

Linearly repetitive Delone sets are rectifiable

José Aliste-Prieto^a, Daniel Coronel^b, Jean-Marc Gambaudo^{c,*}

^a Centro de Modelamiento Matemático, Universidad de Chile, Blanco Encalada 2120 7to. piso, Santiago, Chile

^b Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Campus San Joaquín, Avenida Vicuña Mackenna 4860, Santiago, Chile

^c Institut Non Linéaire de Nice–Sophia Antipolis, UMR CNRS 7335, Université de Nice – Sophia Antipolis, 1361, Route des Lucioles, 06560 Valbonne, France

Received 3 November 2011; accepted 20 July 2012

Available online 20 August 2012

Abstract

We show that every linearly repetitive Delone set in the Euclidean d -space \mathbb{R}^d , with $d \geq 2$, is equivalent, up to a bi-Lipschitz homeomorphism, to the integer lattice \mathbb{Z}^d . In the particular case when the Delone set X in \mathbb{R}^d comes from a primitive substitution tiling of \mathbb{R}^d , we give a condition on the eigenvalues of the substitution matrix which ensures the existence of a homeomorphism with bounded displacement from X to the lattice $\beta\mathbb{Z}^d$ for some positive β . This condition includes primitive Pisot substitution tilings but also concerns a much broader set of substitution tilings.

© 2012 L'Association Publications de l'Institut Henri Poincaré. Published by Elsevier B.V. All rights reserved.

1. Introduction

Let (Z, δ) be a metric space. A subset X of Z is called a *Delone set* or *separated net* if there exist $r, R > 0$ such that each ball of radius R in Z contains at least one point of X and each ball of radius r in Z contains at most one point of X . Let X_1 and X_2 be two Delone sets in Z . We say that they are *bi-Lipschitz equivalent* if there exist a homeomorphism $\phi : X_1 \rightarrow X_2$ and a constant $K > 0$ such that

$$\frac{1}{K}\delta(x, x') \leq \delta(\phi(x), \phi(x')) \leq K\delta(x, x')$$

holds for all x and x' in X_1 . The map ϕ is then called a *bi-Lipschitz homeomorphism* between X_1 and X_2 . We say that a homeomorphism $\phi : X_1 \rightarrow X_2$ is a *bounded displacement* if

$$\sup_{x \in X_1} \delta(\phi(x), x) < \infty.$$

Clearly a bounded displacement between two Delone sets is a bi-Lipschitz homeomorphism.

In the case when the ambient metric space (Z, δ) is the d -dimensional Euclidean space \mathbb{R}^d , $d \geq 2$, endowed with the Euclidean distance, the problem to know whether two Delone sets are bi-Lipschitz equivalent was raised by Gromov in [6], and boiled down in Toledo's review [18] to the following question for the 2-dimensional Euclidean space:

* Corresponding author.

E-mail addresses: jaliste@dim.uchile.cl (J. Aliste-Prieto), acoronel@mat.puc.cl (D. Coronel), gambaudo@unice.fr (J.-M. Gambaudo).

Is every separated net in \mathbb{R}^2 bi-Lipschitz equivalent to \mathbb{Z}^2 ? Counterexamples to this question were given independently by Burago and Kleiner [2] and McMullen [11]. McMullen also showed that when relaxing the bi-Lipschitz condition to a Hölder one, all separated nets in \mathbb{R}^d are equivalent. Later, Burago and Kleiner [3] gave a sufficient condition for a separated net to be bi-Lipschitz equivalent to \mathbb{Z}^2 and asked the following question: *If one forms a separated net in the plane by placing a point in the center of each tile of a Penrose tiling, is the resulting net bi-Lipschitz equivalent to \mathbb{Z}^2 ?* They studied the more general question of knowing whether a separated net arising from a cut-and-project tiling is bi-Lipschitz equivalent to \mathbb{Z}^2 (recall that the Penrose tiling is also a cut-and-project tiling [4]) and solved it in some cases that do not include the case of Penrose tilings, thus leaving the former question open.

More recently, Solomon [15] gave a positive answer in the case of Penrose tilings by using the fact that they can be constructed using substitutions (see for instance [7]). In fact, Solomon proved that each separated net arising from a primitive substitution tiling in \mathbb{R}^2 is bi-Lipschitz to \mathbb{Z}^2 . Moreover, as an application of the work of Laczkovich [10], he showed that for every substitution tiling of \mathbb{R}^d of Pisot type there is a bounded displacement between its associated separated net and $\beta\mathbb{Z}^d$ for some $\beta > 0$ (see Section 2 for more details).

During the same period and surprisingly rather independently, Delone sets in \mathbb{R}^d , have been used in mathematical physics as models of solid materials. In particular, after the discovery of quasicrystals at the beginning of the 80s [14], a strong impulse has been devoted to model these quasi-periodic materials by appropriate Delone sets, introducing in this way the notion of “repetitive” Delone sets.

Later, Lagarias and Pleasants focused on “linearly repetitive” Delone sets [9] as a subclass of repetitive Delone sets that models all known examples of quasicrystals. This class includes all Delone sets arising from self-similar tilings (it contains in particular the Penrose tiling drawn with triangles and the Penrose tiling drawn with “thick” and “thin” rhombi [7]) but is actually much broader (see [16,12] and [17]).

In this paper, we make a connection between these two fields of research by using the second point of view (Delone sets and quasicrystals) to improve some known results concerning separated nets which are bi-Lipschitz equivalent to \mathbb{Z}^d or obtained from \mathbb{Z}^d by a bounded displacement.

On one hand, we prove that for any $d \geq 2$, every linearly repetitive Delone set in \mathbb{R}^d is bi-Lipschitz equivalent to \mathbb{Z}^d . On the other hand, we show that Delone sets arising from a class of substitution tilings of \mathbb{R}^d , which is larger than the class of Pisot-type tilings, are obtained from \mathbb{Z}^d by a bounded displacement.

From now on, we will prefer the denomination “Delone sets”, more widely used in the literature when the ambient metric space is the d -dimensional Euclidean space, than “separated nets”.

2. Definitions and results

2.1. Repetitive Delone sets

Let $d \geq 2$ and X be a Delone set in \mathbb{R}^d . We denote by $B(x, r)$ the closed ball around x of radius r in \mathbb{R}^d . A set of the form $X \cap B(x, r)$ with $x \in X$ is called a *patch (with size r)* of X centered at x . A Delone set X is called *repetitive* if for each $r > 0$, there exists $M > 0$ such that for each point z in \mathbb{R}^d , and for every patch with size r , $X \cap B(x, r)$, there exists y in $X \cap B(z, M)$, such that:

$$X \cap B(y, r) = (X \cap B(x, r)) + y - x.$$

The smallest such M is denoted by $M_X(r)$ and it is called the *repetitivity function* of X (see [9]). If there exists $L > 0$ such that $M_X(r) \leq Lr$, then X is called *linearly repetitive*.

Our first result is the following:

Theorem 2.1. *Every linearly repetitive Delone set in \mathbb{R}^d is bi-Lipschitz equivalent to \mathbb{Z}^d .*

Remark 1. Of course Theorem 2.1 is trivial when the dimension $d = 1$ since, in this case, every Delone set (with no extra assumptions) is bi-Lipschitz equivalent to \mathbb{Z} .

2.2. Substitution tilings

Our second result concerns tilings arising from primitive substitutions. For more details about substitutions and tilings, see for instance [16]. Let $d \geq 2$ and Λ be a closed subset of the Euclidean space \mathbb{R}^d . A *tiling* of Λ is an

at most countable collection $\mathcal{T} = (t_j)_{j \in J}$ of closed subsets of Λ that cover Λ and have pairwise disjoint interiors. The sets t_j are called *tiles* and, in this paper, all tiles are supposed to be homeomorphic to the unit closed ball in \mathbb{R}^d . Tiles may also be *colored*, which means that formally they carry a label or color with them. There are several notions of equivalence between tiles which depend on the tilings under consideration. Let \mathcal{E} be a group of isometries of \mathbb{R}^d containing all translations: two tiles p and q are \mathcal{E} -equivalent (or in short q is an \mathcal{E} -copy of p) if q is the image of p by an isometry in \mathcal{E} . If furthermore p and q are colored, then they must have the same color.

Let $\mathcal{P} = \{p_1, \dots, p_k\}$ be a finite collection of tiles. A tiling \mathcal{T} of $\Lambda \subset \mathbb{R}^d$ is \mathcal{E} -generated by \mathcal{P} if every tile in \mathcal{T} is an \mathcal{E} -copy of some tile in \mathcal{P} . The tiles in \mathcal{P} are called *prototiles*. The set of all tilings of Λ that are \mathcal{E} -generated by \mathcal{P} is denoted by $\Omega_{\mathcal{E}, \mathcal{P}}(\Lambda)$. When $\Lambda = \mathbb{R}^d$, we write $\Omega_{\mathcal{E}, \mathcal{P}}$ instead of $\Omega_{\mathcal{E}, \mathcal{P}}(\mathbb{R}^d)$.

Given $\lambda > 1$ and a subset $U \subset \mathbb{R}^d$, let $\lambda U := \{\lambda x \mid x \in U\}$ be the dilation of U by λ . Similarly, if \mathcal{M} is a collection of tiles, the dilation of \mathcal{M} by λ is the collection $\lambda \mathcal{M} := \{\lambda t \mid t \in \mathcal{M}\}$. Clearly, if \mathcal{M} is a tiling, then $\lambda \mathcal{M}$ is also a tiling. A *substitution rule* (with dilation factor $\lambda > 1$) is a collection $\mathcal{S} = (\mathcal{S}_{p_i})_{i=1}^k$, where for each $i \in \{1, \dots, k\}$, \mathcal{S}_{p_i} is a tiling of λp_i which is \mathcal{E} -generated by \mathcal{P} . Thus, each \mathcal{S}_{p_i} gives the rule of how to decompose λp_i into prototiles, for each i . Every substitution induces a natural map $\mathcal{I}_{\mathcal{S}}$ on $\Omega_{\mathcal{E}, \mathcal{P}}$, which first dilates tiles by λ and then replaces dilated tiles by a patch of prototiles according to the substitution rule, for more details see [Appendix B](#).

A tiling \mathcal{T} in $\Omega_{\mathcal{E}, \mathcal{P}}$ is said to be *admissible* for \mathcal{S} if it belongs to

$$\Omega_{\mathcal{S}} := \bigcap_{k \geq 0} \mathcal{I}_{\mathcal{S}}^k(\Omega_{\mathcal{E}, \mathcal{P}}).$$

For any given substitution, we can associate an integer matrix, in which each element counts how many tiles of a given type appear in a tile of another type after dilation and substitution. Depending on the definition of tile-type, we may obtain different matrices. For our purposes here, we consider the following definition (see also [15]): Given a substitution rule \mathcal{S} , we say that two tiles p_i and p_j have the same *type* (or are \mathcal{S} -equivalent) if there exists an isometry O of \mathbb{R}^d such that $O(p_i) = p_j$ and $O(\mathcal{S}_{p_i}) = \mathcal{S}_{p_j}$. Let $\mathcal{Q} = \{q_1, \dots, q_n\}$ be the set of tile-types of all prototiles. The *substitution matrix* is then defined as the $n \times n$ matrix $M_{\mathcal{S}} = (m_{i,j})_{i,j}$, where each $m_{i,j}$ is the number of tiles of type q_j that belong to \mathcal{S}_p where p is any prototile of type q_i . The definition of tile-type implies that $m_{i,j}$ does not depend on p and thus $M_{\mathcal{S}}$ is well-defined.

Finally, we recall some basic definitions of Perron–Frobenius theory needed to state our result. A matrix M is *primitive* if there exists $n > 0$ such that all the elements of M^n are positive. By Perron–Frobenius theorem, every primitive matrix M has a largest positive real eigenvalue μ , the *Perron eigenvalue*, and moreover it has no other eigenvalue with the same modulus as μ . It is easy to check that the Perron eigenvalue of $M_{\mathcal{S}}$ for a given substitution rule \mathcal{S} is $\mu = \lambda^d$ when $M_{\mathcal{S}}$ is primitive.

Given a tiling \mathcal{T} in $\Omega_{\mathcal{E}, \mathcal{P}}$, define $X_{\mathcal{T}}$ to be the set of barycenters of all the tiles in \mathcal{T} . It is clear that $X_{\mathcal{T}}$ is a Delone set in \mathbb{R}^d , and we call it the *Delone set induced by \mathcal{T}* .

Theorem 2.2. *Suppose that \mathcal{S} is a substitution rule with dilation factor λ , that the substitution matrix $M_{\mathcal{S}}$ is primitive and that*

$$r(M) := \max\{|\eta| \mid \eta \neq \mu \text{ is an eigenvalue of } M\} < \lambda,$$

where μ is the Perron eigenvalue of $M_{\mathcal{S}}$. Then there exists $\beta > 0$ such that for every tiling \mathcal{T} in $\Omega_{\mathcal{S}}$, the Delone set $X_{\mathcal{T}}$ induced by \mathcal{T} is obtained from $\beta \mathbb{Z}^d$ by a bounded displacement.

Remark 2. A substitution rule \mathcal{S} is of *Pisot type* if $r(M_{\mathcal{S}}) < 1$ (compare with the definition in [15]).

We finish this section by introducing some useful notations. For any $d \geq 1$, we denote by μ_d the Lebesgue measure in \mathbb{R}^d . For every subset U of \mathbb{R}^d , we denote its boundary by ∂U . Furthermore, for every subset U of \mathbb{R}^d and every Delone set X in \mathbb{R}^d , we denote by $\mathcal{N}(X, U)$ the number of points of X inside U .

3. Proof of Theorem 2.1

In [3], Burago and Kleiner gave a sufficient condition for a Delone set in \mathbb{R}^2 to be bi-Lipschitz equivalent to \mathbb{Z}^2 . This condition concerns the speed of convergence to an asymptotic density of the number of points of X inside larger

and larger balls. As we will see, it turns out that an analog condition works in every dimension $d \geq 2$. First, we need some definitions. A *cube* C with size $l(C) > 0$ in \mathbb{R}^d is a subset of the form $C = x + [0, l(C)]^d$, where $x \in \mathbb{R}^d$. Let $X \subset \mathbb{R}^d$ be a Delone set. Given $\rho > 0$ and a cube $C \subset \mathbb{R}^d$ with size $l(C)$, define $e_\rho(C)$ to be the *density deviation*

$$e_\rho(C) = \max\left(\frac{\rho\mu_d(C)}{\mathcal{N}(X, C)}, \frac{\mathcal{N}(X, C)}{\rho\mu_d(C)}\right). \tag{3.1}$$

Next, for $k \in \mathbb{N}$, define $E_\rho(k)$ as the supremum of the quantities $E_\rho(C)$, where C ranges over all cubes with size $l(C) = k$ and vertices at \mathbb{Z}^d . The condition reads as follows.

Theorem 3.1. *Let $d \geq 2$ and $X \subset \mathbb{R}^d$ be a Delone set. Suppose that there is $\rho > 0$ such that the product*

$$\prod_{m=1}^{+\infty} E_\rho(2^m)$$

converges. Then X is bi-Lipschitz equivalent to \mathbb{Z}^d .

Remark 3. When $d = 2$, Theorem 3.1 corresponds to the main theorem of [3]. In the proof of [3], the authors solve a prescribed volume form equation. To prove this result for $d > 2$ we use a very useful construction by Rivière and Ye [13], which actually simplifies the original proof of [3]. The proof of Theorem 3.1 is given in Appendix A.

Linearly repetitive Delone sets are good candidates to satisfy the condition of Theorem 3.1 as suggested by the following result of Lagarias and Pleasants:

Theorem 3.2. *(See [9].) Let $d \geq 2$ and X be a linearly repetitive Delone set in \mathbb{R}^d . Then there exist positive constants $\rho(X)$ and $\delta(X) < 1$, such that for any cube C with size $l(C)$, we have:*

$$|\mathcal{N}(X, C) - \rho(X)\mu_d(C)| = O(l(C)^{d-\delta(X)}). \tag{3.2}$$

Remark 4. Lagarias and Pleasants proved a stronger version of the above theorem by giving similar estimates for the occurrences of every patch in X . We will not use this stronger version here.

Proof of Theorem 2.1. Let X be a linearly repetitive Delone set in \mathbb{R}^d . By Theorem 3.1, it is enough to show that there exists $\rho > 0$ such that the product $\prod_{m=1}^{\infty} E_\rho(2^m)$ converges. Indeed, Theorem 3.2, tells us that there exist $\rho(X)$, M , δ and l_0 , all positive, such that

$$\left| \frac{\mathcal{N}(X, C)}{\mu_d(C)} - \rho(X) \right| \leq Ml(C)^{-\delta},$$

for every cube C in \mathbb{R}^d with side $l(C) \geq l_0$. Since $\rho(X) > 0$, a simple computation shows that there exist constants $M', l_1 > 0$ such that

$$\max\left(\left| \frac{\rho(X)\mu_d(C)}{\mathcal{N}(X, C)} - 1 \right|, \left| \frac{\mathcal{N}(X, C)}{\rho(X)\mu_d(C)} - 1 \right|\right) \leq M'l(C)^{-\delta}$$

for every cube C with side $l(C) \geq l_1$ and thus

$$1 \leq e_{\rho(X)}(C) \leq 1 + M'l(C)^{-\delta}.$$

Taking the supremum we get

$$1 \leq E_{\rho(X)}(l(C)) \leq 1 + M'l(C)^{-\delta}, \quad \text{for all } l(C) \geq l_1. \tag{3.3}$$

It follows that $\sum_{m=1}^{\infty} \log E(2^m)$ converges which implies that $\prod_{m=0}^{\infty} E(2^m)$ also converges and the conclusion now follows. \square

4. Proof of Theorem 2.2

First let us introduce some notations. A *unit cube* in \mathbb{R}^d is a cube with size 1 whose vertices have integer coordinates. We denote by \mathcal{U} the set of all subsets of \mathbb{R}^d which coincide with a finite union of unit cubes. Similarly, for all $\delta > 0$, we denote by \mathcal{U}_δ the set of all subsets of \mathbb{R}^d which coincide with a finite union of cubes with size δ whose vertices have coordinates in $\delta\mathbb{Z}^d$. In [10], Laczkovich obtained the following very elegant characterization of Delone sets that can be obtained from a lattice $\beta\mathbb{Z}^d$ by a bounded displacement.

Theorem 4.1. (See [10].) *For any $d \geq 2$ and for every Delone set X in \mathbb{R}^d and every $\alpha > 0$, the following statements are equivalent:*

(1) *There exists $K > 0$ such that for every subset $U \in \mathcal{U}$,*

$$|\mathcal{N}(X, U) - \alpha\mu_d(U)| \leq K\mu_{d-1}(\partial U); \tag{4.1}$$

(2) *There is a bounded displacement from X onto $\alpha^{-1/d}\mathbb{Z}^d$.*

Remark 5. Notice that by rescaling Theorem 4.1 works as well if we prove Eq. (4.1) for all subsets U in some \mathcal{U}_δ .

Using Laczkovich’s characterization and the above remark, Theorem 2.2 turns to be a straightforward corollary of the following result, whose proof follows arguments introduced in [1] and is given in Appendix B.

Theorem 4.2. *Let $d > 0$ and let \mathcal{S} be a substitution rule with dilation factor λ and primitive substitution matrix $M_{\mathcal{S}}$. If $r(M_{\mathcal{S}}) < \lambda$, then there exist $\delta > 0$, $K > 0$ and $\alpha > 0$ such that for every tiling \mathcal{T} in $\Omega_{\mathcal{S}}$,*

$$|\mathcal{N}(X_{\mathcal{T}}, U) - \alpha\mu_d(U)| \leq K\mu_{d-1}(\partial U) \tag{4.2}$$

for every subset $U \in \mathcal{U}_\delta$.

Acknowledgements

After a first version of this work was sent to arXiv, we learned from M. Baake that D. Frettlöh and A. Garber [5] obtained a version of Theorem 2.1 when $d = 2$.

This work is part of the project *CrystalDyn* supported by the “Agence Nationale de la Recherche” (ANR-06-BLAN-0070-01). D. Coronel and J. Aliste-Prieto respectively acknowledge support from Fondecyt post-doctoral Grants 3100092 and 3100097.

The authors are very grateful to Dong Ye who, very generously, introduced them to his work with Tristan Rivière.

Appendix A. Proof of Theorem 3.1

Burago and Kleiner proved that Theorem 3.1 is a consequence of the following proposition that they prove in dimension 2.

Proposition A.1. *Let $u : \mathbb{R}^d \rightarrow \mathbb{R}$ be a positive function which is constant on each open unit cube with vertices in \mathbb{Z}^d , and let $\rho > 0$ be given. Assume that u and $1/u$ are bounded. For any cube C in \mathbb{R}^d , let $e(C)$ be the quantity*

$$\max \left\{ \frac{\rho}{\frac{1}{|C|} \int_C u}, \frac{\frac{1}{|C|} \int_C u}{\rho} \right\},$$

where $|C| = \mu_d(C)$ is the Lebesgue measure of the cube C . Define an “error” function $E : \mathbb{N} \rightarrow \mathbb{R}$ by letting $E(k)$ be the supremum of $e(\cdot)$ taken over the collection of cubes of the form $[i_1, i_1 + k] \times \cdots \times [i_d, i_d + k]$, where $(i_1, \dots, i_d) \in \mathbb{Z}^d$. If the product

$$\prod_{i=1}^{+\infty} E(2^i)$$

converges, then there exists a bi-Lipschitz homeomorphism $\Psi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ with $\det(\nabla\Psi) = u$ a.e.

We prove this proposition in any dimension. Our proof is shorter than the specific one done in the two-dimensional case in [3] and relies on the following lemma¹ proved by D. Ye and T. Rivière:

Lemma A.2. (See [13].) *Let $D = [0, 1]^d$, $A = [0, 1]^{d-1} \times [0, 1/2]$ and $B = [0, 1]^{d-1} \times [1/2, 1]$. Let $\alpha > 0$ and $\beta > 0$ such that $\alpha + \beta = 1$. There exists a bi-Lipschitz homeomorphism Φ from D into itself such that:*

- (i) $\Phi(x) = x, x \in \partial D$;
- (ii) $\det(\nabla\Phi) \equiv 2\alpha$ in A and $\det(\nabla\Phi) \equiv 2\beta$ in B ;
- (iii) $\|\nabla(\Phi - Id)\|_{L^\infty(D)} \leq C_\eta|\beta - \alpha|$;
- (iv) $\|\nabla(\Phi^{-1} - Id)\|_{L^\infty(D)} \leq C_\eta|\beta - \alpha|$,

where $0 < \eta \leq \alpha, 0 < \eta \leq \beta$ and C_η only depends on η .

In order to prove Proposition A.1, we adapt to our context a constructive method developed in [13]. Let $\bar{n} = (n_1, \dots, n_d)$ be a point in \mathbb{Z}^d and $m > 0$ be a positive integer. Consider the cube:

$$C_{\bar{n},m} = \prod_{l=1}^d [n_l, n_l + 2^m].$$

In each of these cubes, for each integer $i, 0 \leq i \leq m$, and each integer vector $\bar{k} = (k_1, \dots, k_d)$ in $\Lambda_{m,i} := \mathbb{Z}^d \cap \prod_{l=1}^d [0, 2^{m-i}]$, we consider the smaller cube:

$$C_{\bar{n},m,i,\bar{k}} = \prod_{l=1}^d [n_l + k_l 2^i, n_l + (k_l + 1) 2^i].$$

Notice that for each $0 \leq i \leq m$, and every pair $\bar{k} \neq \bar{k}'$, the cubes $C_{\bar{n},m,i,\bar{k}}$ and $C_{\bar{n},m,i,\bar{k}'}$ have disjoint interiors. Moreover,

$$\bigcup_{\bar{k} \in \Lambda_{m,i}} C_{\bar{n},m,i,\bar{k}} = C_{\bar{n},m}.$$

Take $\epsilon = (\epsilon_1, \dots, \epsilon_d) \in \{0, 1\}^d$, let $1 \leq p \leq d$ and denote by $A_{\bar{n},m,i,\bar{k}}^p(\epsilon), B_{\bar{n},m,i,\bar{k}}^p(\epsilon)$ and $D_{\bar{n},m,i,\bar{k}}^p(\epsilon)$ the following subsets of $C_{\bar{n},m,i,\bar{k}}$:

$$\begin{aligned} A_{\bar{n},m,i,\bar{k}}^p(\epsilon) &= \prod_{l=1}^{p-1} [n_l + k_l 2^i, n_l + (k_l + 1) 2^i] \times \left[n_p + k_p 2^i, n_p + \left(k_p + \frac{1}{2}\right) 2^i \right] \\ &\quad \times \prod_{l=p+1}^d \left[n_l + \left(k_l + \frac{\epsilon_l}{2}\right) 2^i, n_l + \left(k_l + \frac{\epsilon_l + 1}{2}\right) 2^i \right], \\ B_{\bar{n},m,i,\bar{k}}^p(\epsilon) &= \prod_{l=1}^{p-1} [n_l + k_l 2^i, n_l + (k_l + 1) 2^i] \times \left[n_p + \left(k_p + \frac{1}{2}\right) 2^i, n_p + (k_p + 1) 2^i \right] \\ &\quad \times \prod_{l=p+1}^d \left[n_l + \left(k_l + \frac{\epsilon_l}{2}\right) 2^i, n_l + \left(k_l + \frac{\epsilon_l + 1}{2}\right) 2^i \right], \end{aligned}$$

¹ Actually (iv) is not explicitly written in [13] but turns to be a straightforward consequence of the construction.

and

$$D_{\bar{n},m,i,\bar{k}}^p(\epsilon) = A_{\bar{n},m,i,\bar{k}}^p(\epsilon) \cup B_{\bar{n},m,i,\bar{k}}^p(\epsilon).$$

Let $u : \mathbb{R}^d \rightarrow \mathbb{R}$ be a positive function which is constant on each open unit cube with vertices in \mathbb{Z}^d and consider the two positive numbers

$$\alpha_{\bar{n},m,i,\bar{k}}^p(\epsilon) = \frac{\int_{A_{\bar{n},m,i,\bar{k}}^p(\epsilon)} u(x) dx}{\int_{D_{\bar{n},m,i,\bar{k}}^p(\epsilon)} u(x) dx},$$

and

$$\beta_{\bar{n},m,i,\bar{k}}^p(\epsilon) = \frac{\int_{B_{\bar{n},m,i,\bar{k}}^p(\epsilon)} u(x) dx}{\int_{D_{\bar{n},m,i,\bar{k}}^p(\epsilon)} u(x) dx}.$$

We clearly have

$$\alpha_{\bar{n},m,i,\bar{k}}^p(\epsilon) + \beta_{\bar{n},m,i,\bar{k}}^p(\epsilon) = 1.$$

Lemma A.3. $\forall \epsilon \in \{0, 1\}^d, \forall p \in \{1, \dots, d\}, \forall \bar{k} \in \Lambda_{m,i}, \forall i \in \{1, \dots, m\}, \forall m > 0,$ and $\forall \bar{n} \in \mathbb{Z}^d,$ we have:

$$\frac{1}{2} \frac{1}{(E(2^{i-1}))^2} \leq \alpha_{\bar{n},m,i,\bar{k}}^p(\epsilon) \leq \frac{1}{2} (E(2^{i-1}))^2$$

and

$$\frac{1}{2} \frac{1}{(E(2^{i-1}))^2} \leq \beta_{\bar{n},m,i,\bar{k}}^p(\epsilon) \leq \frac{1}{2} (E(2^{i-1}))^2.$$

Consequently, if u satisfies the hypothesis of Proposition A.1, then there exists $0 < \eta^* < 1$ such that

$$\eta^* \leq \alpha_{\bar{n},m,i,\bar{k}}^p(\epsilon)$$

and

$$\eta^* \leq \beta_{\bar{n},m,i,\bar{k}}^p(\epsilon).$$

Proof. The very definition of $E(2^i)$ shows that for all $C_{\bar{n},m,i,\bar{k}}$ we have:

$$\frac{1}{E(2^i)} \rho |C_{\bar{n},m,i,\bar{k}}| \leq \int_{C_{\bar{n},m,i,\bar{k}}} u(x) dx \leq E(2^i) \rho |C_{\bar{n},m,i,\bar{k}}|.$$

These estimates remain true for a given i , for every finite collection R_i of cubes $C_{\bar{n},m,i,\bar{k}}$ with disjoint interiors:

$$\frac{1}{E(2^i)} \rho |R_i| \leq \int_{R_i} u(x) dx \leq E(2^i) \rho |R_i|.$$

We remark that when $1 \leq i \leq m, A_{\bar{n},m,i,\bar{k}}^p(\epsilon)$ and $B_{\bar{n},m,i,\bar{k}}^p(\epsilon)$ are precisely finite union of cubes $C_{\bar{n},m,i-1,\bar{k}}$ with disjoint interiors. The proof of the lemma follows easily. \square

Up to homothety and rotation, we are in a situation to apply Lemma A.2 and get a bi-Lipschitz homeomorphism $\Phi_{\bar{n},m,i,\bar{k}}^p(\epsilon)$ from $D_{\bar{n},m,i,\bar{k}}^p(\epsilon)$ into itself such that:

- (i) $\Phi_{\bar{n},m,i,\bar{k}}^p(\epsilon)(x) = x, x \in \partial D_{\bar{n},m,i,\bar{k}}^p(\epsilon);$
- (ii) $\det(\nabla \Phi_{\bar{n},m,i,\bar{k}}^p(\epsilon)) = 2\alpha_{\bar{n},m,i,\bar{k}}^p(\epsilon)$ in $A_{\bar{n},m,i,\bar{k}}^p(\epsilon);$
- (iii) $\det(\nabla \Phi_{\bar{n},m,i,\bar{k}}^p(\epsilon)) = 2\beta_{\bar{n},m,i,\bar{k}}^p(\epsilon)$ in $B_{\bar{n},m,i,\bar{k}}^p(\epsilon);$

- (iv) $\|\nabla(\Phi_{\bar{n},m,i,\bar{k}}^P(\epsilon) - Id)\|_{L^\infty(D_{\bar{n},m,i,\bar{k}}^P(\epsilon))} \leq \frac{1}{2}C_{\eta^*}((E(2^{i-1}))^2 - (E(2^{i-1}))^{-2});$
- (v) $\|\nabla(\Phi_{\bar{n},m,i,\bar{k}}^P(\epsilon)^{-1} - Id)\|_{L^\infty(D_{\bar{n},m,i,\bar{k}}^P(\epsilon))} \leq \frac{1}{2}C_{\eta^*}((E(2^{i-1}))^2 - (E(2^{i-1}))^{-2}).$

We denote by $\Phi_{\bar{n},m,i,\bar{k}}^P$ the bi-Lipschitz homeomorphism from $C_{\bar{n},m,i,\bar{k}}$ into itself defined by $\Phi_{\bar{n},m,i,\bar{k}}^P(x) = \Phi_{\bar{n},m,i,\bar{k}}^P(\epsilon)(x)$ if $x \in D_{\bar{n},m,i,\bar{k}}^P(\epsilon)$, and consider the bi-Lipschitz homeomorphism obtained by composition:

$$\Phi_{\bar{n},m,i,\bar{k}} = \Phi_{\bar{n},m,i,\bar{k}}^d \circ \Phi_{\bar{n},m,i,\bar{k}}^{d-1} \circ \dots \circ \Phi_{\bar{n},m,i,\bar{k}}^1.$$

Since the Jacobians of the $\Phi_{\bar{n},m,i,\bar{k}}^P$ are constant by parts, a simple calculation shows that

$$\det(\nabla\Phi_{\bar{n},m,i,\bar{k}}) = 2^d \frac{\int_{C_{\bar{n},m,i-1,\bar{k}'}} u(x) dx}{\int_{C_{\bar{n},m,i,\bar{k}}} u(x) dx},$$

for any sub-cube $C_{\bar{n},m,i-1,\bar{k}'}$ in $C_{\bar{n},m,i,\bar{k}}$. It is also direct to check that:

$$\|\nabla\Phi_{\bar{n},m,i,\bar{k}}\|_{L^\infty(C_{\bar{n},m,i,\bar{k}})} \leq K_{i,d} \quad \text{and} \quad \|\nabla\Phi_{\bar{n},m,i,\bar{k}}^{-1}\|_{L^\infty(C_{\bar{n},m,i,\bar{k}})} \leq K_{i,d},$$

where

$$K_{i,d} = \left(1 + \frac{1}{2}C_{\eta^*}((E(2^{i-1}))^2 - (E(2^{i-1}))^{-2})\right)^d.$$

We denote by $\Phi_{\bar{n},m,i}$ the bi-Lipschitz homeomorphism from $C_{\bar{n},m}$ into itself defined by $\Phi_{\bar{n},m,i}(x) = \Phi_{\bar{n},m,i,\bar{k}}(x)$ if $x \in C_{\bar{n},m,i,\bar{k}}$, and we consider the bi-Lipschitz homeomorphism obtained by composition:

$$\Psi_{\bar{n},m} = \Phi_{\bar{n},m,m} \circ \Phi_{\bar{n},m,m-1} \circ \dots \circ \Phi_{\bar{n},m,1}.$$

The map $\Psi_{\bar{n},m}$ is a bi-Lipschitz homeomorphism on $C_{\bar{n},m}$ which satisfies:

- $\Psi_{\bar{n},m} = Id$ on $\partial C_{\bar{n},m}$;
- $\det(\nabla\Psi_{\bar{n},m}) = \frac{u}{|C_{\bar{n},m}| \int_{C_{\bar{n},m}} u(x) dx}$ in each unit cube with integer vertices in $C_{\bar{n},m}$;
- $\|\nabla\Phi_{\bar{n},m}\|_{L^\infty(C_{\bar{n},m})} \leq \prod_{i=1}^m K_{i,d}$ and $\|\nabla\Phi_{\bar{n},m}^{-1}\|_{L^\infty(C_{\bar{n},m})} \leq \prod_{i=1}^m K_{i,d}$.

Lemma A.4. *If the product $\prod_{i=1}^{+\infty} E(2^i)$ converges, then the product $\prod_{i=1}^{+\infty} K_{i,d}$ converges too.*

Proof. It is enough to show that the product

$$\prod_{i=1}^{+\infty} \left(1 + \frac{1}{2}C_{\eta^*}((E(2^{i-1}))^2 - (E(2^{i-1}))^{-2})\right)$$

converges and this last product converges if the series:

$$\sum_{i=1}^{+\infty} \ln\left(1 + \frac{1}{2}C_{\eta^*}((E(2^{i-1}))^2 - (E(2^{i-1}))^{-2})\right)$$

converges. This will be true if

$$\sum_{i=1}^{+\infty} ((E(2^{i-1}))^2 - (E(2^{i-1}))^{-2}) < +\infty,$$

and thus if

$$\sum_{i=1}^{+\infty} ((E(2^{i-1})) - (E(2^{i-1}))^{-1}) < +\infty. \tag{*}$$

On the one hand, the convergence of the product $\prod_{i=1}^{+\infty} E(2^i)$ implies that the series

$$\sum_{i=1}^{+\infty} \ln(1 + (E(2^{i-1}) - 1))$$

converges which in turn implies that

$$\sum_{i=1}^{+\infty} (E(2^{i-1}) - 1) < +\infty.$$

On the other hand, for i_0 large enough

$$\sum_{i=i_0}^{+\infty} ((E(2^{i-1})) - (E(2^{i-1}))^{-1}) \leq 2 \sum_{i=i_0}^{+\infty} ((E(2^{i-1})) - 1). \quad \square$$

We extend each $\Psi_{\bar{n},m}$ by the Identity out of $C_{\bar{n},m}$ to get a bi-Lipschitz homeomorphism of \mathbb{R}^d .

It follows from Lemma A.4 that the sequence of bi-Lipschitz constants of the bi-Lipschitz homeomorphisms $\Psi_{\bar{n},m}$ is bounded and thus, using Azerlâ–Ascoli theorem, there exists an accumulation point Φ which is a bi-Lipschitz homeomorphism on \mathbb{R}^d and which satisfies:

$$\det(\nabla\Phi) = \frac{u}{\rho}$$

in each unit cube with integer vertices in \mathbb{R}^d . By post-composing Φ with a homothety with ratio $\rho^{\frac{1}{d}}$ we get an answer to Proposition A.1.

The reminder of the proof of Theorem 3.1, which uses the Hall marriage lemma, follows exactly [3] and the dimension is irrelevant, see Section 4 in [3].

Appendix B. Proof of Theorem 4.2

We first state Theorem B.1 below, which is a generalization of the main theorem in [1].

Theorem B.1. *Let $d \geq 2$ and let S be a substitution rule with dilation factor λ and primitive substitution matrix M_S . If $r(M_S) < \lambda$, then there exist $\delta > 0$, $K > 0$ and $\alpha > 0$ such that for every tiling \mathcal{T} in Ω_S ,*

$$|\mathcal{N}(\mathcal{T}, U) - \alpha \mu_d(U)| \leq K \mathcal{L}(\mathcal{T}, \partial U)$$

for every $U \in \mathcal{U}_\delta$, where $\mathcal{N}(\mathcal{T}, U)$ stands for the number of tiles of \mathcal{T} that are contained in U and $\mathcal{L}(\mathcal{T}, \partial U)$ stands for the number of tiles of \mathcal{T} that intersect ∂U .

Let us show how to prove Theorem 4.2 assuming Theorem B.1. We need to compare the number of tiles that intersect the boundary of a subset $U \in \mathcal{U}_\delta$ with the measure of its boundary.

For a subset A of \mathbb{R}^d , define

$$r_{\min}(A) := \sup\{r > 0 \mid \text{there exists } x \in A \text{ s.t. } B(x, r) \subset A\}.$$

Given a tiling \mathcal{T} of \mathbb{R}^d , define

$$r_{\mathcal{T}} = \inf\{r_{\min}(t) \mid t \in \mathcal{T}\}$$

and

$$R_{\mathcal{T}} = \sup\{\text{diam}(t) \mid t \in \mathcal{T}\}/2.$$

If $0 < r_{\mathcal{T}} < R_{\mathcal{T}} < +\infty$, then we say that \mathcal{T} is *locally finite* and define $K_{\mathcal{T}} := \lfloor 4^d R_{\mathcal{T}}^d r_{\mathcal{T}}^{-d} \rfloor$. Notice that the substitution tilings we consider in this paper are locally finite. A simple computation yields (see [1] for details):

Lemma B.2. *Let $d \geq 2$ and \mathcal{T} be a locally finite tiling of \mathbb{R}^d , then each ball of radius smaller than or equal to $2R_{\mathcal{T}}$ meets at most $K_{\mathcal{T}}$ tiles of \mathcal{T} .*

Corollary B.3. *Let $d \geq 2$, \mathcal{T} be a locally finite tiling of \mathbb{R}^d and $\delta > 0$. Then, for every subset $U \in \mathcal{U}_{\delta}$ we have:*

$$\mathcal{L}(\mathcal{T}, \partial U) \leq K_{\mathcal{T}} \left(\frac{\sqrt{d-1}}{2R_{\mathcal{T}}} + 1 \right)^{d-1} \delta^{-(d-1)} \mu_{d-1}(\partial U).$$

Proof. Each facet of ∂U is covered by less than $(\lfloor \sqrt{d-1}/(2R_{\mathcal{T}}) \rfloor + 1)^{d-1}$ $(d-1)$ -cubes of size $2R_{\mathcal{T}}/\sqrt{d-1}$ having pairwise disjoint $(d-1)$ -interiors. Each of these cubes is included in a d -ball of radius $2R_{\mathcal{T}}$ and thus intersect at most $K_{\mathcal{T}}$ tiles of \mathcal{T} . The number of facets of ∂U is equal to $\delta^{-(d-1)} \mu_{d-1}(\partial U)$. \square

Since there is exactly one point of $X_{\mathcal{T}}$ in the center of each tile of \mathcal{T} , it follows that

$$0 \leq \mathcal{N}(X_{\mathcal{T}}, U) - \mathcal{N}(\mathcal{T}, U) \leq \mathcal{L}(\mathcal{T}, \partial U). \tag{B.1}$$

Combining this inequality with Theorem B.1 and Corollary B.3 we obtain that there exists a constant $K > 0$ such that

$$|\mathcal{N}(X_{\mathcal{T}}, U) - \alpha \mu_d(U)| < K \mu_{d-1}(\partial U) \tag{B.2}$$

for every for every $U \in \mathcal{U}_{\delta}$, which proves Theorem 4.2.

We now turn to the proof of Theorem B.1. First let us choose the adequate scale δ . Let \mathcal{T} be a tiling in $\Omega_{\mathcal{S}}$. We say that δ fits with \mathcal{T} if it is chosen large enough such that for any subset $U \in \mathcal{U}_{\delta}$:

- U contains a tile of \mathcal{T} ;
- for any connected component \mathcal{C} of ∂U , $K_{\mathcal{T}} < \mathcal{N}(\mathcal{T}, \mathcal{C})$;
- two distinct connected components of ∂U cannot intersect the same tile of \mathcal{T} .

From now, δ will be chosen to fit with \mathcal{T} . The proof of Theorem B.1 is divided in four key arguments.

• The first argument is a topological simplification. We show that it suffices to consider U in \mathcal{U}_{δ} such that U and ∂U are connected. Indeed, for any $U \in \mathcal{U}_{\delta}$, there is a finite collection V_1, \dots, V_n of pairwise disjoint connected elements in \mathcal{U}_{δ} such that:

$$U = \bigcup_{i=1}^n V_i.$$

We denote by $\partial V_{i,0}, \dots, \partial V_{i,p(i)}$ the connected components of ∂V_i , and fix $\partial V_{i,0}$ to be the connected component that bounds the component of the complementary of V_i with infinite diameter. For $j = 1, \dots, p(i)$, $\partial V_{i,j}$ is the boundary of an element $V_{i,j} \in \mathcal{U}_{\delta}$ which is connected and whose interior does not intersect V_i . We denote by \hat{V}_i the union:

$$\hat{V}_i = V_i \cup V_{i,1} \cup \dots \cup V_{i,p(i)},$$

it belongs to \mathcal{U}_{δ} is connected and its boundary $\partial V_{i,0}$ is connected. If Theorem B.1 works for a δ that fits with \mathcal{T} and all $U \in \mathcal{U}_{\delta}$ which are connected and have a connected boundary, we obtain that there exists a constant $K > 0$ such that, on one hand:

$$|\mathcal{N}(\mathcal{T}, \hat{V}_i) - \alpha \mu_d(\hat{V}_i)| < K \mathcal{L}(\mathcal{T}, \partial V_{i,0}), \quad \forall i \in \{1, \dots, n\}, \tag{B.3}$$

and on the other hand:

$$|\mathcal{N}(\mathcal{T}, V_{i,j}) - \alpha \mu_d(V_{i,j})| < K \mathcal{L}(\mathcal{T}, \partial V_{i,j}), \quad \forall i \in \{1, \dots, n\}, \forall j \in \{1, \dots, p(i)\}. \tag{B.4}$$

Since

$$\mathcal{N}(\mathcal{T}, \hat{V}_i) = \mathcal{N}(\mathcal{T}, V_i) + \sum_{j=1}^{p(i)} (\mathcal{N}(\mathcal{T}, V_{i,j}) + \mathcal{L}(\mathcal{T}, \partial V_{i,j}))$$

and

$$\mu_d(\hat{V}_i) = \mu_d(V_i) + \sum_{j=1}^{p(i)} \mu_d(V_{i,j}),$$

we get

$$|\mathcal{N}(\mathcal{T}, V_i) - \alpha \mu_d(V_i)| < (K + 1) \left(\sum_{j=0}^{p(i)} \mathcal{L}(\mathcal{T}, \partial V_{i,j}) \right), \quad \forall i \in \{1, \dots, n\}.$$

Summing on all the connected components of U we get

$$|\mathcal{N}(\mathcal{T}, U) - \alpha \mu_d(U)| < (K + 1) \mathcal{L}(\mathcal{T}, \partial U).$$

From now we will restrict ourselves to the case when U and ∂U are connected. The remainder of the proof follows the same lines of the proof of the main result in [1].

• The second argument is a geometrical one and is developed in the following two lemmas. These lemmas are proved in [1] for the two-dimensional case. The generalization of the first one to higher dimension is straightforward and therefore we skip it, while the proof of the second one requires slight modifications so is given for the convenience of the reader.

Lemma B.4. *Let \mathcal{T} be a locally finite tiling and γ be a simple curve. Then,*

$$\text{diam}(\gamma) \leq 2R_{\mathcal{T}}t \mathcal{L}(\mathcal{T}, \gamma),$$

where $\mathcal{L}(\mathcal{T}, \gamma)$ denotes the number of tiles that γ intersects.

Lemma B.5. *Let \mathcal{T} be a locally finite tiling and \mathcal{C} be a compact arc connected subset of \mathbb{R}^d . Then, for every locally finite tiling \mathcal{T}' satisfying $R_{\mathcal{T}'} > R_{\mathcal{T}}t$, and $K_{\mathcal{T}'} < \mathcal{L}(\mathcal{T}', \mathcal{C}) \leq \mathcal{L}(\mathcal{T}, \mathcal{C})$, we have:*

$$\mathcal{L}(\mathcal{T}', \mathcal{C}) \leq (2K_{\mathcal{T}'} + 1) \frac{R_{\mathcal{T}}t}{R_{\mathcal{T}'}} \mathcal{L}(\mathcal{T}, \mathcal{C}),$$

where $K_{\mathcal{T}'}$ is the constant defined in Lemma B.2.

Proof. Given $y \in \mathbb{R}^d$, set

$$C_y := \{t \in \mathcal{T}' \mid t \cap B_{2R_{\mathcal{T}'}}(y) \neq \emptyset\} \quad \text{and} \quad \hat{C}_y := \bigcup_{t \in C_y} t.$$

We construct a finite subset $\mathcal{Y} = \{y_i\}_{i=1}^p$ of \mathcal{C} as follows. First, fix any point of \mathcal{C} to be y_1 . Next, suppose that y_1, \dots, y_j have been chosen in such a way that, for each $i \in \{1, \dots, j\}$, the point y_i does not belong to \hat{C}_{y_k} for every $k \in \{1, \dots, i-1\}$. Then, if the sets $\{\hat{C}_{y_i}\}_{i=0}^j$ cover \mathcal{C} , we set $p = j$ and the construction is completed; otherwise, we choose any point in the intersection of \mathcal{C} and the complement of $\bigcup_{i=1}^j \hat{C}_{y_i}$ as y_{j+1} and continue iterating the construction. Since $\mathcal{L}(\mathcal{T}', \mathcal{C})$ is finite (this is because \mathcal{T}' is locally finite and \mathcal{C} is compact) and, in each iteration we add at least one tile of \mathcal{T}' intersecting \mathcal{C} to the region covered by $\{\hat{C}_{y_i}\}_{i=0}^j$, the construction stops after finitely many iterations. Thus, we have constructed a set $\mathcal{Y} = \{y_i\}_{i=1}^p$ with the following properties:

- (1) $\bigcup_{y \in \mathcal{Y}} \hat{C}_y$ covers \mathcal{C} ;
- (2) $d(y_i, y_j) > 2R_{\mathcal{T}'}$ for every $i, j \in \{1, \dots, p\}$ with $i \neq j$.

Lemma B.2 implies that for each $i \in \{1, \dots, p\}$, the set C_{y_i} contains at most $K_{\mathcal{T}'}$ tiles of \mathcal{T}' . From (1), we deduce that

$$\mathcal{L}(\mathcal{T}', \mathcal{C}) \leq pK_{\mathcal{T}'}. \tag{B.5}$$

By hypothesis, $p > 1$.

Now define $\mathcal{B} = \{B_i = B_{y_i}(R_{\mathcal{T}'} - R_{\mathcal{T}t})\}_{i=1}^p$. From (2) it is clear that the balls in \mathcal{B} are pairwise disjoint and that the minimal distance between two distinct balls in \mathcal{B} is (strictly) greater than $2R_{\mathcal{T}t}$. Fix $i \in \{1, \dots, p\}$. Since $p > 1$, there is a point $z \in \mathcal{C}$ that belongs to the complement of B_i . It follows that there is a path $\gamma_i : [0, 1] \rightarrow \mathcal{C}$ such that $\gamma_i(0) = y_i$ and $\gamma_i(1) = z$. It is clear that the map $\eta : t \mapsto \|\gamma_i - \gamma(t)\|$ is continuous, $\eta(0) = 0$ and $\eta(1) > R_{\mathcal{T}'} - R_{\mathcal{T}t}$. By continuity, we deduce that there exists $t_i \in [0, 1]$ such that $\gamma_i([0, t_i])$ is included in B_i and $\|\gamma_i(t_i) - y_i\| = R_{\mathcal{T}'} - R_{\mathcal{T}t}$. Applying Lemma B.4 we get

$$R_{\mathcal{T}'} - R_{\mathcal{T}t} \leq \text{diam}(\gamma_i([0, t_i])) \leq 2R_{\mathcal{T}t} \mathcal{L}(\mathcal{T}, \gamma_i([0, t_i]))$$

for all $i \in \{1, \dots, p\}$. Adding these inequalities yields

$$p(R_{\mathcal{T}'} - R_{\mathcal{T}t}) \leq 2R_{\mathcal{T}t} \sum_{i=1}^p \mathcal{L}(\mathcal{T}, \gamma_i([0, t_i])). \tag{B.6}$$

Now, observe that since the image of $\gamma_i([0, t_i])$ is included in B_i for each i , and the distance between B_i and B_j with $i \neq j$ is greater than $2R_{\mathcal{T}t}$, each tile in \mathcal{T} that intersects $\gamma_i([0, t_i])$ does not intersect $\gamma_j([0, t_j])$ for every $j \neq i$. On the other hand, the image of γ_i is included in \mathcal{C} . Hence, we deduce that

$$p(R_{\mathcal{T}'} - R_{\mathcal{T}t}) \leq 2R_{\mathcal{T}} \mathcal{L}(\mathcal{T}, \mathcal{C}). \tag{B.7}$$

Finally, combining (B.7) and (B.5) we get

$$\mathcal{L}(\mathcal{T}', \mathcal{C}) \leq 2K_{\mathcal{T}'} \frac{R_{\mathcal{T}t}}{R_{\mathcal{T}'} - R_{\mathcal{T}t}} \mathcal{L}(\mathcal{T}, \mathcal{C}). \tag{B.8}$$

To finish the proof, fix $c > 0$ arbitrarily and consider the following two cases. First, suppose that $R_{\mathcal{T}'} / R_{\mathcal{T}} \leq 1 + c$. Since $\mathcal{L}(\mathcal{T}', \mathcal{C}) \leq \mathcal{L}(\mathcal{T}, \mathcal{C})$, it follows that

$$\mathcal{L}(\mathcal{T}', \mathcal{C}) \leq (1 + c) \frac{R_{\mathcal{T}t}}{R_{\mathcal{T}'}} \mathcal{L}(\mathcal{T}, \mathcal{C}). \tag{B.9}$$

Now suppose that $R_{\mathcal{T}'} / R_{\mathcal{T}} > 1 + c$. It is easy to check that

$$(1 + c)(R_{\mathcal{T}'} - R_{\mathcal{T}t}) > cR_{\mathcal{T}'}$$

Replacing this inequality in (B.8) we get

$$\mathcal{L}(\mathcal{T}', \mathcal{C}) \leq 2K_{\mathcal{T}'} \frac{R_{\mathcal{T}t}}{R_{\mathcal{T}'}} \left(1 + \frac{1}{c}\right) \mathcal{L}(\mathcal{T}, \mathcal{C}). \tag{B.10}$$

Combining (B.10) and (B.9) yields

$$\mathcal{L}(\mathcal{T}', \mathcal{C}) \leq \max \left\{ 1 + c, 2K_{\mathcal{T}'} \left(1 + \frac{1}{c}\right) \right\} \frac{R_{\mathcal{T}t}}{R_{\mathcal{T}'}} \mathcal{L}(\mathcal{T}, \mathcal{C}).$$

An easy computation shows that the last bound is optimal when $c = 2K_{\mathcal{T}'}$ and the conclusion follows. \square

- The third argument is a combinatorial one, and is related to the notion of *hierarchical decompositions* for substitution tilings as studied in [1], which we recall now. Let \mathcal{S} be a substitution rule with dilation factor λ and \mathcal{T} be a tiling in $\Omega_{\mathcal{E}, \mathcal{P}}$. The substitution map $\mathcal{I}_{\mathcal{S}}$ is defined as follows: Each tile t in \mathcal{T} is first *dilated* by λ and then replaced by $O(\mathcal{S}_{p_i})$, where t is an \mathcal{E} -copy of p_i and O is the isometry in \mathcal{E} sending p_i to t . The union of all these tiles is clearly a tiling, which we denote $\mathcal{I}_{\mathcal{S}}(\mathcal{T})$. Similarly, we also can define $\mathcal{J}_{\mathcal{S}}^{(l)} : \Omega_{\mathcal{E}, \lambda^{(l+1)}\mathcal{P}} \rightarrow \Omega_{\mathcal{E}, \lambda^l\mathcal{P}}$ in a natural way. Recall that $\Omega_{\mathcal{S}}$ is defined as

$$\Omega_{\mathcal{S}} = \bigcap_{k \geq 0} \mathcal{I}_{\mathcal{S}}^k(\Omega_{\mathcal{E}, \mathcal{P}}).$$

It is plain to check that the map $\mathcal{I}_{\mathcal{S}}$ is onto when restricted to $\Omega_{\mathcal{S}}$. This implies that for each admissible tiling \mathcal{T} , there is a sequence $(\mathcal{T}^l)_{l \geq 0}$ of tilings, called a *hierarchical sequence* of \mathcal{T} , such that

- $\mathcal{T}^0 = \mathcal{T}$;
- for each $l \geq 0$, $\mathcal{T}^l \in \Omega_{\mathcal{E}, \lambda^l\mathcal{P}}$ and $\mathcal{J}_{\mathcal{S}}^{(l)}(\mathcal{T}^{l+1}) = \mathcal{T}^l$.

Remark 6. For every tiling \mathcal{T} in Ω_S and all $l \geq 0$, it is easy to see that $r_{\mathcal{T}^l} = \lambda^l r_{\mathcal{T}} > 0$ and $R_{\mathcal{T}^l} = \lambda^l R_{\mathcal{T}} < +\infty$. Thus, $K_{\mathcal{T}^l} = K_{\mathcal{T}}$ for all $l \geq 0$.

Proposition B.6. (See [1].) Let \mathcal{T} be an admissible tiling for S and let $(\mathcal{T}^l)_{l \geq 0}$ be a hierarchical sequence of \mathcal{T} and suppose that $\delta > 0$ fits \mathcal{T} . For every subset $U \in \mathcal{U}_\delta$ such that U and ∂U are connected, there exists a finite collection U_0, \dots, U_{m-1} of closed subsets of U such that:

- (i) $U_{\mathcal{T}} =: \bigcup_{i=0}^{m-1} U_i = \bigcup_{t \in u_{\mathcal{T}}} t$ where $u_{\mathcal{T}} = \{t \in \mathcal{T} \mid t \subset U\}$;
- (ii) all U_l 's have pairwise disjoint interiors;
- (iii) for each $l \in \{0, \dots, m-1\}$, U_l is a union of tiles in \mathcal{T}^l , and does not contain a tile in \mathcal{T}^{l+1} ;
- (iv) U does not contain a tile of \mathcal{T}^m .

The collection $\{U_0, \dots, U_{m-1}\}$ is called a hierarchical decomposition of U associated with \mathcal{T} . Moreover,

$$\mathcal{N}(\mathcal{T}^l, U_l) \leq \|M_S\|_1 \mathcal{L}(\mathcal{T}^{l+1}, \partial U), \tag{B.11}$$

for all $l \in \{0, \dots, m-1\}$, where $\|M_S\|_1$ is the maximum absolute column sum of the substitution rule M , and

$$\lambda^{m-l-1} \leq \frac{R_{\mathcal{T}}}{r_{\mathcal{T}}} \mathcal{L}(\mathcal{T}^l, \partial U), \tag{B.12}$$

for all $l \in \{0, \dots, m-1\}$.

Proof. Despite the fact that in [1], the proof of Proposition B.6 is given in the particular case when U is a bounded connected domain of the plane, it works here exactly along the same lines. \square

• The last argument is an algebraic one. It consists into applying Perron–Frobenius theory to the substitution matrix M_S . Recall that we are assuming that M is primitive. Thus, Perron–Frobenius theorem asserts that M_S has a largest real eigenvalue $\mu > 0$, the *Perron eigenvalue*, which is simple and larger than one. Recall that $r(M)$ is the modulus of the second largest eigenvalue of M , that is,

$$r(M) = \max\{|\eta| \mid \eta \neq \mu \text{ is an eigenvalue of } M\}.$$

Now, if v is a right-eigenvector associated with the eigenvalue μ with positive coefficients and we let e be the unit vector $(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})$, then we have the following well-known consequence of Perron–Frobenius theorem, see for instance [8].

Proposition B.7. For all $\rho > r(M)$, there exist $K = K(\rho) > 0$ and $\alpha > 0$ such that

$$\|M_S^l e - \alpha \mu^l v\|_1 \leq K \rho^l \tag{B.13}$$

for all integer $l > 0$.

Recall that \mathcal{Q} is the set of types of prototiles. Given $q_i, q_j \in \mathcal{Q}$, $m_{i,j}$ counts the number of tiles of type q_i in S_p where p is any prototile in q_j . The relation with Perron–Frobenius theory comes from the fact that if $m_{i,j}^{(l)}$ is the corresponding element of M_S^l , then $m_{i,j}^{(l)}$ counts the number of tiles of type q_i in a prototile of type q_j after applying the substitution to it l times.

Proof of Theorem B.1. Let $\{U_0, \dots, U_{m-1}\}$ be the hierarchical decomposition of U given by Proposition B.6. Since $\mathcal{N}(\mathcal{T}, U) \geq 1$, we have that $m \geq 1$. Recall that $\mathcal{Q} = \{q_1, \dots, q_n\}$ denote the set of \mathcal{S} -equivalent classes of tiles in \mathcal{P} . The \mathcal{S} -equivalent relation can be extended to the set of tiles \mathcal{E} -equivalent to some tile in $\lambda^l \mathcal{P}$, for every integer $l \geq 0$. We have that a tile $\lambda^l t$ is \mathcal{S} -equivalent to $\lambda^l p$ for some $p \in \mathcal{P}$ if and only if t is \mathcal{S} -equivalent to p . Thus, the number of \mathcal{S} -equivalence classes of the tiles in \mathcal{P} and $\lambda^l \mathcal{P}$ is the same. Denote by $\{\lambda^l q_1, \dots, \lambda^l q_n\}$ the set of \mathcal{S} -equivalence classes of tiles in $\lambda^l \mathcal{P}$ and by $\{\lambda^l p_1, \dots, \lambda^l p_n\}$ a set of representatives of these classes.

Observe that

$$\mathcal{N}(\mathcal{T}, U) = \sum_{l=0}^{m-1} \sum_{t^l \in U_l} \mathcal{N}(\mathcal{T}, t^l) = \sum_{l=0}^{m-1} \sum_{i=1}^n \sum_{\substack{t^l \in U_l \\ t^l \in \lambda^l q_i}} \mathcal{N}(\mathcal{T}, t^l, \lambda^l q_i), \tag{B.14}$$

where $\mathcal{N}(\mathcal{T}, t^l, \lambda^l q_i)$ denotes the number of tiles of \mathcal{T} included in the tile $t^l \in \lambda^l q_i$. Using the matrix M_S , we deduce

$$\mathcal{N}(\mathcal{T}, t^l, \lambda^l q_i) = \sum_{j=1}^n m_{i,j}^l,$$

where $m_{i,j}^l$ denotes the (i, j) -element of the matrix M_S^l . Moreover, it is straightforward to check, using the fact that any tile in λq_i is tiled by $O(\mathbb{R}^d)$ -copies of tiles in q_j , for $j = 1, \dots, n$, and that M_S counts these tiles, that

$$w = (\mu_d(p_1), \dots, \mu_d(p_n))$$

is a right-eigenvector for M_S with eigenvalue λ^d , which is the Perron–Frobenius eigenvalue of M_S . Thus, as a direct corollary of Proposition B.7, we get:

Corollary B.8. *For all $\rho > r(M_S)$, there are K_0 and $\alpha > 0$ such that for all $l \in \{0, \dots, m\}$, all $i \in \{1, \dots, n\}$, we have*

$$|\mathcal{N}(\mathcal{T}, t^l, \lambda^l p_i) - \alpha \mu_d(\lambda^l p_i)| \leq K_0 \rho^l.$$

On the other hand

$$\mu_d(U_{\mathcal{T}}) = \sum_{l=0}^{m-1} \sum_{\substack{t^l \in \mathcal{T}^l \\ t^l \subset U_l}} \mu_d(t^l) = \sum_{l=0}^{m-1} \sum_{i=1}^n \sum_{\substack{t^l \sim \lambda^l p_i \in \mathcal{T}^l \\ t^l \subset U_l}} \mu_d(\lambda^l p_i). \tag{B.15}$$

Multiplying (B.15) by α and then subtracting to (B.14) we get

$$\mathcal{N}(\mathcal{T}, U) - \alpha \mu_d(U_{\mathcal{T}}) = \sum_{l=0}^{m-1} \sum_{i=1}^n \sum_{\substack{t^l \sim \lambda^l p_i \in \mathcal{T}^l \\ t^l \subset U_l}} (\mathcal{N}(\mathcal{T}, t^l, \lambda^l q_i) - \alpha \mu_d(\lambda^l p_i)). \tag{B.16}$$

Applying Corollary B.8 to (B.16) yields

$$|\mathcal{N}(\mathcal{T}, U) - \alpha \mu_d(U_{\mathcal{T}})| \leq \sum_{l=0}^{m-1} \sum_{i=1}^n \sum_{\substack{t^l \sim \lambda^l p_i \in \mathcal{T}^l \\ t^l \subset U_l}} K_0 \rho^l = \sum_{l=0}^{m-1} \mathcal{N}(\mathcal{T}^l, U_l) K_0 \rho^l. \tag{B.17}$$

After combining (B.11) and (B.17), we get

$$|\mathcal{N}(\mathcal{T}, U) - \alpha \mu_d(U_{\mathcal{T}})| \leq K_0 \|M\|_1 \sum_{l=0}^{m-1} \mathcal{L}(\mathcal{T}^{l+1}, \partial U) \rho^l. \tag{B.18}$$

We want to apply Lemma B.5 to give an upper bound of $\mathcal{N}(\mathcal{T}^{l+1}, \partial U)$ in terms of $\mathcal{N}(\mathcal{T}, \partial U)$ for all $l \in \{0, \dots, m - 1\}$. First, observe that the sequence $\mathcal{N}(\mathcal{T}^l, \partial U)$, $l \in \mathbb{N}$, is decreasing. Define l_0 to be the largest $l \in \{1, \dots, m\}$ such that $\mathcal{N}(\mathcal{T}^l, \partial U) > K_{\mathcal{T}}$. Now, we split the sum in (B.24) into two parts

$$\sum_{l=0}^{l_0-1} \mathcal{L}(\mathcal{T}^{l+1}, \partial U) \rho^l + \sum_{l=l_0}^{m-1} \mathcal{L}(\mathcal{T}^{l+1}, \partial U) \rho^l. \tag{B.19}$$

We apply Lemma B.5 to each term in the first sum, which gives

$$\mathcal{L}(\mathcal{T}^{l+1}, \partial U) \leq (2K_{\mathcal{T}^{l+1}} + 1) \frac{R_{\mathcal{T}^l}}{R_{\mathcal{T}^{l+1}}} \mathcal{L}(\mathcal{T}, \partial U)$$

for all $l \in \{0, \dots, l_0 - 1\}$. Since $K_{\mathcal{T}l} = K_{\mathcal{T}}$ and $R_{\mathcal{T}l}/R_{\mathcal{T}} = \lambda^{-l}$ for every $l \geq 0$, it follows that

$$\sum_{l=0}^{l_0-1} \mathcal{L}(\mathcal{T}^{l+1}, \partial U) \rho^l \leq (2K_{\mathcal{T}} + 1) \lambda^{-1} \mathcal{L}(\mathcal{T}, \partial U) \sum_{l=0}^{l_0-1} \left(\frac{\rho}{\lambda}\right)^l. \tag{B.20}$$

Now we estimate the second sum in (B.19). Since $\rho < \lambda$ and the sequence $\mathcal{N}(\mathcal{T}^l, \partial U)$, $l \in \mathbb{N}$, is decreasing, we have

$$\sum_{l=l_0}^{m-1} \mathcal{L}(\mathcal{T}^{l+1}, \partial U) \rho^l \leq \mathcal{L}(\mathcal{T}^{l_0}, \partial U) \sum_{l=l_0}^{m-1} \lambda^l \leq K_{\mathcal{T}} \left(\frac{\lambda^m - \lambda^{l_0}}{\lambda - 1}\right). \tag{B.21}$$

From (B.12) we get $\lambda^m \leq \lambda R_{\mathcal{T}t}/r_{\mathcal{T}t} \mathcal{L}(\mathcal{T}, \partial U)$. Setting $N_{\mathcal{T}t} := (K_{\mathcal{T}} R_{\mathcal{T}t}/r_{\mathcal{T}t})(\lambda/(\lambda - 1))$ and replacing into (B.21) we get

$$\sum_{l=l_0}^{m-1} \mathcal{L}(\mathcal{T}^{l+1}, \partial U) \rho^l \leq N_{\mathcal{T}t} \mathcal{L}(\mathcal{T}, \partial U). \tag{B.22}$$

Combining (B.20) and (B.23), we get

$$\sum_{l=0}^{m-1} \mathcal{L}(\mathcal{T}^{l+1}, \partial U) \rho^l \leq \left((2K_{\mathcal{T}} + 1) \sum_{l=0}^{l_0-1} \left(\frac{\rho}{\lambda}\right)^l + N_{\mathcal{T}t} \right) \mathcal{L}(\mathcal{T}, \partial U). \tag{B.23}$$

Using B.24 we conclude that

$$|\mathcal{N}(\mathcal{T}, U) - \alpha \mu_d(U_{\mathcal{T}})| \leq K_0 \|M\|_1 (2K_{\mathcal{T}} + 1) \left((2K_{\mathcal{T}} + 1) \sum_{l=0}^{l_0-1} \left(\frac{\rho}{\lambda}\right)^l + N_{\mathcal{T}t} \right) \mathcal{L}(\mathcal{T}, \partial U). \tag{B.24}$$

Since we can choose $r(M) < \rho < \lambda$, for all δ that fits with \mathcal{T} , there exists a constant \hat{K} which depends only on \mathcal{T} such that

$$|\mathcal{N}(\mathcal{T}, U) - \alpha \mu_d(U_{\mathcal{T}})| \leq \hat{K} \mathcal{L}(\mathcal{T}, \partial U). \tag{B.25}$$

On the other hand, we have

$$|\mu_d(U) - \mu_d(U_{\mathcal{T}})| \leq (2R_{\mathcal{T}})^d \mathcal{L}(\mathcal{T}, \partial U), \tag{B.26}$$

for all $U \in \mathcal{U}_{\delta}$. Combining these last two equations, we get that for all δ that fits with \mathcal{T} , there exists a constant K which depends only on \mathcal{T} such that:

$$|\mathcal{N}(\mathcal{T}, U) - \alpha \mu_d(U)| \leq K \mathcal{L}(\mathcal{T}, \partial U). \quad \square \tag{B.27}$$

References

[1] J. Aliste-Prieto, D. Coronel, J.-M. Gambaudo, Rapid convergence to frequency for substitution tilings of the plane, *Comm. Math. Phys.* 306 (2) (2011) 365–380.
 [2] D. Burago, B. Kleiner, Separated nets in Euclidean space and Jacobians of bi-Lipschitz maps, *Geom. Funct. Anal.* 8 (2) (1998) 273–282.
 [3] D. Burago, B. Kleiner, Rectifying separated nets, *Geom. Funct. Anal.* 12 (1) (2002) 80–92.
 [4] N.G. de Bruijn, Algebraic theory of Penrose’s nonperiodic tilings of the plane. I, *Nederl. Akad. Wetensch. Indag. Math.* 43 (1) (1981) 39–52;
 N.G. de Bruijn, Algebraic theory of Penrose’s nonperiodic tilings of the plane. II, *Nederl. Akad. Wetensch. Indag. Math.* 43 (1) (1981) 53–66.
 [5] D. Frettlöh, A. Garber, private communication.
 [6] M. Gromov, Asymptotic invariants of infinite groups, in: *Geometric Group Theory*, vol. 2, Sussex, 1991, in: *London Math. Soc. Lecture Note Ser.*, vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295.
 [7] B. Grünbaum, G.C. Shephard, *Tilings and Patterns*, W.H. Freeman and Company, New York, 1989.
 [8] R.A. Horn, Ch.R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1990, corrected reprint of the 1985 original.
 [9] J.C. Lagarias, P.A.B. Pleasants, Repetitive Delone sets and quasicrystals, *Ergodic Theory Dynam. Systems* 23 (3) (2003) 831–867.
 [10] M. Łaczkovich, Uniformly spread discrete sets in \mathbb{R}^d , *J. London Math. Soc.* (2) 46 (1) (1992) 39–57.
 [11] C.T. McMullen, Lipschitz maps and nets in Euclidean space, *Geom. Funct. Anal.* 8 (1998) 304–314.
 [12] N. Priebe, B. Solomyak, Characterization of planar pseudo-self-similar tilings, *Discrete Comput. Geom.* 26 (3) (2001) 289–306.
 [13] T. Rivière, D. Ye, Resolutions of the prescribed volume form equation, *NoDEA* 3 (1996) 323–369.

- [14] D. Shechtman, I. Blech, D. Gratias, J.W. Cahn, Metallic phase with long range orientational order and no translational symmetry, *Phys. Rev. Lett.* 53 (20) (1984) 1951–1954.
- [15] Y. Solomon, Substitution tilings and separated nets with similarities to the integer lattice, *Israel J. Math.* 181 (1) (2011) 445–460.
- [16] B. Solomyak, Nonperiodicity implies unique composition for self-similar translationally finite tilings, *Discrete Comput. Geom.* 20 (2) (1998) 265–279.
- [17] B. Solomyak, Pseudo-self-affine tilings in \mathbb{R}^d , in: *Teor. Predst. Din. Sist. Komb. i Algoritm. Metody.* 13, *Zap. Nauchn. Sem. St.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 326 (2005) 198–213, 282–283, translation in: *J. Math. Sci. (N.Y.)* 140 (3) (2007) 452–460.
- [18] D. Toledo, Geometric group theory, 2: Asymptotic invariants of finite groups by M. Gromov, *Bull. Amer. Math. Soc.* 33 (1996) 395–398.