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Global weighted estimates for the gradient of solutions to nonlinear elliptic equations

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Abstract

We consider nonlinear elliptic equations of p-Laplacian type that are not necessarily of variation form when the nonlinearity is allowed to be discontinuous and the boundary of the domain can go beyond the Lipschitz category. Under smallness in the BMO nonlinearity and sufficient flatness of the Reifenberg domain, we obtain the global weighted L^q estimates with $q \in (p, \infty)$ for the gradient of weak solutions.

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1. Introduction

This paper concerns the global higher regularity results of elliptic problems involving discontinuous operators in divergence form of p-Laplacian type. In particular, we are interested in establishing an optimal Calderón–Zygmund type theory for the gradient of weak solutions to such divergence structure nonlinear problems with the discontinuous nonlinearity in the nonsmooth bounded domain. More precisely, we want to find minimal additional assumptions to the primary structural conditions on the nonlinearity and the boundary of the domain under which the gradient of solutions is as integrable as the nonhomogeneous term in the weighted L^q spaces for the full range $q \in (p, \infty)$.

There have been research activities on the gradient estimates in the weighted L^q spaces regarding elliptic and parabolic problems, see [20,21,27]. One main advantage for these weighted L^q estimates is to provide higher regularity results in Morrey and Hölder spaces by taking appropriate weight functions and applying the Sobolev–Morrey Embedding Theorem. Our work was motivated by a series of works [20,21,27] by Mengesha and Phuc where the authors obtained the global weighted gradient estimates for nonlinear elliptic operators either with linear growth as

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in [20] or of variational type as in [21,27]. The main approach in those works is to make use of harmonic analysis tools such as the maximal function operator, as done in [6,8,9,12]. Here we extend their results to wider class of nonlinear operators of nonvariational type by treating polynomial growth of rate $p-1 \in (0,\infty)$. It is worth noticing that no maximal function is associated with the proof in this paper. We will use Harmonic Analysis free approach to nonlinear Calderón–Zygmund estimates which was first introduced in [2,22], developed [3,4] and later adapted to nonsmooth domains for boundary regularity results as well as in the setting of Orlicz spaces, see [5,7,10,31]. We also would like to point out that this work is a natural extension of the local gradient estimates in the Lebesgue spaces in the recent paper [9] to the global gradient estimates in the weighted Lebesgue spaces.

This paper is organized as follows. We state some background and the main result in the next section. In Section 3 we obtain both local and global comparison estimates from perturbation results. In Section 4 we establish the a priori estimates of the main result from the a priori regularity assumption. In Section 5, we complete the proof of the main theorem by removing the a priori regularity assumption from a standard approximation procedure.

2. Background and main result

Let $1<p<\infty$ be fixed and Ω be a bounded open domain in $\mathbb{R}^n$, $n \geqslant 2$, with its nonsmooth boundary $\partial\Omega$. We then consider the following Dirichlet boundary value problem for a divergence structure nonlinear PDE:

$$\begin{cases} \operatorname{div} \mathbf{a}(Du,x) = \operatorname{div}\big(|F|^{p-2}F\big) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $F=(f^1,\dots,f^n) \in L^p(\Omega;\mathbb{R}^n)$ is the given vector-valued function and

$$\mathbf{a}=\mathbf{a}(\xi,x)=\big(a^1,\dots,a^n\big)(\xi,x):\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$$

is a Carathéodory function, namely, it is measurable with respect to x for each ξ and continuous with respect to ξ for each x. We denote $Du=D_xu$ to mean the gradient vector with respect to the variables $x=(x_1,\dots,x_n)$. Let us hereafter assume that $\mathbf{a}$ satisfies the following primary structure conditions:

$$\gamma|\xi|^{p-2}|\eta|^2 \leqslant \big\langle D_\xi \mathbf{a}(\xi,x)\eta,\eta\big\rangle \tag{2.2}$$

and

$$\big|\mathbf{a}(\xi,x)\big| + |\xi|\big|D_\xi\mathbf{a}(\xi,x)\big| \leqslant \Lambda|\xi|^{p-1}, \tag{2.3}$$

for each $\xi,\eta\in\mathbb{R}^n$, for almost every $x\in\mathbb{R}^n$, and for some positive constants γ, Λ, where $D_\xi\mathbf{a}(\xi,x)$ is the Jacobian matrix of $\mathbf{a}(\xi,x)$ with respect to ξ and $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{R}^n$. We would like to point out that these primary conditions (2.2)–(2.3) imply the following monotonicity condition:

$$\begin{cases} \big\langle \mathbf{a}(\xi,x)-\mathbf{a}(\eta,x),\xi-\eta\big\rangle \\ \qquad \geqslant \tilde{\gamma}\begin{cases} |\xi-\eta|^p & \text{if } p\geqslant 2, \\ |\xi-\eta|^2\big(1+|\xi|+|\eta|\big)^{p-2} & \text{if } 1<p<2, \end{cases} \end{cases} \tag{2.4}$$

where $\tilde{\gamma}$ depends only on γ, n and p.

The function $u\in W_0^{1,p}(\Omega)$ is a weak solution of (2.1) if it satisfies

$$\int_\Omega \big\langle \mathbf{a}(Du,x),D\varphi\big\rangle\,dx = \int_\Omega \big\langle |F|^{p-2}F,D\varphi\big\rangle\,dx \quad \text{for any } \varphi\in W_0^{1,p}(\Omega).$$

According to *Minty–Browder method in L^p*, there exists a unique weak solution u of (2.1) with the estimate

$$\|Du\|_{L^p(\Omega)} \leqslant c\|F\|_{L^p(\Omega)} \tag{2.5}$$

where c is a constant, depending only on ν, Λ, p and Ω.

We now introduce the weighted Lebesgue space. To do this, we start with a concept of the so-called Muckenhoupt class. A positive locally integrable function w on $\mathbb{R}^n$ is said to be a weight. Then the weight w belongs to a Muckenhoupt class A_s, denoted by $w \in A_s$, for some $1 < s < \infty$, if

$$[w]_s := \sup\left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{\frac{-1}{s-1}}\,dx\right)^{s-1} < +\infty, \tag{2.6}$$

where the supremum is taken over all the balls B in $\mathbb{R}^n$. We remark that the $\{A_s\}_{1<s<\infty}$ classes are nested, that is, $A_{s_1} \subset A_{s_2}$ if $1 < s_1 \leqslant s_2 < \infty$. Let

$$w_\alpha(x) = |x|^\alpha, \quad x \in \mathbb{R}^n.$$

Then $w_\alpha \in A_s$ if and only if $-n < \alpha < n(s-1)$. This w_α is a typical weight which can be used in this paper.

We denote

$$A_\infty = \bigcup_{1<s<\infty} A_s. \tag{2.7}$$

The weighted Lebesgue measure w is defined by

$$w(E) = \int_E w(x)\,dx, \tag{2.8}$$

for any bounded measurable set $E \subset \mathbb{R}^n$.

We need the following properties of A_s weight later in this paper.

Lemma 2.1. *(See [20,21,29].)*

(1) *If $w \in A_s$ for some $1 < s < \infty$, then $w \in A_{s-\underline{s}}$ for some $\underline{s} = \underline{s}(s) > 0$ small.*
(2) *Let $w \in A_s$ for some $1 < s < \infty$ and E be a measurable subset of a ball $B \subset \mathbb{R}^n$. Then*

$$w(B) \leqslant [w]_s\left(\frac{|B|}{|E|}\right)^s w(E). \tag{2.9}$$

(3) *A weight $w \in A_\infty$ if and only if there exist positive constants α and t such that*

$$w(E) \leqslant \alpha\left(\frac{|E|}{|B|}\right)^t w(B), \tag{2.10}$$

for every ball B and every measurable subset E of $B \subset \mathbb{R}^n$.

Let U be a bounded domain in $\mathbb{R}^n$ and let $1 < q < \infty$. Then given a weight $w \in A_s$ for some $1 < s < \infty$, the weighted Lebesgue space $L^q_w(U)$ is the set of all measurable functions $v : U \to \mathbb{R}$ satisfying

$$\int_U \left|v(x)\right|^q w(x)\,dx < +\infty. \tag{2.11}$$

This weighted Lebesgue space $L^q_w(U)$ is a Banach space equipped with the following norm

$$\|v\|_{L^q_w(U)} = \left(\int_U |v|^q w(x)\,dx\right)^{\frac{1}{q}}. \tag{2.12}$$

Hereafter, we always assume that $1 < q < \infty$ and $w \in A_q$. Under these assumptions one can easily check that

$$L^{pq}_w(U) \subset L^p(U), \tag{2.13}$$

which implies the existence of a weak solution under the assumption $F \in L^{pq}_w(U)$.

Our main purpose in this paper is to establish the optimal global $W^{1,q}(\Omega)$-estimate regarding the nonlinear elliptic problem (2.1). More precisely, we want to find a minimal regularity requirement on the nonlinearity and a lower level of geometric assumption on the boundary of the bounded domain to ensure that the gradient of the weak solution of (2.1) is as regular as the nonhomogeneous term F in the weighted Lebesgue space $L^q_w(\Omega)$, by essentially proving that

$$|F|^p \in L^q_w(\Omega) \quad \Longrightarrow \quad |Du|^p \in L^q_w(\Omega) \tag{2.14}$$

for the full range $1 < q < \infty$.

To state the main result we first need to describe the main assumptions on the nonlinearity $\mathbf{a} = \mathbf{a}(\xi, x)$ and on the domain Ω. Let $\rho > 0$ and $y \in \mathbb{R}^n$. We denote

$$B_\rho(y) = \big\{x \in \mathbb{R}^n \colon |x - y| < \rho\big\}.$$

In order to measure the oscillation of $\frac{\mathbf{a}(\xi,x)}{|\xi|^{p-1}}$ in the variables x over $B_\rho(y)$ in the BMO sense, uniformly in ξ, we introduce the following function:

$$\theta\big(\mathbf{a}; B_\rho(y)\big)(x) = \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|\mathbf{a}(\xi, x) - \bar{\mathbf{a}}_{B_\rho(y)}(\xi)|}{|\xi|^{p-1}}, \tag{2.15}$$

where

$$\bar{\mathbf{a}}_{B_\rho(y)}(\xi) = \fint_{B_\rho(y)} \mathbf{a}(\xi, x)\, dx = \frac{1}{|B_\rho(y)|} \int_{B_\rho(y)} \mathbf{a}(\xi, x)\, dx$$

is the integral average of $\mathbf{a}(\xi, x)$ in the variables x over $B_\rho(y)$ for the fixed $\xi \in \mathbb{R}^n$.

The function $\theta(\mathbf{a}; B_\rho(y))$ provides a uniform measurement over $B_\rho(y)$ of how far $\frac{\mathbf{a}(\xi,x)}{|\xi|^{p-1}}$ is from its integral average $\frac{\bar{\mathbf{a}}_{B_\rho(y)}(\xi)}{|\xi|^{p-1}}$, uniformly in $\xi \in \mathbb{R}^n \setminus \{0\}$.

Definition 2.2. We say that the vector field $\mathbf{a}$ is (δ, R)-vanishing if

$$\sup_{0<\rho\leqslant R} \sup_{y\in\mathbb{R}^n} \fint_{B_\rho(y)} \theta\big(\mathbf{a}; B_\rho(y)\big)(x)\, dx \leqslant \delta. \tag{2.16}$$

In order to measure the deviation of $\partial\Omega$ from being an $(n-1)$-dimensional affine space at each scale $\rho > 0$, we use the following so-called Reifenberg flatness.

Definition 2.3. We say that Ω is (δ, R)-Reifenberg flat if for every $x \in \partial\Omega$ and every $\rho \in (0, R]$, there exists a coordinate system $\{y_1, \ldots, y_n\}$, which can depend on ρ and x so that $x = 0$ in this coordinate system and that

$$B_\rho(0) \cap \{y_n > \delta\rho\} \subset B_\rho(0) \cap \Omega \subset B_\rho(0) \cap \{y_n > -\delta\rho\}.$$

Remark 2.4. We remark that throughout this paper $\delta > 0$ is a small positive constant, say, $0 < \delta < \frac{1}{8}$, being determined later so that (2.14) holds true for all $q \in (1, \infty)$. On the other hand, R can be any number which is bigger than 1 by the scaling invariance of the problem (2.1), see Lemma 2.5. We would like to refer to [15,23,24,14,28,30] and references therein, where the notions about *Bounded Mean Oscillation* and *Reifenberg flatness* are extensively discussed, respectively.

We need to check that the problem (2.1) is invariant under a proper scaling and normalization. To do this we have to confirm that the primary assumptions on $\mathbf{a}(\xi, x)$ and $\partial\Omega$ are still invariant under such scaling and normalization with the same uniform constants γ, Λ and δ. The following lemma ensures this invariance.

Lemma 2.5. *For each $\lambda > 1$ and $0 < r < 1$, let us define the rescaled maps*:

$$\tilde{\mathbf{a}}(\xi, x) = \frac{\mathbf{a}(\lambda\xi, rx)}{\lambda^{p-1}}, \qquad \tilde{\Omega} = \left\{\frac{1}{r}x \colon x \in \Omega\right\}, \qquad \tilde{u}(x) = \frac{u(rx)}{\lambda r}, \qquad \tilde{F}(x) = \frac{F(rx)}{\lambda}.$$

Then

(1) $\tilde{u} \in W_0^{1,p}(\tilde{\Omega})$ *is the weak solution of*

$$\operatorname{div} \tilde{\mathbf{a}}(D\tilde{u}, x) = \operatorname{div}\big(|\tilde{F}|^{p-2}\tilde{F}\big) \quad \text{in } \tilde{\Omega}.$$

(2) $\tilde{\mathbf{a}}$ *satisfies the primary structural assumptions* (2.2)–(2.3) *with the same constant* γ, Λ.
(3) $\tilde{\mathbf{a}}$ *is* $(\delta, \frac{R}{r})$*-vanishing.*
(4) $\tilde{\Omega}$ *is* $(\delta, \frac{R}{r})$*-Reifenberg flat.*

Proof. The proof follows from a direct computation. We also refer to those of Lemmas 2.7 and 2.8 in [8] and Lemma 3.3 in [9]. □

We now state the main result.

Theorem 2.6. *Let* $u \in W_0^{1,p}(\Omega)$ *be the weak solution of* (2.1)*, where* $1 < p < \infty$*. Assume that* $w \in A_q$ *and* $|F|^p \in L_w^q(\Omega)$ *for some* $q \in (1, \infty)$*. Then there exists a constant* $\delta = \delta(\gamma, \Lambda, n, p, q, [w]_q) > 0$ *such that if* $\mathbf{a}$ *is* (δ, R)*-vanishing and* Ω *is* (δ, R)*-Reifenberg flat, then there holds* $|Du|^p \in L_w^q(\Omega)$ *with the estimate*

$$\int_\Omega |Du|^{pq} w(x)\, dx \leqslant c \int_\Omega |F|^{pq} w(x)\, dx, \tag{2.17}$$

where $c = c(\gamma, \Lambda, n, p, q, [w]_q, |\Omega|) > 0$.

We remark that if we take $w(x) = 1$ in the main result, then (2.17) is reduced to the global Calderón–Zygmund estimate for the problem (2.1). Even in this special case, this sharp integrability result can provide a global version of the work [9] where a local Calderón–Zygmund theory was established under a similar regularity assumption on the nonlinearity. In the same spirit of that in Chapter 5 of the recent paper [20], we can find a finer regularity in Morrey spaces and Hölder spaces from Theorem 2.6.

3. Approximations and comparison maps

In this section we shall first compare a weak solution u of (2.1) to a weak solution h of the corresponding homogeneous boundary-value problem in a simply scaled domain. We then compare this solution h to a weak solution v of a reference problem in a smaller domain with the flat boundary, for which we have the Lipschitz regularity up to the flat boundary.

For the sake of convenience and simplicity, we employ the letter c through this section to denote any constant which can be explicitly computed in terms of known quantities such as $\gamma, \Lambda, n, p, q, [w]_q$ and the geometry of related domains. Thus the exact value denoted by c may change from line to line in a given computation.

We start with interior comparison estimates in $B_6 \Subset \Omega$, by considering a weak solution $u \in W^{1,p}(B_6)$ of

$$\operatorname{div} \mathbf{a}(Du, x) = \operatorname{div}\big(|F|^{p-2}F\big) \quad \text{in } B_6, \tag{3.1}$$

which means

$$\int_{B_6} \big\langle \mathbf{a}(Du, x), D\varphi \big\rangle\, dx = \int_{B_6} |F|^{p-2}\langle F, D\varphi\rangle\, dx$$

for each $\varphi \in W_0^{1,p}(B_6)$.

The following lemma is a sort of comparison estimates when dealing with interior Calderón–Zygmund theory, see for instance [9].

Lemma 3.1. *Let* $u \in W^{1,p}(B_6)$ *be a weak solution of* (3.1) *under the assumptions* (2.2)–(2.3) *and*

$$\fint_{B_6} |Du|^p\, dx \leqslant 1.$$

Then, there exists a constant $n_1 = n_1(\gamma, \Lambda, n, p) > 1$ so that for $0 < \epsilon < 1$ fixed, one can find a small $\delta = \delta(\epsilon) > 0$ such that if

$$\fint_{B_\rho} \theta(\mathbf{a}; B_\rho)\, dx \leqslant \delta \quad \text{for any } \rho \in (0, 6]$$

and

$$\fint_{B_6} |F|^p\, dx \leqslant \delta^p$$

for such small δ, then there exists a weak solution $v \in W^{1,p}(B_4)$ of

$$\operatorname{div} \bar{\mathbf{a}}_{B_4}(Dv) = 0 \quad \text{in } B_4$$

such that

$$\fint_{B_2} \left|D(u - v)\right|^p dx \leqslant \epsilon^p \quad \text{and} \quad \|Dv\|_{L^\infty(B_3)} \leqslant n_1.$$

We next want to find a boundary version of Lemma 3.1 with an approximation scheme allowing to approximate a portion of the Reifenberg flat boundary by an $(n-1)$-dimensional hyperplane. We use standard geometric notations:

$$B_\rho^+ = B_\rho \cap \{x_n > 0\}, \qquad \Omega_\rho = B_\rho \cap \Omega, \qquad T_\rho = B_\rho \cap \{x_n = 0\}, \qquad \partial_w \Omega_\rho = \partial\Omega \cap B_\rho.$$

By a dilation argument and from the Reifenberg flatness assumption of the domain, we assume the following geometric setting:

$$B_\rho^+ \subset \Omega_\rho \subset B_\rho \cap \{x_n > -2\rho\delta\} \quad \text{for any } \rho \in (0, 6]. \tag{3.2}$$

From the BMO smallness of the nonlinearity, we further assume that

$$\fint_{B_\rho^+} \left|\theta\left(\mathbf{a}; B_\rho^+\right)\right| dx \leqslant \delta \quad \text{for any } \rho \in (0, 6], \tag{3.3}$$

and

$$\fint_{\Omega_6} |F|^p\, dx \leqslant \delta^p. \tag{3.4}$$

We consider a weak solution $u \in W^{1,p}(\Omega_6)$ of

$$\begin{cases} \operatorname{div} \mathbf{a}(Du, x) = \operatorname{div}\left(|F|^{p-2}F\right) & \text{in } \Omega_6, \\ u = 0 & \text{on } \partial_w \Omega_6, \end{cases} \tag{3.5}$$

with the assumption

$$\fint_{\Omega_6} |Du|^p\, dx \leqslant 1. \tag{3.6}$$

We then let $h \in W^{1,p}(B_5)$ be the unique weak solution of

$$\begin{cases} \operatorname{div} \mathbf{a}(Dh, x) = 0 & \text{in } \Omega_5, \\ h = u & \text{on } \partial\Omega_5. \end{cases} \tag{3.7}$$

From the standard L^p-estimate for (3.7) and (3.6), we find

$$\fint_{\Omega_5} |Dh|^p\, dx \leqslant c \fint_{\Omega_5} |Du|^p\, dx \leqslant c. \tag{3.8}$$

The next lemma is a self-improving gradient integrability for nonlinear elliptic problems of p-Laplacian type near the boundary, as is well known, see for instance [11,17].

Lemma 3.2. *Let $h \in W^{1,p}(B_5)$ be the weak solution of* (3.7) *under the assumptions with* (3.2)–(3.4) *and* (3.6). *Then there exists a positive constant $\sigma_1 = \sigma_1(\gamma, \Lambda, n, p)$ such that*

$$h \in L^{p+\sigma_1}(\Omega_4)$$

with the uniform bound

$$\fint_{\Omega_4} |Dh|^{p+\sigma_1}\, dx \leqslant c. \tag{3.9}$$

Proof. From the Reifenberg flatness condition (3.2), we find

$$\left|\Omega \cap B_\rho(y)\right| \geqslant \left(\frac{1-\delta}{2}\right)^n \left|B_\rho(y)\right| \quad \big(y \in B_6,\ \rho \in (0,6)\big),$$

since we may as well assume $0 < \delta < \frac{1}{8}$. This measure density condition implies that $B_5 \setminus \Omega_5$ satisfies the uniform capacity density condition. Then according to the standard higher integrability result for (3.7), see for instance [17], we find

$$\left(\fint_{\Omega_4} |Dh|^{p+\sigma_1}\, dx\right)^{\frac{1}{p+\sigma_1}} \leqslant c\left(1 + \left(\fint_{\Omega_5} |Dh|^p\, dx\right)^{\frac{1}{p}}\right) \tag{3.10}$$

for some small positive constant σ_1 depending on γ, Λ, n, and p. The lemma follows from (3.8) and (3.10). □

Remark 3.3. We would like to point out that the above higher integrability is a useful tool when treating nonlinear elliptic and parabolic problems, see for instance [1,2,16,22] and references therein. It is also very useful when we make a systematic analysis of the gradients of solutions of nonlinear elliptic and parabolic problems near the boundary of the irregular domain which is assumed to satisfy the Reifenberg flatness condition, as in this work. In [11,17] the authors proved that this higher integrability holds under very general geometric condition that the complement of the domain satisfies the uniform capacity density condition, which turns out to be essentially sharp and cannot be relaxed in this direction. Needless to say, the uniform capacity density condition is weaker than our Reifenberg flatness condition. We refer to [11,25,26] and references therein for a further discussion on local and global integrability of gradients in nonlinear problems.

We next consider a weak solution $\bar{h} \in W^{1,p}(\Omega_4)$ of

$$\begin{cases} \operatorname{div} \bar{\mathbf{a}}_{B_4^+}(D\bar{h}) = 0 & \text{in } \Omega_4, \\ \bar{h} = 0 & \text{on } \partial_w \Omega_4, \end{cases} \tag{3.11}$$

and its limiting problem:

$$\begin{cases} \operatorname{div} \bar{\mathbf{a}}_{B_4^+}(Dv) = 0 & \text{in } B_4^+, \\ v = 0 & \text{on } T_4. \end{cases} \tag{3.12}$$

The following classical lemma provides Lipschitz regularity for the reference problem (3.12), see for instance [13,18,19]. This lemma is useful in order to prove the desired L^{pq} estimates for the gradients of the problem (3.5), whose limit case is indeed given by the Lipschitz estimates for (3.12).

Lemma 3.4. *Let $v \in W^{1,p}(B_4^+)$ be a weak solution of* (3.12) *under the primary assumptions* (2.2)–(2.3). *Then we have*

$$\|Dv\|^p_{L^\infty(B_3^+)} \leqslant c \fint_{B_4^+} |Dv|^p\, dx.$$

Remark 3.5. Let $\bar{v}$ and $\overline{Dv}$ be the zero extensions of v and Dv from B_4^+ to B_4, respectively. Observe that $v = 0$ on $T_4 = B_4 \cap \{x_n = 0\}$ in the trace sense to see that $\bar{v} \in W^{1,p}(B_4)$ and $\overline{Dv} = D\bar{v}$ a.e. in B_4. Consequently, we have

$$\|D\bar{v}\|^p_{L^\infty(\Omega_3)} \leqslant c \fint_{B_4^+} |Dv|^p \, dx. \tag{3.13}$$

The following lemma says that a solution of (3.11) is very close to a solution of (3.12) in L^p under the Reifenberg flatness condition (3.2). The proof is based on the compactness method.

Lemma 3.6. *Let $\bar{h} \in W^{1,p}(\Omega_4)$ be a weak solution of (3.11) under the assumption*

$$\fint_{\Omega_4} |D\bar{h}|^p \, dx \leqslant 1. \tag{3.14}$$

Then for any $\epsilon > 0$ fixed, there exists a small $\delta = \delta(\epsilon, \gamma, \Lambda, n, p) > 0$ such that if

$$B_4^+ \subset \Omega_4 \subset B_4 \cap \{x_n > -8\delta\} \tag{3.15}$$

holds for such δ, then there exists a weak solution $v \in W^{1,p}(B_4^+)$ of (3.12) with

$$\fint_{B_4^+} |Dv|^p \, dx \leqslant 1$$

such that

$$\int_{B_4^+} |\bar{h} - v|^p \, dx \leqslant \epsilon^p. \tag{3.16}$$

Proof. If not, there exist $\epsilon_0 > 0$, $\{\bar{h}_k\}_{k=1}^\infty$ and $\{\Omega_4^k\}_{k=1}^\infty$ such that $\bar{h}_k$ is a weak solution of

$$\begin{cases} \operatorname{div} \bar{\mathbf{a}}_{B_4^+}(D\bar{h}_k) = 0 & \text{in } \Omega_4^k, \\ \bar{h}_k = 0 & \text{on } \partial_w \Omega_4^k, \end{cases} \tag{3.17}$$

with

$$\fint_{\Omega_4^k} |D\bar{h}_k|^p \, dx \leqslant 1, \tag{3.18}$$

$$B_4^+ \subset \Omega_4^k \subset B_4 \cap \left\{ x_n > -\frac{8}{k} \right\}, \tag{3.19}$$

but

$$\int_{B_4^+} |\bar{h}_k - v|^p \, dx > \epsilon_0^p \tag{3.20}$$

for any weak solution v of

$$\begin{cases} \operatorname{div} \bar{\mathbf{a}}_{B_4^+}(Dv) = 0 & \text{in } B_4^+, \\ v = 0 & \text{on } T_4, \end{cases} \tag{3.21}$$

with

$$\fint_{B_4^+} |Dv|^p \, dx \leqslant 1. \tag{3.22}$$

From (3.18)–(3.19) and the fact that $\bar{h}_k = 0$ on $\partial_w \Omega_4^k$, we see that $\{\bar{h}_k\}_{k=1}^{\infty}$ is uniformly bounded in $W^{1,p}(B_4^+)$. So there is a subsequence, which we still denote as $\{\bar{h}_k\}$, and $\bar{h}_0 \in W^{1,p}(B_4^+)$ such that

$$\bar{h}_k \rightharpoonup \bar{h}_0 \quad \text{weakly in } W^{1,p}(B_4^+) \quad \text{and} \quad \bar{h}_k \to \bar{h}_0 \quad \text{strongly in } L^p(B_4^+). \tag{3.23}$$

We next observe from (3.17), (3.19) and (3.23) that $\bar{h}_0 = 0$ on T_4 in the trace sense. We then let $k \to \infty$ on (3.18)–(3.19) and use *the method of Browder and Minty*, see [8] for the details, to discover that $\bar{h}_0 \in W^{1,p}(B_4^+)$ is a weak solution of

$$\begin{cases} \operatorname{div} \bar{\mathbf{a}}_{B_4^+}(D\bar{h}_0) = 0 & \text{in } B_4^+, \\ \bar{h}_0 = 0 & \text{on } T_4. \end{cases} \tag{3.24}$$

We derive from (3.18), (3.23) and weak lower semicontinuity property that

$$\fint_{B_4^+} |D\bar{h}_0|^p \, dx \leqslant \liminf_{k\to\infty} \fint_{\Omega_4^k} |D\bar{h}_k|^p \, dx \leqslant 1. \tag{3.25}$$

We then reach a contradiction to (3.20) by taking k sufficiently large and comparing (3.21)–(3.22) with (3.24)–(3.25). This completes the proof. □

We compare a solution u of (3.5) with a solution v of (3.12) and use the perturbation argument to have the following L^p estimates for gradients.

Lemma 3.7. *Let $u \in W^{1,p}(\Omega_6)$ be a weak solution of (3.5) with the assumption (3.6). Then, there exists a constant $n_2 = n_2(\gamma, \Lambda, n, p) > 1$ so that for any $\epsilon > 0$ fixed, one can find a small $\delta = \delta(\epsilon) > 0$ such that if (3.2), (3.3) and (3.4) hold for such small δ, then there exists a weak solution $v \in W^{1,p}(B_4^+)$ of (3.12) such that*

$$\fint_{\Omega_2} \left|D(u - \bar{v})\right|^p dx \leqslant \epsilon^p$$

and

$$\|D\bar{v}\|_{L^\infty(\Omega_3)} \leqslant n_2,$$

where $\bar{v}$ is the zero extension of v from B_4^+ to B_4.

Proof. The proof will be divided into two cases.

Case 1. $1 < p < 2$.
Let h be the weak solution of (3.7). We then take the test function $\varphi = u - h$ for (3.5) and (3.7), to derive

$$\fint_{\Omega_5} \langle \mathbf{a}(Du, x) - \mathbf{a}(Dh, x), Du - Dh \rangle dx = \fint_{\Omega_5} \langle |F|^{p-2} F, Du - Dh \rangle dx.$$

But then, Young's inequality with σ and (3.4) imply that for any $\sigma > 0$,

$$\begin{aligned} \fint_{\Omega_5} \langle |F|^{p-2} F, Du - Dh \rangle dx &\leqslant \sigma \fint_{\Omega_5} \left|D(u-h)\right|^p dx + c(\sigma) \fint_{\Omega_5} |F|^p \, dx \\ &\leqslant \sigma \fint_{\Omega_5} \left|D(u-h)\right|^p dx + c(\sigma)\delta^p. \end{aligned}$$

Hence we have

$$\fint_{\Omega_5} \langle \mathbf{a}(Du, x) - \mathbf{a}(Dh, x), Du - Dh \rangle dx \leqslant \sigma \fint_{\Omega_5} \left|D(u-h)\right|^p dx + c(\sigma)\delta^p. \tag{3.26}$$

Using Young's inequality with $\tau_1 > 0$ and the monotonicity (2.4), we estimate

$$\begin{aligned}\fint_{\Omega_5} \left|D(u-h)\right|^p dx &= \fint_{\Omega_5} \left(1+|Du|+|Dh|\right)^{\frac{p(2-p)}{2}} \left[\left(1+|Du|+|Dh|\right)^{\frac{p(p-2)}{2}} \left|D(u-h)\right|^p\right] dx \\ &\leqslant \tau_1 \fint_{\Omega_5} \left(1+|Du|+|Dh|\right)^p dx + c(\tau_1) \fint_{\Omega_5} \left(1+|Du|+|Dh|\right)^{p-2} \left|D(u-h)\right|^2 dx \\ &\leqslant \tau_1 \fint_{\Omega_5} \left(1+|Du|+|Dh|\right)^p dx + \frac{c(\tau_1)}{\gamma} \fint_{\Omega_5} \left\langle \mathbf{a}(Du,x) - \mathbf{a}(Dh,x), Du - Dh \right\rangle dx \\ &\leqslant c_1\tau_1 + \frac{c(\tau_1)}{\gamma}\left(\sigma \fint_{\Omega_5} \left|D(u-h)\right|^p dx + c(\sigma)\delta^p\right)\end{aligned}$$

for some constant $c_1 = c_1(\gamma, \Lambda, n, p)$, where we have used (3.8) and (3.26) to find the last inequality. Consequently, we have

$$\fint_{\Omega_5} \left|D(u-h)\right|^p dx \leqslant c_1\tau_1 + \frac{c(\tau_1)}{\gamma}\sigma \fint_{\Omega_5} \left|D(u-h)\right|^p dx + \frac{1}{\gamma} c(\tau_1) c(\sigma) \delta^p. \tag{3.27}$$

We first take

$$\tau_1 = \frac{1}{10 \cdot 3^{p-1} \cdot c_1} \left(\frac{2}{5}\right)^n \epsilon^p.$$

A direct computation yields $c(\tau_1) = c\epsilon^{p-2}$. We next take $\sigma = c\epsilon^{2-p}$, in order to get

$$\frac{c(\tau_1)}{\gamma}\sigma = \frac{1}{2}.$$

Another computation yields that $c(\sigma) = c\epsilon^{\frac{p-2}{p-1}}$. Then we have

$$\frac{c(\tau_1)}{\gamma} c(\sigma) = c\epsilon^{p-2}\epsilon^{\frac{p-2}{p-1}} = c\epsilon^{\frac{p(p-2)}{p-1}}.$$

Therefore, we discover

$$\fint_{\Omega_2} \left|D(u-h)\right|^p dx \leqslant 2\left(\frac{5}{2}\right)^n \fint_{\Omega_5} \left|D(u-h)\right|^p dx \leqslant \frac{1}{5 \cdot 3^{p-1}}\epsilon^p + c\epsilon^{\frac{p(p-2)}{p-1}}\delta^p. \tag{3.28}$$

Now let $\bar{h}$ be the unique weak solution of

$$\begin{cases} \operatorname{div} \bar{\mathbf{a}}_{B_4^+}(D\bar{h}) = 0 & \text{in } \Omega_4, \\ \bar{h} = h & \text{on } \partial\Omega_4. \end{cases} \tag{3.29}$$

We then take the test function $\varphi = h - \bar{h}$ for (3.7) and (3.29), to write the resulting expression as

$$A = B \tag{3.30}$$

for

$$A = \fint_{\Omega_4} \left\langle \bar{\mathbf{a}}_{B_4^+}(Dh) - \bar{\mathbf{a}}_{B_4^+}(D\bar{h}), Dh - D\bar{h} \right\rangle dx$$

and

$$B = \fint_{\Omega_4} \left\langle \bar{\mathbf{a}}_{B_4^+}(Dh) - \mathbf{a}(Dh,x), Dh - D\bar{h} \right\rangle dx.$$

We first estimate A by almost exactly the same way in the estimation for (3.27). After some calculations we find that for any $\tau_2 > 0$,

$$\fint_{\Omega_2} \left|D(h-\bar{h})\right|^p\,dx \leqslant c \fint_{\Omega_4} \left|D(h-\bar{h})\right|^p\,dx \leqslant c\tau_2 + c(\tau_2)|A|. \tag{3.31}$$

We next estimate B:

$$\begin{aligned}
|B| &\leqslant \fint_{\Omega_4} \left|\bar{\mathbf{a}}_{B_4^+}(Dh) - \mathbf{a}(Dh,x)\right| |Dh - D\bar{h}|\,dx \\
&\overset{(2.15)}{\leqslant} \fint_{\Omega_4} \theta\left(\mathbf{a}, B_4^+\right)|Dh|^{p-1}|Dh - D\bar{h}|\,dx \\
&\overset{\text{Young's inequality}}{\leqslant} \sigma \fint_{\Omega_4} \left|D(h-\bar{h})\right|^p\,dx + c(\sigma) \fint_{\Omega_4} \theta^{\frac{p}{p-1}} |Dh|^p\,dx \\
&\leqslant \sigma \fint_{\Omega_4} \left|D(h-\bar{h})\right|^p\,dx + c(\sigma)\left(\fint_{\Omega_4} \theta^{\frac{p(p+\sigma_1)}{(p-1)\sigma_1}}\,dx\right)^{\frac{\sigma_1}{p+\sigma_1}} \left(\fint_{\Omega_4} |Dh|^{p+\sigma_1}\,dx\right)^{\frac{p}{p+\sigma_1}} \\
&\overset{(3.3),\ (3.9)}{\leqslant} \sigma \fint_{\Omega_4} \left|D(h-\bar{h})\right|^p\,dx + c(\sigma)\delta^{\sigma_3}
\end{aligned} \tag{3.32}$$

for any $\sigma > 0$ and for some $\sigma_3 = \sigma_3(\gamma, \Lambda, n, p) > 0$. We then combine (3.30), (3.31), and (3.32) to discover

$$\fint_{\Omega_4} \left|D(h-\bar{h})\right|^p\,dx \leqslant c_2\tau_2 + \sigma c(\tau_2) \fint_{\Omega_4} \left|D(h-\bar{h})\right|^p\,dx + c(\tau_2)c(\sigma)\delta^{\sigma_3} \tag{3.33}$$

for any $\sigma, \tau_2 > 0$ and for some c_2 and σ_3, depending only on γ, Λ, n and p.

We take

$$\tau_2 = \frac{1}{5 \cdot 3^{p-1} \cdot 2^{n+1} \cdot c_2} \epsilon^p.$$

A direct computation yields $c(\tau_2) = c\epsilon^{p-2}$. We next take $\sigma = c\epsilon^{2-p}$, in order to get

$$c(\tau_2)\sigma = \frac{1}{2}.$$

Another computation yields that $c(\sigma) = c\epsilon^{\frac{p-2}{p-1}}$. Then we have

$$c(\tau_2)c(\sigma) = c\epsilon^{p-2}\epsilon^{\frac{p-2}{p-1}} = c\epsilon^{\frac{p(p-2)}{p-1}}.$$

Thus, we find

$$\fint_{\Omega_2} \left|D(h-\bar{h})\right|^p\,dx \leqslant 2^{n+1} \fint_{\Omega_4} \left|D(h-\bar{h})\right|^p\,dx \leqslant \frac{1}{5 \cdot 3^{p-1}}\epsilon^p + c\epsilon^{\frac{p(p-2)}{p-1}}\delta^{\sigma_3} \tag{3.34}$$

for some $\sigma_3 = \sigma_3(\gamma, \Lambda, n, p) > 0$.

Now from the standard L^p estimates for (3.7) and (3.29), we deduce

$$\fint_{\Omega_4} |D\bar{h}|^p\,dx \leqslant c \fint_{\Omega_4} |Dh|^p\,dx \leqslant c \fint_{\Omega_4} |Du|^p\,dx.$$

We then recall the assumption (3.6) to conclude from the invariance property under a proper normalization, see Lemma 2.5, that

$$\fint_{\Omega_4} |D\bar{h}|^p\,dx \leqslant 1. \tag{3.35}$$

Then according to Lemma 3.6, there exists a weak solution v of

$$\begin{cases} \operatorname{div} \bar{\mathbf{a}}_{B_4^+}(Dv) = 0 & \text{in } B_4^+, \\ v = 0 & \text{on } T_4, \end{cases} \tag{3.36}$$

with

$$\fint_{\Omega_4} |Dv|^p \, dx \leqslant 1, \tag{3.37}$$

such that

$$\int_{B_4^+} |\bar{h} - v|^p \, dx \leqslant \eta \epsilon^p, \tag{3.38}$$

where $\eta > 0$ is to be determined. We now recall Lipschitz regularity for (3.36), see Lemma 3.4, to find from (3.37) that

$$\|Dv\|_{L^\infty(B_3^+)} \leqslant n_2,$$

for some universal constant $n_2 = n_2(\gamma, \Lambda, n, p) > 1$. Let $\bar{v}$ be the zero extension of v from B_4^+ to B_4 and recall Remark 3.5 to further find that

$$\|D\bar{v}\|_{L^\infty(\Omega_3)} \leqslant n_2. \tag{3.39}$$

According to a direct computation, we see that $\bar{v}$ is a weak solution of

$$\begin{cases} \operatorname{div} \bar{\mathbf{a}}_{B_4^+}(D\bar{v}) = -D_n\big(\overline{a^n}_{B_4^+}\big(Dv\big(x',0\big)\big)\chi_{\{x_n<0\}}\big) & \text{in } \Omega_4, \\ \bar{v} = 0 & \text{on } \partial_w \Omega_4, \end{cases} \tag{3.40}$$

where χ is standard characteristic function. Choose a cutoff function $\zeta \in C_0^\infty(B_3)$ such that

$$0 \leqslant \zeta \leqslant 1, \quad \zeta \equiv 1 \text{ on } B_2, \quad |D\zeta| \leqslant 2. \tag{3.41}$$

We then take the test function $\varphi = \zeta^p(\bar{h} - \bar{v})$ for (3.40) and (3.29), to write the resulting expression as

$$A = B \tag{3.42}$$

for

$$A = \fint_{\Omega_3} \big\langle \bar{\mathbf{a}}_{B_3^+}(D\bar{h}) - \bar{\mathbf{a}}_{B_4^+}(D\bar{v}), D\big(\zeta^p(\bar{h} - \bar{v})\big)\big\rangle \, dx$$

and

$$B = \fint_{\Omega_3 \setminus B_3^+} \overline{a^n}_{B_4^+}\big(Dv\big(x',0\big)\big) D_n\big(\zeta^p(\bar{h} - \bar{v})\big) \, dx.$$

We write

$$\begin{aligned} A &= \fint_{\Omega_3} \zeta^p \big\langle \bar{\mathbf{a}}_{B_4^+}(D\bar{h}) - \bar{\mathbf{a}}_{B_4^+}(D\bar{v}), D\bar{h} - D\bar{v}\big\rangle \, dx + \fint_{\Omega_3} p\zeta^{p-1}(\bar{h} - \bar{v})\big\langle \bar{\mathbf{a}}_{B_4^+}(D\bar{h}) - \bar{\mathbf{a}}_{B_4^+}(D\bar{v}), D\zeta\big\rangle \, dx \\ &= A_1 + A_2. \end{aligned} \tag{3.43}$$

Again, we estimate A_1 to find that for any $\tau_3 > 0$,

$$\fint_{\Omega_3} \zeta^p \big|D(\bar{h} - \bar{v})\big|^p \, dx \leqslant c\tau_3 + c(\tau_3) A_1. \tag{3.44}$$

We now estimate A_2 as follows:

$$
\begin{aligned}
|A_2| &\leqslant c \fint_{\Omega_3} \big|\bar{\mathbf{a}}_{B_4^+}(D\bar{h}) - \bar{\mathbf{a}}_{B_4^+}(D\bar{v})\big||\bar{h}-\bar{v}|\,dx \\
&\overset{(2.2)}{\leqslant} c \fint_{\Omega_3} \big(|D\bar{h}|^{p-1} + |D\bar{v}|^{p-1}\big)|\bar{h}-\bar{v}|\,dx \\
&\leqslant c\bigg(\fint_{\Omega_3} |D\bar{h}|^p\,dx + \fint_{\Omega_3} |D\bar{v}|^p\,dx\bigg)^{\frac{p-1}{p}} \bigg(\fint_{\Omega_3} |\bar{h}-\bar{v}|^p\,dx\bigg)^{\frac{1}{p}} \\
&\overset{(3.35),\,(3.39)}{\leqslant} c\bigg(\fint_{\Omega_3} |\bar{h}-\bar{v}|^p\,dx\bigg)^{\frac{1}{p}} \\
&\overset{(3.2)}{\leqslant} c\bigg(\fint_{B_3^+} |\bar{h}-v|^p\,dx + \int_{\Omega_3\setminus B_3^+} |\bar{h}-\bar{v}|^p\,dx\bigg)^{\frac{1}{p}} \\
&\overset{(3.38)}{\leqslant} c\bigg(\eta\epsilon^p + \int_{\Omega_3\setminus B_3^+} |\bar{h}|^p\,dx\bigg)^{\frac{1}{p}} \\
&\leqslant c\bigg(\eta\epsilon^p + \bigg(\int_{\Omega_3\setminus B_3^+} |\bar{h}|^{\frac{np}{n-p}}\,dx\bigg)^{\frac{n-p}{np}} \bigg(\int_{\Omega_3\setminus B_3^+} 1\,dx\bigg)^{\frac{1}{n}}\bigg) \\
&\overset{\text{Sobolev inequality, (3.2)}}{\leqslant} c\bigg(\eta\epsilon^p + \bigg(\int_{\Omega_4} |D\bar{h}|^p\,dx\bigg)^{\frac{1}{p}} \delta^{\frac{1}{n}}\bigg) \\
&\overset{(3.35)}{\leqslant} c\big(\eta\epsilon^p + \delta^{\frac{1}{n}}\big).
\end{aligned} \tag{3.45}
$$

Here we assumed $1 < p < n$. The case $p \geqslant n$ is trivial in the above estimates.

We next estimate B as follows:

$$
\begin{aligned}
|B| &\leqslant c \fint_{\Omega_3\setminus B_3^+} \big|\overline{a^n}_{B_4^+}\big(Dv(x',0)\big)\big|\big|D_n\big(\zeta^p(\bar{h}-\bar{v})\big)\big|\,dx \\
&\overset{(2.3),\,(3.41)}{\leqslant} c \int_{\Omega_3\setminus B_3^+} \big|Dv(x',0)\big|^{p-1}\big(|\bar{h}| + |D\bar{h}|\big)\,dx \\
&\overset{(3.39)}{\leqslant} \int_{\Omega_3\setminus B_3^+} \big(|h| + |Dh|\big)\,dx \\
&\leqslant c\bigg(\int_{\Omega_3\setminus B_3^+} \big(|\bar{h}|^p + |D\bar{h}|^p\big)\,dx\bigg)^{\frac{1}{p}} \bigg(\int_{\Omega_3\setminus B_3^+} 1\,dx\bigg)^{1-\frac{1}{p}} \\
&\overset{\text{Poincarè's inequality}}{\leqslant} c\bigg(\int_{\Omega_3} |D\bar{h}|^p\,dx\bigg)^{\frac{1}{p}} \bigg(\int_{\Omega_3\setminus B_3^+} 1\,dx\bigg)^{1-\frac{1}{p}} \\
&\overset{(3.2),\,(3.35)}{\leqslant} c\delta^{1-\frac{1}{p}}.
\end{aligned} \tag{3.46}
$$

In view of (3.42)–(3.46), we conclude that

$$\fint_{\Omega_3} \zeta^p \left|D(\bar{h}-\bar{v})\right|^p dx \leqslant c\tau_3 + c(\tau_3)\left(\eta\epsilon^p + \delta^{\frac{1}{n}} + \delta^{1-\frac{1}{p}}\right).$$

Then it follows from (3.41) that

$$\fint_{\Omega_2} \left|D(\bar{h}-\bar{v})\right|^p dx \leqslant c_3\tau_3 + c(\tau_3)\left(\eta\epsilon^p + \delta^{\frac{1}{n}} + \delta^{1-\frac{1}{p}}\right). \tag{3.47}$$

We take

$$\tau_3 = \frac{1}{5\cdot 3^{p-1}\cdot c_3}\epsilon^p.$$

A direct computation yields $c(\tau_3) = c\epsilon^{p-2}$. Take $\eta = c\epsilon^{2-p}$ so that

$$c(\tau_3)\eta = \frac{1}{5\cdot 3^{p-1}}.$$

As a consequence, we have

$$\fint_{\Omega_2} \left|D(\bar{h}-\bar{v})\right|^p dx \leqslant \frac{2}{5\cdot 3^{p-1}}\epsilon^p + c\epsilon^{p-2}\delta^{\sigma_4}, \tag{3.48}$$

where $\sigma_4 = \min\{\frac{1}{n}, 1-\frac{1}{p}\}$.

We finally combine (3.28), (3.34) and (3.48), to discover

$$\fint_{\Omega_2} \left|D(u-\bar{v})\right|^p dx \leqslant \frac{4}{5}\epsilon^p + c\left(\epsilon^{\frac{p(p-2)}{p-1}}\delta^p + \epsilon^{\frac{p(p-2)}{p-1}}\delta^{\sigma_3} + \epsilon^{p-2}\delta^{\sigma_4}\right) = \epsilon^p,$$

by taking δ so that the last identity holds.

Case 2. $p \geqslant 2$.

The proof of the case 2 is similar to that of the case 1, even simpler in this degenerate case. In fact, from the monotonicity (2.4), we can directly find the counterparts of (3.27), (3.33) and (3.47) without selecting τ_i $(i = 1, 2, 3)$. It's worth noting that $\eta > 0$ in (3.38) can be chosen so that it is independent of ϵ. □

4. Global a priori estimates

In this section we will establish the a priori estimate

$$\int_{\Omega} |Du|^{pq} w(x)\,dx \leqslant c \int_{\Omega} |F|^{pq} w(x)\,dx, \tag{4.1}$$

for every $q \in (1,\infty)$, under the a priori assumption:

$$\int_{\Omega} |Du|^{pq} w(x)\,dx < +\infty. \tag{4.2}$$

To do this, we further assume that F, $\mathbf{a}$ and $\partial\Omega$ to be of class C^∞, in order to get

$$|Du| \in L^\infty(\Omega).$$

These assumptions shall be removed in the next section by the standard approximation scheme. Hereafter we write

$$\bar{p} = pq - \mu_0 \ (> p),$$

where μ_0 is a positive number which will be determined below in Remark 4.3. We now denote λ_0 to mean the integral average of $|Du|^{\overline{p}} + \frac{1}{\delta^{\overline{p}}}|F|^{\overline{p}}$ over Ω with respect to the weight measure $w(\cdot)$. That is,

$$\lambda_0 = \frac{1}{w(\Omega)} \int_{\Omega} \left[|Du|^{\overline{p}} + \frac{1}{\delta^{\overline{p}}}|F|^{\overline{p}} \right] w(x)\,dx. \tag{4.3}$$

We point out that $\delta \in (0, \frac{1}{8})$ is now fixed, but to be selected later in this section.

Although our problem is nonlinear, our problem (2.1) is invariant under scaling and normalization, see Lemma 2.5. Then for the sake of simplicity we may assume that

$$[w]_q \cdot \left(\frac{16}{7}\right)^{nq} \cdot \frac{w(\Omega)}{w(B_1(y) \cap \Omega)} \cdot \lambda_0 < 1 \quad \text{for all } y \in \Omega, \tag{4.4}$$

as we can control the problem (2.1) with a proper normalization and dilation so that the inequality in (4.4) holds true.

Given a fixed point $y \in E(1) := \{x \in \Omega\colon |Du(x)| > 1\}$ and for each $r > 0$, we define a continuous function $\Theta = \Theta_y : (0, \infty) \mapsto \mathbb{R}$ by

$$\Theta(r) = \frac{1}{w(\Omega_r(y))} \int_{\Omega_r(y)} \left[|Du|^{\overline{p}} + \frac{1}{\delta^{\overline{p}}}|F|^{\overline{p}} \right] w(x)\,dx, \tag{4.5}$$

where $\Omega_r(y) = \Omega \cap B_r(y) = \Omega_r + y$.

Remark 4.1. (See [29].) Observe that the measure $w(\cdot)$ is a nonnegative regular Borel measure on $\mathbb{R}^n$, is finite on bounded sets and has a doubling property, to find that for almost every $y \in E(1)$

$$\Theta(0) = \lim_{r \to 0} \Theta(r) = \left|Du(y)\right|^{\overline{p}} + \frac{1}{\delta^{\overline{p}}}\left|F(y)\right|^{\overline{p}} > 1. \tag{4.6}$$

Since the given domain Ω is bounded, we also see that

$$\lim_{r \to \infty} \Theta(r) = \frac{1}{w(\Omega)} \int_{\Omega} \left[|Du|^{\overline{p}} + \frac{1}{\delta^{\overline{p}}}|F|^{\overline{p}} \right] w(x)\,dx = \lambda_0 < 1.$$

On the other hand, for $r \geqslant 1$,

$$\begin{aligned} \Theta(r) &= \frac{1}{w(\Omega_r(y))} \int_{\Omega_r(y)} \left[|Du|^{\overline{p}} + \frac{1}{\delta^{\overline{p}}}|F|^{\overline{p}} \right] w(x)\,dx \\ &\leqslant \frac{w(B_r(y))}{w(\Omega_r(y))} \frac{w(\Omega)}{w(B_1(y))} \frac{1}{w(\Omega)} \int_{\Omega} \left[|Du|^{\overline{p}} + \frac{1}{\delta^{\overline{p}}}|F|^{\overline{p}} \right] w(x)\,dx \\ &= \frac{w(B_r(y))}{w(\Omega_r(y))} \frac{w(\Omega)}{w(B_1(y))} \lambda_0. \end{aligned}$$

We also see from Lemma 2.1 and Definition 2.3 that

$$\begin{aligned} \frac{w(B_r(y))}{w(\Omega_r(y))} &\leqslant [w]_q \cdot \left(\frac{|B_r(y)|}{|B_r(y) \cap \Omega|} \right)^q \\ &\leqslant [w]_q \cdot \left(\frac{2}{1-\delta} \right)^{nq} \leqslant [w]_q \cdot \left(\frac{16}{7} \right)^{nq}. \end{aligned}$$

But then (4.4) implies that

$$\Theta(r) < 1, \quad \forall r \geqslant 1. \tag{4.7}$$

Hence we deduce from (4.6) and (4.7) that for almost every $y \in E(1)$ there exists a number $\tilde{r} = \tilde{r}(y) \in (0, 1)$ such that

$$\Theta(\tilde{r}) = 1 \quad \text{and} \quad \Theta(r) < 1 \quad \text{for all } r \geqslant \tilde{r}.$$

We now apply *Vitali's covering lemma*, to obtain the following lemma.

Lemma 4.2. *There exists a family of disjoint* $\{\Omega_{r_k}(y_k)\}_{k\geqslant 1}$ *with* $y_k \in E(1)$ *and* $0 < r_k = r_k(y_k) < 1$ *such that*

$$\Theta_{y_k}(r_k) = 1, \tag{4.8}$$

$$\Theta_{y_k}(r) < 1 \quad (r > r_k), \tag{4.9}$$

and

$$E(1) \subset \bigcup_{k\geqslant 1} \Omega_{5r_k}(y_k) \cup \textit{negligible set}. \tag{4.10}$$

Remark 4.3. Recall that a weight $w \in A_q$ for $q > 1$. Then in view of Lemma 2.1 we observe that $w \in A_{\underline{q}}$ for some constant $\underline{q} = \underline{q}(n, q, [w]_q) \in (1, q)$. We next take a small positive number μ_0 so that

$$0 < \mu_0 < p(q - \underline{q}) < p(q-1) < pq - 1. \tag{4.11}$$

We now claim that

$$\fint_{\Omega_r(y)} |Du|^p\,dx \leqslant \big([w]_{\frac{pq-\mu_0}{p}}\big)^{\frac{p}{pq-\mu_0}}$$

under the assumption

$$\frac{1}{w(\Omega_r(y))} \int_{\Omega_r(y)} |Du|^{pq-\mu_0} w(x)\,dx \leqslant 1, \tag{4.12}$$

for any number $r > 0$ and for any point $y \in \Omega$.

Indeed, we use Hölder's inequality to estimate

$$\begin{aligned}
\fint_{\Omega_r(y)} |Du|^p\,dx &= \fint_{\Omega_r(y)} |Du|^p \cdot w(x)^{\frac{p}{pq-\mu_0}} \cdot w(x)^{-\frac{p}{pq-\mu_0}}\,dx \\
&\overset{(4.11)}{\leqslant} \bigg(\fint_{\Omega_r(y)} |Du|^{pq-\mu_0} w(x)\,dx\bigg)^{\frac{p}{pq-\mu_0}} \bigg(\fint_{\Omega_r(y)} w(x)^{-\frac{p}{p(q-1)-\mu_0}}\,dx\bigg)^{\frac{p(q-1)-\mu_0}{pq-\mu_0}} \\
&\overset{(4.12)}{\leqslant} \bigg(\frac{w(\Omega_r(y))}{|\Omega_r(y)|}\bigg)^{\frac{p}{pq-\mu_0}} \bigg(\fint_{\Omega_r(y)} w(x)^{-\frac{p}{p(q-1)-\mu_0}}\,dx\bigg)^{\frac{p(q-1)-\mu_0}{pq-\mu_0}} \\
&\overset{(2.8)}{\leqslant} \bigg(\fint_{\Omega_r(y)} w(x)\,dx\bigg)^{\frac{p}{pq-\mu_0}} \bigg(\fint_{\Omega_r(y)} w(x)^{-\frac{1}{\frac{pq-\mu_0}{p}-1}}\,dx\bigg)^{(\frac{pq-\mu_0}{p}-1)\frac{p}{pq-\mu_0}} \\
&\overset{(4.11),\,(2.6)}{\leqslant} \big([w]_{\frac{pq-\mu_0}{p}}\big)^{\frac{p}{pq-\mu_0}}.
\end{aligned}$$

By the same reason we have

$$\fint_{\Omega_r(y)} |F|^p\,dx \leqslant \big([w]_{\frac{pq-\mu_0}{p}}\big)^{\frac{p}{pq-\mu_0}} \delta^p$$

under the assumption

$$\frac{1}{w(\Omega_r(y))} \int_{\Omega_r(y)} |F|^{pq-\mu_0} w(x)\,dx \leqslant \delta^{pq-\mu_0}. \tag{4.13}$$

Therefore, under the assumptions (4.12) and (4.13), and with the invariance property under a proper normalization, see Lemma 2.5, we assume that

$$\fint_{\Omega_r(y)} |Du|^p\,dx \leqslant 1 \quad \text{and} \quad \fint_{\Omega_r(y)} |F|^p\,dx \leqslant \delta^p.$$

We now recall $\overline{p} = pq - \mu_0$ and want to estimate the size of each member $\Omega_{r_k}(y_k)$ of the covering of the upper level set $E(1)$, given in Lemma 4.2.

Lemma 4.4. *Under the same hypothesis and results as in Lemma 4.2, we have*

$$w\big(\Omega_{r_k}(y_k)\big) \leqslant c\left(\int_{\Omega_{r_k}(y_k)\cap\{|Du|>\frac{1}{2}\}} |Du|^{\overline{p}} w(x)\,dx + \frac{1}{\delta^{\overline{p}}}\int_{\Omega_{r_k}(y_k)\cap\{|F|>\frac{\delta}{2}\}} |F|^{\overline{p}} w(x)\,dx\right).$$

Proof. It follows from (4.8) that

$$\Theta_{r_k}(y_k) = 1.$$

Then we have

$$w\big(\Omega_{r_k}(y_k)\big) = \int_{\Omega_{r_k}(y_k)} \left[|Du|^{\overline{p}} + \frac{1}{\delta^{\overline{p}}}|F|^{\overline{p}}\right] w(x)\,dx.$$

Now we split the two integrals in the right-hand side of the above inequality to estimate

$$\begin{aligned} w\big(\Omega_{r_k}(y_k)\big) \leqslant{} & \int_{\Omega_{r_k}(y_k)\cap\{|Du|>\frac{1}{2}\}} |Du|^{\overline{p}} w(x)\,dx + \frac{1}{2^{\overline{p}}} w\big(\Omega_{r_k}(y_k)\big) \\ & + \frac{1}{\delta^{\overline{p}}}\int_{\Omega_{r_k}(y_k)\cap\{|F|>\frac{\delta}{2}\}} |F|^{\overline{p}} w(x)\,dx + \frac{1}{2^{\overline{p}}} w\big(\Omega_{r_k}(y_k)\big). \end{aligned}$$

This estimate and (4.11) yield conclusion. □

Lemma 4.5. *Let $\lambda > 1$ and $N = \max\{n_1, n_2\}$, where n_1 and n_2 are given in Lemmas 3.1 and 3.7, respectively. Under the same notation and results as in Lemma 4.4, we have*

$$w\big(\big\{x \in \Omega\colon \big|Du(x)\big| > 2N\lambda\big\}\big) \leqslant c\frac{\epsilon^{pt}}{\lambda^{\overline{p}}}\left(\int_{\{x\in\Omega\colon |Du|>\frac{\lambda}{2}\}} |Du|^{\overline{p}} w(x)\,dx + \frac{1}{\delta^{\overline{p}}}\int_{\{x\in\Omega\colon |F|>\frac{\delta\lambda}{2}\}} |F|^{\overline{p}} w(x)\,dx\right),$$

where the positive constant t is given in Lemma 2.1.

Proof. In Lemma 4.2, we have found a family of disjoint covers $\{\Omega_{r_k}(y_k)\}_{k\geqslant 1}$ with $y_k \in E(1)$ and $r_k \in (0,1)$. We will estimate the upper level set on each fixed member $\Omega_{r_k}(y_k)$ of this covering, arguing on the comparison estimates as in Lemmas 3.1 and 3.7 from the scaling invariant property of the problem (2.1).

We first consider the interior case $B_{30r_k}(y_k) \subseteq \Omega$. Define the scaled functions

$$\tilde{u}_k(x) = \frac{1}{5r_k} u(y_k + 5r_k x), \qquad \tilde{F}_k(x) = F(y_k + 5r_k x), \qquad \tilde{\mathbf{a}}_k(\xi, x) = \mathbf{a}_k(\xi, y_k + 5r_k x)$$

for $x \in B_6$ and $\xi \in \mathbb{R}^n$. Then $\tilde{u}_k$ is a weak solution of

$$\operatorname{div} \tilde{\mathbf{a}}_k(D\tilde{u}_k, x) = \operatorname{div}\big(|\tilde{F}_k|^{p-2}\tilde{F}_k\big) \quad \text{in } B_6.$$

In view of Lemma 4.2 and Remark 4.3, we are under the hypotheses of Lemma 3.1, which implies that there exists a constant $n_1 = n_1(\gamma, \Lambda, n, p) > 1$ so that for any $\epsilon > 0$ fixed, we find a small $\delta = \delta(\epsilon, \gamma, \Lambda, n, p, q, [w]_q) > 0$ and a weak solution $\tilde{v}_k$ of

$$\operatorname{div}\big(\overline{\tilde{\mathbf{a}}_k}_{B_4}(D\tilde{v}_k)\big) = 0 \quad \text{in } B_4$$

such that

$$\|D\tilde{v}_k\|_{L^\infty(B_3)} \leqslant n_1$$

and

$$\fint_{B_2} \big|D(\tilde{u}_k - \tilde{v}_k)\big|^p \, dx \leqslant \epsilon^p.$$

Now we define

$$v_k(x) = 5r_k \tilde{v}_k\left(\frac{x}{5r_k}\right) \quad \big(x \in B_{20r_k}(y_k)\big).$$

By change of variables, we see that

$$\|Dv_k\|_{L^\infty(B_{15r_k}(y_k))} \leqslant n_1 \tag{4.14}$$

and

$$\fint_{B_{10r_k}(y_k)} \big|D(u - v_k)\big|^p \, dx \leqslant \epsilon^p. \tag{4.15}$$

Consequently, in this interior case, we have

$$\begin{aligned}
\frac{|\{x \in B_{10r_k}(y_k)\colon |Du| > 2n_1\}|}{|B_{10r_k}(y_k)|} &\leqslant \frac{|\{x \in B_{10r_k}(y_k)\colon |D(u - v_k)| > n_1\}|}{|B_{10r_k}(y_k)|} + \frac{|\{y \in B_{10r_k}(y_k)\colon |Dv_k| > n_1\}|}{|B_{10r_k}(y_k)|} \\
&\overset{(4.14)}{\leqslant} \frac{1}{n_1^p} \fint_{B_{10r_k}(y_k)} \big|D(u - v_k)\big|^p \, dx \\
&\overset{(4.15)}{\leqslant} \frac{1}{n_1^p} \epsilon^p.
\end{aligned} \tag{4.16}$$

We now consider the boundary case $B_{30r_k}(y_k) \nsubseteq \Omega$. In this case, for simplicity, we assume that Ω is $(\delta, 108)$-Reifenberg flat. Then we find an appropriate coordinate system with

$$y_k = x_k$$

that

$$B^+_{108r_k} \subset \Omega_{108r_k} \subset B_{108r_k} \cap \{x_n > -216 r_k \delta\}$$

and

$$\Omega_{5r_k}(x_k) \subset \Omega_{36r_k} = \Omega_{36r_k}(0) \subset \Omega_{108r_k}(0) \subset \Omega_{120r_k}(x_k).$$

We define the scaled functions

$$\tilde{u}_k(x) = \frac{1}{18r_k} u(18r_k x), \qquad \tilde{F}_k(x) = F(18r_k x), \qquad \tilde{\mathbf{a}}_k(\xi, x) = \mathbf{a}_k(\xi, 18r_k x)$$

for $x \in \Omega_6$ and $\xi \in \mathbb{R}^n$. Then $\tilde{u}_k$ is a weak solution of

$$\operatorname{div}\big(\tilde{\mathbf{a}}_k(D\tilde{u}_k, x)\big) = \operatorname{div}\big(|\tilde{F}_k|^{p-2}\tilde{F}_k\big) \quad \text{in } \Omega_6.$$

Then thanks to Lemma 2.5, Lemma 4.2 and Remark 4.3, we apply Lemma 3.7, to ascertain that there exists a constant $n_2 = n_2(\gamma, \Lambda, n, p) > 1$ so that for any $\epsilon > 0$ fixed, we find a small $\delta = \delta(\epsilon, \gamma, \Lambda, n, p, q, [w]_q) > 0$ and a weak solution $\tilde{v}_k$ of

$$\operatorname{div}\big(\overline{\tilde{\mathbf{a}}_k}_{B_4^+}(D\tilde{v}_k)\big) = 0 \quad \text{in } B_4^+ \text{ with } \tilde{v}_k = 0 \text{ on } T_4$$

such that

$$\|D\bar{\tilde{v}}_k\|_{L^\infty(\Omega_3)} \leqslant n_2$$

and

$$\fint_{\Omega_2} \big|D(\tilde{u}_k - \bar{\tilde{v}}_k)\big|^p \, dx \leqslant \epsilon^p,$$

where $\bar{\tilde{v}}_k$ is the zero extension of $\tilde{v}_k$. Now we define

$$v_k(x) = 18 r_k \tilde{v}_k\left(\frac{x}{18 r_k}\right) \quad (x \in \Omega_{72 r_k}).$$

By change of variables, eventually we see that

$$\|D\bar{v}_k\|_{L^\infty(\Omega_{54 r_k})} \leqslant n_2 \tag{4.17}$$

and

$$\fint_{\Omega_{36 r_k}} \big|D(u - \bar{v}_k)\big|^p \, dx \leqslant \epsilon^p, \tag{4.18}$$

where $\bar{v}_k$ is the zero extension of v_k which is a weak solution of

$$\operatorname{div} \bar{\mathbf{a}}_{B_{72 r_k}^+}(Dv_k) = 0 \quad \text{in } B_{72 r_k}^+ \text{ with } v_k = 0 \text{ on } T_{72 r_k}.$$

Therefore, in this boundary case, we compute

$$\begin{aligned} \frac{|\{x \in \Omega_{36 r_k}\colon |Du| > 2n_2\}|}{|\Omega_{36 r_k}|} &\leqslant \frac{|\{x \in \Omega_{36 r_k}\colon |D(u - v_k)| > n_2\}|}{|\Omega_{36 r_k}|} + \frac{|\{x \in \Omega_{36 r_k}\colon |Dv_k| > n_1\}|}{|\Omega_{36 r_k}|} \\ &\overset{(4.17)}{\leqslant} \frac{1}{n_2^p} \fint_{\Omega_{36 r_k}} \big|D(u - v_k)\big|^p \, dx \\ &\overset{(4.18)}{\leqslant} \frac{1}{n_2^p} \epsilon^p. \end{aligned} \tag{4.19}$$

Now we recall the measure density condition of a Reifenberg flat domain discussed previously in the proof Lemma 3.2 and our setting that $N = \max\{n_1, n_2\}$. We then combine (4.16) in the interior case with (4.19) in the boundary case, to discover

$$\frac{|\{x \in \Omega_{5 r_k}(y_k)\colon |Du(x)| > 2N\}|}{|\Omega_{r_k}(y_k)|} \leqslant c\epsilon^p,$$

where we point out that the constant $c = c(\gamma, \Lambda, n, p)$ is independent of k. This estimate and (2.10) in Lemma 2.1 yield

$$w\big(\big\{x \in \Omega_{5 r_k}(y_k)\colon \big|Du(x)\big| > 2N\big\}\big) \leqslant c\epsilon^{pt} w\big(\Omega_{r_k}(y_k)\big). \tag{4.20}$$

Recalling Lemma 4.2 and Lemma 4.4 and using (4.20), we estimate

$$\begin{aligned} w\big(\big\{x \in \Omega\colon \big|Du(x)\big| > 2N\big\}\big) &\leqslant \sum_{k \geqslant 1} w\big(\big\{x \in \Omega_{5 r_k}(y_k)\colon \big|Du(x)\big| > 2N\big\}\big) \\ &\leqslant c\epsilon^{pt} \sum_{k \geqslant 1} w\big(\Omega_{r_k}(y_k)\big) \end{aligned}$$

$$\leqslant c\epsilon^{pt}\sum_{k\geqslant 1}\biggl(\int\limits_{\Omega_{r_k}(y_k)\cap\{|Du|>\frac{1}{2}\}}|Du|^{\overline{p}}w\,dx+\frac{1}{\delta^{\overline{p}}}\int\limits_{\Omega_{r_k}(y_k)\cap\{|F|>\frac{\delta}{2}\}}|F|^{\overline{p}}w\,dx\biggr)$$

$$\leqslant c\epsilon^{pt}\biggl(\int\limits_{\Omega\cap\{|Du|>\frac{1}{2}\}}|Du|^{\overline{p}}w\,dx+\frac{1}{\delta^{\overline{p}}}\int\limits_{\Omega\cap\{|F|>\frac{\delta}{2}\}}|F|^{\overline{p}}w\,dx\biggr).$$

Notice that the Dirichlet problem (2.1) is invariant under normalization, as one can take $\lambda>1$ and $r=1$ in Lemma 2.5, to discover that for each $\lambda>1$

$$w\bigl(\bigl\{x\in\Omega\colon\ \bigl|Du(x)\bigr|>2N\lambda\bigr\}\bigr)\leqslant c\epsilon^{pt}\frac{1}{\lambda^{\overline{p}}}\biggl(\int\limits_{\Omega\cap\{|Du|>\frac{\lambda}{2}\}}|Du|^{\overline{p}}w\,dx+\frac{1}{\delta^{\overline{p}}}\int\limits_{\Omega\cap\{|F|>\frac{\delta\lambda}{2}\}}|F|^{\overline{p}}w\,dx\biggr).$$

This completes the proof. □

Before proving the a priori estimate (4.1), we recall the following classical measure theory on weighted Lebesgue spaces.

Lemma 4.6. *Let Ω be a bounded domain in $\mathbb{R}^n$. Let w be an A_q weight for some $1<q<\infty$. Then for all nonnegative function $g\in L^{\beta}_{w}(\Omega)$ and any $\beta>\alpha>1$,*

$$\int\limits_{\Omega}|g|^{\beta}w(x)\,dx=\beta\int\limits_{0}^{\infty}\lambda^{\beta-1}w\bigl(\bigl\{x\in\Omega\colon\ g(x)>\lambda\bigr\}\bigr)\,d\lambda$$

$$=(\beta-\alpha)\int\limits_{0}^{\infty}\lambda^{\beta-\alpha-1}\biggl(\int\limits_{\{|g|>\lambda\}}|g|^{\alpha}w(x)\,dx\biggr)\,d\lambda.$$

We are now set to give a complete proof of the a priori estimate (4.1).

Proof. We first recall that the letter c means a universal positive constant being dependent only on $\gamma,\Lambda,n,p,q,[w]_q$ and the geometry of the domain. The proof proceeds with Lemmas 4.5 and 4.6.

According to the first identity formula in Lemma 4.6 when $g=|Du|$ and $\beta=pq$, we find

$$\begin{aligned}\int\limits_{\Omega}|Du|^{pq}w(x)\,dx\ &=\ pq\int\limits_{0}^{\infty}\lambda^{pq-1}w\bigl(\bigl\{x\in\Omega\colon\ |Du|>\lambda\bigr\}\bigr)\,d\lambda\\ &=\ pq\int\limits_{0}^{2N}\lambda^{pq-1}w\bigl(\bigl\{x\in\Omega\colon\ |Du|>\lambda\bigr\}\bigr)\,d\lambda+pq\int\limits_{2N}^{\infty}\lambda^{pq-1}w\bigl(\bigl\{x\in\Omega\colon\ \bigl|Du(x)\bigr|>\lambda\bigr\}\bigr)\,d\lambda\\ &=:I_1+I_2.\end{aligned}$$

Estimate of I_1: A direct computation finds

$$I_1\leqslant c\cdot w(\Omega).$$

Estimate of I_2: It follows from the change of variables and Lemma 4.5 that

$$\begin{aligned}I_2&=pq\int\limits_{2N}^{\infty}\lambda^{pq-1}\ w\bigl(\bigl\{x\in\Omega\colon\ |Du|>\lambda\bigr\}\bigr)\,d\lambda\\ &\leqslant c\int\limits_{1}^{\infty}\lambda^{pq-1}w\bigl(\bigl\{x\in\Omega\colon\ |Du|>2N\lambda\bigr\}\bigr)\,d\lambda\end{aligned}$$

$$\leqslant c\epsilon^{pt}\int_1^\infty \frac{\lambda^{pq-1}}{\lambda^{\overline{p}}}\left[\int_{\Omega\cap\{|Du|>\frac{\lambda}{2}\}} |Du|^{\overline{p}} w(x)\,dx\right]d\lambda + c\frac{\epsilon^{pt}}{\delta^{\overline{p}}}\int_1^\infty \frac{\lambda^{pq-1}}{\lambda^{\overline{p}}}\left[\int_{\Omega\cap\{|F|>\frac{\delta\lambda}{2}\}} |F|^{\overline{p}} w(x)\,dx\right]d\lambda.$$

Apply the second identity formula in Lemma 4.6 when $g = |Du|$ and $g = |F|$ respectively, $\beta = pq$ and $\alpha = \overline{p} = pq - \mu_0$, to derive

$$I_2 \leqslant c\epsilon^{pt}\int_\Omega |Du|^{pq} w(x)\,dx + c(\delta,\epsilon)\int_\Omega |F|^{pq} w(x)\,dx.$$

We combine the estimate of I_1 with the estimate of I_2, to derive

$$\int_\Omega |Du|^{pq} w(x)\,dx \leqslant c + c\epsilon^{pt}\int_\Omega |Du|^{pq} w(x)\,dx + c(\delta,\epsilon)\int_\Omega |F|^{pq} w(x)\,dx.$$

We then use the a priori assumption (4.2) and select $\epsilon > 0$ small enough in order to get $0 < c\epsilon^{pt} < 1$, thereby determining $\delta = \delta(\gamma, \Lambda, n, p, q, [w]_q) > 0$ due to Lemmas 3.1 and 3.7, to finally get the required one

$$\int_\Omega |Du|^{pq} w(x)\,dx \leqslant c\left(\int_\Omega |F|^{pq} w(x)\,dx + 1\right)$$

for some positive and universal constant $c = c(\gamma, \Lambda, n, p, q, [w]_q, |\Omega|)$. □

5. Approximation procedure

In the previous section we have established the a priori estimate (4.1) under the a priori regularity assumption (4.2). In this section we will complete our proof of the main result, Theorem 2.6, by removing the assumption (4.2). Since this procedure is similar to those made in the previous papers [5,10], we make a brief sketch for the completeness of the proof.

Proof of Theorem 2.6. In view of Lemma 4.2 in [6] and by a standard approximation of a Lipschitz domain by smooth domains, we can extract a sequence of smooth domains Ω^m with the uniform (δ, R)-*Reifenberg flatness property* such that

$$\Omega^m \subset \Omega^{m+1} \subset \Omega \quad \text{and} \quad d_H(\partial\Omega^m, \partial\Omega) \to 0 \quad \text{as } m \to \infty, \tag{5.1}$$

where d_H denotes the Hausdorff distance. We next select a sequence of smooth nonlinearities $\mathbf{a}_k \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$ satisfying the basic structural conditions (2.2)–(2.4) and the regularity requirement (2.16) such that

$$\mathbf{a}_k(\xi, \cdot) \to \mathbf{a}(\xi, \cdot) \quad \text{strongly in } L^a(\mathbb{R}^n) \text{ as } k \to \infty, \tag{5.2}$$

for each $1 < a < \infty$ and uniformly at each ξ.

We also choose a sequence $\{F_k\}_{k=1}^\infty$ of smooth functions on $C^\infty(\Omega; \mathbb{R}^n)$ such that

$$F_k \to F \quad \text{strongly in } L_w^{pq}(\Omega; \mathbb{R}^n) \text{ as } k \to \infty. \tag{5.3}$$

Then, since $w \in A_q$,

$$F_k \to F \quad \text{strongly in } L^p(\Omega; \mathbb{R}^n) \text{ as } k \to \infty.$$

Now we fix any sufficiently large positive integer m. Then according to a standard theory for a nonlinear elliptic equation on the fixed smooth domain Ω^m with smooth data, $\mathbf{a}_k$ and F_k, there exists a unique weak solution $u_k \in W_0^{1,p}(\Omega^m)$ of

$$\begin{cases} \operatorname{div} \mathbf{a}_k(Du_k, x) = \operatorname{div}(|F_k|^{p-2}F_k) & \text{in } \Omega^m, \\ u_k = 0 & \text{on } \partial\Omega^m, \end{cases} \tag{5.4}$$

with the Lipschitz regularity $Du_k \in L^\infty(\Omega^m; \mathbb{R}^n)$, see [13,18]. Needless to say, this weak solution satisfies our a priori regularity assumption

$$|Du_k|^p \in L_w^q(\Omega^m). \tag{5.5}$$

Then as in Section 4 with the assumption (5.5), we have

$$\int_{\Omega^m} |Du_k|^{pq} w(x)\,dx \leqslant c \int_{\Omega^m} |F_k|^{pq} w(x)\,dx$$

where the constant c is independent of k. Then it follows from (5.1) and (5.3) that

$$\int_{\Omega^m} |Du_k|^{pq} w(x)\,dx \leqslant c \int_{\Omega} |F|^{pq} w(x)\,dx < \infty. \tag{5.6}$$

Therefore, $\{u_k\}_{k=1}^{\infty}$ is uniformly bounded in $W_0^{1,pq}(\Omega^m, w(x)\,dx)$ which is the weighted Sobolev space with a weight w like (2.11). So there is a subsequence, which we still denote as $\{u_k\}$, and u^m in $W_0^{1,pq}(\Omega^m, w(x)\,dx)$ such that

$$\begin{cases} u_k \rightharpoonup u^m & \text{weakly in } W_0^{1,pq}(\Omega^m, w(x)\,dx) \subset W_0^{1,p}(\Omega^m), \\ u_k \to u^m & \text{strongly in } L_w^{pq}(\Omega^m) \subset L^p(\Omega^m), \end{cases} \tag{5.7}$$

as $k \to \infty$. Returning to (5.4) and letting $k \to \infty$, it follows from (5.2), (5.3) and (5.7) that u^m is the unique weak solution of

$$\begin{cases} \operatorname{div} \mathbf{a}(Du^m, x) = \operatorname{div}(|F|^{p-2}F) & \text{in } \Omega^m, \\ u^m = 0 & \text{on } \partial\Omega^m. \end{cases} \tag{5.8}$$

In addition, we observe from the weak lower semicontinuity in (5.6) and (5.7) that

$$\int_{\Omega^m} |Du^m|^{pq} w(x)\,dx \leqslant c \int_{\Omega} |F|^{pq} w(x)\,dx. \tag{5.9}$$

We next let $\bar{u}^m$ be the zero extension of u^m from Ω^m to Ω. That is,

$$\bar{u}^m(x) = \begin{cases} u^m(x) & \text{if } x \in \Omega^m, \\ 0 & \text{if } x \in \Omega \setminus \Omega^m. \end{cases} \tag{5.10}$$

Then, from (5.1), (5.9) and (5.10), we discover that $\bar{u}^m \in W_0^{1,p}(\Omega)$ with

$$\int_{\Omega} |D\bar{u}^m|^{pq} w(x)\,dx \leqslant c \int_{\Omega} |F|^{pq} w(x)\,dx < \infty. \tag{5.11}$$

Hence we have, up to subsequences, that

$$\begin{cases} \bar{u}^m \rightharpoonup \bar{u} & \text{weakly in } W_0^{1,pq}(\Omega, w(x)\,dx) \subset W_0^{1,p}(\Omega), \\ \bar{u}^m \to \bar{u} & \text{strongly in } L_w^{pq}(\Omega) \subset L^p(\Omega), \end{cases} \tag{5.12}$$

for some $\bar{u} \in W_0^{1,pq}(\Omega, w(x)\,dx) \subset W_0^{1,p}(\Omega)$ as $m \to \infty$. Then by (5.11) and (5.12), we have

$$\int_{\Omega} |D\bar{u}|^{pq} w(x)\,dx \leqslant c \int_{\Omega} |F|^{pq} w(x)\,dx. \tag{5.13}$$

On the other hand, it follows from (5.1), (5.8), (5.12), and the uniqueness of the weak solution of the original problem (2.1) that $\bar{u} = u$. This completes the proof. □

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