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Standing waves for nonlinear Schrödinger equations with singular potentials

Ondes stationnaires pour les équations nonlinéaires de Schrödinger avec potentiels singulaires

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Abstract

We study semiclassical states of nonlinear Schrödinger equations with anisotropic type potentials which may exhibit a combination of vanishing and singularity while allowing decays and unboundedness at infinity. We give existence of spike type standing waves concentrating at the singularities of the potentials.

Résumé

Nous étudions les états semi-classiques des équations de Schrödinger non linéaires avec potentiels de type anisotropiques qui peuvent tendre vers zéro à l'infini, pour lesquels des phénomènes d'évanescence et de singularité sont possibles. Nous donnons l'existence d'ondes stationnaires se concentrant aux singularités des potentiels.

Keywords: Nonlinear Schrödinger equations; Singularities of potentials; Decaying and unbounded potentials

1. Introduction

This paper is concerned with standing waves for nonlinear Schrödinger equations

$$i\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2}\Delta\psi-V(x)\psi+K(x)|\psi|^{p-1}\psi=0,$$

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where $\hbar$ denotes the Planck constant, i is the imaginary unit. The equation arises in many fields of physics, in particular, when we describe Bose–Einstein condensates (refer [18,19]) and the propagation of light in some nonlinear optical materials (refer [20]). In this paper we are concerned with the existence of standing waves of the nonlinear Schrödinger equation for small $\hbar$. For small $\hbar > 0$, these standing wave solutions are refereed as semiclassical states. Here a solution of the form $\psi(x,t) = \exp(-iEt/\hbar)v(x)$ is called a standing wave. Then, a function $\psi(x,t) \equiv \exp(-iEt/\hbar) v(x)$ is a standing wave solution if and only if the function v satisfies

$$\frac{\hbar^2}{2}\Delta v - \big(V(x) - E\big)v + K(x)|v|^{p-1}v = 0, \quad x \in \mathbb{R}^n.$$

With a simple re-scaling and renaming the potential $V - E$ to be V we work on the following version of the equation in this paper

$$\begin{cases} -\varepsilon^2 \Delta v + V(x)v = K(x)v^p, & v > 0,\ x \in \mathbb{R}^n, \\ v \in W^{1,2}(\mathbb{R}^n), & \lim_{|x|\to\infty} v(x) = 0. \end{cases} \tag{1}$$

Here ε is a small parameter, V, K are nonnegative potentials, and p is subcritical $1 < p < \frac{n+2}{n-2}$ with $2^* = \frac{2n}{n-2}$ the critical exponent for $n \geqslant 3$. In recent years intensive works have been done in understanding solutions structure of Eq. (1) as $\varepsilon \to 0$. One of the most characteristic feature is that the semiclassical bound states exhibit concentration behaviors as $\varepsilon \to 0$ (see the classical work [15] by Floer and Weinstein, [21,13,17,14,3,7] and the recent monograph [2] and references therein). In particular, some recent works have been devoted to the cases where the potentials may have vanishing points and may be decaying to zero at infinity [1,4–6,8–11,22]. In [1] ground state solutions in the associated weighted Sobolev spaces are obtained for positive potentials with decay at infinity. In [4,5] decaying potentials are also considered and spike solutions which concentrate at points of positive potential values are given. Then in [6] ground state solutions concentrating near zeroes of potentials were constructed in the weighted Sobolev spaces.

In this paper we consider concentration solutions which both concentrate near zeroes of the potential V and singularities of the potentials V and K for Eq. (1). The potentials V may decay at infinity and K may be unbounded at infinity. One feature of our results is that the solutions we construct have small magnitudes comparing with the spike solutions concentrating at points of positive potential values. Another feature is that these solutions may have very different limiting equations under quite different scalings.

We assume that V satisfies

(V) $V \in L^1_{\rm loc}(\mathbb{R}^n) \cap C(\mathbb{R}^n \setminus \mathcal{S}_0, [0,\infty))$ for a bounded Lebesgue measure zero set $\mathcal{S}_0$, $\mathcal{Z} \cap \overline{\mathcal{S}_0} = \emptyset$ where $\mathcal{Z} = \{x \in \mathbb{R}^n \setminus \mathcal{S}_0 \mid V(x) = 0\}$; $\liminf_{\operatorname{dist}(x,\mathcal{S}_0)\to 0} V(x) \in (0,\infty]$; $\liminf_{|x|\to\infty} |x|^2 V(x) \geqslant 4\lambda > 0$ for some $\lambda > 0$.

We assume that K satisfies

(K) $K \in L^{q_0}_{\rm loc}(\mathbb{R}^n)$ for some $q_0 > \frac{2n}{2n-(p+1)(n-2)}$, $n \geqslant 3$; $K \in C(\mathbb{R}^n \setminus \mathcal{S}, [0,\infty))$ for a bounded and Lebesgue measure zero set $\mathcal{S}$; $\limsup_{|x|\to\infty} K(x)|x|^{-\gamma_\infty} < \infty$ for some $\gamma_\infty > 0$.

If $\mathcal{S}$ is a singleton, for example $\mathcal{S} = \{0\}$, the first condition of (K) holds if $\limsup_{|x|\to 0} |x|^\gamma K(x) < \infty$ for some $\gamma < \frac{2n-(p+1)(n-2)}{2}$.

Our main existence result is the following one.

Theorem 1. *Let $n \geqslant 3$. Suppose that* (V) *and* (K) *hold. Let $A \subset \mathcal{Z} \cup \mathcal{S}$ be an isolated compact subset of $\mathcal{Z} \cup \mathcal{S}$ such that $\overline{A \cap \mathcal{S}_0} \cap \overline{\mathcal{S}_0 \setminus A} = \emptyset$ and*

$$\lim_{0<\operatorname{dist}(x,A)\to 0} V^{\frac{2n-(p+1)(n-2)}{2}}(x)/K^2(x) = 0.$$

Then for ε sufficiently small, (1) *has a positive solution $w_\varepsilon \in W^{1,2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that*

$$\lim_{\epsilon\to 0} \|w_\varepsilon\|_\infty = 0 \quad \textit{and} \quad \liminf_{\varepsilon\to 0} \varepsilon^{\frac{-2}{p+1}} \|w_\varepsilon\|_\infty > 0. \tag{2}$$

Moreover, for each $\delta > 0$, *there are constants* $C, c > 0$ *such that*

$$w_\varepsilon(x) \leqslant C \exp\left(-\frac{c}{\varepsilon}\right)\left(1 + \operatorname{dist}\left(x, A^\delta\right)\right)^{-\sqrt{\lambda}/\varepsilon}, \quad x \notin A^\delta, \tag{3}$$

where $A^\delta \equiv \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, A) \leqslant \delta\}$.

We also study the asymptotic profile of the concentrating solutions given above. It turns out the asymptotic behavior depends on the local behavior of the potentials V and K near the concentrating set A. We give some detained results in Section 3.

Our scheme for a proof of the existence of localized solutions in Theorem 1 is as follow. Since there are singular points of K, and V may converges to 0 at infinity, we consider a truncated equation on a bounded ball $B(0, \mu)$ with a truncated K^μ instead of K and a homogeneous Dirichlet boundary condition. For the truncated problem, we consider a minimization problem with two constraints, where we should delve an appropriate weight and an appropriate exponent for a constraint. The existence of a minimizer u_ε^μ follows from our well chosen setting. Then, we get an upper bound estimate and a lower bound estimate for the minimum. One crucial step in our argument is to obtain an exponential decay, uniform for large μ, of u_ε^μ on a certain set. The decay estimate is derived by combining Moser iterations, standard elliptic estimates, Caffarelli–Kohn–Nirenberg inequalities and comparison principles. Then, we show that a scaled minimizer w_ε^μ is a solution of the truncated problem for small $\varepsilon > 0$ and large $\mu > 0$, and that a weak limit w_ε of $\{w_\varepsilon^\mu\}_\mu$ is a desired solution of our original problem.

We close up the introduction with some discussions of known results and comparison with our new results. During the last twenty years there have been intensive work on semiclassical states of standing waves for the nonlinear Schrödinger equations with potentials. Spike solutions concentrating at points of positive potentials values for V and K have been given in the pioneer work [15] and subsequent works e.g., [1,4,5,21,22]. These solutions are shown to have nice asymptotic behavior in the sense that if $w_\varepsilon(x)$ is a solution of (1) and x_0 is the concentration point then $W_\varepsilon(x) = w_\varepsilon(\varepsilon(x - x_0))$ converges uniformly to a least energy solution of the following limiting equation $\Delta U + V(x_0)U = K(x_0)U^p$, $U > 0$, $x \in \mathbb{R}^n$. Thus U is the limiting profile of semiclassical limits in this case. In [9–11] the authors have studied the critical frequency case for which at the concentration point x_0 the potential V is zero, i.e., $V(x_0) = 0$. We have found that for this case the limiting equations are abundant and appear in different families and that under different scalings the limiting profile of the semiclassical states have small magnitudes, i.e., $\|w_\varepsilon\|_{L^\infty}$ tends to zero as $\varepsilon \to 0$. The new phenomenon we present in the current paper here is that the concentration for spike solutions can be at zeroes and singularities of the potentials for both V and K. It depends on the zero set of a weighted potential involving both V and K. Comparing with the results in [9,10] we construct small solutions even when $V(x_0)$ is positive. On the other hand our new results here allow us to construct solutions concentrating at singular points of V and K. We also investigate the limiting profile of these localized solutions. Under more precise information on the local behaviors of V and K near the concentration points we derive a variety of limiting equations. One simple example covered by our results is when $V(x) = |x|^\tau$ for $|x|$ small, $V(x) \geqslant |x|^{-2}$ for $|x|$ large and $K(x) = |x|^{-\gamma}$ for $|x|$ small, $K(x) \leqslant |x|^{\gamma_\infty}$ for $|x|$ large for some $\gamma_\infty > 0$. Then for $\tau > -2$, $\gamma \in (-\tau(2n - (p+1)(n-2))/4, (2n - (p+1)(n-2))/2)$, our result applies to give the existence of a localized solution w_ε which under a suitable scaling converges to a least energy solution of Eq. (27) (see the statement of Theorem 11).

The proof of Theorem 1 is given in Sections 2 and 3 is devoted to asymptotic analysis of the localized solutions as $\varepsilon \to 0$.

2. Proof of Theorem 1

For a proof of our main results we further elaborate the minimization techniques in [9,11] to construct the spike solutions concentrating near zeroes and singularities of the potentials.

Let $A \subset \mathcal{Z} \cup \mathcal{S}$ be the isolated set assumed in the theorem. We choose $\delta \in (0, 1)$ such that $A^{8\delta} \cap ((\mathcal{Z} \cup \mathcal{S}) \setminus A) = A^{8\delta} \cap (\mathcal{S}_0 \setminus A) = \emptyset$, where for $d > 0$, $A^d = \{x \in \mathbb{R}^n \,|\, \operatorname{dist}(x, A) \leqslant d\}$. We denote in the following $A_\varepsilon^d = \{x \in \mathbb{R}^n \mid \varepsilon x \in A^d\}$ for $\varepsilon, d > 0$. Let E_ε be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_\varepsilon = \left(\int \varepsilon^2 |\nabla u|^2 + V(x)u^2\right)^{1/2}.$$

We define a truncated function for $\mu > 0$

$$K_\mu(x) = \begin{cases} \min\{K(x), \mu\} & \text{if } x \in \mathbb{R}^n \setminus A^{4\delta}, \\ K(x) & \text{if } x \in A^{4\delta}. \end{cases}$$

We consider the following problem for $\mu > 0$

$$\begin{cases} \varepsilon^2 \Delta v - V(x)v + K_\mu(x)v^p = 0, \quad v > 0,\ x \in B(0,\mu), \\ v(x) = 0 \quad \text{on } \partial B(0,\mu). \end{cases} \tag{4}$$

Define $E_\varepsilon^\mu \equiv \overline{(C_0^\infty(B(0,\mu)), \|\cdot\|_\varepsilon)}$ and choose $R_0 > 0$ so that $\mathcal{Z} \cup \mathcal{S}_0 \cup \mathcal{S} \subset B(0, R_0/2)$. Note that

$$\frac{n}{2} < \frac{2n}{2n-(n-2)(p+1)}, \qquad \frac{n}{2} < \frac{p+1}{p-1}.$$

From now, we fix a number $\beta > 0$ satisfying

$$\frac{n}{2} \leqslant \beta < \min\left\{\frac{2n}{2n-(n-2)(p+1)}, \frac{p+1}{p-1}\right\}. \tag{5}$$

For a sufficiently large $\alpha > 0$ which will be specified later, we define χ_ε by

$$\chi_\varepsilon^\mu(x) = \begin{cases} \varepsilon^{-\alpha} & \text{if } |x| \leqslant R_0 \text{ and } x \notin (\mathcal{Z} \cup \mathcal{S})^{4\delta}, \\ \varepsilon^{-\alpha}(\max\{1, K_\mu(x)\})^\beta & \text{if } x \in ((\mathcal{Z} \cup \mathcal{S}) \setminus A)^{4\delta}, \\ \varepsilon^{-2\alpha}|x|^\alpha & \text{if } |x| \geqslant R_0, \\ 0 & \text{if } x \in A^{4\delta}. \end{cases}$$

We define $\tilde\beta \equiv \max\{\beta(p-1), 2\}$. Defining

$$\Phi_\varepsilon^\mu(u) \equiv \int_{\mathbb{R}^n} K_\mu |u|^{p+1}\,dx \quad \text{and} \quad \Psi_\varepsilon^\mu(u) \equiv \int_{\mathbb{R}^n} \chi_\varepsilon^\mu |u|^{\tilde\beta}\,dx,$$

we consider the following minimization problem

$$M_\varepsilon^\mu = \inf\left\{\|u\|_\varepsilon^2 \;\middle|\; \Phi_\varepsilon^\mu(u) = 1,\ \Psi_\varepsilon^\mu(u) \leqslant 1,\ u \in E_\varepsilon^\mu\right\}. \tag{6}$$

Since (K) holds and $\chi_\varepsilon^\mu = 0$ in $A^{4\delta}$, we see that $\Phi_\varepsilon^\mu, \Psi_\varepsilon^\mu \in C^1(E_\varepsilon^\mu)$.

Lemma 2. $\lim_{\varepsilon\to 0} \varepsilon^{-n\frac{p-1}{p+1}} M_\varepsilon^\mu = 0$ *uniformly for large* $\mu > 0$.

Proof. Note that

$$M_\varepsilon^\mu \leqslant \inf_{u \in C_0^\infty(A^{4\delta})} \frac{\int \varepsilon^2|\nabla u|^2 + V(x)u^2\,dx}{(\int K(x)|u|^{p+1}\,dx)^{\frac{2}{p+1}}}.$$

Letting $v(x) = u(\varepsilon x)$ we have

$$M_\varepsilon^\mu \leqslant \varepsilon^{\frac{n(p-1)}{p+1}} \inf_{v \in C_0^\infty(A_\varepsilon^{4\delta})} \frac{\int |\nabla v|^2 + V(\varepsilon x)v^2\,dx}{(\int K(\varepsilon x)|v|^{p+1}\,dx)^{\frac{2}{p+1}}}.$$

For any $x_0 \in A^{4\delta} \setminus A$, we can take $r > 0$ with $B(x_0, r) \subset A^{4\delta} \setminus A$ such that $V(x) \leqslant 2V(x_0)$ for $x \in B(x_0, r)$, and $K(x) \geqslant \frac{1}{2}K(x_0)$ for $x \in B(x_0, r)$. Then, we see easily that

$$\begin{aligned} M_\varepsilon^\mu &\leqslant \varepsilon^{\frac{n(p-1)}{p+1}} \inf_{v \in C_0^\infty(B(x_0/\varepsilon, r/\varepsilon))} \frac{\int |\nabla v|^2 + V(\varepsilon x)v^2\,dx}{(\int K(\varepsilon x)|v|^{p+1}\,dx)^{\frac{2}{p+1}}} \leqslant \varepsilon^{\frac{n(p-1)}{p+1}} \inf_{v \in C_0^\infty(B(0, r/\varepsilon))} \frac{\int |\nabla v|^2 + 2V(x_0)v^2\,dx}{(\int \frac{1}{2}K(x_0)|v|^{p+1}\,dx)^{\frac{2}{p+1}}} \\ &= \varepsilon^{\frac{n(p-1)}{p+1}} \big(2V(x_0)\big)^{\frac{2n-(p+1)(n-2)}{2(p+1)}} \big(K(x_0)/2\big)^{-\frac{2}{p+1}} \inf\left\{\frac{\int |\nabla v|^2 + v^2\,dx}{(\int |v|^{p+1}\,dx)^{\frac{2}{p+1}}} \;\middle|\; v \in C_0^\infty\big(B\big(0, \sqrt{2V(x_0)}r/\varepsilon\big)\big)\right\}. \end{aligned}$$

Thus, it follows that there exists $C > 0$ such that for any $x_0 \in A^{4\delta} \setminus A$ sufficiently close to A,

$$\lim_{\varepsilon\to 0} \varepsilon^{-n\frac{p-1}{p+1}} M_\varepsilon^\mu \leqslant C V^{\frac{2n-(p+1)(n-2)}{2(p+1)}}(x_0)/K^{\frac{2}{p+1}}(x_0).$$

Then, the estimate follows. □

Lemma 3. *The minimization M_ε^μ is achieved by a nonnegative $u_\varepsilon^\mu \in E_\varepsilon^\mu$ which satisfies for some $\eta_\varepsilon^\mu > 0 \geqslant \xi_\varepsilon^\mu$*

$$-\varepsilon^2 \Delta u_\varepsilon^\mu + V(x) u_\varepsilon^\mu = \eta_\varepsilon^\mu K_\mu(x)\big(u_\varepsilon^\mu\big)^p + \xi_\varepsilon^\mu \chi_\varepsilon^\mu(x)\big(u_\varepsilon^\mu\big)^{\tilde\beta-1} \quad \text{in } B(0,\mu). \tag{7}$$

Proof. From the choice of β and the definition K^μ and χ_ε^μ, we see that M_ε^μ is achieved by a minimizer $u_\varepsilon^\mu \in E_\varepsilon^\mu$ which can be assumed to be nonnegative and satisfies Eq. (7) with Lagrange multipliers α_ε^μ and β_ε^μ. Following an argument from [9] we have $\eta_\varepsilon^\mu > 0 \geqslant \xi_\varepsilon^\mu$. □

Lemma 4.

$$\lim_{\varepsilon\to 0} \varepsilon^{-n\frac{p-1}{p+1}} \eta_\varepsilon^\mu = 0 \quad \textit{uniformly for large } \mu > 0.$$

Proof. By contradiction we assume there exist $\eta_0 > 0$, $\mu_m \to \infty$, $\varepsilon_m \to 0$ such that $\liminf_{m\to\infty} \varepsilon_m^{-n\frac{p-1}{p+1}} \eta_m^{\mu_m} \geqslant \eta_0 > 0$. Let $u_m = u_{\varepsilon_m}^{\mu_m}$ and $\eta_m = \eta_{\varepsilon_m}^{\mu_m}$. For small $b > 0$, we define a cut-off function $\phi_b(x)$ which is 1 for x satisfying $\mathrm{dist}(x, \mathbb{R}^n \setminus A^{4\delta}) > b$ and 0 for $x \notin A^{4\delta}$, and $|\nabla\phi| \leqslant 2/b$. Multiplying Eq. (7) by $\phi_b u_m$ and integrating over the space, and using the fact that $\inf_{\{x\in A^{4\delta} | d(x,\mathbb{R}^n\setminus A^{4\delta})\leqslant b\}} V(x) \geqslant c_0 > 0$ (independent of small b), we get that for some $C > 0$,

$$\varepsilon_m^{-n\frac{p-1}{p+1}} \eta_m \int\limits_{\{x | d(x,\mathbb{R}^n\setminus A^{4\delta})>b\}} K_{\mu_m} u_m^{p+1} \leqslant C \varepsilon_n^{-n\frac{p-1}{p+1}} \int \big(\varepsilon_m^2 |\nabla u_m|^2 + V(x) u_m^2\big).$$

We see from Lemma 2 that the right-hand side in above inequality converges to 0 as $n \to \infty$. Then we see that

$$\lim_{m\to\infty} \int\limits_{\{x | d(x,\mathbb{R}^n\setminus A^{4\delta})>b\}} K_{\mu_m} u_m^{p+1} = 0. \tag{8}$$

On the other hand, let ψ be another cut-off function satisfying that $\psi(x) = 1$ for $x \in A^{4\delta} \setminus A^{3\delta}$ and $\psi(x) = 0$ for $x \in A^{2\delta}$ or $x \notin A^{5\delta}$, and $|\nabla\psi| \leqslant 2/\delta$. Note that $\tilde\beta < p + 1$. Then, using (8), the fact $\Psi_{\varepsilon_m}^{\mu_m}(u_m) \leqslant 1$ and Hölder inequality, we see that $\int_{\mathbb{R}^n} K_{\mu_m}(\psi u_m)^{p+1} \to 1$ as $m \to \infty$. Consider $w_m(x) = \varepsilon_m^{\frac{n}{p+1}} \psi(\varepsilon_m x) u_m(\varepsilon_m x)$ and by using Lemma 2 we have $w_m \to 0$ in $H^1(\mathbb{R}^n)$. By embedding theorems and the fact $\inf_{x\in A^{5\delta}\setminus A^{2\delta}} V(x) > 0$, it follows that $\int K_{\mu_m}(\varepsilon_m x) w_m^{p+1} \to 0$. This contradicts that $\int K_{\mu_n}(\varepsilon_m x) w_m^{p+1} = \int K_{\mu_m}(x)(\psi(x) u_m(x))^{p+1} \to 1$ as $m \to \infty$. □

Lemma 5. *If $\alpha > 0$ is sufficiently large,*

$$\liminf_{\varepsilon\to 0} \varepsilon^{-2} \eta_\varepsilon^\mu > 0 \quad \textit{uniformly for large } \mu > 0.$$

Proof. Arguing indirectly, we assume for a subsequence (still denoted by ε) $\varepsilon^{-2}\eta_\varepsilon^\mu \to 0$. Choose a cut-off function ϕ satisfying $\phi(x) = 1$ for $x \in A^{4\delta}$ and $\phi(x) = 0$ for $x \notin A^{5\delta}$. Then for some constant $C > 0$ independent of small $\varepsilon > 0$ and large $\mu > 0$, it follows that

$$\begin{aligned}
\int \big|\nabla\big(\phi u_\varepsilon^\mu\big)\big|^2\,dx &\leqslant 2\int \phi^2 \big|\nabla u_\varepsilon^\mu\big|^2 + |\nabla\phi|^2 \big(u_\varepsilon^\mu\big)^2\,dx \\
&\leqslant C\int \big|\nabla u_\varepsilon^\mu\big|^2 + \varepsilon^{-2} V \big(u_\varepsilon^\mu\big)^2\,dx = C\varepsilon^{-2} M_\varepsilon^\mu \\
&\leqslant C\varepsilon^{-2}\eta_\varepsilon^\mu \to 0 \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Then, by Hölder's inequality and Sobolev embedding, we see that

$$\int_{A^{4\delta}} K_\mu\big(u_\varepsilon^\mu\big)^{p+1} = \int_{\mathbb{R}^n} K\big(\phi u_\varepsilon^\mu\big)^{p+1} \to 0 \quad \text{as } \varepsilon \to 0.$$

By the constraint $\Psi_\varepsilon^\mu(u_\varepsilon^\mu) \leqslant 1$, we see that

$$\int_{\{x\notin(\mathcal{Z}\cup\mathcal{S})^{4\delta} \mid |x|\leqslant R_0\}} \big(u_\varepsilon^\mu\big)^{\tilde\beta}\,dx \leqslant \varepsilon^\alpha, \tag{9}$$

$$\int_{\{x\in(\mathcal{Z}\cup\mathcal{S}\setminus A)^{4\delta}\}} K_\mu^\beta\big(u_\varepsilon^\mu\big)^{\tilde\beta}\,dx \leqslant \varepsilon^\alpha \tag{10}$$

and

$$\int_{\{|x|\geqslant R_0\}} |x|^\alpha\big(u_\varepsilon^\mu\big)^{\tilde\beta}\,dx \leqslant \varepsilon^{2\alpha}. \tag{11}$$

Note that $\tilde\beta < p+1$. Then, using Hölder's inequality, we see that for some $C>0$, independent of $\mu,\varepsilon>0$,

$$\begin{aligned}
&\int_{\{x\in(\mathcal{Z}\cup\mathcal{S}\setminus A)^{4\delta}\}} K_\mu\big(u_\varepsilon^\mu\big)^{p+1}\,dx \\
&\leqslant \bigg(\int_{\{x\in(\mathcal{Z}\cup\mathcal{S}\setminus A)^{4\delta}\}} (K_\mu)^\beta\big(u_\varepsilon^\mu\big)^{\beta(p-1)}\,dx\bigg)^{1/\beta}\bigg(\int_{\{x\in(\mathcal{Z}\cup\mathcal{S}\setminus A)^{4\delta}\}} \big(u_\varepsilon^\mu\big)^{2\beta/(\beta-1)}\bigg)^{(\beta-1)/\beta} \\
&\leqslant C\bigg(\int_{\{x\in(\mathcal{Z}\cup\mathcal{S}\setminus A)^{4\delta}\}} (K_\mu)^\beta\big(u_\varepsilon^\mu\big)^{\beta(p-1)}\,dx\bigg)^{1/\beta}\bigg(\int_{\{x\in(\mathcal{Z}\cup\mathcal{S}\setminus A)^{4\delta}\}} \big(u_\varepsilon^\mu\big)^{2^*}\bigg)^{2/2^*}.
\end{aligned}$$

Thus, if $\tilde\beta = \beta(p-1)$, it follows that

$$\lim_{\varepsilon\to 0} \int_{\{x\in(\mathcal{Z}\cup\mathcal{S}\setminus A)^{4\delta}\}} K_\mu\big(u_\varepsilon^\mu\big)^{p+1}\,dx = 0.$$

When $\tilde\beta = 2$, we deduce from conditions (5) and (10) that for some $C>0$,

$$\begin{aligned}
&\int_{\{x\in(\mathcal{Z}\cup\mathcal{S}\setminus A)^{4\delta}\}} (K_\mu)^\beta\big(u_\varepsilon^\mu\big)^{\beta(p-1)}\,dx \\
&\leqslant \int_{\{x\in(\mathcal{Z}\cup\mathcal{S}\setminus A)^{4\delta} \mid u_\varepsilon^\mu(x)\leqslant\varepsilon^{\alpha/2}\}} (K_\mu)^\beta\big(u_\varepsilon^\mu\big)^{\beta(p-1)}\,dx + \int_{\{x\in(\mathcal{Z}\cup\mathcal{S}\setminus A)^{4\delta} \mid u_\varepsilon^\mu(x)\geqslant\varepsilon^{\alpha/2}\}} (K_\mu)^\beta\big(u_\varepsilon^\mu\big)^{\beta(p-1)}\,dx \\
&\leqslant \varepsilon^{\alpha\beta(p-1)/2} \int_{\{x\in(\mathcal{Z}\cup\mathcal{S}\setminus A)^{4\delta}\}} (K_\mu)^\beta\,dx + \varepsilon^{\alpha\beta(p-1)/2} \int_{\{x\in(\mathcal{Z}\cup\mathcal{S}\setminus A)^{4\delta} \mid u(x)_\varepsilon^\mu\geqslant\varepsilon^{\alpha/2}\}} (K_\mu)^\beta\big(u_\varepsilon^\mu/\varepsilon^{\alpha/2}\big)^{\beta(p-1)}\,dx \\
&\leqslant \varepsilon^{\alpha\beta(p-1)/2}C + \varepsilon^{\alpha\beta(p-1)/2-\alpha} \int_{\{x\in(\mathcal{Z}\cup\mathcal{S}\setminus A)^{4\delta}\}} (K_\mu)^\beta\big(u_\varepsilon^\mu\big)^2\,dx \\
&\leqslant \varepsilon^{\alpha\beta(p-1)/2}C + \varepsilon^{\alpha\beta(p-1)/2}.
\end{aligned}$$

Thus, we see that

$$\lim_{\varepsilon\to 0} \int_{\{x\in(\mathcal{Z}\cup\mathcal{S}\setminus A)^{4\delta}\}} K_\mu\big(u_\varepsilon^\mu\big)^{p+1}\,dx = 0$$

uniformly for $\mu > 0$. Using Hölder's inequality, condition (K), (9) and (11), we see that if $\alpha > 0$ is sufficiently large,

$$\lim_{\varepsilon\to 0} \int\limits_{\{x\notin(\mathcal{Z}\cup\mathcal{S})^{4\delta} \mid |x|\leqslant R_0\}} K_\mu \big(u_\varepsilon^\mu\big)^{p+1}\,dx = 0$$

and

$$\lim_{\varepsilon\to 0} \int\limits_{\{|x|\geqslant R_0\}} K_\mu \big(u_\varepsilon^\mu\big)^{p+1}\,dx = 0$$

uniformly for large $\mu > 0$. Then, we get

$$\lim_{\varepsilon\to 0} \int K_\mu \big(u_\varepsilon^\mu\big)^{p+1}\,dx = 0,$$

which contradicts the constraint $\int K_\mu (u_\varepsilon^\mu)^{p+1}\,dx = 1$. This completes the proof. □

We denote $w_\varepsilon^\mu \equiv (\eta_\varepsilon^\mu)^{\frac{1}{p-1}} u_\varepsilon^\mu$. Then, we see that

$$-\varepsilon^2 \Delta w_\varepsilon^\mu + V w_\varepsilon^\mu \leqslant K_\mu \big(w_\varepsilon^\mu\big)^p \quad \text{in } B(0,\mu). \tag{12}$$

From Lemmas 2 and 4, we see that

$$\lim_{\varepsilon\to 0} \varepsilon^{-n} \int\limits_{\mathbb{R}^n} \varepsilon^2 \big|\nabla w_\varepsilon^\mu\big|^2 + V\big(w_\varepsilon^\mu\big)^2 = 0 \quad \text{uniformly for large } \mu > 0. \tag{13}$$

Then, it follows from Sobolev embedding and the fact $\Psi_\varepsilon^\mu(u_\varepsilon^\mu) \leqslant 1$ that for any $q \in [\tilde{\beta}, 2^*)$ and $r \in (0,\delta)$,

$$\lim_{\varepsilon\to 0} \varepsilon^{-n} \int\limits_{\mathbb{R}^n\setminus A^r} \big(w_\varepsilon^\mu\big)^q = 0 \quad \text{uniformly for large } \mu > 0, \tag{14}$$

where $2^* = 2n/(n-2)$ for $n \geqslant 3$.

Lemma 6. *For any $r \in (0,\delta)$, there exist $c, C > 0$, independent of large $\mu > 0$, such that for small $\varepsilon > 0$,*

$$w_\varepsilon^\mu(x) \leqslant C\exp(-c/\varepsilon) \quad \textit{for } |x| \leqslant R_0 \quad \textit{and} \quad \operatorname{dist}(x, \mathcal{Z}\cup\mathcal{S}) > r.$$

Proof. Note that in the set $B(0,R_0)\setminus\{x \mid \operatorname{dist}(x,\mathcal{Z}\cup\mathcal{S}) > r\}$, V has a positive lower bound and K is continuous (thus has a upper bound). We may use (13) and (14) together with elliptic estimates (refer [16]) and a maximum principle argument similar to [9] (Lemma 2.6, 2.7 there) to deduce the estimate. □

Lemma 7. *There exist $c, C > 0$, independent of large $\mu > 0$, such that for small $\varepsilon > 0$,*

$$w_\varepsilon^\mu(x) \leqslant C\exp(-c/\varepsilon) \quad \textit{for } x \in \big((\mathcal{Z}\cup\mathcal{S})\setminus A\big)^\delta.$$

Proof. Denoting $w_\varepsilon^\mu \equiv (\eta_\varepsilon^\mu)^{\frac{1}{p-1}} u_\varepsilon^\mu$, we see that

$$-\varepsilon^2 \Delta w_\varepsilon^\mu + V(x) w_\varepsilon^\mu \leqslant K_\mu \big(w_\varepsilon^\mu\big)^p \quad \text{in } B(0,\mu). \tag{15}$$

Let $\phi \in C_0^\infty((\mathcal{Z}\cup\mathcal{S}\setminus A)^{2\delta})$ be a cut-off function such that $\phi(x) = 1$ for $x \in (\mathcal{Z}\cup\mathcal{S}\setminus A)^\delta$ and $|\nabla\phi| \leqslant 4/\delta$. Multiplying both sides of (15) through by $(w_\varepsilon^\mu)^{2l+1}\phi^2$ with $l \geqslant 0$, we see that

$$\frac{\varepsilon^2}{l+1} \int\limits_{\mathbb{R}^n} \big|\nabla\big(w_\varepsilon^\mu\big)^{l+1}\phi\big|^2\,dx \leqslant \frac{\varepsilon^2}{l+1} \int\limits_{\mathbb{R}^n} \big(w_\varepsilon^\mu\big)^{2l+2} |\nabla\phi|^2\,dx + \int\limits_{\mathbb{R}^n} K_\mu \big(w_\varepsilon^\mu\big)^{p-1} \big(w_\varepsilon^\mu\big)^{2l+2} \phi^2\,dx.$$

Then, by the Sobolev inequality and Hölder's inequality, it follows that for some $c > 0$, independent of $\phi, l, \varepsilon, \mu$

$$\frac{c\varepsilon^2}{l+1}\big\|\big(w_\varepsilon^\mu\big)^{2l+2}\phi^2\big\|_{L^{n/(n-2)}} \leqslant \frac{\varepsilon^2}{l+1}\int_{\mathbb{R}^n}\big(w_\varepsilon^\mu\big)^{2l+2}|\nabla\phi|^2\,dx + \Big(\int_{\mathrm{supp}(\phi)}(K_\mu)^\beta\big(w_\varepsilon^\mu\big)^{\beta(p-1)}\,dx\Big)^{1/\beta}\big\|\big(w_\varepsilon^\mu\big)^{2l+2}\phi^2\big\|_{L^{\beta/(\beta-1)}}. \tag{16}$$

If $\beta(p-1)\geqslant 2$, it follows that

$$\int_{\mathrm{supp}(\phi)}(K_\mu)^\beta\big(w_\varepsilon^\mu\big)^{\beta(p-1)}\,dx \leqslant \varepsilon^\alpha.$$

When $\beta(p-1)<2$, it follows that for some $C>0$, independent of small $\varepsilon>0$ and large $\mu>0$,

$$\begin{aligned}
&\int_{\mathrm{supp}(\phi)}(K_\mu)^\beta\big(w_\varepsilon^\mu\big)^{\beta(p-1)}\,dx \\
&= \int_{\{x\in\mathrm{supp}(\phi)\mid w_\varepsilon^\mu(x)\leqslant\varepsilon^{\alpha/2}\}}(K_\mu)^\beta\big(w_\varepsilon^\mu\big)^{\beta(p-1)}\,dx + \int_{\{x\in\mathrm{supp}(\phi)\mid w_\varepsilon^\mu(x)>\varepsilon^{\alpha/2}\}}(K_\mu)^\beta\big(w_\varepsilon^\mu\big)^{\beta(p-1)}\,dx \\
&\leqslant C\varepsilon^{\alpha\beta(p-1)/2} + \varepsilon^{\alpha\beta(p-1)/2}\int_{\{x\in\mathrm{supp}(\phi)\mid w_\varepsilon^\mu(x)>\varepsilon^{\alpha/2}\}}(K_\mu)^\beta\big(w_\varepsilon^\mu/\varepsilon^{\alpha/2}\big)^{\beta(p-1)}\,dx \\
&\leqslant C\varepsilon^{\alpha\beta(p-1)/2} + \varepsilon^{\alpha\beta(p-1)/2}\int_{\mathrm{supp}(\phi)}(K_\mu)^\beta\big(w_\varepsilon^\mu/\varepsilon^{\alpha/2}\big)^2\,dx \\
&\leqslant C\varepsilon^{\alpha\beta(p-1)/2} + \varepsilon^{\alpha\beta(p-1)/2}\big(\eta_\varepsilon^\mu\big)^{2/(p-1)} \leqslant (C+1)\varepsilon^{\alpha\beta(p-1)/2},
\end{aligned}$$

where we used the fact $\Psi_\varepsilon^\mu(u_\varepsilon^\mu)\leqslant 1$. Then we deduce that there exists $C,c>0$, independent of l,ε,μ, satisfying

$$\big\|\big(w_\varepsilon^\mu\big)^{2l+2}\phi^2\big\|_{L^{n/(n-2)}} \leqslant C\exp(-c/\varepsilon) + C(l+1)\varepsilon^{\frac{\alpha\min\{1,\beta(p-1)/2\}}{\beta}-2}\big\|\big(w_\varepsilon^\mu\big)^{2l+2}\phi^2\big\|_{L^{\beta/(\beta-1)}}. \tag{17}$$

Note that $\frac{\beta}{\beta-1}<\frac{n}{n-2}$. Then, by Hölder inequality again there is a constant C_1 only depending on n, β and δ such that

$$\big\|\big(w_\varepsilon^\mu\big)^{2l+2}\phi^2\big\|_{L^{n/(n-2)}} \leqslant C\exp(-c/\varepsilon) + C_1(l+1)\varepsilon^{\frac{\alpha\min\{1,\beta(p-1)/2\}}{\beta}-2}\big\|\big(w_\varepsilon^\mu\big)^{2l+2}\phi^2\big\|_{L^{n/(n-2)}}. \tag{18}$$

We take large $\alpha>0$ so that $\frac{\alpha\min\{1,\beta(p-1)/2\}}{\beta}>2$. This implies that for any large $q>0$, there exists $C,c>0$ such that for small ε, independent of $\mu>0$,

$$\int_{(\mathcal{Z}\cup\mathcal{S}\setminus A)^\delta}\big(w_\varepsilon^\mu\big)^q\,dx \leqslant C\exp(-c/\varepsilon).$$

Applying an elliptic estimate [16] to (15), we see that for any $s>2$ and $t>n/2$, there exists a constant $C>0$, independent of $\varepsilon,\mu>0$, satisfying

$$\big\|w_\varepsilon^\mu\big\|_{L^\infty((\mathcal{Z}\cup\mathcal{S}\setminus A)^\delta)} \leqslant \big\|w_\varepsilon^\mu\big\|_{L^s((\mathcal{Z}\cup\mathcal{S}\setminus A)^{2\delta})} + \Big(\int_{(\mathcal{Z}\cup\mathcal{S}\setminus A)^{2\delta}}\big(K^\mu\big)^t\big(w_\varepsilon^\mu\big)^{pt}\,dx\Big)^{1/t}.$$

Note that $K^\mu\leqslant K$ and $K\in L^{q_0}_{\mathrm{loc}}$ for some $q_0>\frac{2n}{2n-(p+1)(n-2)}>\frac{n}{2}$. We take $t\in(n/2,q_0)$. This implies that $K^t\in L^s_{\mathrm{loc}}$ for some $s>1$. Thus, we see from Hölder inequality that for some C, independent of large μ and small $\varepsilon>0$,

$$\big\|w_\varepsilon^\mu\big\|_{L^\infty((\mathcal{Z}\cup\mathcal{S}\setminus A)^\delta)} \leqslant C\exp(-c/\varepsilon).$$

Thus, for some $C,c>0$, independent of large $\mu>0$ and small $\varepsilon>0$, we see that

$$w_\varepsilon^\mu(x)\leqslant C\exp(-c/\varepsilon)\quad \text{for } x\in\big((\mathcal{Z}\cup\mathcal{S})\setminus A\big)^\delta. \qquad \square$$

The last two lemmas show the following estimate

Lemma 8. *For any $r \in (0, \delta)$, there exist $c, C > 0$, independent of large $\mu > 0$, such that for small $\varepsilon > 0$,*

$$w_\varepsilon^\mu(x) \leqslant C \exp(-c/\varepsilon) \quad \text{for } |x| \leqslant R_0 \quad \text{and} \quad \operatorname{dist}(x, A) > r.$$

Lemma 9. *There exist $c, C > 0$, independent of large $\mu > 0$, such that for small $\varepsilon > 0$,*

$$w_\varepsilon^\mu(x) \leqslant C \exp(-c/\varepsilon)|x/R_0|^{-\sqrt{\lambda}/\varepsilon} \quad \text{for } R_0 \leqslant |x| \leqslant \mu.$$

Proof. First we see from condition (K) that there is a constant $C > 0$, independent of large $\mu > 0$, satisfying

$$-\varepsilon^2 \Delta w_\varepsilon^\mu(x) + V(x) w_\varepsilon^\mu(x) \leqslant C|x|^{\gamma_\infty} \big(w_\varepsilon^\mu(x)\big)^p \quad \text{on } B(0,\mu) \setminus B(0, R_0). \tag{19}$$

For any $\varphi \in C_0^\infty(\mathbb{R}^n \setminus B(0, R_0), [0,1])$, $a > 0$ and $b \geqslant 0$, we multiply both sides of (19) through by $|x|^a (w_\varepsilon^\mu)^{2b+1}\varphi^2$ and integrate by parts. Then, we deduce that

$$\begin{aligned}
\varepsilon^2 \int |x|^a \big|\nabla \big(w_\varepsilon^\mu\big)^{b+1}\varphi\big|^2 dx &\leqslant \varepsilon^2 \int |x|^a \big(w_\varepsilon^\mu\big)^{2b+2} |\nabla\varphi|^2 + a|x|^{a-1} \big|\nabla\big(w_\varepsilon^\mu\big)^{b+1}\big| \big(w_\varepsilon^\mu\big)^{b+1}\varphi^2\, dx \\
&\quad + C_1(b+1) \int |x|^{a+\gamma_\infty} \big(w_\varepsilon^\mu\big)^{p+2b+1} \varphi^2\, dx \\
&\leqslant \varepsilon^2 \int |x|^a \big(w_\varepsilon^\mu\big)^{2b+2} |\nabla\varphi|^2 + a\varepsilon^2 \int |x|^{a-1} \big(w_\varepsilon^\mu\big)^{2b+2} |\nabla\varphi|\varphi\, dx \\
&\quad + a\varepsilon^2 \int |x|^{a/2} \big|\nabla\big(w_\varepsilon^\mu\big)^{b+1}\varphi\big| |x|^{a/2-1} \big(w_\varepsilon^\mu\big)^{b+1}\varphi\, dx \\
&\quad + C_1(b+1) \int |x|^{a+\gamma_\infty} \big(w_\varepsilon^\mu\big)^{p+2b+1} \varphi^2\, dx \\
&\leqslant \varepsilon^2 \int |x|^a \big(w_\varepsilon^\mu\big)^{2b+2} |\nabla\varphi|^2 + a\varepsilon^2 \int |x|^{a-1} \big(w_\varepsilon^\mu\big)^{2b+2} |\nabla\varphi|\varphi\, dx \\
&\quad + a\varepsilon^2 \bigg(\frac{1}{2a} \int |x|^a \big|\nabla\big(w_\varepsilon^\mu\big)^{b+1}\varphi\big|^2 + \frac{a}{2} \int |x|^{a-2} \big(w_\varepsilon^\mu\big)^{2b+2} \varphi^2\, dx\bigg) \\
&\quad + C_1(b+1) \int |x|^{a+\gamma_\infty} \big(w_\varepsilon^\mu\big)^{p+2b+1} \varphi^2\, dx.
\end{aligned}$$

This implies that

$$\begin{aligned}
\varepsilon^2 \int |x|^a \big|\nabla \big(w_\varepsilon^\mu\big)^{b+1}\varphi\big|^2 dx &\leqslant 2\varepsilon^2 \int |x|^a \big(w_\varepsilon^\mu\big)^{2b+2} |\nabla\varphi|^2 + 2a\varepsilon^2 \int |x|^{a-1} \big(w_\varepsilon^\mu\big)^{2b+2} |\nabla\varphi|\varphi\, dx \\
&\quad + a^2\varepsilon^2 \int |x|^{a-2} \big(w_\varepsilon^\mu\big)^{2b+2} \varphi^2\, dx + 2C_1(b+1) \int |x|^{a+\gamma_\infty} \big(w_\varepsilon^\mu\big)^{p+2b+1} \varphi^2\, dx.
\end{aligned}$$

Then, we see from Caffarelli–Kohn–Nirenberg inequality [12] that for some $C_2 > 0$, depend only on n, a and b,

$$\begin{aligned}
&\bigg(\int |x|^{2an/(n-2)} \big|\big(w_\varepsilon^\mu\big)^{b+1}\varphi\big|^{2n/(n-2)}\, dx\bigg)^{(n-2)/2} \\
&\quad \leqslant C_2 \int |x|^a \big(w_\varepsilon^\mu\big)^{2b+2} |\nabla\varphi|^2 + a|x|^{a-1} \big(w_\varepsilon^\mu\big)^{2b+2} |\nabla\varphi|\varphi\, dx \\
&\qquad + C_2 a^2 \int |x|^{a-2} \big(w_\varepsilon^\mu\big)^{2b+2} \varphi^2\, dx + C_2 \frac{b+1}{\varepsilon^2} \int |x|^{a+\gamma_\infty} \big(w_\varepsilon^\mu\big)^{p+2b+1} \varphi^2\, dx.
\end{aligned} \tag{20}$$

Suppose that $\operatorname{supp}(\varphi) \subset B(y, 1) \subset \mathbb{R}^n \setminus B(0, R_0)$. From Lemma 4 and the constraint

$$\int_{\mathbb{R}^n} \chi_\varepsilon^\mu \big(w_\varepsilon^\mu\big)^{\tilde\beta}\, dx \leqslant \big(\eta_\varepsilon^\mu\big)^{\frac{p+1}{p-1}},$$

we see that for small $\varepsilon > 0$,

$$\int_{\{|x|\geqslant R_0\}} |x|^{\alpha}\big(w_\varepsilon^\mu\big)^{\tilde\beta}\,dx \leqslant \varepsilon^{2\alpha}.$$

Note that if $\beta(p-1)\geqslant 2$, there exist some $C>0$, independent of small $\varepsilon>0$ and large $\mu>0$, satisfying

$$\begin{aligned}
&\int |x|^{a+\gamma_\infty}\big(w_\varepsilon^\mu\big)^{p+2b+1}\varphi^2\,dx\\
&\quad\leqslant \bigg(\int_{\mathrm{supp}(\varphi)} |x|^{\beta a+\beta\gamma_\infty}\big(w_\varepsilon^\mu\big)^{\beta(p-1)}\,dx\bigg)^{1/\beta}\bigg(\int\big(\big(w_\varepsilon^\mu\big)^{2b+2}\varphi^2\big)^{\beta/(\beta-1)\,dx}\bigg)^{(\beta-1)/\beta}\\
&\quad\leqslant R_0^{a+\gamma_\infty-\alpha/\beta}\bigg(\int_{\mathrm{supp}(\varphi)} |x|^{\alpha}\big(w_\varepsilon^\mu\big)^{\beta(p-1)}\,dx\bigg)^{1/\beta} C R_0^{-2a}\bigg(\int |x|^{2an/(n-2)}\big(\big(w_\varepsilon^\mu\big)^{2b+2}\varphi^2\big)^{n/(n-2)}\,dx\bigg)^{(n-2)/n}\\
&\quad\leqslant C\varepsilon^{2\alpha/\beta}R_0^{\gamma_\infty-a-\alpha/\beta}\bigg(\int |x|^{2an/(n-2)}\big(\big(w_\varepsilon^\mu\big)^{2b+2}\varphi^2\big)^{n/(n-2)}\,dx\bigg)^{(n-2)/n}.
\end{aligned}$$

Note also that if $\beta(p-1)<2$, there exist some $C>0$, independent of small $\varepsilon>0$ and large $\mu>0$, satisfying

$$\begin{aligned}
&\int |x|^{a+\gamma_\infty}\big(w_\varepsilon^\mu\big)^{p+2b+1}\varphi^2\,dx\\
&\quad\leqslant \bigg(\int_{\mathrm{supp}(\varphi)} |x|^{\beta a+\beta\gamma_\infty}\big(w_\varepsilon^\mu\big)^{\beta(p-1)}\,dx\bigg)^{1/\beta}\bigg(\int\big(\big(w_\varepsilon^\mu\big)^{2b+2}\varphi^2\big)^{\beta/(\beta-1)\,dx}\bigg)^{(\beta-1)/\beta}\\
&\quad\leqslant C\bigg(\int_{\mathrm{supp}(\varphi)} |x|^{\frac{2(a+\gamma_\infty)}{p-1}}\big(w_\varepsilon^\mu\big)^{2}\,dx\bigg)^{(p-1)/2}\bigg(\int\big(\big(w_\varepsilon^\mu\big)^{2b+2}\varphi^2\big)^{\beta/(\beta-1)\,dx}\bigg)^{(\beta-1)/\beta}\\
&\quad\leqslant C R_0^{a+\gamma_\infty-\alpha(p-1)/2}\bigg(\int_{\mathrm{supp}(\varphi)} |x|^{\alpha}\big(w_\varepsilon^\mu\big)^{2}\,dx\bigg)^{(p-1)/2} C R_0^{-2a}\bigg(\int |x|^{\frac{2an}{n-2}}\big(\big(w_\varepsilon^\mu\big)^{2b+2}\varphi^2\big)^{\frac{n}{n-2}}\,dx\bigg)^{(n-2)/n}\\
&\quad\leqslant C^2\varepsilon^{\alpha(p-1)}R_0^{\gamma_\infty-a\alpha(p-1)/2}\bigg(\int |x|^{\frac{2an}{n-2}}\big(\big(w_\varepsilon^\mu\big)^{2b+2}\varphi^2\big)^{\frac{n}{n-2}}\,dx\bigg)^{(n-2)/n}.
\end{aligned}$$

Taking a large $\alpha>0$, we see from (20) that for some constant $C>0$, independent of small $\varepsilon>0$ and large $\mu>0$,

$$\begin{aligned}
\bigg(\int |x|^{2an/(n-2)}\big|\big(w_\varepsilon^\mu\big)^{b+1}\varphi\big|^{2n/(n-2)}\,dx\bigg)^{(n-2)/2} &\leqslant C\int |x|^{a}\big(w_\varepsilon^\mu\big)^{2b+2}|\nabla\varphi|^2 + a|x|^{a-1}\big(w_\varepsilon^\mu\big)^{2b+2}|\nabla\varphi|\varphi\,dx\\
&\quad + Ca^2\int |x|^{a-2}\big(w_\varepsilon^\mu\big)^{2b+2}\varphi^2\,dx. \qquad (21)
\end{aligned}$$

Now we note that for any $c,d,e>0$,

$$\int_{\mathrm{supp}(\varphi)} |x|^{d}\big(w_\varepsilon^\mu\big)^{2b+2}\,dx \leqslant \big(|y|-1\big)^{-c}\int_{\mathrm{supp}(\varphi)} |x|^{d+c}\big(w_\varepsilon^\mu\big)^{2b+2}\,dx.$$

From the inequality

$$\int_{\{|x|\geqslant R_0\}} |x|^{\alpha}\big(w_\varepsilon^\mu\big)^{\tilde\beta}\,dx \leqslant \varepsilon^{2\alpha},$$

we deduce via a finite number of iterations of (21) that for any $a,b,c>0$, we can choose a large $\alpha>0$ so that

$$\int_{\{x\,|\,|y-x|<1/2\}} |x|^{a}\big(w_\varepsilon^\mu\big)^{2b+2}\,dx \leqslant \varepsilon^{c}|y|^{-c}. \qquad (22)$$

Then, applying [16, Theorem 9.20] to (19), we can choose a large $\alpha > 0$ so that for a constant $C > 0$,

$$w_\varepsilon^\mu(x) \leqslant C\left(\frac{\varepsilon}{|x|}\right)^{(\gamma_\infty+2)/(p-1)} \quad \text{for any } |x| \geqslant R_0.$$

We may assume from condition (V) that $V(x) \geqslant 3\lambda|x|^{-2}$ for $|x| \geqslant R_0$. Then, setting $\psi_\varepsilon(r) = (r/\varepsilon)^{-\sqrt{\lambda}/\varepsilon}$, we deduce from condition (V) that for small $\varepsilon > 0$,

$$-\varepsilon^2\Delta\psi_\varepsilon + V\psi_\varepsilon \geqslant \frac{\lambda}{r^2}\psi_\varepsilon, \quad r \geqslant R_0.$$

Then, it follows that for small $\varepsilon > 0$,

$$-\varepsilon^2\Delta\psi_\varepsilon + V\psi_\varepsilon \geqslant K_\mu(x)\big(w_\varepsilon^\mu\big)^{p-1}\psi_\varepsilon \quad \text{in } \mathbb{R}^n \setminus B(0, R_0).$$

Note that for some $C, c > 0$, $\max_{x\in\partial B(0,R_0)} w_\varepsilon(x) \leqslant C\exp(-\frac{c}{\varepsilon})$. Then, by the maximum principle, we get that for some $C, c > 0$, independent of small $\varepsilon > 0$ and large $\mu > 0$,

$$w_\varepsilon(x) \leqslant C\exp\left(-\frac{c}{\varepsilon}\right)\left(\frac{R_0}{\varepsilon}\right)^{\sqrt{\lambda}/\varepsilon}\psi_\varepsilon(x) \quad \text{for } x \in B(0,\mu) \setminus B(0, R_0). \tag{23}$$

This completes the proof. □

Now we complete a proof for our main theorem.

Completion of Proof of Theorem 1. Note that $u_\varepsilon^\mu(x) = (\eta_\varepsilon^\mu)^{-1/(p-1)}w_\varepsilon^\mu(x)$. From Lemma 5, we see that for some $C > 0$, independent of small $\varepsilon > 0$ and large $\mu > 0$, $u_\varepsilon^\mu(x) \leqslant C\varepsilon^{-2/(p-1)}w_\varepsilon^\mu(x)$, $x \in B(0,\mu)$. Then, by Lemmas 8 and 9, we see the estimate (3). This implies that $\int_{\mathbb{R}^n}\chi_\varepsilon(u_\varepsilon^\mu)^{p+1}\,dx < 1$ for sufficiently small $\varepsilon > 0$, independent of large $\mu > 0$. Then, we see that w_ε^μ satisfies Eq. (4). It is easy to see from condition (K) and the decay property in Lemma 9 that for fixed $\epsilon > 0$, $\{\|w_\varepsilon^\mu\|_{L^\infty} \mid \mu > 0 \text{ large}\}$ is bounded away from 0.

Next we claim that $\lim_{\varepsilon\to 0}\|w_\varepsilon^\mu\|_{L^\infty} = 0$ uniformly for large $\mu > 0$. Indeed, note that for small $\varepsilon > 0$, independent of large $\mu > 0$,

$$\varepsilon^2\Delta w_\varepsilon^\mu - Vw_\varepsilon^\mu + K^\mu\big(w_\varepsilon^\mu\big)^p = 0 \quad \text{in } B(0,\mu).$$

We define $W_\varepsilon^\mu(x) \equiv w_\varepsilon^\mu(\varepsilon x)$. Then, we see that

$$\Delta W_\varepsilon^\mu - V(\varepsilon x)W_\varepsilon^\mu + K^\mu(\varepsilon x)\big(W_\varepsilon^\mu\big)^p = 0 \quad \text{in } B(0,\mu/\varepsilon). \tag{24}$$

We see from Lemma 3 that

$$\lim_{\varepsilon\to 0}\int_{B(0,\mu/\varepsilon)} K^\mu(\varepsilon x)\big(W_\varepsilon^\mu\big)^{p+1}\,dx = 0.$$

This implies that

$$\lim_{\varepsilon\to 0}\int\big|\nabla W_\varepsilon^\mu\big|^2 + V(\varepsilon x)\big(W_\varepsilon^\mu\big)^2\,dx = \lim_{\varepsilon\to 0}\int K^\mu(\varepsilon x)\big(W_\varepsilon^\mu\big)^{p+1}\,dx = 0.$$

This implies that $\lim_{\varepsilon\to 0}\int(W_\varepsilon^\mu)^{2n/(n-2)}\,dx = 0$. For $R > 0$ and $x_0 \in \mathbb{R}^n$, we take $\phi \in C_0^\infty(B(x_0,R))$ such that $\phi(x) = 1$ for $|x - x_0| \leqslant R - 1$. Multiplying both sides of (24) through by $\max\{(W_\varepsilon^\mu)^{2s+1}, l\}\phi^2$, and taking $l \to \infty$, we obtain that

$$\int\big|\nabla\big(W_\varepsilon^\mu\big)^{s+1}\phi\big|^2\,dx \leqslant \int\big(W_\varepsilon^\mu\big)^{2s+2}|\nabla\phi|^2\,dx + (s+1)\int K^\mu(\varepsilon x)\big(W_\varepsilon^\mu\big)^{p+1+2s}\phi^2\,dx. \tag{25}$$

Since $\lim_{\varepsilon\to 0}\int(W_\varepsilon^\mu)^{2n/(n-2)}\,dx = 0$, it follows from condition (K) that if $s = s_1(p, q_0) > 0$ is small,

$$\lim_{\varepsilon\to 0}\int\big|\nabla\big(W_\varepsilon^\mu\big)^{s_1+1}\phi\big|^2\,dx = 0$$

uniformly for large $\mu > 0$. Then,it follows from Sobolev embedding that

$$\lim_{\varepsilon\to 0} \int\limits_{B(x_0,R-1)} \left(W_\varepsilon^\mu\right)^{2n(1+s_1)/(n-2)}\,dx = 0 \quad \text{uniformly for large } \mu > 0.$$

Then, using this and (25) again, we deduce that if $s = s_2(s_1, p, q_0) > s_1$,

$$\lim_{\varepsilon\to 0} \int\limits_{B(x_0,R-2)} \left(W_\varepsilon^\mu\right)^{2n(1+s_2)/(n-2)}\,dx = 0 \quad \text{uniformly for large } \mu > 0.$$

We note that $0 \leqslant K^\mu \leqslant K \in L^{q_0}_{\mathrm{loc}}$ and $q_0 > 2n/(2n - (p+1)(n-2)) > n/2$. Let $q \in (n/2, q_0)$. Then, iterating above process finite times, we conclude that for each $r > 0$ and $x_0 \in \mathbb{R}^n$,

$$\lim_{\varepsilon\to 0} \int\limits_{B(x_0,r)} \left(K^\mu(\varepsilon x)\left(W_\varepsilon^\mu\right)^p\right)^q\,dx = 0 \quad \text{uniformly for large } \mu > 0.$$

By an elliptic estimate [16, Theorem 8.25], we see that

$$\lim_{\varepsilon\to 0}\left\| W_\varepsilon^\mu\right\|_{L^\infty} = 0 \quad \text{uniformly for large } \mu > 0. \tag{26}$$

We can assume that w_ε^μ converges weakly to some $w_\varepsilon \in E_\varepsilon$ as $\mu \to \infty$. Then, we get a solution $w_\varepsilon > 0$ satisfying Eq. (1). From the uniform decay (26), we see that $\lim_{\varepsilon\to 0}\|w_\varepsilon\|_{L^\infty} = 0$.

The decaying property (3) follows from Lemmas 8 and 9. From the decaying property (3), we see that the solution $u_\varepsilon \in E_\varepsilon$ belongs to $L^2(\mathbb{R}^n)$. This implies that $u_\varepsilon \in W^{1,2}(\mathbb{R}^n)$.

The second property of (2) in the theorem is proved by the following argument. Let $w_\varepsilon = \varepsilon^{2/(p-1)} v_\varepsilon$. Multiplying the equation by v_ε and integrating over on $\mathbb{R}^n$ we obtain that

$$\begin{aligned}\int |\nabla v_\varepsilon|^2 + \frac{V}{\varepsilon^2} v_\varepsilon^2\,dx &\leqslant \|v_\varepsilon\|_{L^\infty}^{(p-1)/2} \int K (v_\varepsilon)^{(p-1)/2} (v_\varepsilon)^2\,dx \\ &\leqslant \|v_\varepsilon\|_{L^\infty}^{(p-1)/2} \left(\int K^{n/2} (v_\varepsilon)^{n(p-1)/4}\,dx\right)^{n/2} \left(\int (v_\varepsilon)^{2n/(n-2)}\,dx\right)^{(n-2)/n}.\end{aligned}$$

Since $K \in L^{n/2}_{\mathrm{loc}}$ and (K) hold, we see from the decay property (3), we see that $\limsup_{\varepsilon\to 0} \int K^{n/2}(v_\varepsilon)^{n(p-1)/4}\,dx < \infty$. Thus, by Sobolev inequality, we see that

$$\liminf_{\varepsilon\to 0} \|v_\varepsilon\|_{L^\infty} > 0.$$

This proves the second property of (2) in the theorem. □

3. Asymptotic behavior of localized solutions

We will study the asymptotic behavior of w_ε for small $\varepsilon > 0$. For a family of functions u_ε with $\varepsilon > 0$, we say the family sub-converges as $\varepsilon \to 0$ if for any sequence $\varepsilon_m \to 0$ there is a subsequence of ε_m along which the sequence of functions converge.

Suppose w_ε is the localized solution concentrating near A, given in Theorem 1. For any positive functions $a(\varepsilon)$ and $b(\varepsilon)$ with $\varepsilon > 0$, we define

$$u_\varepsilon(x) \equiv \left(a(\varepsilon)\right)^{\frac{2}{p-1}} \left(b(\varepsilon)\right)^{-\frac{2}{p-1}} w_\varepsilon\left(a(\varepsilon)x\right).$$

Then, it follows that

$$\Delta u_\varepsilon - V\left(a(\varepsilon)x\right)\left(\frac{a(\varepsilon)}{\varepsilon}\right)^2 u_\varepsilon + K\left(a(\varepsilon)x\right)\left(\frac{b(\varepsilon)}{\varepsilon}\right)^2 (u_\varepsilon)^p = 0 \quad \text{in } \mathbb{R}^n.$$

Without loss of generality we can assume that $0 \in A \subset \mathcal{Z} \cup \mathcal{S}$. For an integer $k \in \mathbb{Z}$ and $t > 0$, we define

$$\ln^k t \equiv \big(\overbrace{\ln \circ \cdots \circ \ln}^{|k|\text{-times}}(t)\big)^{k/|k|}, \quad k \neq 0,$$

and $\ln^0 t = 1$ for any $t > 0$. We consider three typical cases:

(A1) the interior A is a bounded domain containing 0;
(A2) $A = \{0\}$ is an isolated point, and for $\tau > -2$, some $k, l \in \mathbb{Z}$, and

$$\gamma \in \big(-\tau\big(2n - (p+1)(n-2)\big)/4, \big(2n - (p+1)(n-2)\big)/2\big)$$

it holds that $\lim_{|x|\to 0} V(x)/|x|^\tau \ln^k(\frac{1}{|x|}) = c > 0$ and $\lim_{|x|\to 0} K(x)|x|^\gamma \ln^l(\frac{1}{|x|}) = d > 0$;
(A3) $A = \{0\}$ and for some $l \in \mathbb{Z}$, $\tau > 0$ and $\gamma < (2n - (p+1)(n-2))/2$, $\lim_{|x|\to 0} V(x)/\exp(-|x|^{-\tau}) = c > 0$ and $\lim_{|x|\to 0} K(x)|x|^\gamma \ln^l(\frac{1}{|x|}) = d > 0$.

In case (A1), taking $a(\varepsilon) = 1$ and $b(\varepsilon) = \varepsilon$, we see that

$$V\big(a(\varepsilon)x\big)\left(\frac{a(\varepsilon)}{\varepsilon}\right)^2 = 0 \quad \text{for } x \in A$$

and that for any small $d > 0$,

$$\lim_{\varepsilon\to 0} V\big(a(\varepsilon)x\big)\left(\frac{a(\varepsilon)}{\varepsilon}\right)^2 = \infty \quad \text{uniformly on } x \in A^{2d} \setminus A^d.$$

In this case, we see also that

$$K\big(a(\varepsilon)x\big)\left(\frac{b(\varepsilon)}{\varepsilon}\right)^2 = K(x).$$

In case (A2), we take

$$a(\varepsilon) = \varepsilon^{\frac{2}{\tau+2}}\big(\ln^{-k}\big(\varepsilon^{-2/(\tau+2)}\big)\big)^{1/(\tau+2)} \quad \text{and} \quad b(\varepsilon) = \varepsilon a(\varepsilon)^{\frac{\gamma}{2}}\left(\ln^l\left(\frac{1}{a(\varepsilon)}\right)\right)^{1/2}.$$

Then, we see that

$$\lim_{\varepsilon\to 0} V\big(a(\varepsilon)x\big)\left(\frac{a(\varepsilon)}{\varepsilon}\right)^2 = c|x|^\tau \quad \text{and} \quad \lim_{\varepsilon\to 0} K\big(a(\varepsilon)x\big)\left(\frac{b(\varepsilon)}{\varepsilon}\right)^2 = d|x|^{-\gamma}$$

locally uniformly in $\mathbb{R}^n$.

In case (A3), we take

$$a(\varepsilon) = \big(\ln \varepsilon^{-2}\big)^{-1/\tau} \quad \text{and} \quad b(\varepsilon) = \varepsilon a(\varepsilon)^{\frac{\gamma}{2}}\left(\ln^l\left(\frac{1}{a(\varepsilon)}\right)\right)^{1/2}.$$

Then, for any $\delta \in (0, 1)$ and $\delta' \in (1, 2)$

$$\lim_{\varepsilon\to 0} V\big(a(\varepsilon)x\big)\left(\frac{a(\varepsilon)}{\varepsilon}\right)^2 = 0 \quad \text{uniformly on } B(0, \delta),$$

$$\lim_{\varepsilon\to 0} V\big(a(\varepsilon)x\big)\left(\frac{a(\varepsilon)}{\varepsilon}\right)^2 = \infty \quad \text{uniformly on } B(0, 2) \setminus B(0, \delta').$$

Moreover, it follows that $\lim_{\varepsilon\to 0} K(a(\varepsilon)x)(\frac{b(\varepsilon)}{\varepsilon})^2 = d|x|^{-\gamma}$ locally uniformly in $\mathbb{R}^n$.

Then, we see the following asymptotic result for each cases (A1)–(A3).

Theorem 10. *Assume $A = \overline{\mathrm{int}(A)}$ and A is a connected component of $\mathcal{Z} \cup \mathcal{S}$. Let w_ε be a localized solution given in Theorem 1. Then $u_\varepsilon(x) = \varepsilon^{-2/(p-1)} w_\varepsilon(x)$ sub-converges point-wisely to a least energy solution U of*

$$\Delta U + K(x)U^p = 0, \quad U > 0,\ x \in int(A); \quad u = 0,\ x \in \partial A.$$

Theorem 11. *Assume $A = \{0\} \in \mathcal{Z} \cup \mathcal{S}$. Assume that* (A2) *is satisfied by functions V and K near* 0. *Let w_ε be a localized solution given in Theorem* 1. *Then $u_\varepsilon(x) := (a(\varepsilon))^{2/(p-1)}(b(\varepsilon))^{-2/(p-1)} w_\varepsilon(a(\varepsilon)x)$ sub-converges uniformly to a least energy solution U of*

$$\Delta U - c|x|^\tau U + d|x|^{-\gamma} U^p = 0, \quad x \in \mathbb{R}^n. \tag{27}$$

Here $a(\varepsilon) = \varepsilon^{\frac{2}{\tau+2}} (\ln^{-k}(\varepsilon^{-2/(\tau+2)}))^{1/(\tau+2)}$ and $b(\varepsilon) = \varepsilon\, a(\varepsilon)^{\gamma/2}\, (\ln^l(\frac{1}{a(\varepsilon)}))^{1/2}$.

Theorem 12. *Assume $A = \{0\} \in \mathcal{Z} \cup \mathcal{S}$. Assume that* (A3) *is satisfied by functions V and K near* 0. *Let w_ε be a localized solution given in Theorem* 1. *Then $u_\varepsilon(x) := (a(\varepsilon))^{2/(p-1)}(b(\varepsilon))^{-2/(p-1)} w_\varepsilon(a(\varepsilon)x)$ sub-converges uniformly to a least energy solution U of*

$$\Delta U + d|x|^{-\gamma} U^p = 0, \quad x \in B_1(0),\ U = 0,\ x \in \partial B_1(0). \tag{28}$$

Here $a(\varepsilon) = (\ln \varepsilon^{-2})^{-1/\tau}$ and $b(\varepsilon) = \varepsilon\, a(\varepsilon)^{\gamma/2}\, (\ln^l(\frac{1}{a(\varepsilon)}))^{1/2}$.

For the proofs of the above theorems, the first can be proved by slight modifications of the arguments in [9], the proof of the third is simpler than that of the second. In the following we give the proof of Theorem 11.

Proof of Theorem 11. Without loss of generality we assume $c = d = 1$. First we show that the limiting equation (27) has a ground state solution U in the space

$$X := \left\{ u \in H^1(\mathbb{R}^n) \;\Big|\; \int \left(|\nabla u|^2 + |x|^\tau u^2\right) dx < \infty \right\}.$$

We consider the following minimization problem:

$$m = \inf_{u \in X \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 + |x|^\tau u^2}{(\int_{\mathbb{R}^n} |x|^{-\gamma} |u|^{p+1})^{2/(p+1)}}.$$

It is standard to show that if $\tau \geqslant 0$ and $\gamma \in [0, (2n - (p+1)(n-2))/2)$, the embedding from X into the weighted $L^{p+1}(\mathbb{R}^n; |x|^{-\gamma})$ is compact. Thus the minimization problem is solved. For $\tau \geqslant 0$ and $\gamma \in (-\tau(2n - (p+1)(n-2))/4, 0)$ or $\tau \in (-2, 0)$ we can argue as follows. For $a = 2n - (p+1)(n-2)/2$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$, it follows from Hölder's inequality and Sobolev inequality that for some $C > 0$,

$$\begin{aligned}
\left(\int |x|^{-\gamma} \varphi^{p+1}\, dx \right)^{2/(p+1)} &= \left(\int |x|^{-\gamma} \varphi^a \varphi^{p+1-a}\, dx \right)^{2/(p+1)} \\
&\leqslant \left(\int |x|^{-2\gamma/a} \varphi^2\, dx \right)^{a/(p+1)} \left(\int \varphi^{2n/(n-2)}\, dx \right)^{(2-a)/(p+1)} \\
&\leqslant C \left(\int |x|^{-2\gamma/a} \varphi^2\, dx \right)^{a/(p+1)} \left(\int |\nabla \varphi|^2\, dx \right)^{\frac{n(2-a)}{(n-2)(p+1)}} \\
&\leqslant C \frac{a}{p+1} \int |x|^{-2\gamma/a} \varphi^2\, dx + C \frac{n(2-a)}{(n-2)(p+1)} \int |\nabla \varphi|^2\, dx.
\end{aligned}$$

Since $-2\gamma/a < \tau$, the embedding from X into the weighted $L^{p+1}(\mathbb{R}^n; |x|^{-\gamma})$ is compact. Thus the minimization problem is solved.

Now there exists a minimizer u of the minimization attaining m. Then $U = m^{\frac{1}{p-1}} u$ is a least energy solution of Eq. (27).

Next, since $\gamma < (2n - (p+1)(n-2))/2$, we observe that $|\cdot|^{-\gamma} \in L^s_{\mathrm{loc}}$ for some $s \geqslant n/2$ and $|x|^{-\gamma} U^p \in L^t(\mathbb{R}^n)$ for some $t > 1$. By a bootstrap argument and an elliptic estimate [16, Theorem 8.25], we deduce that $U \in L^\infty(\mathbb{R}^n)$. Note that $-\gamma < \tau(2n - (p+1)(n-2))/4 < \tau$. Then, by comparison principle there exist $C, c > 0$ such that for $\tau \geqslant 0$ we have $U(x) \leqslant C \exp(-c|x|)$ for all $x \in \mathbb{R}^n$, and for $\tau \in (-2, 0)$ we have $U(x) \leqslant C \exp(-c|x|^{\frac{2+\tau}{2}})$ for all $x \in \mathbb{R}^n$.

Now let $w_\varepsilon(x)$ be a sequence of localized solutions concentrating at $A = \{0\}$ as given in Theorem 1. Define

$$u_\varepsilon(x) := \left(a(\varepsilon)\right)^{2/(p-1)} \left(b(\varepsilon)\right)^{-2/(p-1)} w_\varepsilon\left(a(\varepsilon)x\right).$$

Here $a(\varepsilon)=\varepsilon^{2/(\tau+2)}(\ln^{-k}(\varepsilon^{-2/(\tau+2)}))^{1/(\tau+2)}$ and $b(\varepsilon)=\varepsilon\, a(\varepsilon)^{\gamma/2}\,(\ln^{l}(\frac{1}{a(\varepsilon)}))^{1/2}$. Then we have

$$\Delta u_\varepsilon - V\big(a(\varepsilon)x\big)\left(\frac{a(\varepsilon)}{\varepsilon}\right)^2 u_\varepsilon + K\big(a(\varepsilon)x\big)\left(\frac{b(\varepsilon)}{\varepsilon}\right)^2 (u_\varepsilon)^p = 0 \quad \text{in } \mathbb{R}^n.$$

Using the fact that u_ε corresponds to local minimizers of M_ε and the exponential decay of U we have

$$\limsup_{\varepsilon\to 0}\int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2 + V\big(a(\varepsilon)x\big)\big(a(\varepsilon)\varepsilon^{-1}\big)^2 u_\varepsilon(x)^2 \leqslant \int_{\mathbb{R}^n} |\nabla U|^2 + |x|^\tau U^2.$$

By the decay property of w_ε, for any $r_1>0$ there exists $C_1, c>0$ such that for $|x|\geqslant r_1/a(\varepsilon)$,

$$u_\varepsilon(x) \leqslant C_1 \exp(-c/\varepsilon)\big(a(\varepsilon)\big)^{2/(p-1)}\big(b(\varepsilon)\big)^{-2/(p-1)}\big(1+a(\varepsilon)|x|\big)^{-\sqrt{\lambda}/\varepsilon}. \tag{29}$$

There exists $r_2>0$ such that for $|x|\leqslant r_\varepsilon := r_2/a(\varepsilon)$, $V(a(\varepsilon)x)(a(\varepsilon)\varepsilon^{-1})^2 \geqslant |x|^\tau/2$. Thus $\|u_\varepsilon\|_{H^1(B_{r_\varepsilon})}$ are uniformly bounded. By this, elliptic estimates and (29) we have $\|u_\varepsilon\|_{L^\infty}$ are uniformly bounded. Using the coercivity of the potential $|x|^\tau$ as $|x|\to\infty$ and elliptic estimates we obtain $\lim_{|x|\to\infty} u_\varepsilon(x)=0$ uniformly for ε.

Next we claim that $\liminf_{\varepsilon\to 0}\|u_\varepsilon\|_{L^\infty}>0$. From $\|u_\varepsilon\|_{H^1(B_{r_\varepsilon})}$ being uniformly bounded there is $C>0$ such that

$$\|u_\varepsilon\|^2_{L^{2^*}(B_{r_\varepsilon})} \leqslant C\int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2 + V\big(a(\varepsilon)x\big)\big(a(\varepsilon)\varepsilon^{-1}\big)^2 u_\varepsilon(x)^2.$$

Then by Hölder inequality and the fact $-\gamma<\tau$, we deduce that for some $C>0$, independent of $\varepsilon>0$,

$$\int_{B_{r_\varepsilon}} K\big(a(\varepsilon)x\big)\big(b(\varepsilon)\varepsilon^{-1}\big)^2 u_\varepsilon^2(x) \leqslant C\int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2 + V\big(a(\varepsilon)x\big)\big(a(\varepsilon)\varepsilon^{-1}\big)^2 u_\varepsilon(x)^2.$$

Note that

$$\int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2 + V\big(a(\varepsilon)x\big)\left(\frac{a(\varepsilon)}{\varepsilon}\right)^2 u_\varepsilon(x)^2 \leqslant \|u_\varepsilon\|^{p-1}_{L^\infty}\int_{\mathbb{R}^n} K\big(a(\varepsilon)x\big)\left(\frac{b(\varepsilon)}{\varepsilon}\right)^2 u_\varepsilon^2(x).$$

If $\liminf_{n\to\infty}\|u_n\|_{L^\infty}=0$, it follows from (29) that for some $C, c>0$,

$$\int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2 + V\big(a(\varepsilon)x\big)\big(a(\varepsilon)\varepsilon^{-1}\big)^2 u_\varepsilon(x)^2 \leqslant C\exp\left(-\frac{c}{\varepsilon}\right).$$

By elliptic estimates, we have $\lim_{\varepsilon\to 0}\varepsilon^{-q}\|u_\varepsilon\|_{L^\infty}=0$ for any $q>0$, which contradicts with property (2) in Theorem 1.

Finally we see from elliptic estimates and the uniform decay at infinity that u_ε sub-converges to a least energy solution of Eq. (27). □

Remark 13. We cover several typical cases of asymptotic behaviors. There are some more cases interesting enough to be examined. We point out one case here. Suppose that $V(x)=\exp(-|x|^{-\tau})$, $K(x)=\exp(-|x|^{-\rho})$ for $|x|\leqslant 1$. If $\tau>\rho>0$, it follows that

$$\lim_{|x|\to 0} V^{\frac{2n-(p+1)(n-2)}{2}}(x)/K^2(x)=0.$$

Thus our main result assures the existence of a localized concentrating solution. However it seems not easy to find appropriate scaling functions $a(\varepsilon)$ and $b(\varepsilon)$ so that $V(a(\varepsilon)x)(\frac{a(\varepsilon)}{\varepsilon})^2$ and $K(a(\varepsilon)x)(\frac{b(\varepsilon)}{\varepsilon})^2$ converge in a suitable sense and there is a nontrivial least energy solution of a certain limiting equation. It would be interesting to study the asymptotic behavior of the localized solution u_ε in this case.

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