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Entire spacelike radial graphs in the Minkowski space, asymptotic to the light-cone, with prescribed scalar curvature ☆

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Abstract

We prove the existence and uniqueness in $\mathbb{R}^{n,1}$ of entire spacelike hypersurfaces contained in the future of the origin O and asymptotic to the light-cone, with scalar curvature prescribed at their generic point M as a negative function of the unit vector pointing in the direction of $\overrightarrow{OM}$, divided by the square of the norm of $\overrightarrow{OM}$ (a dilation invariant problem). The solutions are seeked as graphs over the future unit-hyperboloid emanating from O (the hyperbolic space); radial upper and lower solutions are constructed which, relying on a previous result in the Cartesian setting, imply their existence.

Résumé

On prouve l'existence et l'unicité dans $\mathbb{R}^{n,1}$ d'hypersurfaces entières de genre espace contenues dans le futur de l'origine O et asymptotes au cône de lumière, dont la courbure scalaire est prescrite au point générique M comme fonction négative du vecteur unité pointant en direction de $\overrightarrow{OM}$, divisée par le carré de la norme du vecteur $\overrightarrow{OM}$ (un problème invariant par homothétie). Les solutions sont cherchées comme graphes sur l'hyperboloïde-unité futur émanant de O (l'espace hyperbolique) ; des solutions supérieure et inférieure radiales sont construites qui, d'après un résultat antérieur en cartésien, impliquent l'existence de telles solutions.

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0. Introduction

The Minkowski space $\mathbb{R}^{n,1}$ is the affine Lorentzian manifold $\mathbb{R}^n \times \mathbb{R}$ endowed with the metric

$$ds^2 = dX'^2 - dX_{n+1}^2, \quad \text{where } dX'^2 = dX_1^2 + \cdots + dX_n^2,$$

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setting $X = (X', X_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$, and time-oriented by $dX_{n+1} > 0$. Distinguishing the origin O of $\mathbb{R}^{n,1}$, let

$$\mathbb{H} = \left\{x \in \mathbb{R}^{n,1},\ |\overrightarrow{Ox}|^2 = |x'|^2 - {x_{n+1}}^2 = -1,\ x_{n+1} > 0\right\},$$

be the future unit-hyperboloid, model of the hyperbolic space in $\mathbb{R}^{n,1}$. If φ is a real function defined on $\mathbb{H}$, we define the *radial graph* of φ by

$$\mathrm{graph}_{\mathbb{H}}\,\varphi = \left\{X \in \mathbb{R}^{n,1},\ \overrightarrow{OX} = e^{\varphi(x)}\overrightarrow{Ox},\ x \in \mathbb{H}\right\}.$$

This is a hypersurface contained in the future open solid cone

$$C^+ = \left\{X \in \mathbb{R}^{n,1},\ X_{n+1} > |X'|\right\}.$$

We say that φ is spacelike if its graph is a spacelike hypersurface, which means that the metric induced on it is Riemannian. Conversely, a spacelike and connected hypersurface in C^+ is the radial graph of a uniquely determined function $\varphi : \mathbb{H} \to \mathbb{R}$. Of course, such a graph may also be considered as the Cartesian graph of some function $u : \mathbb{R}^n \to \mathbb{R}$

$$\mathrm{graph}_{\mathbb{R}^n}\, u = \left\{\left(x', u(x')\right),\ x' \in \mathbb{R}^n\right\},$$

and the correspondence between the two representations is bijective passing from the Cartesian chart $X = (X', X_{n+1})$ restricted to C^+, to the polar chart $(x, \rho) \in \mathbb{H} \times (0, \infty)$ of C^+ defined by:

$$\rho = \sqrt{-|\overrightarrow{OX}|^2}, \quad \overrightarrow{Ox} = \frac{1}{\rho}\overrightarrow{OX}.$$

Recall that the principal curvatures $(\kappa_1, \ldots, \kappa_n)$ at a point of a spacelike hypersurface are the eigenvalues of its shape endomorphism dN, where N is the future oriented unit normal field, and the mth mean curvature (denoted by H_m) is the mth elementary symmetric function of its principal curvatures: $H_m = \sigma_m(\kappa_1, \ldots, \kappa_n)$. For each real $\lambda > 0$, the cone C^+ is globally invariant under the ambient dilation $X \mapsto \lambda X$ of $\mathbb{R}^{n,1}$ and the above mth mean curvature is $(-m)$-homogeneous; specifically, it transforms like $H_m(\lambda X) = \lambda^{-m} H_m(X)$. It is thus natural to pose, as in [6, Theorem 1], the following inverse problem for H_m: given a positive function $h > 0$ on $\mathbb{H}$ tending to 1 at infinity, find a spacelike hypersurface Σ in C^+, asymptotic to ∂C^+ at infinity, such that, for each point $X \in \Sigma$, the mth mean curvature of Σ at X is given by:

$$\widetilde{H_m} := \frac{1}{\binom{n}{m}} H_m(X) = \frac{1}{(-|\overrightarrow{OX}|^2)^{\frac{m}{2}}}\left[h(x)\right]^m, \quad \text{with } \overrightarrow{Ox} = \frac{\overrightarrow{OX}}{\sqrt{-|\overrightarrow{OX}|^2}}. \tag{1}$$

By construction, this problem is dilation invariant; moreover, as explained below, the positivity of h makes it elliptic. Actually, introducing the positivity cone [9] of σ_m:

$$\Gamma_m = \left\{\kappa \in \mathbb{R}^n,\ \forall i = 1, \ldots, m,\ \sigma_i(\kappa) > 0\right\},$$

and recalling McLaurin's inequalities (satisfied on Γ_m):

$$0 < (\widetilde{H_m})^{\frac{1}{m}} \leqslant (\widetilde{H_{m-1}})^{\frac{1}{m-1}} \leqslant \cdots \leqslant \widetilde{H_2}^{\frac{1}{2}} \leqslant \widetilde{H_1},$$

we note that, if a hypersurface $\Sigma = \mathrm{graph}_{\mathbb{R}^n}\, u$ solves (1) with the asymptotic condition, then the time-function u must assume a minimum on Σ and, as readily checked (using e.g. [3, p. 245]), the principal curvatures of Σ at such a minimum point of u must lie in Γ_m. Now Eq. (1) combined with McLaurin's inequalities forces the principal curvatures of Σ to stay in Γ_m *everywhere*. Let us call any spacelike hypersurface of C^+ having this property, m-admissible; accordingly, a function $\varphi : \mathbb{H} \to \mathbb{R}$ (resp. $u : \mathbb{R}^n \to \mathbb{R}$) is called m-admissible, provided $\mathrm{graph}_{\mathbb{H}}\,\varphi$ (resp. $\mathrm{graph}_{\mathbb{R}^n}\, u$) is so. The condition of m-admissibility is local (and open); one may thus speak of a function $\varphi : \mathbb{H} \to \mathbb{R}$ being m-admissible *at a point* (hence nearby) whenever $\mathrm{graph}_{\mathbb{H}}\,\varphi$ is so at that point. We will seek the solution hypersurface Σ as the radial graph of some m-admissible function $\varphi : \mathbb{H} \to \mathbb{R}$ vanishing at infinity (to comply with the asymptotic condition). Eq. (1) then reads

$$F_m(\varphi) = h, \tag{2}$$

with the radial operator F_m defined by:

$$F_m(\varphi) = e^{\varphi}\left[\widetilde{H_m}(X)\right]^{\frac{1}{m}}, \quad X \in \mathrm{graph}_{\mathbb{H}}\,\varphi.$$

For briefness, we will not compute here explicitly the general expression of the operator F_m (keeping it for a further study)—its restriction to radial functions will suffice (see Section 3.3 below). We will rely instead on the well-known corresponding Cartesian expression (see e.g. [2]) combined with a few basic properties of F_m recorded in the next section (and proved with elementary arguments).

Furthermore, we will essentially restrict to the case $m = 2$ (and freely say 'admissible', for short, instead of '2-admissible'). Since H_2 is related to the scalar curvature S by $S = -2H_2$, our present study is really about the prescription of the scalar curvature, at a generic point X of a radial graph, as a negative function of $x \in \mathbb{H}$ (with x given as in (1)) divided by the square of the norm of $\overrightarrow{OX}$. Aside from the origin O of the ambient space $\mathbb{R}^{n,1}$, we will distinguish a point o in $\mathbb{H}$ and set $r = r(x)$ for the hyperbolic distance from o to $x \in \mathbb{H}$; accordingly, a function on $\mathbb{H}$ will be called *radial* whenever it factors through a function of r only. Our main result is the following:

Theorem 1. *For $\alpha \in (0, 1)$, let $h : \mathbb{H} \to (0, \infty)$ be a function of class $C^{2,\alpha}$ with*

$$\lim_{r(x)\to+\infty} h(x) = 1.$$

Assume that the functions h^- and h^+ defined on $\mathbb{R}^+$ by

$$h^-(r) = \sup_{r(x)=r} h(x) \quad \text{and} \quad h^+(r) = \inf_{r(x)=r} h(x)$$

satisfy

$$\int_0^{+\infty} \left(h^- - 1\right)_+ dr < +\infty, \qquad \int_0^{+\infty} \left(1 - h^+\right)_+ dr < +\infty,$$

where $(h^- - 1)_+$ (resp. $(1 - h^+)_+$) means the positive part of $h^- - 1$ (resp. $1 - h^+$). Then the equation

$$F_2(\varphi) = h \tag{3}$$

has a unique admissible solution of class $C^{4,\alpha}$ such that $\lim_{r(x)\to+\infty} \varphi(x) = 0$.

Remark 1. From Lemma 4 below, anytime the function h is radial, the integral convergence conditions of Theorem 1 appears necessary for the existence of bounded solutions.

An analogous problem in the Euclidean setting is solved for the Gauss curvature in [6, Théorème 1], and in [13,5] some related problems are studied. In the Lorentzian setting, the prescription of the mean curvature for entire graphs is studied in [1] and that of the Gauss curvature in [11,8,4]. In [3], the scalar curvature is prescribed in Cartesian coordinates $x_{n+1} = u(x_1, \ldots, x_n)$.

The outline of the paper is as follows. In Section 1, we prove that there exists at most one solution vanishing at infinity for Eq. (2) with $m \in \{1, \ldots, n\}$. In Section 2, relying on [3], we prove the existence of a solution when $m = 2$, provided upper and lower barriers are known. The latter are constructed, as radial functions, in Section 3.

1. Uniqueness

We first require a few basic properties of the operator F_m. It is a non-linear second order scalar differential operator defined on m-admissible real functions on $\mathbb{H}$. The dilation invariance of (1) implies the identity:

$$F_m(\psi + c) \equiv F_m(\psi), \tag{4}$$

for every m-admissible function $\psi : \mathbb{H} \to \mathbb{R}$ and constant c; linearizing at ψ yields

$$dF_m(\psi)(1) \equiv 0.$$

Furthermore, we have:

Lemma 1. *For each m-admissible function ψ, the linear differential operator $dF_m(\psi)$ is elliptic everywhere on $\mathbb{H}$, with positive-definite symbol.*

Summarizing for later use, the expression of $dF_m(\psi)$, in the chart $x' \in \mathbb{R}^n$ of $\mathbb{H}$, at a fixed m-admissible function ψ reads like:

$$\delta\psi \mapsto dF_m(\psi)(\delta\psi) = \sum_{1\leqslant i,j\leqslant n} B_{ij} \frac{\partial^2}{\partial x'_i \partial x'_j}(\delta\psi) + \sum_{i=1}^n B_i \frac{\partial}{\partial x'_i}(\delta\psi), \tag{5}$$

with the $n \times n$ matrix (B_{ij}) symmetric positive definite (depending on ψ, of course, like the B_i's). We now proceed to proving Lemma 1.

Proof. We require the Cartesian operator $v \mapsto G_m(v) := F_m(\psi)$ defined on m-admissible functions $v : \mathbb{R}^n \to \mathbb{R}$ by:

$$\mathrm{graph}_{\mathbb{R}^n}\, v = \mathrm{graph}_{\mathbb{H}}\, \psi. \tag{6}$$

The ellipticity of $dG_m(v)$ and the positive-definiteness of its symbol are well-known [10,14,2]. Its expression thus starts out like

$$dG_m(v)(\delta v) = \sum_{1\leqslant i,j\leqslant n} A_{ij} \frac{\partial^2}{\partial X'_i \partial X'_j}(\delta v) + \text{ lower order terms,}$$

with the matrix (A_{ij}) symmetric positive definite. The m-admissible function ψ on $\mathbb{H}$ such that (6) holds, is related to v, in the chart $x' = (x_1, \dots, x_n) \in \mathbb{R}^n$, by:

$$v(X') = \sqrt{1+|x'|^2}\exp\bigl[\psi(x')\bigr], \quad \text{with } \overrightarrow{OX'} = e^{\psi(x')}\overrightarrow{Ox'}.$$

Varying ψ by $\delta\psi$ thus yields for the corresponding variation δv of v the following expression: $\delta v(X') = w(X')\delta\psi(x')$, with

$$w(X') = \left[v - \sum_{i=1}^n X'_i \frac{\partial v}{\partial X'_i}\right](X').$$

Since the graph lies in C^+ and it is spacelike, we have $v(X') > |X'|$ and (using Schwarz inequality)

$$\sum_{i=1}^n X'_i \frac{\partial v}{\partial X'_i} < |X'|,$$

therefore $w > 0$. Moreover, up to lower order terms, we have:

$$\frac{\partial^2}{\partial X'_i \partial X'_j}(\delta v)(X') = w(X') \sum_{1\leqslant i,j\leqslant n} \frac{\partial^2}{\partial x'_k \partial x'_l}(\delta\psi)(x') \frac{\partial x'_k}{\partial X'_i}\frac{\partial x'_l}{\partial X'_j}$$

with $x'_k = \frac{X'_k}{\sqrt{v^2(X')-|X'|^2}}$. We thus find in (5):

$$B_{kl} = w(X') \sum_{1\leqslant i,j\leqslant n} A_{ij} \frac{\partial x'_k}{\partial X'_i}\frac{\partial x'_l}{\partial X'_j}$$

and the ellipticity of $\delta\psi \mapsto dF_m(\psi)(\delta\psi)$ follows. □

We need also a more specific (ellipticity) property of the operator F_m, namely:

Lemma 2. *For each couple (φ_0, φ_1) of m-admissible real functions on $\mathbb{H}$ and each point $x_0 \in \mathbb{H}$ where $\varphi = \varphi_1 - \varphi_0$ assumes a local extremum, the whole segment $t \in [0,1] \to \varphi_t = \varphi_0 + t\varphi$ consists of m-admissible functions at the point x_0.*

Proof. The analogue of Lemma 2 is fairly standard in the Cartesian setting, using the expression of the operator G_m introduced in the proof of Lemma 1 (see [2]) together with the well-known fact: $\forall \kappa \in \Gamma_m, \forall i \in \{1,\dots,n\}$, $\frac{\partial \sigma_m}{\partial \kappa_i}(\kappa) > 0$. Here, we will simply reduce the proof to that setting (and let the reader complete the argument). Let us first normalize the situation at an extremum point $x_0 \in \mathbb{H}$ of φ. From (4), we may assume $\varphi(x_0) = 0$. Moreover, we may assume that φ has a local minimum at x_0 (if not, exchange φ_0 and φ_1). Finally, setting $\mathrm{graph}_{\mathbb{H}}\, \varphi_a = \mathrm{graph}_{\mathbb{R}^n}\, u_a$ for $a = 0, 1$, and performing if necessary a suitable Lorentz transform (hyperbolic rotation), we may take $x_0 = (0, 1) \in \mathbb{R}^n \times \mathbb{R}$ thus with $u_a(0) = 1$. For $t \in [0, 1]$ and near x_0, set $\Sigma_t = \mathrm{graph}_{\mathbb{R}^n}\, u_t$ for the hypersurface $\mathrm{graph}_{\mathbb{H}}\, \varphi_t$. We must prove that Σ_t is m-admissible at x_0. For $X_t \in \mathbb{R}^{n,1}$ lying in Σ_t, we have: $\overrightarrow{OX_t} = e^{t\varphi(x)}\overrightarrow{OX_0}$ with $\overrightarrow{Ox} = \overrightarrow{OX_0}/\sqrt{-|\overrightarrow{OX_0}|^2}$. In the Cartesian setting, we thus have (sticking to the $\mathbb{R}^n$-valued charts used in the preceding proof):

$$u_t\big(X_t'\big) = e^{t\varphi(x')}u_0\big[e^{-t\varphi(x')}X_t'\big],$$

here with $x' = X_0'/\sqrt{u_0^2(X_0') - |X_0'|^2}$, $X_t' = e^{t\varphi(x')}X_0'$, and $(X_0', u_0(X_0')) \in \mathrm{graph}_{\mathbb{R}^n}\, u_0$; moreover, the lemma boils down to proving that u_t is m-admissible at $X_t' = 0$. A routine calculation yields at $X_t' = 0$ the equalities:

$$\frac{\partial u_t}{\partial X_{ti}'}(0) = \frac{\partial u_0}{\partial X_{0i}'}(0), \qquad \frac{\partial^2 u_t}{\partial X_{ti}'\partial X_{tj}'}(0) = \frac{\partial^2 u_0}{\partial X_{0i}'\partial X_{0j}'}(0) + t\frac{\partial^2 \varphi}{\partial x_i'\partial x_j'}(0),$$

where, in the second one, the matrix $[\partial^2\varphi/\partial x_i'\partial x_j'(0)]_{1\leqslant i,j\leqslant n}$ is non-negative. We readily infer [2] that, for each $t \in [0, 1]$, the principal curvatures $\kappa_{1t} \leqslant \cdots \leqslant \kappa_{nt}$ of the hypersurface Σ_t at x_0 (each repeated according to its multiplicity) satisfy: $\forall i \in \{1, \ldots, n\}$, $\kappa_{it} \geqslant \kappa_{i0}$. The latter implies that the n-tuple $(\kappa_{1t}, \ldots, \kappa_{nt})$ lies in the cone Γ_m, since $(\kappa_{10}, \ldots, \kappa_{n0}) \in \Gamma_m$. □

Theorem 2. *The operator F_m is one-to-one on m-admissible functions of class C^2 vanishing at infinity.*

Proof. Let us argue by contradiction. Let φ_0, φ_1 be two m-admissible C^2 functions vanishing at infinity and having the same image by F_m. For $t \in [0, 1]$, set $\varphi_t = \varphi_0 + t\varphi$ with $\varphi = \varphi_1 - \varphi_0$. Since φ vanishes at infinity, if $\varphi \not\equiv 0$, it assumes a non-zero local extremum (a maximum, say, with no loss of generality) at some point $x_0 \in \mathbb{H}$. By Lemma 2, the whole segment $t \in [0, 1] \to \varphi_t$ is m-admissible in a neighborhood Ω of x_0 where φ thus satisfies the second order linear equation $L\varphi = 0$ with $L1 = 0$ and the operator L given by $L = \int_0^1 dF_m(\varphi_t)\,dt$. Combining Lemma 1 above with Hopf's strong Maximum Principle (see [7]), we get $\varphi \equiv \varphi(x_0)$ throughout Ω. By connectedness, we infer $\varphi \equiv \varphi(x_0) \neq 0$ on the whole of $\mathbb{H}$, contradicting $\lim_{r(x)\to+\infty}\varphi = 0$. So, indeed, we must have $\varphi \equiv 0$, in other words F_m is one-to-one. □

2. Existence of a solution reduced to that of upper and lower solutions

Theorem 3. *Let $h : \mathbb{H} \to \mathbb{R}$ be a function of class $C^{2,\alpha}$, for some $\alpha \in (0, 1)$, such that there exists $\varphi^- \in C^{4,\alpha}(\mathbb{H})$ with $\mathrm{graph}_{\mathbb{H}}\, \varphi^-$ strictly convex and spacelike, and $\varphi^+ \in C^2(\mathbb{H})$ with $\mathrm{graph}_{\mathbb{H}}\, \varphi^+$ spacelike, satisfying*

$$F_2(\varphi^-) \geqslant h, \quad F_2(\varphi^+) \leqslant h \quad \text{and} \quad \lim_{r(x)\to+\infty} \varphi^{\pm} = 0.$$

Then the equation

$$F_2(\varphi) = h$$

has a unique admissible solution of class $C^{4,\alpha}$ such that $\lim_{r(x)\to+\infty}\varphi(x) = 0$. Moreover φ satisfies the pinching:

$$\varphi^- \leqslant \varphi \leqslant \varphi^+.$$

Remark 2. Since φ is a bounded function, the hypersurface $M = \mathrm{graph}_{\mathbb{H}}(\varphi)$ is entire. More precisely, denoting by $\varphi_{\min}$ and $\varphi_{\max}$ two constants such that $\varphi_{\min} \leqslant \varphi \leqslant \varphi_{\max}$, the function $u : \mathbb{R}^n \to \mathbb{R}$ such that $\mathrm{graph}_{\mathbb{R}^n}(u) = \mathrm{graph}_{\mathbb{H}}(\varphi)$ satisfies $u_{\min} \leqslant u \leqslant u_{\max}$ where $u_{\min}$ (resp. $u_{\max}$) is such that $\mathrm{graph}_{\mathbb{R}^n}(u_{\min}) = \mathrm{graph}_{\mathbb{H}}(\varphi_{\min})$ (resp. $\mathrm{graph}_{\mathbb{R}^n}(u_{\max}) = \mathrm{graph}_{\mathbb{H}}(\varphi_{\max})$). Noting that the graphs of $u_{\min}$ and $u_{\max}$ are hyperboloids, we see that the inequality $u \geqslant u_{\min}$ implies that M is entire, and the inequality $u \leqslant u_{\max}$ implies that M is asymptotic to the lightcone.

Proof. The asserted uniqueness follows from Theorem 2; so let us focus on the existence part. A straightforward comparison principle, using (5) and Lemma 2, implies $\varphi^- \leqslant \varphi^+$ on $\mathbb{H}$. Let $u^-, u^+ : \mathbb{R}^n \to \mathbb{R}$ be such that $\text{graph}_{\mathbb{R}^n}(u^\pm) = \text{graph}_{\mathbb{H}}(\varphi^\pm)$. Set H for the function on $\mathbb{R}^{n,1}$ defined by:

$$H(X) = \frac{\binom{n}{2}}{|X_{n+1}|^2 - |X'|^2}\left[h\left(\frac{X}{\sqrt{|X_{n+1}|^2 - |X'|^2}}\right)\right]^2. \tag{7}$$

The spacelike functions u^- and u^+ satisfy:

$$H_2[u^-] \geqslant H(\cdot, u^-), \quad H_2[u^+] \leqslant H(\cdot, u^+), \quad u^- \leqslant u^+ \quad \text{and} \quad \lim_{|x'|\to\infty}\left[u^\pm(x') - |x'|\right] = 0,$$

where $H_2[u^\pm]$ stands for the second mean curvature of the graph of $u^\pm$. Theorem 1.1 in [3] asserts the existence of a function $u : \mathbb{R}^n \to \mathbb{R}$, belonging to $C^{4,\alpha}$, spacelike, such that $H_2[u] = H(\cdot, u)$ in $\mathbb{R}^n$, $\lim_{|x'|\to+\infty} u(x') - |x'| = 0$, and $u^- \leqslant u \leqslant u^+$. The function $\varphi : \mathbb{H} \to \mathbb{R}$ such that $\text{graph}_{\mathbb{H}}(\varphi) = \text{graph}_{\mathbb{R}^n}(u)$ is a solution of our original problem. □

3. Construction of radial upper and lower solutions

In the sequel of the paper, we first solve the Dirichlet problem on a bounded set in $\mathbb{H}$ (Section 3.1) then proceed to proving the existence and uniqueness of an entire solution in the radial case and study its properties (Sections 3.2 and 3.3); finally, we construct the required radial barriers (Section 3.4).

3.1. The Dirichlet problem

Theorem 4. *Given $\alpha \in (0,1)$, let Ω be a uniformly convex bounded open subset of $\mathbb{H}$ with $C^{2,\alpha}$ boundary, $h : \Omega \to \mathbb{R}$ be a positive function of class $C^{2,\alpha}$, and $\varphi_0 : \overline{\Omega} \to \mathbb{R}$ be a spacelike function of class $C^{2,\alpha}$ whose radial graph is strictly convex. Then the Dirichlet problem*

$$F_2(\varphi) = h \quad \text{in } \Omega, \qquad \varphi = \varphi_0 \quad \text{on } \partial\Omega, \tag{8}$$

has a unique admissible solution of class $C^{4,\alpha}$.

Proof. We first prove uniqueness, by contradiction: let φ_0, φ_1 be two admissible solutions of (8), and, for $t \in [0,1]$, set $\varphi_t = \varphi_0 + t\varphi$ with $\varphi = \varphi_1 - \varphi_0$. Since φ vanishes on $\partial\Omega$, if $\varphi \not\equiv 0$, it assumes a non-zero local extremum. Following the arguments of the proof of Theorem 2 we obtain a contradiction with the Hopf's strong Maximum Principle. Let us focus now on the existence part. Setting $x = (x', \sqrt{1 + |x'|^2}) \in \mathbb{R}^n \times \mathbb{R}$, and

$$\Omega' = \left\{e^{\varphi_0(x)}x',\ x \in \Omega\right\}, \qquad u_0\left(e^{\varphi_0(x)}x'\right) = e^{\varphi_0(x)}\sqrt{1 + |x'|^2},$$

problem (8) is equivalent to the Dirichlet problem:

$$H_2[u] = H(\cdot, u) \quad \text{in } \Omega', \qquad u = u_0 \quad \text{on } \partial\Omega', \tag{9}$$

where H_2 is the scalar curvature operator acting on spacelike graphs defined on $\Omega' \subset \mathbb{R}^n$, and H is defined on $\Omega' \times \mathbb{R}$ by (7).

Let us consider the Banach space

$$E = \left\{\bar{v} \in C^{2,\alpha}(\overline{\Omega'}),\ \bar{v} = 0 \text{ on } \partial\Omega'\right\},$$

and the open convex subset of E

$$U = \left\{\bar{v} \in E,\ \sup_{\overline{\Omega'}}\left|D(\bar{v} + u_0)\right| < 1\right\}.$$

We first note that for every $\bar{v} \in U$, $\text{graph}_{\mathbb{R}^n}(\bar{v} + u_0)$ belongs to the dependence set K of $\text{graph}_{\mathbb{R}^n} u_0$. Here, by definition, $X \in \mathbb{R}^{n,1}$ belongs to K if for every $\xi \in \mathbb{R}^{n,1}$ with $\langle \xi, \xi \rangle \leqslant 0$ and $\xi \neq 0$, the ray $X + \mathbb{R}.\xi$ meets $\text{graph}_{\mathbb{R}^n} u_0$. The set K is a compact subset of the open cone C^+.

For each $(\bar{v}, t) \in U \times [0,1]$, we know from [2,15] that the Dirichlet problem

$$H_2[u] = tH(\cdot, \bar{v} + u_0) + (1-t)H_2[u_0] \quad \text{in } \Omega', \qquad u = u_0 \quad \text{on } \partial\Omega' \tag{10}$$

has a unique admissible solution (belonging to $C^{4,\alpha}$). We define the map

$$T : [0,1] \times U \to E,$$
$$(t, \bar{v}) \mapsto \bar{u}$$

where $\bar{u}$ is such that $u = \bar{u} + u_0$ is the admissible solution of (10).

For each $t \in [0,1]$ the fixed points of $T(t,\cdot)$ are under control: indeed, suppose $T(t, \underline{u}) = \underline{u}$, then the function $u = \underline{u} + u_0$ solves the Dirichlet problem

$$H_2[u] = \tilde{H}(\cdot, u) \quad \text{in } \Omega', \qquad u = u_0 \quad \text{on } \partial\Omega' \tag{11}$$

where

$$\tilde{H}(\cdot, u) = tH(\cdot, u) + (1-t)H_2[u_0]. \tag{12}$$

The following *a priori* estimates are carried out in [3, p.251]: there exist $\vartheta \in (0,1)$ and $C > 0$ such that

$$\sup_{\overline{\Omega'}} |Du| < 1 - \vartheta \quad \text{and} \quad \|u\|_{2,\alpha,\overline{\Omega'}} < C. \tag{13}$$

The constants ϑ, C only depend on $\operatorname{diam}(\Omega')$, $\inf_K \tilde{H}$, $\|\tilde{H}\|_{2,K}$, $\|u_0\|_{4,\overline{\Omega'}}$, and on a positive lower bound on the minimum eigenvalue of D^2u_0 on $\overline{\Omega'}$. The expression of $\tilde{H}$ implies that they are independent of the parameter $t \in [0,1]$.

In order to prove that $T(1,\cdot)$ has a fixed point, we now consider the (nonempty) convex subset of the Banach space E:

$$U_{\vartheta,C} = \left\{ \bar{v} \in U,\ \left| D(\bar{v} + u_0) \right| < 1 - \vartheta \text{ and } \|\bar{v} + u_0\|_{2,\alpha,\overline{\Omega'}} < C \right\},$$

and the map $T : [0,1] \times \overline{U}_{\vartheta,C} \to E$. Then the following properties hold:

(i) T is continuous with compact image due to the above estimates on the solutions of the Dirichlet problem (10);
(ii) $T(0,\cdot) \equiv 0$ by definition;
(iii) for every $t \in [0,1]$, $T(t,\cdot)$ does not have any fixed point on $\partial U_{\vartheta,C}$, since each fixed point of $T(t,\cdot)$ belongs to $U_{\vartheta,C}$ by the definitions of ϑ and C.

An elementary version of the Leray–Schauder theorem (due to Browder and Potter [12]) implies that $T(1,\cdot)$ has a fixed point, which proves that (8) has a solution. □

3.2. Existence and uniqueness of entire radial solutions

The aim of this section is to prove the following result:

Theorem 5. *For $\alpha \in (0,1)$, let $h : \mathbb{R}^+ \to \mathbb{R}$ be a positive function of class $C^{2,\alpha}$ constant on some neighborhood of* 0 *and let φ_0 be a real number. Recall $r = r(x)$ denotes the hyperbolic distance of $x \in \mathbb{H}$ from a fixed origin $o \in \mathbb{H}$. The problem*:

$$F_2(\varphi)(x) = h(r) \quad \text{for all } x \in \mathbb{H}, \qquad \varphi(o) = \varphi_0, \tag{14}$$

admits a unique admissible radial solution $\varphi : \mathbb{H} \to \mathbb{R}$ of class $C^{4,\alpha}$.

Proof. *Existence*: let B_i denote the ball in $\mathbb{H}$ with center o and radius $i \in \mathbb{N}^*$, and φ_i be the admissible solution of the Dirichlet problem:

$$F_2(\varphi) = h, \qquad \varphi_{|\partial B_i} = 0, \tag{15}$$

given by Theorem 4. By radial symmetry and uniqueness, φ_i is a radial function: $\varphi_i(x) = f_i(r)$ for some function $f_i : [0, i] \to \mathbb{R}$. By uniqueness again, for $j > i$, the function $\varphi_j - \varphi_i$ must be constant on B_i. Therefore $f_j'(r) \equiv f_i'(r)$ for $r \in [0, i]$. We may thus define g on $\mathbb{R}^+$ by $g = f_i'$ on each $[0, i]$. Now the function φ defined by

$$\varphi(x) = \varphi_0 + \int_0^r g(u)\, du$$

is a radial solution of (14).

Uniqueness: assume that φ_1 and φ_2 are admissible radial solutions of (14): $\varphi_1(x) = f_1(r)$, $\varphi_2(x) = f_2(r)$ where f_1, f_2 are functions $\mathbb{R}^+ \to \mathbb{R}$. For each real $R > 0$, set

$$\varphi_{1,R}(x) = -\int_r^R {f_1}'(u)\, du \quad \text{and} \quad \varphi_{2,R}(x) = -\int_r^R {f_2}'(u)\, du.$$

The functions $\varphi_{1,R}$ and $\varphi_{2,R}$ are both admissible solutions of the Dirichlet problem (15) on B_R. As such, they must coincide on B_R, hence ${f_1}' = {f_2}'$ on $[0, R]$, which implies the desired result. □

3.3. *Properties of the radial solutions*

The following lemma describes the monotonicity of a solution φ of Eq. (14) depending on the sign of $h - 1$:

Lemma 3. *Let $h : \mathbb{R}^+ \to \mathbb{R}$ and $\varphi : \mathbb{H} \to \mathbb{R}$ be as in Theorem* 5, *and let $f : \mathbb{R}^+ \to \mathbb{R}$ be such that $\varphi(x) = f[r(x)]$, $\forall x \in \mathbb{H}$.*

(i) *If $h \leqslant 1$, then f is non-increasing*; *in particular, if $\varphi_0 = 0$, the function φ is non-positive.*
(ii) *If $h \geqslant 1$, then f is non-decreasing*; *in particular, if $\varphi_0 = 0$, the function φ is non-negative.*

Proof. Here, we need to calculate explicitly the expression of Eq. (14) in the radial case. Set $e_1, \ldots, e_{n+1}$, for the standard orthonormal basis of the vector space $\mathbb{R}^{n,1}$. Fix $x \in \mathbb{H}$ and take, with no loss of generality,

$$o = e_{n+1} = (0, \ldots, 0, 1), \qquad x = (\sinh r, 0, \ldots, 0, \cosh r)$$

with r, the hyperbolic distance between o and x. Consider the orthonormal basis of $T_x\mathbb{H}$ defined by:

$$\partial_r = \cosh r e_1 + \sinh r e_{n+1}, \quad \text{and} \quad \partial_\vartheta = e_\vartheta, \quad \vartheta = 2, \ldots, n,$$

and the vectors, tangent to $M = \operatorname{graph}_{\mathbb{H}} \varphi$ at $e^{\varphi(x)}x$, induced by the embedding $x \in \mathbb{H} \to e^{\varphi(x)}x \in M$, given by:

$$u_r = e^f (f'x + \partial_r), \qquad u_\vartheta = e^f \partial_\vartheta, \quad \vartheta = 2, \ldots, n.$$

The future oriented unit normal to M at $e^{\varphi(x)}x$ is the vector:

$$N(r) = \frac{f'}{\sqrt{1 - f'^2}} \partial_r + \frac{1}{\sqrt{1 - f'^2}} x. \tag{16}$$

Let S be the shape endomorphism of M at $e^{\varphi(x)}x$, with respect to the future unit normal $N(r)$. Using the formulas

$$D_{\partial_r} \overline{\partial_r}(x) = x, \qquad D_{\partial_\vartheta} \overline{\partial_r}(x) = \frac{1}{\tanh r} \partial_\vartheta$$

where D denotes the canonical flat connection of $\mathbb{R}^{n,1}$ and $\overline{\partial_r}$ the unit radial vector field of $\mathbb{H}$ with respect to the point o, we readily get:

$$S(u_r) = dN(\partial_r) = \frac{e^{-f}}{\sqrt{1 - f'^2}} \left(\frac{f''}{1 - f'^2} + 1 \right) u_r,$$

and, for $\vartheta = 2, \ldots, n$,

$$S(u_\vartheta) = dN(\partial_\vartheta) = \frac{e^{-f}}{\sqrt{1-f'^2}}\left(\frac{f'}{\tanh r} + 1\right)u_\vartheta.$$

The principal curvatures of M at $r > 0$ are thus equal to:

$$\frac{e^{-f}}{\sqrt{1-f'^2}}\left(\frac{f''}{1-f'^2} + 1\right) \text{ (simple)}, \qquad \frac{e^{-f}}{\sqrt{1-f'^2}}\left(\frac{f'}{\tanh r} + 1\right) \text{ (multiplicity } n-1\text{)}.$$

Setting $s = s(r)$ for the hyperbolic distance from o to $N(r)$, we infer from (16):

$$s(r) = r + \operatorname{Argth}(f'). \tag{17}$$

In terms of the new radial unknown $s(r)$, for $r > 0$, the principal curvatures read

$$\left(e^{-f}\cosh(r-s)s', e^{-f}\frac{\sinh s}{\sinh r}, \ldots, e^{-f}\frac{\sinh s}{\sinh r}\right), \tag{18}$$

and the equation $F_2(\varphi) = h$ reads

$$2s'\cosh(r-s)\sinh r\sinh s = nh^2\sinh^2 r - (n-2)\sinh^2 s. \tag{19}$$

We now prove the first statement of the lemma. Since $f' = \tanh(s-r)$, we must prove: $s \leqslant r$ on $[0, +\infty)$. Suppose first $h < 1$. Since $s(0) = 0$ and $s'(0) = h(0) < 1$ (from (19)), there exists $r_0 > 0$ such that $s \leqslant r$ on $[0, r_0]$. Moreover, we get from (19):

$$s' \leqslant \frac{1}{2\cosh(r-s)}\left(n\frac{\sinh r}{\sinh s} - (n-2)\frac{\sinh s}{\sinh r}\right).$$

We observe that the function $s(r) = r$ is a solution of the ODE:

$$s' = \frac{1}{2\cosh(r-s)}\left(n\frac{\sinh r}{\sinh s} - (n-2)\frac{\sinh s}{\sinh r}\right)$$

on $[r_0, +\infty)$. So the comparison theorem for solutions of ordinary differential equations implies $s \leqslant r$ on $[r_0, +\infty)$. Suppose only $h \leqslant 1$, fix $A > 0$ and consider $h_\delta = h - \delta$, where δ is some small positive constant such that $h_\delta > 0$ on $[0, A]$. Denoting by φ_δ and s_δ the corresponding solutions of (14) and (19) on the ball of radius A, the function $s_\delta - r$ is non-positive; we now prove that $s_\delta - r$ converges uniformly to $s - r$ as δ tends to zero, which will yield the desired result. Set B_A for the ball of radius A in $\mathbb{H}$ centered at o and $U = \{\psi \in C^{2,\alpha}(\overline{B_A}),\ \psi + \varphi \text{ is admissible in } \overline{B_A}, \psi_{|\partial B_A} = 0\}$; consider the auxiliary map:

$$\Phi : \psi \in U \to \Phi(\psi) := F_2(\psi + \varphi) \in C^\alpha(\overline{B_A}).$$

Since $\Phi(0) = h$ and since, classically [7] (recalling (5)), the linearized map $d\Phi(0)$ is an isomorphism from $\{\xi \in C^{2,\alpha}(\overline{B_A}),\ \xi_{|\partial B_A} = 0\}$ to $C^\alpha(\overline{B_A})$, the inverse function theorem implies: $\forall\varepsilon > 0, \exists\delta_0 > 0, \forall\delta \in (0, \delta_0)$, the solution $\psi_\delta \in U$ of $F_2(\psi_\delta + \varphi) = h_\delta$ satisfies $|\psi_\delta|_{2,\alpha} \leqslant \varepsilon$. Since $\varphi_\delta = \psi_\delta + \varphi - \psi_\delta(o)$, we obtain $|\varphi_\delta - \varphi|_{2,\alpha} \leqslant 2\varepsilon$, which implies the convergence of φ_δ to φ in C^1 and thus the uniform convergence of s_δ to s.

The proof of statement (ii) is analogous and thus omitted. □

Our next lemma provides a simple necessary and sufficient condition for an entire radial solution to be bounded.

Lemma 4. *Let $h : \mathbb{R}^+ \to \mathbb{R}$ and $\varphi : \mathbb{H} \to \mathbb{R}$ be as in Theorem* 5.

(i) *Assume $h \leqslant 1$, and $\lim_{r\to\infty} h = 1$. Then*

$$\lim_{r(x)\to+\infty} \varphi(x) > -\infty \quad \textit{if and only if} \quad \int_0^{+\infty} (1-h)\,dr \textit{ converges.}$$

(ii) *Assume* $h \geqslant 1$, *and* $\lim_{r\to\infty} h = 1$. *Then*

$$\lim_{r(x)\to+\infty} \varphi(x) < +\infty \quad \textit{if and only if} \quad \int_0^{+\infty} (h-1)\,dr \textit{ converges.}$$

Proof. Let us prove statement (i), thus assuming $h \leqslant 1$, with $\lim_{r\to\infty} h = 1$. We stick to the notations used in the proof of Lemma 3. From (17), we get at once:

$$\varphi(x) = \varphi_0 - \int_0^{r(x)} \tanh\bigl(u - s(u)\bigr)\,du. \tag{20}$$

Statement (i) amounts to proving that $\int_0^{+\infty} \tanh(u - s(u))\,du$ converges if and only if so does $\int_0^{+\infty} (1-h)\,dr$. We split the proof of this fact into five steps.

Step 1. The solution s of (19) is an increasing function.

Let us consider in the (r,s) plane the curve $\mathcal{C}$ with equation:

$$nh^2 \sinh^2 r = (n-2)\sinh^2 s, \quad r, s \geqslant 0.$$

The slope of its tangent at $(0,0)$ is $\sqrt{\frac{n}{n-2}}h(0)$. Since the solution s satisfies $s(0) = 0$ and $s'(0) = h(0)$, we infer that the graph of s stays under the curve $\mathcal{C}$ near 0. Noting that the following vector field, associated to the differential equation (19):

$$(r,s) \mapsto \bigl(2\cosh(r-s)\sinh r \sinh s,\ nh^2 \sinh^2 r - (n-2)\sinh^2 s\bigr),$$

is horizontal on $\mathcal{C}$, and that the height s of the curve $\mathcal{C}$ is increasing with r, we conclude that the solution s of (19) remains trapped below $\mathcal{C}$. In other words $nh^2 \sinh^2 r \geqslant (n-2)\sinh^2 s$ for all r, and (19) implies: $s' \geqslant 0$.

Step 2. $r - s$ has a limit at $+\infty$.

By contradiction, assume $\liminf(r-s) < \limsup(r-s) = \delta$. Thus there exists a sequence $r_k \to +\infty$ such that $r_k - s(r_k) \to \delta$ and $s'(r_k) = 1$. Denoting $s(r_k)$ by s_k, we get from Eq. (19):

$$1 = \frac{1}{2\cosh(r_k - s_k)}\left[nh^2(r_k)\frac{\sinh r_k}{\sinh s_k} - (n-2)\frac{\sinh s_k}{\sinh r_k}\right]. \tag{21}$$

We distinguish two cases:

First case: $\delta < +\infty$. We then have $s_k \to +\infty$, $\frac{\sinh r_k}{\sinh s_k} \sim e^{r_k - s_k} \sim e^{\delta}$ and $\frac{\sinh s_k}{\sinh r_k} \sim e^{s_k - r_k} \sim e^{-\delta}$ as k tends to infinity (here and below, the equivalence $\sim$ between two quantities means that their quotient has limit 1). So (21) yields

$$1 = \frac{1}{2\cosh\delta}\bigl[ne^{\delta} - (n-2)e^{-\delta}\bigr].$$

Using $e^{\delta} \geqslant e^{-\delta}$ we get $1 \geqslant \frac{e^{\delta}}{\cosh\delta}$, which is absurd.

Second case: $\delta = +\infty$. First assuming that s_k is not bounded, and since s is an increasing function (Step 1), we have: $s_k \to +\infty$, $\frac{\sinh r_k}{\sinh s_k} \sim e^{r_k - s_k} \to +\infty$ and $\frac{\sinh s_k}{\sinh r_k} \sim e^{s_k - r_k} \to 0$ as k tends to infinity. Eq. (21) yields

$$1 \sim \frac{n}{2\cosh(r_k - s_k)} e^{r_k - s_k},$$

which is absurd since $\cosh(r_k - s_k) \sim \frac{e^{r_k - s_k}}{2}$. If we now assume s_k bounded, since s is an increasing function with $s'(0) > 0$, we get that s_k converges to $l > 0$, and, since $\frac{\sinh s_k}{\sinh r_k} \to 0$, we obtain from (21):

$$1 \sim \frac{n}{2\cosh(r_k - s_k)} \frac{\sinh r_k}{\sinh l},$$

with $\sinh r_k \sim \frac{e^{r_k}}{2}$, $\cosh(r_k - s_k) \sim \frac{e^{r_k - s_k}}{2} \sim \frac{e^{-l}}{2} e^{r_k}$; so $1 = \frac{n}{2}\frac{e^{l}}{\sinh l}$, which is absurd.

Step 3. $r - s$ tends to 0 at infinity.

Having proved that $r - s$ converges, let us set $\delta = \lim_{r\to+\infty} r - s$ and prove by contradiction that $\delta = 0$. There are two cases:

First case: $0 < \delta < +\infty$. We get $s \to +\infty$, hence $\frac{\sinh r}{\sinh s} \sim e^{r-s} \sim e^{\delta}$, $\frac{\sinh s}{\sinh r} \sim e^{s-r} \sim e^{-\delta}$ as r tends to infinity, and thus, from (19):

$$s' \to \frac{1}{2\cosh\delta}\big[ne^{\delta} - (n-2)e^{-\delta}\big].$$

The latter expression is larger than 1, which contradicts $r \geqslant s$.

Second case: $\delta = +\infty$. We first note that $\frac{\sinh s}{\sinh r} \to 0$ (if s is bounded this is trivial; if s is not bounded, $s \to +\infty$ since s is increasing, and we have $\frac{\sinh s}{\sinh r} \sim e^{s-r} \to 0$ since $r - s \to +\infty$). Moreover we have $\liminf nh^2 \frac{\sinh r}{\sinh s} \geqslant n$ since $r \geqslant s$. We thus infer from Eq. (19):

$$s' \sim \frac{n}{2\cosh(r-s)} \frac{\sinh r}{\sinh s}.$$

Assuming $s \to +\infty$, we get $\frac{\sinh r}{\sinh s} \sim e^{r-s}$ and $\cosh(r-s) \sim \frac{e^{r-s}}{2}$, hence $s' \sim n$, which is impossible since $s \leqslant r$.

Finally, assuming s bounded yields $s \to l > 0$; since $r - s \to +\infty$, we infer $\cosh(r-s) \sim \frac{e^{r-s}}{2}$ and $\sinh r \sim \frac{e^r}{2}$, hence from (19), $e^{-s}s' \sim \frac{n}{2}\frac{1}{\sinh l}$ and thus $s' \sim \frac{n}{2}\frac{e^l}{\sinh l}$, which contradicts the boundedness assumption on s.

Step 4. $\lim_{r(x)\to+\infty} \varphi(x) > -\infty$ if and only if $\varepsilon(r) := r - s$ is integrable on $[0, +\infty)$.

This is straightforward from (20) combined with $\tanh(u - s(u)) \sim \varepsilon(u)$ which holds as $u \to +\infty$ due to Step 3.

Step 5. ε is integrable on $[0, +\infty)$ if and only if $\beta := 1 - h^2$ is integrable on $[0, +\infty)$.

First observation: $\lim_{r\to\infty} s' = 1$. Indeed, at infinity, we have $r - s \to 0$, so $s \to +\infty$, hence:

$$\frac{\sinh r}{\sinh s} \sim e^{r-s} \sim 1, \qquad \frac{\sinh s}{\sinh r} \sim e^{s-r} \sim 1,$$

and (19) yields $s' \to 1$.

Using Step 3, the assumptions on h and the preceding observation, we get

$$\varepsilon(r) \to 0, \quad \beta(r) \to 0, \quad \text{and} \quad \varepsilon'(r) = 1 - s'(r) \to 0$$

as r tends to infinity. Plugging the definitions of ε and β in (19) and using the expansions

$$\cosh\varepsilon = 1 + \mathrm{o}(\varepsilon), \qquad \sinh(r - \varepsilon) = \sinh r\big(1 - \varepsilon + \mathrm{o}(\varepsilon)\big),$$

yields

$$(n-1)\varepsilon + \varepsilon' + \mathrm{o}(\varepsilon) = \frac{n}{2}\beta. \tag{22}$$

Fixing a real $\delta > 0$, there readily exists $r_\delta > 0$ such that, for all $r \geqslant r_\delta$,

$$\varepsilon' + (n - 1 - \delta)\varepsilon \leqslant \frac{n}{2}\beta, \tag{23}$$

and

$$\varepsilon' + (n - 1 + \delta)\varepsilon \geqslant \frac{n}{2}\beta. \tag{24}$$

Integrating (23), we get, for $r \geqslant r_\delta$,

$$\varepsilon(r) \leqslant e^{-(n-1-\delta)r}\left[C(r_\delta) + \frac{n}{2}\int_{r_\delta}^{r} \beta(u)e^{(n-1-\delta)u}\,du\right].$$

Integrating again and using Fubini Theorem yields, with δ such that $n - 1 - \delta > 0$,

$$\int_{r_\delta}^{+\infty} \varepsilon(r)\,dr \leqslant C'(r_\delta) + \frac{n}{2}\int_{r_\delta}^{+\infty} \beta(u)e^{(n-1-\delta)u}\left(\int_u^{+\infty} e^{-(n-1-\delta)r}dr\right)du,$$

$$\leqslant C'(r_\delta) + \frac{n}{2(n-1-\delta)}\int_{r_\delta}^{+\infty}\beta(u)\,du.$$

We conclude that ε is integrable provided $\beta = 1-h^2$ is integrable.

Analogously, using (24), we get

$$\varepsilon(r) \geqslant e^{-(n-1+\delta)r}\left[C(r_\delta) + \frac{n}{2}\int_{r_\delta}^{r} \beta(u)e^{(n-1+\delta)u}\,du\right],$$

and

$$\int_{r_\delta}^{+\infty} \varepsilon(r)\,dr \geqslant C'(r_\delta) + \frac{n}{2}\int_{r_\delta}^{+\infty} \beta(u)e^{(n-1+\delta)u}\left(\int_u^{+\infty} e^{-(n-1+\delta)r}dr\right)du,$$

$$\geqslant C'(r_\delta) + \frac{n}{2(n-1+\delta)}\int_{r_\delta}^{+\infty}\beta(u)\,du.$$

Taking $\delta > 0$ arbitrary, we find that β is integrable if ε is integrable.

The proof of statement (ii) is analogous and thus omitted. □

3.4. Construction of radial barriers

Lemma 5. *Let $h : \mathbb{H} \to \mathbb{R}$ be a positive and continuous function on the hyperbolic space such that*

$$\lim_{r(x)\to+\infty} h(x) = 1$$

and such that the functions h^- and h^+ defined on $\mathbb{R}^+$ by

$$h^-(r) = \sup_{r(x)=r} h(x) \quad \textit{and} \quad h^+(r) = \inf_{r(x)=r} h(x)$$

satisfy

$$\int_0^{+\infty} (h^- - 1)_+ dr < +\infty, \qquad \int_0^{+\infty} (1-h^+)_+ dr < +\infty,$$

where $(h^- - 1)_+$ (resp. $(1-h^+)_+$) means the positive part of $h^- - 1$ (resp. $1-h^+$). Then there exist $\varphi^-, \varphi^+ \in C^\infty(\mathbb{H})$, with strictly convex spacelike graphs, satisfying:

$$F_2(\varphi^-) \geqslant h, \quad F_2(\varphi^+) \leqslant h \quad \textit{and} \quad \lim_{r\to+\infty} \varphi^\pm = 0.$$

Proof. First, considering $1 + (h^- - 1)_+$ instead of h^- and $1 - (1-h^+)_+$ instead of h^+, we may suppose without loss of generality that h^- and h^+ are two continuous functions such that: $\forall x \in \mathbb{H}$, with $r = r(x)$,

$$h^-(r) \geqslant h(x) \geqslant h^+(r) > 0, \tag{25}$$

$$h^- \geqslant 1 \geqslant h^+, \qquad \lim_{r\to+\infty} h^-(r) = \lim_{r\to+\infty} h^+(r) = 1, \tag{26}$$

and

$$\int_0^{+\infty} (h^- - 1)\,dr < +\infty, \qquad \int_0^{+\infty} (1-h^+)\,dr < +\infty. \tag{27}$$

If we now consider

$$h^- + \frac{\varepsilon_0}{r^2} \quad \text{if } r \geqslant 1, \qquad h^- + \varepsilon_0 \quad \text{if } r \leqslant 1$$

instead of h^-, and

$$h^+ - \frac{\varepsilon_0}{r^2} \quad \text{if } r \geqslant 1, \qquad h^+ - \varepsilon_0 \quad \text{if } r \leqslant 1$$

instead of h^+, where ε_0 is chosen sufficiently small such that $\inf h^+ > \varepsilon_0$, we may moreover assume the following:

$$h^- \geqslant \max(1,h) + \frac{\varepsilon_0}{r^2} \quad \text{and} \quad h^+ \leqslant \min(1,h) - \frac{\varepsilon_0}{r^2} \quad \text{if } r \geqslant 1.$$

We now prove that we can approximate $h^\pm$ by smooth functions $g^\pm$ such that

$$\left|h^\pm - g^\pm\right| \leqslant \min\left(\frac{\varepsilon_0}{r^2}, \varepsilon_0\right). \tag{28}$$

For each $i \in \mathbb{N}$, let us denote by g_i^- a smooth function on $[0, i+1]$ such that $|h^- - g_i^-| \leqslant \frac{\varepsilon_0}{(i+1)^2}$ on $[0, i+1]$. Let $\vartheta \in C_c^\infty(\mathbb{R})$ such that $0 \leqslant \vartheta \leqslant 1$, $\vartheta(x) = 1$ if $|x| \leqslant \frac{1}{4}$ and $\vartheta(x) = 0$ if $|x| \geqslant \frac{3}{4}$. We define g^- on $[i, i+1]$ by

$$g^- = \vartheta_i g_i^- + (1-\vartheta_i) g_{i+1}^-,$$

where $\vartheta_i = \vartheta(. - i)$. By construction, we have $g^- = g_i^-$ on a neighborhood of i. The function g^- is thus smooth on $[0, +\infty)$, and satisfies on $[i, i+1]$:

$$\left|g^- - h^-\right| \leqslant \vartheta_i \left|g_i^- - h^-\right| + (1-\vartheta_i)\left|g_{i+1}^- - h^-\right| \leqslant \frac{\varepsilon_0}{(i+1)^2},$$

which implies the estimate (28). We may thus assume that (25), (26) and (27) hold, where $h^\pm$ are two smooth functions on $\mathbb{R}^+$. Considering $\vartheta \sup_{\mathbb{R}^+} h^- + (1-\vartheta)h^-$ instead of h^-, and $\vartheta \inf_{\mathbb{R}^+} h^+ + (1-\vartheta)h^+$ instead of h^+, we may also assume that the functions $h^\pm$ are constant on some neighborhood of 0. Let φ^- and φ^+ be smooth radial functions given by Theorem 5 (with some arbitrary initial condition φ_0) such that $F_2(\varphi^\pm) = h^\pm$. From Lemma 4, subtracting constants if necessary, we obtain $\lim_{r\to+\infty} \varphi^\pm(r) = 0$. □

Now, we can complete the proof of Theorem 1 as follows. Lemma 5 provides two barriers which tend to 0 at infinity; by Theorem 3, we get an entire solution of Eq. (3) pinched between these barriers, and thus tending to 0 at infinity, so the existence part of Theorem 1 is proved. Uniqueness was proved in Theorem 2.

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