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On the complex structure of positive solutions to Matukuma-type equations

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Abstract

In this article we consider the Matukuma type equation

$$\Delta u + K(r)u^p = 0 \quad \text{in } \mathbb{R}^N \tag{0.1}$$

for positive radially symmetric solutions. We assume that $N > 2$, $p > 1$ and $K(r) \geqslant 0$, for all $r \geqslant 0$. When K satisfies some appropriate monotonicity assumption, the set of positive solutions of (0.1) is well understood. In this work we propose a constructive approach to start the analysis of the structure of the set of positive solutions when this monotonicity assumption fails. We construct some functions K so that the equation exhibits a very complex structure. This function K depends on a set of four parameters: p, N and the limits at zero and infinity of certain quotient describing the growth of K.

Keywords: Radial solutions; Matukuma equation; Critical and subcritical exponent

1. Introduction

We are concerned with the structure of positive radial solutions of the equation

$$\Delta u + K(r)u^p = 0 \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N > 2$, $p > 1$, and $K(r) \geqslant 0$ for all $r \geqslant 0$, which was proposed by Matukuma [15] as a model Celestial Mechanics for the dynamics of a cluster of stars, where u is the gravitational potential and $K(r)u^p$ is the density of stars, see also Li [10]. This equation has been extensively studied in last decades, and through the work of many authors some *simple* solution structures have been unraveled under certain monotonicity conditions on K. In this article, we propose a novel constructive approach to show that, when these conditions are not satisfied, the set of radial solutions of (1.1) can be *extremely complex*. This provides the first rigorous establishment of the complexity

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of the positive radial solution set of (1.1), and opens the door to the further investigation of this already well-studied equation.

In the well studied case $K \equiv 1$, when (1.1) is known as Emden–Fowler equation, the solution set is characterized by the Sobolev critical number $2^* = (N+2)/(N-2)$:

(C) *Subcritical* case: If $1 < p < 2^*$, then each positive radial solution $u(r)$ of (1.1) is a *crossing solution* that vanishes at some $r \in (0, \infty)$.
(S) *Supercritical* case: If $p > 2^*$, then each positive radial solution $u(r)$ is a *slowly decaying solution* that remains positive in $[0, \infty)$ and $\lim_{r\to\infty} r^{N-2}u(r) = \infty$.
(F) *Critical* case: If $p = 2^*$, then each positive radial solution $u(r)$ is a *fast decaying solution* that remains positive in $[0, \infty)$ and $\lim_{r\to\infty} r^{N-2}u(r) = c$, for certain $c > 0$.

When K is given by a pure power function $K = r^\ell$, the conclusions of (C), (S) and (F) remain valid, provided that the number 2^* is shifted to the new "critical" value

$$2^*_\ell = \frac{N+2+2\ell}{N-2},$$

as proved by Ni and Nussbaum [17]. Note that for $K = r^\ell$, the "growth rate function" of K defined by

$$P(r) = \frac{rK'(r)}{K(r)}$$

equals the constant exponent ℓ. If the growth rate function $P(r)$ is not a constant, then the solution structure may drastically change. It is expected that the degree of change of the structure must be related to the deviation of P from a constant. This correlation is however very delicate and has only been understood under some strong conditions such as

(H) $P(r)$ is non-increasing and non-constant over $(0, \infty)$.

This condition is satisfied in particular by

$$K(r) = \frac{1}{1+r^2},$$

first proposed by Matukuma [15] in his original paper. Assuming (H) and writing

$$\sigma = \lim_{r\to 0} P(r) \quad \text{and} \quad \ell = \lim_{r\to\infty} P(r),$$

then the Sobolev critical number is shifted and then teared apart to an interval (p_∞, p_0) where

$$p_0 = \frac{N+2+2\sigma}{N-2} \quad \text{and} \quad p_\infty = \max\left\{1, \frac{N+2+2\ell}{N-2}\right\}.$$

As demonstrated by Yanagida and Yotsutani [23], under the additional condition $p_0 > 1$, the half line $(1, \infty)$ is divided into three sub-intervals

$$(1, \infty) = (1, p_\infty] \cup (p_\infty, p_0) \cup [p_0, \infty). \tag{1.2}$$

If $p \in (1, p_\infty]$, then the conclusion of (C) applies. If $p \in [p_0, \infty)$, then the conclusion of (S) applies. However, when $p \in (p_\infty, p_0)$, the conclusion of (F) is *no longer valid*, but an emergent structure occurs. Denote by $u(r; \gamma)$ the unique solution to the initial value problem

$$u'' + \frac{N-1}{r}u' + K(r)u^p = 0, \tag{1.3}$$

$$u(0) = \gamma > 0, \qquad u'(0) = 0, \tag{1.4}$$

It was proved in [23] that, when condition (H) holds and $p \in (p_\infty, p_0)$,

(M) there exists a unique number $\gamma_1(p) \in (0, \infty)$ such that $u(r;\gamma)$ is a slowly decaying solution for $\gamma \in (0, \gamma_1(p))$, $u(r;\gamma)$ is a fast decaying solution for $\gamma = \gamma_1(p)$, and $u(r;\gamma)$ is a crossing solution for $\gamma \in (\gamma_1(p), \infty)$.

In a more recent work, studying of weighted p-Laplacian García-Huidobro, Manásevich and Yarur [8] formulated the growth rate of K by means of a different function

$$m(r) = \frac{2r^N K(r)}{(N-2)\int_0^r s^{N-1} K(s)\,ds}.$$

Assuming

($\tilde{\mathrm{H}}$) $m(r)$ is non-increasing and non-constant over $(0, \infty)$,

the authors defined

$$\rho_0 = \lim_{r\to 0} m(r) \quad \text{and} \quad \rho_\infty = \lim_{r\to\infty} m(r) \tag{1.5}$$

and the critical numbers $p_0 = \rho_0 - 1$ and $p_\infty = \max\{1, \rho_\infty - 1\}$, then it holds again that $p_0 > p_\infty$ and it was proved in [8] that the structure (M) appears in the interval (p_0, p_∞), while (C) and (S) holds in $(1, p_0]$ and $[p_\infty, \infty)$, respectively. Asymptotically, the function $m(r)$ relates to $P(r)$ by

$$\rho_0 = \frac{2}{N-2}(N+\sigma) \quad \text{and} \quad \rho_\infty = \frac{2}{N-2}(N+\ell).$$

In [8] it was shown that the two conditions ($\tilde{\mathrm{H}}$) and (H) are not mutually inclusive, and an example was given for which condition ($\tilde{\mathrm{H}}$) holds while (H) is not.

As a motivation for our current study and the delicacy of the main results, we present a formal discussion on how the hypothesis (H) yields conclusion (M). It is known that when the equation has a subcritical growth, that is

$$1 < p < \frac{N+2+2P(r)}{N-2}, \quad \forall r > 0, \tag{1.6}$$

then all solutions of (1.3)–(1.4) are crossing solutions. Denoting by $R(\gamma)$ the first $r > 0$ where $u(r,\gamma)$ vanishes, it results that $R(\gamma)$ is a strictly decreasing homeomorphism between $(0,\infty)$ and $(0,\infty)$. Now, if $p \in (p_\infty, p_0)$, then there is a constant $r_s > 0$ such that (1.6) holds for $0 < r < r_s$. If γ is chosen sufficiently large, then $u(r,\gamma)$ must vanish somewhere within $(0, r_s)$ and thus u is a crossing solution. On the other hand, there is another constant $r_S \geqslant r_s$ such that

$$p > \frac{N+2+2P(r)}{N-2} \quad \forall r > r_S. \tag{1.7}$$

Now, if γ is sufficiently small, then $u(r,\gamma)$ will not vanish in $(0, r_S)$. For $r > r_S$, because the equation has a supercritical growth, u must stay positive for all $r > 0$. Using an asymptotic analysis it can be shown that u is indeed a slowly decaying solution.

The existence of a fast decaying solutions can be done by a simple topological argument, however the proof of the uniqueness of the fast decaying solution is highly non-trivial. For the Matukuma equation, the uniqueness was first proved by Yanagida in [20], marking a major breakthrough in the study of the Matukuma type equations. Many authors have contributed to the understanding of this equation and the various properties of its solutions, we mention the work by García-Huidobro, Kufner, Manásevich and Yarur [7], Kawano, Yanagida and Yotsutani [9], Li and Ni [11–13], Ni and Yotsutani [18], and Yanagida and Yotsutani [21,22].

If $p_0 < p_\infty$, then condition (H) cannot hold. In this case, it does not seem possible that the solution structure can be characterized as simple as in (M). If $p \in (p_0, p_\infty)$, then there is a small $r_s > 0$ such that (1.7) holds for all $0 < r < r_s$. Therefore, each solution $u(r,\gamma)$ of (1.3)–(1.4) stays positive for all $r \in (0, r_s)$ and $\gamma > 0$. It is then a forbidden task to determine whether or not u will vanish in (r_s, ∞), no matter what type of behavior the growth rate function $P(r)$ has.

The situation occurring here is of reminiscent the problem

$$\Delta u + u^p + u^q = 0, \quad p < 2^* < q \tag{1.8}$$

first considered by Lin and Ni [14] and further investigated by Bamón, del Pino and Flores [1], Flores [6] and recently Campos [2]. In this problem the behavior of the non-linearity is also super-critical for small $r > 0$ and sub-critical for large r and the analogous of some of the solutions we obtain for (1.1) has been obtained for (1.8). However, some of our results do not have known analogous in (1.8).

In this article we initiate the study of the structure of the set of positive solutions of Eq. (1.1) when either (H) or ($\tilde{\mathrm{H}}$) is not satisfied. Our purpose is to construct a family of functions K depending on various parameters

$$K(r) = K(\rho_0, \rho_\infty, p, N; r),$$

such that the limits (1.5) exist but, $\rho_0 < \rho_\infty$ or equivalently $p_0 < p_\infty$ and so K does not satisfy hypothesis (H) or ($\tilde{\mathrm{H}}$). Depending on the values of the parameters we will find that the solution set of (1.3)–(1.4) with that specific K exhibits a very complex structure in deep contrast with the case when (H) or ($\tilde{\mathrm{H}}$) holds. At this point we would like to mention the discussion given in [23], where the case of some functions K not satisfying (H) is considered through numerical calculations. Further understanding of the problem was obtained in the more recent paper [16], potentially showing the possibility of complex structure when the monotonicity condition fails.

Before stating our main theorem we specialize the meaning of slowly decaying solutions and we also specify the case of singular solutions.

(SS) $u(r;\gamma)$ is a *slowly decaying solution*, if $u(r;\gamma) > 0$ in $[0,\infty)$ and

$$\lim_{r\to\infty} r^{\alpha_\infty} u(r;\gamma) = c,$$

(SF) $u(r;\gamma)$ is a *singular-fast solution* of (1.3) if it is positive, it satisfies (1.3) for all $r > 0$ and

$$\lim_{r\to 0} r^{\alpha_0} u(r) = c \quad \text{and} \quad \lim_{r\to\infty} r^{N-2} u(r) = c'.$$

Here c and c' are positive constants and

$$\alpha_i = \frac{2}{p-1}\frac{(N-2)(\rho_i - 2)}{4}, \quad i = 0, \infty.$$

We will see that under our assumption that we give later we will have that $\alpha_\infty < N-2$, justifying the name in (SS), see Remark 3.1.

In the description of the complexity the set of solutions of (1.3)–(1.4) we use a function constructed out of K and the solution u under analysis. We define

$$\varphi(u,r) = \alpha u(r) + \frac{\int_0^r \sqrt{K(s)}\,ds}{\sqrt{K(r)}} u'(r), \tag{1.9}$$

where $\alpha = 2/(p-1)$. Given a positive solution u to Eqs. (1.3)–(1.4), we think the number of zeroes of φ as a measure of its complexity, however the complete meaning of this function φ will be very clear later when we adequately transform our problem into the phase plane.

Our first theorem is about the simultaneous existence of a singular-fast solution and a slowly decaying solution, giving a simplified version of our main result.

Theorem 1.1. *For every $(k,\ell) \in \mathbb{N}\times\mathbb{N}$ there exists a continuum $\mathcal{C}(k,\ell) \subset (2,\infty)^2 \times (1,\infty)$ so that for every $(\rho_0, \rho_\infty, p) \in \mathcal{C}(k,\ell)$ and*

$$\frac{2(\rho_0 - 1)}{\rho_0 - 2} > N > 2, \tag{1.10}$$

there exists $K = K(\rho_0, \rho_\infty, p, N)$ satisfying (1.5) and for which Eqs. (1.3)–(1.4) possesses two solutions u_1 and u_2 such that

1. *u_1 is a slowly decaying solution with $\varphi(u_1, r)$ changing sign $2(\ell - 1)$ times for $0 < r < 1$ and $\varphi(u_1, r) \equiv 0$ for $1 < r$.*
2. *u_2 is a singular-fast solution with $\varphi(u_2, r)$ changing sign $2k - 1$ times for $r > 1$ and $\varphi(u_2, r) \equiv 0$ for $0 < r \leqslant 1$.*

Actually, the functions K that we explicitly construct not only allows for the two solutions given above but for a very rich structure of positive solutions. This is the main result of this article.

Theorem 1.2. *For the family of functions K found in Theorem* 1.1 *the structure of the set of positive solutions of Eqs.* (1.3)–(1.4) *is determined in terms of the initial conditions as follows*:

1. *There is an increasing sequence $\{\gamma_n^1\}$ and a decreasing sequence $\{\gamma_n^2\}$, both converging to $\bar{\gamma}$ such that the solutions $u(r;\gamma_n^i)$ are fast decaying for every $n \in \mathbb{N}$ and $i = 1, 2$, and $u(r;\gamma_n^i)$ converges to $u(r;\bar{\gamma}) = u_1$, uniformly in $\mathbb{R}_+$ as $n \to \infty$. Here u_1 denotes the slowly decaying solution given in Theorem* 1.1.
2. *There is an unbounded increasing sequence $\{\gamma_n^*\}$ such that $u(r;\gamma_n^*)$ is a fast decaying solution, for all $n \in \mathbb{N}$, and $u(r;\gamma_n^*)$ converges to u_2, uniformly on intervals of the form $[r_0, \infty)$ with $r_0 > 0$. The function u_2 is the singular fast solution u_2 given in Theorem* 1.1.
3. *For all other initial conditions $\gamma > 0$, $u(r;\gamma)$ is a crossing solution.*

The situation described in Theorem 1.2 does not have an analogous in case of Eq. (1.8). In particular, we may precisely ask for a combination of parameters p, q and N so that the three possibilities given above occur. In the same line we may ask about results of this type for the equation

$$\Delta u + f(u) = 0, \quad \text{in } \mathbb{R}^N, \tag{1.11}$$

with f given by $f(u) = u^p$ if $0 \leqslant u < 1$ and $f(u) = u^q$ if $u \geqslant 1$, where $1 < p < (N+2)/(N-2) < q$. When the role of p and q are reversed the structure of positive solutions was completely described by Erbe and Tang in [4], see also [19] and [3].

As we see from the explicit formula of the function K in Section 2, see Remark 2.3 and also (2.8), we can control the asymptotic behavior of K, depending on the value of N. For example if we consider $N = 2\rho_0/(\rho_0 - 2)$ then $K(0)$ is finite and $\lim_{r\to\infty} K(r) = \infty$.

The proof of our theorems relies on a change of variables that transforms Matukuma equation into another equation with $K = 1$, but a *variable dimension*, see (2.1). By particularizing the variable dimension to a step function we obtain a very simple problem with three parameters: two values for the step function, N_0 and N_∞, and p. For this simplified problem we have to find two solutions that give rise to functions u_1 and u_2 in Theorem 1.1, and then we find the various initial conditions described in Theorem 1.2.

Even though this step dimension problem looks very simple, there are still various difficulties to control the parameters involved in order to get the desired result. This existence part is solved by using degree arguments in a three-dimensional set. The computation of the degree requires various estimations, some of them non-trivial in our opinion.

Remark 1.1. For the function K we find that the growth rate function m has a simple behavior: it is increasing near the origin and decreasing at infinity, having exactly one maximum point. It would be interesting to find K such that the function m or P is increasing for all $r > 0$ and the complex structure described in our theorems persists.

Remark 1.2. $\mathcal{C}(\ell, k)$ is a continuum in $\mathbb{R}^3$ that we suspect is a curve, but at this point we are not able to prove it. $\mathcal{C}(\ell, k)$ is the intersection of two subset of $\mathbb{R}^3$, that we call $\mathcal{S}_\ell$ and $\mathcal{S}^k$ corresponding to the solution set of single equations. Even though we cannot prove it, we think that these sets are two-dimensional surfaces in $\mathbb{R}^3$. See Proposition 6.1 and discussion in Section 6.

Remark 1.3. The situation described in Theorems 1.1 and 1.2 corresponds to points in the space (ρ_0, ρ_∞, p) that allow for the most complex behavior of solutions in terms of the initial conditions. There are several other simpler cases, but also very interesting, that are discussed at the end of Section 5 and in Section 6.

Remark 1.4. In the original Matukuma model for Celestial Mechanics, see [15] and [10], the function u represents the gravitational potential and $\int_{\mathbb{R}^3} K(r)u^p\,dx$ is the total mass. It is interesting to observe that all fast decaying solutions constructed in Theorem 1.2 give finite total mass, as can be easily proved in view of Remark 2.3 and constraint (3.2). In this sense all these solutions are physically meaningful.

This paper is organized as follows. In Section 2 we transform our problem into the step dimension problem. We give precise formulas for the change of variables and the function K in terms of the parameters of the step dimension problem. We end the section recalling critical exponents and the behavior of the solutions for the problem with constant dimension. In Section 3 we setup the equations that define the set $\mathcal{C}(k,\ell)$ and we discuss the constraint we impose in the three-dimensional space. In Section 4 we use degree theory in order to solve the equations defining $\mathcal{C}(k,\ell)$ and in Section 5 we complete the proof of Theorems 1.1 and 1.2 and we mention some open problems suggested by our results. Finally, in Section 6 we discuss the construction of other functions K that allows the existence of other type of solutions, namely positive solutions that simultaneously are singular at the origin and slowly decaying at infinity.

2. The equation and change of variables

In this section we define a change of variables that transforms the Matukuma equation into an equation with 'variable dimension'. This change of variable is defined in terms of a differential equation, that we solve in a particular case of interest, that is for a dimension function which is constant, except at a point where it jumps. Given this dimension function we construct the function K we are interested in.

In general terms, let us assume that $n(\tau)$ is a given positive function and that $v(\tau)$ is a solution of the variable dimension equation

$$v''(\tau)+\frac{n(\tau)-1}{\tau}v'(\tau)+v^p(\tau)=0. \tag{2.1}$$

If $g:[0,\infty)\to[0,\infty)$ is a diffeomorphism and u is defined as $u(r)=v(g(r))$ then u satisfies

$$u''(r)+\left[-\frac{g''(r)}{g'(r)}+\frac{n(g(r))-1}{g(r)}g'(r)\right]u'(r)+\left(g'(r)\right)^2u^p(r)=0.$$

Thus, if the function g satisfies

$$-\frac{g''(r)}{g'(r)}+\frac{n(g(r))-1}{g(r)}g'(r)=\frac{N-1}{r}, \tag{2.2}$$

then u is a solution to the differential equation of Matukuma type

$$u''(r)+\frac{N-1}{r}u'(r)+K(r)u^p(r)=0, \tag{2.3}$$

where K is given by

$$K(r)=\left(g'(r)\right)^2 \quad\text{and}\quad g(r)=\int_0^r\sqrt{K(s)}\,ds. \tag{2.4}$$

In what follows we exhibit a solution to Eq. (2.2) when the dimension function is given by

$$n(\tau)=\begin{cases}N_0 & \text{if } \tau\leqslant 1,\\ N_\infty & \text{if } \tau>1,\end{cases} \tag{2.5}$$

for $N_0>N_\infty>2$. We prove the following lemma

Lemma 2.1. *Given $N>2$, there is a solution g of* (2.2) *with $n(\tau)$ as in* (2.5)*, which is a diffeomorphism from* $[0,\infty)$ *onto* $[0,\infty)$*. Furthermore, the function K given by* (2.4) *satisfies* (1.5) *with*

$$\rho_0=\frac{2N_0}{N_0-2} \quad \textit{and} \quad \rho_\infty=\frac{2N_\infty}{N_\infty-2}.$$

Proof. Assuming that $g(1)=1$ and for $g'(1)=c$ to be chosen, we see that if g satisfies (2.2) then

$$\frac{g'(r)}{g(r)^{N_0-1}}r^{N-1}=c, \quad r\in(0,1] \tag{2.6}$$

and

$$\frac{g'(r)}{g(r)^{N_\infty-1}}r^{N-1}=c,\quad r\in[1,\infty).\tag{2.7}$$

Integrating (2.7) we find that

$$\frac{1}{g(r)^{N_\infty-2}}=c\frac{N_\infty-2}{N-2}\left(\frac{1}{r^{N-2}}-1\right)+1.$$

In order that g is onto $[0,\infty)$ we have to impose $c=(N-2)/(N_\infty-2)$ and then the function g is determined by

$$\frac{1}{g(r)}=\begin{cases}(\frac{N_0-2}{N_\infty-2}(\frac{1}{r^{N-2}}-1)+1)^{1/(N_0-2)} & \text{if } 0<r\leqslant 1,\\ \frac{1}{r^{(N-2)/(N_\infty-2)}} & \text{if } r>1,\end{cases}$$

and we define $g(0)=0$. This function g satisfies Eq. (2.2) and it is a diffeomorphism from $[0,\infty)$ onto itself.

We can obtain an explicit expression for $K(r)$ and for the function $m(r)$ defined in the introduction. From (2.4), (2.6) and (2.7) we have

$$r^{N-1}K(r)=\begin{cases}c\frac{d}{dr}\frac{g^{N_0}(r)}{N_0} & \text{if } 0\leqslant r<1,\\ c\frac{d}{dr}\frac{g^{N_\infty}(r)}{N_\infty} & \text{if } r>1,\end{cases}\tag{2.8}$$

and then we obtain the function $m(r)$ in terms of the parameters N_0, N_∞ and N: for $0\leqslant r<1$ we have

$$m(r)=\frac{2N_0}{N_0-2+(N_\infty-N_0)r^{N-2}}$$

and for $r>1$ we have

$$m(r)=\frac{2N_\infty}{N_\infty-2}\frac{N_0r^{(N-2)N_\infty/(N_\infty-2)}}{N_\infty-N_0+N_0r^{(N-2)N_\infty/(N_\infty-2)}}.$$

We observe that the function m satisfies (1.5), with ρ_0 and ρ_∞ as defined in the statement of the lemma. $\square$

Remark 2.1. We observe that the function $u(r)=v(g(r))$ is actually of class C^2, even though v and g are only C^1. At points away from the jump we have

$$u''(r)=\big(g'(r)\big)^2v''\big(g(r)\big)+v'\big(g(r)\big)g''(r)$$

and then, using the equation satisfied by v and the one satisfied by g we find that

$$u''(r)=-\big(g'(r)\big)^2v^p\big(g(r)\big)-\frac{N-1}{r}v'\big(g(r)\big)g'(r).$$

Since the right hand side is continuous for all r, we obtain that u is of class C^2.

Remark 2.2. Considering the asymptotic behavior of g at the origin we can easily see that the function $u(r)=v(g(r))$ satisfies $u'(0)=0$ whenever $N>(N_0+2)/2$, which is an hypothesis in Theorems 1.1 and 1.2.

Remark 2.3. Obviously we may write K explicitly in terms of the original parameters ρ_0, ρ_∞ and N. We obtain

$$K(r)=C_\infty r^{-\rho_\infty+(\rho_\infty-2)N/2}\quad \text{if } r\geqslant 1\quad\text{and}\tag{2.9}$$

$$K(r)=C_\infty r^{-\rho_0+(\rho_0-2)N/2}\left\{\frac{\rho_\infty-2}{\rho_0-2}+r^{N-2}\left(\frac{\rho_\infty-\rho_0}{\rho_0-2}\right)\right\}^{-(\rho_0+2)/2}\tag{2.10}$$

if $r\leqslant 1$. Here the constant C_∞ is given by $C_\infty=((N-2)(\rho_\infty-2))^2/16$.

From here we can make explicit the asymptotic behavior of K for different values of the parameters. There are two interesting cases: If $N=N_0$ we have $K(0)$ finite and $\lim_{r\to\infty}K(r)=\infty$, while if $N=N_\infty$ we have $\lim_{r\to\infty}K(r)$ exists and $\lim_{r\to 0}K(r)=\infty$.

We end this section recalling some basic facts about the equation with constant dimension

$$v''(\tau) + \frac{\nu-1}{\tau}v'(\tau) + v^p(\tau) = 0, \tag{2.11}$$

which takes place for $0 < r < 1$ and $r > 1$ for the step-dimension problem. In studying this equation it is very useful to make the Emden–Fowler transformation $x(t) = \tau^\alpha v(\tau)$, with $\tau = e^t$ and $\alpha = 2/(p-1)$. We obtain the autonomous system

$$x'' + ax' - bx + x^p = 0, \tag{2.12}$$

where $a = \nu - 2 - 2\alpha$ and $b = \alpha(\nu - 2 - \alpha)$. The basic critical exponents for this equation are in increasing order: $\nu/(\nu-2)$, where b changes sign and consequently for $p > \nu/(\nu-2)$ a critical point $\mathbf{P} = (b^{\alpha/2}, 0)$ appears. This critical point is a repeller until p reaches the exponent $(\nu+2)/(\nu-2)$, after which it becomes an attractor. In the interval $(\nu/(\nu-2), (\nu+2)/(\nu-2))$ we find another critical exponent, namely

$$\frac{\nu + 2\sqrt{\nu-1}}{\nu - 4 + 2\sqrt{\nu-1}},$$

that determines the value of p so that to the right the critical point P starts being a spiral. If $2 < \nu \leqslant 10$ this property is kept for all p, while for $\nu > 10$ the point P ceases of being a spiral for p larger than

$$\frac{\nu - 2\sqrt{\nu-1}}{\nu - 4 - 2\sqrt{\nu-1}}.$$

When we apply the Emden–Fowler transformation to step-dimension problem (2.1), with $n(r)$ as in (2.5), then we obtain system

$$x'' + \tilde{a}x' - \tilde{b}x + x^p = 0, \tag{2.13}$$

which is non-autonomous and behaves like (2.12) with $\nu = N_0$ if $t < 0$ and with $\nu = N_\infty$ if $t > 0$. Here $\tilde{a}(t) = n(\tau) - 2 - 2\alpha$, $\tilde{b}(t) = \alpha(n(\tau) - 2 - \alpha)$ and $\tau = e^t$.

In the forthcoming sections, we refer to system (2.12) with $\nu = N_0$ as S_0, and with $\nu = N_\infty$ as S_∞ and their coefficients as $a_\infty = N_\infty - 2 - 2\alpha$, $b_\infty = \alpha(N_\infty - 2 - \alpha)$, $a_0 = N_0 - 2 - 2\alpha$ and $b_0 = \alpha(N_0 - 2 - \alpha)$.

The following lemma will be useful later

Lemma 2.2. *Assume $N_0 \neq N_\infty$ and that x_0 and x_∞ are orbits of the systems S_0 and S_∞ respectively.*

1) *If these orbits cross at a point $(\bar{x}, \bar{y})$ then the crossing is transversal or $\bar{y} = 0$ or $\bar{y} = \alpha\bar{x}$.*
2) *If $x_0'(t_0) = x_0'(t_1) = 0$, $x_0'(t) \neq 0$ and $x_0(t) > 0$ for $t \in (t_0, t_1)$ then the orbit x_∞ may cross x_0 at most once in the interval (t_0, t_1). Similar statement can be made for x_∞.*

Proof. 1) Assuming the orbits x_0 and x_∞ cross then, after time shift if necessary we have $(x_0(\bar{t}), x_0'(\bar{t})) = (x_\infty(\bar{t}), x_\infty'(\bar{t}))$. If the crossing occurs with $x_0'(\bar{t}) \neq 0$ and is not transversal, we get $x_0''(\bar{t}) = x_\infty''(\bar{t})$. Then we use the equation for x_0 and x_∞ to obtain $-x_0'(\bar{t}) + \alpha x_0(\bar{t}) = 0$.

2) First we see that the orbit x_0 does not cross $y = \alpha x$ for $t \in (t_0, t_1)$. If this is the case it should cross it twice, but it is easy to see that the crossing is always from left to right ($\ddot{x} > 0$). If there are two or more crossing points with $t \in (t_0, t_1)$ then a continuity argument leads us to a tangent crossing contradicting 1. □

3. Setting up the equations for the existence of double connections for the Step-Dimension problem

In this section we define the domain of parameters and the equations that determine various types of connections for the step-dimension problem in the phase plane. We want to find combinations between the parameters (p, N_0, N_∞) so that system (2.13) has connections between the origin $\mathbf{O} = (0, 0)$ and the critical points of the systems S_0 and S_∞.

It will be convenient to define $\beta = 2 + 2\alpha$ and work with β instead of p. Our first assumptions on the point (β, N_0, N_∞) are

$$2 + \varepsilon_1 \leqslant N_\infty \leqslant \beta \leqslant N_0, \tag{3.1}$$

where $\varepsilon_1 > 0$ is a small number that will be determined later, see Lemma 4.5. Inequalities (3.1) guarantee that p is supercritical for (2.13) when $t < 0$ and subcritical for $t > 0$. We further assume

$$N_\infty + 2(\sqrt{N_\infty - 1} - 1) \geqslant \beta \quad \text{and} \tag{3.2}$$

$$N_0 - 2(\sqrt{N_0 - 1} - 1) \quad \leqslant \beta \quad \text{if } N_0 > 10. \tag{3.3}$$

We see that these inequalities imply that systems S_0 and S_∞ have critical points $\mathbf{P}_0 = (b_0^{\alpha/2}, 0)$ and $\mathbf{P}_\infty = (b_\infty^{\alpha/2}, 0)$, respectively, and both are spiral or center. In what follows we write $P_0 = b_0^{\alpha/2}$ and $P_\infty = b_\infty^{\alpha/2}$.

Remark 3.1. We observe that after (3.2) we have $b_\infty > 0$ and then α_∞, defined in the introduction, satisfies $\alpha_\infty < N - 2$.

Let us consider the orbit $x_0(t)$ of system S_0 emanating from the origin. Under our assumptions on (β, N_0) and $\beta < N_0$, the orbit x_0 spirals towards $\mathbf{P}_0$. Given $k \in \mathbb{N}$ we denote by $q_k^0 = q_k^0(\beta, N_0)$, the value at which the orbit x_0 crosses the x-axis, in the interval $(0, P_0)$, for the k-th time. Thus

$$q_1^0 < q_2^0 < \cdots < q_k^0 \quad \text{and} \quad \lim_{k\to\infty} q_k^0 = P_0.$$

On the other hand we consider the orbit x_∞ entering towards the origin, for system S_∞. Under our assumptions on (β, N_∞) and $\beta > N_\infty$ the orbit x_∞ spirals back towards $\mathbf{P}_\infty$. Given $\ell \in \mathbb{N}$ we denote by $q_\ell^\infty = q_\ell^\infty(\beta, N_\infty)$, the value at which the orbit x_∞ crosses the x-axis, in the interval (P_∞, ∞), for the ℓ-th time. We observe that

$$q_1^\infty > q_2^\infty > \cdots > q_\ell^\infty \quad \text{and} \quad \lim_{\ell\to\infty} q_\ell^\infty = P_\infty.$$

In order to get connections for system (2.13) from $\mathbf{O}$ to $\mathbf{P}_\infty$ and simultaneously, from $\mathbf{P}_\infty$ to $\mathbf{O}$, we set up the following equations

$$P_\infty - q_k^0 = 0, \qquad P_0 - q_\ell^\infty = 0,$$

for given $(k, \ell) \in \mathbb{N}^2$. Next we introduce another assumption on (β, N_0, N_∞) which is very convenient in our analysis. If we consider system S_∞ with the critical exponent $p = (N_\infty + 2)/(N_\infty - 2)$ then the largest x-value of the homoclinic orbit is

$$X_\infty = \big((p+1)b_\infty/2\big)^{1/(p-1)}. \tag{3.4}$$

We would like that for (β, N_0, N_∞) the inequality $P_0 < X_\infty$ holds. We actually assume that for some small constant $\varepsilon_2 > 0$ to be determined later, we have

$$N_0 - 1 - \frac{\beta}{\beta - 2}(N_\infty - 2) \leqslant \varepsilon_2, \tag{3.5}$$

which becomes $P_0 \leqslant X_\infty$ when $\varepsilon_2 = 0$. When $\beta = N_\infty$ we define $q_\ell^\infty = X_\infty$ for all $\ell \in \mathbb{N}$, and we proceed analogously for system S_0.

Finally, we consider that (N_0, N_∞) satisfies

$$\varepsilon_3 + 2 \leqslant N_0 \quad \text{and} \quad N_\infty \leqslant M, \tag{3.6}$$

where $\varepsilon_3 > 0$ is a small constant and $M > 0$ is a large constant to be determined later.

Now we set up the problem in terms of finding the zeroes of a function in an appropriate set. We define Ω as the subset of $\mathbb{R}^3$ given by

$$\Omega = \big\{(\beta, N_0, N_\infty)/(3.1)\text{–}(3.6) \text{ holds}\big\}.$$

In Ω we define the function $F^{k,\ell} : \Omega \to \mathbb{R}^2$ in the following way. Given $(\beta, N_0, N_\infty) \in \Omega$ then

$$F_1^k(\beta, N_0, N_\infty) = P_\infty - q_k^0, \tag{3.7}$$

$$F_2^\ell(\beta, N_0, N_\infty) = P_0 - q_\ell^\infty. \tag{3.8}$$

Then the problem we want to study can be viewed as solving the equation

$$F^{k,\ell}(\beta, N_0, N_\infty) = (0,0), \qquad (\beta, N_0, N_\infty) \in \Omega. \tag{3.9}$$

Before continuing with the study of Eq. (3.9) we would like to describe the set Ω in a more efficient way, eliminating redundant constraints. Then we also describe the sets using faces and a simple representation.

Lemma 3.1. *Given $\varepsilon_1 > 0$ there is $\varepsilon_2 > 0$ such that constraints* (3.2) *and* (3.3) *are redundant, that is*

$$\Omega = \{(\beta, N_0, N_\infty)/(3.1),\ (3.5)\ \textit{and}\ (3.6)\ \textit{holds}\}.$$

Proof. First we see that (3.2) is redundant. Fixing N_∞ we consider the function

$$\hat\beta(N_0) = \frac{2(N_\infty - 2)}{N_0 - N_\infty + 1 - \varepsilon_2} + 2,$$

which represents equality in (3.5). We observe that for $N_0 > N_\infty + 2(\sqrt{N_\infty - 1} - 1)$ the inequality

$$\hat\beta\big(N_\infty + 2(\sqrt{N_\infty - 1} - 1)\big) < N_\infty + 2(\sqrt{N_\infty - 1} - 1), \tag{3.10}$$

implies (3.2) is satisfied, since $\hat\beta$ is decreasing. After some computation we see that (3.10) is equivalent to

$$-N_\infty(1 - \varepsilon_2) - 4(1 + \varepsilon_2) < 2(N_\infty - 5 - \varepsilon_2)\sqrt{N_\infty - 1}.$$

If $\varepsilon_2 = 0$ then this inequality is true for $N_\infty > 2$, reaching equality at $N_\infty = 2$. Thus, given ε_1 we may choose ε_2 so that this inequality remains true. To complete the argument, we see that if $2 < N_0 \leqslant N_\infty + 2(\sqrt{N_\infty - 1} - 1)$ then clearly (3.2) also holds, because of (3.1).

Next we see that (3.3) is redundant. We observe that this constraint is considered only when $N_0 > 10$. According to constraint (3.1) and (3.5), given N_∞ we must have $N_0 \in (N_\infty, N_\infty + 1 + \varepsilon_2)$, consequently inequality (3.3) plays a role only for $N_\infty > 9 - \varepsilon_2$. For this range of N_∞ we then easily see that

$$N_0 - 2(\sqrt{N_0 - 1} - 1) < N_\infty \qquad \text{if } N_0 \in (N_\infty, N_\infty + 1 + \varepsilon_2),$$

completing the proof, since by (3.1) $N_\infty \leqslant \beta$ holds. □

We still would like to describe the set Ω in a more explicit way. According to the lemma just proved, this set is actually defined by 6 constraints, each of them giving rise to a 'face'. It is convenient to think N_∞ as a vertical variable, then the top and the bottom corresponds to (f5) and (f0), associated to the constraints $N_\infty \leqslant M$ and $N_\infty \geqslant 2 + \varepsilon_1$, respectively. Then the faces (f1), (f2) and (f3) corresponding to the constraints (3.5), $N_\infty \leqslant \beta$ and $\beta \leqslant N_0$, respectively. And finally the face (f4) given by $\varepsilon_3 + 2 \leqslant N_0$. It is important to define also the edge opposite to (f1), that is the set (e) given by the union of (f3) $\cap$ (f2) and (f4) $\cap$ (f2). Our goal is to prove the following

Theorem 3.1. *For every $(k,\ell) \in \mathbb{N}^2$, Eq.* (3.9) *in Ω has a continuum of solutions, that is a branch, of solutions $\mathcal{C}(k,\ell)$ emanating out from the face* (f2) *and reaching again the boundary of Ω at the relative interior of the faces* $\{(\text{f3}) \cup (\text{f4})\} \setminus (e)$.

4. Computing the degree of F and proof of main theorems

The proof of Theorem 3.1 is based on a degree theoretic argument. As a first step we analyze the sign of the components of $F^{k,\ell}$ on the faces of Ω. We start with the sign of F_2^ℓ on the face (f1).

Lemma 4.1. *If $(\beta, N_0, N_\infty) \in \Omega$ and $N_0 - 1 - \frac{\beta}{\beta-2}(N_\infty - 2) = \varepsilon_2$ then*

$$F_2^\ell(\beta, N_0, N_\infty) > 0. \tag{4.1}$$

Proof. We observe that when $\beta = N_\infty$ we have $F_2^\ell(\beta, N_\infty + 1, N_\infty) = 0$ and then, since $\varepsilon_2 > 0$, we obtain the desired inequality

$$F_2^\ell(N_\infty, N_\infty + 1 + \varepsilon_2, N_\infty) > 0.$$

When $\beta < N_\infty$, we assume first $\varepsilon_2 = 0$. We consider system S_∞ whose energy

$$E_\infty := \frac{\dot{x}^2}{2} - b_\infty \frac{x^2}{2} + \frac{x^{p+1}}{p+1}$$

increases along the trajectories since p is sub-critical for S_∞. Then $q_\ell^\infty < X_\infty$, for all $\ell \in \mathbb{N}$, where X_∞ was defined in (3.4). Thus, in order to obtain (4.1) it is sufficient to have that $P_0 \geqslant X_\infty$, which, after a short computation is equivalent to

$$N_0 - 1 - \frac{\beta}{\beta - 2}(N_\infty - 2) \geqslant 0.$$

Thus, from the hypothesis, the result follows with $\varepsilon_2 > 0$. □

Next we see the sign of F_2^ℓ on the edge (e) opposite to (f1). Actually we have the following

Lemma 4.2. *If $(\beta, N_0, N_\infty) \in \Omega$, $\beta = N_\infty$ and $N_0 < N_\infty + 1$, then*

$$F_2^\ell(\beta, N_0, N_\infty) < 0. \tag{4.2}$$

In particular, this inequality holds on the edge (e).

Proof. As $\beta = N_\infty$ we have $q_1^\infty = X_\infty$ and then we easily find that $N_0 < N_\infty + 1$. This implies $P_0 - q_1^\infty < 0$, from where (4.2) follows. □

Our next lemma studies the sign of F_1^k when N_∞ is large, that is, we take care of face (f5). Precisely we have

Lemma 4.3. *Given $(k, \ell) \in \mathbb{N}^2$ there exists $M > 0$ so that for every $(\beta, N_0, N_\infty) \in \Omega$, with $N_\infty = M$ we have*

$$F_1^k(\beta, N_0, N_\infty) > 0.$$

Proof. We consider system S_0 and we make the change of variables $y(t) = \theta(mt)$ with $m^2 b_0 = 1$ and $\theta = m^\alpha$, where x is a solution of S_0. Then we have

$$\ddot{y} = -a_0 m \dot{y} + y - y^p.$$

We observe that this system has a critical point at $\mathbf{P}_m = (1, 0)$. We define $y_1 = y_1(N_0)$ as the first time the orbit $y(t)$ emanating from zero hits the x-axis in the segment joining the origin with $\mathbf{P}_m$. We claim that

$$\lim_{N_\infty \to \infty} y_1 = 0, \tag{4.3}$$

where we observe that $N_0 \to \infty$ as $N_\infty \to \infty$ by (3.1). To prove the claim we start using the Dulac integral to get

$$\frac{y_1^2}{2} - \frac{y_1^{p+1}}{p+1} = m a_0 A, \tag{4.4}$$

where A is the area enclosed by this orbit from $\mathbf{O}$ to $(0, y_1)$ together with the segment, along the x-axis, joining $(0, y_1)$ back to the origin $\mathbf{O}$.

First we estimate the area A. Since the orbit $y(t)$ is located inside the zero-energy region

$$E := \frac{\dot{y}^2}{2} - \frac{y^2}{2} + \frac{y^{p+1}}{p+1} = 0,$$

we may estimate A simply by the area of a rectangle containing this region

$$A \leqslant 2\left(\frac{p+1}{2}\right)^{1/(p-1)} \sqrt{\frac{p-1}{p+1}}.$$

We notice that, being inside Ω, if $N_\infty \to \infty$ then $p \to 1$ and consequently

$$A \leqslant \sqrt{p-1}\frac{e^{1/2}}{\sqrt{2}}\bigl(1 + \mathrm{o}(1)\bigr). \tag{4.5}$$

Second we estimate ma_0. We use constraints (3.1) and (3.2) to obtain

$$N_0 - 2 - \alpha \geqslant \alpha - 2(\sqrt{N_\infty - 1} - 1) \tag{4.6}$$

and constraints (3.1) and (3.5) to find that

$$N_0 - 2 - 2\alpha \leqslant \frac{p-1}{2}(2 + 2\alpha).$$

Here we assume that N_∞ is large enough so that p is close to 1, to absorb the constant ε_2. We also see that

$$\lim_{N_\infty \to \infty} \frac{2}{\alpha}(\sqrt{N_\infty - 1} - 1) = 0 \tag{4.7}$$

and consequently we have

$$ma_0 \leqslant (p-1)\frac{1+\alpha}{\alpha}\big(1 + \mathrm{o}(1)\big). \tag{4.8}$$

Finally, since $y_1 < 1$, we see that

$$\frac{y_1^2}{2} - \frac{y_1^{p+1}}{p+1} \geqslant \frac{y_1^2}{2}\left(\frac{p-1}{p+1}\right). \tag{4.9}$$

Putting together (4.5)–(4.9) we then see that for a constant $c > 0$

$$y_1^2 \leqslant c\sqrt{p-1}$$

from where claim (4.3) follows.

Having in mind (4.3), in order to complete the proof of the lemma in case $k = 1$, that is $P_\infty - q_1^0 > 0$, we just need to show that θP_∞ is bounded away from zero. In order to prove this we use the definition of P_∞ and θ, and that $N_\infty \geqslant 2N_0/(p+1)$, as follows from (3.5) for ε_2 small. We find that

$$(\theta P_\infty)^{p-1} \geqslant 1 - \frac{p-1}{p+1}\left(1 + \frac{2+\alpha}{N_0 - 2 - \alpha}\right). \tag{4.10}$$

From here, (4.6) and (4.7) we find a constant $c > 0$ such that $\theta P_\infty \geqslant e^{-c}$, for N_∞ large enough.

To complete the proof for all k, let y_k be the point in the segment joining the origin with $(1, 0)$, where the orbit emanating from the origin hits the x-axis for the k-th time. Then by the Dulac integral we find

$$\frac{y_k^2}{2} - \frac{y_k^{p+1}}{p+1} \leqslant kma_0 A, \tag{4.11}$$

where A is the area of the first hit. The argument from here can be applied without change. □

We finally want to estimate the sign of F_1^k on the face (f0). In order to do this we need an estimate, which was proved in [5] in a more general context. We include the proof here for completeness.

Lemma 4.4. *For system* (2.12)*, in the super-critical case, the maximum value for x for the orbit emanating from the origin satisfies*

$$x^{p-1} \leqslant \alpha(\nu - 1). \tag{4.12}$$

Proof. We consider the curve

$$x' = \alpha x - \frac{x^p}{\nu - 1}, \tag{4.13}$$

which divides the right half plane in an upper region R^+ and lower region R^-. We observe that this curve corresponds to the points where $u''(r) = 0$.

Let $(x(t), x'(t))$ be the orbit emanating from $(0, 0)$. We first claim that there exists t_0 such that the orbit remains in R^+ for all $t \leqslant t_0$. In fact, if $y(x) = x(t)$ and (x, y) is a point where the trajectory $(x(t), x'(t))$ crosses (4.13), we have from (2.12) that

$$\frac{dy}{dx} = (1 + 2\alpha) - \alpha(\alpha + 1)\frac{x}{y}.$$

On the other hand, defining $z = y/x$ and using (4.13), we have that the slope of (4.13) at that point is

$$m = \alpha - \frac{px^{p-1}}{\nu - 1} = -2 + pz.$$

Then, at the point of intersection we have

$$\Delta \equiv \frac{dy}{dx} - m < 0 \quad \text{if and only if} \quad pz^2 - (3 + 2\alpha)z + \alpha(\alpha + 1) > 0.$$

Solving the quadratic equation we find that $\Delta < 0$ if and only if $z \notin [\alpha(\alpha + 1)/(\alpha + 2), \alpha]$. Using Eq. (4.13) we find that if the trajectory crosses (4.13) with $x^{p-1} < x_1^{p-1} = \alpha(\nu - 1)/p$ then it crosses from R^- to R^+. From here we see that the trajectory stays in R^- for $t \leqslant \bar{t}$ for some $\bar{t}$. Thus, in terms of the function u we find then that $u''(r) > 0$ for r close to 0, contradicting the fact that $u'(r) < 0$, while $u(r) > 0$. This proves the claim.

Using Eq. (4.13) again we find that the crossing occurs from R^+ to R^-, if it occurs when $x^{p-1} \geqslant x_1^{p-1} = \alpha(\nu - 1)/p$.

Now we prove the estimate (4.12). We just need to prove the inequality while the trajectory stays in the first quadrant. Let us define the energy-like function

$$e(t) = \frac{(x')^2}{2} + \frac{\alpha x^{p+1}}{2(\nu - 1)} - \frac{(\alpha x)^2}{2}.$$

Given $(x(t), x'(t)) \in R^+$ we have that

$$e'(t) = x'\left\{-ax' + \left(b - \alpha^2\right)x + \left(\frac{\alpha(p + 1) - 2(\nu - 1)}{2(\nu - 1)}\right)x^p\right\},$$

from where it follows that $e'(t) < 0$ when $(x(t), x'(t)) \in R^+$. In fact, the curve

$$x' = \alpha x + \frac{\alpha(p + 1) - 2(\nu - 1)}{2a(\nu - 1)}x^p,$$

which corresponds to $e' = 0$, stays below the curve (4.13), because

$$\frac{\alpha(p + 1) - 2(\nu - 1)}{2a(\nu - 1)} = \frac{-(\nu - 2 - \alpha)}{(\nu - 1)a} < \frac{-1}{(\nu - 1)}.$$

On the other hand, we have that the points $(0, 0)$ and $(x, 0)$, for $x \geqslant \bar{x}$, have energy e greater than or equal to zero. Thus the trajectory crosses (4.13), entering into R^- in the first quadrant. The trajectory remains then in R^- as we showed above. This completes the proof of the lemma. □

Now we estimate F_1^k on the face (f0).

Lemma 4.5. *Given $(k, \ell) \in \mathbb{N}^2$ and $\varepsilon_3 > 0$ there exists $\varepsilon_1 > 0$ so that for every $(\beta, N_0, N_\infty) \in \Omega$, with $N_0 \geqslant 2 + \varepsilon_3$, we have*

$$F_1^k(\beta, N_0, N_\infty) < 0 \quad \textit{if } N_\infty = 2 + \varepsilon_1.$$

Proof. It is enough to prove that $P_\infty < q_1^0$, and for this we start with some energy estimates for system S_0. We consider the energy

$$E_0 := \frac{\dot{x}^2}{2} - b_0\frac{x^2}{2} + \frac{x^{p+1}}{p + 1},$$

which is decreasing along the trajectories, since p is super-critical for S_0. Now let $\bar{x} = (\alpha(N_0 - 1))^{1/(p-1)}$ and q_0^0 be the point where the orbit crosses the x-axis for the first time. Then we have

$$q_1^0 < P_0 < q_0^0 < \bar{x},$$

where the last inequality is a consequence of Lemma 4.4. We observe that P_0 is the minimum value of the function

$$e(y) = -b_0\frac{y^2}{2} + \frac{y^{p+1}}{p+1}$$

and that $P_\infty < P_0$. Then, using the energy decay along trajectories, we just need to show

$$e(P_\infty) > e(\bar{x}). \tag{4.14}$$

Computing $e(\bar{x})$ and $e(P_\infty)$ we see that (4.14) is equivalent to

$$\left(\frac{N_\infty - 2 - \alpha}{N_0 - 1}\right)^2 < \left(\frac{(p+1)(N_0 - 2 - \alpha) - 2(N_0 - 1)}{(p+1)(N_0 - 2 - \alpha) - 2(N_\infty - 2 - \alpha)}\right)^{p-1}.$$

Recalling that if $N_\infty \to 2$ then $\alpha \to 0$ and $p \to \infty$, we see that the left-hand side converges to 0, while we right-hand side converges to $e^{-2(N_0-1)/(N_0-2)}$. In fact, if we denote by S the right-hand side, we find

$$S = \left(1 - \frac{1}{p-1}\frac{2(N_0 - 1) + \mathrm{o}(1)}{N_0 - 2 + \mathrm{o}(1)}\frac{p-1}{p+1}\right)^{p-1},$$

where $\lim_{N_\infty \to 2} \mathrm{o}(1) = 0$. From here we obtain the desired limit. $\square$

Now we are in a position of giving a proof to Theorem 3.1. The idea is to use degree theory, taking advantage of the various estimates we have made for the sign of the components of $F^{k,\ell}$ and the decomposition of Ω as given below.

Proof of Theorem 3.1. We start by decomposing the domain Ω in a convenient way, in order to a apply a homotopy argument. We consider the set in $\mathbb{R}^3$

$$B = \{(x, y, z) \mid 0 \leqslant z \leqslant 1, x \geqslant 0, y \geqslant 0, x + y \leqslant 1\}$$

and the subsets $B_t = \{(x, y, z) \in B \mid y\cos(t\pi/2) - x\sin(t\pi/2) = 0\}$, for $t \in [0, 1]$. Clearly $\bigcup_{t\in[0,1]} B_t = B$ and $B_t \cap B_s = E$ for all $s \neq t$, where $E = \{(0, 0, z) \mid 0 \leqslant z \leqslant 1\}$. The set E corresponds to one vertical edge in the set B and it can be considered as a pivot in the decomposition of B in terms of the B_t, $t \in [0, 1]$.

The boundary of the set B can be seen as the union of five faces. The top and bottom faces (F5) and (F0) are associated to the constraints $1 \geqslant z$ and $0 \leqslant z$, respectively. The face (F1) is associated to the constraint $x + y \leqslant 1$, while (F2) and (F3) correspond to the constraints $x \geqslant 0$ and $y \geqslant 0$, respectively. The edge (E) corresponds to the intersection of the faces (F2) and (F3).

Then we consider a homeomorphism $\varphi : \Omega \to B$ so that the following conditions on the faces hold:

$$\varphi\big((\mathrm{f}i)\big) = (\mathrm{F}i), \quad i = 0, 1, 2, 5, \qquad \varphi\big((\mathrm{f3}) \cup (\mathrm{f4})\big) = (\mathrm{F3}) \quad \text{and} \quad \varphi(e) = E.$$

If we define $\Omega_t = \varphi^{-1}(B_t)$, for $t \in [0, 1]$, then we have the decomposition $\Omega = \bigcup_{t\in[0,1]} \Omega_t$, having as a pivot the edge (e).

Now, for a given $(k, \ell) \in \mathbb{N}$, we define the function $G^{k,\ell} : B \to \mathbb{R}^2$ as $G^{k,\ell} = F^{k,\ell} \circ \varphi$. If we let $G_t^{k,\ell} = G^{k,\ell}|_{B_t}$, the restriction of $G^{k,\ell}$ to B_t, then the Brouwer degree of $G_t^{k,\ell}$ with respect to $(0, 0)$ on the set B_t is well defined and it holds

$$\deg\big(G_t^{k,\ell}, B_t, (0, 0)\big) \neq 0, \quad \text{for all } t \in [0, 1].$$

In fact, according to Lemmas 4.3 and 4.5, the first component of $G_t^{k,\ell}$ is positive at the top of B_t and negative on the bottom. While, according to Lemmas 4.1 and 4.2, the second component of $G_t^{k,\ell}$ is negative on the edge (E) and positive on the opposite side.

From here, and the homotopy properties of the degree, the existence of the continuum $\mathcal{C}(k, \ell)$ follows. This set emanates out from the face (f2), that corresponds to the case $t = 0$. As $t \in [0, 1]$ increases, the set of solutions $\mathcal{C}(k, \ell)$ is defined. $\square$

Remark 4.1. In the proof of Theorem 3.1 we see that the branch $\mathcal{C}(k,\ell)$ cannot touch the boundary faces (f0), (f1), (f5) nor the edge (e). It has to touch eventually the faces (f3) or (f4). We do not know which of these faces is reached, but we suspect the branch goes towards (f4).

We observe that the face (f4) is associated to the small parameters $\varepsilon_1, \varepsilon_2$ and ε_3, that are dominated by ε_3 as seen in Lemma 4.5 and Lemma 3.1. As these parameters go to zero, we believe that the corresponding branch $\mathcal{C}(k,\ell)$ connects with the corner point $(2,2,2)$.

5. Proof of the main results

The main results are consequences of Theorem 3.1 on the solution set for the equation $F^{k,l}=(0,0)$ on Ω.

Proof Theorem 1.1. In Theorem 3.1, for every $(k,\ell)\in\mathbb{N}^2$, we found a continuum $\mathcal{C}(k,\ell)$ such that

$$P_\infty = q_k^0 \quad \text{and} \quad P_0 = q_\ell^\infty. \tag{5.1}$$

From the first equality, there exists an orbit x_0 for system S_0, that emanates from the origin and such that $x_0(0)=P_\infty$. Thus,

$$v_1(\tau) := \begin{cases} x_0(\log(\tau))\tau^{-\alpha} & \text{if } \tau \leqslant 1, \\ P_\infty \tau^{-\alpha} & \text{if } \tau > 1, \end{cases}$$

is a solution to (2.1) and then $u_1(r)=v_1(g(r))$ satisfies (1.3)–(1.4), with $\gamma = \lim_{t\to-\infty} e^{-\alpha t}x_0(t)$. Since $u_1(r) = P_\infty g(r)^{-\alpha}$ for large r, using the asymptotic behavior of $g(r)=r^{(N-2)/(N_\infty-2)}$ we see that u_1 is a slowly decaying solution. Next we observe that for $t<0$ we have $x_0'(t)=\alpha v_1(\tau)+\tau v_1'(\tau)=\varphi(u_1,r)$, taking into account the changes of variables we have performed. Thus, by definition of q_k^0, the function $\varphi(u_1,r)$ vanishes $2k-1$ times before $t=0$ and $\varphi(u_1,r)\equiv 0$ for $t>0$.

From the second equality in (5.1), there exists an orbit x_∞ for system S_∞, that ends at the origin and such that $x_\infty(0)=P_0$. Thus,

$$v_2(\tau) := \begin{cases} P_0 \tau^{-\alpha} & \text{if } \tau \leqslant 1, \\ x_\infty(\log(\tau))\tau^{-\alpha} & \text{if } \tau > 1 \end{cases}$$

is a solution of (2.1) and $u_2(r)=v_2(g(r))$ satisfies (1.3). Since, $u_2(r)=P_0 g(r)^{-\alpha}$ for small r, using the behavior of g near the origin we see that u_2 is singular solution. As above, we obtain that $\varphi(u_2,r)\equiv 0$ for $t<0$ and $\varphi(u_2,r)$ vanishes $2(\ell-1)$ for $t>0$, using the definition of q_ℓ^∞. $\square$

In the proof of Theorem 1.2 we repeatedly use the following construction. Let $x_0(t)$ be the positive orbit of system S_0, emanating from the origin. Given $T\in\mathbb{R}$, we let x_∞ be the orbit of system S_∞ with initial condition $x_\infty(0)=x_0(T)$, $x_\infty'(0)=x_0'(T)$. Then we define

$$\hat{x}(t) = \begin{cases} x_0(t+T) & \text{if } t \leqslant 0, \\ x_\infty(t) & \text{if } t > 0, \end{cases}$$

which is a solution of the step dimension problem in the phase plane (2.13). After our changes of variables we find that

$$u_T(r) = \big(g(r)\big)^{-\alpha}\hat{x}\big(\log\big(g(r)\big)\big)$$

is a solution of (1.3)–(1.4) with initial condition $\gamma=\gamma(T)$. Moreover, the initial condition $\gamma(T)$ is strictly increasing in T. In fact, if $v_T(\tau)=\tau^{-\alpha}\hat{x}(\log(\tau))$ then $T_1<T_2$ and τ small we have

$$v_{T_1}(\tau) = e^{\alpha(T_1-T_2)} v_{T_2}\big(\tau e^{T_1-T_2}\big), \tag{5.2}$$

from where $v_{T_1}(0) < v_{T_2}(0)$.

Proof Theorem 1.2. For parameters in $\mathcal{C}(k,\ell)$, let x_0 be the orbit of S_0 emanating from the origin and let x_∞ be the orbit of S_∞ ending at the origin and spiraling back towards $\mathbf{P}_\infty$. Since $P_\infty = q_k^0$, together with the fact that the orbits of S_0 and S_∞ cross transversally outside the x-axis, as proved in Lemma 2.2, we have exactly two infinite sequences of intersection points. One of such sequences is charicterized by sequences of numbers $\{t_n^1\}$ and $\{s_n^1\}$, increasing and decreasing respectively, so that

$$x_0(t_n^1) = x_\infty(s_n^1) \quad \text{and} \quad x_0'(t_n^1) = x_\infty'(s_n^1) > 0,$$

for all $n \in \mathbb{N}$. The other sequence of points is characterized by sequences of numbers $\{t_n^2\}$ and $\{s_n^2\}$, both decreasing and such that

$$x_0(t_n^2) = x_\infty(s_n^2) \quad \text{and} \quad x_0'(t_n^2) = x_\infty'(s_n^2) < 0,$$

for all $n \in \mathbb{N}$. We observe that $s_n^2 < s_n^1 < s_{n+1}^2$, $t_n^1 < t_{n+1}^1$ and $t_n^2 > t_{n+1}^2$ for all $n \in \mathbb{N}$. Furthermore $\lim_{n\to\infty} t_n^1 = \lim_{n\to\infty} t_n^2 = \bar{t}$ and $\lim_{n\to\infty} s_n^1 = \lim_{n\to\infty} s_n^2 = -\infty$.

Using the construction given above we find the sequences of initial values $\gamma_n^1 = \gamma(t_n^i)$, $n \in \mathbb{N}$ and $i = 1, 2$, giving all the solutions of (1.3)–(1.4) described in part 1 of Theorem 1.2. By linearizing the flow of S_∞ near the origin we find that the positive eigenvalue is $\alpha - (N_\infty - 2)$ so that the solutions entering the origin decay like $\tau^{-(N_\infty - 2)}$. Therefore, using the definition of g in our change of variables, we conclude that the solutions of part 1 given above, are all fast decaying solutions.

The convergence of $u(r; \gamma_n^i)$ to u_1 given in Theorem 1.1, will be established by proving that the sequences $\{x_n^i\}$, in the phase plane defined by

$$x_n(t) := \begin{cases} x_0(t + t_n^i) & \text{if } t \leqslant 0, \\ x_\infty(t + s_n^i) & \text{if } t > 0, \end{cases}$$

converge in compact subsets of $\mathbb{R}$ to $\bar{x}$, the solution in the phase plane associated to u_1. By the convergence properties of the sequences $\{t_n^i\}$ and $\{s_n^i\}$, we see that $x_\infty(t + s_n^i)$ converges to P_∞ for t in compact subsets of $\{t > 0\}$ and $x_0(t + t_n^i)$ converges to $x_0(t)$ for t in a compact set of $\{t \leqslant 0\}$. Thus $x_n \to \bar{x}$ in compact subsets of $\mathbb{R}$, from where the uniform convergence of $u(r; \gamma_n^i)$ to u_1 follows.

To obtain the second part, we start from equality $P_0 = q_\ell^\infty$ and proceed in an analogous way. □

Even though the conclusion of Theorem 1.2 describes in a very precise fashion the possible solutions of (1.3)–(1.4) for the functions K we have constructed, there are still many open questions that one would like to answer.

Given $\gamma > 0$ such that $u(r; \gamma)$ is a crossing solution, we denote by $R(\gamma)$ the first r such that $u(r; \gamma) = 0$. Under the hypothesis and notation of Theorem 1.2, prove or disprove the following facts:

1) The function $R : (\gamma_n^1, \gamma_{n+1}^1) \to \mathbb{R}$ possesses exactly one minimum point $\bar{\gamma}_n^1$ and if $R_n^1 = R(\bar{\gamma}_n^1)$, the sequence $\{R_n^1\}$ is increasing.

2) The function $R : (\gamma_{n+1}^2, \gamma_n^2) \to \mathbb{R}$ possesses exactly one minimum point $\bar{\gamma}_n^2$ and if $R_n^2 = R(\bar{\gamma}_n^2)$, the sequence $\{R_n^2\}$ is increasing.

3) The function $R : (\gamma_n^*, \gamma_{n+1}^*) \to \mathbb{R}$ possesses exactly one minimum point $\bar{\gamma}_n^*$ and if $R_n^* = R(\bar{\gamma}_n^*)$, the sequence $\{R_n^*\}$ is increasing.

Before continuing we briefly present the situation occurring when only one component of $F^{k,\ell} = 0$ vanishes. The following proposition can be proved by the same analysis used to prove Theorems 3.1 and 1.2.

Proposition 5.1.

1. *For every $\ell \in \mathbb{N}$ there is a set $\mathcal{S}_\ell \subset \Omega$ such that $F_2^\ell(\beta, N_0, N_\infty) = 0$ if $(\beta, N_0, N_\infty) \in \mathcal{S}_\ell$. In this case part* 1 *in Theorem* 1.2 *occurs.*
2. *For every $k \in \mathbb{N}$ there is a set $\mathcal{S}^k$ such that $F_1^k(\beta, N_0, N_\infty) = 0$ for all $(\beta, N_0, N_\infty) \in \mathcal{S}^k$. In this case part* 2 *in Theorem* 1.2 *occurs.*

As mentioned in the introduction, we think that this sets $\mathcal{S}^k$ and $\mathcal{S}_\ell$ are two-dimensional surfaces.

We end this section discussing the case when $F_2^\ell(\beta, N_0, N_\infty) \neq 0$ for $(\beta, N_0, N_\infty) \in \Omega$. In this case there exists infinitely many solutions of Eq. (1.3)–(1.4) with Dirichlet boundary condition on some R. In other words, Eq. (1.1) with Dirichlet boundary condition on a fixed ball, has infinitely many solutions. More precisely, we have

Proposition 5.2. *If the parameters* $(\beta, N_0, N_\infty) \in \Omega$ *are such that* $F_2^\ell(\beta, N_0, N_\infty) \neq 0$ *for all* $\ell \in \mathbb{N}$, *then for all large initial condition* γ *the solution to Eqs.* (1.3)–(1.4) *is a crossing solution. Moreover, we find an unbounded monotone sequence of initial condition* $\{\gamma_n\}$ *such that the crossing point* $R(\gamma_n)$ *satisfies* $R(\gamma_{2n+1}) < R^* < R(\gamma_{2n})$ *and* $\lim_{n\to\infty} R(\gamma_n) = R^*$. *Moreover, there are infinitely positive solutions of* (1.3)–(1.4) *in* $[0, R^*)$ *such that* $u(R^*) = 0$.

Proof. Consider the three orbits x_∞^i, $i = 1, 2, 3$, of system S_∞ passing through $(P_0 - \delta, 0)$, $(P_0, 0)$ and $(P_0 + \delta, 0)$, respectively. By hypothesis, here we may choose $\delta > 0$ small so that these three solutions cross the y-axis in finite time. If $(x_\infty^2(0), (x_\infty^2)'(0)) = (P_0, 0)$, we let t^* be the first positive t such that $x_\infty^2(t^*) = 0$.

Let x_0 be the orbit of S_0 emanating from the origin that spirals towards $\mathbf{P}_0$. Using the transversality of crossing proved in Lemma 2.2, we find an increasing unbounded sequence $\{t_n\}$ such that

$$x_0(t_n) = x_\infty^2(t_n) \quad \text{and} \quad x_0'(t_n) = \left(x_\infty^2\right)'(t_n).$$

We observe that $x_0'(t_n)$ alternates sign, so by the construction given just before the proof of Theorem 1.2, we find the sequence of initial conditions $\{\gamma(t_n)\}$ which is increasing and unbounded. Moreover $\lim_{n\to\infty} R(\gamma_n) = R^*$ and $R(\gamma(t_{2n+1})) < R^* < R(\gamma(t_{2n}))$ for n large. By continuity of $R(\cdot)$ we establish the existence of infinitely many solutions to (1.3)–(1.4) with $u(R^*) = 0$.

Finally, we observe that for large time $(x_0(t), x_0'(t))$ stays between the orbits x_∞^1 and x_∞^3. Thus, the orbit $\bar{x}_\infty$ of S_∞ defined by the initial conditions

$$x_0(t) = \bar{x}_\infty(0), \qquad x_0'(t) = \bar{x}_\infty'(0)$$

crosses the y-axis in finite time. This shows that for all γ large the solution to (1.3)–(1.4) is a crossing solution. □

Remark 5.1. If $(\beta, N_0, N_\infty) \in \Omega$ are such that $F_1^k(\beta, N_0, N_\infty) \neq 0$ and $F_2^\ell(\beta, N_0, N_\infty) \neq 0$ then Eqs. (1.3)–(1.4) has at most finitely many fast decaying solutions and all other solutions are crossing.

It will be also interesting, and very challenging, to understand maximum and minimum points of the function $R(\gamma)$ in the case of Proposition 5.2, following the line of the open problems mentioned above.

6. Construction of other type of functions *K*

In this section we provide the construction of a different class of function K that allows to find another type of solution of Eq. (1.1). This example further suggests that the structure of the solution set for (1.1) is hard to be described from general properties of K.

(SiS) $u(r)$ is a *singular-slowly decaying solution*, if u satisfies (1.3) for all $r > 0$ and

$$\lim_{r\to 0} r^{\alpha_0} u(r) = c_1 \quad \text{and} \quad \lim_{r\to\infty} r^{\alpha_\infty} u(r;\gamma) = c_2,$$

for certain $c_i > 0$, $i = 1, 2$ and with α_0 and α_∞ as defined in the introduction.

In what follows find a family of functions K so that there exists of a singular-slowly decaying solution u to Eq. (1.3). Observe that for the families of K functions discussed in our main results equation (1.3) does not have singular-slowly decaying solution because $P_0 \neq P_\infty$.

Proposition 6.1. *There are families of functions* K *such that Eq.* (1.3) *possesses singular-slowly decaying solutions.*

Proof. We construct the function K associated to the dimension function $n(\tau)$ given by

$$n(\tau) = \begin{cases} N_0 & \text{if } 0 \leqslant \tau \leqslant 1, \\ N_1 & \text{if } 1 < \tau \leqslant \tau_1, \\ N_\infty & \text{if } \tau > \tau_1. \end{cases}$$

Then, proceeding as in Section 2, we may find a diffeomorphism $g : [0, \infty) \to [0, \infty)$ and associated function K defined as in (2.4) so that whenever v is a solution of the variable dimension equation (2.1) with $n(\tau)$ as above, the function $u(r) = v(g(r))$ satisfies Eq. (2.3). Moreover, it can be proven that this K satisfies (1.5).

Consider now N_0 and N_1 so that $2 < N_1 < N_0 < N_1 + 1$ and $p = (N_1 + 2)/(N_1 - 2)$. Let x_1 be the orbit of (2.12) with $\nu = N_1$ such that $x_1(0) = P_0$, $x_1'(0) = 0$. Since $N_0 < N_1 + 1$ this is a positive periodic orbit. Let t_1 be a first positive t such that $x_1'(t_1) = 0$ and define N_∞ such that $P_\infty = x_1(t_1)$. Then we define the function

$$\hat{x}(t) = \begin{cases} P_0 & \text{if } t \leqslant 0, \\ x_1(t) & \text{if } 0 < t \leqslant t_1, \\ P_\infty & \text{if } t > t_1, \end{cases}$$

that satisfies (2.1). Finally, we define $u(r) = (g(r))^{-\alpha}\hat{x}(\log(g(r)))$, which is the singular-slowly decaying solution we are looking for. □

Remark 6.1. If T is the period of the periodic orbit x_1 given in the proof above, then we can take $t_1 + kT$, $k \in \mathbb{N}$, instead of t_1 in the above construction, to produce infinitely many singular-slowly decaying solution of (1.3).

Remark 6.2. If we take N_1 close N_0 in the above argument then P_0 is close to P_∞ and so we have that the critical points $\mathbf{P}_0$ and $\mathbf{P}_\infty$ are both spirals for systems S_0 and S_∞, respectively. In this situation it can be proved the existence of an unbounded sequence of fast decaying solution to Eqs. (1.3)–(1.4).

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