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A regularity result for a solid–fluid system associated to the compressible Navier–Stokes equations

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Abstract

In this paper we deal with a fluid-structure interaction problem for a compressible fluid and a rigid structure immersed in a regular bounded domain in dimension 3. The fluid is modelled by the compressible Navier–Stokes system in the barotropic regime with no-slip boundary conditions and the motion of the structure is described by the usual law of balance of linear and angular moment.

The main result of the paper states that, for small initial data, we have the existence and uniqueness of global smooth solutions as long as no collisions occur. This result is proved in two steps; first, we prove the existence and uniqueness of local solution and then we establish some a priori estimates independently of time.

Résumé

Dans cet article, nous considérons un problème d'interaction fluide-structure entre un fluide compressible et une structure rigide évoluant à l'intérieur d'un domaine borné et régulier en dimension 3. Le fluide est décrit par le système de Navier–Stokes compressible barotrope avec des conditions de non-glissement sur le bord et le mouvement de la structure est régi par les lois de conservation des moments linéaire et angulaire.

Nous montrons, pour des données initiales petites, l'existence et l'unicité de solutions globales régulières tant qu'il n'y a pas de chocs. Ce résultat est obtenu en deux temps ; tout d'abord, nous prouvons l'existence et l'unicité de solutions locales puis nous démontrons des estimations a priori indépendamment du temps.
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Keywords: Fluid–structure interaction; Compressible fluid; Strong solutions; Navier–Stokes equations

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1. Introduction

1.1. Statement of problem

We consider a rigid structure immersed in a viscous compressible fluid. At time t, we denote by $\Omega_S(t)$ the domain occupied by the structure. The structure and the fluid are contained in a fixed bounded domain $\Omega \subset \mathbb{R}^3$. We suppose that the boundaries of $\Omega_S(0)$ and Ω are smooth (C^4 for instance) and that

$$d\big(\Omega_S(0), \partial\Omega\big) > 0. \tag{1}$$

For any $t > 0$, we denote by $\Omega_F(t) = \Omega \setminus \overline{\Omega_S(t)}$ the region occupied by the fluid. The time evolution of the Eulerian velocity u and the density ρ in the fluid are governed by the compressible Navier–Stokes equations and the continuity equation: $\forall t > 0,\ \forall x \in \Omega_F(t)$

$$\begin{cases} (\rho_t + \nabla \cdot (\rho u))(t,x) = 0, \\ (\rho u_t + \rho(u \cdot \nabla)u)(t,x) - \nabla \cdot (2\mu\epsilon(u) + \mu'(\nabla \cdot u)\,\mathrm{Id})(t,x) + \nabla p(t,x) = 0, \end{cases} \tag{2}$$

where $\epsilon(u) = \frac{1}{2}(\nabla u + \nabla u^t)$ denotes the symmetric part of the gradient. The viscosity coefficients μ and μ' are real constants which are supposed to satisfy

$$\mu > 0, \qquad \mu + \mu' \geqslant 0. \tag{3}$$

We suppose that we are in a barotropic regime where a constitutive law gives the relation between the pressure of the fluid p and the density ρ. Thus, we suppose that

$$p = P(\rho) \quad \text{where } P \in C^\infty\big(\mathbb{R}_+^*\big),\ P(\rho) > 0 \text{ and } P'(\rho) > 0,\ \forall \rho > 0. \tag{4}$$

For instance, $P(\rho) := \rho^\gamma$ with $\gamma > 0$ is admissible.

Concerning the compressible fluids, a local in time result of existence and uniqueness of a smooth solution was proved in [21]. In [18], the authors proved the existence and uniqueness of a regular solution for small initial data and external forces.

Next, for isentropic fluids ($P(\rho) = \rho^\gamma, \gamma > 0$), the global existence of a weak solution for small initial data was proved in [13] (for $\gamma = 1$) and in [14] (for $\gamma > 1$). Also for an isentropic fluid, the first global result for large data was proved in [15] (see also [16]) (with $\gamma \geqslant 9/5$ for dimension $N = 3$ and with $\gamma > N/2$ for $N \geqslant 4$). Finally, this last result was improved in [9] (see also [11]) (with $\gamma > N/2$ for $N \geqslant 3$).

At time t, the motion of the rigid structure is given by the position $a(t) \in \mathbb{R}^3$ of the center of mass and by a rotation (orthogonal) matrix $Q(t) \in \mathbb{M}_{3\times 3}(\mathbb{R})$. Without loss of generality, we can suppose that

$$a(0) = 0 \quad \text{and} \quad Q(0) = \mathrm{Id}. \tag{5}$$

At time t, the domain occupied by the structure $\Omega_S(t)$ is defined by

$$\Omega_S(t) = \chi_S\big(t, \Omega_S(0)\big), \tag{6}$$

where χ_S denotes the flow associated to the motion of the structure:

$$\chi_S(t,y) = a(t) + Q(t)y, \quad \forall y \in \Omega_S(0),\ \forall t > 0. \tag{7}$$

We notice that, for each $t > 0$, $\chi_S(t,\cdot) : \Omega_S(0) \to \Omega_S(t)$ is invertible and

$$\chi_S(t,\cdot)^{-1}(x) = Q(t)^{-1}\big(x - a(t)\big), \quad \forall x \in \Omega_S(t).$$

Thus, the Eulerian velocity of the structure is given by

$$(\chi_S)_t(t,\cdot) \circ \chi_S(t,\cdot)^{-1}(x) = \dot{a}(t) + \dot{Q}(t)Q(t)^{-1}\big(x - a(t)\big), \quad \forall x \in \Omega_S(t).$$

Since $\dot{Q}(t)Q(t)^{-1}$ is skew-symmetric, for each $t > 0$, we can represent this matrix by a unique vector $\omega(t) \in \mathbb{R}^3$ such that

$$\dot{Q}(t)Q(t)^{-1}y = \omega(t) \wedge y, \quad \forall y \in \mathbb{R}^3.$$

Reciprocally, if ω belongs to $L^2(0,T)$, then there exists a unique matrix $Q \in H^1(0,T)$ such that $Q(0) = \mathrm{Id}$ and which satisfies this formula.

Thus, the Eulerian velocity u_S of the structure is given by

$$u_S(t,x) = \dot a(t) + \omega(t) \wedge \big(x - a(t)\big), \quad \forall x \in \Omega_S(t). \tag{8}$$

For the equations of the structure, we denote by $m > 0$ the mass of the rigid structure and $J(t) \in \mathbb{M}_{3\times 3}(\mathbb{R})$ its tensor of inertia at time t. This tensor is given by

$$J(t)b \cdot \tilde b = \int\limits_{\Omega_S(0)} \rho_{0,S}(y)\big(b \wedge Q(t)y\big) \cdot \big(\tilde b \wedge Q(t)y\big)\, dy, \quad \forall b, \tilde b \in \mathbb{R}^3, \tag{9}$$

where $\rho_{0,S} > 0$ is the initial density of the structure. One can prove that

$$J(t)b \cdot b \geqslant C_J |b|^2 > 0 \quad \text{for all } b \in \mathbb{R}^3 \setminus \{0\}, \tag{10}$$

where C_J is independent of $t > 0$. The equations of the structure motion are given by the balance of linear and angular momentum. We have, for all $t \in (0,T)$

$$\begin{cases} m\ddot a = \int_{\partial\Omega_S(t)} (2\mu\epsilon(u) + \mu'(\nabla\cdot u)\,\mathrm{Id} - p\,\mathrm{Id})n\, d\gamma, \\ J\dot\omega = (J\omega)\wedge\omega + \int_{\partial\Omega_S(t)} (x-a)\wedge((2\mu\epsilon(u) + \mu'(\nabla\cdot u)\,\mathrm{Id} - p\,\mathrm{Id})n)\, d\gamma. \end{cases} \tag{11}$$

In these equations, n is the outward unit normal to $\partial\Omega_S(t)$. On the boundary of the fluid, the Eulerian velocity has to satisfy a no-slip boundary condition. Therefore, we have, for all $t > 0$

$$\begin{cases} u(t,x) = 0, \quad \forall x \in \partial\Omega, \\ u(t,x) = \dot a(t) + \omega(t)\wedge\big(x - a(t)\big), \quad \forall x \in \partial\Omega_S(t). \end{cases} \tag{12}$$

The system is completed by the following initial conditions:

$$u(0,\cdot) = u_0 \quad \text{in } \Omega_F(0), \qquad \rho(0,\cdot) = \rho_0 \quad \text{in } \Omega_F(0), \qquad a(0) = 0, \quad \dot a(0) = a_0, \quad \omega(0) = \omega_0, \tag{13}$$

which satisfy

$$a_0, \omega_0 \in \mathbb{R}^3, \quad \rho_0, u_0 \in H^3\big(\Omega_F(0)\big), \qquad \rho_0(x) > 0, \quad \forall x \in \Omega_F(0). \tag{14}$$

Since we will deal with smooth solutions, we will also need some compatibility conditions to be satisfied:

$$u_0 = a_0 + \omega_0 \wedge x \quad \text{on } \partial\Omega_S(0), \qquad u_0 = 0 \quad \text{on } \partial\Omega \tag{15}$$

and

$$\begin{aligned} &\frac{1}{\rho_0}\nabla\cdot\big(2\mu\epsilon(u_0) + \mu'(\nabla\cdot u_0)\,\mathrm{Id}\big) - \frac{1}{\rho_0}\nabla P(\rho_0) \\ &\quad = \frac{1}{m}\int\limits_{\partial\Omega_S(0)} \big(2\mu\epsilon(u_0) + \mu'(\nabla\cdot u_0)\,\mathrm{Id} - P(\rho_0)\,\mathrm{Id}\big)n\, d\gamma + \big(J(0)^{-1}\big(J(0)\omega_0\big)\wedge\omega_0\big)\wedge x \\ &\qquad + J(0)^{-1}\Bigg(\int\limits_{\partial\Omega_S(0)} x \wedge \big(\big(2\mu\epsilon(u_0) + \mu'(\nabla\cdot u_0)\,\mathrm{Id} - P(\rho_0)\,\mathrm{Id}\big)n\big)\, d\gamma\Bigg)\wedge x + \omega_0\wedge(\omega_0\wedge x) \quad \text{on } \partial\Omega_S(0), \\ &\nabla\cdot\big(2\mu\epsilon(u_0) + \mu'(\nabla\cdot u_0)\,\mathrm{Id}\big) - \nabla P(\rho_0) = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{16}$$

These two conditions are formally obtained by differentiating system (12) with respect to time and taking $t = 0$. To do this, we consider the second equation of (12) on a fix domain by setting $x = \chi_S(t,y)$, with $y \in \partial\Omega_S(0)$.

Let us now recall some of the most relevant results in problems of fluid–structure interaction. In the below lines, when we refer to a global result, we mean before collision (in the case of a rigid solid) or before interpenetration of the structure (in the case of an elastic solid).

• Incompressible fluids:

As long as rigid solids are concerned, a local result was proved in [12], while the existence of global weak solutions is proved in [5] and [7] (with variable density) and [19] (2D, with variable density); in this last paper, the existence of a solution is proved even beyond collisions. Later, the existence and uniqueness of strong global solutions in 2D was proved in [20] as well as the local in time existence and uniqueness of strong solutions in 3D.

When talking about elastic solids, a first existence result of weak solution was proved in [8], when the elastic deformation is given by a finite sum of modes. A local existence result of a strong solution for an elastic plate was proved in [1] (2D). The local existence of a strong solution is proved in [6]. In [4] and [3] (with variable density), the authors proved the global existence of a weak solution.

• Compressible fluids:

Concerning rigid solids, the global existence of a weak solution was proved in [7] for $\gamma \geqslant 2$ and in [10] for $\gamma > N/2$. For elastic solids, in [2] the author proved the global existence of a weak solution in $3D$ for $\gamma > 3/2$.

In this paper, we will prove the existence and uniqueness of smooth global solutions for small initial data (Theorem 3). We can also prove the same result for initial data close to a stationary solution $(\rho, u, a, \omega) = (\rho_e, 0, 0, 0)$ and for special right-hand sides (see Remark 6 for more details).

We give a lemma which allows to extend the flow χ_S by a flow χ defined on the global domain Ω.

Lemma 1. *Let $T \in (0, +\infty)$ and $(a, \omega) \in (H^3(0, T) \cap W^{2,\infty}(0, T)) \times (H^2(0, T) \cap W^{1,\infty}(0, T))$ be given. We suppose that (a, Q) satisfies* (5) *where $Q \in H^3(0, T)$ is the rotation matrix associated to ω. We consider the associated flow χ_S, the Eulerian velocity u_S and the domain defined by* (6) *to* (8). *We suppose that there exists $\alpha > 0$ such that*

$$\forall t \in [0, T], \quad d\big(\Omega_S(t), \partial\Omega\big) \geqslant \alpha > 0. \tag{17}$$

Then, we can extend the flow χ_S by a flow $\chi \in H^3(0, T; C^\infty(\overline{\Omega}))$ such that

- $\chi(t, y) = a(t) + Q(t)y$ *for every y such that* $\operatorname{dist}(y, \partial\Omega_S(0)) < \alpha/4$ *and every $t \in (0, T)$.*
- $\chi(t, y) = y$ *for every y such that* $\operatorname{dist}(y, \partial\Omega) < \alpha/4$ *and for every $t \in (0, T)$.*
- *For all $t \in (0, T)$, $\chi(t, \cdot)$ is invertible from Ω onto Ω and from $\Omega_F(0)$ onto $\Omega_F(t)$ and*

$$(t, x) \in (0, T) \times \Omega \to \chi(t, \cdot)^{-1}(x)$$

  belongs to $H^3(0, T; C^\infty(\overline{\Omega}))$.
- *The Eulerian velocity $v \in H^2(0, T; C^\infty(\Omega))$ associated to χ satisfies*

$$\begin{cases} v(t, x) = u_S(t, x), & \forall x \in \Omega_S(t),\ \forall t \in (0, T),\\ v(t, x) = 0, & \forall x \in \partial\Omega,\ \forall t \in (0, T). \end{cases}$$

- *For $p \in \mathbb{N}$ there exists a constant $C_1 > 0$ just depending on p and Ω such that*

$$\big\|\chi_t(t, x)\big\|_{H^p(\Omega)} \leqslant C_1\big(\big|\dot{a}(t)\big|^2 + \big|\omega(t)\big|^2\big)^{1/2}, \quad \forall t \in (0, T). \tag{18}$$

Proof. We consider a cut-off function which satisfies

$$\xi \in C^\infty(\overline{\Omega}), \quad \xi(y) = 1, \quad \operatorname{dist}(y, \partial\Omega) > \alpha/2, \quad \xi(y) = 0, \quad \operatorname{dist}(y, \partial\Omega) < \alpha/4. \tag{19}$$

Let define the Eulerian velocity in the whole $(0, T) \times \Omega$ by

$$v(t, y) := \xi(y)\big(\dot{a}(t) + \omega(t) \wedge \big(y - a(t)\big)\big), \quad t \in (0, T),\ y \in \Omega. \tag{20}$$

Next, we can define the flow χ associated to v. For all $y \in \Omega$, $\chi(\cdot, y)$ is defined on $(0, T)$ as the solution of the ordinary differential equation

$$\begin{cases} \chi_t(t, y) = v\big(t, \chi(t, y)\big),\\ \chi(0, y) = y. \end{cases} \tag{21}$$

In $\{y\colon \operatorname{dist}(y, \partial\Omega) < \alpha/4\}$, since $v = 0$, we have according to the uniqueness of the flow that $\chi(t, y) = y$. Analogously, if we take a point in the set $\{y\colon \operatorname{dist}(y, \partial\Omega_S(0)) < \alpha/4\}$, $\chi(t, y) = \chi_s(t, y) = a(t) + Q(t)y$. The last point comes from (19), (20) and (21). □

We define now a flow which transforms the moving domains into the initial one:

Corollary 2. *Under the same hypotheses as in the previous lemma, one can define an inverse flow ψ extending*

$$\chi_S^{-1}(t,x) = Q(t)^{-1}\bigl(x - a(t)\bigr), \quad t \in (0,T),\ x \in \Omega_S(t),$$

to the whole domain Ω and which satisfies:

- $\psi(t,x) = Q(t)^{-1}(x - a(t))$, $\forall x \in \Omega_S(t)$, $\forall t \in (0,T)$.
- $\psi(t,x) = x$, $\forall x \in \partial\Omega$, $\forall t \in (0,T)$.
- *The velocity $w := \psi_t$ associated to ψ belongs to $H^2(0,T;C^\infty(\Omega))$ and satisfies*

$$\begin{cases} w(t,x) = -Q(t)^{-1}(\dot a(t) + \omega \wedge (x - a(t))), & \forall x \in \Omega_S(t),\ \forall t \in (0,T), \\ w(t,x) = 0, & \forall x \in \partial\Omega,\ \forall t \in (0,T). \end{cases}$$

- *For $p \in \mathbb{N}$ there exists a constant $C_2 > 0$ just depending on p and Ω such that*

$$\bigl\|\psi_t(t,x)\bigr\|_{H^p(\Omega)} \leqslant C_2\bigl(\bigl|\dot a(t)\bigr|^2 + \bigl|\omega(t)\bigr|^2\bigr)^{1/2}, \quad \forall t \in (0,T). \tag{22}$$

1.2. Main result

Let us now introduce some notation which we will employ all along the paper. Let $h \in (0,+\infty]$ and $(\dot a, \omega) \in (H^2(0,h) \cap W^{1,\infty}(0,h))^2$ be given. First we define

$$Z_h := \bigl\{(s,x):\ s \in (0,h),\ x \in \Omega_F(s)\bigr\}, \qquad \Sigma_h := \bigl\{(s,x):\ s \in (0,h),\ x \in \partial\Omega_S(s)\bigr\}. \tag{23}$$

Then, for r, $p \geqslant 0$ natural numbers, we introduce

$$L_h^2\bigl(L^2\bigr) := L^2(Z_h) = \left\{u \text{ measurable} :\ \int_0^h \int_{\Omega_F(s)} |u|^2\,dx\,ds < +\infty\right\},$$

$$L_h^2\bigl(H^p\bigr) := \left\{u \in L_h^2\bigl(L^2\bigr):\ \int_0^h \|u\|^2_{H^p(\Omega_F(s))}\,ds < +\infty\right\},$$

$$H_h^r\bigl(H^p\bigr) := \left\{u \in L_h^2\bigl(L^2\bigr):\ \int_0^h \sum_{\beta=0}^r \bigl\|\partial_t^\beta u\bigr\|^2_{H^p(\Omega_F(s))}\,ds < +\infty\right\},$$

with associated norms given by the definition. On the other hand, we define

$$C_h^0\bigl(L^2\bigr) := \bigl\{u \text{ such that } \hat u(s,x) := u(s,x)1_{\Omega_F(s)} \in C^0\bigl([0,h];L^2(\Omega)\bigr)\bigr\}$$

and

$$C_h^r\bigl(H^p\bigr) := \bigl\{u:\ \partial_t^\beta \partial_x^\alpha u \in C_h^0\bigl(L^2\bigr),\ \forall 0 \leqslant \beta \leqslant r,\ \forall 0 \leqslant |\alpha| \leqslant p\bigr\}$$

with associated norms given by

$$\|u\|_{C_h^0(L^2)} := \max_{t\in(0,h)} \bigl\|u(t)\bigr\|_{L^2(\Omega_F(t))} = \max_{t\in(0,h)} \bigl\|\hat u(t)\bigr\|_{L^2(\Omega)}$$

and

$$\|u\|_{C_h^r(H^p)} := \sum_{\beta=0}^r \max_{t\in(0,h)} \bigl\|\partial_t^\beta u(t)\bigr\|_{H^p(\Omega_F(t))}.$$

It is clear that for regular functions, the norm $\|\cdot\|_{C_h^r(H^p)}$ coincides with the norm $\|\cdot\|_{W_h^{r,\infty}(H^p)}$. In the sequel, we will always use the second notation.

The main goal of our paper is to prove the following theorem:

Theorem 3. *Let $\bar\rho$ be the mean-value of ρ_0 in $\Omega_F(0)$. We suppose that* (1), (3) *and* (4) *are satisfied. We suppose that the initial conditions satisfy* (5), (14) *and the compatibility conditions* (15)–(16). *Then there exists a constant $\delta > 0$ such that, if*

$$\|\rho_0-\bar\rho\|_{H^3(\Omega_F(0))}+\|u_0\|_{H^3(\Omega_F(0))}+|a_0|+|\omega_0|<\delta, \tag{24}$$

the system of Eqs. (2), (11), (12) *and* (13) *admits a unique solution (ρ,u,a,ω) defined on $(0,T)$ for all T such that* (17) *is satisfied. Moreover, this solution belongs to the following space*

$$\rho\in L^2_T(H^3)\cap C^0_T(H^3)\cap H^1_T(H^2)\cap C^1_T(H^2)\cap H^2_T(L^2),$$
$$u\in L^2_T(H^4)\cap C^0_T(H^3)\cap C^1_T(H^1)\cap H^2_T(L^2),\quad \dot a\in H^2(0,T)\cap C^1([0,T]),\quad \omega\in H^2(0,T)\cap C^1([0,T])$$

and there exists a positive constant C_1 independent of T such that

$$\begin{aligned}&\|\rho-\bar\rho\|_{L^2_T(H^3)}+\|\rho-\bar\rho\|_{L^\infty_T(H^3)}+\|\rho-\bar\rho\|_{W^{1,\infty}_T(H^2)}+\|\rho-\bar\rho\|_{H^2_T(L^2)}+\|\rho-\bar\rho\|_{H^1_T(H^2)}\\&\quad+\|u\|_{L^2_T(H^4)}+\|u\|_{L^\infty_T(H^3)}+\|u\|_{W^{1,\infty}_T(H^1)}+\|u\|_{H^2_T(L^2)}+\|\dot a\|_{W^{1,\infty}_T}+\|\dot a\|_{H^2_T}+\|\omega\|_{W^{1,\infty}_T}+\|\omega\|_{H^2_T}\\&\quad\leqslant C_1\big(\|\rho_0-\bar\rho\|_{H^3(\Omega_F(0))}+\|u_0\|_{H^3(\Omega_F(0))}+|a_0|+|\omega_0|\big).\end{aligned} \tag{25}$$

Remark 4. In Theorem 3, we prove that for $\delta>0$ small enough depending on the initial distance $d(\Omega_S(0),\partial\Omega)$, there exists a unique regular global in time solution of the system of Eqs. (2), (11), (12) and (13) provided that the structure does not touch $\partial\Omega$ (see (17)). Observe that this condition will always be satisfied on an interval $(0,T_{\min})$, where $T_{\min}>0$ only depends on C_1, δ and $d(\partial\Omega,\overline{\Omega}_S(0))$, as easily seen from estimate (25).

Remark 5. This condition on T is natural for fluid-structure interaction problems (see, for instance, [7], [5] and [20]). For results concerning the existence of weak solutions of this type of systems after a collision has occurred, we refer to [19] and [10].

Remark 6. As proved in [18], there exists a unique stationary solution $(\rho_e(x),0,0,0)$ of (2), (12), where, on the fluid, the compressible Navier–Stokes equation is completed with a right-hand side of the form $f_i(x)=\partial_i\phi(x)$ $(i=1,2,3)$ satisfying (24) for the H^3-norm. Then, we can prove that Theorem 3 also holds with $\bar\rho$ replaced by ρ_e and for a right-hand side of the above kind.

2. Intermediate results

Let us recall that $\bar\rho$ is the mean-value of ρ_0:

$$\bar\rho=\frac{1}{V(\Omega_F(0))}\int_{\Omega_F(0)}\rho_0, \tag{26}$$

where $V(\Omega_F(0))$ stands for the volume of $\Omega_F(0)$.

Now, for $0\leqslant h\leqslant+\infty$, we define the space

$$\begin{aligned}X(0,h)=\big\{(\rho,u,a,\omega)\colon\ &(\rho,u)\in C^0_h(H^3)\cap L^2_h(H^3)\cap C^1_h(H^2)\cap H^1_h(H^2)\cap H^2_h(L^2)\\&\times\big(L^2_h(H^4)\cap C^0_h(H^3)\cap H^1_h(H^2)\cap C^1_h(H^1)\cap H^2_h(L^2)\big),\ (\dot a,\omega)\in\big(C^1_h\cap H^2_h\big)^2\big\}\end{aligned}$$

endowed with the following norm:

$$\begin{aligned}N_{0,h}(\rho,u,a,\omega)=\big(&\|\rho\|^2_{L^\infty_h(H^3)}+\|\rho\|^2_{L^2_h(H^3)}+\|\rho\|^2_{W^{1,\infty}_h(H^2)}+\|\rho\|^2_{H^1_h(H^2)}+\|\rho\|^2_{H^2_h(L^2)}+\|u\|^2_{L^2_h(H^4)}\\&+\|u\|^2_{L^\infty_h(H^3)}+\|u\|^2_{H^1_h(H^2)}+\|u\|^2_{W^{1,\infty}_h(H^1)}+\|u\|^2_{H^2_h(L^2)}+\|\dot a\|^2_{W^{1,\infty}_h}+\|\dot a\|^2_{H^2_h}\\&+\|\omega\|^2_{W^{1,\infty}_h}+\|\omega\|^2_{H^2_h}\big)^{1/2}.\end{aligned} \tag{27}$$

By abuse of notation, we will denote for each $0\leqslant h\leqslant+\infty$

$$X(h,h)=\{(\rho_h,u_h,a_h,\omega_h):\ \rho_h\in H^3(\Omega_F(h)),\ u_h\in H^3(\Omega_F(h)),$$
$$\times\big(L^2_h(H^4)\cap C^0_h(H^3)\cap H^1_h(H^2)\cap C^1_h(H^1)\cap H^2_h(L^2)\big),\ (\dot a,\omega)\in\big(C^1_h\cap H^2_h\big)^2\}.$$

The proof of Theorem 3 is divided in two steps: first, a local existence result and next a priori estimates for the system. A suitable combination of both will yield the desired global result.

Thus, we first formulate a local existence and uniqueness result for small initial data.

Proposition 7. *Let* (1) *be satisfied and let* $h\geqslant 0$:

- *For* $h=0$, *let* $(\rho_0,u_0,a_0,\omega_0)$ *satisfy* (14)–(16).
- *For* $h>0$, *we suppose that* (2), (11), (12) *and* (13) *admits a unique solution* (ρ,u,a,ω) *in* $X(0,h)$ *satisfying* (17) *for some* $\alpha>0$ *independent of* h *and for* T *replaced by* h.

Then, there exist constants δ_0, $\tau>0$ *and* $C_2>0$ *independent of* h (*for* $h=0$, *we mean independently of the initial condition*) *such that, if* $N_{h,h}(\rho-\bar\rho,u,a,\omega)\leqslant\delta_0$, *problem* (61) *has a unique solution* (ρ,u,a,ω) *defined on* $(h,h+\tau)$ *satisfying*

$$(\rho,u,a,\omega)\in X(h,h+\tau),\quad N_{h,h+\tau}(\rho-\bar\rho,u,a,\omega)\leqslant C_2N_{h,h}(\rho-\bar\rho,u,a,\omega).$$

We present now the a priori estimates:

Proposition 8. *We suppose that there exists* $T>0$ *such that the problem* (2), (11), (12) *and* (13) *admits a solution* (ρ,u,a,ω) *in* $X(0,T)$. *Then there exist constants* $0<\delta_1\leqslant\delta_0$ *and* $C_3>0$ *independent of* T *such that if* $N_{0,T}(\rho-\bar\rho,u,a,\omega)\leqslant\delta_1$ *then*

$$N_{0,T}(\rho-\bar\rho,u,a,\omega)\leqslant C_3N_{0,0}(\rho-\bar\rho,u,a,\omega).$$

These two propositions will be proved in the next sections. They allow to prove Theorem 3 in the following way:

Proof of Theorem 3. We will apply iteratively Propositions 7 and 8. We suppose that

$$N_{0,0}(\rho-\bar\rho,u,a,\omega)\leqslant\min\left(\delta_0,\frac{\delta_1}{C_2},\frac{\delta_1}{C_3\sqrt{1+C_2^2}}\right).$$

According to Proposition 7 for $h=0$, one can define on $(0,\tau)$ a solution $(\rho^*,u,a,\omega)\in X(0,\tau)$ such that

$$N_{0,\tau}(\rho-\bar\rho,u,a,\omega)\leqslant C_2N_{0,0}(\rho-\bar\rho,u,a,\omega)\leqslant\delta_1\leqslant\delta_0.$$

Since $N_{\tau,\tau}(\rho-\bar\rho,u,a,\omega)\leqslant N_{0,\tau}(\rho-\bar\rho,u,a,\omega)$, we can apply again Proposition 7. Our solution can be extended on $(\tau,2\tau)$ and $N_{\tau,2\tau}(\rho-\bar\rho,u,a,\omega)\leqslant C_2N_{0,\tau}(\rho-\bar\rho,u,a,\omega)$. Thus

$$N^2_{0,2\tau}(\rho-\bar\rho,u,a,\omega)=\big(N^2_{0,\tau}+N^2_{\tau,2\tau}\big)(\rho-\bar\rho,u,a,\omega)\leqslant\big(1+C_2^2\big)N^2_{0,\tau}(\rho-\bar\rho,u,a,\omega).$$

Thanks to Proposition 8 for $T=\tau$ and the choice of $N_{0,0}(\rho-\bar\rho,u,a,\omega)$, we deduce

$$N_{0,2\tau}(\rho-\bar\rho,u,a,\omega)\leqslant C_3\sqrt{1+C_2^2}\,N_{0,0}(\rho-\bar\rho,u,a,\omega)\leqslant\delta_1.$$

This allows to repeat this process and obtain the existence of a regular solution as long as (17) is satisfied. □

Now, we state a result which will be useful in our analysis:

Lemma 9. *Let* $t>0$. *There exists* $C>0$ *independent of* t *such that for all* u *with* $\epsilon(u)\in L^2(\Omega_F(t))$ *and* $u\in L^2(\Omega_F(t))$, *we have that* $u\in H^1(\Omega_F(t))$ *and*

$$\|u\|_{H^1(\Omega_F(t))}\leqslant C\big(\|\epsilon(u)\|_{L^2(\Omega_F(t))}+\|u\|_{L^2(\Omega_F(t))}\big).\tag{28}$$

Consequently, if $u_{|\partial\Omega}=0$, *we have that*

$$\|u\|_{H^1(\Omega_F(t))}\leqslant C\|\epsilon(u)\|_{L^2(\Omega_F(t))},\tag{29}$$

for $C>0$ *independent of* t.

Proof. All along this proof, C will stand for a positive constant independent of t. For the first inequality, we use that

$$\partial_{x_j x_k} u(t,x) = \frac{1}{2}\big(\partial_{x_j}\epsilon(u)_{ik} + \partial_{x_k}\epsilon(u)_{ij} - \partial_{x_i}\epsilon(u)_{jk}\big).$$

First, observe that

$$\|v\|_{L^2(\Omega_F(t))} \leqslant C\|\nabla v\|_{H^{-1}(\Omega_F(t))},$$

where C only depends on the size of $\Omega_F(t) \subset \Omega$. We apply this inequality for $v := \nabla u$ and so we deduce that $\nabla u \in L^2(\Omega_F(t))$ and

$$\|\nabla u\|_{L^2(\Omega_F(t))} \leqslant C\big\|\epsilon(u)\big\|_{L^2(\Omega_F(t))}.$$

Then, inequality (28) readily follows. In order to prove (29) we observe that, thanks to $u_{|\partial\Omega} = 0$, we have that

$$\|u\|_{L^2(\Omega_F(t))} \leqslant C\|\nabla u\|_{L^2(\Omega_F(t))}.$$

Here, C depends on the regularity of $\Omega_F(t)$ (which is that of $\Omega_F(0)$) and the size of Ω. □

3. A local existence result: proof of Proposition 7

This section is devoted to the proof of Proposition 7. We only consider the case $h = 0$; the case $h > 0$ follows directly from the arguments below.

3.1. Statement of the problem in a fixed domain

The system of equations can be written on the reference domains $\Omega_S(0)$ and $\Omega_F(0)$ with the help of the flow defined by Lemma 1. We consider (ρ, u, a, ω) which satisfies the system (2)-(11)-(12) with the hypothesis (3)–(4). We suppose that (17) is satisfied for some $\alpha > 0$ and that the functions ρ, u, a and ω are regular enough. Let us define the functions $\tilde{u}$, $\tilde{\rho}$ and $\tilde{p}$ on $V_T := (0,T) \times \Omega_F(0)$ by

$$\tilde{u}(t,y) = u\big(t,\chi(t,y)\big), \quad \tilde{\rho}(t,y) = \rho\big(t,\chi(t,y)\big) - \bar{\rho}, \quad \tilde{p}(t,y) = P\big(\tilde{\rho}(t,y) + \bar{\rho}\big), \quad \forall (t,y) \in V_T. \tag{30}$$

We have the following formulas, $\forall (t,y) \in V_T$,

$$\tilde{\rho}_t(t,y) = \rho_t\big(t,\chi(t,y)\big) + \big(\big((\nabla\chi)^{-1}\chi_t\big)\cdot\nabla\tilde{\rho}\big)(t,y),$$
$$(\nabla\tilde{u})_{ij}(t,y) = \partial_{y_j}\tilde{u}_i = \big(\nabla u\big(t,\chi(t,y)\big)\nabla\chi(t,y)\big)_{ij}.$$

Thus we get, for the first equation of system (2),

$$\tilde{\rho}_t + \big((\nabla\chi)^{-1}(\tilde{u} - \chi_t)\big)\cdot\nabla\tilde{\rho} + (\tilde{\rho} + \bar{\rho})\,\mathrm{tr}\big(\nabla\tilde{u}(\nabla\chi)^{-1}\big) = 0, \quad \text{in } V_T. \tag{31}$$

Next, the second equation of (2) becomes, $\forall i = 1,2,3$, in V_T

$$\begin{aligned}(\tilde{\rho} + \bar{\rho})(\tilde{u}_i)_t &+ (\tilde{\rho} + \bar{\rho})(\tilde{u}_j - \chi_{j,t})\big(\nabla\tilde{u}(\nabla\chi)^{-1}\big)_{ij} - \mu\partial_{y_l}\big(\partial_{y_k}\tilde{u}_i(\nabla\chi)^{-1}_{kj}\big)(\nabla\chi)^{-1}_{lj}\\ &- (\mu + \mu')\partial_{y_l}\big(\partial_{y_k}\tilde{u}_j(\nabla\chi)^{-1}_{kj}\big)(\nabla\chi)^{-1}_{li} + (\nabla\chi)^{-1}_{ki}\partial_{y_k}\tilde{p} = 0.\end{aligned} \tag{32}$$

In this equation and in what follows, we implicitly sum over repeated indexes. Eqs. (11) become:

$$\begin{cases} m\ddot{a} = \int_{\partial\Omega_S(0)}(\mu(\nabla\tilde{u}Q^t + Q(\nabla\tilde{u})^t) + \mu'\,\mathrm{tr}(\nabla\tilde{u}Q^t)\,\mathrm{Id} - \tilde{p}\,\mathrm{Id})Qn\,d\gamma,\\ J\dot{\omega} = (J\omega)\wedge\omega + \int_{\partial\Omega_S(0)}(Qy)\wedge((\mu(\nabla\tilde{u}Q^t + Q(\nabla\tilde{u})^t) + \mu'\,\mathrm{tr}(\nabla\tilde{u}Q^t)\,\mathrm{Id} - \tilde{p}\,\mathrm{Id})Qn)\,d\gamma,\end{cases} \tag{33}$$

where we have denoted $A^{-t} = (A^t)^{-1}$ At last, the boundary conditions (12) become

$$\begin{cases} \tilde{u}(t,y) = 0, \quad \forall y \in \partial\Omega,\\ \tilde{u}(t,y) = \dot{a}(t) + \omega(t)\wedge(Qy), \quad \forall y \in \partial\Omega_S(0).\end{cases} \tag{34}$$

We denote (m, J, μ, μ') instead of $(m/\bar\rho, J/\bar\rho, \mu/\bar\rho, \mu'/\bar\rho)$ and we rewrite the resulting system as follows:

$$\begin{cases} \tilde\rho_t + ((\nabla\chi)^{-1}(\tilde u - \chi_t))\cdot\nabla\tilde\rho + \bar\rho\nabla\cdot\tilde u = g_0(\tilde\rho,\tilde u, a,\omega) & \text{in } V_T,\\ \tilde u_t - \nabla\cdot\sigma(\tilde u,\tilde\rho) = g_1(\tilde\rho,\tilde u,a,\omega) & \text{in } V_T,\\ m\ddot a = \int_{\partial\Omega_S(0)}\sigma(\tilde u,\tilde\rho)n\,d\gamma + g_2(\tilde\rho,\tilde u,a,\omega) & \text{in } (0,T),\\ J\dot\omega = \int_{\partial\Omega_S(0)}(Qy)\wedge(\sigma(\tilde u,\tilde\rho)n)\,d\gamma + g_3(\tilde\rho,\tilde u,a,\omega) & \text{in } (0,T),\\ \tilde u = 0 & \text{on } (0,T)\times\partial\Omega,\\ \tilde u = \dot a + \omega\wedge(Qy) & \text{on } (0,T)\times\partial\Omega_S(0),\\ \tilde\rho(0,\cdot) = \rho_0 - \bar\rho,\ \tilde u(0,\cdot) = u_0 & \text{in } \Omega_F(0),\\ a(0) = 0,\quad \dot a(0) = a_0,\quad \omega(0) = \omega_0, \end{cases} \tag{35}$$

with

$$g_0(\tilde\rho,\tilde u,a,\omega) = \bar\rho\,\mathrm{tr}\big(\nabla\tilde u\big(\mathrm{Id} - (\nabla\chi)^{-1}\big)\big) - \tilde\rho\,\mathrm{tr}\big(\nabla\tilde u(\nabla\chi)^{-1}\big), \tag{36}$$

$$\begin{aligned} (g_1)_i(\tilde\rho,\tilde u,a,\omega) = {} & -(\tilde u_j - \chi_{j,t})\big(\nabla\tilde u(\nabla\chi)^{-1}\big)_{ij} + \mu\left(\frac{\bar\rho}{\tilde\rho+\bar\rho} - 1\right)\partial_{y_l}\big(\partial_{y_k}\tilde u_i(\nabla\chi)^{-1}_{kj}\big)(\nabla\chi)^{-1}_{lj}\\ & + \mu\big[\partial_{y_l}\big(\partial_{y_k}\tilde u_i\big((\nabla\chi)^{-1}_{kj} - \delta_{kj}\big)\big)(\nabla\chi)^{-1}_{lj} + \partial^2_{y_l y_j}\tilde u_i\big((\nabla\chi)^{-1}_{lj} - \delta_{lj}\big)\big]\\ & + (\mu+\mu')\left(\frac{\bar\rho}{\tilde\rho+\bar\rho} - 1\right)\partial_{y_l}\big(\partial_{y_k}\tilde u_j(\nabla\chi)^{-1}_{kj}\big)(\nabla\chi)^{-1}_{li}\\ & + (\mu+\mu')\big[\partial_{y_l}\big(\partial_{y_k}\tilde u_j\big((\nabla\chi)^{-1}_{kj} - \delta_{kj}\big)\big)(\nabla\chi)^{-1}_{li} + \partial^2_{y_l y_j}\tilde u_j\big((\nabla\chi)^{-1}_{li} - \delta_{li}\big)\big]\\ & - \big((\nabla\chi)^{-1}_{ki} - \delta_{ki}\big)\frac{P'(\tilde\rho+\bar\rho)\partial_{y_k}\tilde\rho}{\tilde\rho+\bar\rho} - \left(\frac{P'(\tilde\rho+\bar\rho)}{\tilde\rho+\bar\rho} - \frac{P'(\bar\rho)}{\bar\rho}\right)\partial_{y_i}\tilde\rho, \end{aligned} \tag{37}$$

$$\begin{aligned} g_2(\tilde\rho,\tilde u,a,\omega) = {} & \mu\int_{\partial\Omega_S(0)}\big[(Q-\mathrm{Id})(\nabla\tilde u)^t Q + (\nabla\tilde u)^t(Q-\mathrm{Id})\big]n\,d\gamma\\ & + \mu'\int_{\partial\Omega_S(0)}\big[\mathrm{tr}\big(\nabla\tilde u\big(Q^t - \mathrm{Id}\big)\big)Q + (\nabla\cdot\tilde u)(Q-\mathrm{Id})\big]n\,d\gamma\\ & - \int_{\partial\Omega_S(0)}\frac{P(\tilde\rho+\bar\rho) - P(\bar\rho)}{\bar\rho}(Q-\mathrm{Id})n\,d\gamma - \int_{\partial\Omega_S(0)}\big(P(\tilde\rho+\bar\rho) - P(\bar\rho) - P'(\bar\rho)\tilde\rho\big)\frac{n}{\bar\rho}\,d\gamma \end{aligned} \tag{38}$$

and

$$\begin{aligned} g_3(\tilde\rho,\tilde u,a,\omega) = {} & \mu\int_{\partial\Omega_S(0)}(Qy)\wedge\big[(Q-\mathrm{Id})(\nabla\tilde u)^t Q + (\nabla\tilde u)^t(Q-\mathrm{Id})\big]n\,d\gamma\\ & + \mu'\int_{\partial\Omega_S(0)}(Qy)\wedge\big[\mathrm{tr}\big(\nabla\tilde u\big(Q^t - \mathrm{Id}\big)\big)Q + \mathrm{tr}(\nabla\tilde u)(Q-\mathrm{Id})\big]n\,d\gamma\\ & + \int_{\partial\Omega_S(0)}\big[(Qy)\wedge\big(p^0\tilde\rho(\mathrm{Id} - Q)n\big)\big]\,d\gamma\\ & + \int_{\partial\Omega_S(0)}\big(P(\bar\rho) + P'(\bar\rho)\tilde\rho - P(\tilde\rho+\bar\rho)\big)(Qy)\wedge Q\frac{n}{\bar\rho}\,d\gamma + (J\omega)\wedge\omega. \end{aligned} \tag{39}$$

Here, we have used (64). Observe that the functions g_i are (at least) quadratic functions of the quantities $\tilde\rho, \tilde u$, $(\nabla\chi)^{-1} - \mathrm{Id}$, $Q - \mathrm{Id}$, χ_t.

3.2. Definition of the fixed point mapping

Let $0 < \tilde{R} < 1$ and $0 < s < 1$ be small enough. We define the space $\tilde{Y}((0,s);\tilde{R})$:

$$\tilde{Y}\big((0,s);\tilde{R}\big) = \big\{(\tilde{\rho}, \tilde{u}_F, a, \omega) \in \tilde{X}(0,s):\ \tilde{u}_F = 0 \text{ on } \partial\Omega_F(0),\ a(0)=0,\ Q(0)=\mathrm{Id},\ \tilde{N}_{0,s}(\tilde{\rho}, \tilde{u}_F, a, \omega) \leqslant \tilde{R}\big\},$$

where

$$\tilde{X}(0,s) = \big\{(\tilde{\rho}, \tilde{u}, a, \omega):\ \tilde{\rho} \in C^0_s\big(H^3\big(\Omega_F(0)\big)\big) \cap C^1_s\big(H^2\big(\Omega_F(0)\big)\big) \cap H^2_s\big(L^2\big(\Omega_F(0)\big)\big),$$
$$\tilde{u} \in L^2_s\big(H^4\big(\Omega_F(0)\big)\big) \cap H^2_s\big(L^2\big(\Omega_F(0)\big)\big),\ a \in H^3_s,\ \omega \in H^2_s\big\}$$

and $\tilde{N}_{0,s}(\tilde{\rho}, \tilde{u}_F, a, \omega)$ is given by the corresponding norm. In this definition, Q is the rotation matrix associated to ω.

Remark 10. The definition of the space $\tilde{Y}((0,s);\tilde{R})$ coincides with

$$Y\big((0,s);R\big) = \big\{\big(\tilde{\rho}\circ\chi^{-1}, \tilde{u}_F\circ\chi^{-1}, a, \omega\big) \in X(0,s)/a(0)=0,\ Q(0)=\mathrm{Id},$$
$$N_{0,s}\big(\tilde{\rho}\circ\chi^{-1}, \tilde{u}_F\circ\chi^{-1}, a, \omega\big) \leqslant R\big\}.$$

This comes from the identities, for all $(t,y) \in (0,s)\times\Omega_F(0)$, if $x=\chi(t,y)$,

$$\partial_t\tilde{u}(t,y) = \partial_t u(t,x) + \big(\chi_t(t,y)\cdot\nabla\big)u(t,x)$$

and

$$\nabla\tilde{u}(t,y) = \nabla u(t,x)\nabla\chi(t,y).$$

In order to prove Proposition 7, we will perform a fixed point argument in the space $\tilde{Y}((0,s);\tilde{R})$. Indeed, we consider the mapping

$$\Lambda : \tilde{Y}\big((0,s);\tilde{R}\big) \to \tilde{Y}\big((0,s);\tilde{R}\big),$$
$$(\hat{\rho}, \hat{u}_F, \hat{a}, \hat{\omega}) \to (\tilde{\rho}, \tilde{u}_F, a, \omega),$$

with $\tilde{u}_F = \tilde{u} - \chi_t$, where $(\tilde{\rho}, \tilde{u}, a, \omega)$ is the solution of

$$\begin{cases} \tilde{\rho}_t + ((\nabla\hat{\chi})^{-1}(\hat{u}-\hat{\chi}_t))\cdot\nabla\tilde{\rho} = g_0(\hat{\rho},\hat{u},\hat{a},\hat{\omega}) - \bar{\rho}\nabla\cdot\tilde{u} & \text{in } V_T,\\ \tilde{u}_t - \nabla\cdot(2\mu\epsilon(\tilde{u}) + \mu'(\nabla\cdot\tilde{u})\,\mathrm{Id}) = g_1(\hat{\rho},\hat{u},\hat{a},\hat{\omega}) - p^0\nabla\tilde{\rho} & \text{in } V_T,\\ m\ddot{a} = \int_{\partial\Omega_S(0)}\sigma(\tilde{u},\tilde{\rho})n\,d\gamma + g_2(\hat{\rho},\hat{u},\hat{a},\hat{\omega}) & \text{in } (0,T),\\ \hat{J}\dot{\omega} = \int_{\partial\Omega_S(0)}(\hat{Q}y)\wedge(\sigma(\tilde{u},\tilde{\rho})n)\,d\gamma + g_3(\hat{\rho},\hat{u},\hat{a},\hat{\omega}) & \text{in } (0,T),\\ \tilde{u} = 0 & \text{on } (0,T)\times\partial\Omega,\\ \tilde{u} = \dot{a} + \omega\wedge(\hat{Q}y) & \text{on } (0,T)\times\partial\Omega_S(0),\\ \tilde{\rho}(0,\cdot) = \rho_0 - \bar{\rho}, \quad \tilde{u}(0,\cdot) = u_0 & \text{in } \Omega_F(0),\\ a(0)=0, \quad \dot{a}(0)=a_0, \quad \omega(0)=\omega_0. \end{cases} \tag{40}$$

Here, $\hat{u} = \hat{u}_F + \hat{\chi}_t$ and $\hat{J}$ is defined by (9) where we replace Q by $\hat{Q}$.

Thanks to Remark 10, if we prove the existence of a fixed point of Λ, the proof of Proposition 7 will be achieved.

3.3. Λ is well defined

Let $(\hat{\rho}, \hat{u}, \hat{a}, \hat{\omega})$ be given in $\tilde{Y}((0,s);\tilde{R})$. Our goal is to prove that the solution of (40) $(\tilde{\rho}, \tilde{u}, a, \omega)$ belongs to $\tilde{Y}((0,s);\tilde{R})$. In order to prove this, we will establish some estimates for $\tilde{\rho}$ regarded as the solution of a transport equation (with right-hand side $g_0 - \bar{\rho}(\nabla\cdot\tilde{u})$) and for $\tilde{u}$ as solution of a heat equation (with right-hand side $g_1 - p^0\nabla\tilde{\rho}$).

In the sequel, we will use the following estimates, coming from Lemma 1:

$$\big\|(\hat{\rho},\hat{u},\hat{a},\hat{\omega})\big\|_{\tilde{X}(0,s)} + \big\|(\nabla\hat{\chi})^{-1} - \mathrm{Id}\big\|_{H^3_s(C^\infty(\overline{\Omega_F(0)}))} + \|\hat{Q} - \mathrm{Id}\|_{H^3_s} + \|\hat{\chi}_t\|_{H^2_s(C^\infty(\overline{\Omega_F(0)}))} \leqslant C\tilde{R}.$$

Taking $\tilde{R} > 0$ small enough, we can suppose that $|\hat{\rho}| < \bar{\rho}/2$ and so every single term of the expression of g_1 (see (37)) makes sense. Then, one can prove that

$$\begin{aligned}&\|g_0(\hat\rho,\hat u,\hat a,\hat\omega)\|_{L^2_s(H^3(\Omega_F(0)))\cap H^1_s(H^1(\Omega_F(0)))}+\|g_1(\hat\rho,\hat u,\hat a,\hat\omega)\|_{L^2_s(H^2(\Omega_F(0)))\cap H^1_s(L^2(\Omega_F(0)))}\\&\quad+\|g_2(\hat\rho,\hat u,\hat a,\hat\omega)\|_{H^1_s}+\|g_3(\hat\rho,\hat u,\hat a,\hat\omega)\|_{H^1_s}\leqslant C\tilde R^2.\end{aligned}\tag{41}$$

Estimates for $\tilde\rho$. Using standard arguments (observe that $\hat u-\hat\chi_t$ has null trace), we can prove using (41) that

$$\begin{aligned}\|\tilde\rho\|_{L^\infty_s(H^3(\Omega_F(0)))}&\leqslant C\big(\|g_0(\hat\rho,\hat u,\hat a,\hat\omega)\|_{L^2_s(H^3(\Omega_F(0)))}+\|\rho_0-\bar\rho\|_{H^3(\Omega_F(0))}+\|\tilde u\|_{L^2_s(H^4(\Omega_F(0)))}\big)\\&\leqslant C\big(\tilde R^2+\|\rho_0-\bar\rho\|_{H^3(\Omega_F(0))}+\tilde N_{0,s}(0,\tilde u,0,0)\big).\end{aligned}\tag{42}$$

Moreover, using the equation of $\tilde\rho$ and (42), we obtain

$$\begin{aligned}&\|\tilde\rho\|_{W^{1,\infty}_s(H^2(\Omega_F(0)))}+\|\tilde\rho\|_{H^2_s(L^2(\Omega_F(0)))}\\&\quad\leqslant C\big(\|g_0(\hat\rho,\hat u,\hat a,\hat\omega)\|_{L^\infty_s(H^2(\Omega_F(0)))\cap H^1_s(L^2(\Omega_F(0)))}+\|\tilde u\|_{L^\infty_s(H^3(\Omega_F(0)))\cap H^1_s(H^1(\Omega_F(0)))}\\&\quad\leqslant C\big(\tilde R^2+\|\rho_0-\bar\rho\|_{H^3(\Omega_F(0))}+\tilde N_{0,s}(0,\tilde u,0,0)\big).\end{aligned}$$

Combining both, we deduce

$$\tilde N_{0,s}(\tilde\rho,0,0,0)\leqslant C\big(\tilde R^2+\|\rho_0-\bar\rho\|_{H^3(\Omega_F(0))}+\tilde N_{0,s}(0,\tilde u,0,0)\big).\tag{43}$$

Estimates for $\tilde u$*, a and* ω. In the rest of this subsection, we denote g_i instead of $g_i(\hat\rho,\hat u,\hat a,\hat\omega)$ for $i=1,2,3$.

• First, we multiply the equation of $\tilde u$ by $\tilde u$, we integrate on V_r for all $r\in(0,s)$ and we integrate by parts. Taking the supremum in r, this gives

$$\begin{aligned}&\frac12\sup_{r\in(0,s)}\Big(|\dot a|^2(r)+|\omega|^2(r)+\int_{\Omega_F(0)}|\tilde u|^2(r)\,dy\Big)+\iint_{V_s}\big(2\mu|\epsilon(\tilde u)|^2+\mu'|\nabla\cdot\tilde u|^2\big)\,dy\,dr\\&\quad\leqslant C\Big(\iint_{V_s}|\tilde u|^2\,dy\,dr+\int_0^s|\omega|^2\,dr+\iint_{V_s}\big(|g_1|^2+|\nabla\tilde\rho|^2\big)\,dy\,dr+\int_{\Omega_F(0)}|u_0|^2\,dx+|a_0|^2+|\omega_0|^2\\&\qquad+\Big(\sup_{r\in(0,s)}|\dot a|\Big)\Big(\int_0^s|g_2|\,dr+\int_0^s\int_{\partial\Omega_S(0)}|\tilde\rho|\,d\gamma\,dr\Big)+\Big(\sup_{r\in(0,s)}|\omega|\Big)\Big(\int_0^s|g_3|\,dr+\int_0^s\int_{\partial\Omega_S(0)}|\tilde\rho|\,d\gamma\,dr\Big)\Big).\end{aligned}\tag{44}$$

Here, we have used that $\hat J$ is coercive (see 10) and satisfies $\|\hat J\|_{L^\infty_t}\leqslant C$.

Thus, using Lemma 9, estimate (43) on $\tilde\rho$ and (41), we deduce that there exists $C>0$ such that

$$\begin{aligned}&\|\tilde u\|_{L^\infty_s(L^2(\Omega_F(0)))}+\|\tilde u\|_{L^2_s(H^1(\Omega_F(0)))}+\|a\|_{W^{1,\infty}_s}+\|\omega\|_{L^\infty_s}\\&\quad\leqslant C\big(\tilde R^2+\|\rho_0-\bar\rho\|_{H^3(\Omega_F(0))}+s^{1/2}\tilde N_{0,s}(0,\tilde u,0,0)+\|u_0\|_{L^2}+|a_0|+|\omega_0|\big).\end{aligned}\tag{45}$$

• Let us multiply the equation of $\tilde u$ by $\tilde u_t$. Arguing as before, this yields

$$\begin{aligned}&\frac12\sup_{r\in(0,s)}\Big(\int_{\Omega_F(0)}\big(2\mu|\epsilon(\tilde u)|^2(r)+\mu'|\nabla\cdot\tilde u|^2(r)\big)\,dy\Big)\\&\quad+\iint_{V_s}|\tilde u_t|^2\,dy\,dr+\int_0^s\int_{\partial\Omega_S(0)}\big(\big(2\mu\epsilon(\tilde u)+\mu'(\nabla\cdot\tilde u)\,\mathrm{Id}\big)n\big)\cdot\tilde u_t\,d\gamma\,dr\\&\quad\leqslant C\Big(\iint_{V_s}\big(|g_1|^2+|\nabla\tilde\rho|^2\big)\,dy\,dr+\int_{\Omega_F(0)}|\nabla u_0|^2\,dy\Big).\end{aligned}\tag{46}$$

Using the boundary condition of $\tilde u$ on $\partial\Omega_S(0)$, we obtain the following for the boundary term:

$$\int_0^s \int_{\partial\Omega_S(0)} \big(\big(2\mu\epsilon(\tilde u)+\mu'(\nabla\cdot\tilde u)\,\mathrm{Id}\big)n\big)\cdot\tilde u_t\,d\gamma\,dr$$
$$=\int_0^s \int_{\partial\Omega_S(0)} \big(\big(2\mu\epsilon(\tilde u)+\mu'(\nabla\cdot\tilde u)\,\mathrm{Id}\big)n\big)\cdot\big(\ddot a+\dot\omega\wedge(\hat Q y)+\omega\wedge(\dot{\hat Q} y)\big)\,d\gamma\,dr.$$

For the two first terms we use the equations of a and ω in (40). This produces the norms $\|\ddot a\|^2_{L^2_s}$ and $\|\dot\omega\|^2_{L^2_s}$. For the third term, we have

$$\left|\int_0^s \int_{\partial\Omega_S(0)} \big(\big(2\mu\epsilon(\tilde u)+\mu'(\nabla\cdot\tilde u)\,\mathrm{Id}\big)n\big)\cdot(\omega\wedge\dot{\hat Q} y)\,d\gamma\,dr\right| \leqslant C\tilde R\|\omega\|_{L^2_s}\|\tilde u\|_{L^2_s(H^2(\Omega_F(0)))}$$
$$\leqslant \tilde R^2\|\tilde u\|^2_{L^2_s(H^2(\Omega_F(0)))}+C\|\omega\|^2_{L^2_s}.$$

Using (45) to estimate $\|\omega\|^2_{L^2_s}$, (41) and (43), we obtain from (46)

$$\begin{aligned}&\|\tilde u\|_{H^1_s(L^2(\Omega_F(0)))}+\|\tilde u\|_{L^\infty_s(H^1(\Omega_F(0)))}+\|a\|_{H^2_s}+\|\omega\|_{H^1_s}\leqslant \tilde R\|\tilde u\|_{L^2_s(H^2(\Omega_F(0)))}\\&\quad+C\big(\tilde R^2+\|\rho_0-\bar\rho\|_{H^3(\Omega_F(0))}+s^{1/2}\tilde N_{0,s}(0,\tilde u,0,0)+\|u_0\|_{H^1(\Omega_F(0))}+|a_0|+|\omega_0|\big).\end{aligned}\tag{47}$$

Now, we regard the equation of $\tilde u$ as a stationary system:

$$\begin{cases}-\mu\Delta\tilde u-(\mu+\mu')\nabla(\nabla\cdot\tilde u)=g_1-p^0\nabla\tilde\rho-\tilde u_t & \text{in } \Omega_F(0),\\ \tilde u=(\dot a+\omega\wedge(\hat Q y))1_{\partial\Omega_S(0)} & \text{on } \partial\Omega_F(0).\end{cases}\tag{48}$$

The solution of this system belongs to H^2 and we have

$$\|\tilde u\|_{H^2(\Omega_F(0))}\leqslant C\big(\|g_1\|_{L^2(\Omega_F(0))}+\|\tilde\rho\|_{H^1(\Omega_F(0))}+\|\tilde u_t\|_{L^2(\Omega_F(0))}+|\dot a|+|\omega|\big).$$

Taking here the L^2_s norm and using (45) to estimate $\|\dot a\|_{L^2_s}+\|\omega\|_{L^2_s}$ and (42) to estimate $\|\tilde\rho\|_{L^2_s(H^1(\Omega_F(0)))}$, we obtain from (47)

$$\begin{aligned}&\|\tilde u\|_{H^1_s(L^2(\Omega_F(0)))}+\|\tilde u\|_{L^2_s(H^2(\Omega_F(0)))}+\|\tilde u\|_{L^\infty_s(H^1(\Omega_F(0)))}+\|a\|_{H^2_s}+\|\omega\|_{H^1_s}\\&\quad\leqslant C\big(\tilde R^2+\|\rho_0-\bar\rho\|_{H^3(\Omega_F(0))}+s^{1/2}\tilde N_{0,s}(0,\tilde u,0,0)+\|u_0\|_{H^1(\Omega_F(0))}+|a_0|+|\omega_0|\big)\end{aligned}\tag{49}$$

(recall that $R>0$ is small enough).

- If we differentiate with respect to time the system satisfied by $(\tilde u,a,\omega)$, we obtain

$$\begin{cases}\tilde u_{tt}-2\mu\nabla\cdot(\epsilon(\tilde u_t))-\mu'\nabla\cdot((\nabla\cdot\tilde u_t)\,\mathrm{Id})=g_{1,t}-p^0\nabla\tilde\rho_t & \text{in } V_T,\\ m\dddot a=\int_{\partial\Omega_S(0)}(2\mu\epsilon(\tilde u_t)+\mu'(\nabla\cdot\tilde u_t)\,\mathrm{Id}-p^0\tilde\rho_t\,\mathrm{Id})n\,d\gamma+g_{2,t} & \text{in } (0,T),\\ \hat J\ddot\omega=-\dot{\hat J}\dot\omega+\int_{\partial\Omega_S(0)}(\hat Q x)\wedge((2\mu\epsilon(\tilde u_t)+(\mu'\nabla\cdot\tilde u_t-p^0\tilde\rho_t)\,\mathrm{Id})n)\,d\gamma & \\ \qquad+\int_{\partial\Omega_S(0)}(\dot{\hat Q} x)\wedge((2\mu\epsilon(\tilde u)+\mu'(\nabla\cdot\tilde u)\,\mathrm{Id}-p^0\tilde\rho\,\mathrm{Id})n)\,d\gamma+g_{3,t} & \text{in } (0,T),\\ \tilde u_t=0 & \text{on } (0,T)\times\partial\Omega,\\ \tilde u_t=\ddot a+\dot\omega\wedge(\hat Q y)+\omega\wedge(\dot{\hat Q} y) & \text{on } (0,T)\times\partial\Omega_S(0).\end{cases}\tag{50}$$

Let us multiply the first equation by $\tilde u_t$ and integrate in space and in time as before. Doing similar computations to those of the previous case, we obtain

$$\sup_{r\in(0,s)}\left(|\ddot a|^2(r)+|\dot\omega|^2(r)+\int_{\Omega_F(0)}|\tilde u_t|^2(r)\,dy\right)+\iint_{Q_s}\big(2\mu\big|\epsilon(\tilde u_t)\big|^2+\mu'|\nabla\cdot\tilde u_t|^2\big)\,dy\,dr$$
$$\leqslant C\left(\iint_{Q_s}\big(|g_{1,t}|^2+|\tilde u_t|^2+|\nabla\tilde\rho_t|^2\big)\,dy\,dr+\int_{\Omega_F(0)}\big|\tilde u_t(0)\big|^2\,dy+\big|\ddot a(0)\big|^2+\big|\dot\omega(0)\big|^2\right.$$

$$+\int_0^s \big(|\omega|^2+|\dot\omega|^2\big)\,dr+\|u\|^2_{L^2_s(H^2(\Omega_F(0)))}+\Big(\sup_{r\in(0,s)}|\ddot a|\Big)\Bigg(\int_0^s |g_{2,t}|\,dr+\int_0^s\int_{\partial\Omega_S(0)}|\tilde\rho_t|\,d\sigma\,dr\Bigg)$$
$$+\Big(\sup_{r\in(0,s)}|\dot\omega|\Big)\Bigg(\int_0^s |g_{3,t}|\,dr+\int_0^s\int_{\partial\Omega_S(0)}\big(|\tilde\rho|+|\tilde\rho_t|\big)\,d\sigma\,dr\Bigg)\Bigg)+\tilde R^2\|\tilde u_t\|^2_{L^2_s(H^2(\Omega_F(0)))}.$$

The last term can be bounded by $\tilde R^2\tilde N^2_{0,s}(0,\tilde u,0,0)$. Using (41), (43) and (49), this implies that

$$\begin{aligned}&\|\tilde u\|_{W^{1,\infty}_s(L^2(\Omega_F(0)))}+\|\tilde u\|_{H^1_s(H^1(\Omega_F(0)))}+\|a\|_{W^{2,\infty}_s}+\|\omega\|_{W^{1,\infty}_s}\\&\quad\leqslant C\big(\tilde R^2+|\ddot a(0)|+|\dot\omega(0)|+\|\tilde u_t(0)\|_{L^2(\Omega_F(0))}+\|\rho_0-\bar\rho\|_{H^3(\Omega_F(0))}+\big(s^{1/2}+\tilde R\big)\tilde N_{0,s}(0,\tilde u,0,0)\\&\quad\quad+\|u_0\|_{H^1(\Omega_F(0))}+|a_0|+|\omega_0|\big).\end{aligned}\tag{51}$$

• Next, if we multiply the first equation of (50) by $\tilde u_{tt}$, we obtain that

$$\begin{aligned}&\iint_{V_s}|\tilde u_{tt}|^2\,dy\,dr+\frac12\sup_{r\in(0,s)}\int_{\Omega_F(0)}\big(2\mu|\epsilon(\tilde u_t)|^2(r)+\mu'|\nabla\cdot\tilde u_t|^2(r)\big)\,dy\\&\quad+\int_0^s\int_{\partial\Omega_S(0)}\big(\big(2\mu\epsilon(\tilde u_t)+\mu'(\nabla\cdot\tilde u_t)\,\mathrm{Id}\big)n\big)\cdot\tilde u_{tt}\,d\gamma\,dr\\&=\iint_{V_s}g_{1,t}\cdot\tilde u_{tt}\,dy\,dr+\frac12\int_{\Omega_F(0)}\big(2\mu|\epsilon(\tilde u_t)|^2(0)+\mu'|\nabla\cdot\tilde u_t|^2(0)\big)\,dy-\iint_{V_s}p^0\nabla\tilde\rho_t\cdot\tilde u_{tt}\,dy\,dr.\end{aligned}$$

On $(0,T)\times\partial\Omega_S(0)$ we have

$$u_{tt}=\dddot a+\ddot\omega\wedge(\hat Qy)+2\dot\omega\wedge(\dot{\hat Q}y)+\omega\wedge(\ddot{\hat Q}y).\tag{52}$$

Thus, we deduce

$$\begin{aligned}&\int_0^s\int_{\partial\Omega_S(0)}\big(\big(2\mu\epsilon(\tilde u_t)+\mu'(\nabla\cdot\tilde u_t)\,\mathrm{Id}\big)n\big)\cdot\tilde u_{tt}\,d\gamma\,dr\\&=\int_0^s\dddot a\cdot(m\dddot a-g_{2,t})\,dr+\int_0^s\dddot a\cdot\int_{\partial\Omega_S(0)}p^0\tilde\rho_t n\,d\gamma\,dr+\int_0^s\ddot\omega\cdot(\hat J\ddot\omega+\dot{\hat J}\dot\omega-g_{3,t})\,dr\\&\quad+\int_0^s\ddot\omega\cdot\int_{\partial\Omega_S(0)}(\hat Qy)\wedge\big(p^0\tilde\rho_t n\big)\,d\gamma\,dr-\int_0^s\ddot\omega\cdot\int_{\partial\Omega_S(0)}(\dot{\hat Q}y)\wedge\big(\big(2\mu\epsilon(\tilde u)+\mu'(\nabla\cdot\tilde u)\,\mathrm{Id}-p^0\tilde\rho\,\mathrm{Id}\big)n\big)\,d\gamma\,dr\\&\quad+\int_0^s\int_{\partial\Omega_S(0)}\big(\big(2\mu\epsilon(\tilde u_t)+\mu'(\nabla\cdot\tilde u_t)\,\mathrm{Id}\big)n\big)\cdot\big(2\dot\omega\wedge\big(\dot{\hat Q}y\big)+\omega\wedge\big(\ddot{\hat Q}y\big)\big)\,d\gamma\,dr.\end{aligned}$$

Then, proceeding as before and using (51), (49) and (43), we obtain that there exists C such that

$$\begin{aligned}&\|\tilde u_{tt}\|_{L^2_s(L^2(\Omega_F(0)))}+\|\tilde u_t\|_{L^\infty_s(H^1(\Omega_F(0)))}+\|\dddot a\|_{L^2_s}+\|\ddot\omega\|_{L^2_s}\\&\quad\leqslant C\big(|\ddot a(0)|+|\dot\omega(0)|+\|u_0\|_{H^1(\Omega_F(0))}+\tilde R^2+\|\tilde u_t(0)\|_{H^1(\Omega_F(0))}+\|\rho_0-\bar\rho\|_{H^3(\Omega_F(0))}\\&\quad\quad+\big(s^{1/2}+\tilde R\big)\tilde N_{0,s}(0,\tilde u,0,0)+|a_0|+|\omega_0|\big).\end{aligned}\tag{53}$$

According to the elliptic equation satisfied by $\tilde u_t$ (see (48)), we have

$$\|\tilde u_t\|_{L^2_s(H^2(\Omega_F(0)))}\leqslant C\big(\|g_{1,t}\|_{L^2_s(L^2(\Omega_F(0)))}+\|\tilde\rho_t\|_{L^2_s(H^1(\Omega_F(0)))}+\|\tilde u_{tt}\|_{L^2_s(L^2(\Omega_F(0)))}+\|\ddot a\|_{L^2_s}+\|\omega\|_{H^1_s}\big).\tag{54}$$

At last, if we consider system (48) satisfied by $\tilde{u}$, since the right-hand side $g_1 - p^0\nabla\tilde{\rho} - \tilde{u}_t$ is estimated in the $L^2_s(H^2(\Omega_F(0))) \cap L^\infty_s(H^1(\Omega_F(0)))$-norm (see (53)–(54)) and the boundary condition is estimated in $L^\infty_s(C^\infty(\partial\Omega_F(0)))$, $\tilde{u}$ is estimated in $L^2_s(H^4(\Omega_F(0))) \cap L^\infty_s(H^3(\Omega_F(0)))$ by

$$\tilde{R}^2 + \|\rho_0 - \bar{\rho}\|_{H^3(\Omega_F(0))} + \big(s^{1/2} + \tilde{R}\big)\tilde{N}_{0,s}(0, \tilde{u}, 0, 0) + \|u_0\|_{H^3(\Omega_F(0))} + |a_0| + |\omega_0|.$$

Thus, we deduce that

$$\tilde{N}_{0,s}(0, \tilde{u}, a, \omega) \leqslant C\big(\tilde{R}^2 + \big(s^{1/2} + \tilde{R}\big)\tilde{N}_{0,s}(0, \tilde{u}, 0, 0) + \tilde{N}_{0,0}(\tilde{\rho}, \tilde{u}, a, \omega)\big).$$

Using that $\tilde{N}_{0,s}(0, \tilde{u}, 0, 0) \leqslant \tilde{N}_{0,s}(\tilde{\rho}, \tilde{u}, a, \omega)$ and since s and $\tilde{R}$ are sufficiently small, we have

$$\tilde{N}_{0,s}(0, \tilde{u}, a, \omega) \leqslant C\big(\tilde{R}^2 + \tilde{N}_{0,0}(\tilde{\rho}, \tilde{u}, a, \omega)\big).$$

Combining with (43) and taking into account that $\tilde{u}_F = \tilde{u} - \chi_t$, we finally get

$$\tilde{N}_{0,s}(\tilde{\rho}, \tilde{u}_F, a, \omega) \leqslant C\tilde{N}_{0,s}(\tilde{\rho}, \tilde{u}, a, \omega) \leqslant C\big(\tilde{R}^2 + \tilde{N}_{0,0}(\tilde{\rho}, \tilde{u}, a, \omega)\big).$$

This allows to assert that, if $\tilde{N}_{0,0}(\tilde{\rho}, \tilde{u}, a, \omega)$ satisfies

$$\tilde{N}_{0,0}(\tilde{\rho}, \tilde{u}, a, \omega) \sim \tilde{R}^2 \tag{55}$$

(which implies that $\tilde{N}_{0,0}(\tilde{\rho}, \tilde{u}, a, \omega)$ is small enough), then $(\tilde{\rho}, \tilde{u}_F, a, \omega)$ belongs to $Y((0, s); \tilde{R})$.

3.4. Fixed point argument

Let us fix $\tilde{R}$ such that (55) is satisfied. We will prove that Λ is a continuous mapping in $Y((0, s); \tilde{R})$ for the norm $\|\cdot\|$ defined by

$$\big\|(\rho, u_F, a, \omega)\big\| = \|\rho\|_{L^\infty_s(H^1(\Omega_F(0)))} + \|u_F\|_{L^2_s(H^2(\Omega_F(0)))} + \|a\|_{H^2_s} + \|\omega\|_{H^1_s}. \tag{56}$$

Once this is done, we apply the Schauder's fixed point theorem (see, for instance, [23]) and we obtain the existence of a fixed point of Λ which solves system (35); then, using the inverse of the flow χ, we obtain a solution of our original system (61), which satisfies the properties of Proposition 7.

In order to prove the continuity, let us consider $(\hat{\rho}_k, \hat{u}_{F,k}, \hat{a}_k, \hat{\omega}_k)$ a sequence converging for the previous norm to some element $(\hat{\rho}, \hat{u}_F, \hat{a}, \hat{\omega})$ of $Y((0, s); \tilde{R})$. We define

$$(\tilde{\rho}_k, \tilde{u}_{F,k}, a_k, \omega_k) := \Lambda(\hat{\rho}_k, \hat{u}_{F,k}, \hat{a}_k, \hat{\omega}_k) \quad \text{and} \quad (\tilde{\rho}, \tilde{u}_F, a, \omega) := \Lambda(\hat{\rho}, \hat{u}_F, \hat{a}, \hat{\omega}).$$

Let us prove that $(R_k, U_k, A_k, \Omega_k) := (\tilde{\rho}_k - \tilde{\rho}, \tilde{u}_k - \tilde{u}, a_k - a, \omega_k - \omega)$ converges to zero for the norm (56). This readily implies that $(\tilde{\rho}_k, \tilde{u}_{F,k}, a_k, \omega_k) \to (\tilde{\rho}, \tilde{u}_F, a, \omega)$.

Estimates for R_k. The equation fulfilled by R_k is the following one:

$$\begin{cases} R_{k,t} + \big((\nabla\hat{\chi}_k)^{-1}(\hat{u}_k - \hat{\chi}_{k,t})\big) \cdot \nabla R_k = \ell_{0,k}, \\ R_k(0, x) = 0, \end{cases}$$

with

$$\ell_{0,k} = g_0(\hat{\rho}_k, \hat{u}_k, \hat{a}_k, \hat{\omega}_k) - g_0(\hat{\rho}, \hat{u}, \hat{a}, \hat{\omega}) - \bar{\rho}\nabla \cdot U_k + \big((\nabla\hat{\chi})^{-1}(\hat{u} - \hat{\chi}_t) - (\nabla\hat{\chi}_k)^{-1}(\hat{u}_k - \hat{\chi}_{k,t})\big) \cdot \nabla\tilde{\rho}.$$

We directly get that

$$\|R_k\|_{L^\infty_s(H^1(\Omega_F(0)))} \leqslant C\|\ell_{0,k}\|_{L^2_s(H^1(\Omega_F(0)))}.$$

Now, we use the expression of g_0 (see (36)) and we deduce

$$\begin{aligned} \|R_k\|_{L^\infty_s(H^1(\Omega_F(0)))} \leqslant C\big(&\|U_k\|_{L^2_s(H^2(\Omega_F(0)))} + \big\|(\nabla\hat{\chi}_k)^{-1} - (\nabla\hat{\chi})^{-1}\big\|_{L^2_s(H^1(\Omega_F(0)))} + \|\hat{u}_k - \hat{u}\|_{L^2_s(H^2(\Omega_F(0)))} \\ &+ \|\hat{\chi}_{k,t} - \hat{\chi}_t\|_{L^2_s(H^1(\Omega_F(0)))} + \|\hat{\rho}_k - \hat{\rho}\|_{L^2_s(H^1(\Omega_F(0)))}\big). \end{aligned} \tag{57}$$

Estimates for (U_k, A_k, Ω_k). We have the equation:

$$\begin{cases} U_{k,t} - \nabla\cdot(2\mu\epsilon(U_k) + \mu'\nabla\cdot U_k\,\mathrm{Id}) = \ell_{1,k} & \text{in } (0,T)\times\Omega_F(0),\\ m\ddot{A}_k = \int_{\partial\Omega_S(0)}(2\mu\epsilon(U_k) + \mu'\nabla\cdot U_k\,\mathrm{Id})n\,d\gamma + \ell_{2,k} & \text{in } (0,T),\\ \hat{J}\dot{\Omega}_k = \int_{\partial\Omega_S(0)}(\hat{Q}y)\wedge((2\mu\epsilon(U_k) + \mu'\nabla\cdot U_k\,\mathrm{Id})n)\,d\gamma + \ell_{3,k} & \text{in } (0,T),\\ U_k = 0 & \text{on } (0,T)\times\partial\Omega,\\ U_k = \dot{A}_k + \Omega_k\wedge(\hat{Q}y) + \omega_k\wedge((\hat{Q}_k - \hat{Q})y) & \text{on } (0,T)\times\partial\Omega_S(0),\\ U_k(0,\cdot) = 0,\quad A_k(0) = 0,\quad \dot{A}_k(0) = 0,\quad \Omega_k(0) = 0, \end{cases} \tag{58}$$

with

$$\ell_{1,k} := g_1(\hat{\rho}_k, \hat{u}_k, a_k, \omega_k) - g_1(\hat{\rho}, \hat{u}, a, \omega) - p^0\nabla R_k,$$

$$\ell_{2,k} = g_2(\hat{\rho}_k, \hat{u}_k, a_k, \omega_k) - g_2(\hat{\rho}, \hat{u}, a, \omega) - p^0\int_{\partial\Omega_S(0)} R_k n\,d\gamma,$$

$$\ell_{3,k} = g_3(\hat{\rho}_k, \hat{u}_k, a_k, \omega_k) - g_3(\hat{\rho}, \hat{u}, a, \omega) - p^0\int_{\partial\Omega_S(0)}(\hat{Q}y)\wedge(R_k n)\,d\gamma$$
$$+ \int_{\partial\Omega_S(0)}\big((\hat{Q}_k - \hat{Q})y\big)\wedge\big(\sigma(\tilde{u}_k, \tilde{\rho}_k)n\big)\,d\gamma + (\hat{J} - \hat{J}_k)\dot{\omega}_k.$$

We multiply the equation of U_k by $U_{k,t}$, we integrate in $(0,r)\times\Omega_F(0)$ and we take the supremum of $r\in(0,s)$. Arguing as for the proof of (47) and using

$$U_{k,t} = \ddot{A}_k + \dot{\Omega}_k\wedge(\hat{Q}y) + \ell_{4,k}$$

with

$$\ell_{4,k} = \Omega_k\wedge(\dot{\hat{Q}}y) + \dot{\omega}_k\wedge\big((\hat{Q}_k - \hat{Q})y\big) + \omega_k\wedge\big((\dot{\hat{Q}}_k - \dot{\hat{Q}})y\big),$$

we obtain

$$\|A_k\|_{H^2_s} + \|\Omega_k\|_{H^1_s} + \|U_k\|_{H^1_s(L^2(\Omega_F(0)))} + \|U_k\|_{L^\infty_s(H^1(\Omega_F(0)))}$$
$$\leqslant \delta\|U_k\|_{L^2_s(H^2(\Omega_F(0)))} + C\big(\|\ell_{1,k}\|_{L^2_s(L^2(\Omega_F(0)))} + \|\ell_{2,k}\|_{L^2_s} + \|\ell_{3,k}\|_{L^2_s} + \|\ell_{4,k}\|_{L^2_s(L^2(\partial\Omega_S(0)))}\big),$$

for $\delta > 0$ small enough. Regarding now the equation of U_k as an elliptic equation, we deduce that $U_k\in L^2_s(H^2(\Omega_F(0)))$ and

$$\|U_k\|_{L^2_s(H^2(\Omega_F(0)))} \leqslant C\big(\|\ell_{1,k} - U_{k,t}\|_{L^2_s(L^2(\Omega_F(0)))} + \|\dot{A}_k\|_{L^2_s} + \|\Omega_k\|_{L^2_s} + \|\hat{Q}_k - \hat{Q}\|_{L^2_s}\big).$$

Thus,

$$\|A_k\|_{H^2_s} + \|\Omega_k\|_{H^1_s} + \|U_k\|_{L^2_s(H^2(\Omega_F(0)))}$$
$$\leqslant C\big(\|\ell_{1,k}\|_{L^2_s(L^2(\Omega_F(0)))} + \|\ell_{2,k}\|_{L^2_s} + \|\ell_{3,k}\|_{L^2_s} + \|\ell_{4,k}\|_{L^2_s(L^2(\partial\Omega_S(0)))} + \|\hat{Q}_k - \hat{Q}\|_{L^2_s}\big). \tag{59}$$

Observe that, for the right-hand side terms, we have the following estimates:

$$\|\ell_{1,k}\|_{L^2_s(L^2(\Omega_F(0)))} \leqslant C\big(\|R_k\|_{L^2_s(H^1(\Omega_F(0)))} + \|\hat{u}_k - \hat{u}\|_{L^2_s(H^2(\Omega_F(0)))} + \|\hat{\rho}_k - \hat{\rho}\|_{L^2_s(H^1(\Omega_F(0)))}$$
$$+ \big\|(\nabla\hat{\chi}_k)^{-1} - (\nabla\hat{\chi})^{-1}\big\|_{L^2_s(L^2(\Omega_F(0)))} + \|\hat{\chi}_{k,t} - \hat{\chi}_t\|_{L^2_s(L^2(\Omega_F(0)))}\big),$$

$$\|\ell_{2,k}\|_{L^2_s} \leqslant C\big(\|R_k\|_{L^2_s(H^1(\Omega_F(0)))} + \|\hat{u}_k - \hat{u}\|_{L^2_s(H^2(\Omega_F(0)))} + \|\hat{\rho}_k - \hat{\rho}\|_{L^2_s(H^1(\Omega_F(0)))} + \|\hat{Q}_k - \hat{Q}\|_{L^2_s}\big),$$

$$\|\ell_{3,k}\|_{L^2_s} \leqslant C\big(\|R_k\|_{L^2_s(H^1(\Omega_F(0)))} + \|\hat{u}_k - \hat{u}\|_{L^2_s(H^2(\Omega_F(0)))} + \|\hat{\rho}_k - \hat{\rho}\|_{L^2_s(H^1(\Omega_F(0)))}$$
$$+ \|\hat{Q}_k - \hat{Q}\|_{L^2_s} + \|\hat{\omega}_k - \hat{\omega}\|_{L^2_s} + \|\hat{J}_k - \hat{J}\|_{L^2_s}\big)$$

and

$$\|\ell_{4,k}\|_{L^2_s(L^2(\partial\Omega_S(0)))} \leqslant C\big(\|\Omega_k\|_{L^2_s} + \|\hat{Q}_k - \hat{Q}\|_{H^1_s}\big).$$

Combining the estimates of $\ell_{j,k}(1 \leqslant j \leqslant 4)$ with (59), we get

$$\begin{aligned}
&\|A_k\|_{H^2_s} + \|\Omega_k\|_{H^1_s} + \|U_k\|_{L^2_s(H^2(\Omega_F(0)))} \\
&\quad \leqslant C\big(\|R_k\|_{L^2_s(H^1(\Omega_F(0)))} + \|\hat{u}_k - \hat{u}\|_{L^2_s(H^2(\Omega_F(0)))} + \|\hat{\rho}_k - \hat{\rho}\|_{L^2_s(H^1(\Omega_F(0)))} \\
&\qquad + \big\|(\nabla\hat{\chi}_k)^{-1} - (\nabla\hat{\chi})^{-1}\big\|_{L^2_s(H^1(\Omega_F(0)))} + \|\hat{\chi}_{k,t} - \hat{\chi}_t\|_{L^2_s(L^2(\Omega_F(0)))} + \|\hat{\omega}_k - \hat{\omega}\|_{L^2_s} \\
&\qquad + \|\hat{J}_k - \hat{J}\|_{L^2_s} + \|\hat{Q}_k - \hat{Q}\|_{L^2_s}\big).
\end{aligned}$$

To conclude, we plug this inequality into (57) and use that $\|R_k\|_{L^2_s(H^1(\Omega_F(0)))} \leqslant Cs^{1/2}\|R_k\|_{L^\infty_s(H^1(\Omega_F(0)))}$:

$$\begin{aligned}
&\|A_k\|_{H^2_s} + \|\Omega_k\|_{H^1_s} + \|U_k\|_{L^2_s(H^2(\Omega_F(0)))} + \|R_k\|_{L^\infty_s(H^1(\Omega_F(0)))} \\
&\quad \leqslant C\big(\|\hat{u}_k - \hat{u}\|_{L^2_s(H^2(\Omega_F(0)))} + \big\|(\nabla\hat{\chi}_k)^{-1} - (\nabla\hat{\chi})^{-1}\big\|_{L^2_s(H^1(\Omega_F(0)))} + \|\hat{\chi}_{k,t} - \hat{\chi}_t\|_{L^2_s(H^1(\Omega_F(0)))} \\
&\qquad + \|\hat{\rho}_k - \hat{\rho}\|_{L^2_s(H^1(\Omega_F(0)))} + \|\hat{\omega}_k - \hat{\omega}\|_{L^2_s} + \|\hat{J}_k - \hat{J}\|_{L^2_s} + \|\hat{Q}_k - \hat{Q}\|_{L^2_s}\big).
\end{aligned} \tag{60}$$

Thanks to the estimate

$$\big\|\big((\nabla\hat{\chi}_k)^{-1} - (\nabla\hat{\chi})^{-1}, \hat{\chi}_{k,t} - \hat{\chi}_t\big)\big\|_{L^2_s(H^1(\Omega_F(0)))} + \big\|(\hat{J}_k - \hat{J}, \hat{Q}_k - \hat{Q})\big\|_{L^2_s} \leqslant C\big(\|\hat{\omega}_k - \hat{\omega}\|_{L^2_s} + \|\hat{a}_k - \hat{a}\|_{H^1_s}\big),$$

we find that $(R_k, U_k, A_k, \Omega_k) \to 0$ for the norm (56).

3.5. Uniqueness

Let us consider two solutions $(\tilde{\rho}_1, \tilde{u}_1, a_1, \omega_1)$ and $(\tilde{\rho}_2, \tilde{u}_2, a_2, \omega_2)$ of system (35) such that

$$\tilde{N}_{0,s}(\tilde{\rho}_j, \tilde{u}_j, a_j, \omega_j) \leqslant \tilde{R}, \quad j = 1, 2.$$

Then, we look at the system fulfilled by the difference, we multiply the equation of $\tilde{\rho}_1 - \tilde{\rho}_2$ by $\tilde{\rho}_1 - \tilde{\rho}_2$, the equation of $\tilde{u}_1 - \tilde{u}_2$ by $\tilde{u}_1 - \tilde{u}_2$, the equation of $a_1 - a_2$ by $\dot{a}_1 - \dot{a}_2$ and the equation of $\omega_1 - \omega_2$ by $\omega_1 - \omega_2$.

Integrating by parts in the second order term of the equation of $\tilde{u}_1 - \tilde{u}_2$ and using classical arguments, we obtain

$$\begin{aligned}
&\frac{d}{dt}\bigg(\int\limits_{\Omega_F(0)} \big(|\tilde{\rho}_1 - \tilde{\rho}_2|^2 + |\tilde{u}_1 - \tilde{u}_2|^2\big)\,dx + |\dot{a}_1 - \dot{a}_2|^2 + |\omega_1 - \omega_2|^2\bigg) + \int\limits_{\Omega_F(0)} |\nabla\tilde{u}_1 - \nabla\tilde{u}_2|^2\,dx \\
&\quad \leqslant C\bigg(\int\limits_{\Omega_F(0)} \big(|\tilde{\rho}_1 - \tilde{\rho}_2|^2 + |\tilde{u}_1 - \tilde{u}_2|^2\big)\,dx + |\dot{a}_1 - \dot{a}_2|^2 + |\omega_1 - \omega_2|^2\bigg).
\end{aligned}$$

Finally, using Gronwall's Lemma we deduce that $\tilde{\rho}_1 = \tilde{\rho}_2$, $\tilde{u}_1 = \tilde{u}_2$, $a_1 = a_2$ and $\omega_1 = \omega_2$.

4. A priori estimates: proof of Proposition 8

The proof of this result is inspired by the works [17] and [18], where the authors dealt with the compressible Navier–Stokes equations.

Let us define p^0 by

$$p^0 := \frac{P'(\bar{\rho})}{\bar{\rho}},$$

where $\bar{\rho}$ is the mean-value of ρ_0 (see (26))

Now, we define $\rho^*(t,x) := \rho(t,x) - \bar\rho$, we change $m/\bar\rho$ into m, $J/\bar\rho$ into J, $\mu/\bar\rho$ into μ and $\mu'/\bar\rho$ into μ' and we find that the system of Eqs. (2), (11), (12) and (13) can be written as follows:

$$\begin{cases} \rho^*_t + u\cdot\nabla\rho^* + \bar\rho\nabla\cdot u = f_0(\rho^*,u,a,\omega) & \text{in } Z_T,\\ u_t - \nabla\cdot\sigma(u,\rho^*) = f_1(\rho^*,u,a,\omega) & \text{in } Z_T,\\ m\ddot a = \int_{\partial\Omega_S(t)}\sigma(u,\rho^*)n\,d\gamma + f_2(\rho^*,u,a,\omega) & \text{in } (0,T),\\ J\dot\omega = \int_{\partial\Omega_S(t)}(x-a)\wedge(\sigma(u,\rho^*)n)\,d\gamma + f_3(\rho^*,u,a,\omega) & \text{in } (0,T),\\ u=0 & \text{on } (0,T)\times\partial\Omega,\\ u=\dot a+\omega\wedge(x-a) & \text{on } \Sigma_T,\\ \rho^*(0,\cdot)=\rho_0-\bar\rho,\ u(0,\cdot)=u_0 & \text{in } \Omega_F(0),\\ a(0)=0,\quad \dot a(0)=a_0,\quad \omega(0)=\omega_0, \end{cases} \tag{61}$$

where

$$\sigma(u,\rho) := 2\mu\epsilon(u) + \mu'(\nabla\cdot u)\,\mathrm{Id} - p^0\rho\,\mathrm{Id} \tag{62}$$

and

$$\begin{cases} f_0(\rho^*,u,a,\omega) = -\rho^*(\nabla\cdot u),\\ f_1(\rho^*,u,a,\omega) = -(u\cdot\nabla)u - \dfrac{\rho^*}{\rho^*+\bar\rho}\nabla\cdot\big(2\mu\epsilon(u)+\mu'(\nabla\cdot u)\,\mathrm{Id}\big) + \left(p^0 - \dfrac{P'(\rho^*+\bar\rho)}{\rho^*+\bar\rho}\right)\nabla\rho^*,\\ f_2(\rho^*,u,a,\omega) = \displaystyle\int_{\partial\Omega_S(t)}\left(p^0\rho^* - \frac{P(\rho^*+\bar\rho)}{\bar\rho}\right)n\,d\gamma,\\ f_3(\rho^*,u,a,\omega) = (J\omega)\wedge\omega + \displaystyle\int_{\partial\Omega_S(t)}(x-a)\wedge\left(\left(p^0\rho^* - \frac{P(\rho^*+\bar\rho)}{\bar\rho}\right)n\right)d\gamma. \end{cases} \tag{63}$$

For the integrals in the expressions of f_2 and f_3, we use that

$$\frac{P(\bar\rho)}{\bar\rho}\int_{\partial\Omega_S(0)}(x-a)\wedge n\,d\gamma = 0 \quad\text{and}\quad \frac{P(\bar\rho)}{\bar\rho}\int_{\partial\Omega_S(0)}n\,d\gamma = 0 \tag{64}$$

and so f_2 and $f_3 - (J\omega)\wedge\omega$ are quadratic terms of ρ^*.

For the sake of simplicity, in this paragraph we will denote our solution by (ρ,u,a,ω) instead of (ρ^*,u,a,ω).

In the following lines, we will give several lemmas where we will present a priori estimates of different nature:

- An estimate of the solid motion in terms of the fluid velocity (Lemma 11).
- Global estimates associated to an elliptic operator (Lemma 12) and energy-type estimates associated to the compressible Stokes system (Lemmas 13 and 14).
- Interior estimates for the compressible Stokes system (Lemma 15).
- Estimates close to the boundary $\partial\Omega_F(0)$. First, we will estimate the tangential derivatives (Lemma 16) and then the normal ones (Lemma 17).
- Global estimates for the Stokes operator (Lemma 18).

In the above lines, the term 'Global' refers to estimates on the whole domain $\Omega_F(t)$.

All this will be proved under the hypothesis that $(\rho,u,a,\omega)\in X(0,T)$. The conclusion of all these lemmas will be the following inequality:

$$\begin{aligned} N_{0,T}(\rho,u,a,\omega) \leqslant C\big(&\|f_0\|_{H^1_T(L^2)} + \|f_0\|_{L^\infty_T(H^2)} + \|f_0\|_{L^2_T(H^3)} + \|f_1\|_{L^2_T(H^2)} + \|f_1\|_{H^1_T(L^2)} + \|f_1\|_{L^\infty_T(H^1)}\\ &+ \|f_2\|_{H^1_T} + \|f_3\|_{H^1_T} + \big\|\partial_t u(0,\cdot)\big\|_{H^1} + \|u_0\|_{H^3} + \big\|\partial_t\rho(0,\cdot)\big\|_{L^2} + \|\rho_0\|_{H^3} + \big|\ddot a(0)\big|\\ &+ |a_0| + \big|\dot\omega(0)\big| + |\omega_0| + N^{3/2}_{0,T}(\rho,u,a,\omega) + N^2_{0,T}(\rho,u,a,\omega)\big) \end{aligned} \tag{65}$$

(recall that $N_{0,T}(\rho,u,a,\omega)$ was defined in (27)). Next, using the equations of u, ρ, a and ω, we see that

$$\begin{aligned} &\big\|\partial_t u(0,\cdot)\big\|_{H^1} \leqslant C\big(\|u_0\|_{H^3} + \|\rho_0\|_{H^2} + \|f_1\|_{L^\infty_T(H^1)}\big),\\ &\big\|\partial_t\rho(0,\cdot)\big\|_{L^2} \leqslant C\big(\|u_0\|_{H^1} + \|f_0\|_{L^\infty_T(L^2)} + N^2_{0,T}(\rho,u,a,\omega)\big),\\ &\big|\ddot a(0)\big| \leqslant C\big(\|u_0\|_{H^2} + \|\rho_0\|_{H^1} + \|f_2\|_{L^\infty_T}\big) \end{aligned}$$

and

$$\left|\dot{\omega}(0)\right| \leqslant C\left(\|u_0\|_{H^2} + \|\rho_0\|_{H^1} + \|f_3\|_{L^\infty_T}\right).$$

Then, from the expression of f_i $(0 \leqslant i \leqslant 3)$, we have

$$\|f_0\|_{H^1_T(L^2)} + \|f_0\|_{L^\infty_T(H^2)} + \|f_0\|_{L^2_T(H^3)} + \|f_1\|_{L^2_T(H^2)} + \|f_1\|_{H^1_T(L^2)} + \|f_1\|_{L^\infty_T(H^1)} + \|f_2\|_{H^1_T} + \|f_3\|_{H^1_T} \leqslant N^2_{0,T}(\rho, u, a, \omega).$$

Finally, using the assumption $N_{0,T}(\rho, u, a, \omega) \leqslant \delta_1$ with δ_1 small enough, we conclude the proof of Proposition 8. Consequently, from now on we concentrate in the proof of inequality (65).

4.1. Technical results

All along the proof of these lemmas and for the sake of simplicity we will adopt the notation $N(0, T)$ instead of $N_{0,T}(\rho, u, a, \omega)$.

Furthermore, C will stand for generic positive constants which do not depend on t but which may depend on $\Omega_S(0)$, $\Omega_F(0)$ and Ω.

The first result is the following one:

Lemma 11. *There exists a positive constant $C > 0$ such that*

$$\|\dot{a}\|_{L^2_T} + \|\omega\|_{L^2_T} \leqslant C\|u\|_{L^2_T(H^1)}. \tag{66}$$

Proof. The value of the velocity vector field on the boundary of the solid is given by (see (61))

$$u(t, x) = \dot{a}(t) + \omega(t) \wedge \left(x - a(t)\right) = u_S(t, x), \quad t \in (0, T),\ x \in \partial\Omega_S(t). \tag{67}$$

Taking the scalar product of this by $(x - a(t))$ and taking the square, we have

$$\left|u(t, x) \cdot \left(x - a(t)\right)\right|^2 = \left|\dot{a}(t)\right|^2\left|x - a(t)\right|^2 \cos^2\left(\dot{a}(t), x - a(t)\right), \quad t \in (0, T),\ x \in \partial\Omega_S(t).$$

Then,

$$\int_{\partial\Omega_S(t)} \left|\dot{a}(t)\right|^2\left|x - a(t)\right|^2\left|\cos\left(\dot{a}(t), x - a(t)\right)\right|^2 dx = \left|\dot{a}(t)\right|^2 \int_{\partial\Omega_S(0)} |y|^2\left|\cos\left(\dot{a}(t), Q(t)y\right)\right|^2 dy. \tag{68}$$

First, we observe that $|y| \geqslant \operatorname{dist}(0, \overline{\partial\Omega_S(0)})$. Thanks to the fact that $\partial\Omega_S(0)$ is a regular closed curve, we have that for any $b \in \mathbb{R}^3$

$$\int_{\partial\Omega_S(0)} \left|\cos(b, y)\right|^2 \geqslant C > 0.$$

Using this in (68), we have (66) for $\|\dot{a}\|_{L^2_T}$. Finally, from (67) and similar arguments, we also deduce this inequality for $\|\omega\|_{L^2_T}$. $\square$

The next result concerns a classical elliptic estimate for u:

Lemma 12. *Let $k = 2, 3$. Then, we have*

$$\|u\|_{L^\infty_T(H^k)} \leqslant C\left(\|u\|_{W^{1,\infty}_T(H^{k-2})} + \|\rho\|_{L^\infty_T(H^{k-1})} + \|\dot{a}\|_{L^\infty_T} + \|\omega\|_{L^\infty_T} + \|f_1\|_{L^\infty_T(H^{k-2})}\right). \tag{69}$$

Lemma 13. *Let $k = 0, 1$. Then, for every $\delta > 0$, there exists a constant $C > 0$ such that*

$$\|u\|_{W_T^{k,\infty}(L^2)} + \|\rho\|_{W_T^{k,\infty}(L^2)} + \|u\|_{H_T^k(H^1)} + \|\dot a\|_{W_T^{k,\infty}} + \|\omega\|_{W_T^{k,\infty}}$$
$$\leqslant \delta\big(\|\rho\|_{H_T^k(L^2)} + \|\dot a\|_{H_T^k} + \|\omega\|_{H_T^k}\big) + C\big(\|f_0\|_{H_T^k(L^2)} + \|f_1\|_{H_T^k(L^2)} + \|f_2\|_{H_T^k} + \|f_3\|_{H_T^k} + \|u_0\|_{L^2}$$
$$+ \|\rho_0\|_{L^2} + |a_0| + |\omega_0| + \big\|\partial_t^k u(0,\cdot)\big\|_{L^2} + \big\|\partial_t^k \rho(0,\cdot)\big\|_{L^2} + \big|\partial_t^{k+1}a(0)\big| + \big|\partial_t^k\omega(0)\big|$$
$$+ N^{3/2}(0,T) + N^2(0,T)\big). \tag{70}$$

Proof. 1) $k=0$. We multiply the equation of ρ by $p^0\rho/\bar\rho$ and the equation of u by u. Integrating in Z_s for $s\in(0,T)$ and adding up both expressions, we obtain:

$$\frac{1}{2}\sup_{s\in(0,T)}\left(|\dot a|^2(s) + |\omega|^2(s) + \int\limits_{\Omega_F(s)}\left(\frac{p^0}{\bar\rho}|\rho|^2 + |u|^2\right)(s)\,dx\right) + \iint\limits_{Z_T}\big(2\mu\big|\epsilon(u)\big|^2 + \mu'|\nabla\cdot u|^2\big)\,dx\,dt$$
$$\leqslant \delta\big(\|u\|_{L_T^2(L^2)} + \|\rho\|^2_{L_T^2(L^2)} + \|\dot a\|^2_{L_T^2} + \|\omega\|^2_{L_T^2}\big) + C\left(\iint\limits_{Z_T}\big(|\nabla\cdot u||\rho|^2 + \big|\nabla\cdot\big(|u|^2u\big)\big|\big)\,dx\,dt\right.$$
$$\left.+\iint\limits_{Z_T}\big(|f_0|^2+|f_1|^2\big)\,dx\,dt + \int\limits_{\Omega_F(0)}\big(|u_0|^2+|\rho_0|^2\big)\,dx + |a_0|^2 + |\omega_0|^2 + \int\limits_0^T\big(|f_2|^2+|f_3|^2\big)\,dt\right). \tag{71}$$

Here, we have put together the integrals coming from the third term in the equation of ρ and the third term in the expression of $\sigma(u,\rho)$ (see (62)) and we have integrated by parts. We have also used the fact that according to the definition (9) of J, we have $(\dot J\omega)\cdot\omega = 0$.

Thanks to Lemma 9, we get the $L_T^2(H^1)$ norm of u in the left-hand side of (71). Moreover, we can estimate the nonlinear term in (71) in the following way:

$$\iint\limits_{Z_T}\big(|\nabla\cdot u||\rho|^2 + \big|\nabla\cdot\big(|u|^2u\big)\big|\big)\,dx\,dt \leqslant C\|u\|_{L_T^\infty(W^{1,\infty})}\big(\|u\|^2_{L_T^2(L^2)} + \|\rho\|^2_{L_T^2(L^2)}\big) \leqslant CN^3(0,T).$$

The definition of $N(0,T)$ was given in (27). Thus, we deduce (70) for $k=0$

2) $k=1$. First, we differentiate (61) with respect to t and we find:

$$\begin{cases}(\rho_t)_t + (u\cdot\nabla)\rho_t + \bar\rho(\nabla\cdot u_t) = G_0 & \text{in } Z_T,\\ (u_t)_t - \nabla\cdot\sigma(u_t,\rho_t) = G_1 & \text{in } Z_T,\\ m\ddot{\ddot a} = \int_{\partial\Omega_S(t)}\sigma(u_t,\rho_t)n\,d\gamma + G_2 & \text{in } (0,T),\\ J\ddot\omega = G_3 + \int_{\partial\Omega_S(t)}(x-a)\wedge(\sigma(u_t,\rho_t))n\,d\gamma & \text{in } (0,T),\\ u_t = \ddot a + \dot\omega\wedge(x-a) + G_4 & \text{in } \Sigma_T,\\ u_t = 0 & \text{in } (0,T)\times\partial\Omega,\end{cases} \tag{72}$$

with

$$G_0 = f_{0,t} - (u_t\cdot\nabla)\rho,\quad G_1 = f_{1,t},\quad G_2 = f_{2,t} + \int\limits_{\partial\Omega_S(t)} u_S\cdot\nabla\big(\sigma(u,p)n\big)\,d\gamma,$$
$$G_3 = f_{3,t} + \int\limits_{\partial\Omega_S(t)} u_S\cdot\nabla\big((x-a)\wedge\big(\sigma(u,p)n\big)\big)\,d\gamma - \dot J\dot\omega - \int\limits_{\partial\Omega_S(t)}\dot a\wedge\big(\sigma(u,\rho)\big)n\,d\gamma,$$
$$G_4 = \omega\wedge\big(\omega\wedge(x-a)\big) - (u_S\cdot\nabla)u.$$

Recall that

$$u_S(t,x) = \dot a(t) + \omega(t)\wedge\big(x-a(t)\big)$$

is the velocity of the solid.

Now, we multiply the equation of ρ_t by $(p^0/\bar\rho)\rho_t$ and the equation of u_t by u_t. Let us see that the boundary terms provide estimates for $\dot\omega$ and $\ddot a$ and the remaining terms are bounded by $C(N^3(0,T)+N^4(0,T))$ and the data:

$$\iint_{\Sigma_s} \big(\sigma(u_t,\rho_t)n\big)\cdot u_t\,d\gamma\,dt = \frac{m}{2}\big(|\ddot a(t)|^2 - |\ddot a(0)|^2\big) - \int_0^s \ddot a G_2\,dt + \iint_{\Sigma_s} \dot\omega\cdot\big((x-a)\wedge\big(\sigma(u_t,\rho_t)n\big)\big)\,d\gamma\,dt + \iint_{\Sigma_s} G_4\cdot\big(\sigma(u_t,\rho_t)n\big)\,d\gamma\,dt. \tag{73}$$

In order to develop the third term in the right-hand side of this identity, we use that

$$\dot\omega\cdot\int_{\partial\Omega_S(t)} (x-a)\wedge\big(\sigma(u_t,\rho_t)n\big)\,d\gamma = (J\ddot\omega)\cdot\dot\omega - G_3\cdot\dot\omega. \tag{74}$$

Observe that we have

$$(J\ddot\omega)\cdot\dot\omega = \frac{1}{2}\frac{d}{dt}\big((J\dot\omega)\cdot\dot\omega\big) - \frac{1}{2}(\dot J\dot\omega)\cdot\dot\omega.$$

Now, we integrate between $t=0$ and $t=s$ in (74) and we use that J is a definite positive matrix (see (10)). We find

$$\iint_{\Sigma_s} \dot\omega\cdot\big((x-a)\wedge\big(\sigma(u_t,\rho_t)n\big)\big)\,d\gamma\,dt \geqslant (C_J/2)|\dot\omega(s)|^2 - C|\dot\omega(0)|^2 - \delta\|\dot\omega\|^2_{L^2_T} - C\int_0^T |G_3|^2\,d\tau - CN^3(0,T), \tag{75}$$

for $\delta>0$ small enough. Thanks to the estimate

$$\|G_4\|_{L^2(\Sigma_T)} \leqslant CN^2(0,T)$$

we have that the last term in identity (73) is bounded by $N^3(0,T)$.

For the term in G_0 we act as follows:

$$\iint_{Z_T} G_0\rho_t\,dx\,dt \leqslant C\big(\|f_{0,t}\|^2_{L^2_T(L^2)} + N^4(0,T)\big) + \delta\|\rho_t\|^2_{L^2_T(L^2)},$$

for $\delta>0$ small enough. The same can be done for the term in G_1.

Similarly as before, we find

$$\frac{1}{2}\sup_{s\in(0,T)}\left(|\ddot a|^2(s) + |\dot\omega|^2(s) + \int_{\Omega_F(s)}\left(\frac{p^0}{\bar\rho}|\rho_t|^2 + |u_t|^2\right)(s)\,dx\right) + \iint_{Z_T}\big(2\mu|\epsilon(u_t)|^2 + \mu'|\nabla\cdot u_t|^2\big)\,dx\,dt$$
$$\leqslant \delta\big(\|u_t\|^2_{L^2_T(L^2)} + \|\rho_t\|^2_{L^2_T(L^2)} + \|\ddot a\|^2_{L^2_T} + \|\dot\omega\|^2_{L^2_T}\big) + C\Bigg(\iint_{Z_T}\big(|\nabla\cdot u||\rho_t|^2 + \big|\nabla\cdot\big(|u_t|^2u\big)\big|\big)\,dx\,dt$$
$$+ \int_{\Omega_F(0)}\big(|\partial_t u(0,\cdot)|^2 + |\partial_t\rho(0,\cdot)|^2\big)\,dx + |\ddot a(0)|^2 + |\dot\omega(0)|^2 + \|f_{0,t}\|^2_{L^2_T(L^2)} + \|f_{1,t}\|^2_{L^2_T(L^2)}$$
$$+ \int_0^T |f_{2,t}|^2\,dt + \int_0^T |f_{3,t}|^2\,dt + N^3(0,T) + N^4(0,T)\Bigg),$$

for $\delta>0$ small enough. In order to prove this last inequality, we have used that

$$\|G_2 - f_{2,t}\|_{L^2_T} + \|G_3 - f_{3,t}\|_{L^2_T} \leqslant CN^2(0,T).$$

Thus, we deduce (70) for $k=1$. □

Lemma 14. *Let $k=0,1$. Then, there exists a constant $C>0$ such that*

$$\|u_t\|_{H^k_T(L^2)}+\|\rho_t\|_{H^k_T(L^2)}+\|u\|_{W^{k,\infty}_T(H^1)}+\|\ddot a\|_{H^k_T}+\|\dot\omega\|_{H^k_T}$$
$$\leqslant C\big(\|u\|_{H^k_T(H^1)}+\|\rho\|_{W^{k,\infty}_T(L^2)}+\|f_0\|_{H^k_T(L^2)}+\|f_1\|_{H^k_T(L^2)}+\|f_2\|_{H^k_T}+\|f_3\|_{H^k_T}$$
$$+\|u_0\|_{H^1}+\big\|\partial_t^k u(0,\cdot)\big\|_{H^1}+N^{3/2}(0,T)+N^2(0,T)\big). \tag{76}$$

Proof. 1) $k=0$. We multiply the equation of ρ by ρ_t and the equation of u by u_t. Integrating in Z_s for $s\in(0,T)$ and adding up both expressions, we obtain:

$$\frac12\int_{\Omega_F(s)}\big(2\mu\big|\epsilon(u)(s)\big|^2+\mu'\big|\nabla\cdot u(s)\big|^2\big)\,dx+\iint_{Z_s}\big(|\rho_t|^2+|u_t|^2\big)\,dx\,dt$$
$$+\iint_{\Sigma_s}u_t\cdot\big(\sigma(u,\rho)n\big)\,d\gamma\,dt+\iint_{Z_s}\big(\bar\rho(\nabla\cdot u)\rho_t-p^0(\nabla\cdot u_t)\rho\big)\,dx\,dt$$
$$\leqslant C\bigg(\iint_{Z_s}\big|\nabla\cdot\big(\big(2\mu\big|\epsilon(u)\big|^2+\mu'|\nabla\cdot u|^2\big)u\big)\big|\,dx\,dt+\int_{\Omega_F(0)}|\nabla u_0|^2\,dx$$
$$+\iint_{Z_s}\big(|f_0|^2+|f_1|^2+\delta\big(|\rho_t|^2+|u_t|^2\big)\big)\,dx\,dt+N^3(0,T)\bigg), \tag{77}$$

for $\delta>0$ small enough. Using the fifth equation in (72), the boundary term in (77) yields

$$\iint_{\Sigma_s}u_t\cdot\big(\sigma(u,\rho)n\big)\,d\gamma\,dt=m\int_0^s|\ddot a|^2\,dt-\int_0^s\ddot a\cdot f_2\,dt+\int_0^s\big((J\dot\omega)\cdot\dot\omega-f_3\cdot\dot\omega\big)\,dt+\iint_{\Sigma_s}G_4\cdot\big(\sigma(u,\rho)n\big)\,d\gamma\,dt$$
$$\geqslant\int_0^s\big((m/2)|\ddot a|^2+(C_J/2)|\dot\omega|^2\big)\,dt-C\bigg(\int_0^s\big(|f_2|^2+|f_3|^2\big)\,dt+N^3(0,T)\bigg).$$

The last integral in the left-hand side of (77) can be estimated as follows:

$$\iint_{Z_s}\big(\bar\rho(\nabla\cdot u)\rho_t-p^0(\nabla\cdot u_t)\rho\big)\,dx\,dt$$
$$\leqslant\frac12\iint_{Z_s}|\rho_t|^2\,dx\,dt+C\iint_{Z_s}|\nabla u|^2\,dx\,dt+\iint_{Z_s}p^0(\nabla\cdot u)\rho_t\,dx\,dt-p^0\int_{\Omega_F(t)}\big((\nabla\cdot u)\rho\big)(t)\big|_{t=0}^{t=s}\,dx$$
$$+\iint_{Z_s}p^0\nabla\cdot\big(\rho(\nabla\cdot u)u\big)\,dx\,dt. \tag{78}$$

Taking the supremum in s in (77), one can readily deduce (76) for $k=0$.

2) $k=1$. As in the proof of Lemma 13, we consider system (72). Let us multiply the equation of ρ_t by ρ_{tt} and the equation of u_t by u_{tt}. This provides:

$$\frac12\int_{\Omega_F(t)}\big(2\mu\big|\epsilon(u_t)(s)\big|^2+\mu'\big|\nabla\cdot u_t(s)\big|^2\big)\,dx+\iint_{Z_s}\big(|\rho_{tt}|^2+|u_{tt}|^2\big)\,dx\,dt$$
$$+\iint_{\Sigma_s}u_{tt}\cdot\big(\sigma(u_t,\rho_t)n\big)\,d\gamma\,dt+\iint_{Z_s}\big(\bar\rho(\nabla\cdot u_t)\rho_{tt}-p^0(\nabla\cdot u_{tt})\rho_t\big)\,dx\,dt$$
$$\leqslant C\bigg(\iint_{Z_s}\big|\nabla\cdot\big(\big(2\mu\big|\epsilon(u_t)\big|^2+\mu'|\nabla\cdot u_t|^2\big)u\big)\big|\,dx\,dt+\iint_{Z_s}|\rho_{tt}||u||\nabla\rho_t|\,dx\,dt$$

$$+\int_{\Omega_F(0)} \left|\partial_t \nabla u(0,\cdot)\right|^2 dx + \iint_{Z_s} \big(|G_0|^2+|G_1|^2+\delta\big(|\rho_{tt}|^2+|u_{tt}|^2\big)\big)\,dx\,dt\Big), \tag{79}$$

for $\delta > 0$ small enough. Let us now compute the boundary terms. On Σ_T, we have:

$$u_{tt} = \ddot{a} + \ddot{\omega} \wedge (x-a) + G_5,$$

with

$$G_5 = 2\dot{\omega} \wedge \big(\omega \wedge (x-a)\big) + \omega \wedge \big[\big(\dot{\omega} \wedge (x-a)\big) + \omega \wedge \big(\omega \wedge (x-a)\big)\big] - (u_{S,t}\cdot\nabla)u \\ - (u_S\cdot\nabla)\big[2u_t + (u_S\cdot\nabla)u\big].$$

Thus,

$$\iint_{\Sigma_s} u_{tt}\cdot(\sigma(u_t,\rho_t)n)\,d\gamma\,dt = m\int_0^s |\ddot{a}|^2 dt - \int_0^s \ddot{a}\cdot G_2\,dt + \iint_{\Sigma_s} \ddot{\omega}\cdot\big((x-a)\wedge\big(\sigma(u_t,\rho_t)\big)n\big)\,d\gamma\,dt \\ + \iint_{\Sigma_s} G_5\cdot\big(\sigma(u_t,\rho_t)n\big)\,d\gamma\,dt. \tag{80}$$

In order to develop the third term in the right-hand side of (80) we use that

$$\ddot{\omega}\cdot\int_{\partial\Omega_S(t)} (x-a)\wedge\big(\sigma(u_t,\rho_t)n\big)\,d\gamma = (J\ddot{\omega})\cdot\ddot{\omega} - G_3\cdot\ddot{\omega}. \tag{81}$$

Now, we integrate between $t=0$ and $t=s$ in (81); we find

$$\iint_{\Sigma_s} \ddot{\omega}\cdot\big((x-a)\wedge\big(\sigma(u_t,\rho_t)n\big)\big)\,d\gamma\,dt \geqslant (C_J/2)\int_0^s |\ddot{\omega}|^2\,dt - C\left(\int_0^s |f_{3,t}|^2\,dt + N^4(0,T)\right), \tag{82}$$

where C_J is such that (10) is satisfied. The last term in the right-hand side of (80) is also bounded by $N^3(0,T) + N^4(0,T)$.

Finally, the last term in the left-hand side of (79) is estimated as the corresponding term in the previous case (see (78)). Taking the supremum in s in (79), we conclude inequality (76) for $k=1$. □

Interior estimates. Before giving the results, we introduce the total time derivative of ρ:

$$\frac{d\rho}{dt} := \rho_t + (u\cdot\nabla)\rho \quad \text{in } Z_T. \tag{83}$$

Let $\zeta_{0,0} \in C_0^\infty(\Omega_F(0))$ and

$$\zeta_0(t,x) := \zeta_{0,0}\big(\psi(t,x)\big), \quad \forall x \in \Omega_F(t),\ t \in (0,T),$$

where ψ is the flow (extending χ_S^{-1}) defined in Corollary 2.

Lemma 15. *For $1 \leqslant k \leqslant 3$, we have*

$$\|\zeta_0\nabla\rho\|_{L^\infty_T(H^{k-1})} + \|\zeta_0\nabla\rho\|_{L^2_T(H^{k-1})} + \|\zeta_0\nabla u\|_{L^\infty_T(H^{k-1})} + \|\zeta_0\nabla u\|_{L^2_T(H^k)} + \left\|\zeta_0\nabla\frac{d\rho}{dt}\right\|_{L^2_T(H^{k-1})} \\ \leqslant C\big(\|u\|_{L^\infty_T(H^{k-1})} + \|u\|_{L^2_T(H^k)} + \|\rho_0\|_{H^k} + \|u_0\|_{H^k} + \|f_0\|_{L^2_T(H^k)} + \|f_1\|_{L^2_T(H^{k-1})} + N^{3/2}(0,T)\big). \tag{84}$$

Proof. In this proof, we denote by D^ℓ all possible derivatives in x of order ℓ. We divide the proof in two steps:

• First, we apply the operator $\zeta_0^2 \partial_{x_j} D^{k-1}$ to the equation of ρ, we multiply it by $\partial_{x_j} D^{k-1}\rho$, we sum up in $j = 1, 2, 3$ and we integrate in Z_s:

$$\sum_{j=1}^{3} \iint_{Z_s} \zeta_0^2 \partial_{x_j} D^{k-1}\big(\rho_t + (u \cdot \nabla)\rho + \bar{\rho}(\nabla \cdot u) - f_0\big)\partial_{x_j} D^{k-1}\rho \, dx\, dt = 0. \tag{85}$$

The first term of this expression coincides with

$$\sum_{j=1}^{3}\left(\frac{1}{2}\int_0^s \frac{d}{dt}\int_{\Omega_F(t)} \zeta_0^2\big|\partial_{x_j} D^{k-1}\rho\big|^2 \, dx\, dt - \iint_{Z_s} \zeta_0\big(\nabla\zeta_{0,0}(\psi)\cdot\psi_t\big)\big|\partial_{x_j} D^{k-1}\rho\big|^2 \, dx\, dt\right). \tag{86}$$

Here, we have used the fact that ζ_0 is compactly supported, so

$$\iint_{Z_s} \nabla \cdot \big(\zeta_0^2\big|\partial_{x_j} D^{k-1}\rho\big|^2 u\big)\, dx\, dt = 0. \tag{87}$$

Thanks to (22), we have that the second integral in (86) is estimated by $CN^3(0,T)$.

As long as the nonlinear term in (85) is concerned, using (87) we have

$$\begin{aligned}
&\sum_{j=1}^{3} \iint_{Z_s} \zeta_0^2 \partial_{x_j} D^{k-1}\big[(u\cdot\nabla)\rho\big]\partial_{x_j} D^{k-1}\rho \, dx\, dt \\
&\quad\leqslant CN^3(0,T) + \sum_{j=1}^{3}\iint_{Z_s} \zeta_0^2\big[(u\cdot\nabla)\partial_{x_j} D^{k-1}\rho\big]\partial_{x_j} D^{k-1}\rho\, dx\, dt \\
&\quad= CN^3(0,T) - \frac{1}{2}\sum_{j=1}^{3}\iint_{Z_s} \nabla\cdot(\zeta_0^2 u)\big|\partial_{x_j} D^{k-1}\rho\big|^2 \, dx\, dt \leqslant CN^3(0,T).
\end{aligned} \tag{88}$$

Putting all this together with (85), we find

$$\begin{aligned}
&\|\zeta_0\nabla\rho\|^2_{L^\infty_T(H^{k-1})} + \bar{\rho}\iint_{Z_s} \zeta_0^2 \nabla\big(D^{k-1}(\nabla\cdot u)\big)\cdot\nabla\big(D^{k-1}\rho\big)\, dx\, dt \\
&\quad\leqslant \delta\big\|\zeta_0 D^k\rho\big\|^2_{L^2_T(L^2)} + C\big(\|\rho_0\|^2_{H^k} + \|f_0\|^2_{L^2_T(H^k)} + N^3(0,T)\big),
\end{aligned} \tag{89}$$

with $\delta > 0$ small enough.

Next, we apply the operator $\frac{\bar{\rho}}{2\mu+\mu'}\zeta_0^2 D^{k-1}$ to the j-th equation of u, we multiply it by $\partial_{x_j} D^{k-1}\rho$, we sum up in $j = 1, 2, 3$ and we integrate in Z_s:

$$\frac{\bar{\rho}}{2\mu+\mu'}\sum_{j=1}^{3}\iint_{Z_s} \zeta_0^2 D^{k-1}\big(u_t - \nabla\cdot\sigma(u,\rho) - f_1\big)_j \partial_{x_j} D^{k-1}\rho\, dx\, dt = 0. \tag{90}$$

We integrate by parts in t in the term concerning u_t:

$$\begin{aligned}
&\iint_{Z_s} \zeta_0^2 D^{k-1}u_{j,t}\partial_{x_j} D^{k-1}\rho\, dx\, dt \\
&\quad= -\iint_{Z_s} \nabla\cdot\big(\big(\zeta_0^2 D^{k-1}u_j\partial_{x_j} D^{k-1}\rho\big)u\big)\, dx\, dt - 2\iint_{Z_s} \zeta_0\big(\nabla\zeta_{0,0}(\psi)\cdot\psi_t\big)D^{k-1}u_j\partial_{x_j} D^{k-1}\rho\, dx\, dt \\
&\qquad - \iint_{Z_s} \zeta_0^2 D^{k-1}u_j\partial_{x_j} D^{k-1}\rho_t\, dx\, dt.
\end{aligned}$$

The first term in the right-hand side is zero thanks to (87) while the second one can be estimated by $CN^3(0,T)$ due to (22). As long as the third term is concerned, we use the equation of ρ in (61); arguing as in (88), we obtain

$$\iint_{Z_s} \zeta_0^2 D^{k-1} u_{j,t} \partial_{x_j} D^{k-1}\rho \,dx\,dt \leqslant \delta \|\zeta_0\rho\|^2_{L^\infty_T(H^k)} + C\big(\|f_0\|^2_{L^2_T(H^k)} + \|u\|^2_{L^2_T(H^k)} + \|u\|^2_{L^\infty_T(H^{k-1})} + N^3(0,T)\big).$$

As long as the elliptic term $-2\mu\nabla\cdot(\epsilon(u)) - \mu'\nabla\cdot((\nabla\cdot u)\,\mathrm{Id})$ is concerned, we rewrite it as

$$-2\mu\nabla\cdot\big(\epsilon(u)\big) - \mu'\nabla\cdot\big((\nabla\cdot u)\,\mathrm{Id}\big) = -\mu\Delta u - (\mu+\mu')\nabla(\nabla\cdot u)$$

and we integrate by parts twice (with respect to x) in the Laplacian term. Observe that there is no boundary term when we integrate in x since ζ_0 has a compact support. Then, from (90) we deduce that

$$\frac{\bar\rho p^0}{2\mu+\mu'}\big\|\zeta_0 D^k\rho\big\|^2_{L^2_T(L^2)} - \bar\rho \iint_{Z_s} \zeta_0^2 \nabla\big(D^{k-1}(\nabla\cdot u)\big)\cdot\nabla\big(D^{k-1}\rho\big)\,dx\,dt$$
$$\leqslant \delta\big(\big\|\zeta_0 D^k\rho\big\|^2_{L^2_T(L^2)} + \|\zeta_0\rho\|^2_{L^\infty_T(H^k)}\big) + C\big(\|u\|^2_{L^\infty_T(H^{k-1})} + \|u\|^2_{L^2_T(H^k)} + \|\rho_0\|^2_{H^k} + \|f_0\|^2_{L^2_T(H^k)}$$
$$+ \|f_1\|^2_{L^2_T(H^{k-1})} + N^3(0,T)\big).$$

This, together with (89) and taking the supremum in s, yields

$$\|\zeta_0\nabla\rho\|_{L^\infty_T(H^{k-1})} + \|\zeta_0\nabla\rho\|_{L^2_T(H^{k-1})}$$
$$\leqslant C\big(\|u\|_{L^\infty_T(H^{k-1})} + \|u\|_{L^2_T(H^k)} + \|\rho_0\|_{H^k} + \|f_0\|_{L^2_T(H^k)} + \|f_1\|_{L^2_T(H^{k-1})} + N^{3/2}(0,T)\big). \tag{91}$$

- Let us now apply the operator $(p^0/\bar\rho)\zeta_0^2 D^k$ to the equation of ρ, we multiply it by $D^k\rho$ and we integrate in Z_s:

$$\frac{p^0}{\bar\rho}\iint_{Z_s} \zeta_0^2 D^k\big(\rho_t + (u\cdot\nabla)\rho + \bar\rho(\nabla\cdot u) - f_0\big) D^k\rho\,dx\,dt = 0.$$

Similarly as before (using (22) and (87)), this gives

$$\|\zeta_0\nabla\rho\|^2_{L^\infty_s(H^{k-1})} + p^0\iint_{Z_s} \zeta_0^2 D^k(\nabla\cdot u) D^k\rho\,dx\,d\tau$$
$$\leqslant \delta\big\|\zeta_0 D^k\rho\big\|^2_{L^2_T(L^2)} + C\big(\|\rho_0\|^2_{H^k} + \|f_0\|^2_{L^2_T(H^k)} + N^3(0,T)\big). \tag{92}$$

Next, we apply the operator $\zeta_0^2 D^k$ to the j-th equation of u, we multiply it by $D^k u_j$, we sum up in $j = 1,2,3$ and we integrate in Z_s:

$$\sum_{j=1}^3 \iint_{Z_s} \zeta_0^2 D^k\big(u_t - \nabla\cdot\sigma(u,\rho) - f_1\big)_j D^k u_j\,dx\,dt = 0.$$

We integrate by parts in x in the second and third terms of the last integral. We deduce that

$$\|\zeta_0 u\|^2_{L^\infty_T(H^k)} + \|\zeta_0 u\|^2_{L^2_T(H^{k+1})} - p^0\iint_{Z_s} \zeta_0^2 D^k(\nabla\cdot u) D^k\rho\,dx\,d\tau$$
$$\leqslant \delta\|\zeta_0\rho\|^2_{L^2_T(H^k)} + C\big(\|u\|^2_{L^2_T(H^k)} + \|u_0\|^2_{H^k} + \|f_1\|^2_{L^2_T(H^{k-1})} + N^3(0,T)\big),$$

for $\delta > 0$ small enough. This, together with (92), yields

$$\|\zeta_0\rho\|_{L^\infty_T(H^k)} + \|\zeta_0 u\|_{L^\infty_T(H^k)} + \|\zeta_0 u\|_{L^2_T(H^{k+1})}$$
$$\leqslant \delta\|\zeta_0\rho\|_{L^2_T(H^k)} + C\big(\|u\|_{L^2_T(H^k)} + \|u_0\|_{H^k} + \|\rho_0\|_{H^k} + \|f_0\|_{L^2_T(H^k)} + \|f_1\|_{L^2_T(H^{k-1})} + N^{3/2}(0,T)\big).$$

Combining this with (91) we obtain the conclusion (84). □

Boundary estimates. Let us now do some estimates close to the boundary. For this, we consider a finite covering $\{\mathcal{O}_i\}_{i=1}^K$ of $\partial\Omega_F(0)$ included in the set $\{y\colon \operatorname{dist}(y, \partial\Omega_F(0)) < \alpha/4\}$. For each $1 \leqslant i \leqslant K$, we consider a C^4 function

$$\begin{aligned}\theta^i : \big(-M_1^i, M_1^i\big) \times \big(-M_2^i, M_2^i\big) &\to \partial\Omega_F(0) \cap \mathcal{O}_i,\\ (\phi_1, \phi_2) &\to \theta^i(\phi_1, \phi_2)\end{aligned} \tag{93}$$

satisfying

$$\partial_{\phi_1}\theta^i \cdot \partial_{\phi_2}\theta^i = 0, \quad \big|\partial_{\phi_1}\theta^i\big| = 1,\ \big|\partial_{\phi_2}\theta^i\big| \geqslant C > 0.$$

Then, we define the mapping

$$\tilde\theta^i(t, \phi_1, \phi_2) = \chi\big(t, \theta^i(\phi_1, \phi_2)\big), \quad \forall (t, \phi_1, \phi_2) \in (0, T) \times \big(-M_1^i, M_1^i\big) \times \big(-M_2^i, M_2^i\big)$$

which sends $(0, T) \times (-M_1^i, M_1^i) \times (-M_2^i, M_2^i)$ in the set $\partial\Omega_F(t) \cap \tilde{\mathcal{O}}_i$, where $\tilde{\mathcal{O}}_i := \chi(t, \mathcal{O}_i)$. Recall that the flow χ was defined in Lemma 1.

Furthermore, we perform the change of variables

$$\begin{cases} x(t, \phi_1, \phi_2, r) = r\tilde n(t, \phi_1, \phi_2) + \tilde\theta^i(t, \phi_1, \phi_2)) \in \tilde{\mathcal{O}}_i \cap \Omega_F(t),\\ r \in (0, r_0^i), \quad (\phi_1, \phi_2) \in l(-M_1^i, M_1^i) \times (-M_2^i, M_2^i), \end{cases} \tag{94}$$

where $n(\phi_1, \phi_2)$ (resp. $\tilde n(t, \phi_1, \phi_2) := \nabla\chi(t, \theta^i(\phi_1, \phi_2))n(\phi_1, \phi_2)$) denotes the inward unit normal vector to $\partial\Omega_F(0)$ (resp. $\partial\Omega_F(t)$).

In what follows, we will establish some estimates of u and ρ in $\Omega_F(t) \cap \tilde{\mathcal{O}}_i$ for each $1 \leqslant i \leqslant K$ (see Lemmas 16, 17 and 18 below). For the sake of simplicity, we drop the index i from now on. In the new variables, we define

$$\begin{cases} e_1(t, \phi_1, \phi_2) = \partial_{\phi_1}\tilde\theta(t, \phi_1, \phi_2) \quad \text{and} \quad e_2(t, \phi_1, \phi_2) = \dfrac{\partial_{\phi_2}\tilde\theta}{|\partial_{\phi_2}\tilde\theta|}(t, \phi_1, \phi_2),\\ \forall (t, \phi_1, \phi_2) \in (0, T) \times (-M_1, M_1) \times (-M_2, M_2). \end{cases}$$

Observe that $e_1 \cdot e_2 = \partial_{\phi_1}\theta \cdot \frac{\partial_{\phi_2}\theta}{|\partial_{\phi_2}\theta|} = 0$, $e_1 \cdot \tilde n = \partial_{\phi_1}\theta \cdot n = 0$ and $e_2 \cdot \tilde n = \frac{\partial_{\phi_2}\theta}{|\partial_{\phi_2}\theta|} \cdot n = 0$ and so we can write

$$\partial_{\phi_1}\tilde n = \alpha e_1 + \beta e_2 \quad \text{and} \quad \partial_{\phi_2}\tilde n = \alpha' e_1 + \beta' e_2, \tag{95}$$

for some $\alpha, \beta, \alpha', \beta' \in H^3(0, T; C^3([-M_1, M_1] \times [-M_2, M_2]))$. Let us compute the Jacobian of this change of variables:

$$\partial_{\phi_1}x = (1 + \alpha r)e_1 + \beta r e_2, \qquad \partial_{\phi_2}x = \alpha' r e_1 + \big(\beta' r + |\partial_{\phi_2}\tilde\theta|\big)e_2 \quad \text{and} \quad \partial_r x = \tilde n, \tag{96}$$

that is to say,

$$\begin{pmatrix} \partial_{\phi_1}x\\ \partial_{\phi_2}x\\ \partial_r x \end{pmatrix} = \begin{pmatrix} 1 + \alpha r & \beta r & 0\\ \alpha' r & \beta' r + |\partial_{\phi_2}\tilde\theta| & 0\\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_1\\ e_2\\ \tilde n \end{pmatrix}.$$

Inversely, we have

$$\begin{pmatrix} \partial_x\phi_1\\ \partial_x\phi_2\\ \partial_x r \end{pmatrix} = \frac{1}{\bar J} \begin{pmatrix} \beta' r + |\partial_{\phi_2}\tilde\theta| & -\beta r & 0\\ -\alpha' r & 1 + \alpha r & 0\\ 0 & 0 & \bar J \end{pmatrix} \begin{pmatrix} e_1\\ e_2\\ \tilde n \end{pmatrix},$$

where $\bar J := (1 + \alpha r)(\beta' r + |\partial_{\phi_2}\tilde\theta|) - \alpha'\beta r^2$ is the Jacobian.

Consequently, we have

$$\nabla_x = \partial_x = \frac{1}{\bar J}(A_1 e_1 + A_2 e_2)\partial_{\phi_1} + \frac{1}{\bar J}(A_3 e_1 + A_4 e_2)\partial_{\phi_2} + \tilde n\partial_r \tag{97}$$

where

$$A_1 = \beta' r + |\partial_{\phi_2}\tilde\theta|, \quad A_2 = -\beta r, \quad A_3 = -\alpha' r \quad \text{and} \quad A_4 = 1 + \alpha r.$$

Now, if we denote

$$\tilde\rho(t, \phi, r) := \rho\big(t, x(t, \phi, r)\big), \qquad \tilde u(t, \phi, r) := u\big(t, x(t, \phi, r)\big) \tag{98}$$

and

$$\tilde{f}_0(t,\phi,r) := f_0\big(t, x(t,\phi,r)\big), \qquad \tilde{f}_1(t,\phi,r) := f_1\big(t, x(t,\phi,r)\big),$$

we can rewrite the equations of (61) in the new variables (ϕ_1, ϕ_2, r) as follows:

$$\begin{cases} \tilde{\rho}_t + \tilde{u}\cdot\left(\frac{1}{\bar{J}}(A_1e_1 + A_2e_2)\partial_{\phi_1}\tilde{\rho} + \frac{1}{\bar{J}}(A_3e_1 + A_4e_2)\partial_{\phi_2}\tilde{\rho} + \tilde{n}\partial_r\tilde{\rho}\right) \\ \quad - \big((r\nabla\chi_t\circ\theta)n + \chi_t\circ\theta\big)\left(\frac{1}{\bar{J}}(A_1e_1 + A_2e_2)\partial_{\phi_1}\tilde{\rho} + \frac{1}{\bar{J}}(A_3e_1 + A_4e_2)\partial_{\phi_2}\tilde{\rho} + \tilde{n}\partial_r\tilde{\rho}\right) \\ \quad + \bar{\rho}\left(\frac{1}{\bar{J}}(A_1e_1 + A_2e_2)\cdot\partial_{\phi_1}\tilde{u} + \frac{1}{\bar{J}}(A_3e_1 + A_4e_2)\cdot\partial_{\phi_2}\tilde{u} + \tilde{n}\cdot\partial_r\tilde{u}\right) = \tilde{f}_0, \\ (t,\phi_1,\phi_2,r) \in (0,T)\times(-M_1,M_1)\times(-M_2,M_2)\times(0,r_0) \end{cases} \tag{99}$$

and

$$\begin{cases} \tilde{u}_t - \frac{\mu}{\bar{J}^2}\Big(\big(A_1^2 + A_2^2\big)\partial^2_{\phi_1\phi_1}\tilde{u} + 2(A_1A_3 + A_2A_4)\partial^2_{\phi_1\phi_2}\tilde{u} + \big(A_3^2 + A_4^2\big)\partial^2_{\phi_2\phi_2}\tilde{u} + \bar{J}^2\partial^2_{rr}\tilde{u}\Big) + P_1(D)\tilde{u} \\ \quad - \big((r\nabla\chi_t\circ\theta)n + \chi_t\circ\theta\big)\left(\frac{1}{\bar{J}}(A_1e_1 + A_2e_2)\partial_{\phi_1}\tilde{u} + \frac{1}{\bar{J}}(A_3e_1 + A_4e_2)\partial_{\phi_2}\tilde{u} + \tilde{n}\partial_r\tilde{u}\right) \\ \quad + \frac{1}{\bar{J}}\partial_{\phi_1}\left(\frac{\mu+\mu'}{\bar{\rho}}\frac{d\tilde{\rho}}{dt} + p^0\tilde{\rho}\right)(A_1e_1 + A_2e_2) + \frac{1}{\bar{J}}\partial_{\phi_2}\left(\frac{\mu+\mu'}{\bar{\rho}}\frac{d\tilde{\rho}}{dt} + p^0\tilde{\rho}\right)(A_3e_1 + A_4e_2) \\ \quad + \partial_r\left(\frac{\mu+\mu'}{\bar{\rho}}\frac{d\tilde{\rho}}{dt} + p^0\tilde{\rho}\right)\tilde{n} = \tilde{f}_1 + \frac{\mu+\mu'}{\bar{\rho}}\nabla_x f_0, \\ (t,\phi_1,\phi_2,r) \in (0,T)\times(-M_1,M_1)\times(-M_2,M_2)\times(0,r_0), \end{cases} \tag{100}$$

where $P_1(D)$ is a first order differential operator in the x variables. In order to obtain this last equation, observe that we have used that

$$u_t - \mu\Delta u + \frac{\mu+\mu'}{\bar{\rho}}\nabla\left(\frac{d\rho}{dt} - f_0\right) + p^0\nabla\rho = f_1$$

and then we have rewritten this in the new variables. In the new variables $\frac{d\tilde{\rho}}{dt}$, which is given by the two first lines in (99), satisfies

$$\frac{d\tilde{\rho}}{dt}(t,\phi,r) = \frac{d\rho}{dt}\big(t, x(t,\phi,r)\big), \quad \forall (t,\phi,r) \in (0,T)\times(-M_1,M_1)\times(-M_2,M_2)\times(0,r_0), \tag{101}$$

where $x(t,\phi,r)$ is given by (94).

Let $\zeta^i_{1,1} \in C_0^\infty(\mathcal{O}_i)$ and

$$\zeta^i_1(t,x) := \zeta^i_{1,1}\big(\psi(t,x)\big), \quad \forall x \in \tilde{\mathcal{O}}_i,\ t \in (0,T),$$

where ψ is the flow (extending χ_S^{-1}) defined in Corollary 2. We will also drop here the index i.

We first estimate the tangential derivatives:

Lemma 16. *Let $k = 1,2,3$. For any $\delta > 0$, there exists a positive constant C such that*

$$\begin{aligned} &\big\|\zeta_1 D^k_\phi\rho\big\|_{L^\infty_T(L^2)} + \big\|\zeta_1 D^k_\phi u\big\|_{L^\infty_T(L^2)} + \big\|\zeta_1 D^k_\phi u\big\|_{L^2_T(H^1)} + \left\|\zeta_1 D^k_\phi\frac{d\rho}{dt}\right\|_{L^2_T(L^2)} \\ &\quad\leqslant \delta\big(\big\|\zeta_1 D^k_\phi\rho\big\|_{L^2_T(L^2)} + \|u\|_{L^2_T(H^2)} + \|\rho\|_{L^2_T(H^1)}\big) + C\big(\|u\|_{L^2_T(H^k)} + \|\omega\|_{L^2_T} + \|\rho_0\|_{H^k} \\ &\qquad + \|u_0\|_{H^k} + \|f_0\|_{L^2_T(H^k)} + \|f_1\|_{L^2_T(H^{k-1})} + N^{3/2}(0,T)\big). \end{aligned} \tag{102}$$

Proof. The proof of this lemma is very similar to that of the second part of Lemma 15. In fact, we first apply the operator $\frac{p^0}{\bar{\rho}}\zeta_1^2 D_\phi^k$ to the equation of ρ in (61) and multiply it by $D_\phi^k\rho$:

$$\frac{p^0}{\bar{\rho}}\iint\limits_{Z_s} \zeta_1^2 D_\phi^k\big(\rho_t + (u\cdot\nabla_x)\rho + \bar{\rho}(\nabla_x\cdot u) - f_0\big)D_\phi^k\rho\,dx\,d\tau = 0. \tag{103}$$

We recall now that (see (97))

$$\nabla_x = B_1\partial_{\phi_1} + B_2\partial_{\phi_2} + B_3\partial_r,$$

where B_j $(j = 1,2,3)$ belongs to $H^3(0,T;C^3([-M_1,M_1]\times[-M_2,M_2]))$. Then, we notice that

$$D_\phi^k\nabla_x h = \nabla_x D_\phi^k h + \sum_{\ell=1}^{k}\binom{k}{\ell}\big(D_\phi^\ell B_1 D_\phi^{k-\ell}\partial_{\phi_1} + D_\phi^\ell B_2 D_\phi^{k-\ell}\partial_{\phi_2} + D_\phi^\ell B_3 D_\phi^{k-\ell}\partial_r\big)h \tag{104}$$

(the same can be done for $D_r^\ell\nabla_x h$). Using this expression, we have the following for the nonlinear term:

$$\iint\limits_{Z_s}\zeta_1^2 D_\phi^k(u\cdot\nabla_x\rho)D_\phi^k\rho\,dx\,d\tau \geqslant \iint\limits_{Z_s}\zeta_1^2\big(u\cdot\nabla_x D_\phi^k\rho\big)D_\phi^k\rho\,dx\,d\tau - CN^3(0,T). \tag{105}$$

As long as the time derivative is concerned, we get

$$\begin{aligned}\iint\limits_{Z_s}\zeta_1^2 D_\phi^k\rho_t D_\phi^k\rho\,dx\,d\tau &= \frac{1}{2}\big(\|\zeta_1 D_\phi^k\rho\|_{L^2}^2(s) - \|\zeta_1(0)D_\phi^k\rho_0\|_{L^2}^2\big) - \iint\limits_{Z_s}\zeta_1\zeta_{1,t}\big|D_\phi^k\rho\big|^2\,dx\,d\tau\\ &\quad - \frac{1}{2}\iint\limits_{Z_s}\zeta_1^2 u\cdot\nabla_x\big(\big|D_\phi^k\rho\big|^2\big)\,dx\,d\tau - \frac{1}{2}\iint\limits_{Z_s}\nabla_x\cdot\big(\zeta_1^2 u\big)\big|D_\phi^k\rho\big|^2\,dx\,d\tau.\end{aligned}$$

Observe that the last terms in the first and second lines can be estimated by $N^3(0,T)$ and that the first term in the second line simplifies with the first term in the right-hand side of (105). Combining with (103), this implies that

$$\begin{aligned}&\|\zeta_1 D_\phi^k\rho\|_{L^2}^2(s) + p^0\iint\limits_{Z_s}\zeta_1^2 D_\phi^k(\nabla_x\cdot u)D_\phi^k\rho\,dx\,d\tau\\ &\quad\leqslant \delta\|\zeta_1 D_\phi^k\rho\|_{L_T^2(L^2)}^2 + C\big(\|\rho_0\|_{H^k}^2 + \|f_0\|_{L_T^2(H^k)}^2 + N^3(0,T)\big).\end{aligned} \tag{106}$$

Then, we apply the operator $\zeta_1^2 D_\phi^k$ to the jth equation of u in (61), multiply it by $D_\phi^k u_j$ and sum up in j:

$$\sum_{j=1}^{3}\iint\limits_{Z_s}\zeta_1^2 D_\phi^k\big(u_t - 2\mu\nabla_x\cdot\epsilon(u) - \mu'\nabla_x(\nabla\cdot u) + p^0\nabla_x\rho - f_1\big)_j D_\phi^k u_j\,dx\,d\tau = 0. \tag{107}$$

Observe that, since ζ_1 has compact support, we have

$$-\iint\limits_{Z_s}\zeta_1^2 D_\phi^k\nabla_x\cdot\epsilon(u)_j D_\phi^k u_j\,dx\,d\tau = (-1)^{k+1}\iint\limits_{Z_s}\nabla_x\cdot\epsilon(u)_j D_\phi^k\big(\zeta_1^2 D_\phi^k u_j\big)\,dx\,d\tau. \tag{108}$$

Integrating by parts on the x variable, we deduce:

$$\begin{aligned}&-\iint\limits_{Z_s}\zeta_1^2 D_\phi^k\nabla_x\cdot\epsilon(u)_j D_\phi^k u_j\,dx\,d\tau\\ &\quad= (-1)^k\iint\limits_{Z_s}\epsilon(u)_j\cdot\nabla_x D_\phi^k\big(\zeta_1^2 D_\phi^k u_j\big)\,dx\,d\tau + (-1)^k\iint\limits_{\Sigma_s}\big(\epsilon(u)_j\cdot n\big)D_\phi^k\big(\zeta_1^2 D_\phi^k u_j\big)\,d\sigma\,d\tau\\ &\quad\geqslant (-1)^k\iint\limits_{Z_s}\epsilon(u)_j\cdot\nabla_x D_\phi^k\big(\zeta_1^2 D_\phi^k u_j\big)\,dx\,d\tau - C\big(\delta\|u\|_{L_T^2(H^2)}^2 + \|\omega\|_{L_T^2}^2\big),\end{aligned} \tag{109}$$

where we have used that $|D_\phi^\ell u| \leqslant C|\omega|$ for all $|\ell| \geqslant 1$.

Let us take $h = \zeta_1^2 D_\phi^k u_j$ in (104). For each term of the sum, we have

$$\left| \iint_{Z_s} \epsilon(u)_j \cdot \left(D_\phi^\ell B_1 D_\phi^{k-\ell} \partial_{\phi_1}\left(\zeta_1^2 D_\phi^k u_j\right) + D_\phi^\ell B_2 D_\phi^{k-\ell} \partial_{\phi_2}\left(\zeta_1^2 D_\phi^k u_j\right) + D_\phi^\ell B_3 D_\phi^{k-\ell} \partial_r\left(\zeta_1^2 D_\phi^k u_j\right)\right) dx\, d\tau \right|$$
$$\leqslant \left| \iint_{Z_s} D_\phi^{k-\ell}\left(D_\phi^\ell B_1 \epsilon(u)_j\right)\partial_{\phi_1}\left(\zeta_1^2 D_\phi^k u_j\right) + D_\phi^{k-\ell}\left(D_\phi^\ell B_2 \epsilon(u)_j\right)\partial_{\phi_2}\left(\zeta_1^2 D_\phi^k u_j\right) dx\, d\tau \right|$$
$$+ \left| \iint_{Z_s} D_\phi^{k-\ell}\left(D_\phi^\ell B_3 \epsilon(u)_j\right)\partial_r\left(\zeta_1^2 D_\phi^k u_j\right) dx\, d\tau \right| \leqslant C\left(\|u\|^2_{L^2_T(H^k)} + \delta \left\|\zeta_1 D_\phi^k u\right\|^2_{L^2_T(H^1)}\right). \tag{110}$$

This implies that

$$(-1)^k \iint_{Z_s} \epsilon(u)_j \cdot \nabla_x D_\phi^k\left(\zeta_1^2 D_\phi^k u_j\right) dx\, d\tau$$
$$\geqslant (-1)^k \iint_{Z_s} \epsilon(u)_j \cdot D_\phi^k \nabla_x\left(\zeta_1^2 D_\phi^k u_j\right) dx\, d\tau - C\left(\|u\|^2_{L^2_T(H^k)} + \delta \left\|\zeta_1 D_\phi^k u\right\|^2_{L^2_T(H^1)}\right)$$
$$\geqslant \iint_{Z_s} D_\phi^k \epsilon(u)_j \cdot \nabla_x\left(\zeta_1^2 D_\phi^k u_j\right) dx\, d\tau - C\left(\|u\|^2_{L^2_T(H^k)} + \delta \left\|\zeta_1 D_\phi^k u\right\|^2_{L^2_T(H^1)}\right). \tag{111}$$

Analogously, we can prove that

$$\iint_{Z_s} D_\phi^k \epsilon(u)_j \cdot \nabla_x\left(\zeta_1^2 D_\phi^k u_j\right) dx\, d\tau$$
$$\geqslant \iint_{Z_s} \epsilon\left(\zeta_1 D_\phi^k u\right)_j \cdot \nabla_x\left(\zeta_1 D_\phi^k u_j\right) dx\, d\tau - C\left(\|u\|^2_{L^2_T(H^k)} + \delta \left\|\zeta_1 D_\phi^k u\right\|^2_{L^2_T(H^1)}\right). \tag{112}$$

So, from (108)–(112), we have for the second term in (107):

$$-\iint_{Z_s} \zeta_1^2 D_\phi^k \nabla_x \cdot \epsilon(u)_j D_\phi^k u_j\, dx\, d\tau \geqslant \iint_{Z_s} \epsilon\left(\zeta_1 D_\phi^k u\right)_j \cdot \nabla_x\left(\zeta_1 D_\phi^k u_j\right) dx\, d\tau$$
$$- C\left(\|u\|^2_{L^2_T(H^k)} + \delta \left\|\zeta_1 D_\phi^k u\right\|^2_{L^2_T(H^1)} + \delta \|u\|^2_{L^2_T(H^2)} + \|\omega\|^2_{L^2_T}\right). \tag{113}$$

The same can also be done for the third term in (107):

$$-\sum_{j=1}^3 \iint_{Z_s} \zeta_1^2 D_\phi^k \partial_{x_j}(\nabla_x \cdot u) D_\phi^k u_j\, dx\, d\tau$$
$$\geqslant \iint_{Z_s} \left|\nabla_x \cdot \left(\zeta_1 D_\phi^k u\right)\right|^2 dx\, d\tau - C\left(\|u\|^2_{L^2_T(H^k)} + \delta \left\|\zeta_1 D_\phi^k u\right\|^2_{L^2_T(H^1)} + \delta \|u\|^2_{L^2_T(H^2)} + \|\omega\|^2_{L^2_T}\right). \tag{114}$$

Eqs. (113) and (114) provide the estimate of $\|\zeta_1 D_\phi^k u\|^2_{L^2_T(H^1)}$ in (102).

Then, using the same arguments, one can prove that

$$p^0 \sum_{j=1}^3 \iint_{Z_s} \zeta_1^2 D_\phi^k \partial_{x_j} \rho D_\phi^k u_j\, dx\, d\tau \geqslant -p^0 \iint_{Z_s} \zeta_1^2 D_\phi^k (\nabla_x \cdot u) D_\phi^k \rho\, dx\, d\tau$$
$$- \delta\left(\|\rho\|^2_{L^2_T(H^1)} + \left\|\zeta_1 D_\phi^k \rho\right\|^2_{L^2_T(L^2)}\right) + C\left(\|\omega\|^2_{L^2_T} + \|u\|^2_{L^2_T(H^k)}\right). \tag{115}$$

Then, inequality (102) is obtained by reassembling (106) (for $\|\zeta_1 D_\phi^k \rho\|_{L^2_T(L^2)}$), (107) (for $\|\zeta_1 D_\phi^k u\|_{L^\infty(L^2)}$), (113)–(114) (for $\|\zeta_1 D_\phi^k u\|^2_{L^2_T(H^1)}$ and, consequently, $\|\zeta_1 D_\phi^k \frac{d\rho}{dt}\|_{L^2_T(L^2)}$) and (115).

Let us now take the derivative with respect to r of Eq. (99), multiply it by $\mu/\bar\rho$ and sum it with Eq. (100) multiplied by $\tilde n$ (which we can symbolically write as $(\mu/\bar\rho)(99)_r + \tilde n \cdot (100)$):

$$\begin{cases} \dfrac{2\mu+\mu'}{\bar\rho}\left(\dfrac{d\tilde\rho}{dt}\right)_r + p^0\tilde\rho_r = -\dfrac{\mu}{\bar J^2}\big[\big(A_1^2+A_2^2\big)\tilde n\cdot\tilde u_{\phi_1\phi_1} + 2(A_1A_3+A_2A_4)\tilde n\cdot\tilde u_{\phi_1\phi_2} \\ \quad + \big(A_3^2+A_4^2\big)\tilde n\cdot\tilde u_{\phi_2\phi_2} - \bar J(A_1e_1+A_2e_2)\cdot\tilde u_{\phi_1 r} - \bar J(A_3e_1+A_4e_2)\cdot\tilde u_{\phi_2 r}\big] \\ \quad - \tilde n\cdot\tilde u_t - \big((r\nabla\chi_t\circ\theta)n + \chi_t\circ\theta\big)\left(\dfrac{1}{\bar J}(A_1e_1+A_2e_2)\partial_{\phi_1}\tilde u + \dfrac{1}{\bar J}(A_3e_1+A_4e_2)\partial_{\phi_2}\tilde u + \tilde n\partial_r\tilde u\right)\cdot\tilde n \\ \quad - P_1(D)\tilde u\cdot\tilde n + \tilde f_1\cdot\tilde n + \dfrac{2\mu+\mu'}{\bar\rho}\tilde f_{0,r}, \quad \phi\in(-M_1,M_1)\times(-M_2,M_2), r\in(0,r_0). \end{cases} \tag{116}$$

Recall that the third line in this expression corresponds to $-u_t(t,x)\cdot\tilde n$. □

Lemma 17. *For $0 \leqslant k+\ell \leqslant 2$, we have*

$$\begin{aligned} &\big\|\zeta_1 D_\phi^k D_r^{\ell+1}\rho\big\|_{L^\infty_T(L^2)} + \big\|\zeta_1 D_\phi^k D_r^{\ell+1}\rho\big\|_{L^2_T(L^2)} + \left\|\zeta_1 D_\phi^k D_r^{\ell+1}\frac{d\rho}{dt}\right\|_{L^2_T(L^2)} \\ &\quad\leqslant C\big(\big\|\zeta_1 D_\phi^{k+1} D_r^\ell u\big\|_{L^2_T(H^1)} + \|u_t\|_{L^2_T(H^{k+\ell})} + \|u\|_{L^2_T(H^{k+\ell+1})} + \|\rho_0\|_{H^{k+\ell+1}} \\ &\quad + \|f_0\|_{L^2_T(H^{k+\ell+1})} + \|f_1\|_{L^2_T(H^{k+\ell})} + N^{3/2}(0,T) + N^2(0,T)\big). \end{aligned} \tag{117}$$

Proof. We take the operator $D_\phi^k D_r^\ell$ in Eq. (116), multiply this by ζ_1 and square the whole expression. Using the definition (98) and (101), the left-hand side term is

$$\begin{aligned} &\iint_{Z_s} \zeta_1^2\left(\left|\frac{2\mu+\mu'}{\bar\rho}\right|^2\left|D_\phi^k D_r^{\ell+1}\frac{d\rho}{dt}\right|^2 + \big(p^0\big)^2\big|D_\phi^k D_r^{\ell+1}\rho\big|^2 + \frac{4\mu+2\mu'}{\bar\rho}p^0 D_\phi^k D_r^{\ell+1}\frac{d\rho}{dt}D_\phi^k D_r^{\ell+1}\rho\right)dx\,d\tau \\ &\quad = \left\|\zeta_1\frac{2\mu+\mu'}{\bar\rho}D_\phi^k D_r^{\ell+1}\frac{d\rho}{dt}\right\|^2_{L^2_s(L^2)} + \big\|\zeta_1 p^0 D_\phi^k D_r^{\ell+1}\rho\big\|^2_{L^2_s(L^2)} \\ &\qquad + \frac{4\mu+2\mu'}{\bar\rho}p^0\left(\zeta_1 D_\phi^k D_r^{\ell+1}\frac{d\rho}{dt}, \zeta_1 D_\phi^k D_r^{\ell+1}\rho\right)_{L^2_T(L^2)}. \end{aligned}$$

Let us see that the last term readily provides the $L^\infty_T(L^2)$ norm of $\zeta_1 D_\phi^k D_r^{\ell+1}\rho$ in terms of $\|\rho_0\|_{H^{k+\ell+1}}$ and $N^{3/2}(0,T)$. First, thanks to the definition of $\frac{d\rho}{dt}$ (see (83)), we have

$$\iint_{Z_s} \zeta_1 D_\phi^k D_r^{\ell+1}\big(\rho_t + (u\cdot\nabla)\rho\big)\zeta_1 D_\phi^k D_r^{\ell+1}\rho\,dx\,d\tau. \tag{118}$$

As long as the first term is concerned, we use (96) and the fact that $|\tilde n_t| + |\tilde\theta_t| + \|\zeta_{1,t}\|_{L^\infty_T(L^\infty)} \leqslant CN(0,T)$ (see (22)) and we obtain

$$\big(\zeta_1 D_\phi^k D_r^{\ell+1}\rho_t\big)(t,x) = \big(\zeta_1 D_\phi^k D_r^{\ell+1}\rho\big)_t(t,x) + h(t,x),$$

where $\|h\|_{L^2_T(L^2)} \leqslant CN^2(0,T)$. Consequently, the first term in (118) is given by

$$\begin{aligned} &\frac12\int_{\Omega_F(s)} \big|\zeta_1 D_\phi^k D_r^{\ell+1}\rho\big|^2(s)\,dx - \frac12\int_{\Omega_F(0)} \big|\zeta_1(0) D_\phi^k D_r^{\ell+1}\rho_0\big|^2\,dx + \iint_{Z_s} h\zeta_1 D_\phi^k D_r^{\ell+1}\rho\,dx\,d\tau \\ &\quad - \frac12\iint_{Z_s}\nabla\cdot\big(\big|\zeta_1 D_\phi^k D_r^{\ell+1}\rho\big|^2 u\big)\,dx\,d\tau \geqslant \big\|\zeta_1 D_\phi^k D_r^{\ell+1}\rho\big\|^2_{L^2}(s) - \big\|\zeta_1(0) D_\phi^k D_r^{\ell+1}\rho_0\big\|^2_{L^2} \end{aligned}$$

$$- \delta \left\| \zeta_1 D_\phi^k D_r^{\ell+1} \rho \right\|_{L_T^2(L^2)}^2 - C N^4(0,T) - \frac{1}{2} \iint\limits_{Z_s} \nabla \cdot \left(\left| \zeta_1 D_\phi^k D_r^{\ell+1} \rho \right|^2 u \right) dx\, d\tau, \tag{119}$$

for $\delta > 0$ small enough. Using (104), the second term in (118) is estimated from below by

$$\frac{1}{2} \iint\limits_{Z_s} \nabla \cdot \left(\left| \zeta_1 D_\phi^k D_r^{\ell+1} \rho \right|^2 u \right) dx\, d\tau - C N^3(0,T).$$

Putting this together with (119), we obtain (117). □

Lemma 18. *For $0 \leqslant k + \ell \leqslant 2$, we have*

$$\begin{aligned} &\left\| \zeta_1 D_\phi^k u \right\|_{L_T^2(H^{2+\ell})} + \left\| \zeta_1 D_\phi^k \rho \right\|_{L_T^2(H^{1+\ell})} \\ &\quad \leqslant C \left(\left\| \zeta_1 D_\phi^k \frac{d\rho}{dt} \right\|_{L_T^2(H^{1+\ell})} + \|u_t\|_{L_T^2(H^{k+\ell})} + \|u\|_{L_T^2(H^{k+\ell+1})} + \|\rho\|_{L_T^2(H^{k+\ell})} + \|\dot a\|_{L_T^2} + \|\omega\|_{L_T^2} \right. \\ &\qquad \left. + \|f_0\|_{L_T^2(H^{1+k+\ell})} + \|f_1\|_{L_T^2(H^{k+\ell})} \right). \end{aligned} \tag{120}$$

Proof. We regard the equation of the fluid in (61) as a stationary Stokes system for each $s \in (0,T)$:

$$\begin{cases} -2\mu \nabla \cdot (\epsilon(u)) - \mu' \nabla \cdot ((\nabla \cdot u)\,\mathrm{Id}) + p^0 \nabla \rho = f_1 - u_t & \text{in } \mathcal{O}_i \cap \Omega_F(t), \\ \nabla \cdot u = \frac{1}{\bar\rho}(f_0 - \frac{d\rho}{dt}) & \text{in } \mathcal{O}_i \cap \Omega_F(t), \\ u = (\dot a + \omega \wedge (x - a)) \mathbf{1}_{\partial \Omega_S(t)} & \text{on } \mathcal{O}_i \cap \partial \Omega_F(t). \end{cases}$$

Recall that $\frac{d\rho}{dt}$ was given in (83). Now, we differentiate the equation of u with respect to $\phi = (\phi_1, \phi_2)$ k times and we multiply by ζ_1. Denoting $(u_k, \rho_k) := \zeta_1 D_\phi^k (u, \rho)$, we have

$$\begin{cases} -2\mu \nabla \cdot (\epsilon(u_k)) - \mu' \nabla \cdot ((\nabla \cdot u_k)\,\mathrm{Id}) + p^0 \nabla (\rho_k) = f_{1,k} & \text{in } \mathcal{O}_i \cap \Omega_F(t), \\ \nabla \cdot u_k = \frac{\zeta_1}{\bar\rho}(D_\phi^k f_0 - D_\phi^k \frac{d\rho}{dt}) + P_k(D) u & \text{in } \mathcal{O}_i \cap \Omega_F(t), \\ u_k = (\zeta_1 D_\phi^k (\dot a + \omega \wedge (x - a))) \mathbf{1}_{\partial \Omega_S(t)} & \text{on } \mathcal{O}_i \cap \partial \Omega_F(t), \end{cases}$$

where

$$f_{1,k} = \zeta_1 D_\phi^k f_1 - \zeta_1 D_\phi^k u_t + P_{k+1}(D) u + P_k(D) \rho,$$

with $P_j(D)$ a differential operator in the x variable of order j. Observe that the $L_T^2(H^4(\mathcal{O}_i \cap \Omega_F(t)))$ norm of the boundary condition can be estimated by $C(\|\dot a\|_{L_T^2} + \|\omega\|_{L_T^2})$. Finally, we use the following classical regularity result for the stationary Stokes problem (see, for instance, [22]):

Let U be a C^{m+2} domain, $m > 0$, $\alpha > 0$, $f \in H^m(U)$, $g_0 \in H^{m+1}(U)$ and $g_1 \in H^{m+3/2}(\partial U)$. Then, the solution (h, π) of

$$\begin{cases} -\alpha \Delta h + \nabla \pi = f & \text{in } U, \\ \nabla \cdot h = g_0 & \text{in } U, \\ h = g_1 & \text{on } \partial U \end{cases}$$

belongs to $H^{m+2}(U) \times H^{m+1}(U)$ and there exists a constant $C > 0$ such that

$$\left\| (h, \pi) \right\|_{H^{m+2}(U) \times H^{m+1}(U)} \leqslant C \left(\|f\|_{H^m(U)} + \|g_0\|_{H^{m+1}(U)} + \|g_1\|_{H^{m+3/2}(\partial U)} \right). \qquad \square$$

4.2. Combination of lemmas and conclusion

In this paragraph we will gather all the technical results previously stated and conclude the proof of Proposition 8.

1) We apply Lemma 13 for $k = 0$:

$$\|u\|_{L^\infty_T(L^2)}+\|\rho\|_{L^\infty_T(L^2)}+\|u\|_{L^2_T(H^1)}+\|\dot a\|_{L^\infty_T}+\|\omega\|_{L^\infty_T}$$
$$\leqslant \delta\big(\|\rho\|_{L^2_T(L^2)}+\|\dot a\|_{L^2_T}+\|\omega\|_{L^2_T}\big)+C\big(\|f_0\|_{L^2_T(L^2)}+\|f_1\|_{L^2_T(L^2)}+\|f_2\|_{L^2_T}+\|f_3\|_{L^2_T}$$
$$+\|u_0\|_{L^2}+\|\rho_0\|_{L^2}+|a_0|+|\omega_0|+N^{3/2}(0,T)+N^2(0,T)\big). \tag{121}$$

2) We apply Lemma 14 for $k=0$:

$$\|u_t\|_{L^2_T(L^2)}+\|\rho_t\|_{L^2_T(L^2)}+\|u\|_{L^\infty_T(H^1)}+\|\ddot a\|_{L^2_T}+\|\dot\omega\|_{L^2_T}$$
$$\leqslant C\big(\|u\|_{L^2_T(H^1)}+\|\rho\|_{L^\infty_T(L^2)}+\|f_0\|_{L^2_T(L^2)}+\|f_1\|_{L^2_T(L^2)}+\|f_2\|_{L^2_T}+\|f_3\|_{L^2_T}$$
$$+\|u_0\|_{H^1}+N^{3/2}(0,T)+N^2(0,T)\big). \tag{122}$$

3) We apply Lemma 13 for $k=1$:

$$\|u\|_{W^{1,\infty}_T(L^2)}+\|\rho\|_{W^{1,\infty}_T(L^2)}+\|u\|_{H^1_T(H^1)}+\|\dot a\|_{W^{1,\infty}_T}+\|\omega\|_{W^{1,\infty}_T}$$
$$\leqslant \delta\big(\|\rho\|_{H^1_T(L^2)}+\|\dot a\|_{H^1_T}+\|\omega\|_{H^1_T}\big)+C\big(\|f_0\|_{H^1_T(L^2)}+\|f_1\|_{H^1_T(L^2)}+\|f_2\|_{H^1_T}+\|f_3\|_{H^1_T}+\|u_0\|_{L^2}$$
$$+\|\rho_0\|_{L^2}+\big|\dot a(0)\big|+|\omega_0|+\big\|\partial_t u(0,\cdot)\big\|_{L^2}+\big\|\partial_t\rho(0,\cdot)\big\|_{L^2}+\big|\partial_t^2 a(0)\big|+\big|\partial_t\omega(0)\big|$$
$$+N^{3/2}(0,T)+N^2(0,T)\big). \tag{123}$$

4) We apply Lemma 11:

$$\|\dot a\|_{L^2_T}+\|\omega\|_{L^2_T}\leqslant C\|u\|_{L^2_T(H^1)}. \tag{124}$$

Putting together estimates (121)–(123) and (124) and taking $\delta>0$ sufficiently small, we have:

$$\|u\|_{W^{1,\infty}_T(L^2)}+\|u\|_{L^\infty_T(H^1)}+\|u\|_{H^1_T(H^1)}+\|\rho\|_{W^{1,\infty}_T(L^2)}+\|\rho_t\|_{L^2_T(L^2)}$$
$$+\|\dot a\|_{W^{1,\infty}_T}+\|\dot a\|_{H^1_T}+\|\omega\|_{W^{1,\infty}_T}+\|\omega\|_{H^1_T}\leqslant \delta\|\rho\|_{L^2_T(L^2)}+C\big(\|f_0\|_{H^1_T(L^2)}+\|f_1\|_{H^1_T(L^2)}+\|f_2\|_{H^1_T}$$
$$+\|f_3\|_{H^1_T}+\big\|\partial_t u(0,\cdot)\big\|_{L^2}+\|u_0\|_{H^1}+\big\|\partial_t\rho(0,\cdot)\big\|_{L^2}+\|\rho_0\|_{L^2}+\big|\partial_t^2 a(0)\big|+|a_0|$$
$$+\big|\partial_t\omega(0)\big|+|\omega_0|+N^{3/2}(0,T)+N^2(0,T)\big). \tag{125}$$

5) We apply Lemma 15 for $k=1$:

$$\|\zeta_0\nabla\rho\|_{L^\infty_T(L^2)}+\|\zeta_0\nabla\rho\|_{L^2_T(L^2)}+\|\zeta_0\nabla u\|_{L^\infty_T(L^2)}+\|\zeta_0\nabla u\|_{L^2_T(H^1)}+\left\|\zeta_0\nabla\frac{d\rho}{dt}\right\|_{L^2_T(L^2)}$$
$$\leqslant C\big(\|u\|_{L^\infty_T(L^2)}+\|u\|_{L^2_T(H^1)}+\|\rho_0\|_{H^1}+\|u_0\|_{H^1}+\|f_0\|_{L^2_T(H^1)}+\|f_1\|_{L^2_T(L^2)}+N^{3/2}(0,T)\big). \tag{126}$$

6) We apply Lemma 16 for $k=1$:

$$\|\zeta_1 D_\phi\rho\|_{L^\infty_T(L^2)}+\|\zeta_1 D_\phi u\|_{L^\infty_T(L^2)}+\|\zeta_1 D_\phi u\|_{L^2_T(H^1)}+\left\|\zeta_1 D_\phi\frac{d\rho}{dt}\right\|_{L^2_T(L^2)}$$
$$\leqslant \delta\big(\|\zeta_1 D_\phi\rho\|_{L^2_T(L^2)}+\|u\|_{L^2_T(H^2)}+\|\rho\|_{L^2_T(H^1)}\big)+C\big(\|u\|_{L^2_T(H^1)}+\|\omega\|_{L^2_T}+\|\rho_0\|_{H^1}+\|u_0\|_{H^1}$$
$$+\|f_0\|_{L^2_T(H^1)}+\|f_1\|_{L^2_T(L^2)}+N^{3/2}(0,T)\big). \tag{127}$$

7) We apply Lemma 17 for $k=\ell=0$:

$$\|\zeta_1 D_r\rho\|_{L^\infty_T(L^2)}+\|\zeta_1 D_r\rho\|_{L^2_T(L^2)}+\left\|\zeta_1 D_r\frac{d\rho}{dt}\right\|_{L^2_T(L^2)}$$
$$\leqslant C\big(\|\zeta_1 D_\phi u\|_{L^2_T(H^1)}+\|u_t\|_{L^2_T(L^2)}+\|u\|_{L^2_T(H^1)}+\|\rho_0\|_{H^1}+\|f_0\|_{L^2_T(H^1)}+\|f_1\|_{L^2_T(L^2)}$$
$$+N^{3/2}(0,T)+N^2(0,T)\big). \tag{128}$$

From (126), (127) and (128), we deduce in particular that we can estimate u in $L^2_T(H^2)$ in the interior and ρ in $L^\infty_T(H^1)$. This last fact comes from the estimate $\|\rho\|_{H^1} \leqslant C\|\nabla\rho\|_{L^2}$, which holds thanks to

$$\frac{d}{dt}\int\limits_{\Omega_F(t)} \rho(t,x)\,dx = 0.$$

Precisely, we have

$$\begin{aligned}&\|\zeta_0\nabla u\|_{L^2_T(H^1)} + \|\rho\|_{L^\infty_T(H^1)} + \|\zeta_0\nabla\rho\|_{L^2_T(L^2)} + \left\|\nabla\frac{d\rho}{dt}\right\|_{L^2_T(L^2)}\\ &\quad\leqslant \delta\big(\|\zeta_1 D_\phi\rho\|_{L^2_T(L^2)} + \|u\|_{L^2_T(H^2)} + \|\rho\|_{L^2_T(H^1)}\big) + C\big(\|u\|_{L^\infty_T(L^2)} + \|u\|_{L^2_T(H^1)} + \|u\|_{H^1_T(L^2)}\\ &\qquad + \|\omega\|_{L^2_T} + \|u_0\|_{H^1} + \|\rho_0\|_{H^1} + \|f_0\|_{L^2_T(H^1)} + \|f_1\|_{L^2_T(L^2)} + N^{3/2}(0,T) + N^2(0,T)\big).\end{aligned}\tag{129}$$

8) We apply Lemma 12 for $k=2$:

$$\|u\|_{L^\infty_T(H^2)} \leqslant C\big(\|u\|_{W^{1,\infty}_T(L^2)} + \|\rho\|_{L^\infty_T(H^1)} + \|\dot a\|_{L^\infty_T} + \|\omega\|_{L^\infty_T} + \|f_1\|_{L^\infty_T(L^2)}\big).\tag{130}$$

9) We apply Lemma 18 for $k=\ell=0$:

$$\begin{aligned}\|\zeta_1 u\|_{L^2_T(H^2)} + \|\zeta_1\rho\|_{L^2_T(H^1)} \leqslant C\Bigg(&\left\|\zeta_1\frac{d\rho}{dt}\right\|_{L^2_T(H^1)} + \|u_t\|_{L^2_T(L^2)} + \|u\|_{L^2_T(H^1)} + \|\rho\|_{L^2_T(L^2)} + \|\dot a\|_{L^2_T}\\ &+ \|\omega\|_{L^2_T} + \|f_0\|_{L^2_T(H^1)} + \|f_1\|_{L^2_T(L^2)}\Bigg).\end{aligned}\tag{131}$$

Using (129) in (130) and (131), we obtain estimates for u in $L^\infty_T(H^2)\cap L^2_T(H^2)$ and for ρ in $L^2_T(H^1)$:

$$\begin{aligned}&\|u\|_{L^\infty_T(H^2)} + \|u\|_{L^2_T(H^2)} + \|\rho\|_{L^\infty_T(H^1)} + \|\rho\|_{L^2_T(H^1)} + \left\|\nabla\frac{d\rho}{dt}\right\|_{L^2_T(L^2)}\\ &\quad\leqslant C\Bigg(\|u\|_{W^{1,\infty}_T(L^2)} + \|u\|_{L^2_T(H^1)} + \|u\|_{H^1_T(L^2)} + \left\|\frac{d\rho}{dt}\right\|_{L^2_T(L^2)} + \|\dot a\|_{L^\infty_T} + \|\dot a\|_{L^2_T} + \|\omega\|_{L^\infty_T} + \|\omega\|_{L^2_T}\\ &\qquad + \|u_0\|_{H^1} + \|\rho_0\|_{H^1} + \|f_0\|_{L^2_T(H^1)} + \|f_1\|_{L^\infty_T(L^2)} + \|f_1\|_{L^2_T(L^2)} + N^{3/2}(0,T) + N^2(0,T)\Bigg).\end{aligned}\tag{132}$$

Therefore, putting this together with (125), we get

$$\begin{aligned}&\|u\|_{W^{1,\infty}_T(L^2)} + \|u\|_{L^\infty_T(H^2)} + \|u\|_{L^2_T(H^2)} + \|u\|_{H^1_T(H^1)} + \|\rho\|_{W^{1,\infty}_T(L^2)} + \|\rho\|_{L^\infty_T(H^1)}\\ &\qquad + \|\rho\|_{H^1_T(H^1)} + \left\|\frac{d\rho}{dt}\right\|_{L^2_T(H^1)} + \|\dot a\|_{W^{1,\infty}_T} + \|\dot a\|_{H^1_T} + \|\omega\|_{W^{1,\infty}_T} + \|\omega\|_{H^1_T}\\ &\quad\leqslant C\big(\|f_0\|_{H^1_T(L^2)} + \|f_0\|_{L^2_T(H^1)} + \|f_1\|_{H^1_T(L^2)} + \|f_1\|_{L^\infty_T(L^2)} + \|f_2\|_{H^1_T}\\ &\qquad + \|f_3\|_{H^1_T} + \big\|\partial_t u(0,\cdot)\big\|_{L^2} + \|u_0\|_{H^1} + \big\|\partial_t\rho(0,\cdot)\big\|_{L^2} + \|\rho_0\|_{H^1}\\ &\qquad + \big|\partial_t^2 a(0)\big| + |a_0| + \big|\partial_t\omega(0)\big| + |\omega_0| + N^{3/2}(0,T) + N^2(0,T)\big).\end{aligned}\tag{133}$$

Observe that we have added the term $\|\rho_t\|_{L^2_T(H^1)}$ in the left-hand side of this inequality. This term comes from the estimate of $\frac{d\rho}{dt}$ in $L^2_T(H^1)$ since the nonlinear term goes to $N^2(0,T)$.

10) We apply Lemma 15 for $k=2$

$$\begin{aligned}&\|\zeta_0\nabla\rho\|_{L^\infty_T(H^1)} + \|\zeta_0\nabla\rho\|_{L^2_T(H^1)} + \|\zeta_0\nabla u\|_{L^\infty_T(H^1)} + \|\zeta_0\nabla u\|_{L^2_T(H^2)} + \left\|\zeta_0\nabla\frac{d\rho}{dt}\right\|_{L^2_T(H^1)}\\ &\quad\leqslant C\big(\|u\|_{L^\infty_T(H^1)} + \|u\|_{L^2_T(H^2)} + \|\rho_0\|_{H^2} + \|u_0\|_{H^2} + \|f_0\|_{L^2_T(H^2)} + \|f_1\|_{L^2_T(H^1)} + N^{3/2}(0,T)\big).\end{aligned}\tag{134}$$

11) We apply Lemma 16 for $k=2$:

$$\begin{aligned}
&\left\|\zeta_1 D_\phi^2 \rho\right\|_{L_T^\infty(L^2)} + \left\|\zeta_1 D_\phi^2 u\right\|_{L_T^\infty(L^2)} + \left\|\zeta_1 D_\phi^2 u\right\|_{L_T^2(H^1)} + \left\|\zeta_1 D_\phi^2 \frac{d\rho}{dt}\right\|_{L_T^2(L^2)} \\
&\quad \leqslant \delta\big(\left\|\zeta_1 D_\phi^2 \rho\right\|_{L_T^2(L^2)} + \|\rho\|_{L_T^2(H^1)}\big) + C\big(\|u\|_{L_T^2(H^2)} + \|\omega\|_{L_T^2} + \|\rho_0\|_{H^2} + \|u_0\|_{H^2} \\
&\qquad + \|f_0\|_{L_T^2(H^2)} + \|f_1\|_{L_T^2(H^1)} + N^{3/2}(0,T)\big). \qquad (135)
\end{aligned}$$

12) We apply Lemma 17 for $k=1$, $\ell=0$:

$$\begin{aligned}
&\|\zeta_1 D_\phi D_r \rho\|_{L_T^\infty(L^2)} + \|\zeta_1 D_\phi D_r \rho\|_{L_T^2(L^2)} + \left\|\zeta_1 D_\phi D_r \frac{d\rho}{dt}\right\|_{L_T^2(L^2)} \\
&\quad \leqslant C\big(\left\|\zeta_1 D_\phi^2 u\right\|_{L_T^2(H^1)} + \|u_t\|_{L_T^2(H^1)} + \|u\|_{L_T^2(H^2)} + \|\rho_0\|_{H^2} + \|f_0\|_{L_T^2(H^2)} + \|f_1\|_{L_T^2(H^1)} \\
&\qquad + N^{3/2}(0,T) + N^2(0,T)\big). \qquad (136)
\end{aligned}$$

Observe that the first term in the right-hand side in (136) can be estimated by the third term in the left-hand side of (135).

13) We apply Lemma 18 for $k=1$, $\ell=0$:

$$\begin{aligned}
\|\zeta_1 D_\phi u\|_{L_T^2(H^2)} + \|\zeta_1 D_\phi \rho\|_{L_T^2(H^1)} \leqslant C\bigg(&\left\|\zeta_1 D_\phi \frac{d\rho}{dt}\right\|_{L_T^2(H^1)} + \|u_t\|_{L_T^2(H^1)} + \|u\|_{L_T^2(H^2)} + \|\rho\|_{L_T^2(H^1)} + \|\dot a\|_{L_T^2} \\
&+ \|\omega\|_{L_T^2} + \|f_0\|_{L_T^2(H^2)} + \|f_1\|_{L_T^2(H^1)}\bigg). \qquad (137)
\end{aligned}$$

The term $\|\zeta_1 \nabla D_\phi \frac{d\rho}{dt}\|_{L_T^2(L^2)}$, which is contained in the first norm of the right-hand side of (137), can be estimated with the help of the third term in the left-hand side of (136) and the fourth term in the left-hand side of (135).

14) We apply Lemma 17 for $k=0$, $\ell=1$:

$$\begin{aligned}
&\left\|\zeta_1 D_r^2 \rho\right\|_{L_T^\infty(L^2)} + \left\|\zeta_1 D_r^2 \rho\right\|_{L_T^2(L^2)} + \left\|\zeta_1 D_r^2 \frac{d\rho}{dt}\right\|_{L_T^2(L^2)} \\
&\quad \leqslant C\big(\|\zeta_1 D_\phi D_r u\|_{L_T^2(H^1)} + \|u_t\|_{L_T^2(H^1)} + \|u\|_{L_T^2(H^2)} + \|\rho_0\|_{H^2} + \|f_0\|_{L_T^2(H^2)} + \|f_1\|_{L_T^2(H^1)} \\
&\qquad + N^{3/2}(0,T) + N^2(0,T)\big). \qquad (138)
\end{aligned}$$

The norm $\|\zeta_1 D_\phi D_r u\|_{L_T^2(H^1)}$ will be bounded thanks to the first term in the left-hand side of (137).

15) We apply Lemma 18 for $k=0$, $\ell=1$:

$$\begin{aligned}
\|\zeta_1 u\|_{L_T^2(H^3)} + \|\zeta_1 \rho\|_{L_T^2(H^2)} \leqslant C\bigg(&\left\|\zeta_1 \frac{d\rho}{dt}\right\|_{L_T^2(H^2)} + \|u_t\|_{L_T^2(H^1)} + \|u\|_{L_T^2(H^2)} + \|\rho\|_{L_T^2(H^1)} + \|\dot a\|_{L_T^2} \\
&+ \|\omega\|_{L_T^2} + \|f_0\|_{L_T^2(H^2)} + \|f_1\|_{L_T^2(H^1)}\bigg). \qquad (139)
\end{aligned}$$

The term $\|\zeta_1 D^2 \frac{d\rho}{dt}\|_{L_T^2(L^2)}$, which is contained in the first norm of the right-hand side of (139), can be estimated with the help of third term in the right-hand side of (138), the third in the left of (136) and the fourth in the left of (135).

Collecting expressions (134)–(139), we deduce

$$\begin{aligned}
&\|u\|_{L_T^2(H^3)} + \left\|D^2 \rho\right\|_{L_T^\infty(L^2)} + \|\rho\|_{L_T^2(H^2)} + \left\|\frac{d\rho}{dt}\right\|_{L_T^2(H^2)} \\
&\quad \leqslant C\bigg(\|u\|_{L_T^\infty(H^1)} + \|u\|_{L_T^2(H^2)} + \|u\|_{H_T^1(H^1)} + \|\rho\|_{L_T^2(H^1)} + \left\|\frac{d\rho}{dt}\right\|_{L_T^2(H^1)} + \|\dot a\|_{L_T^2} + \|\omega\|_{L_T^2} \\
&\qquad + \|u_0\|_{H^2} + \|\rho_0\|_{H^2} + \|f_0\|_{L_T^2(H^2)} + \|f_1\|_{L_T^2(H^1)} + N^{3/2}(0,T) + N^2(0,T)\bigg). \qquad (140)
\end{aligned}$$

Using (133), for the moment we have

$$\begin{aligned}
&\|u\|_{W_T^{1,\infty}(L^2)} + \|u\|_{L_T^\infty(H^2)} + \|u\|_{L_T^2(H^3)} + \|u\|_{H_T^1(H^1)} + \|\rho\|_{W_T^{1,\infty}(H^1)} + \|\rho\|_{L_T^\infty(H^2)} \\
&\quad + \|\rho\|_{H_T^1(H^2)} + \left\|\frac{d\rho}{dt}\right\|_{L_T^2(H^2)} + \|\dot a\|_{W_T^{1,\infty}} + \|\dot a\|_{H_T^1} + \|\omega\|_{W_T^{1,\infty}} + \|\omega\|_{H_T^1} \\
&\leqslant C\big(\|f_0\|_{H_T^1(L^2)} + \|f_0\|_{L_T^2(H^2)} + \|f_1\|_{L_T^2(H^1)} + \|f_1\|_{H_T^1(L^2)} + \|f_1\|_{L_T^\infty(L^2)} + \|f_2\|_{H_T^1} + \|f_3\|_{H_T^1} \\
&\quad + \big\|\partial_t u(0,\cdot)\big\|_{L^2} + \|u_0\|_{H^2} + \big\|\partial_t \rho(0,\cdot)\big\|_{L^2} + \|\rho_0\|_{H^2} + \big|\ddot a(0)\big| + |a_0| + \big|\dot\omega(0)\big| + |\omega_0| \\
&\quad + N^{3/2}(0,T) + N^2(0,T)\big).
\end{aligned} \tag{141}$$

In order to estimate the norm $\|\rho\|_{W_T^{1,\infty}(H^1)}$ we have used the equation of ρ and $\|f_0\|_{L_T^\infty(H^1)} \leqslant C(\|f_0\|_{L_T^2(H^2)} + \|f_0\|_{H_T^1(L^2)})$.

16) We apply Lemma 14 for $k=1$:

$$\begin{aligned}
&\|u_t\|_{H_T^1(L^2)} + \|\rho_t\|_{H_T^1(L^2)} + \|u\|_{W_T^{1,\infty}(H^1)} + \|\ddot a\|_{H_T^1} + \|\dot\omega\|_{H_T^1} \\
&\leqslant C\big(\|u\|_{H_T^1(H^1)} + \|\rho\|_{W_T^{1,\infty}(L^2)} + \|f_0\|_{H_T^1(L^2)} + \|f_1\|_{H_T^1(L^2)} + \|f_2\|_{H_T^1} + \|f_3\|_{H_T^1} \\
&\quad + \big\|\partial_t u(0,\cdot)\big\|_{H^1} + N^{3/2}(0,T) + N^2(0,T)\big).
\end{aligned} \tag{142}$$

Now, we regard the equation satisfied by u_t as a stationary elliptic equation (see system (72)):

$$\begin{cases} -2\mu\nabla\cdot(\epsilon(u_t)) - \mu'\nabla\cdot((\nabla\cdot u_t)\,\mathrm{Id}) = -u_{tt} - p^0\nabla\rho_t + f_{1,t} & \text{in } (0,T)\times\Omega_F(t), \\ u_t = (\ddot a + \dot\omega\wedge(x-a) + \omega\wedge(\omega\wedge(x-a)) - (u_S\cdot\nabla)u)\mathbf{1}_{\partial\Omega_S(t)} & \text{in } (0,T)\times\partial\Omega_F(t). \end{cases}$$

Then, we have

$$\|u\|_{H_T^1(H^2)} \leqslant C\big(\|u\|_{H_T^2(L^2)} + \|\rho\|_{H_T^1(H^1)} + \|f_1\|_{H_T^1(L^2)} + \|\dot a\|_{H_T^1} + \|\omega\|_{H_T^1} + N^2(0,T)\big). \tag{143}$$

17) We apply Lemma 12 for $k=3$:

$$\|u\|_{L_T^\infty(H^3)} \leqslant C\big(\|u\|_{W_T^{1,\infty}(H^1)} + \|\rho\|_{L_T^\infty(H^2)} + \|\dot a\|_{L_T^\infty} + \|\omega\|_{L_T^\infty} + \|f_1\|_{L_T^\infty(H^1)}\big). \tag{144}$$

Combining the three estimates (142)–(144), we get

$$\begin{aligned}
&\|u\|_{W_T^{1,\infty}(H^1)} + \|u\|_{L_T^\infty(H^3)} + \|u\|_{H_T^2(L^2)} + \|u\|_{H_T^1(H^2)} + \|\rho_t\|_{H_T^1(L^2)} + \|\ddot a\|_{H_T^1} + \|\dot\omega\|_{H_T^1} \\
&\leqslant C\big(\|u\|_{H_T^1(H^1)} + \|\rho\|_{W_T^{1,\infty}(L^2)} + \|\rho\|_{L_T^\infty(H^2)} + \|\rho\|_{H_T^1(H^1)} + \|\dot a\|_{L_T^\infty} + \|\dot a\|_{H_T^1} \\
&\quad + \|\omega\|_{H_T^1} + \|\omega\|_{L_T^\infty} + \|f_0\|_{H_T^1(L^2)} + \|f_1\|_{H_T^1(L^2)} + \|f_1\|_{L_T^\infty(H^1)} \\
&\quad + \|f_2\|_{H_T^1} + \|f_3\|_{H_T^1} + \big\|\partial_t u(0,\cdot)\big\|_{H^1} + N^{3/2}(0,T) + N^2(0,T)\big).
\end{aligned} \tag{145}$$

18) We apply Lemma 15 for $k=3$:

$$\begin{aligned}
&\|\zeta_0\nabla\rho\|_{L_T^\infty(H^2)} + \|\zeta_0\nabla\rho\|_{L_T^2(H^2)} + \|\zeta_0\nabla u\|_{L_T^\infty(H^2)} + \|\zeta_0\nabla u\|_{L_T^2(H^3)} + \left\|\zeta_0\nabla\frac{d\rho}{dt}\right\|_{L_T^2(H^2)} \\
&\leqslant C\big(\|u\|_{L_T^\infty(H^2)} + \|u\|_{L_T^2(H^3)} + \|\rho_0\|_{H^3} + \|u_0\|_{H^3} + \|f_0\|_{L_T^2(H^3)} + \|f_1\|_{L_T^2(H^2)} + N^{3/2}(0,T)\big).
\end{aligned} \tag{146}$$

19) We apply Lemma 16 for $k=3$:

$$\begin{aligned}
&\big\|\zeta_1 D_\phi^3\rho\big\|_{L_T^\infty(L^2)} + \big\|\zeta_1 D_\phi^3 u\big\|_{L_T^\infty(L^2)} + \big\|\zeta_1 D_\phi^3 u\big\|_{L_T^2(H^1)} + \left\|\zeta_1 D_\phi^3\frac{d\rho}{dt}\right\|_{L_T^2(L^2)} \\
&\leqslant \delta\big(\big\|\zeta_1 D_\phi^3\rho\big\|_{L_T^2(L^2)} + \|\rho\|_{L_T^2(H^1)}\big) + C\big(\|u\|_{L_T^2(H^3)} + \|\omega\|_{L_T^2} + \|\rho_0\|_{H^3} + \|u_0\|_{H^3} \\
&\quad + \|f_0\|_{L_T^2(H^3)} + \|f_1\|_{L_T^2(H^2)} + N^{3/2}(0,T)\big).
\end{aligned} \tag{147}$$

20) We apply Lemma 17 for $k=2$, $\ell=0$:

$$\begin{aligned}&\|\zeta_1 D_\phi^2 D_r\rho\|_{L_T^\infty(L^2)}+\|\zeta_1 D_\phi^2 D_r\rho\|_{L_T^2(L^2)}+\left\|\zeta_1 D_\phi^2 D_r\frac{d\rho}{dt}\right\|_{L_T^2(L^2)}\\&\quad\leqslant C\big(\|\zeta_1 D_\phi^3 u\|_{L_T^2(H^1)}+\|u_t\|_{L_T^2(H^2)}+\|u\|_{L_T^2(H^3)}+\|\rho_0\|_{H^3}+\|f_0\|_{L_T^2(H^3)}+\|f_1\|_{L_T^2(H^2)}\\&\qquad+N^{3/2}(0,T)+N^2(0,T)\big).\end{aligned}\tag{148}$$

Observe that the norm $\|\zeta_1 D_\phi^3 u\|_{L_T^2(H^1)}$ is estimated with the third term in the left of (147).

21) We apply Lemma 18 for $k=2,\ \ell=0$:

$$\begin{aligned}\|\zeta_1 D_\phi^2 u\|_{L_T^2(H^2)}+\|\zeta_1 D_\phi^2\rho\|_{L_T^2(H^1)}\leqslant C\bigg(&\left\|\zeta_1 D_\phi^2\frac{d\rho}{dt}\right\|_{L_T^2(H^1)}+\|u_t\|_{L_T^2(H^2)}+\|u\|_{L_T^2(H^3)}+\|\rho\|_{L_T^2(H^2)}+\|\dot a\|_{L_T^2}\\&+\|\omega\|_{L_T^2}+\|f_0\|_{L_T^2(H^3)}+\|f_1\|_{L_T^2(H^2)}\bigg).\end{aligned}\tag{149}$$

As for (139), the term $\|\zeta_1\nabla D_\phi^2\frac{d\rho}{dt}\|_{L_T^2(L^2)}$ (which is contained in the first norm of the right-hand side of (149)) is bounded by the third term in the left of (148) and the fourth term in the left of (147).

22) We apply Lemma 17 for $k=1,\ \ell=1$:

$$\begin{aligned}&\|\zeta_1 D_\phi D_r^2\rho\|_{L_T^\infty(L^2)}+\|\zeta_1 D_\phi D_r^2\rho\|_{L_T^2(L^2)}+\left\|\zeta_1 D_\phi D_r^2\frac{d\rho}{dt}\right\|_{L_T^2(L^2)}\\&\quad\leqslant C\big(\|\zeta_1 D_\phi^2 D_r u\|_{L_T^2(H^1)}+\|u_t\|_{L_T^2(H^2)}+\|u\|_{L_T^2(H^3)}+\|\rho_0\|_{H^3}+\|f_0\|_{L_T^2(H^3)}+\|f_1\|_{L_T^2(H^2)}\\&\qquad+N^{3/2}(0,T)+N^2(0,T)\big).\end{aligned}\tag{150}$$

The first term in the left of (149) will serve to absorb the first in the right of (150).

23) We apply Lemma 18 for $k=1,\ \ell=1$:

$$\begin{aligned}\|\zeta_1 D_\phi u\|_{L_T^2(H^3)}+\|\zeta_1 D_\phi\rho\|_{L_T^2(H^2)}\leqslant C\bigg(&\left\|\zeta_1 D_\phi\frac{d\rho}{dt}\right\|_{L_T^2(H^2)}+\|u_t\|_{L_T^2(H^2)}+\|u\|_{L_T^2(H^3)}+\|\rho\|_{L_T^2(H^2)}+\|\dot a\|_{L_T^2}\\&+\|\omega\|_{L_T^2}+\|f_0\|_{L_T^2(H^3)}+\|f_1\|_{L_T^2(H^2)}\bigg).\end{aligned}\tag{151}$$

Observe that $\|\zeta_1 D^2 D_\phi\frac{d\rho}{dt}\|_{L_T^2(L^2)}$ is estimated with the help of the third term in the left of (150), the third in the left of (148) and the fourth in the left of (147).

24) We apply Lemma 17 for $k=0,\ \ell=2$:

$$\begin{aligned}&\|\zeta_1 D_r^3\rho\|_{L_T^\infty(L^2)}+\|\zeta_1 D_r^3\rho\|_{L_T^2(L^2)}+\left\|\zeta_1 D_r^3\frac{d\rho}{dt}\right\|_{L_T^2(L^2)}\\&\quad\leqslant C\big(\|\zeta_1 D_\phi D_r^2 u\|_{L_T^2(H^1)}+\|u_t\|_{L_T^2(H^2)}+\|u\|_{L_T^2(H^3)}+\|\rho_0\|_{H^3}+\|f_0\|_{L_T^2(H^3)}+\|f_1\|_{L_T^2(H^2)}\\&\qquad+N^{3/2}(0,T)+N^2(0,T)\big).\end{aligned}\tag{152}$$

The term $\|\zeta_1 D_\phi D_r^2 u\|_{L_T^2(H^1)}$ is estimated with the help of the first term in the left of (151).

25) We apply Lemma 18 for $k=0,\ \ell=2$:

$$\begin{aligned}\|\zeta_1 u\|_{L_T^2(H^4)}+\|\zeta_1\rho\|_{L_T^2(H^3)}\leqslant C\bigg(&\left\|\zeta_1\frac{d\rho}{dt}\right\|_{L_T^2(H^3)}+\|u_t\|_{L_T^2(H^2)}+\|u\|_{L_T^2(H^3)}+\|\rho\|_{L_T^2(H^2)}+\|\dot a\|_{L_T^2}\\&+\|\omega\|_{L_T^2}+\|f_0\|_{L_T^2(H^3)}+\|f_1\|_{L_T^2(H^2)}\bigg).\end{aligned}\tag{153}$$

The term $\|\zeta_1 D^3\frac{d\rho}{dt}\|_{L_T^2(L^2)}$ (contained in the first norm of the right-hand side of (153)) is estimated with the third terms in the left-hand sides of (152), (150) and (148) and the fourth term in the left of (147).

Combining estimates (146)–(153), we get:

$$
\begin{aligned}
&\|u\|_{L^2_T(H^4)} + \|\rho\|_{L^2_T(H^3)} + \big\|D^3\rho\big\|_{L^\infty_T(L^2)} + \left\|D^3\frac{d\rho}{dt}\right\|_{L^2_T(L^2)} \\
&\quad \leqslant C\bigg(\|u\|_{L^2_T(H^3)} + \|u\|_{L^\infty_T(H^2)} + \|u\|_{H^1_T(H^2)} + \|\rho\|_{L^2_T(H^2)} + \left\|\frac{d\rho}{dt}\right\|_{L^2_T(H^2)} + \|\dot a\|_{L^2_T} \\
&\qquad + \|\omega\|_{L^2_T} + \|u_0\|_{H^3} + \|\rho_0\|_{H^3} + \|f_0\|_{L^2_T(H^3)} + \|f_1\|_{L^2_T(H^2)} + N^{3/2}(0,T) + N^2(0,T)\bigg). \qquad (154)
\end{aligned}
$$

Together with (145), this estimate yields

$$
\begin{aligned}
&\|u\|_{W^{1,\infty}_T(H^1)} + \|u\|_{L^\infty_T(H^3)} + \|u\|_{L^2_T(H^4)} + \|u\|_{H^2_T(L^2)} + \|u\|_{H^1_T(H^2)} \\
&\qquad + \|\rho\|_{L^2_T(H^3)} + \big\|D^3\rho\big\|_{L^\infty_T(L^2)} + \|\rho\|_{H^2_T(L^2)} + \left\|D^3\frac{d\rho}{dt}\right\|_{L^2_T(L^2)} + \|\ddot a\|_{H^1_T} + \|\dot\omega\|_{H^1_T} \\
&\quad \leqslant C\bigg(\|u\|_{L^2_T(H^3)} + \|u\|_{H^1_T(H^1)} + \|\rho\|_{H^1_T(H^1)} + \|\rho\|_{L^2_T(H^2)} + \|\rho\|_{W^{1,\infty}_T(L^2)} + \|\rho\|_{L^\infty_T(H^2)} + \left\|\frac{d\rho}{dt}\right\|_{L^2_T(H^2)} \\
&\qquad + \|\dot a\|_{L^\infty_T} + \|\dot a\|_{H^1_T} + \|\omega\|_{H^1_T} + \|\omega\|_{L^\infty_T} + \|f_0\|_{L^2_T(H^3)} + \|f_0\|_{H^1_T(L^2)} + \|f_1\|_{L^2_T(H^2)} + \|f_1\|_{H^1_T(L^2)} \\
&\qquad + \|f_1\|_{L^\infty_T(H^1)} + \|f_2\|_{H^1_T} + \|f_3\|_{H^1_T} + \|u_0\|_{H^3} + \|\rho_0\|_{H^3} + \big\|\partial_t u(0,\cdot)\big\|_{H^1} \\
&\qquad + N^{3/2}(0,T) + N^2(0,T)\bigg). \qquad (155)
\end{aligned}
$$

Finally, we combine this estimate with (141) and we conclude inequality (65). Similarly as we did for $\|\rho_t\|_{L^\infty(H^1)}$ (just after (141)), the norm $\|\rho_t\|_{L^\infty(H^2)}$ is estimated using the equation of ρ and thanks to $f_0 \in L^\infty_T(H^2)$.

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