

Equilibrium states for non-uniformly expanding maps: Decay of correlations and strong stability

A. Castro, P. Varandas*

Departamento de Matemática, Universidade Federal da Bahia, Av. Ademar de Barros s/n, 40170-110 Salvador, Brazil

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Abstract

We study the rate of decay of correlations for equilibrium states associated to a robust class of non-uniformly expanding maps where no Markov assumption is required. We show that the Ruelle–Perron–Frobenius operator acting on the space of Hölder continuous observables has a spectral gap and deduce the exponential decay of correlations and the central limit theorem. In particular, we obtain an alternative proof for the existence and uniqueness of the equilibrium states and we prove that the topological pressure varies continuously. Finally, we use the spectral properties of the transfer operators in space of differentiable observables to obtain strong stability results under deterministic and random perturbations.

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1. Introduction

The thermodynamical formalism was brought from statistical mechanics to dynamical systems by the pioneering works of Sinai, Ruelle and Bowen [39,11,12] in the mid seventies. Indeed, the correspondence between one-dimensional lattices and uniformly hyperbolic maps, via Markov partitions, allowed to translate and introduce several notions of Gibbs measures and equilibrium states in the realm of dynamical systems. Nevertheless, although uniformly hyperbolic dynamics arise in physical systems (see e.g. [25]) they do not include some relevant classes of systems including the Manneville–Pomeau transformation (phenomena of intermittency), Hénon maps and billiards with convex scatterers. We note that all the previous systems present some non-uniformly hyperbolic behavior and its relevant measure satisfies some weak Gibbs property. Moreover, an extension of the thermodynamical formalism beyond the scope of uniform hyperbolicity reveals fundamental difficulties. Even in the non-uniformly hyperbolic context, where there are no zero Lyapunov exponents and there exists a non-uniform geometric theory of invariant manifolds, the absence of finite generating Markov partitions constitutes an obstruction to use the same strategy pushed forward before. Nevertheless, more recently there have been established many evidences that non-uniformly hyperbolic dynamical systems admit countable and generating Markov partitions. This is now parallel to the development of a thermodynamical formalism of gases with infinitely many states, a hard subject not yet completely understood. We refer the reader to [16,31,32] for recent progress in this direction.

* Corresponding author.

E-mail addresses: armando@impa.br (A. Castro), paulo.varandas@ufba.br (P. Varandas).

So, despite the effort of many authors, a general picture is still far from complete. Some of the recent contributions concerning the existence and uniqueness of equilibrium states in a context of non-uniform hyperbolicity include [21, 13, 38, 15, 16, 43, 26, 14, 33, 29, 30, 40, 37, 17, 37, 41, 34]. Many of these papers deal with dynamical systems with neutral periodic points, unimodal maps, perturbations of hyperbolic transformations and shifts with countable many symbols, some of the relevant sources of examples of non-hyperbolic systems. However, a deep study on the statistical properties of the equilibrium states, as the mixing properties, limit theorems, strong stability under deterministic and random perturbations or regularity of the topological pressure is usually obtained as a consequence of the spectral properties of the Ruelle–Perron–Frobenius operator. This functional analytic approach has gained special interest in the last few years and produced new and interesting results even in the uniformly hyperbolic setting (see e.g. [9, 23, 8]). Just for completeness let us mention that, since the (semi)conjugacy between uniformly hyperbolic dynamical systems and the symbolic dynamics is only Hölder continuous, the strategy developed in the seventies did not allowed to understand the statistical properties in the space of smooth observables. Important and recent extensions of this functional analytic approach to the setting of non-uniform hyperbolicity include e.g. the works [44, 28, 18–20, 4, 5, 36].

In this article we study the strong statistical properties of some equilibrium states built in [41] for a large class of non-uniformly expanding local homeomorphisms that may not admit a Markov partition. Using a characterization of equilibrium states as weak Gibbs measures absolutely continuous with respect to conformal reference measures, the authors proved roughly that every local homeomorphism with coexistence of expanding and contraction exhibit a form of average expansion. This enables to use Birkhoff’s method of projective cones applied to the Ruelle–Perron–Frobenius operator acting on suitable Banach spaces to obtain the existence of a unique equilibrium state for any Hölder continuous potential with low variation and that it satisfies strong statistical properties. Natural examples are obtained by bifurcation of expanding homeomorphisms and subshifts of finite type and allows intermittency phenomena. Even in the absence of Markov partition we establish that the Ruelle–Perron–Frobenius transfer operator has a spectral gap in the Banach spaces of both Hölder continuous and smooth observables. This was inspired and extends the work of Matheus and Arbieto [1] that considered local diffeomorphisms under some slightly different assumptions but where the existence of a finite Markov partition played an important role. In consequence, we get an alternative proof for the existence and uniqueness of equilibrium states in [41], obtain exponential decay of correlations and prove a central limit theorem. Moreover, we prove that in this non-uniformly expanding setting the topological pressure varies continuously with respect to the dynamics and the potential.

At this point one could think the stability of the equilibrium states under deterministic and random perturbations could follow directly from the spectral gap property. We refer the reader to [24] for perturbation theory of smooth families of quasi-compact operators. However this is not the case since the transfer operators acting on the space of Hölder continuous potentials may not vary continuously on the dynamical system as illustrated in Example 4.14. Nevertheless we prove that the densities of the equilibrium states with respect to the conformal measures are Hölder continuous and vary uniformly with the dynamics. Strong statistical and stochastic stability results hold in the space of differentiable observables and are proved after careful analysis of the action of the transfer operators in those functional spaces. We obtain a spectral stability under random perturbations. Namely, the spectral components of the Ruelle–Perron–Frobenius operator associated to general random perturbations of the transformation and the potential varies continuously and converges to the spectral components of Ruelle–Perron–Frobenius of the unperturbed dynamical system outside of a disk containing zero in the spectrum.

Finally, let us also mention that the program to understand to statistical and stochastic properties of the equilibrium states for this class of multidimensional non-uniformly expanding transformations is under way. Some of the very interesting remaining questions are to understand if one can obtain further regularity of the topological pressure and the density of the equilibrium states with respect to conformal measures along parametrized families of potentials (e.g. real analytic) and the study of zeta functions. Such program has been carried out with success for uniformly hyperbolic and some partially hyperbolic and one-dimensional non-uniformly expanding dynamical systems. See e.g. [2, 35, 22, 6, 7] and the references therein. Just to mention some recent developments, in a joint work with T. Bomfim [10], we prove the differentiability of thermodynamical quantities as topological pressure, invariant densities, conformal measures and measures of maximal entropy despite the lack of continuity of the Ruelle–Perron–Frobenius operator with respect to the dynamics.

This paper is organized as follows. In Section 2, we recall some definitions and make the precise statements of our main results and some preliminary results are given in Section 3. The proof of the spectral gap for the Ruelle–Perron–Frobenius operator in the space of Hölder continuous observables, continuity of the topological pressure,

uniform continuity of the densities of equilibrium states with respect to conformal measures and exponential decay of correlations are given in Section 4. In Section 5 we show that Ruelle–Perron–Frobenius operator acting on the space of smooth observables also admits a spectral gap and obtain the strong stability of the equilibrium states under deterministic and random perturbations. Finally, some examples are given in Section 6.

2. Statement of the main results

2.1. Setting

Let M be compact and connected Riemannian manifold of dimension m with distance d . Let $f : M \rightarrow M$ be a local homeomorphism and assume that there exists a continuous function $x \mapsto L(x)$ such that, for every $x \in M$ there is a neighborhood U_x of x so that $f_x : U_x \rightarrow f(U_x)$ is invertible and

$$d(f_x^{-1}(y), f_x^{-1}(z)) \leq L(x)d(y, z), \quad \forall y, z \in f(U_x).$$

In particular every point has the same finite number of preimages $\deg(f)$ which coincides with the degree of f . For all our results we assume that f and ϕ satisfy conditions (H1), (H2), and (P) stated below. Assume there exist constants $\sigma > 1$ and $L \geq 1$, and an open region $\mathcal{A} \subset M$ such that

- (H1) $L(x) \leq L$ for every $x \in \mathcal{A}$ and $L(x) < \sigma^{-1}$ for all $x \notin \mathcal{A}$, and L is close to 1: the precise condition is given in (3.1) and (3.2).
- (H2) There exists a finite covering \mathcal{U} of M by open domains of injectivity for f such that \mathcal{A} can be covered by $q < \deg(f)$.

The first condition means that we allow expanding and contracting behavior to coexist in M : f is uniformly expanding outside \mathcal{A} and not too contracting inside \mathcal{A} . In the case that \mathcal{A} is empty then f is uniformly expanding. The second one requires that every point has at least one preimage in the expanding region. An observable $g : M \rightarrow \mathbb{R}$ is α -Hölder continuous if the Hölder constant

$$|g|_\alpha = \sup_{x \neq y} \frac{|g(x) - g(y)|}{d(x, y)^\alpha}$$

is finite. As usual, we endow the space $C^\alpha(M, \mathbb{R})$ of Hölder continuous observables with the norm $\|\cdot\|_\alpha = \|\cdot\|_0 + |\cdot|_\alpha$. We assume that the potential $\phi : M \rightarrow \mathbb{R}$ is Hölder continuous and that

$$(P) \quad \sup \phi - \inf \phi < \varepsilon_\phi \text{ and } |e^\phi|_\alpha < \varepsilon_\phi e^{\inf \phi}$$

for some $\varepsilon_\phi > 0$ satisfying Eq. (4.1), depending on the constants L, σ, q and $\deg(f)$. The previous is an open condition on the potential, relative to the Hölder norm, and it is satisfied e.g. by constant functions. In particular we consider measures of maximal entropy. The second condition above means that $\exp(\phi)$ is contained in a small cone of Hölder continuous as discussed after Theorem 4.1.

2.2. Existence and uniqueness of equilibrium states

Let us first recall some necessary definitions. Given a continuous map $f : M \rightarrow M$ and a potential $\phi : M \rightarrow \mathbb{R}$, the variational principle for the pressure asserts that

$$P_{\text{top}}(f, \phi) = \sup \left\{ h_\mu(f) + \int \phi d\mu : \mu \text{ is } f\text{-invariant} \right\}$$

where $P_{\text{top}}(f, \phi)$ denotes the topological pressure of f with respect to ϕ and $h_\mu(f)$ denotes the metric entropy. An equilibrium state for f with respect to ϕ is an invariant measure that attains the supremum in the right-hand side above.

The equilibrium states constructed in [41] are absolutely continuous with respect to an expanding, conformal and non-lacunary Gibbs measure ν . Let us recall these definitions and the notions involved. A probability measure ν , not

necessarily invariant, is *conformal* if there exists a function $\psi : M \rightarrow \mathbb{R}$ such that $\nu(f(A)) = \int_A e^{-\psi} d\nu$ for every measurable set A such that $f|_A$ is injective. Let $S_n\phi = \sum_{j=0}^{n-1} \phi \circ f^j$ denote the n th Birkhoff sum of a function ϕ . The *basin of attraction* of an f -invariant, ergodic probability measure μ is the set $B(\mu)$ of points $x \in M$ such that the probability measures $\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}$ converge weakly to μ when $n \rightarrow \infty$. We build over the following theorem which is a direct consequence of the results in [41].

Theorem 2.1. *Let $f : M \rightarrow M$ be a local homeomorphism with Lipschitz continuous inverse satisfying (H1), (H2) and $\phi : M \rightarrow \mathbb{R}$ a Hölder continuous potential such that $\sup \phi - \inf \phi < \log \deg(f) - \log q$. Then, there exists a finite number of ergodic equilibrium states $\mu_1, \mu_2, \dots, \mu_k$ for f with respect to ϕ , and they are absolutely continuous with respect to some conformal expanding measure ν . Moreover, the union of the basins of attraction $B(\mu_i)$ contains ν -almost every point.*

Observe that despite the characterization that equilibrium states are absolutely continuous invariant measures no information was known e.g. on the continuity of the topological pressure and density functions. Here we shall address these questions, the uniqueness of the equilibrium states and also the strong stability of the equilibrium states. Since our assumption (P) implies that the potential ϕ has small variation then it fits in the assumption of the previous theorem. We will build over the aforementioned result with a completely different functional analytic approach.

2.3. Statement of the main results

In this section we recall some necessary definitions and state our main results. The Ruelle–Perron–Frobenius transfer operator \mathcal{L}_ϕ associated to $f : M \rightarrow M$ and $\phi : M \rightarrow \mathbb{R}$ is the linear operator defined on a Banach space $X \subset C^0(M, \mathbb{R})$ of continuous functions $\varphi : M \rightarrow \mathbb{R}$ by

$$\mathcal{L}_\phi \varphi(x) = \sum_{f(y)=x} e^{\phi(y)} \varphi(y).$$

Since f is a local homeomorphism it is clear that $\mathcal{L}_\phi \varphi$ is continuous for every continuous φ and, furthermore, \mathcal{L}_ϕ is indeed a bounded operator relative to the norm of uniform convergence in $C^0(M, \mathbb{R})$ because $\|\mathcal{L}_\phi\| \leq \deg(f) e^{\sup |\phi|}$. Analogously, \mathcal{L}_ϕ preserves the Banach space $C^\alpha(M, \mathbb{R})$, $0 < \alpha < 1$ of Hölder continuous observables. Moreover, it is not hard to check that \mathcal{L}_ϕ is a bounded linear operator in the Banach space $C^r(M, \mathbb{R}) \subset C^0(M, \mathbb{R})$ ($r \geq 1$) endowed with the norm $\|\cdot\|_r$ whenever f is a C^r -local diffeomorphism and $\phi \in C^r(M, \mathbb{R})$. We say that the Ruelle–Perron–Frobenius operator \mathcal{L}_ϕ acting on a Banach space X has the *spectral gap property* if there exists a decomposition of its spectrum $\sigma(\mathcal{L}_\phi) \subset \mathbb{C}$ as follows: $\sigma(\mathcal{L}_\phi) = \{\lambda_1\} \cup \Sigma_1$ where λ_1 is a leading eigenvalue for \mathcal{L}_ϕ with one-dimensional associated eigenspace and $\Sigma_1 \subsetneq \{z \in \mathbb{C} : |z| < \lambda_1\}$.

The first result is a spectral gap for the Ruelle–Perron–Frobenius operator in the space of Hölder continuous observables, which is enough to derive the uniqueness and further regularity of the density of the equilibrium state with respect to the conformal measure.

Theorem A. *Let $f : M \rightarrow M$ be a local homeomorphism with Lipschitz continuous inverse and $\phi : M \rightarrow \mathbb{R}$ be a Hölder continuous potential satisfying (H1), (H2) and (P). Then the Ruelle–Perron–Frobenius has a spectral gap property in the space of Hölder continuous observables, there exists a unique equilibrium state μ for f with respect to ϕ and the density $d\mu/d\nu$ is Hölder continuous.*

Let us mention that the previous result holds for more general compact invariant subsets $K \subset M$ (with the induced topology) also under the assumption that every point has constant number of preimages in K and at least one preimage in the expanding region, as considered in [41]. Since we will be interested in further extensions to differentiable dynamics as discussed below we will not prove or use this fact here. Let us give two important consequences of the previous result.

Corollary 1. *The equilibrium state μ has exponential decay of correlations for Hölder continuous observables: there exists some constant $0 < \tau < 1$ such that for all $\varphi \in L^1(\nu)$, $\psi \in C^\alpha(M)$ there exists $K(\varphi, \psi) > 0$ satisfying*

$$\left| \int_M (\varphi \circ f^n) \psi \, d\mu - \int_M \varphi \, d\mu \int_M \psi \, d\mu \right| \leq K(\varphi, \psi) \cdot \tau^n, \quad \text{for every } n \geq 1.$$

As a byproduct of the previous theorem we also obtain a Central Limit Theorem.

Corollary 2. *Let φ be a Hölder continuous function and set*

$$\sigma_\varphi^2 := \int v^2 \, d\mu + 2 \sum_{j=1}^\infty v \cdot (v \circ f^j) \, d\mu, \quad \text{where } v = \varphi - \int \varphi \, d\mu.$$

Then $\sigma_\varphi < \infty$ and $\sigma_\varphi = 0$ iff $\varphi = u \circ f - u$ for some $u \in L^1(\mu)$. Furthermore, if $\sigma_\varphi > 0$ then the following convergence on distribution

$$\mu \left(x \in M: \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left(\varphi(f^j(x)) - \int \varphi \, d\mu \right) \in A \right) \rightarrow \frac{1}{\sigma_\varphi \sqrt{2\pi}} \int_A e^{-\frac{t^2}{2\sigma_\varphi^2}} \, dt,$$

holds as $n \rightarrow \infty$ for every interval $A \subset \mathbb{R}$.

The stability of the equilibrium state under deterministic perturbations is more subtle. In fact, the Ruelle–Perron–Frobenius operator $\mathcal{L}_{f,\phi}$ acting on the space of Hölder continuous observables is continuous on the potential ϕ but in general it may not vary continuously with the underlying dynamics f , as shown in Example 4.14. Nevertheless we could obtain further that the Hölder continuous densities of the equilibrium states with respect to the conformal measures vary continuously with the dynamics in the C^0 -topology and that the topological pressure varies continuously, which gives a nontrivial extension of the weak* stability results in [41].

Theorem B. *Let \mathcal{F} be a family of local homeomorphisms with Lipschitz inverse and let \mathcal{W} be some family of Hölder continuous potentials satisfying (H1), (H2) and (P) with uniform constants. Then the topological pressure function $\mathcal{F} \times \mathcal{W} \ni (f, \phi) \rightarrow P_{\text{top}}(f, \phi)$ is continuous. Moreover, the invariant density function*

$$\begin{aligned} \mathcal{F} \times \mathcal{W} &\rightarrow C^\alpha(M, \mathbb{R}), \\ (f, \phi) &\mapsto \frac{d\mu_{f,\phi}}{d\nu_{f,\phi}} \end{aligned}$$

is continuous whenever $C^\alpha(M, \mathbb{R})$ is endowed with the C^0 -topology.

2.3.1. Stronger stability results

Now we pay attention to the stability of the equilibrium states under both deterministic and an arbitrary random perturbations. To obtain stronger statistical stability results we will admit that the dynamics is C^r -differentiable ($r \geq 1$) and give a detailed study of the spectral properties for the Ruelle–Perron–Frobenius operator acting on the space $C^r(M, \mathbb{R})$. Associated to $\phi \in C^r(M, \mathbb{R})$ consider the condition:

$$(P') \quad \sup \phi - \inf \phi < \varepsilon_\phi \quad \text{and} \quad \max_{s \leq r} \|D^s \phi\|_0 < \varepsilon'_\phi$$

for some $\varepsilon'_\phi > 0$ expressed precisely in Eq. (4.2) and depending on $L, \sigma, q, \deg(f), \varepsilon_\phi$ and r . This is an open condition on the set of potentials, satisfied by constant potentials, and a natural generalization of condition (P) to the differentiable setting.

Theorem C. *Given an integer $r \geq 1$, let \mathcal{F}^r be a family of C^r -local diffeomorphisms and let \mathcal{W}^r be a family of C^r -potentials satisfying (H1), (H2) and (P') with uniform constants. Then the topological pressure $\mathcal{F}^r \times \mathcal{W}^r \ni (f, \phi) \rightarrow P_{\text{top}}(f, \phi)$ and the invariant density*

$$\begin{aligned} \mathcal{F}^r \times \mathcal{W}^r &\rightarrow C^r(M, \mathbb{R}), \\ (f, \phi) &\mapsto \frac{d\mu_{f,\phi}}{dv_{f,\phi}} \end{aligned}$$

vary continuously in the C^r -topology. Moreover, the conformal measure function

$$\begin{aligned} \mathcal{F}^r \times \mathcal{W}^r &\rightarrow \mathcal{M}(M), \\ (f, \phi) &\mapsto \nu_{f,\phi} \end{aligned}$$

is continuous in the weak* topology. In consequence, the equilibrium measure $\mu_{f,\phi}$ varies continuously in the weak* topology.

Finally we will describe our results on the stability of the spectra of the Ruelle–Perron–Frobenius operator under random perturbations. Given $r \in \mathbb{N}$, and families \mathcal{F}^r of local diffeomorphisms and \mathcal{W}^r of C^r -observables satisfying (H1), (H2) and (P') with uniform constants, a *random perturbation* of $f \in \mathcal{F}$ is a family θ_ε , $0 < \varepsilon \leq 1$ of probability measures in $\mathcal{F}^r \times \mathcal{W}^r$ such that there exists a family $V_\varepsilon(f, \phi)$, $0 < \varepsilon \leq 1$ of neighborhoods of (f, ϕ) , depending monotonically on ε and satisfying

$$\text{supp } \theta_\varepsilon \subset V_\varepsilon(f, \phi) \quad \text{and} \quad \bigcap_{0 < \varepsilon \leq 1} V_\varepsilon(f, \phi) = \{(f, \phi)\}.$$

This dynamics can be codified by considering the skew product map

$$\begin{aligned} F : \mathcal{F}^{\mathbb{N}} \times M &\rightarrow \mathcal{F}^{\mathbb{N}} \times M, \\ (\underline{f}, x) &\mapsto (\sigma(\underline{f}), f_1(x)) \end{aligned}$$

where $\underline{f} = (f_1, f_2, \dots)$ and $\sigma : \mathcal{F}^{\mathbb{N}} \rightarrow \mathcal{F}^{\mathbb{N}}$ is the shift to the left. Associated to this random dynamical system consider the *integrated Ruelle–Perron–Frobenius operator* \mathcal{L}_ε given by

$$\mathcal{L}_\varepsilon \varphi(x) = \int (\mathcal{L}_{f,\phi} \varphi)(x) d\theta_\varepsilon(f). \tag{2.1}$$

We say that (f, ϕ) has C^r -spectral stability under the random perturbation if the operator \mathcal{L}_ε in the Banach space $C^r(M, \mathbb{R})$ has the spectral gap property and the leading eigenvalue λ_ε and associated eigenfunction h_ε vary continuously with ε and accumulate, as $\varepsilon \rightarrow 0$, respectively on the leading eigenvalue and eigenfunction of the unperturbed operator. We prove the following spectral stability under random perturbations.

Theorem D. *Let $(\theta_\varepsilon)_\varepsilon$ be any random perturbation of $(f, \phi) \in \mathcal{F}^r \times \mathcal{W}^r$. Then (f, ϕ) has C^r -spectral stability under the random perturbation $(\theta_\varepsilon)_\varepsilon$.*

Some comments are in order. Weaker stochastic stability results were previously obtained in [41] under a non-degeneracy assumption. Namely, assuming that all $f \in \mathcal{F}$ are non-singular with respect to a fixed conformal measure it follows that there are stationary measures μ^ε absolutely continuous with respect to the conformal measure ν and that converge to the equilibrium state μ in the weak* topology as the noise level ε tends to zero. Here we obtain spectral stability under arbitrary random perturbations.

3. Preliminaries

In this section we provide some preparatory results needed for the proof of the main results. Namely, we study the combinatorics of the orbits, hyperbolic times and some pressure estimates.

3.1. Combinatorial estimates for orbits

Here we give a description of the orbits of points according to the visit to the possibly not expanding region \mathcal{A} using an auxiliary partition \mathcal{P} built using (H2).

Lemma 3.1. *There exists a partition \mathcal{P} of M of domains of injectivity for f with cardinality at most $\#\mathcal{U}$ and such that $\bigcup\{U \in \mathcal{U}: U \cap \mathcal{A} \neq \emptyset\} = \bigcup\{P \in \mathcal{P}: P \cap \mathcal{A} \neq \emptyset\}$. In particular there are at most $q < \deg(f)$ elements of \mathcal{P} that cover \mathcal{A} .*

Proof. Pick an enumeration $\{U_i\}$ of the open covering \mathcal{U} given by (H2) in such a way that the region \mathcal{A} is covered by the first q elements of \mathcal{U} . Consider the partition \mathcal{P} given by $P_1 = U_1$ and, recursively, $P_{i+1} = U_{i+1} \setminus (\bigcup_{j=1}^i P_j)$ for $i = 1 \dots \#\mathcal{U} - 1$. It is clear that $\#\mathcal{P} \leq \#\mathcal{U}$. Moreover, $f|_{P_i}$ is injective for every nonempty P_i since by construction $P_i \subset U_i$ and

$$\bigcup\{U \in \mathcal{U}: U \cap \mathcal{A} \neq \emptyset\} = \bigcup_{j=1}^q U_j = \bigcup_{j=1}^q P_j = \bigcup\{P \in \mathcal{P}: P \cap \mathcal{A} \neq \emptyset\}.$$

Since the last statement in the lemma is immediate from the construction this finishes the proof of the lemma. \square

Since the region \mathcal{A} is contained in q elements of the partition \mathcal{P} we can assume without any loss of generality that \mathcal{A} is contained in the first q elements of \mathcal{P} . For all x we can associate an *itinerary* $i(x) \in (i_0, \dots, i_{n-1}) \in \{1, \dots, \#\mathcal{P}\}^n$ by $i_j = \ell$ if and only if $f^j(x) \in P_\ell$. Given $\gamma \in (0, 1)$ and $n \geq 1$, let us consider also the set $I(\gamma, n)$ of all itineraries (i_0, \dots, i_{n-1}) so that $\#\{0 \leq j \leq n - 1: i_j \leq q\} > \gamma n$.

Lemma 3.2. *Given $\varepsilon > 0$ there exists $\gamma_0 \in (0, 1)$ such that*

$$c_\gamma := \limsup_{n \rightarrow \infty} \frac{1}{n} \log \#I(\gamma, n) < \log q + \varepsilon$$

for every $\gamma \in (\gamma_0, 1)$.

Proof. See [41, Lemma 3.1]. \square

We are in a position to state our precise condition on the constant L in assumption (H1) and the constant c in the definition of hyperbolic times. First note that if $\sup \phi - \inf \phi < \log \deg(f) - \log q$ as in Theorem 2.1 then it follows from Lemma 3.2 that one may find $\gamma < 1$ such that $c_\gamma < \log \deg(f) - \sup \phi + \inf \phi$. We assume that L is close enough to 1, and $c > 0$ and $0 < \varepsilon_\phi < \log \deg(f) - \log q$ are so that

$$\sigma^{-(1-\gamma)} L^\gamma < e^{-2c} < 1 \tag{3.1}$$

and

$$e^{\varepsilon_\phi} \cdot \left(\frac{(\deg(f) - q)\sigma^{-\alpha} + qL^\alpha[1 + (L - 1)^\alpha]}{\deg(f)} \right) < 1. \tag{3.2}$$

The first condition is to guarantee the existence of infinitely many hyperbolic times with respect to the reference measure in the proposition below. The second technical condition roughly means that f has some average backward contraction and will be used to obtain the invariance of a cone of functions under the Ruelle–Perron–Frobenius operator in Proposition 4.1.

3.2. Ruelle–Perron–Frobenius operators and conformal measures

Recall that the Ruelle–Perron–Frobenius transfer operator $\mathcal{L}_\phi : C^0(M, \mathbb{R}) \rightarrow C^0(M, \mathbb{R})$ associated to $f : M \rightarrow M$ and $\phi : M \rightarrow \mathbb{R}$ is the linear operator defined on the space $C^0(M, \mathbb{R})$ of continuous functions $\varphi : M \rightarrow \mathbb{R}$ by

$$\mathcal{L}_\phi \varphi(x) = \sum_{f(y)=x} e^{\phi(y)} \varphi(y).$$

In fact, $\mathcal{L}_\phi \varphi$ is continuous since f is a local homeomorphism and φ is continuous. Moreover, it is not hard to check that \mathcal{L}_ϕ is a bounded operator, relative to the norm of uniform convergence in $C^0(M, \mathbb{R})$ and $\|\mathcal{L}_\phi\| \leq \deg(f)e^{\sup|\phi|}$. Consider also the dual operator $\mathcal{L}_\phi^* : \mathcal{M}(M) \rightarrow \mathcal{M}(M)$ acting on the space $\mathcal{M}(M)$ of Borel measures in M by

$\int \varphi d(\mathcal{L}_\phi^* \eta) = \int (\mathcal{L}_\phi \varphi) d\eta$ for every $\varphi \in C^0(M, \mathbb{R})$. Let $r(\mathcal{L}_\phi)$ be the spectral radius of \mathcal{L}_ϕ . In our context conformal measures associated to the spectral radius always exist as stated in the next proposition, whose proof can be found in the proofs of Theorem B and Theorem 4.1 in [41].

Proposition 3.3. *If f is topologically exact and satisfies (H1), (H2) and ϕ satisfies $\sup \phi - \inf \phi < \log \deg(f) - \log q$ then there exists an expanding conformal measure such that $\mathcal{L}_\phi^* \nu = \lambda \nu$ and $\text{supp}(\nu) = \overline{H}$, where $\lambda = r(\mathcal{L}_\phi) \geq \deg(f) e^{\inf \phi}$. Moreover, ν is a non-lacunary Gibbs measure and has a Jacobian with respect to f given by $J_\nu f = \lambda e^{-\phi}$.*

Just for completeness let us mention that one key ingredient is that our assumptions guarantee we obtain volume expansion with respect to the conformal measure, that is, $J_\nu f(x) \geq \deg(f) e^{\inf \phi - \sup \phi} > e^{c\gamma}$ for all $x \in M$. This is enough to guarantee that ν -almost every point spends at most a fraction γ of time inside the domain \mathcal{A} where f may fail to be expanding. Notice also that $\lambda = \int \mathcal{L}_\phi 1 d\nu$.

Finally, we collect the main estimates concerning the pressure of the invariant sets H and H^c , which play a key role in the construction of equilibrium states.

Proposition 3.4. *$P_{\text{top}}(f, \phi) = P_H(f, \phi) = \log \lambda > P_{H^c}(f, \phi)$, where λ denotes the spectral radius of the Ruelle–Perron–Frobenius \mathcal{L}_ϕ acting on the space of continuous observables. In consequence, any equilibrium state is an expanding measure.*

Proof. See Proposition 6.1, Lemma 6.4 and Lemma 6.5 in [41]. \square

3.3. Regularity of the observables

Here we study a relation between Hölder and locally Hölder continuous functions. We say that $\varphi : M \rightarrow \mathbb{R}$ is (C, α) -Hölder continuous in balls of radius δ if

$$|\varphi(x) - \varphi(y)| \leq C d(x, y)^\alpha$$

for every $y \in B(x, \delta)$ and $x \in M$. Our first auxiliary lemma for the regularity of observables is as follows.

Lemma 3.5. *Given $1 \leq \zeta \leq 2$ and $\delta > 0$, if $\varphi : M \rightarrow \mathbb{R}$ is (C, α) -Hölder continuous in balls of radius δ then it is $(C(1 + r^\alpha), \alpha)$ -Hölder continuous in balls of radius $(1 + r)\delta \leq \zeta \delta$, with $0 < r \leq 1$.*

Proof. Since M is connected then given $y, z \in M$ so that $d(y, z) < (1 + r)\delta$ by considering a geodesic arc connecting y and z in M there exists w so that $d(z, w) = \delta$ and $d(w, y) < rd(z, w) < \delta$. Therefore

$$\begin{aligned} |\varphi(z) - \varphi(y)| &\leq |\varphi(z) - \varphi(w)| + |\varphi(w) - \varphi(y)| \leq C d(z, w)^\alpha + C d(w, y)^\alpha \\ &\leq C(1 + r^\alpha) d(z, w)^\alpha \leq C(1 + r^\alpha) d(z, y)^\alpha, \end{aligned}$$

which proves the lemma. \square

The next lemma asserts that every locally Hölder continuous observable is indeed Hölder continuous. Moreover, we give an estimate for the Hölder constant.

Lemma 3.6. *Let N be a compact and connected metric space. Given $\delta > 0$ there exists $m \geq 1$ (depending only on δ) such that the following holds: if $\varphi : N \rightarrow \mathbb{R}$ is (C, α) -Hölder continuous in balls of radius δ then it is (Cm, α) -Hölder continuous.*

Proof. Fix $\delta > 0$ and let $\mathcal{B} = \{B(x_i, \delta/3)\}_{i=1 \dots s}$ be a finite covering of N . We can assume, without loss of generality, that $x_j \in B(x_{j+1}, \delta)$ for every $j = 1, \dots, s - 1$. Our hypothesis guarantee that if $x, w \in N$ with $d(x, w) < \delta$ we have $|\varphi(x) - \varphi(w)| \leq C d(x, w)^\alpha$. Hence, if $d(x, w) \geq \delta$ then it is not hard to use the triangular inequality to get

$$|\varphi(x) - \varphi(w)| \leq (s + 2)C\delta^\alpha \leq C(s + 2) d(x, w)^\alpha.$$

Thus it is enough to take $m \geq s + 2$ in the lemma. \square

3.4. Positive operators and cones

In this subsection we shall recall some results concerning the theory of projective metrics on cones and positive operators due to G. Birkhoff. Despite the great generality of this theory we shall concentrate on cones and positive operators on Banach spaces. We refer the reader to [27,3] for detailed presentations.

Let \mathcal{B} be a Banach space. A subset $\Lambda \subset \mathcal{B} - \{0\}$ is a *cone* if $r \cdot v \in \Lambda$ for all $v \in \Lambda$ and $r \in \mathbb{R}^+$. The cone Λ is *closed* if $\bar{\Lambda} = \Lambda \cup \{0\}$, and Λ is *convex* if $v + w \in \Lambda$ for all $v, w \in \Lambda$. Notice that a convex cone Λ with $\Lambda \cap (-\Lambda) = \emptyset$ determines a *partial ordering* \preceq on \mathcal{B} given by:

$$w \preceq v \quad \text{iff} \quad v - w \in \Lambda \cup \{0\}.$$

In the sequel, our cones Λ are assumed to be closed, convex and $\Lambda \cap (-\Lambda) = \emptyset$. Given a cone Λ and two vectors $v, w \in \Lambda$, we define $\Theta(v, w) = \Theta_\Lambda(v, w)$ by

$$\Theta(v, w) = \log \frac{B_\Lambda(v, w)}{A_\Lambda(v, w)},$$

where $A_\Lambda(v, w) = \sup\{r \in \mathbb{R}^+ : r \cdot v \preceq w\}$ and $B_\Lambda(v, w) = \inf\{r \in \mathbb{R}^+ : w \preceq r \cdot v\}$. The (pseudo-)metric Θ is called the *projective metric* of Λ (or Λ -*metric* for brevity). Defining the equivalence relation $v \sim w$ iff $w = r \cdot v$ for some $r \in \mathbb{R}^+$, then Θ induces a metric on the quotient Λ / \sim . The following key result is due to Birkhoff, which can be found e.g. in [42, Proposition 2.3].

Theorem 3.7. *Let Λ_i be a closed convex cone (with $\Lambda_i \cap (-\Lambda_i) = \emptyset$) in a Banach space \mathcal{B}_i , for $i = 1, 2$. If $\mathcal{L} : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is a linear operator such that $\mathcal{L}(\Lambda_1) \subset \Lambda_2$ and $\Delta = \text{diam}_{\Theta_{\Lambda_2}}(\mathcal{L}\Lambda_1) < \infty$ then*

$$\Theta_{\Lambda_2}(\mathcal{L}v, \mathcal{L}w) \leq (1 - e^{-\Delta}) \cdot \Theta_{\Lambda_1}(v, w),$$

for any $v, w \in \Lambda_1$.

In consequence of the previous theorem, if the diameter of the cone $\mathcal{L}(\Lambda_1)$ is finite in Λ_2 then \mathcal{L} is a contraction in the projective metric which enables us to prove that it admits a unique fixed point.

3.5. Combinatorial lemma on preimages matching

Here we establish an auxiliary lemma to bound for the distance of preimages associated to different functions in \mathcal{F} which will play a key role in the proof of the stability results. Let $V_\varepsilon(f) \subset \mathcal{F}$ be an open neighborhood of $f \in \mathcal{F}$.

Lemma 3.8. *Given $n \geq 1$, $\underline{f}, \underline{g} \in \mathcal{F}^{\mathbb{N}}$ and $x, y \in M$ there exists bijection between the sets of preimages $\{z \in M : \underline{f}^n(z) = x\}$ and $\{z \in M : \underline{g}^n(z) = y\}$. Moreover, for every $n \in \mathbb{N}$ there exists $\varepsilon(n) > 0$ such that for every $0 < \varepsilon \leq \varepsilon(n)$ the distance between paired n -preimages is such that if $d(x, y) < \varepsilon$ and $\underline{g} \in V_\varepsilon(\underline{f})$ then*

$$d(x_i^{(n)}, y_i^{(n)}) \leq L^n d(x, y) + \sum_{j=1}^n L^{n-j+1} \|f_j - g_j\|_\alpha,$$

for every $i = 1 \dots \text{deg}(f)^n$.

Proof. Let $\hat{\mathcal{U}}$ be a finite open cover by balls obtained using domains of invertibility for f and let $2\hat{\delta}$ be the Lebesgue number of the covering $\hat{\mathcal{U}}$. If $\varepsilon > 0$ is small enough the constant $2\hat{\delta}$ can be taken uniform for every $\tilde{f} \in V_\varepsilon(f)$. Let $x, y \in M$ satisfy $d(x, y) < \varepsilon$ and take $\underline{f} = (f_i)_{i \in \mathbb{N}}$ and $\underline{g} = (g_i)_{i \in \mathbb{N}}$ with $\underline{g} \in V_\varepsilon(\underline{f})$. We will prove the result recursively.

First notice that the sets $\{z \in M : f_1(z) = x\}$ and $\{z \in M : g_1(z) = y\}$ have the same cardinality $\text{deg}(f)$ and thus there exists a one-to-one correspondence. Moreover, reducing $\varepsilon > 0$ if necessary, we obtain that the paired enumerations $\{x_i\}$ and $\{y_i\}$ of such elements verify

$$\begin{aligned} d(g_1(y_i), g_1(x_i)) &= d(y, g_1(x_i)) \leq d(y, x) + d(x, g_1(x_i)) \\ &= d(y, x) + d(f_1(x_i), g_1(x_i)) \leq d(y, x) + \|f_1 - g_1\|_\alpha < \hat{\delta}. \end{aligned}$$

Since $g_1 \in \mathcal{F}$ then it satisfies (H1), (H2) and so $d(x_i, y_i) \leq L[d(y, x) + \|f_1 - g_1\|_\alpha]$ for every i . The same argument as above applied to the pairs $x_i = f_2(x_j^{(2)})$ and $y_i = g_2(y_j^{(2)})$ proves that $d(g_2(y_j^{(2)}), g_2(x_j^{(2)})) \leq d(x_i, y_i) + \|f_2 - g_2\|_\alpha$ and, consequently,

$$\begin{aligned} d(y_j^{(2)}, x_j^{(2)}) &\leq L[d(x_i, y_i) + \|f_2 - g_2\|_\alpha] \\ &\leq L^2[d(x, y) + \|f_1 - g_1\|_\alpha] + L\|f_2 - g_2\|_\alpha \\ &= L^2d(x, y) + L^2\|f_1 - g_1\|_\alpha + L\|f_2 - g_2\|_\alpha, \end{aligned}$$

which can be taken also smaller than $\hat{\delta}$ provided that we reduce ε . Using the same reasoning recursively, if $d(x, y) < \varepsilon(n)$ small so that the corresponding paired enumerations of preimages $(x_i^{(k)})$ and $(y_i^{(k)})$, $i = 1 \dots \deg(f)^k$ in the sets $\{z \in M: \underline{f}^k(z) = x\}$ and $\{z \in M: \underline{g}^k(z) = y\}$ are $\hat{\delta}$ -close for every $1 \leq k \leq n - 1$. Moreover, applying the previous reasoning it follows that

$$d(x_i^{(n)}, y_i^{(n)}) \leq L^n d(x, y) + \sum_{j=1}^n L^{n-j+1} \|f_j - g_j\|_\alpha$$

as claimed. This finishes the proof of the lemma. \square

We also get a simple expression for the distance of n -preimages associated to the same close functions in \mathcal{F} and the same base point in M .

Corollary 3.9. *Given $n \in \mathbb{N}$ there exists $\varepsilon(n) > 0$ such that for any $f_1, f_2 \in \mathcal{F}$ with $\|f_1 - f_2\|_\alpha < \varepsilon(n)$ the following property holds: given $x \in M$ and paired preimages $(x_{1i}^{(n)})$ and $(x_{2i}^{(n)})$ by f_1^n and f_2^n , respectively, then*

$$d(x_{1i}^{(n)}, x_{2i}^{(n)}) \leq nL^n \|f_1 - f_2\|_\alpha \quad \text{for all } i.$$

Proof. This is a direct consequence of the previous lemma, by considering $x = y$ and the sequences of functions $\underline{f} = (f_1, f_1, f_1, \dots)$ and $\underline{g} = (f_2, f_2, f_2, \dots)$ in $V_\varepsilon(f)^\mathbb{N}$ with ε small. \square

We finish this section by proving that paired preimages associated to any close points have similar behavior with respect to the region \mathcal{A} . More precisely,

Lemma 3.10. *Let f satisfy assumptions (H1) and (H2). Then there exists $\delta > 0$ so that for every ball B of radius δ has at most $q < \deg(f)$ connected components in $f^{-1}(B)$ that intersect \mathcal{A} . In particular, if $d(x, y) < \delta$ then there are at most q pairs of paired preimages by f associated to x and y that belong to \mathcal{A} .*

Proof. Assume that $\delta_0 > 0$ is small so that every inverse branch is well defined in a ball of radius δ_0 . Since $\sharp(\{f^{-1}(x)\} \cap \mathcal{A}) \leq q$ then for every $x \in M$ there exists $0 < \delta_x < \delta_0$ so that $f^{-1}(B(x, \delta_x))$ has at most q connected components that intersect \mathcal{A} . By compactness of M pick a finite subcover $\mathcal{B} = \{B(x_i, \delta_i)\}_{i \in I}$, set 2δ to be Lebesgue number of \mathcal{B} and assume, without loss of generality, that $\delta < \delta_0$. Therefore, by construction, given any ball B of radius δ it follows that $B \subset B(x_i, \delta_i)$ for some $i \in I$. In consequence, the number of connected components satisfy

$$\sharp c.c.(f^{-1}(B) \cap \mathcal{A}) \leq \sharp c.c.(f^{-1}(B(x_i, \delta_i)) \cap \mathcal{A}) \leq q.$$

This finishes the proof of the lemma. \square

4. Ruelle–Perron–Frobenius operator in $C^\alpha(M, \mathbb{R})$: Spectral gap and statistical consequences

In this section we prove that the action of the transfer operator in the space of Hölder continuous observables has the spectral gap property. In consequence, we provide an alternative proof for the existence and uniqueness of equilibrium states as well as further statistical properties: exponential decay of correlations and central limit theorem. We also get that the densities of the unique equilibrium state with respect to the conformal measures are Hölder and vary continuously in a uniform way with the dynamical system. Finally, the topological pressure also varies continuously in this non-uniformly expanding setting.

4.1. Invariant cones for the transfer operator in $C^\alpha(M, \mathbb{R})$

To prove that the Ruelle–Perron–Frobenius operator has a spectral gap in the space of Hölder continuous observables one first introduce some notations. Recall that the Hölder constant of $\varphi \in C^\alpha(M, \mathbb{R})$ is

$$|\varphi|_\alpha = \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{d(x, y)^\alpha}$$

and set $|\varphi|_{\alpha, \delta}$ as the least constant $C > 0$ such that $|\varphi(x) - \varphi(y)| \leq Cd(x, y)^\alpha$ for all points x, y such that $d(x, y) < \delta$. Now, consider the cone of locally Hölder continuous observables

$$A_{\kappa, \delta} = \left\{ \varphi \in C^0(M, \mathbb{R}) : \varphi > 0 \text{ and } \frac{|\varphi|_{\alpha, \delta}}{\inf \varphi} \leq \kappa \right\}.$$

Throughout, let $\delta > 0$ be fixed and given by Lemma 3.10. Fix also m given by Lemma 3.5 associated to balls of radius δ . We are now in a position to state the precise condition on the constants ε_ϕ and ε'_ϕ on (P) and (P') respectively. Then taking into account (3.2) we assume:

$$e^{\varepsilon_\phi} \cdot \left(\frac{(\deg(f) - q)\sigma^{-\alpha} + qL^\alpha[1 + (L - 1)^\alpha]}{\deg(f)} \right) + \varepsilon_\phi 2mL^\alpha \text{diam}(M)^\alpha < 1 \tag{4.1}$$

and

$$[1 + \varepsilon'_\phi] \cdot e^{\varepsilon_\phi} \cdot \left(\frac{(\deg(f) - q)\sigma^{-\alpha} + qL^\alpha[1 + (L - 1)^\alpha]}{\deg(f)} \right) < 1. \tag{4.2}$$

Notice that having (3.2) it is possible to consider ε'_ϕ satisfying the later condition. Our main result in this section is as follows.

Theorem 4.1. *Assume that f satisfies (H1), (H2) and that ϕ satisfies (P). Then there exist $\delta > 0$ and $0 < \hat{\lambda} < 1$ such that $\mathcal{L}_\phi(A_{\kappa, \delta}) \subset \Lambda_{\hat{\lambda}\kappa, \delta}$ for every large positive constant κ .*

Proof. Take $\kappa > 0$ and let $\varphi \in A_{\kappa, \delta}$ be given. Moreover, given $x \in M$ we consider the set $(x_j)_{j=1 \dots \deg(f)}$ of the preimages by f of the point x , that is $f(x_j) = x$, and let $K = |\varphi|_{\alpha, \delta}$ be the α -Hölder constant of φ on balls of radius δ . We will prove that there exists $0 < \hat{\lambda} < 1$ such that $\mathcal{L}_\phi \varphi \in \Lambda_{\hat{\lambda}\kappa, \delta}$ provided that κ is large enough. Indeed, if $d(x, w) < \delta$ then

$$\frac{|\mathcal{L}_\phi \varphi(x) - \mathcal{L}_\phi \varphi(w)|}{\inf_{z \in M \{ \mathcal{L}_\phi \varphi(z) \}} d(x, w)^\alpha} \leq \frac{\sum_{j=1}^{\deg(f)} |\varphi(x_j)e^{\phi(x_j)} - \varphi(w_j)e^{\phi(w_j)}|}{\deg(f) \cdot e^{\inf \phi} \inf \varphi \cdot d(x, w)^\alpha} \tag{4.3}$$

$$\leq \frac{\sum_{j=1}^{\deg(f)} |e^{\phi(x_j)}(\varphi(x_j) - \varphi(w_j))|}{\deg(f) \cdot e^{\inf \phi} \inf \varphi \cdot d(x, w)^\alpha} \tag{4.4}$$

$$+ \frac{\sum_{j=1}^{\deg(f)} |\varphi(w_j)(e^{\phi(x_j)} - e^{\phi(w_j)})|}{\deg(f) \cdot e^{\inf \phi} \inf \varphi \cdot d(x, w)^\alpha}.$$

We subdivide the sum in (4.3) according to the possible backward contraction of the inverse branches of f . Using Lemma 3.5 we obtain that φ is \tilde{K} -Hölder on balls of radius $L\delta$ with $\tilde{K} = K(1 + (L - 1)^\alpha)$. Since $K \leq \kappa \inf g$ and $\sup \phi - \inf \phi < \varepsilon_\phi$ we get that (4.3) is bounded from above by

$$e^{\varepsilon_\phi} \frac{(\deg(f) - q)\sigma^{-\alpha} + qL^\alpha[1 + (L - 1)^\alpha]}{\deg(f)} \kappa.$$

For estimating (4.4) we first note that since φ is Hölder continuous then $\sup \varphi \leq \inf \varphi + m|\varphi|_{\alpha,\delta} \text{diam}(M)^\alpha$. Therefore, using $\varphi \in \Lambda_{\kappa,\delta}$ we get

$$\begin{aligned} (4.4) &\leq \frac{\sup \varphi \cdot |e^\phi|_\alpha \cdot L^\alpha}{\inf \varphi \cdot e^{\inf \phi}} \leq \frac{\inf \varphi + m\|\varphi\|_{\alpha,\delta} \text{diam}(M)^\alpha}{\inf \varphi} \frac{|e^\phi|_\alpha}{e^{\inf \phi}} L^\alpha \\ &\leq \frac{|e^\phi|_\alpha}{e^{\inf \phi}} L^\alpha [1 + m\kappa \text{diam}(M)^\alpha] \leq \frac{|e^\phi|_\alpha}{e^{\inf \phi}} 2mL^\alpha \kappa \text{diam}(M)^\alpha. \end{aligned}$$

Using (P) we have $|e^\phi|_\alpha < \varepsilon_\phi e^{\inf \phi}$ and our previous choice of ε_ϕ yields that $\|\mathcal{L}_\phi \varphi\|_{\alpha,\delta} \leq \hat{\lambda} \kappa \inf(\mathcal{L}_\phi \varphi)$. This completes the proof of the theorem. \square

Observe that assumption (P) can be rewritten as $\sup(\phi) - \inf(\phi) < \varepsilon_\phi$ and $e^\phi \in \Lambda_{\varepsilon_\phi}$. To prove that the cone has finite diameter in $\Lambda_{\kappa,\delta}$ we compute an explicit expression for the projective metric.

Lemma 4.2. *The $\Lambda_{\kappa,\delta}$ -cone metric Θ_κ is given by $\Theta_\kappa(\varphi, \psi) = \log \frac{B_\kappa(\varphi, \psi)}{A_\kappa(\varphi, \psi)}$, where*

$$A_\kappa(\varphi, \psi) = \inf_{0 < d(x,y) < \delta, z \in M} \frac{\kappa|x - y|^\alpha \psi(z) - (\psi(x) - \psi(y))}{\kappa|x - y|^\alpha \varphi(z) - (\varphi(x) - \varphi(y))},$$

and

$$B_\kappa(\varphi, \psi) = \sup_{0 < d(x,y) < \delta, z \in M} \frac{\kappa|x - y|^\alpha \psi(z) - (\psi(x) - \psi(y))}{\kappa|x - y|^\alpha \varphi(z) - (\varphi(x) - \varphi(y))}.$$

Proof. By definition, $A\varphi \leq \psi$ if and only if $\psi(x) - A\varphi(x) \geq 0$ for every $x \in M$ and $\|\psi - A\varphi\|_{\alpha,\delta} \leq \kappa \inf(\psi - A\varphi)$. In particular one gets

$$A \leq \min \left\{ \inf_x \frac{\psi(x)}{\varphi(x)}, \inf_{0 < d(x,y) < \delta, z \in M} \frac{\kappa|x - y|^\alpha \psi(z) - (\psi(x) - \psi(y))}{\kappa|x - y|^\alpha \varphi(z) - (\varphi(x) - \varphi(y))} \right\}.$$

We will prove that minimum in the right-hand side is always attained by the second term. Pick $x_0 \in M$ such that $\inf_x \frac{\psi(x)}{\varphi(x)} = \frac{\psi(x_0)}{\varphi(x_0)}$. Then it is immediate that

$$\lim_{x \rightarrow x_0} \frac{\kappa|x - x_0|^\alpha \psi(x_0) - (\psi(x) - \psi(x_0))}{\kappa|x - x_0|^\alpha \varphi(x_0) - (\varphi(x) - \varphi(x_0))} \leq \frac{\psi(x_0)}{\varphi(x_0)},$$

which guarantees that

$$A_\kappa(\varphi, \psi) = \inf_{0 < d(x,y) < \delta, z \in M} \frac{\kappa|x - y|^\alpha \psi(z) - (\psi(x) - \psi(y))}{\kappa|x - y|^\alpha \varphi(z) - (\varphi(x) - \varphi(y))}.$$

Similar computations lead to the expression for $B_\kappa(\varphi, \psi)$. \square

Proposition 4.3. *Given $0 < \hat{\lambda} < 1$, the cone $\Lambda_{\hat{\lambda}\kappa,\delta}$ has finite $\Lambda_{\kappa,\delta}$ -diameter.*

Proof. For all $\varphi \in \Lambda_{\hat{\lambda}\kappa,\delta}$ by definition we have $|\varphi|_{\alpha,\delta} \leq \hat{\lambda} \kappa \inf \varphi$ and, consequently, $\sup \varphi \leq [1 + m\hat{\lambda}\kappa(\text{diam } M)^\alpha] \inf \varphi$. So, using the previous expression for the projective metric, given $\varphi, \psi \in \Lambda_{\hat{\lambda}\kappa,\delta}$ one can easily check that

$$\begin{aligned} \Theta_\kappa(\varphi, \psi) &\leq \log\left(\frac{\kappa \cdot \sup \varphi + \hat{\lambda}\kappa \inf \varphi}{\kappa \cdot \inf \varphi - \hat{\lambda}\kappa \inf \varphi} \cdot \frac{\kappa \cdot \sup \psi + \hat{\lambda}\kappa \inf \psi}{\kappa \cdot \inf \psi - \hat{\lambda}\kappa \inf \psi}\right) \\ &\leq \log\left(\frac{\kappa(1 + m\hat{\lambda}\kappa(\text{diam } M)^\alpha)(1 + \hat{\lambda}) \inf \varphi}{\kappa \cdot (1 - \hat{\lambda}) \cdot \inf \varphi}\right) \\ &\quad + \log\left(\frac{\kappa(1 + m\hat{\lambda}\kappa(\text{diam } M)^\alpha)(1 + \hat{\lambda}) \inf \psi}{\kappa \cdot (1 - \hat{\lambda}) \cdot \inf \psi}\right) \\ &\leq 2 \log\left(\frac{1 + \hat{\lambda}}{1 - \hat{\lambda}}\right) + 2 \log(1 + mc \text{diam}(M)^\alpha) \end{aligned}$$

for some positive constant c . This implies the finite Θ_κ -diameter of $\Lambda_{\hat{\lambda}\kappa, \delta}$ and finishes the proof of the lemma. \square

4.2. Consequences of the spectral gap in $C^\alpha(M, \mathbb{R})$

Now we shall deduce the existence of equilibrium states and some of their ergodic properties.

4.2.1. Existence of equilibrium states

Using the spectral gap property in the space of Hölder continuous observables we get the existence of a unique Hölder continuous invariant density h .

Proposition 4.4. *There exists a unique density $h \in C^\alpha(M, \mathbb{R})$ such that $\mathcal{L}_\phi h = \lambda h$. In particular, $\mu = h\nu$ an equilibrium state for f with respect to ϕ . Finally, the density $d\mu/d\nu$ is bounded away from zero and infinity and Hölder continuous.*

Proof. Consider the normalized operator $\tilde{\mathcal{L}}_\phi = \lambda^{-1}\mathcal{L}_\phi$, where λ is the spectral radius of \mathcal{L} and write Λ^+ for the cone of strictly positive continuous functions on M . Since $\Lambda_{\kappa, \delta} \subset \Lambda^+$ then the projective metrics satisfy $\Theta^+(\varphi, \psi) \leq \Theta_\kappa(\varphi, \psi)$ for any $\varphi, \psi \in \Lambda_{\kappa, \delta}$, where

$$\Theta^+(\varphi, \psi) = \log\left(\frac{\sup_{x \in M}\{\varphi(x)/\psi(x)\}}{\inf_{y \in M}\{\varphi(y)/\psi(y)\}}\right).$$

By the previous proposition, $\tilde{\mathcal{L}}_\phi(\Lambda_{\kappa, \delta})$ has finite diameter in $\Lambda_{\kappa, \delta}$ for any sufficiently large κ . Therefore, as discussed at the end of Subsection 3.4, $\tilde{\mathcal{L}}_\phi$ is a contraction in the Θ_κ -metric and there exists $0 < \tau < 1$ such that for any $\varphi, \psi \in \Lambda_{\kappa, \delta}$ and $n, k \geq 1$

$$\Theta^+(\tilde{\mathcal{L}}_\phi^{n+k}(\varphi), \tilde{\mathcal{L}}_\phi^n(\psi)) \leq \Theta_\kappa(\tilde{\mathcal{L}}_\phi^{n+k}(\varphi), \tilde{\mathcal{L}}_\phi^n(\psi)) \leq \Delta \tau^n, \tag{4.5}$$

where Δ is the Θ_κ -diameter of the cone $\Lambda_{\hat{\lambda}\kappa, \delta}$. This proves that $(\tilde{\mathcal{L}}_\phi^n \varphi)_n$ is a Cauchy sequence in the projective metric. For the reference measure ν we have that $\int \tilde{\mathcal{L}}_\phi \varphi d\nu = \int \varphi d\nu, \forall \varphi \in C^0(M, \mathbb{R})$. Given $\varphi \in \Lambda_{\kappa, \delta}$ with $\int \varphi d\nu = 1$ it is clear that $\sup \varphi \geq 1$ and $\inf \varphi \leq 1$. Together with the remark that any $\varphi \in \Lambda_{\kappa, \delta}$ satisfies $\sup \varphi \leq [1 + m\kappa \text{diam}(M)^\alpha] \inf \varphi$ this shows that

$$\frac{1}{R_1} \leq \inf \varphi \leq 1 \leq \sup \varphi \leq R_1 \tag{4.6}$$

where $R_1 = 1 + m\kappa \text{diam}(M)^\alpha$. Write $\varphi_n = \tilde{\mathcal{L}}_\phi^n(\varphi)$. First notice that $(\varphi_n)_n$ is an equi-Hölder sequence since $|\varphi_n(x) - \varphi_n(y)| \leq \kappa \inf \varphi d(x, y)^\alpha \leq \kappa d(x, y)^\alpha$ for all $d(x, y) < \delta$ and all n , which proves that all φ_n are κm -Hölder continuous.

From the previous discussion we know that $\int \varphi_n d\nu = 1$ for every n and, consequently, the sequence φ_n is uniformly bounded from above and below. In fact, observe first that $\int \varphi_k d\nu = \int \varphi_l d\nu = 1$ implies $\inf \frac{\varphi_k}{\varphi_l} \leq 1 \leq \sup \frac{\varphi_k}{\varphi_l}$. Therefore, from (4.5) we get

$$e^{-\Delta \tau^n} < \frac{\sup_{x \in M}\{\varphi_k(x)/\varphi_l(x)\}}{\inf_{y \in M}\{\varphi_k(y)/\varphi_l(y)\}} = e^{\theta^+(\varphi_k, \varphi_l)} < e^{\Delta \tau^n}$$

for every k and $l \geq n$. In consequence,

$$e^{-\Delta\tau^n} < \inf \frac{\varphi_k}{\varphi_l} \leq 1 \leq \sup \frac{\varphi_k}{\varphi_l} < e^{\Delta\tau^n} \tag{4.7}$$

and $(\varphi_k)_k$ is a Cauchy sequence in the C^0 -norm. In fact,

$$\sup |\varphi_k - \varphi_l| \leq \sup \left(|\varphi_l| \left| \frac{\varphi_k}{\varphi_l} - 1 \right| \right) \leq R_1 (e^{\Delta\tau^n} - 1) \leq 3R_1 \Delta\tau^n \tag{4.8}$$

for every $k, l \geq n$ and any $n \geq -\log(\Delta)/\log(\tau)$. This yields that (φ_k) converges uniformly to some function h in $\Lambda_{\kappa,\delta}$ satisfying $\int h d\nu = 1$ and, consequently, κm -Hölder continuous. It follows from a standard argument that $\mu = \int h d\nu$ is an f -invariant probability measure. Furthermore, the sequence $(\tilde{\mathcal{L}}_\phi^n(\psi))_n$ converges to the same limit for any normalized function $\psi \in \Lambda_{\kappa,\delta}$. Indeed, if this was not the case then the same arguments used before are enough to conclude that the sequence

$$\psi_n := \begin{cases} \varphi_n, & \text{if } n \text{ is odd,} \\ \tilde{\mathcal{L}}_\phi^n(\psi), & \text{otherwise} \end{cases}$$

is Cauchy and, consequently, converges. This shows that the functions $\tilde{\mathcal{L}}_\phi^n(\varphi)$ and $\tilde{\mathcal{L}}_\phi^n(\psi)$ must have the same limit and proves the uniqueness of the Hölder invariant density $h \in C^\alpha(M, \mathbb{R})$ such that $\tilde{\mathcal{L}}_\phi h = h$. By Theorem B and Lemma 6.5 in [41] we know that equilibrium states coincide with invariant probability measures absolutely continuous with respect to ν . Hence, $\mu = h\nu$ is an equilibrium state for f with respect to ϕ . This finishes the proof of the proposition. \square

Here we provide further information on the velocity of convergence to the invariant density in the space of Hölder continuous observables. More precisely,

Corollary 4.5. *Set $\varphi \in \Lambda_{\kappa,\delta}$ be such that $\int \varphi d\nu = 1$ and let h denote the Θ_κ -limit of $\varphi_n = \tilde{\mathcal{L}}_\phi^n(\varphi)$. Then, φ_n converges exponentially fast to h in the Hölder norm.*

Proof. It follows from (4.7) and (4.8) that $|\varphi_n - h|_\infty \leq 3R_1 \Delta\tau^n$ and

$$e^{-\Delta\tau^n} \leq \inf \frac{\varphi_n}{h} \leq 1 \leq \sup \frac{\varphi_n}{h} \leq e^{\Delta\tau^n} \tag{4.9}$$

for every $n \in \mathbb{N}$. Now we claim that $B_\kappa(h, \varphi_n) \geq 1$. In fact this is immediate in the case that $\varphi_n \equiv h$. Assume otherwise, by contradiction, and notice that $B_\kappa(h, \varphi_n) < 1$ implies $\varphi_n \neq h$. Using (4.9), there exists a point $z = z_n \in M$ such that $\varphi_n(z) > h(z)$. Take x_0 such that $\varphi_n(x_0) - h(x_0) = \min\{\varphi_n - h\}$. Therefore, if $0 < d(w, x_0) < \delta$ we obtain that

$$\frac{\varphi_n(w) - \varphi_n(x_0)}{d(w, x_0)^\alpha} \geq \frac{h(w) - h(x_0)}{d(w, x_0)^\alpha}.$$

In consequence

$$B_\kappa(h, \varphi_n) \geq \frac{\varphi_n(z_n) - (h(w) - h(x_0))/\kappa d(w, x_0)^\alpha}{h(z_n) - (\varphi_n(w) - \varphi_n(x_0))/\kappa d(w, x_0)^\alpha} \geq 1.$$

Analogously, one concludes that $A_\kappa(h, \varphi_n) \leq 1$. Using the definition of Θ_κ and the exponentially fast Θ_κ -convergence of φ_n we get $e^{-\Delta\tau^n} < A_\kappa(h, \varphi_n) \leq 1 \leq B_\kappa(h, \varphi_n) \leq e^{\Delta\tau^n}$, $\forall n \in \mathbb{N}$. For notational simplicity, given $x \neq y$, set $H_h(x, y) = (h(x) - h(y))/\kappa d(x, y)^\alpha$ and H_{φ_n} be the corresponding expression for φ_n . The previous estimates imply that $e^{\Delta\tau^n} H_h(x, y) - H_{\varphi_n}(x, y) \leq e^{\Delta\tau^n} \varphi_n(z) - h(z)$. In particular

$$H_h(x, y) - H_{\varphi_n}(x, y) < \varphi_n(z) - h(z) + (e^{\Delta\tau^n} - 1) \cdot (\varphi_n(z) - H_{\varphi_n}(x, y)) \leq 5R_1 \Delta\tau^n$$

for every large n . Since the other inequality follows from completely analogous computations one deduces that $|h - \varphi_n|_{\alpha,\delta} \leq 5R_1 \Delta\tau^n$ for every large n . Therefore, $|h - \varphi_n|_\alpha \leq 5mR_1 \Delta\tau^n$ which together with the previous estimate $\|h - \varphi_n\|_\infty \leq 3R_1 \Delta\tau^n$ proves the corollary. \square

The strict invariance of the cone $\Lambda_{\kappa,\delta}$ is now enough to obtain a spectral gap property for the normalized operator $\tilde{\mathcal{L}}_\phi = \lambda_\phi^{-1} \mathcal{L}_\phi$.

Theorem 4.6 (Spectral gap). *There exists $0 < r_0 < 1$ such that the operator $\tilde{\mathcal{L}}_\phi$ acting on the space $C^\alpha(M, \mathbb{R})$ admits a decomposition of its spectrum given by $\Sigma = \{1\} \cup \Sigma_0$, where Σ_0 contained in a ball $B(0, r_0)$.*

Proof. Let E_1 be the one-dimensional eigenspace relative to the eigenvalue 1, and let $E_0 := \{\varphi \in C^\alpha(M, \mathbb{R}) : \int \varphi \, d\nu = 0\}$. Observe that $\int h \, d\nu = 1$, the subspaces E_0, E_1 are $\tilde{\mathcal{L}}_\phi$ -invariant and $C^\alpha(M, \mathbb{R}) = E_1 \oplus E_0$: given $\varphi \in C^\alpha(M, \mathbb{R})$ just write $\varphi = \int \varphi \, d\nu \cdot h + [\varphi - \int \varphi \, d\nu \cdot h]$. Therefore, to obtain the spectral gap property it is enough to prove that $\tilde{\mathcal{L}}_\phi^n|_{E_0}$ is a contraction for any large n .

Take $\kappa \geq 1$ large such that $\Lambda_{\kappa, \delta}$ is preserved by $\tilde{\mathcal{L}}_\phi$. Pick $\varphi \in E_0$ with norm less or equal to 1 and notice that $\varphi + 2 \in \Lambda_{\kappa, \delta}$ because $|\varphi + 2|_{\alpha, \delta} = |\varphi|_{\alpha, \delta} \leq 1$ and also $1 \leq \kappa \inf |\varphi + 2|$. Therefore $\tilde{\mathcal{L}}_\phi^n(\varphi + 2)$ converges to $\int (\varphi + 2) \, d\nu \cdot h = 2h$ and

$$\|\tilde{\mathcal{L}}_\phi^n(\varphi)\| = \|\tilde{\mathcal{L}}_\phi^n(\varphi + 2) - \tilde{\mathcal{L}}_\phi^n(2)\| \leq \|\tilde{\mathcal{L}}_\phi^n(\varphi + 2) - 2h\| + \|\tilde{\mathcal{L}}_\phi^n(2) - 2h\| \leq 20KR_1 \Delta \tau^n,$$

is exponentially contracted. This concludes the proof of the theorem. \square

A first consequence of the spectral gap is the following strong convergence.

Corollary 4.7. *The equilibrium state μ coincides with the limit of the push-forwards $(f^j)_* \nu$ of the conformal measure ν .*

Proof. First recall that $\mathcal{L}^* \nu = \lambda \nu$. Thus, given any $\varphi \in C^0(M)$ it follows that $\int \varphi \, d(f_*^j \nu) = \int \varphi \circ f^j \, d\nu = \int \varphi(\lambda^{-j} \tilde{\mathcal{L}}^j 1) \, d\nu$ which converges to $\int \varphi h \, d\nu = \int \varphi \, d\mu$ as j tends to infinity. Since φ is arbitrary this proves that $\mu = \lim f_*^j \nu$ as claimed. \square

4.2.2. Uniqueness of equilibrium states and exponential decay of correlations

In this subsection we show that there is a unique equilibrium state for f with respect to ϕ and derive good mixing properties.

Theorem 4.8. *The equilibrium state $\mu = \mu_\phi$ has exponential decay of correlations for Hölder observables: there exists $0 < \tau < 1$ such that for all $\varphi \in L^1(\nu)$, $\psi \in C^\alpha(M)$ there is $K(\varphi, \psi) > 0$ such that*

$$\left| \int (\varphi \circ f^n) \psi \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu \right| \leq K(\varphi, \psi) \cdot \tau^n, \quad \forall n \geq 1.$$

Proof. First we write the correlation function

$$C_{\varphi, \psi}(n) := \int (\varphi \circ f^n) \psi \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu = \int (\varphi \circ f^n) \psi h \, d\nu - \int \varphi \, d\mu \int \psi \, d\mu.$$

It is no restriction to assume that $\int \psi \, d\mu = 1$. Then, using that h is bounded away from zero and infinity we get

$$\left| \int (\varphi \circ f^n) \psi h \, d\nu - \int \varphi \, d\mu \int \psi \, d\mu \right| = \left| \int \varphi \left(\frac{\tilde{\mathcal{L}}_\phi^n(\psi h)}{h} - 1 \right) d\mu \right| \leq \left\| \frac{\tilde{\mathcal{L}}_\phi^n(\psi h)}{h} - 1 \right\|_0 \cdot \|\varphi\|_1$$

where $\|\varphi\|_1 = \int |\varphi| \, d\mu$. If $\psi h \in \Lambda_{\kappa, \delta}$ for some sufficiently large κ as in Theorem 4.1 then it follows from (4.8) that the first term in the right-hand side above satisfies

$$\left\| \frac{\tilde{\mathcal{L}}_\phi^n(\psi h)}{h} - 1 \right\|_0 \leq 2R_1 \left\| \frac{1}{h} \right\|_0 (e^{\Delta \tau^n} - 1) \leq C \tau^n,$$

for some positive constant C and so $|\int (\varphi \circ f^n) \psi \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu| \leq K(\varphi, \psi) \tau^n$. In general write $\psi h = g$ where $g = g_B^+ - g_B^-$ and $g_B^\pm = \frac{1}{2}(|g| \pm g) + B$ for $B > 0$ large so that $g_B^\pm \in \Lambda_{\kappa, \delta}$ and apply the latter estimates to g_B^\pm . By linearity, the same estimate holds for g for some constant $K(\varphi, \psi) \geq K(\varphi, g_B^+) + K(\varphi, g_B^-)$. This concludes the proof of the exponential decay of correlations. \square

As a consequence we remove the topologically mixing assumption from [41] and still deduce that there exists a unique equilibrium state and it is exact.

Corollary 4.9. *The probability measure μ is exact and the unique equilibrium state for f with respect to ϕ .*

Proof. Let $\varphi \in L^1(\mu)$ be such that $\varphi = \varphi_n \circ f^n$ for some measurable functions φ_n . Given any $\psi \in C^\alpha(M)$ it follows from the previous theorem that

$$\left| \int \left(\varphi - \int \varphi d\mu \right) \psi d\mu \right| = \left| \int (\varphi_n \circ f^n) \psi d\mu - \int \varphi d\mu \int \psi d\mu \right| \leq K(\varphi_n, \psi) \tau^n,$$

where the constant $K(\varphi_n, \psi)$ depends only on the value of $\int \varphi_n d\mu = \int \varphi d\mu$ and $\|\psi\|_\alpha$. Hence $K(\varphi_n, \psi)$ does not depend on n and, consequently, $\int (\varphi - \int \varphi d\mu) \psi d\mu = 0$, for all Hölder continuous ψ . This implies that $\varphi = \int \varphi d\mu$ for μ -almost every x , proving that μ is exact. In consequence, $\mu = h\nu$ is an ergodic probability measure whose basin of attraction contains ν -almost every point. Therefore the uniqueness of the equilibrium state follows from Theorem 2.1. \square

4.2.3. Central limit theorem

Here we obtain a central limit theorem from the strong mixing properties. Let \mathcal{F} be the Borel sigma-algebra of M and $\mathcal{F}_n := f^{-n}(\mathcal{F})$ be a non-increasing family of σ -algebras. Recall that a function $\xi : M \rightarrow \mathbb{R}$ is \mathcal{F}_n -measurable iff $\xi = \xi_n \circ f^n$ for some measurable ξ_n . Let $L^2(\mathcal{F}_n) = \{\xi \in L^2(\mu) : \xi \text{ is } \mathcal{F}_n\text{-measurable}\}$ and note that $L^2(\mathcal{F}_n) \supset L^2(\mathcal{F}_{n+1})$ for each $n \geq 0$. Given $\varphi \in L^2(\mu)$, we denote by $\mathbb{E}(\varphi|\mathcal{F}_n)$ the L^2 -orthogonal projection of φ to $L^2(\mathcal{F}_n)$. The strategy now is to apply a general result due to Gordin by proving that the $L^2(\mathcal{F}_n)$ components $\mathbb{E}(\varphi|\mathcal{F}_n)$ of any observable φ are summable.

Lemma 4.10. *For every α -Hölder continuous function φ with $\int \varphi d\mu = 0$ there is $R_0 = R_0(\varphi)$ such that $\|\mathbb{E}(\varphi|\mathcal{F}_n)\|_2 \leq R_0 \tau^n$ for all $n \geq 0$.*

Proof. Observe that since $\|\psi\|_1 \leq \|\psi\|_2$ and $\int \varphi d\mu = \int \varphi h d\nu = 0$ it follows that

$$\begin{aligned} \|\mathbb{E}(\varphi|\mathcal{F}_n)\|_2 &= \sup \left\{ \int \xi \varphi d\mu : \xi \in L^2(\mathcal{F}_n), \|\xi\|_2 = 1 \right\} \\ &= \sup \left\{ \int (\psi \circ f^n) \varphi d\mu : \psi \in L^2(\mu), \|\psi\|_2 = 1 \right\} \leq K(\varphi, \psi) \tau^n, \end{aligned}$$

which proves the lemma. \square

Now the central limit theorem in Corollary 2 follows from the following abstract result due to Gordin (see e.g. [42]).

Theorem 4.11. *Let (M, \mathcal{F}, μ) be a probability space, $f : M \rightarrow M$ be a measurable map such that μ is f -invariant and ergodic. Consider $\varphi \in L^2(\mu)$ such that $\int \varphi d\mu = 0$ and denote by \mathcal{F}_n the non-increasing sequence of sigma-algebras $\mathcal{F}_n = f^{-n}(\mathcal{F})$, $n \geq 0$. If $\sum_{n=0}^\infty \|\mathbb{E}(\varphi|\mathcal{F}_n)\|_2 < \infty$ then σ_φ is finite, and $\sigma_\varphi = 0$ iff $\varphi = u \circ f - u$ for some $u \in L^2(\mu)$. Moreover, if $\sigma_\varphi > 0$ then for any interval $A \subset \mathbb{R}$*

$$\mu \left(x \in M : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left(\varphi(f^j(x)) - \int \varphi d\mu \right) \in A \right) \rightarrow \frac{1}{\sigma_\varphi \sqrt{2\pi}} \int_A e^{-\frac{t^2}{2\sigma^2}} dt,$$

as n tends to infinity.

4.2.4. Uniform continuity of the densities for the equilibrium states

Here we shall prove the first stability result for the equilibrium state: the density of the equilibrium state with respect to the corresponding conformal measure varies continuously in the C^0 -norm. This is not immediate since the Ruelle–Perron–Frobenius operator in general does not vary continuously with the dynamical system in the space of Hölder continuous observables as discussed in Example 4.14. Nevertheless, we could get the continuity of the density function which is the main result of this section.

Proposition 4.12. *Let \mathcal{F} be a family of local homeomorphisms and \mathcal{W} be a family of potentials satisfying (H1), (H2) and (P) with uniform constants. Then the topological pressure $\mathcal{F} \times \mathcal{W} \ni (f, \phi) \mapsto \log \lambda_{f,\phi} = P_{\text{top}}(f, \phi)$ and the density function*

$$\mathcal{F} \times \mathcal{W} \rightarrow (C^\alpha(M, \mathbb{R}), \|\cdot\|_0),$$

$$(f, \phi) \mapsto \frac{d\mu_{f,\phi}}{dv_{f,\phi}}$$

are continuous.

Proof. Recall that Proposition 3.4 implies that $P_{\text{top}}(f, \phi) = \log \lambda_{f,\phi}$ where $\lambda_{f,\phi}$ is the spectral radius of the operator $\mathcal{L}_{f,\phi}$. Moreover, it follows from the proof of Corollary 4.5 that for any $\varphi \in \Lambda_{\kappa,\delta}$ satisfying $\int \varphi d\nu = 1$ one has in particular

$$\left\| \lambda_{f,\phi}^{-n} \mathcal{L}_{f,\phi}^n \varphi - \frac{d\mu_{f,\phi}}{dv_{f,\phi}} \right\|_0 \leq 3R_1 \Delta \tau^n, \tag{4.10}$$

for all n . Notice the previous reasoning applies to $\varphi \equiv 1 \in \Lambda_{\kappa,\delta}$. Moreover, since the spectral gap property estimates depend only on the constants L, σ and $\deg(f)$ it follows that all transfer operators $\mathcal{L}_{\tilde{f},\tilde{\phi}}$ preserve the cone $\Lambda_{\kappa,\delta}$ for all pairs $(\tilde{f}, \tilde{\phi})$ and that the constants R_1 and Δ can be taken uniform in a small neighborhood \mathcal{U} of (f, ϕ) . Furthermore, one has that $\int \lambda_{f,\phi}^{-1} \mathcal{L}_{f,\phi} d\nu_{f,\phi} = 1$ and so the convergence

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|\tilde{\mathcal{L}}_{\tilde{f},\tilde{\phi}}^n(1)\|_0 = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \|\lambda_{\tilde{f},\tilde{\phi}}^{-n} \mathcal{L}_{\tilde{f},\tilde{\phi}}^n(1)\|_0 = 0$$

given by Proposition 4.4 and Corollary 4.5 can be taken uniform in \mathcal{U} . This is the key ingredient to obtain the continuity of the topological pressure and density function. Indeed, let $\varepsilon > 0$ be fixed and take $n_0 \in \mathbb{N}$ such that

$$\left| \frac{1}{n_0} \log \|\mathcal{L}_{\tilde{f},\tilde{\phi}}^{n_0}(1)\|_0 - \log(\lambda_{\tilde{f},\tilde{\phi}}) \right| < \frac{\varepsilon}{3},$$

for all $\tilde{f} \in \mathcal{U}$. Moreover, using $P_{\text{top}}(f, \phi) = \log \lambda_{f,\phi}$ by triangular inequality we get

$$\begin{aligned} |P_{\text{top}}(f, \phi) - P_{\text{top}}(\tilde{f}, \phi)| &\leq \left| \frac{1}{n_0} \log \|\mathcal{L}_{\tilde{f},\tilde{\phi}}^{n_0}(1)\|_0 - \log(\lambda_{\tilde{f},\tilde{\phi}}) \right| + \left| \frac{1}{n_0} \log \|\mathcal{L}_{f,\phi}^{n_0}(1)\|_0 - \log(\lambda_{f,\phi}) \right| \\ &\quad + \left| \frac{1}{n_0} \log \|\mathcal{L}_{f,\phi}^{n_0}(1)\|_0 - \frac{1}{n_0} \log \|\mathcal{L}_{\tilde{f},\tilde{\phi}}^{n_0}(1)\|_0 \right|. \end{aligned}$$

Now, it is not hard to check that, for n_0 fixed, the function $\mathcal{U} \rightarrow C^0(M, \mathbb{R})$ given by

$$\tilde{f} \mapsto \mathcal{L}_{\tilde{f},\tilde{\phi}}^{n_0} 1 = \sum_{\tilde{f}^{n_0}(y)=x} e^{\mathcal{S}_{n_0}\phi(y)}$$

is continuous. Consequently, there exists a neighborhood $\mathcal{V} \subset \mathcal{U}$ of f such that $|\frac{1}{n_0} \log \|\mathcal{L}_{f,\phi}^{n_0}(1)\|_0 - \frac{1}{n_0} \log \|\mathcal{L}_{\tilde{f},\tilde{\phi}}^{n_0}(1)\|_0| < \varepsilon/3$ for every $\tilde{f} \in \mathcal{V}$. Altogether this proves that $|P_{\text{top}}(f, \phi) - P_{\text{top}}(\tilde{f}, \phi)| < \varepsilon$ for all $\tilde{f} \in \mathcal{V}$. Since ε was chosen arbitrary we obtain that both the leading eigenvalue and topological pressure functions vary continuously with the dynamics f . Finally, by Eq. (4.10) above applied to $\varphi \equiv 1$ and triangular inequality we obtain that

$$\left\| \frac{d\mu_{\tilde{f},\tilde{\phi}}}{dv_{\tilde{f},\tilde{\phi}}} - \frac{d\mu_{f,\phi}}{dv_{f,\phi}} \right\|_0 \leq 6R_1 \Delta \tau^n + \left\| \lambda_{\tilde{f},\tilde{\phi}}^{-n} \mathcal{L}_{\tilde{f},\tilde{\phi}}^n 1 - \lambda_{f,\phi}^{-n} \mathcal{L}_{f,\phi}^n 1 \right\|_0$$

for all n . Hence, proceeding as before one can make the right-hand side above as close to zero as desired provided that \tilde{f} is sufficiently close to f . This proves the continuity of the density function and finishes the proof of the proposition. \square

We will finish this section with some comments on the non-continuous dependence of the Ruelle–Perron–Frobenius operators, acting on the space of Hölder continuous observables, as a function of the dynamics f .

Remark 4.13. Notice first that Hölder continuous observables are Lipschitz continuous with respect to the metric $d(\cdot, \cdot)^\alpha$. Hence, for simplicity we provide below an example of discontinuity of the Ruelle–Perron–Frobenius operator $\mathcal{L}_f : \text{Lip}(M) \rightarrow \text{Lip}(M)$ with respect to the dynamics f , where $\text{Lip}(M)$ are the space of continuous observables such that

$$\text{Lip}(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} < \infty.$$

Example 4.14. The key idea of the following surprisingly simple example of discontinuity of Ruelle–Perron–Frobenius operator with respect to the dynamics is that the operator of composition $\varphi \rightarrow \varphi \circ g$ acting in the space of Lipschitz functions does not vary continuously with g . Consider the expanding dynamics f_n on the circle $S^1 \simeq \mathbb{R}/[-1/2, 1/2)$ given by that $f_n(x) = 2(x + \frac{1}{10n}) \pmod{1}$. Obviously, f_n converges to f , $f(x) = 2x \pmod{1}$ in the C^∞ -topology. Now, take a periodic Lipschitz function φ in the circle such that $\varphi(x) = |x|$ say, for $|x| \leq 1/8$ and $\varphi(x) = 0$ for $1/2 \geq |x| \geq 1/5$. Just take the potential $\phi \equiv 0$ and write $\mathcal{L}_n, \mathcal{L}$ for the Perron–Frobenius operators corresponding to f_n, f respectively. Therefore, taking $0 < x_n < y_n < 1/10n$, we obtain that

$$\begin{aligned} \text{Lip}((\mathcal{L}_n - \mathcal{L})(\varphi)) &\geq \frac{|\mathcal{L}_n(\varphi)(y_n) - \mathcal{L}_n(\varphi)(x_n) + \mathcal{L}(\varphi)(x_n) - \mathcal{L}(\varphi)(y_n)|}{y_n - x_n} \\ &= \frac{||y_n/2 - 1/10n| - |x_n/2 - 1/10n| + |x_n/2| - |y_n/2||}{y_n - x_n} \\ &= \frac{|-y_n - x_n|}{y_n - x_n} = 1 = \text{Lip}(\varphi). \end{aligned}$$

Thus $\mathcal{L}_n : \text{Lip}(S^1, \mathbb{R}) \rightarrow \text{Lip}(S^1, \mathbb{R})$ does not converge to \mathcal{L} even in the strong operator topology. In particular, \mathcal{L}_n does not converge to \mathcal{L} in the norm topology.

5. Ruelle–Perron–Frobenius in $C^r(M, \mathbb{R})$: Spectral gap and strong stability results

Throughout this section we assume that f is a C^r -local diffeomorphism ($r \geq 1$) and the potential ϕ belongs to $C^r(M)$. Here we restrict the analysis of the transfer operator to the space of smooth observables.

5.1. Spectral gap for the transfer operator in $C^r(M, \mathbb{R})$

Here we shall assume that f is a C^r ($r \geq 1$) local diffeomorphism on a compact manifold M satisfying (H1) and (H2) and $\phi \in C^r(M, \mathbb{R})$ satisfies (P'). In fact, we require $L \geq 1$ to be close to 1 such that

$$\mathcal{E}_r := e^{\varepsilon\phi} \frac{qL^r + (\text{deg}(f) - q)\sigma^{-1}}{\text{deg}(f)} < 1, \tag{5.1}$$

which we use as counterpart of (3.2) in this differentiable setting. We prove that the transfer operator $\mathcal{L}_\phi : C^r(M, \mathbb{R}) \rightarrow C^r(M, \mathbb{R})$ has a spectral gap. The strategy is to show \mathcal{L}_ϕ -invariance of the cones of smooth observables

$$\Lambda_\kappa^r = \left\{ \varphi \in C^r(M, \mathbb{R}) : \varphi > 0 \text{ and } \frac{\|D^s \varphi\|_0}{\inf \varphi} \leq \frac{\kappa}{c_s^{(r)}} \text{ for } s = 1 \dots r \right\},$$

for some constants $c_s^{(r)}$ with $s = 1 \dots r$ defined recursively using the corresponding constants for the cones corresponding to smaller differentiability. The choice of the constants $c_s^{(r)}$ is made in order to guarantee that observables in Λ_κ^r associated to large κ belong to some cones $\Lambda_{\kappa_i}^{r-i}$ for some large constants $(\kappa_i)_{i=1 \dots r-1}$ where the Ruelle–Perron–Frobenius operator acts as a contraction in the projective metric. The precise construction is described in what follows. For $r = 1$ just consider

$$\Lambda_\kappa^1 = \left\{ \varphi \in C^r(M, \mathbb{R}) : \varphi > 0 \text{ and } \frac{\|D\varphi\|_0}{\inf \varphi} \leq \kappa \right\},$$

which corresponds to the previous cone with $c_1^{(1)} = 1$. For $r = 2$ we obtain that the cone Λ_κ^2 can be written as

$$\Lambda_\kappa^2 = \left\{ \varphi \in C^2(M, \mathbb{R}) : \varphi > 0, \frac{\|D\varphi\|_0}{\inf \varphi} \leq \frac{(1 - \mathcal{E}_2)\kappa}{2e^{\varepsilon_\phi} \max_x \|D^2 f^{-1}(x)\|} \text{ and } \frac{\|D^2 \varphi\|_0}{\inf \varphi} \leq \kappa \right\},$$

with $c_2^{(2)} = 1$ and $c_1^{(2)} = 2(1 - \mathcal{E}_2)^{-1} e^{\varepsilon_\phi} \max_x \|D^2 f^{-1}(x)\|$. Assuming that we have defined the positive constants $(c_s^{(r-1)})_{s=1 \dots r-1}$ associated to the cones Λ_κ^{r-1} of C^{r-1} -observables we define the constants $c_s^{(r)}$ as follows. Set

$$\begin{cases} c_r^{(r)} = 1; \\ c_{r-1}^{(r)} = r!(1 - \mathcal{E}_r)^{-1} e^{\varepsilon_\phi} \max_{1 \leq j \leq r-1} \max_x \|D^{r-1} f^{-1}(x)\|^j; \\ c_{r-t}^{(r)} = c_{r-t+1}^{(r)} \cdot c_{r-t}^{(r-1)}, \text{ for } t = 2 \dots r - 1. \end{cases}$$

Roughly, the choice of $c_{r-1}^{(r)}$ is made in order to guarantee that at most $r!$ terms arising in the computation of higher order derivatives of the observable $\mathcal{L}_\phi \varphi$ are dominated by the term involving $D^r \varphi$, while the recursive choice of the constants $c_s^{(r)}$ with $s < r$ guarantees that the cones corresponding to smaller differentiability are contracted. Hence, our main result in this section is as follows.

Theorem 5.1. *There exists a positive constant $\varepsilon'_\phi > 0$ (depending only on f and r) such that if ϕ satisfies condition (P') given by*

$$\sup \phi - \inf \phi < \varepsilon_\phi \quad \text{and} \quad \max_{s \leq r} \|D^s \phi\|_0 < \varepsilon'_\phi$$

then there are $\kappa_0 > 0$ and $0 < \hat{\lambda} < 1$ such that $\mathcal{L}_\phi(\Lambda_\kappa^r) \subset \Lambda_{\hat{\lambda}\kappa, \delta}^r$ for every $\kappa \geq \kappa_0$.

Proof. We shall prove the theorem recursively on the differentiability r . First set $r = 1$ and consider $\varphi \in \Lambda_\kappa^1$ for $\kappa > 0$ large. Given $x \in M$ let $(x_j)_j$ denote the set of preimages by f of the point x and denote by f_i^{-1} corresponding the local inverse branch for the function f in a neighborhood of x with $f_i(x_i) = x$. It is not hard to check that $|\mathcal{L}_\phi \varphi(x)| \leq e^{\varepsilon_\phi} \inf |\mathcal{L}_\phi \varphi|$ for every $x \in M$. Moreover,

$$D(\mathcal{L}_\phi \varphi)(x) = \sum_{j=1}^{\deg(f)} e^{\phi(x_j)} D\varphi(x_j) Df_j^{-1}(x) + \sum_{j=1}^{\deg(f)} \varphi(x_j) e^{\phi(x_j)} D\phi(x_j) Df_j^{-1}(x) \tag{5.2}$$

and, consequently, $\|D(\mathcal{L}_\phi \varphi)(x)\|$ is bounded by

$$\sum_{j=1}^{\deg(f)} |e^{\phi(x_j)}| \|D\varphi(x_j) Df_j^{-1}(x)\| + \sum_{j=1}^{\deg(f)} |\varphi(x_j) e^{\phi(x_j)}| \|D\phi(x_j) Df_j^{-1}(x)\|.$$

By our assumptions (H1) and (H2) it follows that the isomorphism $\|Df_j^{-1}(\cdot)\| \leq L$ for $j = 1 \dots q$ and is indeed a contraction for $j > q$. Thus, using (P') and that $\sup \varphi \leq \inf \varphi + \|D\varphi\|_0 \text{diam}(M)$ we get

$$\begin{aligned} \frac{\|D(\mathcal{L}_\phi \varphi)(x)\|}{\inf |\mathcal{L}_\phi \varphi|} &\leq \frac{\sum_{j=1}^q L |e^{\phi(x_j)}| \|D\varphi(x_j)\|_0 + \sum_{j>q} \sigma^{-1} |e^{\phi(x_j)}| \|D\varphi(x_j)\|_0}{\deg(f) e^{\inf \phi} \inf \varphi} \\ &\quad + \frac{\sum_{j=1}^q L |\varphi(x_j) e^{\phi(x_j)}| \|D\phi(x_j)\|_0 + \sum_{j>q} \sigma^{-1} |\varphi(x_j) e^{\phi(x_j)}| \|D\phi(x_j)\|_0}{\deg(f) e^{\inf \phi} \inf \varphi} \\ &\leq e^{\varepsilon_\phi} \frac{qL + \sigma^{-1}(\deg(f) - q) \|D\varphi\|_0}{\deg(f)} + e^{\varepsilon_\phi} \|D\phi(x_j)\|_0 \frac{\sup \varphi}{\inf \varphi} \frac{qL + \sigma^{-1}(\deg(f) - q)}{\deg(f)} \\ &\leq \mathcal{E}_1 \cdot \kappa + \mathcal{E}_1 \varepsilon'_\phi (1 + \|D\varphi\|_0 \text{diam}(M)) \end{aligned}$$

which can be taken smaller than $\tilde{\lambda} \kappa$, for some constant $0 < \tilde{\lambda} < 1$ by our choice of ε_ϕ in (5.1) provided that ε'_ϕ is sufficiently small. In consequence we obtain that $\|D(\mathcal{L}_\phi \varphi)\|_0 \leq \tilde{\lambda} \kappa \inf |\mathcal{L}_\phi \varphi|$, which proves the theorem in the case that $r = 1$.

We now consider the case $r = 2$. Fix $\kappa > 0$ and consider $\varphi \in \Lambda_\kappa^2$. Differentiating (5.2) by means of the chain rule we obtain sums involving the seven terms

$$\begin{aligned} & D^2\phi(x_j)[Df_j^{-1}(x)]^2 e^{\phi(x_j)}\varphi(x_j), \\ & D\phi(x_j)D^2f_j^{-1}(x)e^{\phi(x_j)}\varphi(x_j), \\ & D\phi(x_j)Df_j^{-1}(x)e^{\phi(x_j)}D\varphi(x_j)Df_j^{-1}(x), \\ & D\phi(x_j)Df_j^{-1}(x)e^{\phi(x_j)}D\phi(x_j)Df_j^{-1}(x), \\ & e^{\phi(x_j)}D\phi(x_j)Df_j^{-1}(x)D\varphi(x_j)Df_j^{-1}(x), \\ & e^{\phi(x_j)}D^2\varphi(x_j)[Df_j^{-1}(x)]^2, \\ & e^{\phi(x_j)}D\varphi(x_j)D^2f_j^{-1}(x). \end{aligned}$$

Our assumption $\max_{s \leq r} \|D^s \phi\|_0 < \varepsilon'_\phi$ with $\varepsilon'_\phi > 0$ small implies that all but the last two previous terms can be taken neglectable. In consequence, proceeding as before we conclude that there exists a uniform constant $C > 0$ depending only on f such that

$$\begin{aligned} \frac{\|D^2(\mathcal{L}_\phi\varphi)(x)\|}{\inf|\mathcal{L}_\phi\varphi|} & \leq C\varepsilon'_\phi + e^{\varepsilon_\phi} \frac{\|D^2\varphi\|_0}{\inf\varphi} \frac{qL^2 + \sigma^{-2}(\deg(f) - q)}{\deg(f)} + e^{\varepsilon_\phi} \frac{\|D\varphi\|_0}{\inf\varphi} \max_{x \in M} \|D^2f^{-1}(x)\| \\ & \leq C\varepsilon'_\phi + \mathcal{E}_2\kappa + \frac{1 - \mathcal{E}_2}{2}\kappa \end{aligned}$$

which can be again taken smaller than $\tilde{\lambda}\kappa$, for some constant $0 < \tilde{\lambda} < 1$, provided that κ is large enough and ε'_ϕ is small. This estimate, which involves the information on the smaller derivatives, proves the strict invariance of the cone Λ_κ^2 under the operator \mathcal{L}_ϕ . The complete statement in the theorem follows by completely analogous computations for the s -derivatives of $\mathcal{L}_\phi g$, with $2 < s \leq r$. In fact, the remaining of the proof can be obtained recursively for $\ell + 1$ using previous information or $s \in \{1, 2, \dots, \ell\}$ by analogous computations of higher order derivatives using the chain rule and estimating dominating terms as above. In fact the number of terms associated containing the derivatives $D^s \varphi$, $s = 1 \dots r$ are clearly less than $r!$ and, by definition of the cones, each of such terms is bounded by $(1 - \mathcal{E}_r)/r!$. Then if ε'_ϕ is small proceeding as above we get that Λ_κ^r is strictly preserved by \mathcal{L}_ϕ , which proves the theorem. \square

Again we use that the smaller cone has finite diameter in the projective metric in the case of the cones for differentiable observables, whose proof can be simply adapted from the one of Proposition 4.3. For that reason we shall omit its proof.

Proposition 5.2. *Given $0 < \hat{\lambda} < 1$, the cone $\Lambda_{\hat{\lambda}\kappa, \delta}^r$ has finite Λ_κ^r -diameter.*

In the next subsection we will use Birkhoff’s theorem to deduce good spectral properties for the Ruelle–Perron–Frobenius operator.

5.2. Strong stability properties

Here we establish the statistical and spectral stochastic stability results. The discussion in Remark 4.13 shows that this property was far from being immediate. In the space of C^r -observables ($r \geq 1$) the Perron–Frobenius vary continuously with the dynamics in the strong (pointwise) operator topology. However, it can also be shown that in general such operators do not vary continuously in norm in the space of bounded linear operators. In fact, the stability results presented here will follow from the careful study of the spectral properties of the transfer operators and will be consequence of the uniformity of the gap spectral for all close dynamical systems and potentials.

Throughout this subsection let \mathcal{F}^r be a family of C^r , $r \geq 1$ local diffeomorphisms and let \mathcal{W}^r be some family of C^r -potentials satisfying (H1), (H2) and (P’) with uniform constants. Let $B(C^r(M, \mathbb{R}), C^r(M, \mathbb{R}))$ denote the space of bounded linear operators on $C^r(M, \mathbb{R})$ endowed with the strong operator topology.

Proposition 5.3. *The Ruelle–Perron–Frobenius operator function*

$$\begin{aligned} \mathcal{F}^r \times \mathcal{W}^r &\rightarrow B(C^r(M, \mathbb{R}), C^r(M, \mathbb{R})), \\ (f, \phi) &\mapsto \mathcal{L}_{f, \phi} \end{aligned}$$

is continuous in the C^r -topology.

Proof. Let $(f, \phi), (\tilde{f}, \tilde{\phi}) \in \mathcal{F}^r \times \mathcal{W}^r$ be given. Then for any fixed $\varphi \in C^r(M, \mathbb{R})$ and $x \in M$ we get that

$$\|\mathcal{L}_{\tilde{f}, \tilde{\phi}}(\varphi)(x) - \mathcal{L}_{f, \phi}(\varphi)(x)\| \leq \sum_{j=1}^{\deg(f)} \|\varphi(\tilde{f}_i^{-1}(x)) \cdot e^{\tilde{\phi}(\tilde{f}_i^{-1}(x))} - \varphi(f_i^{-1}(x)) \cdot e^{\phi(f_i^{-1}(x))}\|$$

where, as before, f_i^{-1} denote the inverse branches of f at x . Moreover, the right-hand side above goes to zero independently of x as $(\tilde{f}, \tilde{\phi})$ converges to (f, ϕ) in the C^1 -topology. Furthermore, $\|D\mathcal{L}_{\tilde{f}, \tilde{\phi}}(\varphi)(x) - D\mathcal{L}_{f, \phi}(\varphi)(x)\|$ is bounded by

$$\sum_{j=1}^{\deg(f)} \|D(\varphi(\tilde{f}_i^{-1}(x)) \cdot e^{\tilde{\phi}(\tilde{f}_i^{-1}(x))}) - D(\varphi(f_i^{-1}(x)) \cdot e^{\phi(f_i^{-1}(x))})\|,$$

which also converges uniformly to zero by standard triangular argument as the element provided that $(\tilde{f}, \tilde{\phi})$ converge to (f, ϕ) . We note that analogous computations hold for higher order derivatives which lead to the statement of the proposition. \square

Now we deduce our functional analysis approach to deduce the important continuity of the topological pressure, a fact unknown in [41].

Proposition 5.4. *The topological pressure function $\mathcal{F}^r \times \mathcal{W}^r \ni (f, \phi) \rightarrow P_{\text{top}}(f, \phi)$ is continuous in the C^r -topology, for $r \geq 1$. Moreover, the densities $\frac{d\mu_{f, \phi}}{dv_{f, \phi}}$ vary continuously with respect to $(f, \phi) \in \mathcal{F}^r \times \mathcal{W}^r$.*

Proof. This proof goes along the same lines of the proof of Proposition 4.12. For that reason we will prove the result by focusing on the main differences. First notice that $\lambda_{f, \phi}$ is the leading eigenvalue and spectral radius of the operator $\mathcal{L}_{f, \phi}$ acting in any space of the Banach spaces C^r with $r \geq 0$. Now, using once more that all transfer operators associated to all $(\hat{f}, \hat{\phi})$ in some neighborhood \mathcal{U} of (f, ϕ) preserve the same cone of functions we obtained, following Proposition 4.4 and Corollary 4.5, that the limit

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|\tilde{\mathcal{L}}_{\hat{f}, \hat{\phi}}^n(1)\|_r = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \|\lambda_{f, \phi}^{-n} \mathcal{L}_{\hat{f}, \hat{\phi}}^n(1)\|_r = 0$$

is uniform for all $(\hat{f}, \hat{\phi})$ in some neighborhood \mathcal{U} of (f, ϕ) . Therefore, by standard triangular inequality together with the continuity of the transfer operators in the C^r -strong topology it follows that given $\epsilon > 0$ there exists n_0 such that

$$\begin{aligned} |P_{\text{top}}(f, \phi) - P_{\text{top}}(\hat{f}, \hat{\phi})| &\leq \left| \frac{1}{n_0} \log \|\mathcal{L}_{\hat{f}, \hat{\phi}}^{n_0}(1)\|_r - \log(\lambda_{\hat{f}, \hat{\phi}}) \right| + \left| \frac{1}{n_0} \log \|\mathcal{L}_{f, \phi}^{n_0}(1)\|_r - \log(\lambda_{f, \phi}) \right| \\ &\quad + \left| \frac{1}{n_0} \log \|\mathcal{L}_{f, \phi}^{n_0}(1)\|_r - \frac{1}{n_0} \log \|\mathcal{L}_{\hat{f}, \hat{\phi}}^{n_0}(1)\|_r \right| < \epsilon \end{aligned}$$

as $(\hat{f}, \hat{\phi})$ converges to (f, ϕ) . This argument shows that the leading eigenvalue, thus the topological pressure, vary continuously. Proceeding as in the proof of Proposition 4.4, noticing that $C^r(M, \mathbb{R}) \subset C^0(M, \mathbb{R})$ and $\int \lambda_{f, \phi}^{-1} \mathcal{L}_{f, \phi} dv = 1$, one obtains from the contraction on the projective metric Θ_κ that

$$\left\| \frac{d\mu_{\hat{f}, \hat{\phi}}}{dv_{\hat{f}, \hat{\phi}}} - \frac{d\mu_{f, \phi}}{dv_{f, \phi}} \right\|_r \leq 6R_1 \Delta \tau^n + \|\lambda_{\hat{f}, \hat{\phi}}^{-n} \mathcal{L}_{\hat{f}, \hat{\phi}}^n 1 - \lambda_{f, \phi}^{-n} \mathcal{L}_{f, \phi}^n 1\|_r$$

where R_1 is a uniform upper bound for the C^0 -norm of the iterates $\lambda_{f,\phi}^{-n} \mathcal{L}_{f,\phi}^n 1$ in a neighborhood of (f, ϕ) , the constant Δ is the diameter of the cone Λ_κ^r , and $0 < \tau < 1$. Using the continuity of the transfer operators given in Proposition 5.3 then for any fixed n the last expression in the right-hand side above can be made arbitrarily small provided that $(\hat{f}, \hat{\phi})$ is sufficiently close to (f, ϕ) . This proves the continuity of the density function and finishes the proof of the proposition. \square

We are now in a position to prove that the equilibrium states are strongly stable under deterministic perturbations.

Proof of Theorem C. The continuity of topological pressure given by Proposition 5.4 together with Theorem D in [41] that the conformal measures ν_{f_n, ϕ_n} converge to $\nu_{f, \phi}$ in the weak* topology as (f_n, ϕ_n) goes to (f, ϕ) in the C^r -topology. Now, using that $\frac{d\mu_{f,\phi}}{d\nu_{f,\phi}}$ is C^r and also varies continuously in the C^r -norm with $(f, \phi) \in \mathcal{F}^r \times \mathcal{W}^r$ it follows that the equilibrium state $\mu_{f,\phi}$ also varies continuously in the weak* topology, which completes the proof of the theorem. \square

Finally we derive the strong stochastic stability of the spectra. Consider any family θ_ε , $0 < \varepsilon \leq 1$ of probability measures in $\mathcal{F}^r \times \mathcal{W}^r$ such that its support $\text{supp } \theta_\varepsilon$ is contained in a neighborhood $V_\varepsilon(f, \phi)$ of (f, ϕ) depending monotonically on ε and satisfying $\bigcap_{0 < \varepsilon \leq 1} V_\varepsilon(f, \phi) = \{(f, \phi)\}$. We refer to $(\theta_\varepsilon)_\varepsilon$ as an arbitrary random perturbation of $(f, \phi) \in \mathcal{F}^r \times \mathcal{W}^r$. We first prove that the stochastic transfer operator $\mathcal{L}_\varepsilon : C^r(M, \mathbb{R}) \rightarrow C^r(M, \mathbb{R})$ given by

$$\mathcal{L}_\varepsilon(\varphi) = \int \mathcal{L}_{\tilde{f}, \tilde{\phi}} \varphi d\theta_\varepsilon(\tilde{f}, \tilde{\phi}) \tag{5.3}$$

is well defined and preserves a cone of C^r -observables.

Lemma 5.5. *The stochastic transfer operator \mathcal{L}_ε defined in (5.3) is well defined. Moreover, there exists $0 < \hat{\lambda} < 1$ so that $\mathcal{L}_\varepsilon(\Lambda_\kappa^r) \subset \Lambda_{\hat{\lambda}\kappa}^r$ for every small ε and every large κ .*

Proof. First we prove that the stochastic transfer operator \mathcal{L}_ε is well defined. Given any fixed $\varphi \in C^r(M, \mathbb{R})$ it follows that $\mathcal{L}_{\tilde{f}, \tilde{\phi}}(\varphi)$ is C^r for all $(\tilde{f}, \tilde{\phi}) \in \mathcal{F}^r \times \mathcal{W}^r$. Moreover, since the constants are taken uniform in the family \mathcal{F}^r and \mathcal{W}^r then it is a consequence of Lebesgue dominated convergence theorem that $\mathcal{L}_\varepsilon(\varphi)$ is also C^r . This proves the first claim in the lemma.

On the other hand, by construction we obtain $0 < \hat{\lambda} < 1$ and κ large so that $\mathcal{L}_{\tilde{f}, \tilde{\phi}}(\Lambda_\kappa^r) \subset \Lambda_{\hat{\lambda}\kappa}^r$ for every $(\tilde{f}, \tilde{\phi})$ in a neighborhood of (f, ϕ) . In particular, if ε is small then this property holds in $V_\varepsilon(f, \phi)$ and, consequently, $\mathcal{L}_\varepsilon(\Lambda_\kappa^r) \subset \Lambda_{\hat{\lambda}\kappa}^r$. This proves the second statement finishes the proof of the lemma. \square

We finish our section by proving our spectral stochastic stability result.

Proof of Theorem D. Let $(f, \phi) \in \mathcal{F}^r \times \mathcal{W}^r$ be fixed. By Proposition 5.3 the transfer operators $\mathcal{L}_{\tilde{f}, \tilde{\phi}}$ acting on the space $C^r(M, \mathbb{R})$ vary continuously with $(\tilde{f}, \tilde{\phi}) \in \mathcal{F}^r \times \mathcal{W}^r$ in the strong operator topology.

Recall also that the dominant eigenvalue for $\mathcal{L}_{f,\phi}$ equals to the spectral radius and has multiplicity one and that both the leading eigenvalue and corresponding eigenspace vary continuously. Moreover, since all transfer operators $\mathcal{L}_{\tilde{f}, \tilde{\phi}}$ preserve the same cone Λ_κ^r for all $(\tilde{f}, \tilde{\phi})$ in a small neighborhood of (f, ϕ) then it follows from the last proposition that the same property holds for \mathcal{L}_ε with ε small. Proceeding as in the later sections we get that \mathcal{L}_ε has a spectral gap for every small ε . In particular, there exists a unique eigenvalue λ_ε , which coincides with the spectral radius of \mathcal{L}_ε , and the eigenspace associated to λ_ε is one-dimensional.

We claim that the spectral radius λ_ε of \mathcal{L}_ε varies continuously for all small ε and that it converges to $\lambda_{f,\phi}$ whenever ε tends to zero. If $\varepsilon > 0$ is small we have that all operators $\lambda_\varepsilon^{-1} \mathcal{L}_\varepsilon$ preserve the same cone of functions Λ_κ^r . Moreover, there exists a conformal measure ν_ε , that is, such that $\mathcal{L}_\varepsilon^* \nu_\varepsilon = \lambda_\varepsilon \nu_\varepsilon$ and it follows from the normalization $\int \lambda_\varepsilon^{-1} \mathcal{L}_\varepsilon 1 d\nu_\varepsilon = 1$ that the convergence $\lim_n \frac{1}{n} \log \|\lambda_\varepsilon^{-n} \mathcal{L}_\varepsilon^n 1\|_r = 0$ is uniform for all small ε . Proceeding as in the proof of Proposition 5.4 we deduce that the functions $\varepsilon \rightarrow \lambda_\varepsilon$ and $\varepsilon \rightarrow d\mu_\varepsilon/d\nu_\varepsilon$ vary continuously for all small ε . In fact, proceeding as before one obtains that

$$\left\| \frac{d\mu_\varepsilon}{dv_\varepsilon} - \frac{d\mu_{f,\phi}}{dv_{f,\phi}} \right\|_r \leq 6R_1\Delta\tau^n + \left\| \lambda_\varepsilon^{-n}\mathcal{L}_\varepsilon^n 1 - \lambda_{f,\phi}^{-n}\mathcal{L}_{f,\phi}^n 1 \right\|_r$$

where R_1 is a uniform upper bound for the C^0 -norm of the iterates $\lambda_\varepsilon^{-n}\mathcal{L}_\varepsilon^n 1$ for all small ε , the constant Δ is the diameter of the cone Λ_k^r and $0 < \tau < 1$. Using that for n fixed the functions $\mathcal{L}_\varepsilon^n 1$ and $\mathcal{L}_{f,\phi}^n 1$ are uniformly close provided that ε is small then one deduces that $\lambda_\varepsilon \rightarrow \lambda_{f,\phi}$ and consequently, that the density $d\mu_\varepsilon/dv_\varepsilon$ converges to $d\mu_{f,\phi}/dv_{f,\phi}$ as $\varepsilon \rightarrow 0$. This concludes the proof of our theorem. \square

6. Examples

In this section we provide some examples and comment on our assumptions.

Example 6.1. Let $f_0 : \mathbb{T}^d \rightarrow \mathbb{T}^d$ be a linear expanding map. Fix some covering \mathcal{U} by domains of injectivity for f_0 and some $U_0 \in \mathcal{U}$ containing a fixed (or periodic) point p . Then deform f_0 on a small neighborhood of p inside U_0 by a pitchfork bifurcation in such a way that p becomes a saddle for the perturbed local diffeomorphism f . In particular, such perturbation can be done in the C^r -topology, for every $r > 0$. By construction, f coincides with f_0 in the complement of P_1 , where uniform expansion holds. Observe that we may take the deformation in such a way that f is never too contracting in P_1 , which guarantees that conditions (H1) and (H2) hold, and that f is still topologically exact. Condition (P') is clearly satisfied by any C^r -potential close to $\phi \equiv 0$. Hence, there exists a unique measure of maximal entropy μ for f , it is absolutely continuous with respect to a conformal measure ν , supported in the whole manifold \mathbb{T}^d and has exponential decay of correlations on the space C^r -observables. Moreover, it follows from our results that the density $d\mu/d\nu$ is C^r and it varies continuously in the $C^{\lfloor r \rfloor}$ -topology with the dynamical system f , where $\lfloor r \rfloor$ denotes the integer part of r . Furthermore, the topological pressure function $P_{\text{top}}(f, \phi)$ varies continuously among the pairs (f, ϕ) that satisfy conditions (H1), (H2) and (P') with uniform constants. Finally, in the case that $r \geq 1$ we have that the maximal entropy measure is strong stable under deterministic perturbations and satisfies a random spectral stability.

In fact, the previous example can be modified to deal with expanding maps with indifferent periodic points in a higher-dimensional setting. A particularly interesting one-dimensional example is given by the Manneville–Pomeau transformation and the family of potentials $\varphi_t = -t \log |Df|$. An intermittency phenomenon occurs at $t = 1$ but no longer occurs whenever t is close to zero as we now discuss with detail.

Example 6.2 (Manneville–Pomeau map). If $\alpha \in (0, 1)$, let $f_\alpha : [0, 1] \rightarrow [0, 1]$ be the $C^{1+\alpha}$ -local diffeomorphism given by

$$f_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2x - 1 & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Observe that conditions (H1) and (H2) are verified and the family $\varphi_{\alpha,t} = -t \log |Df_\alpha|$ of C^α -potentials do satisfy condition (P) for all $|t| \leq t_0$ small and $\alpha \in (0, 1)$ since

$$|\varphi_{\alpha,t}(x) - \varphi_{\alpha,t}(y)| = |t \log |Df_\alpha(x)| - t \log |Df_\alpha(y)|| = |t| \log \frac{|Df_\alpha(x)|}{|Df_\alpha(y)|} \leq |t| \log(2 + \alpha).$$

Hence, we obtain that for all $|t| \leq t_0$ there exists a unique equilibrium state μ_t , it is absolutely continuous with respect to a conformal measure ν_t and has exponential decay of correlations in the space of Hölder observables. Moreover, $d\mu_t/d\nu_t$ is Hölder continuous and it varies continuously in the C^0 -norm for all $|t| \leq t_0$.

Moreover no transition occurs once one considers the order of contact α of the indifferent fixed point to increase. Indeed, if α is arbitrary large then it follows from our previous reasoning that there exists a small interval $I_\alpha = [-t_\alpha, t_\alpha]$ containing zero such that the topological pressure $\mathbb{R}^+ \times [-t_\alpha, t_\alpha] \ni (\alpha, t) \mapsto P_{\text{top}}(f_\alpha, \varphi_{\alpha,t})$ varies continuously. Moreover, there is a unique equilibrium state for the C^α -potential $\varphi_{\alpha,t}$ with $|t| \leq t_\alpha$ and it is $C^{\lfloor \alpha \rfloor}$ -strong stable under deterministic perturbations: for every $(\alpha, t) \in \mathbb{R}^+ \times [-t_\alpha, t_\alpha]$ there exists a unique equilibrium state $\mu_{\alpha,t}$ absolutely continuous with respect to a conformal measure $\nu_{\alpha,t}$, its density $d\mu_{\alpha,t}/d\nu_{\alpha,t}$ is $C^{\lfloor \alpha \rfloor}$ and varies continuously with (α, t) . Finally, since our strong random spectral stability result applies for general random perturbations one can consider e.g. θ_ε to be the uniform distribution in the one-parameter family of pairs

$\{(f_\alpha, \varphi_{\alpha, t_\alpha}) : \alpha \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)\} \subset \mathcal{F} \times \mathcal{W}$. In particular, the random dynamical system associated considers random orbits using maps with indifferent fixed points with different contact orders. Here our results yield that the random Ruelle–Perron–Frobenius operator \mathcal{L}_ε has the spectral gap property and that its spectral radius λ_ε converges to $\exp(P_{\text{top}}(f_\alpha, \varphi_{\alpha, t_\alpha}))|_{\alpha=\alpha_0}$ as ε tends to zero.

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