eta_n(s)\,ds,$$

$$F_n=-c\int_{-\infty}^{\infty}\psi(s,x)\eta_n'(s)e^{-\lambda s}\,ds.$$

Observe that in D_n, we can assume that the integral in y ranges on $|y-x|\leqslant 1$ (because we assume that J has support contained in the unit ball). Then $|(y-x)\cdot e|\leqslant 1$ and since η is nondecreasing

$$\begin{aligned}&\int_{\mathbb{R}^N} J(x-y)e^{-\lambda(x-y)\cdot e}\int_{-\infty}^{\infty}\psi(\tau,y)e^{-\lambda\tau}\eta_n\big(\tau-(y-x)\cdot e\big)\,d\tau\,dy\\ &\quad\geqslant \int_{\mathbb{R}^N} J(x-y)e^{-\lambda(x-y)\cdot e}\int_{-\infty}^{\infty}\psi(\tau,y)e^{-\lambda\tau}\eta_n(\tau-1)\,d\tau\,dy\end{aligned}$$

$$= \int_{\mathbb{R}^N} J(x-y)e^{-\lambda(x-y)\cdot e} \int_{-\infty}^{\infty} \psi(\tau+1,y)e^{-\lambda(\tau+1)}\eta_n(\tau)\,d\tau\,dy$$

$$\geqslant e^{-\lambda} \int_{\mathbb{R}^N} J(x-y)e^{-\lambda(x-y)\cdot e} \int_{-\infty}^{\infty} \psi(\tau,y)e^{-\lambda\tau}\eta_n(\tau)\,d\tau\,dy$$

because $\psi(\cdot,x)$ is nondecreasing. It follows that

$$D_n \leqslant \left(1-e^{-\lambda}\right)J_\lambda U_n(\cdot,\lambda).$$

Thus, from (8.8) and since $F_n \leqslant 0$

$$\varepsilon\Delta U_n + J_\lambda U_n - U_n + f_u(x,0)U_n - c\lambda U_n \leqslant \left(1-e^{-\lambda}\right)J_\lambda U_n(\cdot,\lambda) + E_n.$$

Write

$$E_n = \int_{-\infty}^{0} \ldots ds + \int_{0}^{\infty} \ldots ds$$

and note that

$$\int_{0}^{\infty} \left|\left(f\left(x,\psi(s,x)\right) - f_u(x,0)\psi(s,x)\right)e^{-\lambda s}\eta_n(s)\right| ds \leqslant C_1$$

with $C_1 \sim 1/\lambda$ as $\lambda \to 0^+$. We estimate the other integral as follows:

$$\int_{-\infty}^{0} \left(f\left(x,\psi(s,x)\right) - f_u(x,0)\psi(s,x)\right)e^{-\lambda s}\,ds \leqslant C_f \int_{-\infty}^{0} \psi(s,x)^2 e^{-\lambda s}\eta_n(s)\,ds$$

$$\leqslant C_f\delta \int_{-\infty}^{0} \psi(s,x)e^{-\lambda s}\eta_n(s)\,ds \leqslant C_f\delta U_n(x,\lambda)$$

where C_f is a constant that depends only on f.

In this way we obtain

$$\varepsilon\Delta U_n + J_\lambda U_n - U_n + f_u(x,0)U_n - c\lambda U_n \leqslant \left(1-e^{-\lambda}\right)J_\lambda U_n(\cdot,\lambda) + C_f\delta U_n + C_1. \tag{8.9}$$

Let $\mu_{\varepsilon,\lambda}$ be the principal eigenvalue of the operator $-(\varepsilon\Delta\phi + J_\lambda\phi - \phi + f_u(x,0)\phi)$, $\phi_{\varepsilon,\lambda}$, the principal eigenfunction and $\phi^*_{\varepsilon,\lambda}$ be the principal eigenfunction for the adjoint operator. Since $\mu_{\varepsilon,\lambda} \to \mu_\lambda$ as $\varepsilon \to 0$ and $\mu_\lambda < 0$, we can assume that $\mu_{\varepsilon,\lambda} < 0$. Multiplying (8.9) by $\phi^*_{\varepsilon,\lambda}$ and integrating over the period $[0,1]^N$ we find

$$(-\mu_{\varepsilon,\lambda} - c\lambda) \int_{[0,1]^N} U_n(x,\lambda)\phi^*_{\varepsilon,\lambda}(x)\,dx \leqslant \left(1-e^{-\lambda}\right) \int_{[0,1]^N} J_\lambda U_n(x,\lambda)\phi^*_{\varepsilon,\lambda}(x)\,dx$$

$$+ C_f\delta \int_{[0,1]^N} U_n(x,\lambda)\phi^*_{\varepsilon,\lambda}(x)\,dx + C_1 \int_{[0,1]^N} \phi^*_{\varepsilon,\lambda}(x)\,dx.$$

But

$$\int_{[0,1]^N} J_\lambda U_n(x,\lambda)\phi^*_{\varepsilon,\lambda}(x)\,dx = \int_{[0,1]^N} (J_\lambda)^*\phi^*_{\varepsilon,\lambda}(x)U_n(x,\lambda)\,dx$$

$$= \int_{[0,1]^N} \left[-\mu_{\varepsilon,\lambda}\phi^*_{\varepsilon,\lambda} + \phi^*_{\varepsilon,\lambda} - f_u(x,0)\phi^*_{\varepsilon,\lambda} - \varepsilon\Delta\phi^*_{\varepsilon,\lambda}\right]U_n(x,\lambda)\,dx.$$

Note that $\phi^*_{\varepsilon,\lambda}$ is uniformly bounded in $C^2([0,1]^N)$ as $\varepsilon \to 0$, see Remark 3.10, a property where use that f is C^3. Using the uniform smoothness of $\phi^*_{\varepsilon,\lambda}$ and the fact that it is uniformly bounded below $\phi^*_{\varepsilon,\lambda}(x) \geqslant c > 0$ as $\varepsilon \to 0$ with $\lambda > 0$ fixed, we see that

$$\int_{[0,1]^N} J_\lambda U_n(x,\lambda)\phi^*_{\varepsilon,\lambda}(x)\,dx \leqslant C \int_{[0,1]^N} U_n(x,\lambda)\phi^*_{\varepsilon,\lambda}(x)\,dx.$$

Therefore

$$(-\mu_{\varepsilon,\lambda} - c\lambda) \int_{[0,1]^N} U_n(x,\lambda)\phi^*_{\varepsilon,\lambda}(x)\,dx \leqslant \big((1-e^{-\lambda})C + C_f\delta\big) \int_{[0,1]^N} U_n(x,\lambda)\phi^*_{\varepsilon,\lambda}(x)\,dx + C_1 \int_{[0,1]^N} \phi^*_{\varepsilon,\lambda}(x)\,dx.$$

Choosing $\delta > 0$ and $\lambda > 0$ sufficiently small we deduce that

$$\int_{[0,1]^N} U_n(x,\lambda)\phi^*_{\varepsilon,\lambda}(x)\,dx \leqslant C$$

and again using that $\phi^*_{\varepsilon,\lambda}$ is uniformly bounded below, we find

$$\int_{[0,1]^N} U_n(x,\lambda)\,dx \leqslant C \tag{8.10}$$

where C is independent of ε and n. Now letting $n \to \infty$, we obtain the conclusion (8.7).

To prove the last part we observe that

$$\lim_{n\to\infty} U_n(x,\lambda) = U(x,\lambda)$$

by monotone convergence where

$$U(x,\lambda) = \int_{-\infty}^{\infty} \psi(s,x)e^{-\lambda s}\,ds.$$

By (8.10), $U(\cdot,\lambda)$ is in $L^1([0,1]^N)$ and is a weak solution of

$$\varepsilon\Delta U + J_\lambda U - U - c\lambda U = \tilde{E} \quad \text{in } \mathbb{R}^N$$

where

$$\tilde{E} = \int_{-\infty}^{\infty} f\big(x,\psi(s,x)\big)e^{-\lambda s}\,ds.$$

Note that

$$\|\tilde{E}\|_{L^p([0,1]^N)} \leqslant C\big\|U(\cdot,\lambda)\big\|_{L^p([0,1]^N)}$$

for all $p \geqslant 1$. Then, using standard elliptic L^p estimates we deduce that $U(\cdot,\lambda) \in L^\infty$ for $0 < \lambda \leqslant \lambda_0$. □

Lemma 8.4. *Suppose $\psi : (-\infty,0] \to [0,\infty)$ is nondecreasing and let $\lambda \in \mathbb{R}$. Then*

$$\psi(s) \leqslant \lambda \frac{e^{\lambda s}}{1-e^{\lambda s}} \int_{-\infty}^{0} \psi(\tau)e^{-\lambda\tau}\,d\tau \quad \forall s \leqslant 0. \tag{8.11}$$

Proof. Let $t \leqslant 0$. Then

$$\psi(t)\int_t^0 e^{-\lambda s}\,ds \leqslant \int_t^0 \psi(s)e^{-\lambda s}\,ds. \qquad \square$$

We prove first the exponential decay of ψ for some constant that depends on ε.

Lemma 8.5. *For any $\lambda < \lambda_\varepsilon(c)$ there is $C_\varepsilon > 0$ such that if ψ is a solution of (8.1) then*

$$\psi(s,x) \leqslant C_\varepsilon e^{\lambda s} \quad \forall x \in \mathbb{R}^N,\ \forall s \in \mathbb{R}. \tag{8.12}$$

Proof. In this proof $\varepsilon > 0$ is fixed and we find $\delta_\varepsilon > 0$ such that if ψ satisfies

$$\max_{x\in[0,1]^N} \psi(0,x) \leqslant \delta_\varepsilon \tag{8.13}$$

then the conclusion (8.12) holds. Given any solution of (8.1) we know already by Lemma 8.3 that $\psi(s,x) \to 0$ as to $-\infty$ uniformly in x, even at an exponential rate, so that (8.13) holds provided we replace $\psi(x,s)$ by $\psi(x,s-\tau)$ with τ sufficiently large.

Let $\eta \in C^\infty(\mathbb{R})$ be such that $\eta(t) = 1$ for $t \leqslant 1$ and $\eta(t) = 0$ for $t \geqslant 2$. For $\lambda \in \mathbb{R}$, $x \in [0,1]^N$, let U be defined by

$$U(x,\lambda) = \int_{-\infty}^{\infty} \psi(s,x)e^{-\lambda s}\eta(s)\,ds \tag{8.14}$$

with values in $[0,\infty]$. At this moment we know from Lemma 8.3 that $U(x,\lambda) < +\infty$ if we take $0 < \lambda \leqslant \lambda_0$ where $\lambda_0 > 0$ is a small fixed number. The objective is to prove that for any λ such that $0 < \lambda c < -\mu_{\varepsilon,\lambda}$

$$\big\|U(\cdot,\lambda)\big\|_{L^\infty([0,1]^N)} < +\infty.$$

Then from (8.11) we obtain the desired conclusion.

Assume that λ is such that $\|U(\cdot,\lambda)\|_{L^\infty([0,1]^N)} < +\infty$. We multiply (8.1) by $\eta(s)e^{-\lambda s}$ and integrate on $(-\infty,\infty)$. We obtain

$$\varepsilon\Delta U + J_\lambda U - U + f_u(x,0)U - c\lambda U = D_\lambda(x) + E_\lambda(x) + F_\lambda(x)$$

where

$$D_\lambda(x) = \int_{\mathbb{R}^N} J(x-y)e^{-\lambda(x-y)\cdot e}\int_{-\infty}^{\infty} \psi(\tau,y)e^{-\lambda\tau}\big[\eta(\tau) - \eta\big(\tau-(y-x)\cdot e\big)\big]\,d\tau\,dy,$$

$$E_\lambda(x) = \int_{-\infty}^{\infty} \big(f\big(x,\psi(s,x)\big) - f_u(x,0)\psi(s,x)\big)e^{-\lambda s}\eta(s)\,ds,$$

$$F_\lambda(x) = -c\int_{-\infty}^{\infty} \psi(s,x)\eta'(s)e^{-\lambda s}\,ds.$$

Thus

$$(L_{\varepsilon,\lambda} - \lambda c)U = D_\lambda + E_\lambda + F_\lambda.$$

Since U is nonnegative, we may apply Lemma 7.1 and deduce

$$\big\|U(\cdot,\lambda)\big\|_{L^\infty} \leqslant C_{\varepsilon,\lambda}\big(\|D_\lambda + E_\lambda + F_\lambda\|_{L^\infty}\big).$$

Write $U = U_1 + U_2$ where

$$U_1 = \int_{-\infty}^{0} \psi(s,x)e^{-\lambda s}\eta(s)\,ds, \qquad U_2 = \int_{0}^{\infty} \psi(s,x)e^{-\lambda s}\eta(s)\,ds. \tag{8.15}$$

Since $U_2 \geqslant 0$, we also have

$$\|U_1\|_{L^\infty([0,1]^N)} \leqslant C_{\varepsilon,\lambda}\|D_\lambda + E_\lambda + F_\lambda\|_{L^\infty([0,1]^N)}.$$

In $D_\lambda(x)$ one can restrict τ to $[-1,4]$. Hence

$$\|D_\lambda\|_{L^\infty([0,1]^N)} \leqslant C$$

and the constant remains bounded as λ varies in a bounded interval of $\mathbb{R}$. Similarly the integral in $F_\lambda(x)$ is restricted to $1 \leqslant \tau \leqslant 2$ and hence

$$\|F_\lambda\|_{L^\infty([0,1]^N)} \leqslant C$$

with C as before. We estimate

$$\begin{aligned}\big|E_\lambda(x)\big| &= \left|\int_{-\infty}^{\infty} \big(f\big(x,\psi(s,x)\big) - f_u(x,0)\psi(s,x)\big)e^{-\lambda s}\eta(s)\,ds\right| \\ &\leqslant C\int_{-\infty}^{-1} \big|\psi(s,x)\big|^2 e^{-\lambda s}\,ds + C.\end{aligned}$$

By (8.11)

$$\big|\psi(s,x)\big| \leqslant C_0 e^{\lambda s}\big\|U_1(\cdot,\lambda)\big\|_{L^\infty} \quad \forall x \in [0,1]^N,\ \forall s \leqslant -1.$$

Hence, using (8.13),

$$\begin{aligned}\big|E_\lambda(x)\big| &\leqslant C\delta_\varepsilon^{1/2}\int_{-\infty}^{-1} \big|\psi(s,x)\big|^{3/2} e^{-\lambda s}\,ds + C \\ &\leqslant C\delta_\varepsilon^{1/2}\big\|U_1(\cdot,\lambda)\big\|_{L^\infty}^{3/2}\int_{-\infty}^{-1} e^{\lambda s/2}\,ds + C = C_{\lambda_0}\delta_\varepsilon^{1/2}\big\|U_1(\cdot,\lambda)\big\|_{L^\infty}^{3/2} + C,\end{aligned}$$

where $C_{\lambda_0} \sim 1/\lambda_0$. Therefore

$$\big\|U_1(\cdot,\lambda)\big\|_{L^\infty([0,1]^N)} \leqslant \delta_\varepsilon^{1/2} C_{\lambda_0} C_{\varepsilon,\lambda}\big\|U_1(\cdot,\lambda)\big\|_{L^\infty}^{3/2} + C_1. \tag{8.16}$$

If we choose $\delta_\varepsilon > 0$ small this implies that there is a gap for $\|U_1(\cdot,\lambda)\|_{L^\infty([0,1]^N)}$. For example we can achieve

$$\text{either } \big\|U_1(\cdot,\lambda)\big\|_{L^\infty([0,1]^N)} \leqslant 2C_1 \quad \text{or } \big\|U_1(\cdot,\lambda)\big\|_{L^\infty([0,1]^N)} \geqslant 3C_1.$$

Indeed, first fix $0 < \lambda_0 < \lambda_1 < \lambda_\varepsilon(c)$. Then we know from Lemma 7.1 that

$$\sup_{\lambda_0 \leqslant \lambda \leqslant \lambda_1} C_{\varepsilon,\lambda} < \infty.$$

Choose $\delta_\varepsilon > 0$ such that

$$\delta_\varepsilon^{1/2}(3C_1)^{1/2} C_{\lambda_0}\Big(\sup_{\lambda_0 \leqslant \lambda \leqslant \lambda_1} C_{\varepsilon,\lambda}\Big) \leqslant \frac{1}{3}.$$

Suppose that $\|U_1(\cdot,\lambda)\|_{L^\infty([0,1]^N)} \leqslant 3C_1$. Then by (8.16)

$$\begin{aligned}\|U_1(\cdot,\lambda)\|_{L^\infty([0,1]^N)} &\leqslant \delta_\varepsilon^{1/2} C_{\lambda_0} C_{\varepsilon,\lambda} \|U_1(\cdot,\lambda)\|_{L^\infty}^{3/2} + C_1\\ &\leqslant \delta_\varepsilon^{1/2} C_{\lambda_0} C_{\varepsilon,\lambda} (3C_1)^{1/2} \|U_1(\cdot,\lambda)\|_{L^\infty} + C_1\\ &\leqslant \frac{1}{3}\|U_1(\cdot,\lambda)\|_{L^\infty} + C_1 \leqslant 2C_1.\end{aligned}$$

Using Lemma 8.3 and increasing C_1 and decreasing δ_ε if necessary, we can assume that

$$\|U_1(\cdot,\lambda_0)\|_{L^\infty} \leqslant 2C_1.$$

Since $\lambda \mapsto \|U_1(\cdot,\lambda)\|_{L^\infty}$ is continuous we see that

$$\|U_1(\cdot,\lambda)\|_{L^\infty} \leqslant 2C_1 \quad \forall \lambda_0 \leqslant \lambda \leqslant \lambda_1. \qquad \square$$

Proof of Proposition 8.1. We argue as in Lemma 8.5. In this proof we take $\rho > 0$ as in Proposition 7.2 and let $0 < \rho' < \rho$. We restrict λ so that it satisfies $(-\mu_{\varepsilon,\lambda} - \rho)/c \leqslant \lambda \leqslant (-\mu_{\varepsilon,\lambda} - \rho')/c$ and take $0 < \varepsilon \leqslant \varepsilon_0$.

Let U be defined by (8.14), and U_1, U_2 defined in (8.15). Following the proof of Lemma 8.5, if ψ satisfies (8.1) and (8.2) then, using Proposition 7.2,

$$\|U_1(\cdot,\lambda)\|_{L^\infty([0,1]^N)} \leqslant \delta^{1/2}\overline{C}\|U_1(\cdot,\lambda)\|_{L^\infty}^{3/2} + C_1,$$

where $\overline{C}$ now remains bounded for any $0 < \varepsilon \leqslant \varepsilon_0$ if λ satisfies $(-\mu_{\varepsilon,\lambda} - \rho)/c \leqslant \lambda \leqslant (-\mu_{\varepsilon,\lambda} - \rho')/c$. Again, choosing $\delta > 0$ small such that

$$\delta^{1/2}(3C_1)^{1/2}\overline{C} \leqslant \frac{1}{3}$$

we obtain

$$\text{either } \|U_1(\cdot,\lambda)\|_{L^\infty([0,1]^N)} \leqslant 2C_1 \quad \text{or } \|U_1(\cdot,\lambda)\|_{L^\infty([0,1]^N)} \geqslant 3C_1.$$

Let $\psi_\tau(s,x) = \psi(s-\tau,x)$ where $\tau > 0$ and $U_{1,\tau}$ denote the corresponding *Laplace transform* as in (8.14), (8.15). By Lemma 8.5

$$\|U_{1,\tau}(\cdot,\lambda)\|_{L^\infty} \to 0 \quad \text{as } \tau \to +\infty.$$

Since $\tau \mapsto \|U_{1,\tau}(\cdot,\lambda)\|_{L^\infty}$ is continuous we see that

$$\|U_{1,0}(\cdot,\lambda)\|_{L^\infty} \leqslant 2C_1.$$

Then by Lemma 8.4 we obtain (8.3). $\square$

9. Proof of the main theorem

In this section we prove Theorem 1.2, by establishing a uniform estimate in $W^{1,p}_{loc}$ of ψ_ε, the convergence of ψ_ε to a function ψ satisfying the equation, and finally establishing that ψ solves the full problem.

Proposition 9.1. *There is $\delta > 0$ such that if ψ_ε is a solution of* (8.1) *satisfying the normalization condition* (8.2)*, then for any $1 \leqslant p < \infty$ and bounded open set D in $\mathbb{R} \times \mathbb{R}^N$ there is a constant C independent of ε as $\varepsilon \to 0$ such that*:

$$\|\psi_\varepsilon\|_{W^{1,p}(D)} \leqslant C. \tag{9.1}$$

Proof. For simplicity we write $\psi = \psi_\varepsilon$ and we use the notation $\psi_{x_i} = \frac{\partial\psi}{\partial x_i}$. We differentiate the equation in (8.1) with respect to x_i and get

$$c\psi_{sx_i} = \varepsilon\Delta\psi_{x_i} + M_{x_i}[\psi] - e_i M[\psi_s] - \psi_{x_i} + f_u(x,\psi)\psi_{x_i} + f_{x_i}(x,\psi) \tag{9.2}$$

where

$$M_{x_i}[\psi](s,x) = \int_{\mathbb{R}^N} J_{x_i}(x-y)\psi\big(s+(y-x)\cdot e, y\big)\,dy$$

$e = (e_1, \dots, e_N)$. We write this as

$$c\psi_{sx_i} + \big(1 - f_u(x,0)\big)\psi_{x_i} = \varepsilon\Delta\psi_{x_i} + M_{x_i}[\psi] - e_i M[\psi_s] + \big(f_u(x,\psi) - f_u(x,0)\big)\psi_{x_i} + f_{x_i}(x,\psi). \tag{9.3}$$

Let $1 \leqslant p < +\infty$ and $\theta > 0$ to be fixed later on. Then

$$\frac{\partial}{\partial s}\big(e^{sp(1-f_u(x,0)-\theta)/c}|\psi_{x_i}|^p\big) = \frac{p}{c}e^{sp(1-f_u(x,0)-\theta)/c}\big(c\psi_{sx_i} + \big(1 - f_u(x,0) - \theta\big)\psi_{x_i}\big)|\psi_{x_i}|^{p-2}\psi_{x_i}.$$

Using (9.3) we obtain

$$\frac{\partial}{\partial s}\big(e^{sp(1-f_u(x,0)-\theta)/c}|\psi_{x_i}|^p\big) = \frac{p}{c}e^{sp(1-f_u(x,0)-\theta)/c}\big(\varepsilon\Delta\psi_{x_i} + M_{x_i}[\psi] - e_i M[\psi_s]$$
$$+ \big(f_u(x,\psi) - f_u(x,0)\big)\psi_{x_i} + f_{x_i}(x,\psi) - \theta\psi_{x_i}\big)|\psi_{x_i}|^{p-2}\psi_{x_i}.$$

We integrate now with respect to x over the period $[0,1]^N$ and estimate the terms on the right hand side.

$$\frac{c}{p}\frac{\partial}{\partial s}\int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c}|\psi_{x_i}|^p\,dx = I_1 + I_2 + I_3 + I_4 + I_5 + I_6$$

where

$$I_1 = \varepsilon\int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c}\Delta\psi_{x_i}|\psi_{x_i}|^{p-2}\psi_{x_i}\,dx,$$
$$I_2 = \int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c}M_{x_i}[\psi]|\psi_{x_i}|^{p-2}\psi_{x_i}\,dx,$$
$$I_3 = -e_i\int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c}M[\psi_s]|\psi_{x_i}|^{p-2}\psi_{x_i}\,dx,$$
$$I_4 = \int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c}\big(f_u(x,\psi) - f_u(x,0)\big)|\psi_{x_i}|^p\,dx,$$
$$I_5 = \int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c}f_{x_i}(x,\psi)|\psi_{x_i}|^{p-2}\psi_{x_i}\,dx,$$
$$I_6 = -\theta\int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c}|\psi_{x_i}|^p\,dx.$$

Integrating by parts we can estimate

$$I_1 = -\varepsilon(p-1)\int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c}|\psi_{x_i}|^{p-2}|\nabla\psi_{x_i}|^2\,dx$$
$$- \varepsilon\int_{[0,1]^N}\nabla\big(e^{sp(1-f_u(x,0)-\theta)/c}\big)\nabla\psi_{x_i}|\psi_{x_i}|^{p-2}\psi_{x_i}\,dx$$
$$\leqslant \frac{\varepsilon|s|p}{c}\int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c}\big|\nabla_x f_u(x,0)\big||\nabla\psi_{x_i}||\psi_{x_i}|^{p-1}\,dx.$$

By Young's inequality

$$I_1 \leqslant \frac{\theta}{5}\int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c}|\psi_{x_i}|^p\,dx + C\varepsilon^p|s|^p\int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c}|\nabla\psi_{x_i}|^p\,dx$$

where C depends on θ and $\|f\|_{C^2}$. In a similar way

$$I_2 \leqslant \frac{\theta}{5} \int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c} |\psi_{x_i}|^p \, dx + C \int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c} \big|M_{x_i}[\psi]\big|^p \, dx,$$

$$I_3 \leqslant \frac{\theta}{5} \int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c} |\psi_{x_i}|^p \, dx + C \int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c} \big|M[\psi_s]\big|^p \, dx,$$

$$I_5 \leqslant \frac{\theta}{5} \int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c} |\psi_{x_i}|^p \, dx + C \int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c} \big|f_{x_i}(x,\psi)\big|^p \, dx.$$

To estimate I_4 we write

$$I_4 \leqslant \sup_y \big| f_u\big(y, \psi(s,y)\big) - f_u(y,0)\big| \int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c} |\psi_{x_i}|^p \, dx.$$

We work with $\delta > 0$ small so that from the normalization condition (8.2) we get

$$\sup_y \big| f_u\big(y, \psi(s,y)\big) - f_u(y,0)\big| \leqslant \frac{\theta}{5} \quad \text{for all } s \leqslant 0.$$

Then

$$I_4 \leqslant \frac{\theta}{5} \int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c} |\psi_{x_i}|^p \, dx.$$

Combining the previous estimates we obtain

$$\begin{aligned} \frac{c}{p}\frac{\partial}{\partial s} \int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c} |\psi_{x_i}|^p \, dx \leqslant{}& C\varepsilon^p |s|^p \int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c} |\nabla \psi_{x_i}|^p \, dx \\ &+ C \int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c} \big|M_{x_i}[\psi]\big|^p \, dx \\ &+ C \int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c} \big|M[\psi_s]\big|^p \, dx \\ &+ C \int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c} \big|f_{x_i}(x,\psi)\big|^p \, dx. \end{aligned} \tag{9.4}$$

Let $t_0 \leqslant t \leqslant 0$. We integrate with respect to s over $[t_0, t]$ and then let $t_0 \to -\infty$. By (8.5), given any $0 < \lambda < \lambda_\varepsilon(c)$ there is C such that

$$\int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c} \big|\psi_{x_i}(s,x)\big|^p \, dx \leqslant \frac{C}{\varepsilon^{p/2}} \int_{[0,1]^N} \exp\big(sp\big(1 - f_u(x,0) - \theta + \lambda c\big)/c\big) \, dx. \tag{9.5}$$

We choose now λ and θ as follows. We fix a large $\Lambda_0 > 0$. We note that since there is a principal periodic eigenfunction $\phi_\lambda \in C_{per}(\mathbb{R}^N)$, $\phi_\lambda > 0$ for

$$J_\lambda * \phi_\lambda - \phi_\lambda + f_u(x,0)\phi_\lambda + \mu_\lambda \phi_\lambda = 0 \quad \text{in } \mathbb{R}^N$$

we must have

$$\gamma \equiv \inf_{\lambda \in [0,\Lambda_0]} \inf_{x \in \mathbb{R}^N} \big(1 - f_u(x,0) - \mu_\lambda\big) = \inf_{\lambda \in [0,\Lambda_0]} \inf_{x \in \mathbb{R}^N} \frac{J_\lambda * \phi_\lambda(x)}{\phi_\lambda(x)} > 0.$$

Since $\mu_{\varepsilon,\lambda} \to \mu_\lambda$ as $\varepsilon \to 0$, for $\varepsilon > 0$ sufficiently small

$$\inf_{x\in\mathbb{R}^N}\big(1 - f_u(x,0) - \mu_{\varepsilon,\lambda}\big) \geqslant \gamma/2 > 0$$

and since for $\lambda = \lambda_\varepsilon(c)$ we have $\lambda c = -\mu_{\varepsilon,\lambda}$ we get

$$\lambda_\varepsilon(c) \geqslant \frac{\gamma}{2c} + \sup_{x\in\mathbb{R}^N}\frac{f_u(x,0)-1}{c}.$$

Take $\lambda > 0$ such that

$$\sup_{x\in\mathbb{R}^N}\frac{f_u(x,0)-1}{c} + \frac{\gamma}{4c} \leqslant \lambda \leqslant \lambda_\varepsilon(c) - \frac{\gamma}{4c}. \tag{9.6}$$

Then choose $\theta = \gamma/8 > 0$ and get

$$\sigma \equiv \inf_{x\in\mathbb{R}^N}\left(\frac{1 - f_u(x,0) - \theta}{c} + \lambda\right) > 0. \tag{9.7}$$

Then from (9.5) we obtain

$$\int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c}\big|\psi_{x_i}(s,x)\big|^p\,dx \leqslant \frac{C}{\varepsilon^{p/2}}e^{p\sigma s} \quad \forall s \leqslant 0,$$

and therefore

$$\lim_{s\to-\infty}\int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c}\big|\psi_{x_i}(s,x)\big|^p\,dx = 0. \tag{9.8}$$

Integrating (9.4) in $[t_0,t]$ with $t_0 \leqslant t \leqslant 0$ and using (9.8) we obtain

$$\frac{c}{p}\int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c}|\psi_{x_i}|^p\,dx \leqslant K_1 + K_2 + K_3 + K_4 \tag{9.9}$$

where

$$K_1 = C\varepsilon^p\int_{-\infty}^t |s|^p\int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c}|\nabla\psi_{x_i}|^p\,dx\,ds,$$

$$K_2 = C\int_{-\infty}^t\int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c}\big|M_{x_i}[\psi]\big|^p\,dx\,ds,$$

$$K_3 = C\int_{-\infty}^t\int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c}\big|M[\psi_s]\big|^p\,dx\,ds,$$

$$K_4 = C\int_{-\infty}^t\int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c}\big|f_{x_i}(x,\psi)\big|^p\,dx\,ds.$$

Next we claim that K_1, K_2, K_3, K_4 remain bounded as $\varepsilon \to 0$. Indeed, by (8.6) and (9.7),

$$\begin{aligned} e^{sp(1-f_u(x,0)-\theta)/c}|\nabla\psi_{x_i}|^p &\leqslant e^{sp(1-f_u(x,0)-\theta)/c}\big|\nabla_x^2\psi\big|^p \\ &\leqslant \frac{C}{\varepsilon^p}e^{sp(1-f_u(x,0)-\theta+\lambda c)/c} \leqslant \frac{C}{\varepsilon^p}e^{sp\sigma}, \end{aligned}$$

for $s \leqslant 0$, $x \in \mathbb{R}^N$ with C independent of ε (note that $\nabla \psi_{x_i}$ is a second order derivative of ψ). Therefore K_1 is bounded as $\varepsilon \to 0$. The other ones can be bounded similarly, using (8.3), (8.4) and the hypotheses $f(x,0)=0$, $f \in C^3$ which imply

$$\left|f_{x_i}(x,u)\right| \leqslant Cu \quad \text{for } 0 \leqslant u \leqslant \delta$$

for some C. Thus from (9.9) we deduce that there exists C independent of ε for ε small such that for all $s \leqslant 0$

$$\int_{[0,1]^N} e^{sp(1-f_u(x,0)-\theta)/c} \left|\psi_{x_i}(s,x)\right|^p dx \leqslant C. \tag{9.10}$$

This together with (8.4) proves the estimate (9.1) for any bounded open set $D \subset (-\infty,0) \times \mathbb{R}^N$. To obtain (9.1) for any bounded open set $D \subset \mathbb{R} \times \mathbb{R}^N$ we proceed similarly as before. We multiply (9.2) by $|\psi_{x_i}|^{p-2}\psi_{x_i}$ and integrate over $[0,1]^N$. Using that ψ has a uniform upper bound we obtain

$$\frac{d}{ds} \int_{[0,1]^N} |\psi_{x_i}|^p \, dx \leqslant C \int_{[0,1]^N} |\psi_{x_i}|^p \, dx.$$

Then, using Gronwall's inequality we deduce for $s \geqslant 0$

$$\int_{[0,1]^N} \left|\psi_{x_i}(s,x)\right|^p dx \leqslant e^{Cs} \int_{[0,1]^N} \left|\psi_{x_i}(0,x)\right|^p dx + C.$$

Since by (9.10) we have a uniform control of the form $\int_{[0,1]^N} |\psi_{x_i}(0,x)|^p \, dx \leqslant C$, we obtain that for all $R > 0$ there exists $C > 0$ independent of ε such that

$$\int_{[0,1]^N} \left|\psi_{x_i}(s,x)\right|^p dx \leqslant C \quad \text{for all } |s| \leqslant R.$$

Using this and (8.4) we obtain the estimate (9.1) for any bounded open set $D \subset \mathbb{R} \times \mathbb{R}^N$. □

Lemma 9.2. *If $c \geqslant c_e^*$ there exists a function $\psi : \mathbb{R} \times \mathbb{R}^N$ which is C^1 in s and Lipschitz continuous and satisfies*

$$c\psi_s = M[\psi] - \psi + f(x,\psi) \quad \forall s \in \mathbb{R},\ x \in \mathbb{R}^N \tag{9.11}$$

and

$$\lim_{s \to -\infty} \psi(s,x) = 0.$$

Furthermore $\psi > 0$ is periodic in x and nondecreasing in s.

Proof. Let $c \geqslant c_e^*$. If $c > c_e^*$ then $c > c_e^*(\varepsilon)$ for $\varepsilon > 0$ small and we let, for small $\varepsilon > 0$, ψ_ε be the solution constructed in Proposition 6.1 with speed c. If $c = c_e^*$ we let ψ_ε be the solution constructed in Proposition 6.1 with speed $c_\varepsilon = c_e^*(\varepsilon)$. In any case we have a solution of (6.1) with speed $c_\varepsilon \to c$, satisfying also (6.2).

Let $\delta > 0$ be from Proposition 9.1 and shift in s so that ψ_ε satisfies

$$\max_{x \in [0,1]^N} \psi_\varepsilon(0,x) = \delta.$$

Then, choosing $p > N$ in Proposition 9.1 we can find a sequence $\varepsilon_n \to 0$ such that $\psi_{\varepsilon_n} \to \psi$ uniformly on compact sets. Using this local uniform convergence we see that the function ψ satisfies (9.11) in the following weak form

$$-c \int_{-\infty}^{\infty} \int_{[0,1]^N} \psi \varphi_s \, dx \, ds = \int_{-\infty}^{\infty} \int_{[0,1]^N} \big(M[\psi] - \psi + f(x,\psi)\big)\varphi \, dx \, ds$$

for all $\varphi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ smooth periodic function with compact support. This implies that ψ is C^1 in s and satisfies (9.11) classically. Since ψ_ε is nondecreasing in s and periodic in x we deduce that ψ is also nondecreasing in s

and periodic in x. Moreover, by Proposition 8.1, if we take $0 < \lambda < \lambda_c$ we have $\psi_\varepsilon(s,x) \leqslant Ce^{\lambda s}$ with C independent of ε. Letting $\varepsilon \to 0$ we find the same inequality for ψ and hence $\lim_{s\to-\infty} \psi(s,x) = 0$.

Finally, we prove that ψ is Lipschitz continuous, which follows the same lines of Proposition 6.1, so we point out the main steps. Let b_i, $i = 1,\dots,N$, denote the canonical basis in $\mathbb{R}^N$. Given $h \in \mathbb{R}$ we define

$$D_i^h \psi(s,x) = \frac{\psi(s,x+b_i h) - \psi(s,x)}{h}.$$

We choose $\lambda, \theta, \sigma > 0$ as in (9.6), (9.7) so that

$$e^{2s(1-f_u(x,0)-\theta)/c} \leqslant e^{2s(\sigma-\lambda)} \quad \forall x \in \mathbb{R}^N,\ s \leqslant 0. \tag{9.12}$$

Then we compute

$$\frac{\partial}{\partial s}\left(e^{2s(1-f_u(x,0)-\theta)/c}\left(D_i^h\psi\right)^2\right) = \frac{2}{c}e^{2s(1-f_u(x,0)-\theta)/c}\Big(M_i\big[\psi^h\big] - e_i M\big[D_s^{-he_i}\psi\big] + \big(f_u(x,\tilde\psi) - f_u(x,0)\big)D_i^h\psi$$
$$+ D_i^h f\big(\cdot,\psi(s,x+b_i h)\big) - \theta D_i^h\psi\Big)D_i^h\psi$$

where $e = (e_1,\dots,e_N)$,

$$M_i[g](s,x) = \int_{\mathbb{R}^N} \frac{J(x+b_i h - y) - J(x-y)}{h} g\big(s + (y-x)\cdot e, y\big)\,dy,$$

$$\psi^h(s,x) = \psi(s - e_i h, x),$$

$$D_s^\tau \psi(s,x) = \frac{\psi(s+\tau,x) - \psi(s,x)}{\tau},$$

and $\tilde\psi(s,x)$ lies between $\psi(s,x)$ and $\psi(s,x+b_i h)$. From here we deduce

$$\frac{\partial}{\partial s}\left(e^{2s(1-f_u(x,0)-\theta)/c}\left(D_i^h\psi\right)^2\right) \leqslant e^{2s(1-f_u(x,0)-\theta)/c}\Big(M_i\big[\psi^h\big]^2 + M\big[D_s^{-e_i h}\psi\big]^2 + \big(D_i^h f\big(\cdot,\psi(s,x+b_i h)\big)\big)^2\Big).$$

Using the exponential decay $\psi(s,x) \leqslant Ce^{\lambda s}$ for all $s \leqslant 0$ and all $x \in \mathbb{R}^N$, and a similar one for ψ_s (cf. (8.4)), we deduce from this and (9.12) that

$$\frac{\partial}{\partial s}\left(e^{2s(1-f_u(x,0)-\theta)/c}\left(D_i^h\psi\right)^2\right) \leqslant Ce^{2\sigma s}.$$

Integrating from $-\infty$ to $s \leqslant 0$, we conclude that there exists C independent of h such that

$$\left|D_i^h\psi(s,x)\right| \leqslant Ce^{\lambda s} \quad \forall x \in \mathbb{R}^N,\ \forall s \leqslant 0.$$

This proves that $\psi(s,\cdot)$ is Lipschitz continuous for all $s \leqslant 0$. An argument similar to the one at the end of Proposition 6.1 shows that it is also Lipschitz continuous for all $s \in \mathbb{R}$. □

We now prove the exponential convergence $\psi(s,x) \to p(x)$ as $s \to +\infty$, uniformly in x, by constructing appropriate subsolutions.

Lemma 9.3. *Let ψ be the function constructed in Lemma 9.2. Then there exist C, $\sigma > 0$ such that*

$$0 \leqslant p(x) - \psi(s,x) \leqslant Ce^{-\sigma s} \quad \text{for all } s \geqslant 0.$$

In particular

$$\lim_{s\to+\infty} \psi(s,x) = p(x) \quad \text{uniformly for } x \in \mathbb{R}^N.$$

Proof. First we note that

$$\psi(s,x) \leqslant p(x) \quad \text{for all } s \in \mathbb{R},\ x \in \mathbb{R}^N.$$

Next we show that $\psi(s,x) \to p(x)$ as $s \to +\infty$ uniformly for $x \in \mathbb{R}^N$. For this we will prove that there exists $\varepsilon_0 > 0$ such that for any $0 < m_0 < 1$ there is $s_0 \in \mathbb{R}$ such that

$$\psi_\varepsilon(s,x) \geqslant m_0 p_\varepsilon(x) \quad \text{for all } x \in \mathbb{R}^N,\ s \geqslant s_0,\ 0 < \varepsilon \leqslant \varepsilon_0. \tag{9.13}$$

The value s_0 depends on m_0 but not on ε.

Recall that we have normalized ψ_ε by

$$\max_{x\in[0,1]^N} \psi_\varepsilon(0,x) = \delta$$

where $\delta > 0$ is from Proposition 9.1. By Lemma 9.2

$$\psi_\varepsilon \to \psi \quad \text{as } \varepsilon \to 0$$

uniformly on compact sets of $\mathbb{R} \times \mathbb{R}^N$. Since $\psi > 0$ in $\mathbb{R}^N \times \mathbb{R}$ and is continuous we see that there is $\varepsilon_0 > 0$ and $a > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$

$$\psi_\varepsilon(0,x) \geqslant 2ap_\varepsilon(x) \quad \forall x \in \mathbb{R}^N.$$

Note that $a < 1$. Then we also have

$$\psi_\varepsilon(s,x) \geqslant 2ap_\varepsilon(x) \quad \forall x \in \mathbb{R}^N,\ s \geqslant 0,$$

because $\psi_\varepsilon(\cdot,x)$ is nondecreasing.

Given $a \leqslant m \leqslant 1$, $R \geqslant 1$, we construct a family of functions

$$v_m(s,x) = \lambda_m(s)p_\varepsilon(x) \quad s \in \mathbb{R},\ x \in \mathbb{R}^N$$

where

$$\lambda_m(s) = a + \frac{(m-a)s}{R+1}\big(1 - \eta(s-R)\big) + (m-a)\eta(s-R)$$

and $\eta \in C^\infty(\mathbb{R})$ is a cut-off function such that $\eta(s) = 0$ for $s \leqslant 0$, $\eta(s) = 1$ for $s \geqslant 1$, $0 \leqslant \eta \leqslant 1$ and $0 \leqslant \eta' \leqslant 2$. Note that $a \leqslant \lambda_m(s) \leqslant m$ for all $s \geqslant 0$.

Fix $0 < m_0 < 1$ and let $a \leqslant m \leqslant m_0$. It can be shown that we can choose $R > 0$ large enough, independently of ε, so that v_m satisfies

$$\varepsilon\Delta v_m + M[v_m] - v_m + f(x,v_m) - c(v_m)_s \geqslant 0$$

for $s \geqslant 1$ and $x \in \mathbb{R}^N$.

Using a sliding argument we obtain that $a \leqslant m \leqslant m_0$

$$\psi_\varepsilon \geqslant v_m \quad \text{for all } s \geqslant 1,\ x \in [0,1]^N.$$

Using this inequality with $m = m_0$ we establish (9.13). Letting $\varepsilon \to 0$ we deduce that

$$\lim_{s\to+\infty} \psi(s,x) = p(x) \quad \text{uniformly for } x \in \mathbb{R}^N.$$

Finally, let us show that there is exponential convergence. For this we construct a subsolution w_m with this property. Indeed, let $\sigma > 0$ to be fixed shortly and $0 \leqslant m \leqslant 1$. We set

$$w_m(s,x) = m\big(1 - e^{-\sigma s}\big)p(x).$$

Choosing S_0 large and $\sigma > 0$ small we obtain that

$$M[w_m] - w_m + f(x,w_m) - c(w_m)_s \geqslant 0 \quad \text{in } [S_0,+\infty) \times \mathbb{R}^N.$$

Let S_1 be such that

$$\psi(s,x) \geqslant \big(1 - e^{-\sigma(S_0+1)}\big)p(x) \quad \forall s \geqslant S_1,\ x \in \mathbb{R}^N.$$

This can be done because we know that $\psi(s,x) \to p(x)$ as $s \to +\infty$ uniformly for $x \in \mathbb{R}^N$.

Using again a sliding argument we can prove that

$$\psi(s,x) \geqslant w_m(s + S_0 - S_1, x) \quad \forall s \geqslant S_1,\ x \in \mathbb{R}^N$$

and all $0 \leqslant m < 1$. Letting $m \to 1$ we find

$$\psi(s,x) \geqslant \left(1 - e^{-\sigma(s+S_0-S_1)}\right)p(x) \quad \text{for all } s \geqslant s_0,\ x \in \mathbb{R}^N,$$

which finishes the proof of the lemma. □

Remark 9.4. The limit $\tilde{p}(x) = \lim_{s\to\infty}\psi(s,x)$ exists by monotonicity, but we cannot assert that it defines a continuous function (we have not proved uniform continuity of $\psi(s,x)$ as $s \to \infty$). One could then argue that $\tilde{p}$ is a bounded measurable solution of the stationary problem and that Theorem 1.1 also asserts the uniqueness of this solution. This would yield pointwise convergence $\lim_{s\to+\infty}\psi(s,x) = p(x)$ for all $x \in \mathbb{R}^N$.

Lastly, to finish the proof of Theorem 1.2 we prove the nonexistence of front for speed $c < c_e^*$.

Lemma 9.5. *Let J and f satisfy (1.3) and (1.4) and let $e \in \mathbb{R}^N$ be a unit vector. Assume $\mu_0 < 0$ and that there exists $\phi \in C_{per}(\mathbb{R}^N)$, $\phi > 0$ satisfying (1.7). Then there exists no pulsating front (ψ, c) connecting 0 and $p(x)$ in the direction e so that $c < c_e^*$.*

Proof. Assume by contradiction that there exists a pulsating front ψ with speed $c < c_e^*$. Then up to a shift ψ is a supersolution of the parabolic problem (1.1) for any initial data $u_0 \geqslant 0$ so that

$$\sup_{\mathbb{R}^N} u_0 < \min_{\mathbb{R}^N} p(x), \qquad \liminf_{r\to+\infty}\inf_{x\cdot e\leqslant r} u_0 > 0, \qquad u_0 = 0 \quad \text{for } x\cdot e \ll -1.$$

Let u be the solution of the parabolic problem (1.1) with initial data u_0 satisfying the above condition then by the maximum principle, we have for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N$,

$$u(t,x) \leqslant \psi(x\cdot e + ct + t_0, x)$$

for some fixed t_0. From Shen and Zhang results, Theorem C in [56], since $c < c_e^*$ we have

$$\liminf_{t\to+\infty}\inf_{x\cdot e+ct\geqslant 0}\left(u(x,t) - p(x)\right) = 0.$$

Thus we get the following contradiction

$$0 = \liminf_{t\to+\infty}\inf_{x\cdot e+ct\geqslant 0}\left(u(x,t) - p(x)\right) \leqslant \liminf_{t\to+\infty}\inf_{x\cdot e+ct\geqslant 0}\left(\psi(x\cdot e + ct + t_0, x) - p(x)\right)$$
$$\leqslant \left(\psi(t_0,x) - p(x)\right) < 0. \qquad \square$$

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Appendix A. Uniform estimates for solutions some regularized problems

In this section we prove Proposition 6.5. The estimates in this proposition divide naturally in 2 parts, one consisting in energy type estimates, and the other one are Schauder type estimates.

Proof of Proposition 6.5 (i). We proceed as in Lemma 2.5 in [9]. Let us denote $\phi_{\kappa,\varepsilon}$ the solution of (6.9). Then multiply Eq. (6.9) by $\partial_s\psi_{\kappa,\varepsilon}$ and integrate over $[-R,R]\times\mathcal{C}$ where $\mathcal{C} := [0,1]^N$. Then it follows that

$$c\int_{[-R,R]\times\mathcal{C}} |\partial_s\psi_{\kappa,\varepsilon}|^2 = \kappa\int_{[-R,R]\times\mathcal{C}} \partial_s\psi_{\kappa,\varepsilon}\partial_{ss}\psi_{\kappa,\varepsilon} + \varepsilon\int_{[-R,R]\times\mathcal{C}} \partial_s\psi_{\kappa,\varepsilon}\Delta_x\psi_{\kappa,\varepsilon}$$
$$+ \int_{[-R,R]\times\mathcal{C}} \partial_s\psi_{\kappa,\varepsilon}(M\psi_{\kappa,\varepsilon} - \psi_{\kappa,\varepsilon}) + \int_{[-R,R]\times\mathcal{C}} \partial_s\psi_{\kappa,\varepsilon} f(s,\psi_{\kappa,\varepsilon}).$$

Excepted the term $\mathcal{I} := \int_{[-R,R]\times\mathcal{C}} \partial_s \psi_{\kappa,\varepsilon}(M\psi_{\kappa,\varepsilon} - \psi_{\kappa,\varepsilon})$, all the term can be estimated as in the proof of Lemma 2.5 in [9], so we only deal with $\mathcal{I}$.

A simple computation shows that

$$\int_{[-R,R]\times\mathcal{C}} \partial_s \psi_{\kappa,\varepsilon}\psi_{\kappa,\varepsilon} = \frac{1}{2}\int_{[-R,R]\times\mathcal{C}} \partial_s(\psi_{\kappa,\varepsilon})^2 = \frac{1}{2}\int_{\mathcal{C}} \big[(\psi_{\kappa,\varepsilon})^2\big]_{-R}^{R}.$$

So it remains to compute

$$I := \int_{[-R,R]\times\mathcal{C}} \partial_s \psi_{\kappa,\varepsilon} M\psi_{\kappa,\varepsilon}.$$

Let us denote $\mathcal{C}_k := k + \mathcal{C}$ where $k \in \mathbb{Z}^N$. With this notation, using the periodicity in x of the function $\psi_{\kappa,\varepsilon}$ we have

$$\begin{aligned} M\psi_{\kappa,\varepsilon} &= \sum_{k\in\mathbb{Z}^N}\int_{k+\mathcal{C}} J(x-y)\psi_{\kappa,\varepsilon}\big(s+(y-x)\cdot e, y\big)\,dy \\ &= \sum_{k\in\mathbb{Z}^N}\int_{\mathcal{C}} J(x-k-y)\psi_{\kappa,\varepsilon}\big(s+(y-x)\cdot e + k\cdot e, y\big)\,dy. \end{aligned}$$

Now using integration by parts it follows that

$$\begin{aligned} I = {}& \int_{\mathcal{C}\times\mathcal{C}} \sum_{k\in\mathbb{Z}^N} J(x-y-k)\big[\psi_{\kappa,\varepsilon}(s,x)\psi_{\kappa,\varepsilon}\big(s+(y-x)\cdot e + k\cdot e, y\big)\big]_{-R}^{R} \\ & - \int_{\mathcal{C}\times\mathcal{C}} \sum_{k\in\mathbb{Z}^N} J(x-y-k)\int_{-R}^{R} \psi_{\kappa,\varepsilon}(s,x)\partial_s\psi_{\kappa,\varepsilon}\big(s+(y-x)\cdot e + k\cdot e, y\big). \end{aligned}$$

Let us make the change of variable $\tau = s + (y-x)\cdot e + k\cdot e$ in the last term of the right hand side. Then we have

$$\begin{aligned} & \int_{\mathcal{C}\times\mathcal{C}}\int_{-R}^{R} \sum_{k\in\mathbb{Z}^N} J(x-y-k)\psi_{\kappa,\varepsilon}(s,x)\partial_s\psi_{\kappa,\varepsilon}\big(s+(y-x)\cdot e + k\cdot e, y\big) \\ &\quad = \int_{\mathcal{C}\times\mathcal{C}} \sum_{k\in\mathbb{Z}^N} J(x-y-k)\int_{-R+(y-x)\cdot e+k\cdot e}^{R+(y-x)\cdot e+k\cdot e} \psi_{\kappa,\varepsilon}\big(\tau+(x-y)\cdot e - k\cdot e, x\big)\partial_s\psi_{\kappa,\varepsilon}(\tau, y). \end{aligned}$$

Let $R \to \infty$. Using that $\psi_{\kappa,\varepsilon} \to p_\varepsilon$ respectively 0 as $s \to \pm\infty$, $\psi_{\kappa,\varepsilon} \geqslant 0$, $\partial_s\psi_{\kappa,\varepsilon} \geqslant 0$ we obtain

$$\int_{\mathbb{R}\times\mathcal{C}} \partial_s\psi_{\kappa,\varepsilon}\psi_{\kappa,\varepsilon} = \frac{1}{2}\int_{\mathcal{C}} p_\varepsilon^2 \tag{A.1}$$

and

$$\begin{aligned} \int_{\mathbb{R}\times\mathcal{C}} \partial_s\psi_{\kappa,\varepsilon} M\psi_{\kappa,\varepsilon} = {}& \int_{\mathcal{C}\times\mathcal{C}} \sum_{k\in\mathbb{Z}^N} J(x-y-k)p_\varepsilon(x)p_\varepsilon(y) \\ & - \int_{\mathcal{C}\times\mathcal{C}} \sum_{k\in\mathbb{Z}^N} J(x-y-k)\int_{-\infty}^{+\infty} \psi_{\kappa,\varepsilon}\big(\tau+(x-y)\cdot e - k\cdot e, x\big)\partial_s\psi_{\kappa,\varepsilon}(\tau, y). \end{aligned}$$

Going back to the definition of $M\psi_{\kappa,\varepsilon}$ and using the symmetry of J we can rewrite the above equality the following way

$$\int_{\mathbb{R}\times\mathcal{C}} \partial_s\psi_{\kappa,\varepsilon} M\psi_{\kappa,\varepsilon} = \int_{\mathcal{C}} J * p_\varepsilon(x)p_\varepsilon(x)\,dx - \int_{\mathbb{R}\times\mathcal{C}} M\psi_{\kappa,\varepsilon}(\tau,y)\partial_\tau\psi_{\kappa,\varepsilon}(\tau,y)\,d\tau\,dy.$$

Thus we have

$$\int_{\mathbb{R}\times\mathcal{C}} \partial_s \psi_{\kappa,\varepsilon} M\psi_{\kappa,\varepsilon} = \frac{1}{2}\int_{\mathcal{C}} J * p_\varepsilon(x) p_\varepsilon(x)\,dx.$$

Set $\tilde{\mathcal{J}}(x,y) := \sum_{k\in\mathbb{Z}^N} J(x-y+k)$, the above equality rewrites as follows

$$\int_{\mathbb{R}\times\mathcal{C}} \partial_s \psi_{\kappa,\varepsilon} M\psi_{\kappa,\varepsilon} = \frac{1}{2}\int_{\mathcal{C}}\int_{\mathcal{C}} \tilde{\mathcal{J}}(x,y) p_\varepsilon(y) p_\varepsilon(x)\,dy\,dx. \tag{A.2}$$

Finally, combining (A.1) and (A.2), we obtain

$$\int_{\mathbb{R}\times\mathcal{C}} \partial_s \psi_{\kappa,\varepsilon}(M\psi_{\kappa,\varepsilon} - \psi_{\kappa,\varepsilon}) = -\frac{1}{4}\int_{\mathcal{C}\times\mathcal{C}} \tilde{\mathcal{J}}(x,y)\big(p_\varepsilon(x) - p_\varepsilon(y)\big)^2\,dx\,dy.$$

Hence,

$$c\int_{\mathbb{R}\times\mathcal{C}} |\partial_s \psi_{\kappa,\varepsilon}|^2 = -\frac{\varepsilon}{2}\int_{\mathcal{C}} |\nabla_x p_\varepsilon|^2 - \frac{1}{4}\int_{\mathcal{C}^2} \tilde{\mathcal{J}}(x,y)\big(p_\varepsilon(x) - p_\varepsilon(y)\big)^2 + \int_{\mathcal{C}} F(x,p_\varepsilon)$$

which proves (i). □

Proof of Proposition 6.5 (ii). Let $\mathcal{K}$ be a compact set of $\mathbb{R}\times\mathbb{R}^N$. Then since $\mathcal{K}$ is bounded, there exists $n\in\mathbb{N}$ and $R>0$ so that $\mathcal{K}\subset(-R_0,R_0)\times n\tilde{Q}$ where $\tilde{Q} := [-1,1]^N$.

Let us denote $\mathcal{E}(u)$ the following energy on the set of periodic function

$$\mathcal{E}(u) := -\frac{\varepsilon}{2}\int_{\mathcal{C}} |\nabla_x u|^2 - \frac{1}{4}\int_{\mathcal{C}^2} \tilde{\mathcal{J}}(x,y)\big(u(x) - u(y)\big)^2 + \int_{\mathcal{C}} F(x,u).$$

From (i), there exists $R\in[R_0,R_0+1]$ so that

$$c\int_{\mathcal{C}} |\partial_s \psi_{\kappa,\varepsilon}|^2(R) \leqslant \mathcal{E}(p_\varepsilon). \tag{A.3}$$

Let us now multiply (6.9) by $\psi_{\kappa,\varepsilon}$ and integrate over $(-R,R)\times\tilde{Q}$. Then we have

$$\begin{aligned}\frac{c}{2}\int_{\tilde{Q}} \big[\psi_{\kappa,\varepsilon}^2\big]_{-R}^R &= \kappa\int_{\tilde{Q}} [\psi_{\kappa,\varepsilon}\partial_s\psi_{\kappa,\varepsilon}]_{-R}^R - \kappa\int_{(-R,R)\times\tilde{Q}} |\partial_s\psi_{\kappa,\varepsilon}|^2 - \varepsilon\int_{(-R,R)\times\tilde{Q}} |\nabla_x\psi_{\kappa,\varepsilon}|^2 \\ &\quad + \int_{(-R,R)\times\tilde{Q}} (M\psi_{\kappa,\varepsilon} - \psi_{\kappa,\varepsilon})\psi_{\kappa,\varepsilon} + \int_{(-R,R)\times\tilde{Q}} f(x,\psi_{\kappa,\varepsilon})\psi_{\kappa,\varepsilon}.\end{aligned}$$

Therefore since $\psi_{\kappa,\varepsilon}$ is uniformly bounded and periodic in x we have

$$\varepsilon\int_{(-R,R)\times\tilde{Q}} |\nabla_x\psi_{\kappa,\varepsilon}|^2 = 2\gamma(R)$$

where

$$\begin{aligned}\gamma(R) &:= -\frac{c}{2}\int_{\mathcal{C}} \big[\psi_{\kappa,\varepsilon}^2\big]_{-R}^R - \kappa\int_{(-R,R)\times\mathcal{C}} |\partial_s\psi_{\kappa,\varepsilon}|^2 + \kappa\int_{\mathcal{C}} [\psi_{\kappa,\varepsilon}\partial_s\psi_{\kappa,\varepsilon}]_{-R}^R \\ &\quad + \int_{(-R,R)\times\mathcal{C}} (M\psi_{\kappa,\varepsilon} - \psi_{\kappa,\varepsilon})\psi_{\kappa,\varepsilon} + \int_{(-R,R)\times\mathcal{C}} f(x,\psi_{\kappa,\varepsilon})\psi_{\kappa,\varepsilon}.\end{aligned}$$

Since $0 \leqslant \psi_{\kappa,\varepsilon} \leqslant p_\varepsilon$, $\partial_s \psi_{\kappa,\varepsilon} \geqslant 0$ and f is uniformly bounded, using Cauchy–Schwartz inequality it follows that

$$\gamma(R) \leqslant |c| \int_{\mathcal{C}} p_\varepsilon^2 + \kappa \int_{\mathcal{C}} p_\varepsilon^2 \int_{\mathcal{C}} |\partial_s \psi_{\kappa,\varepsilon}|^2(R,x) + 2R \int_{\mathcal{C}} (J * p_\varepsilon) p_\varepsilon + 2R \|f\|_\infty \int_{\mathcal{C}} p_\varepsilon.$$

Thus, since $c > 0$ by (A.3) we have

$$\gamma(R) \leqslant |c| \int_{\mathcal{C}} p_\varepsilon^2 + \frac{\kappa \mathcal{E}(p_\varepsilon)}{|c|} \int_{\mathcal{C}} p_\varepsilon^2 + 2R \int_{\mathcal{C}} (J * p_\varepsilon) p_\varepsilon + 2R \|f\|_\infty \int_{\mathcal{C}} p_\varepsilon.$$

Hence the estimate (ii) follows by periodicity. □

The proof of Proposition 6.5 (iii) is based on the next 2 lemmas. The first one is a version of a result of [4], on gradient estimates for elliptic regularizations of semilinear parabolic equations. The result in [4] is based on Bernstein type estimates and is nonlinear in nature, while the estimates below have a linear character, and are based on a technique of Brandt [13] (see also [14,43] and [37, Chap. 3]).

Given $R > 0$ let

$$Q_R = \{(t,x) \in \mathbb{R} \times \mathbb{R}^N \colon |t| < R,\ |x_i| < R\ \forall i = 1, \ldots, N\}.$$

Lemma A.1. *Suppose $u \in C^2(Q_R)$ satisfies*

$$\Delta_x u + \varepsilon u_{tt} + u_t = f(x,t) \quad \text{in } Q_R$$

where $0 < \varepsilon \leqslant 1$, $f \in L^\infty(Q_R)$. Then

$$\left|\partial_{x_i} u(0,0)\right| \leqslant \left(\frac{2(N+1)}{R} + 2\right) \sup_{Q_R} |u| + \frac{R}{2} \sup_{Q_R} |f| \tag{A.4}$$

for all $i = 1, \ldots, N$, where C is independent of R, ε.

Proof. Let us write $x = (x_1, x') \in \mathbb{R}^N$ with $x_1 \in \mathbb{R}$, $x' \in \mathbb{R}^{N-1}$. Define

$$\tilde{Q} = \{(t, x_1, x') \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1} \colon 0 < x_1 < R,\ |x_i| < 1\ \forall i = 2, \ldots, N,\ |t| < 1\}$$

and

$$v(t, x_1, x') = \frac{1}{2}\big(u(t, x_1, x') - u(t, -x_1, x')\big)$$

for $(t, x_1, x') \in \tilde{Q}$. Let us write

$$Lv = \Delta_x v + \varepsilon v_{tt} + v_t.$$

Then L is an elliptic operator and satisfies the maximum principle. We have

$$Lv(t, x_1, x') = \frac{1}{2}\big(f(t, x_1, x') - f(t, -x_1, x')\big) \quad \text{for } (t, x_1, x') \in \tilde{Q}$$

and

$$|v| \leqslant \sup_{Q_R} |u| \quad \text{in } \tilde{Q}.$$

Let

$$\bar{v}(t, x_1, x') = A x_1 (R - x_1) + B\left(x_1^2 + |x'|^2 + t^2\right)$$

where

$$B = \frac{1}{R^2} \sup_{Q_R} |u|$$

and

$$A = \frac{1}{2}\Big(\sup_{Q_R} |f| + B(2N + 2\varepsilon + 2R)\Big).$$

With these choices we see that

$$|v| \leqslant \bar{v} \quad \text{on } \partial \tilde{Q}$$

and

$$L\bar{v} \leqslant -\sup_{Q_R} |f| \quad \text{in } \tilde{Q}.$$

By the maximum principle $\bar{v} - v \geqslant 0$ in $\tilde{Q}$. Similarly $\bar{v} + v \geqslant 0$ in $\tilde{Q}$ and therefore

$$|v| \leqslant \bar{v} \quad \text{in } \tilde{Q}.$$

This implies

$$\big|\partial_{x_1} v(0,0)\big| \leqslant AR$$

and gives (A.4) for $i = 1$. The same proof replacing x_1 by any of the other variables $x_2, \ldots, x_n$ yields (A.4). □

Lemma A.2. *Suppose $u \in C^2(Q_2)$ satisfies*

$$u_t - \Delta_x u - \varepsilon u_{tt} = f(x,t) \quad \text{in } Q_2$$

where $\varepsilon > 0$ and $f \in L^\infty(Q_2)$. Then for some $0 < \alpha < 1$ there is a constant C independent of ε such that

$$\sup_{|x| \leqslant 1,\, t_1, t_2 \in [-1,1]} \frac{|u(x,t_1) - u(x,t_2)|}{|t_1 - t_2|^\alpha} \leqslant C\Big(\sup_{Q_2} |f| + \sup_{Q_2} |u|\Big).$$

Proof. Let us write

$$M = \sup_{Q_2} |f| + \sup_{Q_2} |u|.$$

By Lemma A.1

$$\sup_{Q_1} |\nabla_x u| \leqslant CM. \tag{A.5}$$

Let $\varphi \in C^1(\mathbb{R}^N)$ have support in the closed ball $\overline{B}_1$ of $\mathbb{R}^N$. Multiplying the equation by $u\varphi$ and integrating in B_2 we find

$$\frac{1}{2}\frac{d}{dt}\int_{B_2} u^2 \varphi\, dx - \varepsilon \frac{d}{dt}\int_{B_2} u u_t \varphi\, dx + \varepsilon \int_{Q_1} u_t^2 \varphi\, dx + \int_{B_2} |\nabla u|^2 \varphi\, dx + \int_{B_2} \nabla u \nabla \varphi u\, dx = \int_{B_2} f u \varphi\, dx.$$

Integrating this from t_0 to t_1 with $-1 \leqslant t_0 < t_1 \leqslant 1$ and using (A.5) gives

$$-\frac{\varepsilon}{2}\frac{d}{dt}\int_{B_2} u^2 \varphi\, dx\bigg|_{t=t_1} + \frac{\varepsilon}{2}\frac{d}{dt}\int_{B_2} u^2 \varphi\, dx\bigg|_{t=t_0} + \varepsilon \int_{t_0}^{t_1}\int_{Q_1} u_t^2 \varphi\, dx = O\big(M^2\big)$$

where $O(M^2)$ is uniform in ε. Integrate now with respect to $t_0 \in [1/2, 2/3]$ and $t_1 \in [5/6, 1]$. We obtain

$$\varepsilon \int_{1/2}^{1}\int_{B_2} g(t) u_t^2 \varphi\, dx\, dt = O\big(M^2\big)$$

where $g(t)$ is a continuous function which is positive in $[1/2, 1]$. Therefore one can always select $t_0 \in [1/2, 1]$, possibly depending on ε, such that

$$\varepsilon \int_{B_2} u_t(t_0)^2 \varphi\, dx = O\big(M^2\big). \tag{A.6}$$

Now multiply the equation by $u_t\varphi$ and integrate in B_2, to obtain

$$\int_{B_2} u_t^2\varphi\,dx - \frac{\varepsilon}{2}\frac{d}{dt}\int_{B_2} u_t^2\varphi\,dx + \frac{1}{2}\frac{d}{dt}\int_{B_2} |\nabla u|^2\varphi\,dx + \int_{B_2} \nabla u\nabla\varphi u_t\,dx = \frac{d}{dt}\int_{B_2} fu\varphi.$$

Integrating with respect to $t \in [-1/2, t_0]$ with t_0 as above yields

$$\int_{-1/2}^{t_0}\int_{B_2} u_t^2\varphi\,dx\,dt - \frac{\varepsilon}{2}\int_{B_2} u_t^2\varphi\,dx\bigg|_{-1/2}^{t_0} + \frac{1}{2}\int_{B_2} |\nabla u|^2\varphi\,dx\bigg|_{-1/2}^{t_0} + \int_{B_2} \nabla u\nabla\varphi u_t\,dx = \int_{B_2} fu\varphi\bigg|_{-1/2}^{t_0}.$$

Using (A.5) and (A.6) we find

$$\int_{-1/2}^{t_0}\int_{B_2} u_t^2\varphi\,dx\,dt + \int_{B_2} \nabla u\nabla\varphi u_t\,dx = O\big(M^2\big). \tag{A.7}$$

But

$$\left|\int_{B_2} \nabla u\nabla\varphi u_t\,dx\right| \leqslant \frac{1}{2}\int_{B_2} |\nabla u|^2\frac{|\nabla\varphi|^2}{\varphi}\,dx + \frac{1}{2}\int_{B_2} \varphi u_t^2\,dx.$$

One can select a function $\varphi \geqslant 0$ with support the ball $|x| \leqslant 1$ and positive in $|x| < 1$ such that $\frac{|\nabla\varphi|^2}{\varphi}$ is bounded. So by (A.5)

$$\left|\int_{B_2} \nabla u\nabla\varphi u_t\,dx\right| \leqslant O\big(M^2\big) + \frac{1}{2}\int_{B_2} \varphi u_t^2\,dx$$

and integrating on $[-1/2, t_0]$ we have

$$\left|\int_{-1/2}^{t_0}\int_{B_2} \nabla u\nabla\varphi u_t\,dx\,dt\right| \leqslant O\big(M^2\big) + \frac{1}{2}\int_{-1/2}^{t_0}\int_{B_2} \varphi u_t^2\,dx\,dt.$$

This combined with (A.7) gives

$$\int_{-1/2}^{t_0}\int_{B_2} \varphi u_t^2\,dx\,dt \leqslant CM^2.$$

We may further restrict φ such that $\varphi \geqslant 1$ in the ball $|x| \leqslant 1/2$ and deduce

$$\int_{Q_{1/2}} u_t^2\,dx\,dt \leqslant CM^2. \tag{A.8}$$

Let $t_1, t_2 \in [-1/4, 1/4]$, with $t_1 \leqslant t_2$. Let $x \in \mathbb{R}^N$ with $|x| \leqslant 1$. Then

$$u(x,t_2) - u(x,t_1) = \int_{t_1}^{t_2} u_t(x,t)\,dt.$$

Now integrate this with respect to x in the ball of center x_0, $|x_0| \leqslant 1/4$ and radius $r = (t_2 - t_1)^{1/(2N)}$:

$$\int_{B(x_0,r)} \big(u(x,t_2) - u(x,t_1)\big)\,dx = \int_{t_1}^{t_2}\int_{B(x_0,r)} u_t(x,t)\,dx\,dt.$$

By the mean value theorem there is some $\bar{x} \in B(x_0, r)$ such that

$$u(\bar{x}, t_2) - u(\bar{x}, t_1) = \frac{C}{r^N} \int_{B(x_0,r)} \big(u(x, t_2) - u(x, t_1)\big)\, dx$$

and therefore, using (A.8)

$$\begin{aligned}\big|u(\bar{x}, t_2) - u(\bar{x}, t_1)\big| &\leqslant \frac{C}{r^N} \int_{t_1}^{t_2} \int_{B(x_0,r)} \big|u_t(x, t)\big|\, dx\, dt \\ &\leqslant \frac{C(t_2 - t_1)^{1/2}}{r^{N/2}} \left(\int_{t_1}^{t_2} \int_{B(x_0,r)} u_t(x, t)^2\, dx\, dt \right)^{1/2} \\ &\leqslant CM(t_2 - t_1)^{1/4}.\end{aligned}$$

Since (A.5) holds we deduce

$$\big|u(x_0, t_2) - u(x_0, t_1)\big| \leqslant CM(t_2 - t_1)^{1/(2N)}. \qquad \square$$

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