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Fading absorption in non-linear elliptic equations

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Abstract

We study the equation $-\Delta u + h(x)|u|^{q-1}u = 0$, $q > 1$, in $\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times \mathbb{R}_+$ where $h \in C(\overline{\mathbb{R}^N_+})$, $h \geqslant 0$. Let $(x_1, \dots, x_N)$ be a coordinate system such that $\mathbb{R}^N_+ = [x_N > 0]$ and denote a point $x \in \mathbb{R}^N$ by (x', x_N). Assume that $h(x', x_N) > 0$ when $x' \neq 0$ but $h(x', x_N) \to 0$ as $|x'| \to 0$. For this class of equations we obtain sharp necessary and sufficient conditions in order that singularities on the boundary do not propagate in the interior.

1. Introduction

In this paper we study solutions of the equation

$$-\Delta u + h(x)|u|^{q-1}u = 0, \tag{1.1}$$

in $\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times \mathbb{R}_+$ where $q > 1$ and $h \in C(\overline{\mathbb{R}^N_+})$, $h \geqslant 0$. (If $x \in \mathbb{R}^N_+$ we write $x = (x', x_N)$ where $x' = (x_1, \dots, x_{N-1})$.)

If $h > 0$ in $\mathbb{R}^N_+$ then boundary singularities of solutions of (1.1) cannot propagate to the interior. This is due to the presence of the absorption term $h|u|^{q-1}u$ and the Keller–Osserman estimates [3] and [7]. In fact, in this case, (1.1) possesses a maximal solution U in $\mathbb{R}^N_+$ and,

$$\lim_{\substack{x_N \to 0 \\ |x| \leqslant M}} U(x) = \infty \quad \forall M > 0. \tag{1.2}$$

A solution satisfying this boundary condition is called a *large solution*. If, in addition, h is bounded away from zero then the large solution is unique. (See [1] for the case of bounded domains. If h is bounded away from zero, the extension to unbounded domains is standard.)

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On the other hand, if h vanishes on a set $F \subset \mathbb{R}^N_+$ which has limit points on $[x_N = 0]$ then a singularity at these limit points may propagate to the interior. By this we mean that there may exist a sequence $\{u_n\}$ of solutions of (1.1) in $\mathbb{R}^N_+$ which converges in

$$\Omega = \mathbb{R}^N_+ \setminus F$$

but tends to infinity at some points of F.

In this paper we shall study the case where h is positive in Ω but may vanish on

$$F = \{(0, x_N) \in \mathbb{R}^N_+ \colon\ x_N > 0\}.$$

Since h is positive in $\mathbb{R}^N_+ \setminus F$ a singularity at the origin may propagate only along the set F. Furthermore a weak singularity, such as that of the Poisson kernel, cannot propagate to the interior because any solution of (1.1) is dominated by the harmonic function with the same boundary behavior. Therefore we must consider only strong singularities, i.e. singularities which cannot occur in the case of a harmonic function but may occur with respect to solutions of (1.1).

Suppose that

$$h(x', x_N) \leqslant h_0(|x'|),$$

where

$$h_0 \in C^1[0, \infty), \qquad h_0(s) > 0 \quad \text{for } s > 0, \qquad h_0(0) = 0.$$

It is clear that, the faster $h_0(s)$ tends to zero as $s \to 0$ the greater the chance that a strong boundary singularity at the origin will propagate to the interior.

Our aim is to determine a sharp criterion for the propagation of singularities with respect to solutions of (1.1) with $h \in C(\overline{\mathbb{R}^N_+})$ such that $h > 0$ in $\mathbb{R}^N_+ \setminus F$. It turns out that such a criterion can be expressed in terms of functions of the form

$$\bar{h}(s) := e^{-\frac{\omega(s)}{s}}. \tag{1.3}$$

We assume that ω satisfies the following conditions:

$$\begin{aligned} &\text{(i)} && \omega \in C(0,\infty) \text{ is a positive nondecreasing function,} \\ &\text{(ii)} && s \mapsto \mu(s) := \tfrac{\omega(s)}{s} \text{ is monotone decreasing on } \mathbb{R}_+, \\ &\text{(iii)} && \lim_{s\to 0} \mu(s) = \infty \end{aligned} \tag{1.4}$$

bounded. We establish the following results.

Theorem 1.1. *Suppose that*

$$\liminf_{\substack{x \to 0 \\ x' \neq 0}} h(x)/\bar{h}(|x'|) > 0 \tag{1.5}$$

where $\bar{h}$ is given by (1.3) *and that* (1.4) *holds.*

Suppose that ω satisfies the Dini condition,

$$\int_0^1 (\omega(t)/t)\, dt < \infty. \tag{1.6}$$

If $\{u_n\}$ is a sequence of positive solutions of (1.1) *in $\mathbb{R}^N_+$ converging* (*pointwise*) *in*

$$\Omega = \mathbb{R}^N_+ \setminus F$$

then the sequence converges in $\mathbb{R}^N_+$ and its limit is a solution of (1.1) *in $\mathbb{R}^N_+$.*

In particular, (1.1) *possesses a maximal solution U in $\mathbb{R}^N_+$ and U is a large solution.*

Theorem 1.2. *Suppose that there exists a constant $c > 0$ such that*

$$h(x) \leqslant c\bar{h}\big(|x'|\big) \quad \forall x \in \mathbb{R}^N_+ \tag{1.7}$$

where $\bar{h}$ is given by (1.3)*. Assume that* (1.4) *and the following additional conditions hold*:

$$\limsup_{j\to\infty} \frac{\mu(a^{-j+1})}{\mu(a^{-j})} < 1 \quad \text{for some } a > 1 \tag{1.8}$$

and

$$\lim_{s\to 0} \mu(s)/|\ln s| = \infty. \tag{1.9}$$

Condition (1.9) *guarantees that, for every real k,* (1.1) *has a solution $u_{0,k}$ with boundary data $k\delta_0$ (where δ_0 denotes the Dirac measure at the origin).*

Under these assumptions, if

$$\int_0^1 \big(\omega(t)/t\big)\, dt = \infty \tag{1.10}$$

then

$$u_{0,\infty} = \lim u_{0,k} \tag{1.11}$$

is a solution of (1.1) *in Ω but*

$$u_{0,\infty}(x) = \infty \quad \forall x \in F.$$

Corollary 1.1. *Suppose that there exists a positive constant c such that*

$$c^{-1}\bar{h}\big(|x'|\big) \leqslant h(x) \leqslant c\bar{h}\big(|x'|\big) \quad \forall x \in \overline{\mathbb{R}^N_+} \tag{1.12}$$

where $\bar{h}$ is given by (1.3) *and satisfies conditions* (1.4), (1.8) *and* (1.9)*. Then the Dini condition* (1.6) *is necessary and sufficient for the existence of a large solution of* (1.1) *in $\mathbb{R}^N_+$. It is also necessary and sufficient for the existence of the strongly singular solution $u_{0,\infty}$.*

Problems concerning the propagation of singularities for semilinear equations with absorption have been studied in [6,9] (elliptic case) and in [5,8,10] (parabolic case). In the elliptic case it was assumed that the absorption term is positive everywhere in the interior of the domain, fading only on the boundary. Consequently singularities could propagate only along the boundary.

In [5] the authors studied the equation

$$\partial_t u - \Delta u + e^{-\frac{1}{t}} u^q = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+ \tag{1.13}$$

and proved that if u is a positive solution with strong singularity at a point on $t = 0$ then u blows up at every point of the initial plane. In [6] the authors studied the corresponding elliptic problem in a domain D where the coefficient of the absorption term is $e^{-\frac{1}{\rho(x)}}$, $\rho(x) = \operatorname{dist}(x, \partial\Omega)$, proving a similar result.

In [8] the authors considered the equation,

$$\partial_t u - \Delta u + e^{-\frac{\omega(t)}{t}} u^q = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+ \tag{1.14}$$

where ω is a positive, continuous and increasing function on $\mathbb{R}_+$. They proved that if $\sqrt{\omega}$ satisfies the Dini condition then there exist solutions with a strong isolated singularity at a point on $t = 0$. Similar sufficient conditions were obtained in [9] and [10] with respect to an elliptic (respectively parabolic) equation, where the absorption term vanishes at the boundary (respectively along the axis $x = 0$, $t > 0$). In addition some necessary conditions were presented, but a considerable gap remained between these and the corresponding sufficient conditions. In the recent preprint [11] the authors provide a rough condition for the propagation of singularities for equation (1.14) when the absorption term vanishes along an ascending curve. Conditions for the non-propagation of singularities are not discussed.

In the present paper, the proof of the sufficiency of the Dini condition (Theorem 1.1) is based on a refinement of the energy estimates technique. Given $R > 0$ denote by x^R the point $(x', x_N) = (0, R)$ and let $D_R = B_R(x^R)$. For $M > 0$ denote by V_M the solution of (1.1) in D_R such that $V_M = M$ on ∂D_R. For $r \in (0, R)$ let $D'_r := \{(x', x_N): |x'| < r, |x_N - x^R| < r\}$. By estimating various energy integrals of V_M over domains D'_r and using a double iteration scheme we show that, for a specific sequence $\{M_j\}$ tending to infinity and a related sequence $\{r_j\}$ decreasing to a positive number b,

$$\sup_j \int_{D'_{r_j}} \left(|\nabla V_{M_j}|^2 + h(x) V_{M_j}^{q+1}\right) dx < \infty$$

for every sufficiently small R. By a standard argument this leads to the conclusion that $V^R := \lim V_M$ is bounded in a neighborhood of $F \cap D_R$ and consequently it is a solution of (1.1) in D_R. Let w_k denote the solution of (1.1) in $\mathbb{R}^N_+$ such that $w_k = k$ on $x_N = 0$. Then $w_k \leqslant V^R$ in D_R and consequently $w = \lim w_k < V^R$ in D_R. This implies that w is finite everywhere on F so that w is a large solution of (1.1) in $\mathbb{R}^N_+$. Using this fact we prove the conclusion of Theorem 1.1, first in the case that $h = \bar{h}$ and then in the general case.

The proof of the necessity of the Dini condition (Theorem 1.2) is based on the analysis of a sequence of boundary value problems whose solutions are dominated by $u_{0,\infty}$ and blow up on F. The sequence of boundary value problems is of the form:

$$-\Delta u_j + a_j u_j^q = 0 \quad \text{in } \Omega_j := \left[|x'| < 2^{-j}\right] \cap [x_N > 0],$$
$$u_j(x) = 0 \quad \text{on } \left[|x'| = 2^{-j}\right] \cap [x_N > 0],$$
$$u_j(x', 0) = \gamma_j(x') \quad \text{for } |x'| \leqslant 2^{-j}$$

where $a_j = \sup_{|x'|<2^{-j}} \bar{h}$. The boundary data γ_j is chosen in such a way that, by a transformation akin to the similarity transformation, each problem is reduced to a boundary value problem in $[|x'| < 1] \cap [x_N > 0]$, which is independent of j. Using this fact and a result of Brada [2] we derive precise upper and lower estimates for u_j. By an iterative technique these estimates lead to the conclusion that $\{u_j\}$ blows up at F.

The methods of the present paper can be applied to many of the problems with fading absorption mentioned before (parabolic and elliptic). In a subsequent paper we shall consider a parabolic problem involving the equation

$$\partial_t u - Lu + h(x)|u|^{q-1}u = 0 \tag{1.15}$$

in a cylindrical domain $D \times \mathbb{R}_+$ where $D \subset \mathbb{R}^N$, $0 \in D$ and the absorption term fades along the axis $x = 0$, $t > 0$. Here L is a linear, second order, uniformly elliptic operator with smooth coefficients which may depend on both space and time variables.

Assuming that $h = \bar{h}(|x|)$, with $\bar{h}$ as in (1.3), we shall study the question of propagation of singularities along this axis, using the tools developed in the present paper.

2. Proof of Theorem 1.1

Given $R > 0$ let $x^R = (0, R)$ and denote by B_R the ball of radius R centered at x^R. We shall prove the following:

Theorem 2.1. *Suppose that $h = \bar{h}$. Then, under the assumptions of Theorem 1.1, for every $R > 0$, (1.1) has a solution V^R in B_R which blows up on ∂B_R, i.e.,*

$$V^R(x) \to \infty \quad \text{as } x \to \partial B_R.$$

Before proving this theorem, we show that it implies Theorem 1.1.

Let v_k denote the (smallest) solution of (1.1) in $\mathbb{R}^N_+$ such that $v_k = k$ on the boundary. This means that

$$v_k = \lim_{r\to\infty} v_{k,r},$$

where $v_{k,r}$ is the solution of (1.1) in $B_r(0) \cap \mathbb{R}^N_+$ such that $v_{k,r} = k$ on

$$\partial B_r(0) \cap [x_N = 0]$$

and $v_{k,r} = 0$ on $\partial B_r(0) \cap [x_N > 0]$. Note that, for fixed k, $v_{k,r}$ increases with respect to r. Put

$$V = \lim_{k\to\infty} v_k.$$

Condition (1.5) implies that there exist positive constants c_1 and R_1 such that

$$h(x) \geqslant c_1 \bar{h}(|x'|) \quad \text{for } |x| < 2R_1. \tag{2.1}$$

Without loss of generality we assume that $c_1 = 1$. Therefore, if V^R is as in Theorem 2.1 then, for $R \in (0, R_1)$, V^R is a supersolution of (1.1) in B_R. Since V_R blows up at ∂B_R we conclude that $v_k \leqslant V^R$ in B_R and consequently

$$V \leqslant V^R \quad \forall R \in (0, R_1).$$

Further this implies that V is locally bounded in the strip $\mathbb{R}^N_+ \cap [x_N < R'_0]$ and therefore, everywhere in $\mathbb{R}^N_+$. By its definition V is the smallest large solution of (1.1) in $\mathbb{R}^N_+$.

Now let us assume that $h = \bar{h}$. In this case we may apply Theorem 2.1 in the half-space $[x_N > a]$, for every real a. In particular we deduce that for every $a > 0$ there exists a large solution of (1.1) in $B_r(a)$ for every $r \in (0, a]$. The smallest large solution is denoted by V^r_a. Let $\{u_n\}$ be a sequence of positive solutions of (1.1) in $\mathbb{R}^N_+$ converging pointwise to u in $\mathbb{R}^N_+ \setminus F$. For every $a > 0$ and $r \in (0, a)$, $u_n < V^r_a$ in $B_r(a)$. Consequently $\{u_n\}$ is bounded in $B_r(a)$ for every r and a as above. This implies that $\{u_n\}$ converges pointwise in $\mathbb{R}^N_+$ and the limit u is a solution of (1.1) in $\mathbb{R}^N_+$.

In the general case, it remains true that $\{u_n\}$ is bounded in $B_r(a)$ for every $a \in (0, R_1)$ and $r \in (0, a)$. This implies that $\{u_n\}$ converges pointwise in the strip $\mathbb{R}^N_+ \cap [x_n < R_1]$ and the limit u is a solution of (1.1) in this strip. This in turn implies that $\{u_n\}$ converges to a solution u in $\mathbb{R}^N_+$ which is the first assertion of Theorem 1.1. By a standard argument, if U is the supremum of all solutions of (1.1) in $\mathbb{R}^N_+$ then there exists a sequence of solutions $\{u_n\}$ that converges to U in $\mathbb{R}^N_+ \setminus F$. Therefore, the first assertion implies that U is a solution of (1.1) in $\mathbb{R}^N_+$.

The proof of Theorem 2.1 is based on estimates of certain energy integrals of solutions of (1.1). In a half-space these integrals are infinite. Therefore we shall estimate integrals over a bounded domain for solutions with arbitrary large boundary data.

Condition (1.6) implies that $\lim_{s\to 0} \omega(s) = 0$ while (1.4) implies that $\lim_{s\to 0} \bar{h}(s) = 0$. We extend both of these functions to $[0, \infty)$ by setting them equal to zero at the origin.

In the course of the proof we denote by c, c', c_i constants which depend only on N, q. The value of the constant may vary from one formula to another. A notation such as $C(b)$ denotes a constant depending on the parameter b as well as on N, q.

2.1. Part 1

Let R, b be positive numbers such that $R/8 < b < R/2$. Denote by U_M, $M > 0$, the solution of (1.1) in $B_R(0)$ such that $U_M = M$ on $\partial B_R(0)$.

Let

$$\Omega_b = \{x = (x', x_N) \in \mathbb{R}^N \colon |x'| < b,\ |x_N| < b\}.$$

We start with an elementary estimate of the energy integral:

$$I_b(M) = \int_{\Omega_b} \left(|\nabla U_M|^2 + h(x) U_M^{q+1}\right) dx. \tag{2.2}$$

Lemma 2.1. *Let h be as in (1.3) and assume (1.4). Then*

$$I_b(M) \leqslant C_1(b) M^{q+1}, \quad C_1(b) = c b^N \bar{h}(8b). \tag{2.3}$$

Proof. Let $v_M := U_M - M$. Multiplying (1.1) (for $u = U_M$) by v_M and integrating by parts we obtain,

$$\int_{B_R(0)} \left(|\nabla U_M|^2 + h(x) U_M^q v_M\right) dx = 0.$$

Therefore

$$I_b(M) \leqslant \int\limits_{B_R(0)} \left(|\nabla U_M|^2 + h(x)U_M^{q+1}\right) dx$$
$$= M \int\limits_{B_R(0)} h(x)U_M^q \, dx \leqslant c' M^{q+1} \bar{h}(R) R^N \leqslant c b^N \bar{h}(8b) M^{q+1}. \qquad \square \tag{2.4}$$

Notation. Put

$$\Omega_b(s) := \left\{x \in \mathbb{R}^N:\ s < \left|x'\right| < b - s,\ |x_N| < b - s\right\} \quad \forall s \in (0, b/2). \tag{2.5}$$

If v is a positive solution of (1.1) in $B_R(0)$, denote

$$J_b(s; v) := \int\limits_{\Omega_b(s)} \left(|\nabla_x v|^2 + \bar{h}\left(\left|x'\right|\right) v^{q+1}\right) dx. \tag{2.6}$$

Finally denote,

$$\varphi_b(s) := \int\limits_{\partial\Omega_b(s)} h(x)^{-\frac{2}{q-1}} \, d\sigma. \tag{2.7}$$

Proposition 2.1. *There exists a constant c such that, for every positive solution v of (1.1) in $B_R(0)$,*

$$J_b(s; v) \leqslant c \left(\int\limits_0^s \varphi_b(r)^{-\frac{q-1}{q+3}} \, dr \right)^{-\frac{q+3}{q-1}} \quad \forall s \in (0, b/2). \tag{2.8}$$

Proof. Put $S_b(s) := \partial\Omega_b(s)$ and denote by $\vec{n} = \vec{n}(x)$ the unit outward normal to $S_b(s)$ at x.

Multiplying Eq. (1.1) by v and integrating by parts over $\Omega_b(s)$ we obtain,

$$\int\limits_{\Omega_b(s)} \left(|\nabla_x v|^2 + \bar{h}\left(\left|x'\right|\right) v^{q+1}\right) dx = \int\limits_{S_b(s)} \frac{\partial v}{\partial \vec{n}} v \, d\sigma. \tag{2.9}$$

We estimate the term on the right-hand side using first Hölder's inequality (for a product of three terms) and secondly Young's inequality:

$$\left| \int\limits_{S_b(s)} v \frac{\partial v}{\partial \vec{n}} \, d\sigma \right| \leqslant \int\limits_{S_b(s)} |\nabla_x v||v| \, d\sigma$$
$$\leqslant \left(\int\limits_{S_b(s)} |\nabla_x v|^2 \, d\sigma \right)^{\frac{1}{2}} \left(\int\limits_{S_b(s)} h(x)|v|^{q+1} \, d\sigma \right)^{\frac{1}{q+1}} \varphi_b(s)^{\frac{q-1}{2(q+1)}}$$
$$\leqslant c_1 \left(\int\limits_{S_b(s)} \left(|\nabla_x v|^2 + h(x) v^{q+1}\right) d\sigma \right)^{\frac{q+3}{2(q+1)}} \varphi_b(s)^{\frac{q-1}{2(q+1)}}. \tag{2.10}$$

Substituting estimate (2.10) into (2.9) we obtain:

$$J_b(s; v) \leqslant c_2 \left(\int\limits_{S_b(s)} \left(|\nabla_x v|^2 + h(x) v^{q+1}\right) d\sigma \right)^{\frac{q+3}{2(q+1)}} \varphi_b(s)^{\frac{q-1}{2(q+1)}}. \tag{2.11}$$

Since

$$-\frac{d}{ds} J_b(s; v) = \int\limits_{S_b(s)} \left(|\nabla_x v|^2 + h(x) v^{q+1}\right) d\sigma,$$

$|x_N| = b - s$, inequality (2.11) is equivalent to

$$J_b(s; v) \leqslant c_3 \varphi_b(s)^{\frac{q-1}{2(q+1)}} \left(-\frac{d}{ds} J_b(s; v)\right)^{\frac{q+3}{2(q+1)}} \quad \forall s \in (0, b/2).$$

Solving this differential inequality, with initial data $J_b(b/2; v) = 0$, we obtain (2.8). □

In continuation we derive a more explicit estimate for h as in (1.3). We need the following technical lemma.

Lemma 2.2. *Let $A > 0$, $m \in \mathbb{N}$, $l \in \mathbb{R}^1$ and let $\omega \in C^1(0, \infty)$ be a positive function satisfying condition* (1.4). *Then there exist $\bar{s} \in (0, 1)$, depending on A, l and ω such that the following inequality holds*:

$$\int_0^s t^{m-1} \omega(t)^l \exp\bigl(-A\mu(t)\bigr)\, dt \geqslant \frac{s^{m+1}\omega(s)^{l-1}}{(m+1)\mu(s)^{-1} + A} \exp\bigl(-A\mu(s)\bigr) \quad \forall s\colon 0 < s < \bar{s}. \tag{2.12}$$

Proof. Due to condition (1.4)(ii) integration by parts yields:

$$\int_0^s t^m \omega(t)^l \exp\bigl(-A\mu(t)\bigr)\, dt = \frac{s^{m+1}}{m+1} \omega(s)^l \exp\bigl(-A\mu(s)\bigr) - \int_0^s \frac{At^{m-1}}{m+1} \exp\bigl(-A\mu(t)\bigr)\omega(t)^{l+1}\, dt$$
$$+ \int_0^s \frac{t^{m+1}}{m+1} \exp\bigl(-A\mu(t)\bigr)\omega'(t)\omega^{l-1}\bigl(A\mu(t) - l\bigr)\, dt. \tag{2.13}$$

Again due to (1.4)(ii), there exists $\bar{s} > 0$ such that

$$A\mu(s) \geqslant l \quad \forall s \in (0, \bar{s}).$$

For later estimates it is convenient to choose $\bar{s}$ in $(0, 1)$.

As $\omega(s)$ is non-decreasing, it follows that, for $0 < s \leqslant \bar{s}$,

$$\left(s + \frac{A\omega(s)}{m+1}\right) \int_0^s t^{m-1} \omega(t)^l \exp\bigl(-A\mu(t)\bigr)\, dt \geqslant \frac{s^{m+1}}{m+1} \omega(s)^l \exp\bigl(-A\mu(s)\bigr).$$

This inequality is equivalent to (2.12). □

Proposition 2.2. *Assume that h is given by* (1.3) *and satisfies* (1.4). *Then there exists a constant $s^* \in (0, b/2)$, depending on N, q and the rate of blow-up of $\mu(s) = \omega(s)/s$ as $s \to 0$, such that*

$$J_b(s; v) \leqslant c b^{N-1} \exp Q(s) \quad \forall s \in \bigl(0, s^*\bigr),$$
$$Q(s) = \frac{2\mu(s)}{q-1} + \frac{q+3}{q-1} \ln \mu(s) - \frac{q+3}{q-1} \ln s, \tag{2.14}$$

for every positive solution v of (1.1) *in $B_R(0)$.*

If, in addition, there exists a positive constant β such that

$$\beta \ln \frac{1}{s} \leqslant \mu(s) \quad 0 < s \leqslant s^*, \tag{2.15}$$

then

$$Q(s) \leqslant Q_0 \mu(s) \quad 0 < s \leqslant s^* \tag{2.16}$$

where

$$Q_0 := \frac{2}{q-1} + \frac{q+3}{(q-1)} + \frac{q+3}{\beta(q-1)}. \tag{2.17}$$

Proof. Denote

$$S_{b,1}(s)=\{x:\ |x'|=s,\ |x_N|<b\}\cup\{x:\ |x'|=b-s,\ |x_N|<b\}$$

and

$$S_{b,2}(s)=\{x:\ s<|x'|<b-s,\ |x_N|=b\}.$$

Then

$$\int_{S_{b,1}}\bar h(|x'|)^{-\frac{2}{q-1}}\,d\sigma=2\gamma_{N-1}(b-s)\big(\bar h(s)^{-\frac{2}{q-1}}s^{N-2}+\bar h(b)^{-\frac{2}{q-1}}(b-s)^{N-2}\big)$$
$$\leqslant 4b^{N-1}\gamma_{N-1}\exp\frac{2\mu(s)}{q-1}\quad 0<s<b/2, \tag{2.18}$$

where γ_{N-1} denotes the area of the unit sphere in $\mathbb{R}^{N-1}$. Further, since μ is monotone decreasing,

$$\int_{S_{b,2}}\bar h(|x'|)^{-\frac{2}{q-1}}\,d\sigma=2\gamma_{N-1}\int_s^{b-s}\exp\frac{2\mu(\rho)}{q-1}\rho^{N-2}\,d\rho$$
$$\leqslant 2(N-1)^{-1}b^{N-1}\gamma_{N-1}\exp\frac{2\mu(s)}{q-1}. \tag{2.19}$$

By (2.18) and (2.19):

$$\varphi_b(s)=\int_{S_b(s)}\bar h(|x'|)^{-\frac{2}{q-1}}\,d\sigma\leqslant cb^{N-1}\exp\frac{2\mu(s)}{q-1},\quad 0<s<b/2,$$

where $c=(4+2(N-1)^{-1})\gamma_{N-1}$. This implies,

$$\int_0^s\varphi_b(r)^{-\frac{q-1}{q+3}}\,dr\geqslant c_1b^{-\frac{(N-1)(q-1)}{q+3}}\int_0^s\exp\left(-\frac{2\mu(r)}{q+3}\right)dr,\quad c_1=c^{-\frac{q-1}{q+3}}. \tag{2.20}$$

Let s^* be the largest number in $(0,b/2)$ such that

- ◇ $s^*\leqslant\bar s$ ($\bar s$ as in Lemma 2.2 for $l=0$, $m=1$ and $A=\frac{2}{q+3}$),
- ◇ $\mu(s^*)\geqslant A^{-1}=(q+3)/2$.

Then (2.20) and (2.12) imply

$$\int_0^s\varphi_b(r)^{-\frac{q-1}{q+3}}\,dr\geqslant c_2b^{-\frac{(N-1)(q-1)}{q+3}}\frac{s^2}{\omega(s)}\exp\left(-\frac{2\mu(s)}{q+3}\right),\quad c_2=c_1(q+3)/6, \tag{2.21}$$

for all $s\in(0,s^*]$. This inequality and (2.8) imply (2.14).

Suppose now that the function $\mu(\cdot)$ given by (1.3) satisfies (2.15). Since $\ln r\leqslant r$ for $r\geqslant 1$, conditions (1.4), (2.14) and (2.15) imply (2.16). □

Next we estimate energy integrals over domains of the form

$$\Omega_b(\tau,\sigma):=\{x=(x',x_N):\ |x'|<\sigma,\ |x_N|<b-\tau\} \tag{2.22}$$

where $0<\sigma<b/2$, $0\leqslant\tau<b$.

Let $\eta\in C^\infty([0,\infty))$ be a monotone decreasing function such that

$$\eta(s)=1\quad\text{if } s<1,\qquad \eta(s)=0\quad\text{if } s>2,\qquad \eta'(s)\leqslant 2 \tag{2.23}$$

and denote

$$\eta_\sigma(s) = \eta(s/\sigma).$$

We shall estimate the integrals,

$$E_b(\tau,\sigma;v) := \int_{\Omega_b(\tau,2\sigma)} \left(\left|\nabla_x\left(\eta_\sigma\left(\left|x'\right|\right)v\right)\right|^2 + h(x)\eta_\sigma\left(\left|x'\right|\right)^2 v^{q+1}\right)dx. \tag{2.24}$$

Proposition 2.3. *Assume condition* (1.4)*. Let* $s^* \in (0, b/2)$ *be as in Proposition* 2.2*. Then the following inequality holds for* $0 < \sigma \leqslant s^*$ *and* $\sigma \leqslant \tau < b$*:*

$$E_b(\tau,\sigma;v) \leqslant c\sigma\left(-\frac{dE_b(\tau,\sigma;v)}{d\tau}\right) + C_2(b)\exp H(\sigma), \tag{2.25}$$

where $C_2(b) := cb^{\frac{2(N-1)}{q+1}}$*,*

$$\begin{aligned} H(\sigma) &= 2\frac{Q(\sigma)+\mu(\sigma)}{q+1} + \frac{(N-1)(q-1)-2(q+1)}{q+1}\ln\sigma \\ &= \frac{2\mu(\sigma)}{q-1} + \frac{2(q+3)}{q^2-1}\ln\mu(\sigma) - c_+^*\ln\sigma \end{aligned} \tag{2.26}$$

and

$$c^* = \frac{2(q+3)+2(q^2-1)-(N-1)(q-1)^2}{q^2-1}.$$

If, in addition, condition (2.15) *holds then there exists a constant* H_0 *depending only on* q *and* β *such that*

$$H(\sigma) \leqslant H_0\mu(\sigma), \tag{2.27}$$

where

$$H_0 = \frac{2}{q-1} + \frac{2(q+3)}{(q-1)(q+1)} + \frac{c_+^*}{\beta}. \tag{2.28}$$

Proof. Multiplying Eq. (1.1) by $\eta_\sigma(|x'|)^2 v$ and integrating by parts over $\Omega_b(\tau,2\sigma)$ we obtain,

$$\int_{\Omega_b(\tau,2\sigma)} \nabla v\cdot\nabla\left(v\eta_\sigma^2\right)dx + \int_{\Omega_b(\tau,2\sigma)} h(x)v^{q+1}\eta_\sigma^2\,dx = \int_{S_b'(\tau,2\sigma)} \frac{\partial v}{\partial\vec{n}}v\eta_\sigma^2\,dx', \tag{2.29}$$

where $S_b'(\tau,\sigma) = \{x\colon |x'| < \sigma,\ |x_N| = b-\tau\}$.

We estimate the first term on the left hand side:

$$\begin{aligned} \int_{\Omega_b(\tau,2\sigma)} \nabla v\cdot\nabla\left(v\eta_\sigma^2\right)dx &= \int_{\Omega_b(\tau,2\sigma)} \left|\nabla(v\eta_\sigma)\right|^2 dx - \int_{\Omega_b(\tau,2\sigma)} v^2|\nabla\eta_\sigma|^2\,dx \\ &\geqslant \int_{\Omega_b(\tau,2\sigma)} \left|\nabla(v\eta_\sigma)\right|^2 dx - 4\sigma^{-2}\int_{\tilde{\Omega}_b(\tau,\sigma)} v^2\,dx, \end{aligned} \tag{2.30}$$

where

$$\tilde{\Omega}_b(\tau,\sigma) := \left\{\sigma < \left|x'\right| < 2\sigma,\ |x_N| < b-\tau\right\}. \tag{2.31}$$

Using Hölder's inequality, conditions (1.3), (1.4) and estimate (2.14) with $s=\sigma$, we obtain:

$$\int_{\tilde{\Omega}_b(\tau,\sigma)} v(x)^2\,dx \leqslant \Bigl(\int_{\tilde{\Omega}_b(\tau,\sigma)} v^{q+1}h(x)\,dx\Bigr)^{\frac{2}{q+1}}\Bigl(\int_{\tilde{\Omega}_b(\tau,\sigma)} h(x)^{-\frac{2}{q-1}}\,dx\Bigr)^{\frac{q-1}{q+1}}$$
$$\leqslant c'\bigl(b^{N-1}\exp Q(\sigma)\bigr)^{\frac{2}{q+1}}\bar{h}(\sigma)^{-\frac{2}{q+1}}\bigl|\tilde{\Omega}_b(\tau,\sigma)\bigr|^{\frac{q-1}{q+1}}$$
$$\leqslant cb^{\frac{2(N-1)}{q+1}}\exp\Bigl(\frac{2Q(\sigma)}{q+1}\Bigr)\exp\Bigl(\frac{2\mu(\sigma)}{q+1}\Bigr)\sigma^{\frac{(N-1)(q-1)}{q+1}} \tag{2.32}$$

for $\sigma < \tau < b$ and $0 < \sigma < \min\{s^*, \frac{b}{3}\}$. The application of (2.14) here is justified because, for τ and σ as above, $\tilde{\Omega}_b(\tau,\sigma) \subset \Omega_b(\sigma)$.

Combining (2.29)–(2.32) we obtain,

$$\int_{\Omega_b(\tau,2\sigma)} \bigl|\nabla(v\eta_\sigma)\bigr|^2\,dx + \int_{\Omega_b(\tau,2\sigma)} h(x)v^{q+1}\eta_\sigma^2\,dx$$
$$\leqslant \int_{S_b'(\tau,2\sigma)} \frac{\partial v}{\partial \vec{n}} v\eta_\sigma^2\,dx' + cb^{\frac{2(N-1)}{q+1}}\exp\Bigl(\frac{2(Q(\sigma)+\mu(\sigma))}{q+1}\Bigr)\sigma^{\frac{(N-1)(q-1)}{q+1}-2}. \tag{2.33}$$

Next, by Hölder's inequality,

$$\Bigl|\int_{S_b'(\tau,2\sigma)} \frac{\partial v}{\partial \vec{n}} v\eta_\sigma^2\,dx'\Bigr| \leqslant \int_{S_b'(\tau,2\sigma)} \Bigl|\frac{\partial}{\partial x_N}\bigl(v\eta_\sigma\bigl(|x'|\bigr)\bigr)\Bigr| v\eta_\sigma\,dx'$$
$$\leqslant \Bigl(\int_{S_b'(\tau,2\sigma)} \Bigl(\frac{\partial}{\partial x_N}(v\eta_\sigma)\Bigr)^2 dx'\Bigr)^{1/2}\Bigl(\int_{S_b'(\tau,2\sigma)} (v\eta_\sigma)^2\,dx'\Bigr)^{1/2}$$

and by Poincaré's inequality in $S_b'(\tau,\sigma)$,

$$\int_{S_b'(\tau,2\sigma)} (v\eta_\sigma)^2\,dx' \leqslant (c_0\sigma)^2 \int_{S_b'(\tau,2\sigma)} \bigl|\nabla_{x'}(v\eta_\sigma)\bigr|^2\,dx'.$$

Therefore

$$\Bigl|\int_{S_b'(\tau,2\sigma)} \frac{\partial v}{\partial \vec{n}} v\eta_\sigma^2\,dx'\Bigr| \leqslant c\sigma \int_{S_b'(\tau,2\sigma)} \bigl|\nabla_x(v\eta_\sigma)\bigr|^2\,dx'. \tag{2.34}$$

Since

$$\frac{dE_b(\tau,\sigma;v)}{d\tau} = -\int_{S_b'(\tau,2\sigma)} \bigl(\bigl|\nabla(v\eta_\sigma)\bigr|^2 + h(x)v^{q+1}\eta_\sigma^2\bigr)\,dx'$$

inequalities (2.33) and (2.34) imply (2.25).

Finally, if (2.15) holds, (2.27) is obtained in the same way as (2.16). □

2.2. *Part 2*

Notation. Given $M > 0$ and $\nu \in (0,1)$, let $s_\nu = s_\nu(M)$ be defined by,

$$\exp\bigl(Q_0\mu\bigl(s_\nu(M)\bigr)\bigr) = \bar{h}\bigl(s_\nu(M)\bigr)^{-Q_0} = M^\nu, \tag{2.35}$$

where Q_0 is given by (2.17).

Lemma 2.3. *Put*

$$\gamma := \frac{2(q+1+\beta)-(N-1)(q-1)}{\beta Q_0(q+1)}, \tag{2.36}$$

where β is a positive number satisfying (2.15) and

$$\nu_0 := \begin{cases} 1 & \text{if } \gamma \leqslant 0, \\ \frac{q-1}{\gamma} & \text{if } \gamma > 0. \end{cases} \tag{2.37}$$

If

$$0 < \nu < \min(\nu_0, 1) \tag{2.38}$$

then,

$$E_b\big(0, s_\nu\big(M'\big); U_M\big) \leqslant 2\big(I_b(M) + C_3(b)M^2 M'^{\,q-1}\big) \quad 1 \leqslant M' \leqslant M, \tag{2.39}$$

where

$$C_3(b) := cb^{\frac{2N+q-1}{q+1}} \bar{h}(8b)^{\frac{2}{q+1}}. \tag{2.40}$$

Proof. Put

$$I_b'(s, M) := \int_{\Omega_b} U_M^2 |\nabla \eta_s|^2 \, dx.$$

Then,

$$\begin{aligned} E_b\big(0, s_\nu\big(M'\big), U_M\big) &\leqslant 2\int_{\Omega_b} \big(\big|\nabla(U_M)\big|^2 \eta_{s_\nu}^2 + h(x) U_M^{q+1} \eta_{s_\nu}^2\big)\, dx + 2\int_{\Omega_b} U_M^2 |\nabla \eta_{s_\nu}|^2 \, dx \\ &\leqslant 2\big(I_b(M) + I_b'(s_\nu, M)\big), \quad s_\nu = s_\nu\big(M'\big). \end{aligned} \tag{2.41}$$

By (2.23), $\nabla \eta_{s_\nu}(|x'|) = 0$ for $|x'| < s_\nu$ and for $|x'| > 2s_\nu$. Therefore, applying Hölder's inequality and using the monotonicity of $\bar{h}$ we obtain

$$\begin{aligned} I_b'\big(s_\nu\big(M'\big), M\big) &\leqslant 4s_\nu^{-2} \int_{\tilde{\Omega}_b(0,s_\nu)} U_M^2 \, dx \\ &\leqslant 4s_\nu^{-2} \Big(\int_{\tilde{\Omega}_b(0,s_\nu)} U_M^{q+1} h \, dx\Big)^{\frac{2}{q+1}} \Big(\int_{\tilde{\Omega}_b(0,s_\nu)} \bar{h}\big(\big|x'\big|\big)^{\frac{2}{1-q}} \, dx\Big)^{\frac{q-1}{q+1}} \\ &\leqslant cs_\nu^{-2} \big(b^N \bar{h}(8b) M^{q+1}\big)^{\frac{2}{q+1}} \bar{h}(s_\nu)^{-\frac{2}{q+1}} s_\nu^{\frac{(N-1)(q-1)}{q+1}} b^{\frac{q-1}{q+1}} \\ &= c\big(b^N \bar{h}(8b)\big)^{\frac{2}{q+1}} b^{\frac{q-1}{q+1}} M^2 s_\nu^{-2+\frac{(N-1)(q-1)}{q+1}} \exp \frac{2\mu(s_\nu)}{q+1}. \end{aligned}$$

By (2.15) and (2.35)

$$s^{-1} \leqslant \exp\big(\mu(s)/\beta\big), \qquad M'^{\,-\nu/Q_0} = \bar{h}(s_\nu) = \exp\big(-\mu(s_\nu)\big).$$

Therefore the previous inequality yields

$$I_b'\big(s_\nu\big(M'\big), M\big) \leqslant c\big(b^N \bar{h}(8b)\big)^{\frac{2}{q+1}} b^{\frac{q-1}{q+1}} M^2 M'^{\,\frac{\nu}{\beta Q_0}(2-\frac{(N-1)(q-1)}{q+1})+\frac{2}{q+1}\frac{\nu}{Q_0}}.$$

Hence

$$I_b'\big(s_\nu\big(M'\big), M\big) \leqslant C_3(b) M^2 M'^{\,\nu\gamma} \tag{2.42}$$

with γ and $C_3(b)$ as in (2.36) and (2.40). By (2.38) $\nu\gamma \leqslant q-1$. Therefore (2.41) and (2.42) imply (2.39). □

Notation. For every $M>0$ and $0\leqslant s\leqslant b/2$ denote,

$$T_b(s,M)=\bigl\{\tau\colon\ s\leqslant\tau<b,\ E_b(\tau,s;U_M)\geqslant 2C_2(b)\exp\bigl(H_0\mu(s)\bigr)\bigr\} \tag{2.43}$$

where $C_2(b)$ is the constant in (2.25) and H_0 is given by (2.28).

Note that $\tau\mapsto E_b(\tau,s;U_M)$ is continuous and non-increasing in the interval $[s,b]$. Therefore, if

$$E_b(s,s;U_M)<2C_2(b)\exp\bigl(H_0\mu(s)\bigr)$$

then $T_b(s,M)=\emptyset$. Put,

$$\tau_b(s,M)=\begin{cases} s & \text{if } T_b(s,M)=\emptyset,\\ \sup T_b(s,M) & \text{otherwise}\end{cases} \tag{2.44}$$

and

$$\tau_{b,\nu}\bigl(M',M\bigr):=\tau_b\bigl(s_\nu\bigl(M'\bigr),M\bigr). \tag{2.45}$$

Since $\lim_{\tau\to b}E_b(\tau,s;U_M)\to 0$ it follows that

$$s_\nu\bigl(M'\bigr)\leqslant\tau_{b,\nu}\bigl(M',M\bigr)<b. \tag{2.46}$$

Furthermore,

$$E_b\bigl(\tau_{b,\nu}\bigl(M',M\bigr),s_\nu\bigl(M'\bigr);U_M\bigr)\leqslant 2C_2(b)\exp\bigl(H_0\mu\bigl(s_\nu\bigl(M'\bigr)\bigr)\bigr) \tag{2.47}$$

and, if $\tau_{b,\nu}(M',M)>s_\nu(M')$ then,

$$E_b\bigl(\tau,s_\nu\bigl(M'\bigr);U_M\bigr)\geqslant 2C_2(b)\exp\bigl(H_0\mu\bigl(s_\nu\bigl(M'\bigr)\bigr)\bigr) \tag{2.48}$$

for every $\tau\in(0,\tau_{b,\nu}(M',M)]$, with equality for $\tau=\tau_{b,\nu}(M',M)$.

Proposition 2.4. (i) *Let*

$$b'_\nu\bigl(M',M\bigr):=b-\tau_{b,\nu}\bigl(M',M\bigr).$$

Then

$$\int\limits_{\Omega_{b'_\nu(M',M)}}\bigl(|\nabla_x U_M|^2+h(x)U_M^{q+1}\bigr)\,dx\leqslant c_0\bigl(b^{N-1}M'^{\,\nu}+C_2(b)M'^{\,\frac{\nu H_0}{Q_0}}\bigr). \tag{2.49}$$

(ii) *Assume that*

$$0<\nu\leqslant\frac{q+1}{4}\min(1,Q_0/H_0), \tag{2.50}$$

where H_0 is given by (2.28) *and Q_0 is given by* (2.17). *Let $a\in(1,2)$ and assume that M' is large enough so that,*

$$C_4(b):=c_0\bigl(b^{N-1}+C_2(b)\bigr)/C_1(b)\leqslant M'^{\,(q+1)/2a} \tag{2.51}$$

where $C_1(b)$ and $C_2(b)$ are the constants in Lemma 2.1 *and Proposition* 2.3 *respectively while c_0 is the constant in* (2.49)*.*

Then

$$I_{b'_\nu(M',M)}(M)=\int\limits_{\Omega_{b'_\nu(M',M)}}\bigl(|\nabla_x U_M|^2+h(x)U_M^{q+1}\bigr)\,dx\leqslant C_1(b)M'^{\,\frac{q+1}{a}}. \tag{2.52}$$

Proof. By (2.35),

$$M'=\exp\biggl(\frac{Q_0}{\nu}\mu\bigl(s_\nu\bigl(M'\bigr)\bigr)\biggr). \tag{2.53}$$

Therefore, by (2.47),

$$E_b\bigl(\tau_{b,\nu}\bigl(M',M\bigr),s_\nu\bigl(M'\bigr);U_M\bigr)\leqslant 2C_2(b)M'^{\,\frac{\nu H_0}{Q_0}}. \tag{2.54}$$

By Proposition 2.2 applied to the estimate of $J_b(s_\nu(M'), U_M)$,

$$J_b\big(s_\nu\big(M'\big), U_M\big) \leqslant cb^{N-1} \exp\big(Q_0\mu\big(s_\nu\big(M'\big)\big)\big) = cb^{N-1}M'^{\nu}. \tag{2.55}$$

Inequality (2.46) implies that $b'_\nu(M', M) \leqslant b - s_\nu(M')$. Therefore

$$\Omega_{b'_\nu(M',M)} \subset \Omega_b\big(\tau_{b,\nu}\big(M', M\big), s_\nu\big(M'\big)\big) \cup \overline{\Omega_b\big(s_\nu\big(M'\big)\big)}$$

(see (2.5) for definition of $\Omega_b(s)$). Consequently

$$I_{b'_\nu(M',M)}(M) \leqslant E_b\big(\tau_{b,\nu}\big(M', M\big), s_\nu\big(M'\big); U_M\big) + J_b\big(s_\nu\big(M'\big), U_M\big).$$

This inequality together with (2.54) and (2.55) imply (2.49).

In view of (2.50) we have,

$$b^{N-1}M'^{\nu} + C_2(b)M'^{\frac{\nu H_0}{Q_0}} \leqslant \big(b^{N-1} + C_2(b)\big)M'^{(q+1)/2a}.$$

If M' satisfies (2.51), this inequality and (2.49) imply (2.52). □

Next we derive an upper bound for $\tau_{b,\nu}(M', M)$ in terms of $s_\nu(M')$.

Lemma 2.4. *Suppose that* $0 < \nu$ *satisfies conditions* (2.38) *and* (2.50) *and that*

$$M \geqslant \exp\left(\frac{Q_0}{\nu}\mu\big(s^*\big)\right) \tag{2.56}$$

where s^* *is as in Proposition* 2.3. *Then*

$$\exp\left(\frac{\tau_{b,\nu}(M', M)}{2cs_\nu(M')}\right) \leqslant c_1\big(I_b(M) + C_3(b)M^2M'^{q-1}\big)C_2(b)^{-1}M'^{-\frac{\nu H_0}{Q_0}}. \tag{2.57}$$

Proof. Since ν satisfies (2.50) and $1 < a < 2$,

$$0 < Q_0(q+1)\left(1 - \frac{1}{2a}\right) \leqslant Q_0(q+1) - H_0\nu.$$

By (2.39),

$$E_b\big(\tau, s_\nu\big(M'\big); M\big) \leqslant E_b\big(0, s_\nu\big(M'\big); M\big) \leqslant 2\big(I_b(M) + C_3(b)M^2M'^{q-1}\big) \quad \forall \tau \in (0, b) \tag{2.58}$$

where $1 < M' < M$.

If $\tau_{b,\nu} \leqslant s_\nu$ inequality (2.57) is trivial. Therefore we may assume that

$$\tau_{b,\nu}\big(M', M\big) > s_\nu\big(M'\big).$$

Temporarily denote

$$F(\tau) = E_b\big(\tau, s_\nu\big(M'\big); M\big).$$

By Proposition 2.3, (2.56) and (2.48),

$$F(\tau) \leqslant 2cs_\nu\big(M'\big)\left(-\frac{dF(\tau)}{d\tau}\right) \quad \forall \tau\colon s_\nu\big(M'\big) < \tau < \tau_{b,\nu}\big(M', M\big). \tag{2.59}$$

Solving this differential inequality with initial condition $F(s_\nu(M'))$ satisfying (2.58) we obtain,

$$E_b\big(\tau, s_\nu\big(M'\big); M\big) \leqslant c_1\big(I_b(M) + C_3(b)M^2M'^{q-1}\big)\exp\left(-\frac{\tau}{2cs_\nu}\right) \tag{2.60}$$

for every $\tau \in [s_\nu(M'), \tau_{b,\nu}(M', M)]$. Combining (2.60) and (2.48) for $\tau = \tau_{b,\nu}(M', M)$ (in which case (2.48) holds with equality) we obtain,

$$2C_2(b)\exp\big(H_0\mu\big(s_\nu\big(M'\big)\big)\big) \leqslant c_1\big(I_b(M) + C_3(b)M^2M'^{q-1}\big)\exp\left(-\frac{\tau_{b,\nu}(M', M)}{2cs_\nu(M')}\right).$$

In view of (2.53) this inequality implies

$$\exp\left(\frac{\tau_{b,\nu}(M', M)}{2cs_\nu(M')}\right) \leqslant c_1\big(I_b(M) + C_3(b)M^2 M'^{q-1}\big)C_2(b)^{-1} \exp\big(-H_0\mu\big(s_\nu\big(M'\big)\big)\big)$$
$$= c_1\big(I_b(M) + C_3(b)M^2 M'^{q-1}\big)C_2(b)^{-1} M'^{-\frac{\nu H_0}{Q_0}}. \qquad \square \tag{2.61}$$

2.3. Part 3

In this part of the proof we apply the previous estimates to a specific sequence $\{M_j\}$ defined below. As before R is an arbitrary positive number and we require that $R/4 < b < R/2$.

Proposition 2.5. *Let*

$$M_j = \exp\big(a^j\big), \qquad s_j := s_\nu(M_j) \tag{2.62}$$

where $s_\nu(\cdot)$ is defined as in (2.35) and

$$1 < a < \min\left(1 + \frac{\nu H_0}{2Q_0}, 2\right). \tag{2.63}$$

Put $u_j = U_{M_j}$. Then there exists $j_0 \in \mathbb{N}$ such that

$$\int\limits_{\Omega_{b/2}} \big(|\nabla_x u_j|^2 + h(x)u_j^{q+1}\big)\,dx \leqslant C_1(b)M_{j_0}^{q+1} \quad \forall j > j_0 \tag{2.64}$$

where $C_1(b) = cb^N \bar{h}(8b)$.

Proof. By (2.62) and (2.35),

$$a^j \nu/Q_0 = \mu(s_j). \tag{2.65}$$

Let j_0 be a positive integer to be determined later on. For each integer $j \geqslant j_0$ we define the set of pairs

$$\big\{b_{i,j}, \tau^{i,j}\colon i = j_0, \ldots, j\big\}$$

by induction as follows:

$$\tau^{j,j} = \tau_{b,\nu}(M_j, M_j), \qquad b_{j,j} = b - \tau^{j,j},$$
$$\tau^{i,j} = \tau_{b_{i+1,j},\nu}(M_i, M_j), \qquad b_{i,j} = b_{i+1,j} - \tau^{i,j}, \quad j_0 \leqslant i < j.$$

Thus

$$b_{i,j} = b - \sum_{k=i}^{j} \tau^{k,j}, \quad j_0 \leqslant i < j.$$

We show below that if j_0 is sufficiently large then

$$\sum_{i=j_0}^{j} \tau^{i,j} < b/2 \quad \forall j > j_0, \tag{2.66}$$

which implies,

$$b/2 < b_{i,j}.$$

Specifically we choose j_0 so that,

$$\begin{aligned}
&\text{(i)} \qquad C_4(b/2) \leqslant M_{j_0}^{(q+1)/2a},\\
&\text{(ii)} \qquad \exp\biggl(\frac{Q_0}{\nu}\mu\bigl(s^*\bigr)\biggr) \leqslant M_{j_0},\\
&\text{(iii)} \qquad C_5(b) := c_1\frac{C_1(b)+C_3(b)}{C_2(b)} \leqslant M_{j_0}^{q+1}
\end{aligned} \tag{2.67}$$

with c_1 as in (2.57). For the definition of $C_1(b), \dots, C_4(b)$ see (2.3), (2.25), (2.40) and (2.51).

We observe that $C_4(b)$ decreases as b increases. Therefore (assuming (2.66)) condition (i) implies,

$$C_4(b_{i,j}) \leqslant M_i^{(q+1)/2a}, \quad j_0 \leqslant i \leqslant j,\ j_0 \leqslant j. \tag{2.68}$$

The left hand side in condition (2.67)(iii) increases as b increases. Therefore

$$C_5(b_{i,j}) \leqslant (q+1)\ln M_i, \quad j_0 \leqslant i \leqslant j,\ j_0 \leqslant j. \tag{2.69}$$

Put $u_j = U_{M_j}$. Assuming that (2.66) holds, we apply Proposition 2.4 to the case where b is replaced by $b_{j_0+1,j}$ and $M' = M_{j_0+1}$, $M = M_j$; we obtain,

$$\int\limits_{\Omega_{b_{j_0,j}}} \bigl(|\nabla_x u_j|^2 + h(x)u_j^{q+1}\bigr)\,dx \leqslant C_1(b)M_{j_0}^{q+1} \tag{2.70}$$

which implies (2.64).

It remains to verify (2.66). To this end we prove the following estimate:

$$\tau^{i,j} \leqslant \bar{c}Q_0(q+1)\frac{\omega(s_i)}{\nu}, \quad j_0 \leqslant i \leqslant j \tag{2.71}$$

where $\bar{c} = 4c$ (c as in (2.57)).

The proof is by induction. We apply Lemma 2.4 in the case where

$$b \text{ is replaced by } b_{i+1,j}, \qquad M' = M_i, \qquad M = M_j, \quad j_0 \leqslant i \leqslant j.$$

For $i = j$ we put $b_{j+1,j} := b$. Note that, for $M \geqslant M_{j_0}$, condition (2.67)(ii) yields (2.56).

Applying Lemma 2.4 and Lemma 2.1 to the case $i = j$ we obtain

$$\exp\frac{\tau^{j,j}}{2cs_j} \leqslant C_5(b)M_j^{q+1-\nu\frac{H_0}{Q_0}}.$$

Consequently, using (2.62) and condition (2.67)(iii)

$$\begin{aligned}
\frac{\tau^{j,j}}{2cs_j} &\leqslant \ln C_5(b) + \left(q+1-\nu\frac{H_0}{Q_0}\right)\ln M_j\\
&\leqslant 2(q+1)\frac{Q_0\mu(s_j)}{\nu}.
\end{aligned} \tag{2.72}$$

For the last inequality recall that $s_j = s_\nu(M_j)$, which implies,

$$\ln M_j = \frac{Q_0\mu(s_j)}{\nu}.$$

Inequality (2.72) implies (2.71) for $i = j$.

Observe that $s_j \downarrow 0$ as $j \uparrow \infty$ and consequently, $\omega(s_j) \downarrow 0$. Therefore if j_0 is sufficiently large we have $\tau^{j,j} < b/2$ and $b_{j,j} > b/2$. By Proposition 2.4,

$$I_{b_{j,j}}(M_j) \leqslant C_1(b_{j,j})M_j^{(q+1)/a} \leqslant C_1(b)M_{j-1}^{q+1}. \tag{2.73}$$

Here we use condition (2.67)(i) and the fact that $b_{j,j} = b - \tau_{b,\nu}(M_j, M_j)$.

Now we apply Lemma 2.4 for $i = j-1$, i.e., when b is replaced by $b_{j,j}$ and $M' = M_{j-1}$, $M = M_j$. This lemma, combined with (2.73), yields

$$\exp\frac{\tau^{j-1,j}}{2cs_{j-1}} \leqslant c_1\big(I_{b_{j,j}}(M_j)+C_3(b_{j,j})M_j^2M_{j-1}^{q-1}\big)C_2(b_{j,j})^{-1}M_{j-1}^{-\nu\frac{H_0}{Q_0}}$$
$$\leqslant c_1\big(C_1(b_{j,j})M_{j-1}^{q+1}+C_3(b_{j,j})M_j^2M_{j-1}^{q-1}\big)C_2(b_{j,j})^{-1}M_{j-1}^{-\nu\frac{H_0}{Q_0}}.$$

By (2.63),

$$M_j^2M_{j-1}^{-\nu\frac{H_0}{Q_0}} \leqslant M_{j-1}^2. \tag{2.74}$$

Therefore, similarly to (2.72), we obtain

$$\frac{\tau^{j-1,j}}{2cs_{j-1}} \leqslant \ln C_5(b_{j,j})+(q+1)\ln M_{j-1}$$
$$\leqslant 2(q+1)\frac{Q_0\mu(s_{j-1})}{\nu}, \tag{2.75}$$

which, in turn, implies (2.71) for $i=j-1$.

This process can be repeated inductively for $i=j-2, j-3, \dots, j_0$ provided that $b_{i+1,j}\geqslant b/2$. For each value of i in this range we first apply Proposition 2.4 to obtain,

$$I_{b_{i+1,j}}(M_j) \leqslant C_1(b_{i+1,j})M_{i+1}^{(q+1)/a} \leqslant C_1(b)M_i^{q+1}. \tag{2.76}$$

After that we apply Lemma 2.4 combined with (2.76) to obtain (2.71) for the respective value of i, always with the same constant $\bar{c}$. Therefore, to complete the proof, it remains to be shown that there exists j_0 such that:

If $j>j_0$, $j_0\leqslant k<j$ and $\tau^{i,j}$ satisfies (2.71) for $k\leqslant i\leqslant j$ then,

$$\sum_{i=k}^{j}\tau^{i,j} < b/2. \tag{2.77}$$

By (2.65) and (1.4)

$$s_i \leqslant (Q_0/\nu)a^{-i}\omega(s_i) \leqslant \ell a^{-i}, \quad \ell := Q_0\omega(s_0)/\nu.$$

Since, by assumption, (2.71) holds for $k\leqslant i\leqslant j$,

$$\sum_{i=k}^{j}\tau_{i,j} \leqslant C(N,q,\nu)\sum_{i=k}^{j}\omega(s_i) \leqslant C(N,q,\nu)\sum_{i=k}^{j}\omega\big(\ell a^{-i}\big).$$

Further, using the monotonicity of ω,

$$\sum_{i=k}^{j}\omega\big(\ell a^{-i}\big) \leqslant \int_k^j \omega\big(\ell a^{-s}\big)\,ds < (\ln a)^{-1}\int_0^{\beta_k}\frac{\omega(r)}{r}\,dr$$

where $\beta_k=\ell a^{-k}$. Because of the Dini condition, the last integral tends to zero when $\beta_k\to 0$. Therefore, if j_0 is sufficiently large (depending only on N, q, ν and a) (2.77) holds for all $k\geqslant j_0$. □

Completion of proof of Theorem 2.1. Since U_M increases as M increases

$$U^R := \lim_{M\to\infty}U_M = \lim_{j\to\infty}u_j.$$

The function V_M defined by

$$V_M(x)=U_M\big(x', x_N+R\big)$$

is a solution of (1.1) in the ball $B_R(x^R)$, where $x^R=(0,R)$. Put

$$V^R := \lim_{M\to\infty}V_M \quad \text{in } B_R\big(x^R\big).$$

We show that V^R is bounded in a neighborhood of the point $(0,R)$.

By interior elliptic estimates, (2.64) implies that

$$\sup_{j_0\leqslant j}\int_{\Omega_{b/3}} |u_j|^2\,dx<\infty. \tag{2.78}$$

Since $h(x)\geqslant 0$, u_j is subharmonic in Ω_b. Therefore (2.78) implies

$$\sup\{u_j(x):\ j_0\leqslant j,\ x\in\Omega_{b/4}\}<\infty. \tag{2.79}$$

Thus U^R is bounded in a neighborhood of the origin which means that V^R is bounded in a neighborhood of $(0, R)$.

For every $r\in(0,R)$, $V^R<V^r$ in $B_r(x^r)$. (Recall that x^r denotes the point $(x',x_N)=(0,r)$.) As V^r is bounded in a neighborhood of $(0,r)$ we conclude that V^R is locally bounded in $B_R\cap[0<x_N<R]$.

Recall that $h=\bar{h}$ is independent of x_N. Therefore applying the same argument in the half-space $[x_N>a]$ we deduce that for every $a\in(0,2R)$ the sequence of solutions $\{V_M\}$ is uniformly bounded in a neighborhood of the point $(0,R+a/2)$. Hence V^R is locally bounded in $B_R\cap[R<x_N]$. In conclusion, V^R is locally bounded in B_R and therefore it is a solution of (1.1) in B_R. □

3. Proof of Theorem 1.2

Put

$$r_j:=2^{-j},\qquad \Omega_j=\{(x',x_N):\ |x'|<r_j,\ 0<x_N\},\qquad j=1,2,\ldots.$$

Further denote,

$$a_j:=\exp(-\mu(r_j)),\qquad A_j=(a_jr_j^2)^{\frac{1}{q-1}} \tag{3.1}$$

and, for $x'\in\mathbb{R}^{N-1}$,

$$\gamma_j(x')=\begin{cases}A_j^{-1}\phi_1(x'/r_{j+1}) & \text{if } |x'|<r_{j+1},\\ 0 & \text{if } |x'|\geqslant r_{j+1}\end{cases} \tag{3.2}$$

where ϕ_1 is the first eigenfunction of the Dirichlet problem to $-\Delta_{y'}$ in B_1^{N-1} normalized by $\phi_1(0)=1$. Recall that $\mu(s)=\omega(s)/s$.

We consider the boundary value problems

$$\begin{aligned}&-\Delta u_j+a_ju_j^q=0\quad\text{in } \Omega_j,\\ &u_j(x)=0\quad\text{on } \{x\in\partial\Omega_j:\ x_N>0\},\\ &u_j(x',0)=\gamma_j(x')\quad\text{for } |x'|\leqslant r_j.\end{aligned} \tag{3.3}$$

In view of (1.4), $\{a_j\}$ is a decreasing sequence converging to zero and

$$a_j=\sup_{s\in(0,r_j)}\exp(-\mu(s)).$$

Therefore, for every $x_N>0$, $\{u_j(0,x_N)\}$ is an increasing sequence and u_j is a subsolution of the problem

$$\begin{aligned}&-\Delta w+h(x)w^q=0\quad\text{in } \Omega_j,\\ &w(x)=0\quad\text{on } \{x\in\partial\Omega_j:\ x_N>0\},\\ &w(x',0)=\gamma_j(x')\quad\text{for } |x'|\leqslant r_j.\end{aligned} \tag{3.4}$$

The proof of Theorem 1.2 is based on the following:

Proposition 3.1. *For every $x_N>0$,*

$$\lim_{j\to\infty}u_j(0,x_N)=\infty.$$

In the next lemma we collect several results of Brada [2] that are used in the proof of this proposition.

Lemma 3.1. *Let a be a positive number, let $q>1$ and let f be a positive function in $L^\infty(B_1^{N-1})$, where B_1^{N-1} denotes the unit ball in $\mathbb{R}^{N-1}$ centered at the origin.*

Consider the problem

$$\begin{aligned}&-\Delta u+bu^q=0 \quad \text{in } D_0,\\ &u(y)=0 \quad \text{for } y\in\partial D_0\text{: } 0<y_N,\\ &u\big(y',0\big)=f\big(y'\big) \quad \text{for } \big|y'\big|\leqslant 1,\end{aligned} \tag{3.5}$$

where

$$D_0=\big\{y=\big(y',y_N\big)\in\mathbb{R}^N\text{: } \big|y'\big|<1,\ 0<y_N\big\}.$$

If u is the solution of this problem then there exists a number $\alpha>0$ such that

$$\lim_{y_N\to\infty}\exp(\sqrt{\lambda_1}y_N)u(y)=\alpha\phi_1\big(y'\big) \tag{3.6}$$

uniformly in B_1^{N-1}. Here λ_1 is the first eigenvalue and ϕ_1 the corresponding eigenfunction of $-\Delta_{y'}$ in B_1^{N-1} normalized by $\phi_1(0)=1$.

The limit α satisfies

$$\alpha\leqslant cb^{-\frac{1}{q-1}}\sup f. \tag{3.7}$$

Proof. By [2, Theorem 4], (3.6) holds for some $\alpha\in\mathbb{R}$. Under our assumptions u is positive so that $\alpha\geqslant 0$. By the remark in [2, p. 357], if $\alpha=0$ then there exists $k>1$ such that

$$\lim_{y_N\to\infty}\exp(\sqrt{\lambda_k}y_N)u(y)=\phi_k\big(y'\big)$$

where ϕ_k an eigenfunction of $-\Delta_{y'}$ in B_1^{N-1} corresponding to the k-th eigenvalue. However this is impossible because ϕ_k changes signs. Thus $\alpha>0$.

Inequality (3.7) is a consequence of [2, Proposition 1]. □

3.1. An estimate of u_j

We start by rescaling problem (3.3). Put

$$y=x/r_j, \qquad \tilde u_j(y)=A_ju_j(r_jy), \tag{3.8}$$

where A_j is given by (3.1). Then $v:=\tilde u_j$ is the solution of the problem

$$\begin{aligned}&-\Delta v+v^q=0 \quad \text{in } D_0,\\ &v(y)=0 \quad \text{for } y\in\partial D_0\text{: } 0<y_N,\\ &v\big(y',0\big)=\tilde\gamma\big(y'\big) \quad \text{for } \big|y'\big|\leqslant 1,\end{aligned} \tag{3.9}$$

where

$$\tilde\gamma\big(y'\big):=\begin{cases}\phi_1(2y') & \text{if } |y'|<\frac{1}{2},\\ 0 & \text{otherwise.}\end{cases} \tag{3.10}$$

Applying Lemma 3.1 to the solution v of (3.9) we obtain,

$$\lim_{y_N\to\infty}\exp(\sqrt{\lambda_1}y_N)v\big(y',y_N\big)=\alpha\phi_1\big(y'\big) \tag{3.11}$$

where α is a positive number depending only on q,N. Consequently there exists $\beta>0$ such that

$$\begin{aligned}\frac{1}{2}\alpha\phi_1\big(y'\big)\exp(-\sqrt{\lambda_1}y_N)&\leqslant A_ju_j(r_jy)\\ &\leqslant 2\alpha\phi_1\big(y'\big)\exp(-\sqrt{\lambda_1}y_N) \quad \forall y_N\geqslant\beta,\ \big|y'\big|\leqslant 1.\end{aligned}$$

This inequality is equivalent to

$$\frac{\alpha}{2A_j}\phi_1\big(x'/r_j\big)\exp(-\sqrt{\lambda_1}x_N/r_j) \leqslant u_j(x) \leqslant \frac{2\alpha}{A_j}\phi_1\big(x'/r_j\big)\exp(-\sqrt{\lambda_1}x_N/r_j) \quad \forall x_N \geqslant \beta r_j,\ \big|x'\big| \leqslant r_j. \tag{3.12}$$

3.2. Comparison of u_j and u_{j-1}

Let τ_j be the number determined by the equation,

$$\frac{\alpha}{2}\exp(-\sqrt{\lambda_1}\tau_j/r_j) = \left(\frac{a_j}{a_{j-1}}\right)^{\frac{1}{q-1}} 2^{-\frac{2}{q-1}} = 2^{-\frac{2}{q-1}}\exp\frac{-\mu(r_j)+\mu(r_{j-1})}{q-1}. \tag{3.13}$$

By (3.1) and (3.2), this is equivalent to

$$\frac{\alpha}{2A_j}\phi_1\big(x'/r_j\big)\exp\left(-\sqrt{\lambda_1}\frac{\tau_j}{r_j}\right) = \gamma_{j-1}\big(x'\big). \tag{3.14}$$

Without loss of generality we may assume that (1.8) holds for $a = 2$. Therefore there exists $\kappa \in (0, 1)$ such that

$$\mu(r_j) - \mu(r_{j-1}) \geqslant \kappa\mu(r_j). \tag{3.15}$$

By (3.13),

$$\sqrt{\lambda_1}\frac{\tau_j}{r_j} = \frac{\mu(r_j)-\mu(r_{j-1})}{q-1} + c(N, q).$$

Therefore, by (3.15) and (1.4), there exist positive numbers c_0, c_1 and j_0 (depending only on κ, N, q) such that

$$\beta r_j < c_0\omega(r_j) \leqslant \tau_j \leqslant c_1\omega(r_j) \tag{3.16}$$

for every $j \geqslant j_0$ (β as in (3.12)).

By (3.12), (3.14) and (3.16)

$$\gamma_{j-1}\big(x'\big) \leqslant u_j\big(x', \tau_j\big), \quad \big|x'\big| \leqslant r_j,\ j \geqslant j_0. \tag{3.17}$$

By the maximum principle, (3.3), (3.17) and the fact that $a_{j-1} > a_j$ imply

$$u_{j-1}\big(x', x_N\big) \leqslant u_j\big(x', x_N + \tau_j\big) \quad \forall j \geqslant j_0,\ x \in \Omega_j. \tag{3.18}$$

3.3. Proof of Proposition 3.1

Let $j_0 \leqslant k < m$. Iterating inequality (3.18) for $j = k+1, \ldots, m$ we obtain,

$$u_k\big(x', x_N\big) \leqslant u_m\left(x', x_N + \sum_{j=k+1}^{m}\tau_j\right) \quad \forall x \in \Omega_m. \tag{3.19}$$

Combining this inequality (for $x' = x_N = 0$) with (3.12) yields

$$\frac{1}{2}\alpha\big(a_k r_k^2\big)^{-\frac{1}{q-1}} = \frac{\alpha}{2A_k} \leqslant u_k(0) \leqslant u_m\left(0, \sum_{j=k+1}^{m}\tau_j\right) \tag{3.20}$$

for every m, k such that $j_0 \leqslant k < m$. By (1.10),

$$\sum_{j=k}^{\infty}\omega(r_j) = \infty.$$

Therefore, by (3.16)

$$\sum_{j=k}^{\infty} \tau_j = \infty. \tag{3.21}$$

Consequently,

$$s_{m,k} := \sum_{j=k+1}^{m} \tau_j \quad \Longrightarrow \quad \lim_{m\to\infty} s_{m,k} = \infty. \tag{3.22}$$

Note that $a_k r_k^2 \to 0$; therefore, by (3.20), for every $M > 0$ there exists j_M such that

$$M < u_m(0, s_{m,k}) \quad j_M \leqslant k < m. \tag{3.23}$$

We claim that

$$\sup u_j(0, x_N) = \infty \quad \forall x_N > 0. \tag{3.24}$$

By negation, assume that

$$\exists s > 0\colon\ \sup u_j(0, s) = K < \infty.$$

By (3.12)

$$\frac{u_j(x', s)}{u_j(0, s)} \leqslant 4\alpha \quad |x'| \leqslant r_j.$$

Here we use the fact that $1 = \phi(0) = \max \phi$. It follows that, for every j such that $2^j > \beta/s$,

$$\sup u_j\big(x', s\big) \leqslant 4\alpha K, \quad |x'| \leqslant r_j.$$

Therefore, by the maximum principle, for every j as above,

$$u_j\big(x', x_N\big) \leqslant 4\alpha K \quad \forall x \in \Omega_j \cap [x_N \geqslant s].$$

In view of (3.22), this contradicts (3.23). □

3.4. *Proof of Theorem 1.2*

Let $P_0(x, y) = c_N x_N |x - y|^{-N}$ be the Poisson kernel for $-\Delta$ in $\mathbb{R}^N_+$. Condition (1.9) implies that, for any positive constants a, R

$$\sup_{|x'|<R} |x'|^{-a} h(x) < \infty. \tag{3.25}$$

For every $q > 1$ choose $a > 0$ such that $q < (N + 1 + a)/(N - 1)$. Then for every $R > 0$,

$$\int_{[|x|<R,\, 0<x_N]} h(x) P_0^q(x, 0) x_N\, dx < C_a \int_{[|x|<R,\, 0<x_N]} |x|^a P_0^q(x, 0) x_N\, dx < \infty.$$

Consequently, for every $k > 0$, the problem

$$\begin{aligned} &-\Delta v + h(x) v^q = 0 \quad \text{in } D_0,\\ &v = 0 \quad \text{on } \partial_\ell D_0 := \big[|x'| = 1,\ x_N > 0\big],\\ &v = k\delta_0 \quad \text{on } [x_N = 0] \end{aligned}$$

possesses a unique solution dominated by the supersolution kP_0 (see [4]).

The function

$$v_{0,\infty} := \lim_{k\to\infty} v_{0,k} \quad \text{in } D_0 \tag{3.26}$$

is a solution of (1.1) in $D_0 \cap [|x'| > 0]$ but it may blow up as $|x'| \to 0$.

Put

$$f(x_N)=\int\limits_{|x'|<1} v_{0,\infty}\big(x',\bar{x}_N\big)\,dx' \quad \forall x_N>0.$$

If $f(a)<\infty$ for some $a>0$ then $v_{0,\infty}$ is finite in $D_0\cap[x_N>a]$ so that $f(x_N)<\infty$ for every $x_N>a$. Thus

$$f(a)<\infty \quad \text{for some } a>0 \quad\Longrightarrow\quad f(x_N)<\infty \quad \forall x_N\geqslant a. \tag{3.27}$$

Let

$$b=\inf\big\{x_N>0\colon\ f(x_N)<\infty\big\}. \tag{3.28}$$

By (3.27)

$$f(x_N)=\infty \quad \forall x_N\in(0,b), \qquad f(x_N)<\infty \quad \forall x_N\in(b,\infty). \tag{3.29}$$

We have to show that $b=\infty$. By negation assume that $b<\infty$. First consider the case $0<b$. Let $a\in(0,b)$ and put $\eta(x');=v_{0,\infty}(x',a)$. Then

$$\int\limits_{|x'|<1}\varphi\eta\,dx'=\infty \quad \forall\varphi\in C\big(\big[\big|x'\big|\leqslant 1\big]\big) \text{ such that } \varphi(0)>0.$$

Thus the measure $\mu_\eta=\eta\,dx'$ is larger than $k\delta_0$ for every $k>0$. The function V given by $V(x)=v_{0,\infty}(x',x_N+a)$ satisfies

$$\begin{aligned}
&-\Delta V+h(x)V^q=0 \quad \text{in } D_0,\\
&V=0 \quad \text{on } \partial_\ell D_0:=\big[\big|x'\big|=1,\ x_N>0\big],\\
&V=\eta \quad \text{on } [x_N=0].
\end{aligned}$$

Therefore $V\geqslant v_{0,\infty}$, i.e.,

$$v_{0,\infty}\big(x',x_N+a\big)\geqslant v_{0,\infty}\big(x',x_N\big).$$

But this implies

$$f(x_N+a)=\infty \quad \forall x_N\in(0,a+b)$$

which contradicts (3.28).

Next assume that $b=0$. In this case,

$$v_{0,\infty}(0,x_N)<\infty \quad \forall x_N>0 \tag{3.30}$$

and consequently $v_{0,\infty}$ is a solution of (1.1) in D_0. Let w_j be the unique solution of the boundary value problem:

$$\begin{aligned}
&-\Delta w_j+a_j w_j^q=0 \quad \text{in } \Omega_j,\\
&w_j=0 \quad \text{on } \partial\Omega_j\cap[x_N>0],\\
&w_j=\infty\delta_0 \quad \text{on } [x_N=0],
\end{aligned} \tag{3.31}$$

where $a_j=h(r_j)$. As usual, this means that $w_j=\lim_{k\to\infty} w_{j,k}$ where $w_{j,k}$ is the solution of the modified problem where the boundary data on $x_N=0$ is $w_{j,k}(x',0)=k\delta_0$. Since $a_j\geqslant h(x)$ in Ω_j it follows that

$$w_j\leqslant v_{0,\infty} \quad \text{in } \Omega_j. \tag{3.32}$$

The function w_j^* given by $w_j^*(x):=A_j w_j(r_j x)$ for $x\in D_0$ is a solution of the problem:

$$\begin{aligned}
&-\Delta w+w^q=0 \quad \text{in } D_0,\\
&w=0 \quad \text{on } \partial_\ell D_0,\\
&w\big(x',0\big)=\infty\delta_0 \quad \text{on } [x_N=0].
\end{aligned} \tag{3.33}$$

The solution of this problem is unique; consequently w_j^* is independent of j and we denote it by w^*.

Let $C := \sup_{|x'|<1/2} w^*(x', 1)$. Then $w_j(y) = A_j^{-1} w^*(y/r_j)$ satisfies

$$w_j(y', r_j) \geqslant cA_j^{-1}, \quad |y'| < r_{j+1}.$$

As $\gamma_j(x') = 0$ for $|x'| > r_{j+1}$ it follows that

$$w_j(y', r_j) \geqslant c\gamma_j(x'), \quad |x'| < r_j.$$

Hence

$$w_j(x', x_N + r_j) \geqslant u_j(x) \quad \text{in } \Omega_j.$$

Therefore, by Proposition 3.1,

$$\lim_{j\to\infty} w_j(0, x_N) = \infty \quad \forall x_N > 0.$$

Hence, by (3.32),

$$v_{0,\infty}(0, x_N) = \infty \quad \forall x_N > 0$$

in contradiction to (3.30). □

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References

[1] C. Bandle, M. Marcus, Asymptotic behavior of solutions and their derivatives, for semilinear, elliptic problems with blowup at the boundary, Ann. Inst. H. Poincare 12 (1995) 155–171.
[2] A. Brada, Comportement asymptotique de solutions d'équations elliptiques semi-linéares dans un cylindre, Asymptot. Anal. 10 (1995) 335–366.
[3] J.B. Keller, On solutions of $\Delta u = f(u)$, Comm. Pure Appl. Math. 10 (1957) 503–510.
[4] M. Marcus, L. Véron, The boundary trace of positive solutions of semilinear elliptic equations: The subcritical case, Arch. Ration. Mech. Anal. 144 (1998) 201–231.
[5] M. Marcus, L. Véron, Initial trace of positive solutions to semilinear parabolic inequalities, Adv. Nonlinear Stud. 2 (2002) 395–436.
[6] M. Marcus, L. Véron, Boundary trace of positive solutions of nonlinear elliptic inequalities, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) III (2004) 481–533.
[7] R. Osserman, On the inequality $\Delta u \geqslant f(u)$, Pacific J. Math. 7 (1957) 1641–1647.
[8] A. Shishkov, L. Véron, The balance between diffusion and absorption in semilinear parabolic equations, Rend. Lincei Mat. Appl. 18 (2007) 59–96.
[9] A. Shishkov, L. Véron, Diffusion versus absorption in semilinear elliptic equations, J. Math. Anal. Appl. 352 (2009) 206–217.
[10] A. Shishkov, L. Véron, Singular solutions of some nonlinear parabolic equations with spatially inhomogeneous absorption, Calc. Var. Partial Differential Equations 33 (2008) 343–375.
[11] A. Shishkov, L. Véron, Propagation of singularities of nonlinear heat flow in fissured media, arXiv:1103.5893v1.