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Well-posedness and scattering for the KP-II equation in a critical space

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Abstract

The Cauchy problem for the Kadomtsev–Petviashvili-II equation $(u_t + u_{xxx} + uu_x)_x + u_{yy} = 0$ is considered. A small data global well-posedness and scattering result in the scale invariant, non-isotropic, homogeneous Sobolev space $\dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$ is derived. Additionally, it is proved that for arbitrarily large initial data the Cauchy problem is locally well-posed in the homogeneous space $\dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$ and in the inhomogeneous space $H^{-\frac{1}{2},0}(\mathbb{R}^2)$, respectively.

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1. Introduction and main results

The Kadomtsev–Petviashvili-II (KP-II) equation

$$\begin{aligned} &\partial_x\big(\partial_t u + \partial_x^3 u + u\partial_x u\big) + \partial_y^2 u = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^2,\\ &u(0,x,y) = u_0(x,y) \quad (x,y)\in\mathbb{R}^2 \end{aligned} \tag{1}$$

has been introduced by B.B. Kadomtsev and V.I. Petviashvili [9] to describe weakly transverse water waves in the long wave regime with small surface tension. It generalizes the Korteweg–de Vries equation, which is spatially one dimensional and thus neglects transversal effects. The KP-II equation has a remarkably rich structure. Let us begin with its symmetries and assume that u is a solution of (1).

(i) *Translation*: Translates of u in x, y and t are solutions.
(ii) *Scaling*: If $\lambda > 0$ then also

$$u_\lambda(t,x,y) = \lambda^2 u\big(\lambda^3 t, \lambda x, \lambda^2 y\big) \tag{2}$$

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is a solution.

(iii) *Galilean invariance*: For all $c \in \mathbb{R}$ the function

$$u_c(t,x,y) = u\big(t, x - cy - c^2 t, y + 2ct\big) \tag{3}$$

satisfies Eq. (1).

The KP-II equation is integrable in the sense that there exists a Lax pair. Formally, there exists an infinite sequence of conserved quantities [18], the two most important being

$$I_0 = \frac{1}{2}\int u^2\,dx\,dy$$

and

$$I_1 = \frac{1}{2}\int (\partial_x u)^2 - \frac{1}{3}u^3 - \big(\partial_x^{-1}\partial_y u\big)^2\,dx\,dy.$$

The conserved quantities besides I_0 seem to be useless for proofs of well-posedness, because of the difficulty to define ∂_x^{-1} and because the quadratic term is indefinite.

There are many explicit formulas for solutions, see [4]. Particular solutions are the line solitons coming from solitons of the Korteweg–de Vries equation, their Galilei transforms, and multiple line soliton solutions with an intricate structure, see [1].

It may be possible to apply the machinery of inverse scattering to solve the initial value problem and to obtain asymptotics for solutions, see [11] for some results in that direction. It is however not clear which classes of initial data can be treated.

The line solitons are among the simplest solutions. An analysis of the spectrum of the linearization and inverse scattering indicate that the line soliton is stable [9,16]. A satisfactory non-linear stability result for the line soliton is an outstanding problem.

In this paper we want to make a modest step towards this challenging question: We prove well-posedness and scattering in a critical space. These results are in remarkable contrast to the situation for the Korteweg–de Vries equation where the critical space is $H^{-\frac{3}{2}}(\mathbb{R})$ and iteration techniques, as employed in the present work, are known [3] to fail for initial data below $H^{-\frac{3}{4}}(\mathbb{R})$. Stability of solitons has been proved by inverse scattering techniques and by convexity arguments using conserved quantities [14] which has no chance to carry over to KP-II because the quadratic part of I_1 is not convex.

We study the Cauchy problem (1) for initial data u_0 in the non-isotropic Sobolev space $H^{-\frac{1}{2},0}(\mathbb{R}^2)$ and in the homogeneous variant $\dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$, respectively, which are defined as spaces of distributions with $-\frac{1}{2}$ generalized x-derivatives in $L^2(\mathbb{R}^2)$, see (4) and (5) at the end of this section. These spaces are natural for KP-II equation because of the following considerations: The homogeneous space $\dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$ is invariant under the scaling symmetry (2) of solutions of the KP-II equation as well as under the action of the Galilei transform (3) for fixed t. Any Fourier multiplier m invariant under scaling and reflection satisfies $m(\xi,\eta) = |\xi|^{-1/2}m(1,\eta/|\xi|^2)$. Galilean invariance now implies that m is independent of η.

While in the super-critical range, i.e. $s < -\frac{1}{2}$, the scaling symmetry suggests ill-posedness of the Cauchy problem (cp. also [10] Theorem 4.2), we will prove global well-posedness and scattering in $\dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$ for small initial data, see Theorem 1.1 and Corollary 1.7, and local well-posedness in $H^{-\frac{1}{2},0}(\mathbb{R}^2)$ and $\dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$ for arbitrarily large initial data, see Theorem 1.2.

The well-posedness of (1) has been thoroughly studied in the last two decades. After a first well-posedness result by S. Ukai [23] in more regular spaces, J. Bourgain established global well-posedness in $L^2(\mathbb{T}^2;\mathbb{R})$ and $L^2(\mathbb{R}^2;\mathbb{R})$ in his seminal paper [2] by combining the Fourier restriction norm method with the L^2 conservation law. N. Tzvetkov [22] improved the local theory within the scale of non-isotropic Sobolev spaces. Local well-posedness in the full subcritical range $s > -\frac{1}{2}$ was obtained by H. Takaoka [19] in the homogeneous spaces and by the first author [7] in the inhomogeneous spaces. Global well-posedness for large, real valued data in $H^{s,0}(\mathbb{R}^2)$ has been pushed down to $s > -\frac{1}{14}$ by Pedro Isaza J. and Jorge Mejía L. [8]. For a more complete account on previous work, we would like to refer the interested reader to the aforementioned papers and references therein.

The first main result of this paper is concerned with small data global well-posedness in $\dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$. For $\delta > 0$ we define

$$\dot{B}_\delta := \{u_0 \in \dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2) \mid \|u_0\|_{\dot{H}^{-\frac{1}{2},0}} < \delta\},$$

and obtain the following:

Theorem 1.1. *There exists $\delta > 0$, such that for all initial data $u_0 \in \dot{B}_\delta$ there exists a solution*

$$u \in \dot{Z}^{-\frac{1}{2}}([0,\infty)) \subset C([0,\infty); \dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2))$$

of the KP-II equation (1) *on* $(0,\infty)$. *If for some $T > 0$ a solution $v \in Z^{-\frac{1}{2}}([0,T])$ on $(0,T)$ satisfies $v(0) = u(0)$, then $v = u|_{[0,T]}$. Moreover, the flow map*

$$F_+ : \dot{B}_\delta \to \dot{Z}^{-\frac{1}{2}}([0,\infty)), \quad u_0 \mapsto u$$

is analytic.

In order to state the second main result of this paper let us define

$$B_{\delta,R} := \{u_0 \in H^{-\frac{1}{2},0}(\mathbb{R}^2) \mid u_0 = v_0 + w_0,\ \|v_0\|_{\dot{H}^{-\frac{1}{2},0}} < \delta,\ \|w_0\|_{L^2} < R\},$$

for $\delta > 0$, $R > 0$. We establish local well-posedness for arbitrarily large initial data, both in $H^{\frac{1}{2},0}(\mathbb{R}^2)$ and $\dot{H}^{\frac{1}{2},0}(\mathbb{R}^2)$:

Theorem 1.2.

(i) *There exists $\delta > 0$ such that for all $R \geqslant \delta$ and $u_0 \in B_{\delta,R}$ there exists a solution*

$$u \in Z^{-\frac{1}{2}}([0,T]) \subset C([0,T]; H^{-\frac{1}{2},0}(\mathbb{R}^2))$$

for $T := \delta^6 R^{-6}$ of the KP-II equation (1) *on* $(0,T)$. *If a solution $v \in Z^{-\frac{1}{2}}([0,T])$ on $(0,T)$ satisfies $v(0) = u(0)$, then $v = u|_{[0,T]}$. Moreover, the flow map*

$$B_{\delta,R} \ni u_0 \mapsto u \in Z^{-\frac{1}{2}}([0,T])$$

is analytic.

(ii) *The statement in part* (i) *remains valid if we replace the space $H^{-\frac{1}{2},0}(\mathbb{R}^2)$ by $\dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$ as well as $Z^{-\frac{1}{2}}([0,T])$ by $\dot{Z}^{-\frac{1}{2}}([0,T])$.*

Remark 1.3. For the definition of the spaces $\dot{Z}^{-\frac{1}{2}}(I)$ and $Z^{-\frac{1}{2}}(I)$ we refer the reader to Definition 2.22 and the subsequent Remark 2.23. In particular, we have the embedding $\dot{Z}^{-\frac{1}{2}}(I) \subset Z^{-\frac{1}{2}}(I)$. Moreover, a solution of the KP-II equation (1) is understood to be a solution of the corresponding operator equation (53), compare Section 4.

Remark 1.4. Due to the time reversibility of the KP-II equation, the above theorems also hold in corresponding intervals $(T,0)$, $-\infty \leqslant T < 0$. We denote the flow map with respect to $(-\infty,0)$ by F_-.

Remark 1.5. For each $u_0 \in H^{-\frac{1}{2},0}(\mathbb{R}^2)$ and $\delta > 0$ there exists $N > 0$ such that $\|P_{\geqslant N} u_0\|_{\dot{H}^{-\frac{1}{2},0}} < \delta$. We obviously have the representation $u_0 = P_{\geqslant N} u_0 + P_{<N} u_0$, thus $u_0 \in B_{\delta,R}$ for some $R > 0$. However, the time of local existence provided by Theorem 1.2 for large data may depend on the profile of the Fourier transform of u_0, not only on its norm.

Remark 1.6. The well-posedness results above are presented purely at the critical level of regularity $s = -\frac{1}{2}$ as this is the most challenging case. As the reader will easily verify by the standard modification of our arguments, the estimates also imply persistence of higher initial regularity.

A consequence of Theorem 1.1 is scattering for small data in $\dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$.

Corollary 1.7. *Let $\delta > 0$ be as in Theorem* 1.1. *For every $u_0 \in \dot{B}_\delta$ there exists $u_\pm \in \dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$ such that*

$$F_\pm(u_0)(t) - e^{tS}u_\pm \to 0 \quad \text{in } \dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2) \text{ as } t \to \pm\infty.$$

The maps

$$V_\pm : \dot{B}_\delta \to \dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2), \quad u_0 \mapsto u_\pm$$

are analytic, respectively. For $u_0 \in L^2(\mathbb{R}^2;\mathbb{R}) \cap \dot{B}_\delta$ we have

$$\|V_\pm(u_0)\|_{L^2} = \|u_0\|_{L^2}.$$

Moreover, the local inverses, the wave operators

$$W_\pm : \dot{B}_\delta \to \dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2), \quad u_\pm \mapsto u(0)$$

exist and are analytic, respectively. For $u_\pm \in L^2(\mathbb{R}^2;\mathbb{R}) \cap \dot{B}_\delta$ we have

$$\|W_\pm(u_\pm)\|_{L^2} = \|u_0\|_{L^2}.$$

Remark 1.8. The proofs of the global well-posedness and scattering results for small data in Theorem 1.1 and Corollary 1.7 do not rely on the sum structure of the resolution spaces used in [7]. The proof of Theorem 1.2, however, is based on a similar construction adapted to the endpoint case $s = -\frac{1}{2}$, see (33).

1.1. Organization of the paper

At the end of this section we introduce some notation. In Section 2 we review function spaces related to the well-posedness theory for non-linear dispersive PDE's, with a focus on the recently introduced U^p space in this context due to D. Tataru and one of the authors, cp. [12,13] and references therein, as well as the closely related V^p space due to N. Wiener [24]. We believe that the techniques are useful and of independent interest. For that reason we devoted a considerable effort to the presentation of the methods even though most of the details are implicitly contained in [12,13]. Proposition 2.20 however seems to be new. In Section 3 we prove bilinear estimates related to the KP-II equation. These are the main ingredients for the proofs of our main results, which are finally presented in Section 4.

1.2. Notation

The non-isotropic Sobolev spaces $H^{s_1,s_2}(\mathbb{R}^2)$ and $\dot{H}^{s_1,s_2}(\mathbb{R}^2)$ are spaces of complex valued temperate distributions, defined via the norms

$$\|u\|_{H^{s_1,s_2}} := \left(\int_{\mathbb{R}^2} \langle\xi\rangle^{2s_1}\langle\eta\rangle^{2s_2} \left|\hat{u}(\xi,\eta)\right|^2 d\xi\, d\eta \right)^{\frac{1}{2}}, \tag{4}$$

$$\|u\|_{\dot{H}^{s_1,s_2}} := \left(\int_{\mathbb{R}^2} |\xi|^{2s_1}|\eta|^{2s_2} \left|\hat{u}(\xi,\eta)\right|^2 d\xi\, d\eta \right)^{\frac{1}{2}}, \tag{5}$$

respectively, where $\langle\xi\rangle^2 = 1 + |\xi|^2$. The n-dimensional Fourier transform is defined as

$$\hat{u}(\mu) = \mathcal{F}u(\mu) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\mu} u(x)\, dx$$

for $u \in L^1(\mathbb{R}^n)$, and extended to $\mathcal{S}'(\mathbb{R}^n)$ by duality. For $1 \leqslant p \leqslant \infty$ we define the dual exponent $1 \leqslant p' \leqslant \infty$ by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

2. Function spaces and dispersive estimates

In this section we discuss properties of function spaces of U^p and V^p type [12,13,24]. In particular, we present embedding results and a rigorous duality statement as well as interpolation properties and an extension lemma for dispersive estimates. Though many aspects of these spaces are well known, the interpolation result of Proposition 2.20 seems to be new.

Let $\mathcal{Z}$ be the set of finite partitions $-\infty = t_0 < t_1 < \cdots < t_K = \infty$ and let $\mathcal{Z}_0$ be the set of finite partitions $-\infty < t_0 < t_1 < \cdots < t_K < \infty$. In the following, we consider functions taking values in $L^2 := L^2(\mathbb{R}^d; \mathbb{C})$, but in the general part of this section L^2 may be replaced by an arbitrary Hilbert space.

Definition 2.1. Let $1 \leqslant p < \infty$. For $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset L^2$ with $\sum_{k=0}^{K-1} \|\phi_k\|_{L^2}^p = 1$ and $\phi_0 = 0$ we call the function $a : \mathbb{R} \to L^2$ given by

$$a = \sum_{k=1}^{K} \mathbb{1}_{[t_{k-1}, t_k)} \phi_{k-1}$$

a U^p-atom. Furthermore, we define the atomic space

$$U^p := \left\{ u = \sum_{j=1}^{\infty} \lambda_j a_j \,\middle|\, a_j\, U^p\text{-atom},\ \lambda_j \in \mathbb{C} \text{ such that } \sum_{j=1}^{\infty} |\lambda_j| < \infty \right\}$$

with norm

$$\|u\|_{U^p} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| \,\middle|\, u = \sum_{j=1}^{\infty} \lambda_j a_j,\ \lambda_j \in \mathbb{C},\ a_j\ U^p\text{-atom} \right\}. \tag{6}$$

Proposition 2.2. *Let* $1 \leqslant p < q < \infty$.

(i) *U^p is a Banach space.*
(ii) *The embeddings $U^p \subset U^q \subset L^\infty(\mathbb{R}; L^2)$ are continuous.*
(iii) *For $u \in U^p$ it holds $\lim_{t \downarrow t_0} \|u(t) - u(t_0)\|_{L^2} = 0$, i.e. every $u \in U^p$ is right-continuous.*
(iv) *$u(-\infty) := \lim_{t \to -\infty} u(t) = 0$, $u(\infty) := \lim_{t \to \infty} u(t)$ exists.*
(v) *The closed subspace U_c^p of all continuous functions in U^p is a Banach space.*

Proof. Part (i) is straightforward. The embedding $U^p \subset U^q$ follows immediately from $\ell^p(\mathbb{N}) \subset \ell^q(\mathbb{N})$. $U^q \subset L^\infty(\mathbb{R}; L^2)$ (including the norm estimate) is obvious for atoms, hence also for general $u \in U^q$, and part (ii) follows. This also proves that convergence in U^q implies uniform convergence, hence part (v). The right-continuity of part (iii) now follows from the definition of atoms. It remains to prove (iv): Let $u = \sum_n \lambda_n a_n$ and $\varepsilon > 0$. There is $n_0 \in \mathbb{N}$ such that $\sum_{n \geqslant n_0+1} |\lambda_n| < \varepsilon$. On the one hand, there exists $T_- < 0$ such that $a_n(t) = 0$ for all $t < T_-$, $n = 1, \ldots, n_0$, which shows $\|u(t)\|_{L^2} < \varepsilon$ for $t < T_-$. On the other hand, there exists $T_+ > 0$ such that $a_n(t) = a_n(t')$ for all $t, t' > T_+$, $n = 1, \ldots, n_0$, which implies $\|u(t) - u(t')\|_{L^2} < 2\varepsilon$ for $t, t' > T_+$. □

The following spaces were introduced by N. Wiener [24].

Definition 2.3. Let $1 \leqslant p < \infty$. We define V^p as the normed space of all functions $v : \mathbb{R} \to L^2$ such that $v(\infty) := \lim_{t \to \infty} v(t) = 0$ and $v(-\infty)$ exists and for which the norm

$$\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \left(\sum_{k=1}^{K} \left\| v(t_k) - v(t_{k-1}) \right\|_{L^2}^p \right)^{\frac{1}{p}} \tag{7}$$

is finite. Likewise, let V_-^p denote the normed space of all functions $v : \mathbb{R} \to L^2$ such that $v(-\infty) = 0$, $v(\infty)$ exists, and $\|v\|_{V^p} < \infty$, endowed with the norm (7).

Proposition 2.4. *Let $1 \leqslant p < q < \infty$.*

(i) *Let $v : \mathbb{R} \to L^2$ be such that*

$$\|v\|_{V_0^p} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}_0} \left(\sum_{k=1}^{K} \left\| v(t_k) - v(t_{k-1}) \right\|_{L^2}^p \right)^{\frac{1}{p}}$$

is finite. Then, it follows that $v(t_0^+) := \lim_{t \downarrow t_0} v(t)$ exists for all $t_0 \in [-\infty, \infty)$ and $v(t_0^-) := \lim_{t \uparrow t_0} v(t)$ exists for all $t_0 \in (-\infty, \infty]$ and moreover,

$$\|v\|_{V^p} = \|v\|_{V_0^p}.$$

(ii) *We define the closed subspace V_{rc}^p ($V_{-,rc}^p$) of all right-continuous V^p functions (V_-^p functions). The spaces V^p, V_{rc}^p, V_-^p and $V_{-,rc}^p$ are Banach spaces.*
(iii) *The embedding $U^p \subset V_{-,rc}^p$ is continuous.*
(iv) *The embeddings $V^p \subset V^q$ and $V_-^p \subset V_-^q$ are continuous.*

Proof. Part (i) essentially can be found in [24], §1. Part (ii) is straightforward, the closedness follows from the fact that V^p convergence implies uniform convergence. Now, let us prove part (iii): Due to Proposition 2.2, parts (iii) and (iv) it remains to show the norm estimate and it suffices to do so for a U^p-atom $a = \sum_{k=1}^K \mathbb{1}_{[t_{k-1}, t_k)} \phi_{k-1}$. Let $\{s_j\}_{j=1}^J \in \mathcal{Z}$. Then, $a(s_j) - a(s_{j-1}) = \phi_{k_j - 1} - \phi_{k_{j-1}-1}$, which is zero if $k_j = k_{j-1}$. It follows

$$\sum_{j=1}^{J} \left\| a(s_j) - a(s_{j-1}) \right\|_{L^2}^p \leqslant 2^p \sum_{k=1}^{K} \|\phi_{k-1}\|_{L^2}^p \leqslant 2^p,$$

which implies $\|a\|_{V^p} \leqslant 2$. Part (iv) is implied by $\ell^p(\mathbb{N}) \subset \ell^q(\mathbb{N})$. □

Proposition 2.5. *Let $v \in V_{-,rc}^p$ such that $\|v\|_{V^p} = 1$. For any $n \in \mathbb{N}_0$*

(i) *there exists $\mathfrak{t}_n \in \mathcal{Z}$ such that $\mathfrak{t}_0 \subset \mathfrak{t}_1 \subset \cdots$ and $\#\mathfrak{t}_n \leqslant 2^{1+np}$,*
(ii) *there exists a right-continuous step-function u_n subordinate to $\mathfrak{t}_n$ such that $\sup_t \|u_n(t)\|_{L^2} \leqslant 2^{1-n}$,*
(iii) *there exists a $v_n \in V_{-,rc}^p$ such that $\sup_t \|v_n(t)\|_{L^2} \leqslant 2^{-n}$,*
(iv) *it holds $v_n = u_{n+1} + v_{n+1}$, $u_0 = 0$, $v_0 = v$.*

Proof. We proceed by induction: For $n = 0$ we define $\mathfrak{t}_n := \{-\infty, \infty\}$, $u_0 = 0$ and $v_0 = v$, hence all the claims are immediate. For $n \in \mathbb{N}$ let $\mathfrak{t}_n := \{-\infty = t_{n,0} < \cdots < t_{n,K_n}\}$ and u_n, v_n be given with the requested properties. Let $k \in \{0, \ldots, K_n - 1\}$. For $j = 0$ we define $t_{n+1,k}^0 := t_{n,k}$. For $j \geqslant 1$ we define

$$t_{n+1,k}^j := \inf\left\{ t \mid t_{n+1,k}^{j-1} < t \leqslant t_{n,k+1} \colon \left\| v(t) - v\left(t_{n+1,k}^{j-1}\right) \right\|_{L^2} > 2^{-n-1} \right\}$$

if this set is non-empty and $t_{n+1,k}^j := t_{n,k+1}$ otherwise.

Now, we relabel all these points $\{t_{n+1,k}^j\}_{j,k}$ as

$$-\infty = t_{n+1,0} < \cdots < t_{n+1,K_{n+1}} = \infty$$

which defines the partition $\mathfrak{t}_{n+1} \in \mathcal{Z}$. We define

$$u_{n+1} := \sum_{k=1}^{K_{n+1}} \mathbb{1}_{[t_{n+1,k-1}, t_{n+1,k})} v_n(t_{n+1,k-1})$$

$$v_{n+1} := v_n - u_{n+1}.$$

For $t \in \mathbb{R}$ there exists k such that $t \in [t_{n+1,k-1}, t_{n+1,k})$ and it holds

$$\left\| v_{n+1}(t) \right\|_{L^2} \leqslant \left\| v_n(t) - v_n(t_{n+1,k-1}) \right\|_{L^2} \leqslant 2^{-n-1}.$$

Moreover, $1 = \|v\|_{V^p}^p \geqslant (\#t_{n+1} - \#t_n) 2^{-(n+1)p}$ and therefore $\#t_{n+1} \leqslant 2^{1+(n+1)p}$. □

Corollary 2.6. *Let $1 \leqslant p < q < \infty$. The embedding $V^p_{-,rc} \subset U^q$ is continuous.*

Proof. Let $v \in V^p_{-,rc}$ with $\|v\|_{V^p} = 1$. Then, by Proposition 2.5 there exist $\mathfrak{t}_n \in \mathcal{Z}$ with $\#\mathfrak{t}_n \leqslant 2^{1+np}$ and associated step-functions u_n with $\sup_t \|u_n(t)\|_{L^2} \leqslant 2^{1-n}$ such that $v(t) = \sum_{n=0}^\infty u_n(t)$. Moreover, $\|u_n\|_{U^q} \leqslant 4 \cdot 2^{n(\frac{p}{q}-1)}$, hence $\sum_n \|u_n\|_{U^q} \leqslant 4(1 - 2^{\frac{p}{q}-1})^{-1}$. The claim follows since U^q is a Banach space. □

Proposition 2.7. *For $u \in U^p$ and $v \in V^{p'}$ and a partition $\mathfrak{t} := \{t_k\}_{k=0}^K \in \mathcal{Z}$ we define*

$$B_{\mathfrak{t}}(u, v) := \sum_{k=1}^K \big\langle u(t_{k-1}), v(t_k) - v(t_{k-1})\big\rangle.$$

Here, $\langle \cdot,\cdot \rangle$ denotes the L^2 inner product. There is a unique number $B(u, v)$ with the property that for all $\varepsilon > 0$ there exists $\mathfrak{t} \in \mathcal{Z}$ such that for every $\mathfrak{t}' \supset \mathfrak{t}$ it holds

$$\big|B_{\mathfrak{t}'}(u, v) - B(u, v)\big| < \varepsilon, \tag{8}$$

and the associated bilinear form

$$B\colon U^p \times V^{p'} : (u, v) \mapsto B(u, v)$$

satisfies the estimate

$$\big|B(u, v)\big| \leqslant \|u\|_{U^p} \|v\|_{V^{p'}}. \tag{9}$$

Proof. First of all, we note the following: Let $\mathfrak{t} = \{t_n\}_{n=0}^N \in \mathcal{Z}$ and let u be a step function $u = \sum_{k=1}^K \mathbb{1}_{[s_{k-1},s_k)}\phi_{k-1}$ subordinate to a partition $\mathfrak{s} \in \mathcal{Z}$ (not necessarily an atom), with $\phi_0 = 0$. For each $t_n \in \mathfrak{t}$, $n < N$, there exists $k_n < K$ such that $s_{k_n} \leqslant t_n < s_{k_n+1}$. Then,

$$B_{\mathfrak{t}}(u, v) = \sum_{n=1}^N \big\langle \phi_{k_{n-1}}, v(t_n) - v(t_{n-1})\big\rangle. \tag{10}$$

Now, if for some n it is $k_{n-1} = k_n$, then

$$\big\langle \phi_{k_{n-1}}, v(t_n) - v(t_{n-1})\big\rangle + \big\langle \phi_{k_n}, v(t_{n+1}) - v(t_n)\big\rangle = \big\langle \phi_{k_{n-1}}, v(t_{n+1}) - v(t_{n-1})\big\rangle$$

which shows that we may remove such t_n from the partition $\mathfrak{t}$ which gives rise to a partition $\mathfrak{t}^* \subset \mathfrak{t}$. In summary, we may write

$$B_{\mathfrak{t}}(u, v) = \sum_{n=1}^{N^*} \big\langle \phi_{k^*_{n-1}}, v(t^*_n) - v(t^*_{n-1})\big\rangle \tag{11}$$

where now $0 \leqslant k^*_0 < \cdots < k^*_{N^*-1} < K$.

Let $\mathfrak{t} \in \mathcal{Z}$ be given. Assume a is a U^p-atom. Obviously, (11) and Hölder's inequality imply

$$\big|B_{\mathfrak{t}}(a, v)\big| \leqslant \|v\|_{V^{p'}},$$

for all $v \in V^{p'}$. Hence,

$$\big|B_{\mathfrak{t}}(u, v)\big| \leqslant \|u\|_{U^p} \|v\|_{V^{p'}},$$

for all $u \in U^p$ and $v \in V^{p'}$.

Now, let $u \in U^p$ and $v \in V^{p'}$ and $\varepsilon > 0$. Let $u = \sum_{l=1}^\infty \lambda_l a_l$ be an atomic decomposition such that $\sum_{l=n+1}^\infty |\lambda_l| < \varepsilon/(2\|v\|_{V^{p'}})$. We define the approximating step function $u_n = \sum_{l=1}^n \lambda_l a_l$ and let $\mathfrak{t} \in \mathcal{Z}$ be the subordinate partition. Then, for all $\mathfrak{t}' \in \mathcal{Z}$ with $\mathfrak{t} \subset \mathfrak{t}'$ it follows as in (11) that

$$\begin{aligned}\left|B_{\mathfrak{t}'}(u,v)-B_{\mathfrak{t}}(u,v)\right| &\leqslant \left|B_{\mathfrak{t}'}(u,v)-B_{\mathfrak{t}'}(u_n,v)\right|+\left|B_{\mathfrak{t}}(u_n,v)-B_{\mathfrak{t}}(u,v)\right|\\ &\leqslant 2\|u-u_n\|_{U^p}\|v\|_{V^{p'}}\\ &\leqslant 2\sum_{l=n+1}^{\infty}|\lambda_l|\|v\|_{V^{p'}}<\varepsilon.\end{aligned}$$

Therefore, for a given $j\in\mathbb{N}$ there exists $\mathfrak{t}^{(j)}\in\mathcal{Z}$ such that for all $\mathfrak{t}'\in\mathcal{Z}$ with $\mathfrak{t}^{(j)}\subset\mathfrak{t}'$

$$\left|B_{\mathfrak{t}'}(u,v)-B_{\mathfrak{t}^{(j)}}(u,v)\right|<2^{-j},$$

and with $t'=\mathfrak{t}^{(j)}\cup\mathfrak{t}^{(j+1)}$ it follows

$$\left|B_{\mathfrak{t}^{(j+1)}}(u,v)-B_{\mathfrak{t}^{(j)}}(u,v)\right|<2^{1-j}.$$

Hence, $\lim_{j\to\infty}B_{\mathfrak{t}^{(j)}}(u,v)=:B(u,v)$ exists and (8) and (9) are satisfied. Property (8) also implies the uniqueness. □

Theorem 2.8. *Let $1<p<\infty$. We have*

$$\left(U^p\right)^*=V^{p'}$$

in the sense that

$$T:V^{p'}\to(U^p)^*,\quad T(v):=B(\cdot,v) \tag{12}$$

is an isometric isomorphism.

Proof. In view of (9) it suffices to show that for each $L\in(U^p)^*$ there is $v\in V^{p'}$ such that $T(v)(u)=L(u)$ and $\|v\|_{V^{p'}}\leqslant\|L\|$. To this end, let $0\neq L\in(U^p)^*$. For t fixed we have $\phi\mapsto -L(\mathbb{1}_{[t,\infty)}\phi)\in(L^2)^*$, hence there exists $\tilde v(t)\in L^2$ such that $L(\mathbb{1}_{[t,\infty)}\phi)=-\langle\phi,\tilde v(t)\rangle$ for all $\phi\in L^2$. Fix a partition $\{t_k\}_{k=0}^K\in\mathcal{Z}_0$ and define $u:=\sum_{k=1}^K\mathbb{1}_{[t_{k-1},t_k)}\phi_{k-1}$ with

$$\phi_{k-1}:=\frac{(\tilde v(t_k)-\tilde v(t_{k-1}))\|\tilde v(t_k)-\tilde v(t_{k-1})\|_{L^2}^{p'-2}}{(\sum_{k=1}^K\|\tilde v(t_k)-\tilde v(t_{k-1})\|_{L^2}^{p'})^{\frac{1}{p}}}.$$

Then, $\|u\|_{U^p}\leqslant 1$ and

$$\begin{aligned}\|L\|&\geqslant\left|\sum_{k=1}^K L(\mathbb{1}_{[t_{k-1},t_k)}\phi_{k-1})\right|=\left|\sum_{k=1}^K L(\mathbb{1}_{[t_{k-1},\infty)}\phi_{k-1})-L(\mathbb{1}_{[t_k,\infty)}\phi_{k-1})\right|\\ &=\left|\sum_{k=1}^K\left\langle\phi_{k-1},\tilde v(t_k)-\tilde v(t_{k-1})\right\rangle\right|=\left(\sum_{k=1}^K\left\|\tilde v(t_k)-\tilde v(t_{k-1})\right\|_{L^2}^{p'}\right)^{\frac{1}{p'}},\end{aligned}$$

which shows that $\|\tilde v\|_{V_0^{p'}}\leqslant\|L\|$ and that $\lim_{s\to\pm\infty}\tilde v(s)$ exists due to Proposition 2.4, part (i). For $v(t):=\tilde v(t)-\tilde v(\infty)$ it follows $v\in V^{p'}$ and

$$\|v\|_{V^{p'}}\leqslant\|L\|.$$

It remains to show that $T(v)(u)=L(u)$ for all $u\in U^p$: For a step function $u=\sum_{k=1}^K\mathbb{1}_{[t_{k-1},t_k)}\phi_{k-1}$ with underlying partition $\mathfrak{t}$ we have

$$\begin{aligned}T(v)(u)=B_{\mathfrak{t}}(u,v)&=\sum_{k=1}^K\left\langle\phi_{k-1},v(t_k)-v(t_{k-1})\right\rangle\\ &=\sum_{k=1}^K L(\mathbb{1}_{[t_{k-1},t_k)}\phi_{k-1})=L(u)\end{aligned}$$

and the claim follows by density and (9). □

Proposition 2.9. *For $1 < p < \infty$ let $u \in U^p$ be continuous and $v, v^* \in V^{p'}$ such that $v(s) = v^*(s)$ except for at most countably many points. Then,*

$$B(u, v) = B(u, v^*).$$

Proof. For $w := v - v^*$ it holds that $w(s) = 0$ except for at most countably many points. We have to show that $B(u, w) = 0$. We may assume $\|u\|_{U^p} = \|w\|_{V^{p'}} = 1$. For $\varepsilon > 0$ there exists $\mathfrak{t} = \{t_k\}_{k=0}^K \in \mathcal{Z}$ such that for every $\mathfrak{t}' \supset \mathfrak{t}$:

$$\left|B_{\mathfrak{t}'}(u, w) - B(u, w)\right| < \varepsilon.$$

Since u is continuous, there exists $\delta > 0$ such that for all $k \in \{1, \dots, K - 1\}$ and $s \in (t_k - \delta, t_k)$ it holds $\|u(s) - u(t_k)\|_{L^2} < \frac{\varepsilon}{K}$. For all $k \in \{1, \dots, K - 1\}$ we choose $t_k^* \in (t_k - \delta, t_k)$ such that $t_k^* > t_{k-1}$ and $w(t_k^*) = 0$ and set

$$\mathfrak{t}' = \mathfrak{t} \cup \left\{t_1^*, \dots, t_{K-1}^*\right\}.$$

Summation by parts yields

$$B_{\mathfrak{t}'}(u, w) = \sum_{k=1}^{K-1} \left\langle u(t_k^*) - u(t_k), w(t_k)\right\rangle.$$

Hence, $|B(u, w)| < |B_{\mathfrak{t}'}(u, w)| + \varepsilon < 2\varepsilon$. □

Proposition 2.10. *Let $1 < p < \infty$, $u \in V_-^1$ be absolutely continuous on compact intervals and $v \in V^{p'}$. Then,*

$$B(u, v) = -\int_{-\infty}^{\infty} \left\langle u'(t), v(t)\right\rangle dt. \tag{13}$$

Proof. Without loss of generality we may assume $\|u\|_{V^1} = \|v\|_{V^{p'}} = 1$. By Corollary 2.6 we have $u \in U^p$, so that the left-hand side of (13) makes sense. From our assumptions on u it follows that $u' \in L^1(\mathbb{R}; L^2)$ with $\|u'\|_{L^1} \leqslant \|u\|_{V^1} = 1$ and that the Fundamental Theorem of Calculus is valid (cf. for example [5], Corollaries 2.9.20 and 2.9.22). Because u is continuous and v is left-continuous except for at most countably many points, it suffices by Proposition 2.9 to consider left-continuous $v \in V^{p'}$. For $\varepsilon > 0$ there exists $\mathfrak{t} = \{t_n\}_{n=0}^N \in \mathcal{Z}$ such that for every $\mathfrak{t}' \supset \mathfrak{t}$ the estimate (8) is satisfied. Furthermore, there exists $T_1 \leqslant t_1$ and $T_2 \geqslant t_{N-1}$ such that $\|v(t) - v(T_1)\|_{L^2} < \varepsilon$ for $t \leqslant T_1$ and $\|v(t)\|_{L^2} < \varepsilon$ for $t \geqslant T_2$. Since v is a left-continuous, regulated function on $[T_1, T_2]$, there exists $\mathfrak{t}' = \{t_n'\}_{n=0}^{N'} \supset \mathfrak{t}$ such that $t_1' = T_1$ and $t_{N'-1}' = T_2$ and

$$\|v - w\|_{L^\infty} < \varepsilon, \quad \text{for } w := \sum_{n=1}^{N'-1} v\big(t_n'\big)\mathbb{1}_{(t_{n-1}', t_n']}.$$

Now, estimate (8) and summation by parts yield

$$\left|-\sum_{n=1}^{N'-1} \left\langle u\big(t_n'\big) - u\big(t_{n-1}'\big), v\big(t_n'\big)\big)\right\rangle - B(u, v)\right| < \varepsilon.$$

By the Fundamental Theorem of Calculus and the definition of w we have for $n \in \{1, \dots, N' - 1\}$:

$$\left\langle u\big(t_n'\big) - u\big(t_{n-1}'\big), v\big(t_n'\big)\right\rangle = \int_{t_{n-1}'}^{t_n'} \left\langle u'(s), w(s)\right\rangle ds.$$

Altogether, we obtain

$$\left|-\int_{-\infty}^{\infty} \left\langle u'(s), v(s)\right\rangle ds - B(u, v)\right| < \|u'\|_{L^1} \|v - w\|_{L^\infty} + \varepsilon < 2\varepsilon,$$

which finishes the proof. □

Remark 2.11. For $u \in U^p$ it is clear that

$$\|u\|_{U^p} = \sup_{v \in V^{p'}:\ \|v\|_{V^{p'}}=1} \left|B(u,v)\right|$$

by Theorem 2.8. Although we will not use it in the sequel, let us remark that for $u \in V^1_-$ which is absolutely continuous on compact intervals it holds

$$\|u\|_{U^p} = \sup_{v \in V^{p'}_c: \|v\|_{V^{p'}}=1} \left|B(u,v)\right|,$$

where $V^{p'}_c$ is the set of all continuous functions in $V^{p'}$ (which is obviously not dense). This may be seen as follows: By Proposition 2.10 we may restrict the supremum to $V^{p'}_{rc}$. Then, we may restrict this further to the dense subset of the right-continuous step-functions $\mathcal{T}_{rc}$. Finally, we may replace $\mathcal{T}_{rc}$ by $V^{p'}_c$ by substituting jumps in a piecewise linear and continuous way with the help of (13).

Remark 2.12. For $v \in V^p$ Theorem 2.8 also implies

$$\|v\|_{V^p} = \sup_{u\ U^{p'}\text{-atom}} \left|B(u,v)\right|$$

for $1 < p < \infty$.

We will use the convention that capital letters denote dyadic numbers, e.g. $N = 2^n$ for $n \in \mathbb{Z}$ and for a dyadic summation we write $\sum_N a_N := \sum_{n\in\mathbb{Z}} a_{2^n}$ and $\sum_{N\geqslant M} a_N := \sum_{n\in\mathbb{Z}:2^n\geqslant M} a_{2^n}$ for brevity. Let $\chi \in C_0^\infty((-2,2))$ be an even, non-negative function such that $\chi(t) = 1$ for $|t| \leqslant 1$. We define $\psi(t) := \chi(t) - \chi(2t)$ and $\psi_N := \psi(N^{-1}\cdot)$. Then, $\sum_N \psi_N(t) = 1$ for $t \neq 0$. We define

$$\widehat{Q_N u} := \psi_N \hat{u}$$

and $\widehat{Q_0 u} = \chi(2\cdot)\hat{u}$, $Q_{\geqslant M} = \sum_{N\geqslant M} Q_N$ as well as $Q_{<M} = I - Q_{\geqslant M}$.

Definition 2.13. Let $s \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$. We define the semi-norms

$$\|u\|_{\dot{B}^s_{p,q}} := \left(\sum_N N^{qs} \|Q_N u\|^q_{L^p(\mathbb{R};L^2)}\right)^{\frac{1}{q}} \quad (q < \infty),$$
$$\|u\|_{\dot{B}^s_{p,\infty}} := \sup_N N^s \|Q_N u\|_{L^p(\mathbb{R};L^2)} \tag{14}$$

for all $u \in \mathcal{S}'(\mathbb{R}; L^2)$ for which these numbers are finite.

Proposition 2.14. *Let $1 < p < \infty$. For any $v \in V^p$, the estimate*

$$\|v\|_{\dot{B}^{\frac{1}{p}}_{p,\infty}} \lesssim \|v\|_{V^p} \tag{15}$$

holds true. Moreover, for any $u \in \mathcal{S}'(\mathbb{R}; L^2)$ such that the semi-norm $\|u\|_{\dot{B}^{\frac{1}{p}}_{p,1}}$ is finite there exists $u(\pm\infty) \in L^2$. Then, $u - u(-\infty) \in U^p$ and the estimate

$$\left\|u - u(-\infty)\right\|_{U^p} \lesssim \|u\|_{\dot{B}^{\frac{1}{p}}_{p,1}} \tag{16}$$

holds true.

Proof. Concerning (15), see e.g. Example 9 in [15], pp. 167–168. Now, the second part follows by duality: Let $u \in \mathcal{S}'(\mathbb{R}; L^2)$ such that $\|u\|_{\dot{B}^{\frac{1}{p}}_{p,1}} < \infty$ and we consider $Q_N u \in L^p(\mathbb{R}; L^2)$, which is smooth. Hence, $Q_N u \in U^p$. Then,

$$\|Q_N u\|_{U^p} = \sup_{\|L\|_{(U^p)^*}=1} \left|L(Q_N u)\right| = \sup_{\|v\|_{V^{p'}}=1} \left|B(Q_N u, v)\right| = \sup_{\|v\|_{V^{p'}}=1} \left| \int_{-\infty}^{\infty} \left\langle Q_N u'(t), v(t)\right\rangle dt \right|$$
$$\leqslant \sup_{\|v\|_{V^{p'}}=1} \|Q_N u\|_{\dot{B}^{\frac{1}{p}}_{p,1}} \|v\|_{\dot{B}^{\frac{1}{p'}}_{p',\infty}} \lesssim N^{\frac{1}{p}} \|Q_N u\|_{L^p},$$

and it follows that $\tilde{u} := \sum_N Q_N u$ converges in U^p and $\|\tilde{u}\|_{U^p} \lesssim \|u\|_{\dot{B}^{\frac{1}{p}}_{p,1}}$. It is $\|u - \tilde{u}\|_{\dot{B}^{\frac{1}{p}}_{p,1}} = 0$, hence $u = \tilde{u} + const$ and the claim follows. □

Now, we focus on the spatial dimension $d = 2$ (i.e. $L^2 = L^2(\mathbb{R}^2; \mathbb{C})$) and consider $S := -\partial_x^3 - \partial_x^{-1}\partial_y^2$. We define the associated unitary operator $e^{tS} : L^2 \to L^2$ to be the Fourier multiplier

$$\widehat{e^{tS}u_0}(\xi, \eta) = \exp\left(it\left(\xi^3 - \frac{\eta^2}{\xi}\right)\right)\widehat{u_0}(\xi, \eta).$$

Definition 2.15. We define

(i) $U^p_S = e^{\cdot S} U^p$ with norm $\|u\|_{U^p_S} = \|e^{-\cdot S} u\|_{U^p}$,
(ii) $V^p_S = e^{\cdot S} V^p$ with norm $\|v\|_{V^p_S} = \|e^{-\cdot S} u\|_{V^p}$,

and similarly the closed subspaces $U^p_{c,S}$, $V^p_{rc,S}$, $V^p_{-,S}$ and $V^p_{-,rc,S}$.

Let us note that for S defined above these spaces are invariant under complex conjugation, because the symbol of S is an odd function.

We define the smooth projections

$$\widehat{P_N u}(\tau, \xi, \eta) := \psi_N(\xi)\hat{u}(\tau, \xi, \eta),$$
$$\widehat{Q^S_M u}(\tau, \xi, \eta) := \psi_M\left(\tau - \xi^3 + \eta^2\xi^{-1}\right)\hat{u}(\tau, \xi, \eta),$$

as well as $\widehat{P_0 u}(\tau, \xi, \eta) := \chi(2\xi)\hat{u}(\tau, \xi, \eta)$, $Q^S_{\geqslant M} := \sum_{N \geqslant M} Q^S_N$, and $Q^S_{<M} := I - Q^S_{\geqslant M}$. Note that we have

$$Q^S_M = e^{\cdot S} Q_M e^{-\cdot S} \tag{17}$$

and similarly for $Q^S_{\geqslant M}$ and $Q^S_{<M} := I - Q^S_{\geqslant M}$.

Definition 2.16. Let $s, b \in \mathbb{R}$ and $1 \leqslant q \leqslant \infty$. We define the semi-norms

$$\|u\|_{\dot{X}^{s,b,q}} := \left(\sum_N N^{2s} \left\|e^{-\cdot S} P_N u\right\|^2_{\dot{B}^b_{2,q}}\right)^{\frac{1}{2}} \tag{18}$$

for all $u \in \mathcal{S}'(\mathbb{R}; L^2)$ for which these numbers are finite.

Remark 2.17. Roughly speaking, the spaces U^2_S and $V^2_{-,S}$ serve as substitutes for the corresponding Bourgain spaces $\dot{X}^{0,\frac{1}{2},1}$ and $\dot{X}^{0,\frac{1}{2},\infty}$ — which shall be defined for the purpose of this remark by the above semi-norms with the normalization $u(-\infty) = 0$ — in the sense that the embeddings

$$\dot{X}^{0,\frac{1}{2},1} \subset U^2_S \subset V^2_{-,S} \subset \dot{X}^{0,\frac{1}{2},\infty}$$

are continuous, cp. [13, Eq. (2.5)]. This is an immediate consequence of Proposition 2.14 and part (iii) of Proposition 2.4.

We may identify $u \in \mathcal{S}'(\mathbb{R}; L^2)$ with a subset of $\mathcal{S}'(\mathbb{R}^3)$ and

$$\|u\|_{\dot{X}^{s,b,q}} = \left(\sum_N N^{2s} \left(\sum_M M^{bq} \|P_N Q_M^S u\|_{L^2(\mathbb{R}^3)}^q \right)^{\frac{2}{q}} \right)^{\frac{1}{2}}$$

with the obvious modification in the case $q = \infty$.

Corollary 2.18. *We have*

$$\|Q_M^S u\|_{L^2(\mathbb{R}^3)} \lesssim M^{-\frac{1}{2}} \|u\|_{V_S^2}, \tag{19}$$

$$\|Q_{\geqslant M}^S u\|_{L^2(\mathbb{R}^3)} \lesssim M^{-\frac{1}{2}} \|u\|_{V_S^2}, \tag{20}$$

$$\|Q_{<M}^S u\|_{V_S^p} \lesssim \|u\|_{V_S^p}, \qquad \|Q_{\geqslant M}^S u\|_{V_S^p} \lesssim \|u\|_{V_S^p}, \tag{21}$$

$$\|Q_{<M}^S u\|_{U_S^p} \lesssim \|u\|_{U_S^p}, \qquad \|Q_{\geqslant M}^S u\|_{U_S^p} \lesssim \|u\|_{U_S^p}. \tag{22}$$

Proof. By (17) and Definition 2.15, we see that (19) follows from

$$\|Q_M v\|_{L^2(\mathbb{R}^3)} \lesssim M^{-\frac{1}{2}} \|v\|_{V^2} \tag{23}$$

and similarly for (20)–(22). Now, (23) is just a reformulation of the Besov embedding (15). Furthermore, (23) implies that

$$\|Q_{\geqslant M} v\|_{L^2(\mathbb{R}^3)} \lesssim \|v\|_{V^2} \sum_{N \geqslant M} N^{-\frac{1}{2}}$$

and (20) follows from $\sum_{N \geqslant M} N^{-\frac{1}{2}} \lesssim M^{-\frac{1}{2}}$. We only need to prove the left inequalities of (21) and (22) because of $Q_{\geqslant M} = I - Q_{<M}$. By scaling it suffices to show (21) and (22) for $M = 1$ only. We have $Q_{<1} v = \phi * v$ for some Schwartz function ϕ. Due to the Riemann–Lebesgue Lemma, $Q_{<1}(\pm\infty) = 0$. For $\{t_k\}_{k=0}^K \in \mathcal{Z}_0$ we apply Minkowski's inequality

$$\begin{aligned}\left(\sum_{k=1}^K \|Q_{<1} v(t_k) - Q_{<1} v(t_{k-1})\|_{L^2}^p \right)^{\frac{1}{p}} &\leqslant \left(\sum_{k=1}^K \left(\int_{\mathbb{R}} |\phi(s)| \|v(t_k - s) - v(t_{k-1} - s)\|_{L^2} \, ds \right)^p \right)^{\frac{1}{p}} \\ &\leqslant \int_{\mathbb{R}} |\phi(s)| \left(\sum_{k=1}^K \|v(t_k - s) - v(t_{k-1} - s)\|_{L^2}^p \right)^{\frac{1}{p}} ds \leqslant \|\phi\|_{L^1(\mathbb{R})} \|v\|_{V^p}\end{aligned}$$

which implies (21). Let us finally prove (22):

$$\|Q_{<1} u\|_{U^p} = \sup_{\|L\|_{(U^p)^*} = 1} |L(\phi * u)| = \sup_{\|v\|_{V^{p'}} = 1} |B(\phi * u, v)|$$

with ϕ as above. For given $\mathfrak{t} = \{t_k\}_{k=0}^K \in \mathcal{Z}$ we obtain

$$\begin{aligned}|B_{\mathfrak{t}}(\phi * u, v)| &\leqslant \left| \sum_{k=1}^{K-1} \langle (\phi * u)(t_{k-1}), v(t_k) - v(t_{k-1}) \rangle \right| \leqslant \int_{\mathbb{R}} |\phi(s)| \left| \sum_{k=1}^{K-1} \langle u(t_{k-1} - s), v(t_k) - v(t_{k-1}) \rangle \right| ds \\ &\leqslant \|\phi\|_{L^1(\mathbb{R})} \|u\|_{U^p} \|v\|_{V^{p'}}.\end{aligned}$$

Since this bound is independent of $\mathfrak{t}$, (22) follows. □

Similarly to [13], Corollary 3.3 or [21], Lemma 4.1 we have the following general extension result, which is well known at least for Bourgain type spaces (cp. [6], Lemma 2.3):

Proposition 2.19. *Let*

$$T_0 : L^2 \times \cdots \times L^2 \to L^1_{\mathrm{loc}}\big(\mathbb{R}^2;\mathbb{C}\big)$$

be a n-linear operator.

(i) *Assume that for some $1 \leqslant p, q \leqslant \infty$*

$$\big\|T_0\big(e^{\cdot S}\phi_1,\dots,e^{\cdot S}\phi_n\big)\big\|_{L^p_t(\mathbb{R};L^q_{x,y}(\mathbb{R}^2))} \lesssim \prod_{i=1}^n \|\phi_i\|_{L^2}.$$

Then, there exists $T : U^p_S \times \cdots \times U^p_S \to L^p_t(\mathbb{R}; L^q_{x,y}(\mathbb{R}^2))$ satisfying

$$\big\|T(u_1,\dots,u_n)\big\|_{L^p_t(\mathbb{R};L^q_{x,y}(\mathbb{R}^2))} \lesssim \prod_{i=1}^n \|u_i\|_{U^p_S},$$

such that $T(u_1,\dots,u_n)(t)(x,y) = T_0(u_1(t),\dots,u_n(t))(x,y)$ a.e.

(ii) *Assume that for some $1 \leqslant p, q \leqslant \infty$*

$$\big\|T_0\big(e^{\cdot S}\phi_1,\dots,e^{\cdot S}\phi_n\big)\big\|_{L^q_x(\mathbb{R};L^p_{t,y}(\mathbb{R}^2))} \lesssim \prod_{i=1}^n \|\phi_i\|_{L^2}.$$

For $r := \min(p,q)$ there exists $T : U^r_S \times \cdots \times U^r_S \to L^q_x(\mathbb{R}; L^p_{t,y}(\mathbb{R}^2))$ satisfying

$$\big\|T(u_1,\dots,u_n)\big\|_{L^q_x(\mathbb{R};L^p_{t,y}(\mathbb{R}^2))} \lesssim \prod_{i=1}^n \|u_i\|_{U^r_S},$$

such that $T(u_1,\dots,u_n)(x)(t,y) = T_0(u_1(t),\dots,u_n(t))(x,y)$ a.e.

Proof. Concerning part (i), we define

$$T(u_1,\dots,u_n)(t)(x,y) = T_0\big(u_1(t),\dots,u_n(t)\big)(x,y).$$

Let $a_1,\dots,a_n$ be U^p_S-atoms given as

$$a_i = \sum_{k_i=1}^{K_i} \mathbb{1}_{[t_{k_i-1,i},t_{k_i,i})} e^{\cdot S}\phi_{k_i-1,i}$$

such that $\sum_{k_i=1}^{K_i} \|\phi_{k_i-1,i}\|^p_{L^2} = 1$ and $\phi_{0,i} = 0$. Then, we use Hölder's inequality

$$\begin{aligned}
\big\|T(a_1,\dots,a_n)\big\|_{L^p_t(\mathbb{R};L^q_{x,y}(\mathbb{R}^2))} &\leqslant \Bigg\| \sum_{k_1,\dots,k_n} \prod_{i=1}^n \mathbb{1}_{[t_{k_i-1,i},t_{k_i,i})} \big\|T_0\big(e^{tS}\phi_{k_1-1,1},\dots,e^{tS}\phi_{k_n-1,n}\big)\big\|_{L^q_{x,y}(\mathbb{R}^2)} \Bigg\|_{L^p_t(\mathbb{R})} \\
&\leqslant \Bigg(\sum_{k_1,\dots,k_n} \big\|T_0\big(e^{tS}\phi_{k_1-1,1},\dots,e^{tS}\phi_{k_n-1,n}\big)\big\|^p_{L^p_t(\mathbb{R};L^q_{x,y}(\mathbb{R}^2))} \Bigg)^{\frac{1}{p}} \\
&\lesssim \Bigg(\sum_{k_1,\dots,k_n} \prod_{i=1}^n \|\phi_{k_i-1,i}\|^p_{L^2(\mathbb{R}^2)} \Bigg)^{\frac{1}{p}} = 1
\end{aligned}$$

and the claim follows. Now, we turn to the proof of part (ii): We define

$$T(u_1,\dots,u_n)(x)(t,y) = T_0\big(u_1(t),\dots,u_n(t)\big)(x,y).$$

Let $a_1,\dots,a_n$ be U^r_S-atoms for $r = \min(p,q)$. Then, by Hölder's and Minkowski's inequality (here we use $r \leqslant p, q$)

$$\begin{aligned}\|T(a_1,\dots,a_n)\|_{L^q_x(\mathbb{R};L^p_{t,y}(\mathbb{R}^2))} &\leqslant \left\|\left(\sum_{k_1,\dots,k_n}\left|T_0\left(e^{tS}\phi_{k_1-1,1},\dots,e^{tS}\phi_{k_n-1,n}\right)\right|^r\right)^{\frac{1}{r}}\right\|_{L^q_x(\mathbb{R};L^p_{t,y}(\mathbb{R}^2))}\\ &\lesssim \left(\sum_{k_1,\dots,k_n}\left\|T_0\left(e^{tS}\phi_{k_1-1,1},\dots,e^{tS}\phi_{k_n-1,n}\right)\right\|^r_{L^q_x(\mathbb{R};L^p_{t,y}(\mathbb{R}^2))}\right)^{\frac{1}{r}}\\ &\lesssim \left(\sum_{k_1,\dots,k_n}\prod_{i=1}^n\|\phi_{k_i-1,i}\|^r_{L^2(\mathbb{R}^2)}\right)^{\frac{1}{r}}=1\end{aligned}$$

and the claim follows. □

Proposition 2.20. *Let $q>1$, E be a Banach space and $T:U^q_S\to E$ be a bounded, linear operator with $\|Tu\|_E\leqslant C_q\|u\|_{U^q_S}$ for all $u\in U^q_S$. In addition, assume that for some $1\leqslant p<q$ there exists $C_p\in(0,C_q]$ such that the estimate $\|Tu\|_E\leqslant C_p\|u\|_{U^p_S}$ holds true for all $u\in U^p_S$. Then, T satisfies the estimate*

$$\|Tu\|_E\leqslant\frac{4C_p}{\alpha_{p,q}}\left(\ln\frac{C_q}{C_p}+2\alpha_{p,q}+1\right)\|u\|_{V^p_S},\quad u\in V^p_{-,rc,S}$$

where $\alpha_{p,q}=(1-\frac{p}{q})\ln(2)$.

Proof. Let $v\in V^p_{-,rc,S}$ be such that $\|v\|_{V^p_S}=1$. Due to Proposition 2.5 there exists $u_n\in U^r$ for all $r\geqslant1$ such that $v=\sum_{n=1}^\infty u_n$ in U^q and $\|u_n\|_{U^r_S}\leqslant 4\cdot2^{n(\frac{p}{r}-1)}$. For $N\in\mathbb{N}$ it follows $\|\sum_{n=1}^N u_n\|_{U^p_S}\leqslant 4N$ and $\|\sum_{n=N+1}^\infty u_n\|_{U^q_S}\leqslant 4\cdot2^{-N(1-\frac{p}{q})}$. We obtain the estimate

$$\|Tv\|_E\leqslant 4C_pN+4C_q2^{-N(1-\frac{p}{q})}.$$

Minimizing with respect to $N\in\mathbb{N}$ gives the desired upper bound. □

Corollary 2.21. *We have*

$$\|u\|_{L^4(\mathbb{R}^3)}\lesssim\|u\|_{U^4_S},\tag{24}$$

$$\|u\|_{L^4(\mathbb{R}^3)}\lesssim\|u\|_{V^p_{-,S}}\quad(1\leqslant p<4),\tag{25}$$

$$\|\partial_x u\|_{L^\infty_x(\mathbb{R};L^2_{t,y}(\mathbb{R}^2))}\lesssim\|u\|_{U^2_S},\tag{26}$$

$$\|P_{N_1}u_1P_{N_2}u_2\|_{L^2(\mathbb{R}^3)}\lesssim\left(\frac{N_1}{N_2}\right)^{\frac12}\|P_{N_1}u_1\|_{U^2_S}\|P_{N_2}u_2\|_{U^2_S}.\tag{27}$$

Moreover, for $N_2\geqslant N_1$ and $u_1,u_2\in V^2_{-,S}$ it holds

$$\|P_{N_1}u_1P_{N_2}u_2\|_{L^2(\mathbb{R}^3)}\lesssim\left(\frac{N_1}{N_2}\right)^{\frac12}\left(\ln\left(\frac{N_2}{N_1}\right)+1\right)^2\|P_{N_1}u_1\|_{V^2_S}\|P_{N_2}u_2\|_{V^2_S}.\tag{28}$$

Proof. Proposition 2.3 of [17] and Lemma 3.2 of [10] show that the estimates (24) and (26) hold true for free solutions. Thus, the claims (24) and (26) follow from Proposition 2.19. Then, (25) follows from Corollary 2.6 and the observation that $v\in V^p_{-,S}$ coincides a.e. with its right-continuous variant. In order to prove (27), let $u_i=e^{tS}\phi_i$ $(i=1,2)$ be free solutions, $\phi_i\in L^2(\mathbb{R}^2)$. With the smooth cutoff in time χ we obtain

$$\|P_{N_1}u_1P_{N_2}u_2\|_{L^2([-1,1]\times\mathbb{R}^2)}\leqslant\|\chi P_{N_1}u_1\chi P_{N_2}u_2\|_{L^2(\mathbb{R}^3)}\lesssim\left(\frac{N_1}{N_2}\right)^{\frac12}\|P_{N_1}\phi_1\|_{L^2}\|P_{N_2}\phi_2\|_{L^2}$$

which is an immediate consequence of [7], Theorem 3.3 (this is implicitly contained in the earlier work [20]). By rescaling it follows

$$\|P_{N_1}u_1 P_{N_2}u_2\|_{L^2(\mathbb{R}^3)} \lesssim \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}} \|P_{N_1}\phi_1\|_{L^2}\|P_{N_2}\phi_2\|_{L^2}$$

and we may apply Proposition 2.19.

Now, the estimate (28) follows from interpolation between (24) and (27) via Proposition 2.20. Indeed, we first consider the operator $T_1 : u_2 \mapsto \tilde{P}_{N_1}u_1\tilde{P}_{N_2}u_2$, where $\tilde{P}_{N_i} = P_{N_i/2} + P_{N_i} + P_{2N_i}$, such that $\tilde{P}_{N_i}P_{N_i} = P_{N_i}$, and obtain the bounds

$$\|T_1\|_{U_S^4\to L^2} \lesssim \|u_1\|_{U_S^2}, \qquad \|T_1\|_{U_S^2\to L^2} \lesssim \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}}\|u_1\|_{U_S^2},$$

by (24) combined with $U_S^2 \subset U_S^4$ and (27). Proposition 2.20 implies

$$\|\tilde{P}_{N_1}u_1\tilde{P}_{N_2}u_2\|_{L^2} \lesssim \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}}\left(\ln\left(\frac{N_2}{N_1}\right)+1\right)\|u_1\|_{U_S^2}\|u_2\|_{V_S^2}. \tag{29}$$

Second, we consider the operator $T_2 : u_1 \mapsto \tilde{P}_{N_1}u_1\tilde{P}_{N_2}u_2$. Then, using (24) and (29) we obtain

$$\|T_2\|_{U_S^4\to L^2} \lesssim \|u_2\|_{V_S^2}, \qquad \|T_2\|_{U_S^2\to L^2} \lesssim \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}}\left(\ln\left(\frac{N_2}{N_1}\right)+1\right)\|u_2\|_{V_S^2},$$

where we also used the embedding $V^2_{-,rc,S} \subset U_S^4$. Now, we apply Proposition 2.20 again to deduce

$$\|\tilde{P}_{N_1}u_1\tilde{P}_{N_2}u_2\|_{L^2} \lesssim \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}}\left(\ln\left(\frac{N_2}{N_1}\right)+1\right)^2\|u_1\|_{V_S^2}\|u_2\|_{V_S^2}$$

for all $u_1, u_2 \in V^2_{-,rc,S}$. Finally, we apply this estimate to $P_{N_i}u_i$ instead of u_i and observe that both sides do not change if we replace $u_i \in V^2_{-,S}$ by its right-continuous variant and (28) follows. □

Definition 2.22. Let $s \leqslant 0$.

(i) Define $\dot{Y}^s$ as the closure of all $u \in C(\mathbb{R}; H^{1,1}(\mathbb{R}^2)) \cap V^2_{-,rc,S}$ such that

$$\|u\|_{\dot{Y}^s} := \left(\sum_N N^{2s}\|P_N u\|^2_{V_S^2}\right)^{\frac{1}{2}} < \infty, \tag{30}$$

in the space $C(\mathbb{R}; \dot{H}^{s,0}(\mathbb{R}^2))$ with respect to the $\|\cdot\|_{\dot{Y}^s}$-norm.

(ii) Define $\dot{Z}^s$ as the closure of all $u \in C(\mathbb{R}; H^{1,1}(\mathbb{R}^2)) \cap U_S^2$ such that

$$\|u\|_{\dot{Z}^s} := \left(\sum_N N^{2s}\|P_N u\|^2_{U_S^2}\right)^{\frac{1}{2}} < \infty, \tag{31}$$

in the space $C(\mathbb{R}; \dot{H}^{s,0}(\mathbb{R}^2))$ with respect to the $\|\cdot\|_{\dot{Z}^s}$-norm.

(iii) Define X as the closure of all $u \in C(\mathbb{R}; H^{1,1}(\mathbb{R}^2)) \cap U_S^2$ such that

$$\|u\|_X := \|u\|_{\dot{Z}^0} + \|u\|_{\dot{X}^{0,1,1}} < \infty, \tag{32}$$

in the space $C(\mathbb{R}; L^2(\mathbb{R}^2))$ with respect to the $\|\cdot\|_X$-norm. Define $Z^s := \dot{Z}^s + X$, with norm

$$\|u\|_{Z^s} = \inf\{\|u_1\|_{\dot{Z}^s} + \|u_2\|_X \mid u = u_1 + u_2\}. \tag{33}$$

Remark 2.23. Let E be a Banach space of continuous functions $f : \mathbb{R} \to H$, for some Hilbert space H. We also consider the corresponding restriction space to the interval $I \subset \mathbb{R}$ by

$$E(I) = \{u \in C(I, H) \mid \exists \tilde{u} \in E\colon\ \tilde{u}(t) = u(t),\ t \in I\}$$

endowed with the norm $\|u\|_{E(I)} = \inf\{\|\tilde{u}\|_E \mid \tilde{u} : \tilde{u}(t) = u(t), t \in I\}$. Obviously, $E(I)$ is also a Banach space.

Proposition 2.24.

(i) *Let* $T > 0$ *and* $u \in \dot{Y}^s([0,T])$, $u(0) = 0$. *Then, for every* $\varepsilon > 0$ *there exists* $0 \leqslant T' \leqslant T$ *such that* $\|u\|_{\dot{Y}^s([0,T'])} < \varepsilon$.
(ii) *Let* $T > 0$ *and* $u \in \dot{Z}^s([0,T])$, $u(0) = 0$. *Then, for every* $\varepsilon > 0$ *there exists* $0 \leqslant T' \leqslant T$ *such that* $\|u\|_{\dot{Z}^s([0,T'])} < \varepsilon$.

Proof. It is enough to consider $s = 0$. Assume $u \in \dot{Y}^0([0,T])$ with $u(0) = 0$ and let $\tilde{u} \in \dot{Y}^0$ be an extension. There exists a decomposition $\tilde{u} = \tilde{u}_h + \tilde{u}_r$ with

$$\tilde{u}_h = \sum_{M_0 \leqslant N \leqslant M_1} P_N \tilde{u}, \qquad \|\tilde{u}_r\|_{\dot{Y}^0} < \varepsilon. \tag{34}$$

Due to the right-continuity of $\tilde{u}_h$ there exists $0 < T_0 \leqslant T$ such that

$$\|\tilde{u}_h\|_{L^\infty([0,T_0];L^2)} < \varepsilon.$$

Moreover, there exists $\mathfrak{t} = \{t_k\}_{k=0}^K \in \mathcal{Z}$ such that $0 \in \mathfrak{t}$ and

$$\left(\sum_{k=1}^K \left\|e^{-t_k S}\tilde{u}_h(t_k) - e^{-t_{k-1}S}\tilde{u}_h(t_{k-1})\right\|_{L^2}^2\right)^{\frac{1}{2}} > \|\tilde{u}_h\|_{V_S^2} - \varepsilon.$$

We define $T' := \min\{t_k \mid t_k > 0\}$ and the continuous extension

$$\tilde{u}_{h,T'} := \mathbb{1}_{[0,T')}\tilde{u}_h + \mathbb{1}_{[T',\infty)}\tilde{u}_h(T'). \tag{35}$$

Then, $\|\tilde{u}_{h,T'}\|_{V_S^2} < \varepsilon$. Finally,

$$\|u_h\|_{\dot{Y}^0([0,T'])} \leqslant \|\tilde{u}_{h,T'}\|_{\dot{Y}^0} \leqslant \left(\sum_{M_0/2 \leqslant N \leqslant 2M_1} \|P_N \tilde{u}_{h,T'}\|_{V_S^2}^2\right)^{\frac{1}{2}} \lesssim \varepsilon.$$

In order to prove part (ii) let us assume that $u \in \dot{Z}^0([0,T])$ with $u(0) = 0$ and let $\tilde{u} \in \dot{Z}^0$ be an extension. We perform a similar decomposition as in (34). Since $\tilde{u}_h \in U_S^2$, we have an atomic decomposition

$$\tilde{u}_h = \sum_{k=1}^\infty \lambda_k e^{\cdot S} a_k \quad \text{such that} \quad \sum_{k=k_0+1}^\infty |\lambda_k| < \varepsilon.$$

There exists $0 < T' \leqslant T$, such that all a_k ($k = 1, \ldots, k_0$) are constant on $[0,T']$. Define $\lambda_0 = \|\sum_{k=1}^{k_0} \lambda_k a_k(0)\|_{L^2}$ and $\phi = \lambda_0^{-1} \sum_{k=1}^{k_0} \lambda_k a_k(0)$ as well as the atom $a_0 = \mathbb{1}_{[0,\infty)}\phi$. Then,

$$\lambda_0 = \left\|u(0) - \sum_{k=k_0+1}^\infty \lambda_k a_k(0)\right\|_{L^2} \leqslant \sum_{k=k_0+1}^\infty |\lambda_k| < \varepsilon.$$

For $f(t) := \lambda_0 e^{tS} a_0(t) + \sum_{k=k_0+1}^\infty \lambda_k e^{tS} a_k(t)$, we define the continuous function $f_{T'} = \mathbb{1}_{[0,T')} f + \mathbb{1}_{[T',\infty)} f(T'-)$. It holds $u_h(t) = \tilde{u}_h(t) = f_{T'}(t)$ for $t \in [0,T']$ and therefore $\|u_h\|_{\dot{Z}^0([0,T'])} \leqslant \|f_{T'}\|_{\dot{Z}^0} \lesssim \varepsilon$. □

3. Bilinear estimates

Let $T \in (0,\infty]$. In the following, we will give estimates on the Duhamel term

$$I_T(u_1,u_2)(t) := \int_0^t \mathbb{1}_{[0,T)} e^{(t-t')S} \partial_x(u_1 u_2)(t')\,dt', \tag{36}$$

which is initially defined on $C(\mathbb{R}; H^{1,1}(\mathbb{R}^2))$, and the estimates will eventually allow us to extend this bilinear operator to larger function spaces.

3.1. The homogeneous case

We start with an estimate on dyadic pieces. For a dyadic number N let $A_N := \{(\tau,\xi,\eta) \mid \frac{1}{2}N \leqslant |\xi| \leqslant 2N\}$.

Proposition 3.1. *There exists $C > 0$, such that for all $T > 0$ and functions $u_{N_1}, v_{N_2}, w_{N_3} \in V^2_{-,S}$ satisfying $\operatorname{supp}\widehat{u_{N_1}} \subset A_{N_1}$, $\operatorname{supp}\widehat{v_{N_2}} \subset A_{N_2}$, $\operatorname{supp}\widehat{w_{N_2}} \subset A_{N_3}$ for dyadic numbers N_1, N_2, N_3 the following holds true:*
If $N_2 \sim N_3$, then

$$\left|\sum_{N_1 \lesssim N_2} \int_0^T\int_{\mathbb{R}^2} u_{N_1} v_{N_2} w_{N_3}\,dx\,dy\,dt\right| \leqslant C\Big(\sum_{N_1 \lesssim N_2} N_1^{-1}\|u_{N_1}\|^2_{V^2_S}\Big)^{\frac{1}{2}} N_2^{-\frac{1}{2}}\|v_{N_2}\|_{V^2_S} N_3^{-\frac{1}{2}}\|w_{N_3}\|_{V^2_S}, \tag{37}$$

and if $N_1 \sim N_2$, then

$$\bigg(\sum_{N_3 \lesssim N_2} N_3 \sup_{\|w_{N_3}\|_{V^2_S}=1}\left|\int_0^T\int_{\mathbb{R}^2} u_{N_1} v_{N_2} w_{N_3}\,dx\,dy\,dt\right|^2\bigg)^{\frac{1}{2}} \leqslant C N_1^{-\frac{1}{2}}\|u_{N_1}\|_{V^2_S} N_2^{-\frac{1}{2}}\|v_{N_2}\|_{V^2_S}. \tag{38}$$

Proof. We define $\tilde u_{N_1} = \mathbb{1}_{[0,T)}u_{N_1}$, $\tilde v_{N_2} = \mathbb{1}_{[0,T)}v_{N_2}$, $\tilde w_{N_3} = \mathbb{1}_{[0,T)}w_{N_3}$. Then, we decompose $\mathrm{Id} = Q_{<M} + Q_{\geqslant M}$, where M will be chosen later, and we divide the integrals on the left-hand side of (37) into eight pieces of the form

$$\int_{\mathbb{R}^3} Q_1^S\tilde u_{N_1} Q_2^S\tilde v_{N_2} Q_3^S\tilde w_{N_3}\,dx\,dy\,dt$$

with $Q_i^S \in \{Q^S_{\geqslant M}, Q^S_{<M}\}$, $i = 1,2,3$. These are well-defined because of the L^4 Strichartz estimate (25) and (21).

Let us first consider the case $Q_i^S = Q^S_{<M}$ for $1 \leqslant i \leqslant 3$. By using Plancherel's Theorem we see

$$\int_{\mathbb{R}^3} Q^S_{<M}\tilde u_{N_1} Q^S_{<M}\tilde v_{N_2} Q^S_{<M}\tilde w_{N_3}\,dx\,dy\,dt = c\big(\widehat{Q^S_{<M}\tilde u_{N_1}} * \widehat{Q^S_{<M}\tilde v_{N_2}} * \widehat{Q^S_{<M}\tilde w_{N_3}}\big)(0). \tag{39}$$

Now, if we let $\mu_i = (\tau_i,\xi_i,\eta_i)$, $i = 1,2,3$, be the Fourier variables corresponding to $\widehat{Q^S_{<M}\tilde u_{N_1}}$, $\widehat{Q^S_{<M}\tilde v_{N_2}}$, and $\widehat{Q^S_{<M}\tilde w_{N_3}}$ respectively, then the only frequencies which contribute to (39) are those for which we have $\mu_1 + \mu_2 + \mu_3 = 0$. For $\lambda_i = \tau_i - \xi_i^3 + \frac{\eta_i^2}{\xi_i}$, $i = 1,2,3$, we have that $|\lambda_i| < M$ within the domain of integration because of the cutoff operator $Q^S_{<M}$. We also have $|\xi_i| \geqslant N_i/2$ due to the cut off operators P_{N_i}. By the well-known resonance identity

$$\lambda_1 + \lambda_2 + \lambda_3 = 3\xi_1\xi_2\xi_3 + \frac{(\xi_2\eta_1 - \eta_2\xi_1)^2}{\xi_1\xi_2\xi_3}, \tag{40}$$

we get

$$\frac{1}{8}N_1N_2N_3 \leqslant |\xi_1||\xi_2||\xi_3| \leqslant \max\big(|\lambda_1|,|\lambda_2|,|\lambda_3|\big) < M \tag{41}$$

within the domain of integration. Therefore, if we set $M = 8^{-1}N_1N_2N_3$ (our notation suppresses the dependence on N_1, N_2, N_3), it follows that

$$\int_{\mathbb{R}^3} Q^S_{<M}\tilde u_{N_1} Q^S_{<M}\tilde v_{N_2} Q^S_{<M}\tilde w_{N_3}\,dx\,dy\,dt = 0.$$

So, let us now consider the case that $Q_i^S = Q^S_{\geqslant M}$ for some $1 \leqslant i \leqslant 3$ and start with the case $i = 1$. Using the L^4 Strichartz estimate (25) we obtain for $Q_2^S, Q_3^S \in \{Q^S_{\geqslant M}, Q^S_{<M}\}$

$$\left|\sum_{N_1 \lesssim N_2}\int_{\mathbb{R}^3} Q^S_{\geqslant M}\tilde u_{N_1} Q_2^S\tilde v_{N_2} Q_3^S\tilde w_{N_3}\,dx\,dy\,dt\right|$$

$$\leqslant \Big\| \sum_{N_1 \lesssim N_2} Q^S_{\geqslant M} \tilde{u}_{N_1} \Big\|_{L^2(\mathbb{R}^3)} \big\| Q^S_2 \tilde{v}_{N_2} \big\|_{L^4(\mathbb{R}^3)} \big\| Q^S_3 \tilde{w}_{N_3} \big\|_{L^4(\mathbb{R}^3)}$$

$$\leqslant C\Big(\sum_{N_1 \lesssim N_2} \frac{1}{N_1 N_2 N_3} \|\tilde{u}_{N_1}\|^2_{V^2_S} \Big)^{\frac{1}{2}} \big\| Q^S_2 \tilde{v}_{N_2} \big\|_{V^2_S} \big\| Q^S_3 \tilde{w}_{N_3} \big\|_{V^2_S}, \tag{42}$$

where we used the L^2-orthogonality and (20) on the first factor. Now, we exploit (21) and

$$\|\mathbb{1}_{[0,T)} f\|_{V^2_S} \leqslant 2\|f\|_{V^2_S}, \quad f \in V^2_S$$

and the claim is proved.

We turn to the case $i = 2$. Using the interpolated bilinear Strichartz estimate (28) and Corollary 2.18, we find for $Q^S_1, Q^S_3 \in \{Q^S_{\geqslant M}, Q^S_{<M}\}$

$$\Big| \int_{\mathbb{R}^3} Q^S_1 \tilde{u}_{N_1} Q^S_{\geqslant M} \tilde{v}_{N_2} Q^S_3 \tilde{w}_{N_3} \, dx\, dy\, dt \Big| \leqslant \big\| Q^S_{\geqslant M} \tilde{v}_{N_2} \big\|_{L^2(\mathbb{R}^3)} \Big(\frac{N_1}{N_3} \Big)^{\frac{1}{4}} \big\| Q^S_1 \tilde{u}_{N_1} \big\|_{V^2_S} \big\| Q^S_3 \tilde{w}_{N_3} \big\|_{V^2_S}$$

$$\leqslant \frac{C}{(N_1 N_2 N_3)^{\frac{1}{2}}} \|v_{N_2}\|_{V^2_S} \Big(\frac{N_1}{N_3} \Big)^{\frac{1}{4}} \|u_{N_1}\|_{V^2_S} \|w_{N_3}\|_{V^2_S}$$

which is easily summed up with respect to $N_1 \lesssim N_2$, because $N_2 \sim N_3$.

Finally, the case $i = 3$ is proved in exactly the same way as $i = 2$ and the proof of (37) is complete.

In order to prove (38), we use the same decomposition as above. The case $i = 1, 2$, i.e. if the modulation on the first or second factor is high, we use the bilinear Strichartz estimate (28) and the claim follows similar to the case $i = 2, 3$ above. It remains to consider the case $i = 3$, where the modulation on the third factor is high. Let P_{N_3} be the projection operator onto the set A_{N_3}, which is symmetric. Therefore, using L^2-orthogonality and (21) we obtain

$$\Big(\sum_{N_3 \lesssim N_2} N_3 \sup_{\|w_{N_3}\|_{V^2_S}=1} \Big| \int_{\mathbb{R}^3} Q^S_1 \tilde{u}_{N_1} Q^S_2 \tilde{v}_{N_2} Q^S_{\geqslant M} \tilde{w}_{N_3} \, dx\, dy\, dt \Big|^2 \Big)^{\frac{1}{2}}$$

$$\leqslant \Big(\sum_{N_3 \lesssim N_2} \big\| P_{A_{N_3}} \big(Q^S_1 \tilde{u}_{N_1} Q^S_2 \tilde{v}_{N_2} \big) \big\|^2_{L^2} \sup_{\|w_{N_3}\|_{V^2_S}=1} N_3 \big\| Q^S_{\geqslant M} \tilde{w}_{N_3} \big\|^2_{L^2} \Big)^{\frac{1}{2}}$$

$$\lesssim (N_1 N_2)^{-\frac{1}{2}} \big\| Q^S_1 \tilde{u}_{N_1} Q^S_2 \tilde{v}_{N_2} \big\|_{L^2}. \tag{43}$$

The claim now follows from the standard L^4 Strichartz estimate (24) and Corollary 2.18. □

Theorem 3.2. *There exists $C > 0$, such that for all $0 < T < \infty$ and for all $u_1, u_2 \in \dot{Z}^{-\frac{1}{2}} \cap C(\mathbb{R}; H^{1,1}(\mathbb{R}^2))$ it holds*

$$\big\| I_T(u_1, u_2) \big\|_{\dot{Z}^{-\frac{1}{2}}} \leqslant C \prod_{j=1}^{2} \|u_j\|_{\dot{Y}^{-\frac{1}{2}}}, \tag{44}$$

and I_T continuously extends to a bilinear operator

$$I_T : \dot{Y}^{-\frac{1}{2}} \times \dot{Y}^{-\frac{1}{2}} \to \dot{Z}^{-\frac{1}{2}}.$$

Proof. Let $u_{1,N_1} := P_{N_1} u_1$, $u_{2,N_2} := P_{N_2} u_2$. By symmetry, it is enough to consider the two terms

$$J_1 := \Big\| \sum_{N_2} \sum_{N_1 \ll N_2} I_T(u_{1,N_1}, u_{2,N_2}) \Big\|_{\dot{Z}^{-\frac{1}{2}}},$$

$$J_2 := \Big\| \sum_{N_2} \sum_{N_1 \sim N_2} I_T(u_{1,N_1}, u_{2,N_2}) \Big\|_{\dot{Z}^{-\frac{1}{2}}}.$$

We start with J_1 and fix N. We may assume $N \sim N_2$ and by Theorem 2.8 and Proposition 2.10

$$\Bigl\| e^{-\cdot S} P_N \sum_{N_1 \ll N_2} I_T(u_{1,N_1}, u_{2,N_2}) \Bigr\|_{U^2} = \sup_{\|v\|_{V^2}=1} \Bigl| \sum_{N_1 \ll N_2} B\bigl(e^{-\cdot S} P_N I_T(u_{1,N_1}, u_{2,N_2}), v\bigr) \Bigr|$$
$$= \sup_{\|v\|_{V^2_S}=1} \Bigl| \sum_{N_1 \ll N_2} \int_0^T \int_{\mathbb{R}^2} u_{1,N_1} u_{2,N_2} \partial_x P_N v \, dx\, dy\, dt \Bigr|.$$

We apply (37) and obtain

$$N^{-\frac{1}{2}} \Bigl\| \sum_{N_1 \ll N_2} P_N I_T(u_{1,N_1}, u_{2,N_2}) \Bigr\|_{U^2_S} \lesssim \Bigl(\sum_{N_1} N_1^{-1} \|u_{1,N_1}\|^2_{V^2_S} \Bigr)^{\frac{1}{2}} N_2^{-\frac{1}{2}} \|u_{2,N_2}\|_{V^2_S}.$$

We easily sum up the squares with respect to $N_2 \sim N$.

Next, we turn to J_2 and fix N_2. We may assume $N \lesssim N_2$ and by Theorem 2.8 and Proposition 2.10

$$\sum_{N \lesssim N_2} N^{-1} \bigl\| e^{-\cdot S} P_N I_T(u_{1,N_1}, u_{2,N_2}) \bigr\|^2_{U^2} = \sum_{N \lesssim N_2} N^{-1} \sup_{\|v\|_{V^2}=1} \bigl| B\bigl(e^{-\cdot S} P_N I_T(u_{1,N_1}, u_{2,N_2}), v\bigr) \bigr|^2$$
$$= \sum_{N \lesssim N_2} N^{-1} \sup_{\|v\|_{V^2_S}=1} \Bigl| \int_0^T \int_{\mathbb{R}^2} u_{1,N_1} u_{2,N_2} P_N \partial_x v \, dx\, dy\, dt \Bigr|^2.$$

We apply (38) and obtain

$$\Bigl\| \sum_{N_2} \sum_{N_1 \sim N_2} I_T(u_{1,N_1}, u_{2,N_2}) \Bigr\|_{\dot{Z}^{-\frac{1}{2}}} \lesssim \sum_{N_2} \sum_{N_1 \sim N_2} \bigl\| I_T(u_{1,N_1}, u_{2,N_2}) \bigr\|_{\dot{Z}^{-\frac{1}{2}}}$$
$$\lesssim \sum_{N_2} \sum_{N_1 \sim N_2} (N_1 N_2)^{-\frac{1}{2}} \|u_{1,N_1}\|_{V^2_S} \|u_{2,N_2}\|_{V^2_S}$$

and the proof is complete. □

Corollary 3.3. *There exists $C > 0$, such that for all $0 < T < \infty$ and for all $u_1, u_2 \in \dot{Z}^{-\frac{1}{2}} \cap C(\mathbb{R}; H^{1,1}(\mathbb{R}^2))$ it holds*

$$\bigl\| I_T(u_1, u_2) \bigr\|_{\dot{Z}^{-\frac{1}{2}}} \leqslant C \prod_{j=1}^{2} \|u_j\|_{\dot{Z}^{-\frac{1}{2}}}, \tag{45}$$

and I_T continuously extends to a bilinear operator

$$I_T : \dot{Z}^{-\frac{1}{2}} \times \dot{Z}^{-\frac{1}{2}} \to \dot{Z}^{-\frac{1}{2}}.$$

A similar statement holds true with $\dot{Z}^s$ replaced by $\dot{Y}^s$.

Proof. This is due to the continuous embedding $\dot{Z}^s \subset \dot{Y}^s$ and Theorem 3.2. □

Corollary 3.4. *Assume that $u_1, u_2 \in \dot{Y}^{-\frac{1}{2}}$. Then, $I_\infty(u_1, u_2) \in \dot{Z}^{-\frac{1}{2}}$ and*

$$\bigl\| I_T(u_1, u_2) - I_\infty(u_1, u_2) \bigr\|_{\dot{Z}^{-\frac{1}{2}}} \to 0 \quad (T \to \infty).$$

In particular, for any $u \in \dot{Y}^{-\frac{1}{2}}$ it exists

$$\lim_{t \to \infty} e^{-tS} I_\infty(u, u)(t) \in \dot{H}^{-\frac{1}{2},0}\bigl(\mathbb{R}^2\bigr). \tag{46}$$

Proof. Without loss of generality we may assume $u_1, u_2 \in C(\mathbb{R}; H^{1,1}(\mathbb{R}^2))$ such that $\|u_1\|_{\dot{Y}^{-\frac{1}{2}}} = \|u_2\|_{\dot{Y}^{-\frac{1}{2}}} = 1$. Estimate (44) implies

$$\sum_N N^{-1} \left\| e^{-\cdot S} P_N I_\infty(u_1, u_2) \right\|_{V_0^2}^2 \leqslant C,$$

and due to Proposition 2.4, part (iv), for all the dyadic pieces the limits at ∞ exist and we have $P_N I_\infty(u_1, u_2) \in V^2_{-,rc,S}$ along with

$$\sum_N N^{-1} \left\| P_N I_\infty(u_1, u_2) \right\|_{V_S^2}^2 \leqslant C,$$

which yields $I_\infty(u_1, u_2) \in \dot{Y}^{-\frac{1}{2}}$ and in particular the convergence (46).

The limits $e^{-tS} u_i(t) \to \phi_i \in \dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$ for $t \to \infty$ exist. Let $\alpha_T : \mathbb{R} \to \mathbb{R}$ be

$$\alpha_T(t) = \begin{cases} 0 & (t < T-1), \\ t+1 & (T-1 \leqslant t < T), \\ 1 & (t \geqslant T). \end{cases} \tag{47}$$

We define $\tilde{u}_i = u_i - \alpha_0 e^{\cdot S} \phi_i$, $i = 1, 2$. Let $\varepsilon > 0$. There exists $T > 0$, such that $\|\alpha_T \tilde{u}_i\|_{\dot{Y}^{-\frac{1}{2}}} < \varepsilon$, which follows by a similar argument as in the proof of Proposition 2.24, part (i). Let $T_2 > T_1 > T$. Then,

$$I_{T_1}(\tilde{u}_1, u_2) - I_{T_2}(\tilde{u}_1, u_2) = I_{T_1}(\alpha_{T_1}\tilde{u}_1, u_2) - I_{T_2}(\alpha_{T_1}\tilde{u}_1, u_2)$$

and for $i = 1, 2$

$$\left\| I_{T_i}\left(\alpha_{T_1}\tilde{u}_1, u_2\right)\right\|_{\dot{Z}^{-\frac{1}{2}}} \lesssim \varepsilon.$$

By a similar argument,

$$\left\| I_{T_1}\left(\alpha_0 e^{\cdot S}\phi_1, \tilde{u}_2\right) - I_{T_2}\left(\alpha_0 e^{\cdot S}\phi_1, \tilde{u}_2\right)\right\|_{\dot{Z}^{-\frac{1}{2}}} \lesssim \varepsilon.$$

On the other hand, by the L^4 Strichartz estimate (25) there exists $T' > 0$ such that $\|\alpha_{T'} e^{\cdot S} P_N \phi\|_{L^4(\mathbb{R}^3)} < \varepsilon \|P_N \phi\|_{L^2}$. For $T_2 > T_1 > T'$

$$\begin{aligned} &\left\| I_{T_1}\left(\alpha_0 e^{\cdot S}\phi_1, \alpha_0 e^{\cdot S}\phi_2\right) - I_{T_2}\left(\alpha_0 e^{\cdot S}\phi_1, \alpha_0 e^{\cdot S}\phi_2\right)\right\|_{\dot{Z}^{-\frac{1}{2}}} \\ &\quad = \left\| I_{T_1}\left(\alpha_{T_1} e^{\cdot S}\phi_1, \alpha_{T_1} e^{\cdot S}\phi_2\right) - I_{T_2}\left(\alpha_{T_1} e^{\cdot S}\phi_1, \alpha_{T_1} e^{\cdot S}\phi_2\right)\right\|_{\dot{Z}^{-\frac{1}{2}}} \lesssim \varepsilon \end{aligned}$$

by the same proof as of Theorem 3.2, using again Proposition 3.1 where now the factor ε comes from (42) and (43). Hence, the family $(I_T(u_1, u_2))_T$ satisfies a Cauchy condition in $\dot{Z}^{-\frac{1}{2}}$, which is a complete space. Therefore, it converges in $\dot{Z}^{-\frac{1}{2}}$ to $I_\infty(u_1, u_2)$. $\square$

3.2. The inhomogeneous case

Proposition 3.5. *Let $\varepsilon > 0$. There exists $C > 0$ such that for all $0 < T \leqslant 1$ and $u_{N_1} \in X$, $v_{N_2} \in U_S^2$, $w_{N_3} \in V^2_{-,S}$ with $\operatorname{supp}\widehat{u_{N_1}} \subset A_{N_1}$, $\operatorname{supp}\widehat{v_{N_2}} \subset A_{N_2}$, $\operatorname{supp}\widehat{w_{N_2}} \subset A_{N_3}$ for dyadic numbers N_1, N_2, N_3 where $N_1 \leqslant 1 \leqslant N_2$ it holds*

$$\left| \int_0^T \int_{\mathbb{R}^2} u_{N_1} v_{N_2} w_{N_3} \, dx \, dy \, dt \right| \leqslant \frac{C(TN_1)^{\frac{1}{4}-\varepsilon}}{(N_2 N_3)^{\frac{1}{2}}} \|u_{N_1}\|_X \|v_{N_2}\|_{U_S^2} \|w_{N_3}\|_{V_S^2}. \tag{48}$$

Proof. We use the same notation as in the proof of Proposition 3.1 and again the left-hand side is well defined. In particular we denote the time truncation of a function u by $\tilde{u}$. Note that obviously

$$\|\mathbb{1}_{[0,T)} u\|_{U_S^2} \leqslant \|u\|_{U_S^2}, \quad u \in U_S^2.$$

In any case we may assume that $N_3 \lesssim N_2$, because otherwise the left-hand side vanishes. In the first case we assume $N_1 N_3^2 \leqslant T^{-1}$. Using the bilinear Strichartz estimate (27), we obtain

$$\left|\int_{\mathbb{R}^3} u_{N_1}\tilde v_{N_2}\tilde w_{N_3}\,dx\,dy\,dt\right| \lesssim \|u_{N_1}\tilde v_{N_2}\|_{L^2(\mathbb{R}^3)}\|\tilde w_{N_3}\|_{L^2(\mathbb{R}^3)} \lesssim T^{\frac12}\left(\frac{N_1}{N_2}\right)^{\frac12}\|u_{N_1}\|_{U^2_S}\|v_{N_2}\|_{U^2_S}\|w_{N_3}\|_{V^2_S}$$

and the claim follows from $\|u_{N_1}\|_{U^2_S}\leqslant \|u_{N_1}\|_X$ and $N_1^{\frac14}\leqslant T^{-\frac14}N_3^{-\frac12}$.

Now, assume that $N_1N_3^2\geqslant T^{-1}$ and we fix $M=8^{-1}N_1N_2N_3$. Recall from the proof of Proposition 3.1 that we have

$$\int_{\mathbb{R}^3} Q^S_{<M}\tilde u_{N_1}Q^S_{<M}\tilde v_{N_2}Q^S_{<M}\tilde w_{N_3}\,dx\,dy\,dt=0.$$

Therefore we can always assume to have high modulation on one of the three factors.

If $Q^S_2,Q^S_3\in\{Q^S_{\geqslant M},Q^S_{<M}\}$ and the modulation on the first factor is high, we apply the bilinear estimate (27) and Corollary 2.18 and obtain

$$\left|\int_{\mathbb{R}^3} Q^S_{\geqslant M}u_{N_1}Q^S_2\tilde v_{N_2}Q^S_3\tilde w_{N_3}\,dx\,dy\,dt\right| \lesssim \left(\frac{N_1}{N_2}\right)^{\frac12}\left\|Q^S_{\geqslant M}u_{N_1}\right\|_{U^2_S}\|v_{N_2}\|_{U^2_S}\left\|Q^S_3\tilde w_{N_3}\right\|_{L^2}.$$

Now, we combine this with $\|Q^S_3\tilde w_{N_3}\|_{L^2}\leqslant T^{\frac12}\|w_{N_3}\|_{V^2_S}$ and

$$\left\|Q^S_{\geqslant M}u_{N_1}\right\|_{U^2_S}\lesssim \left\|Q^S_{\geqslant M}u_{N_1}\right\|_{\dot X^{0,\frac12,1}}\lesssim (N_1N_2N_3)^{-\frac12}\|u_{N_1}\|_{\dot X^{0,1,1}}$$

and (48) follows, because $N_2^{\frac12}\gtrsim N_3^{\frac12}\geqslant T^{-\frac14}N_1^{-\frac14}$. If $Q^S_1,Q^S_3\in\{Q^S_{\geqslant M},Q^S_{<M}\}$ and the modulation on the second factor is high, an application of the interpolated estimate (28) yields

$$\begin{aligned}\left|\int_{\mathbb{R}^3} Q^S_1u_{N_1}Q^S_{\geqslant M}\tilde v_{N_2}Q^S_3\tilde w_{N_3}\,dx\,dy\,dt\right| &\lesssim \left(\frac{N_1}{N_3}\right)^{\frac{1-\varepsilon}{2}}\|u_{N_1}\|_{V^2_S}\|w_{N_3}\|_{V^2_S}\left\|Q^S_{\geqslant M}\tilde v_{N_2}\right\|_{L^2(\mathbb{R}^3)}\\ &\lesssim \frac{1}{N_1^{\frac{\varepsilon}{2}}N_2^{\frac12}N_3^{1-\frac{\varepsilon}{2}}}\|u_{N_1}\|_{V^2_S}\|w_{N_3}\|_{V^2_S}\|v_{N_2}\|_{V^2_S}\end{aligned}$$

which shows the claim in this case, because $N_3^{\frac{1-\varepsilon}{2}}\geqslant (TN_1)^{-\frac14+\frac{\varepsilon}{2}}$.

Finally, if $Q^S_1,Q^S_2\in\{Q^S_{\geqslant M},Q^S_{<M}\}$ and the modulation on the third factor is high, we invoke estimate (27) and obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^3} Q^S_1u_{N_1}Q^S_2\tilde v_{N_2}Q^S_{\geqslant M}\tilde w_{N_3}\,dx\,dy\,dt\right| &\lesssim \left(\frac{N_1}{N_2}\right)^{\frac12}\left\|Q^S_1u_{N_1}\right\|_{U^2_S}\left\|Q^S_1\tilde v_{N_2}\right\|_{U^2_S}\left\|Q^S_{\geqslant M}\tilde w_{N_3}\right\|_{L^2(\mathbb{R}^3)}\\ &\lesssim \frac{1}{N_2N_3^{\frac12}}\|u_{N_1}\|_{U^2_S}\|v_{N_2}\|_{U^2_S}\|w_{N_3}\|_{V^2_S}\end{aligned}$$

which completes the proof, because $N_2^{\frac12}\gtrsim N_3^{\frac12}\geqslant (TN_1)^{-\frac14}$. □

Theorem 3.6. *There exists $C>0$, such that for all functions $u_1,u_2\in Z^{-\frac12}\cap C(\mathbb{R};H^{1,1}(\mathbb{R}^2))$ it holds*

$$\left\|I_1(u_1,u_2)\right\|_{\dot Z^{-\frac12}}\leqslant C\prod_{j=1}^{2}\|u_j\|_{Z^{-\frac12}},\tag{49}$$

and I_1 continuously extends to a bilinear operator

$$I_1: Z^{-\frac12}\times Z^{-\frac12}\to \dot Z^{-\frac12}\subset Z^{-\frac12}.$$

Proof. We decompose $u_j = v_j + w_j$, $v_j \in \dot{Z}^{-\frac{1}{2}}$ and $w_j \in X$, $j = 1, 2$. Due to $\|P_{\geqslant 1}u\|_{\dot{Z}^{-\frac{1}{2}}} \lesssim \|P_{\geqslant 1}u\|_X$ and Corollary 3.3, it remains to prove

$$\left\|I_1(P_{<1}w_1, v_2)\right\|_{\dot{Z}^{-\frac{1}{2}}} \leqslant C\|w_1\|_X\|v_2\|_{\dot{Z}^{-\frac{1}{2}}}, \tag{50}$$

$$\left\|I_1(P_{<1}w_1, P_{<1}w_2)\right\|_{\dot{Z}^{-\frac{1}{2}}} \leqslant C\|w_1\|_X\|w_2\|_X. \tag{51}$$

We start with a proof of (51). By Theorem 2.8 and Proposition 2.10,

$$\begin{aligned} N^{-\frac{1}{2}}\left\|P_N I_1(P_{<1}w_1, P_{<1}w_2)\right\|_{U_S^2} &\lesssim N^{\frac{1}{2}}\|P_{<1}w_1 P_{<1}w_2\|_{L^1([0,1];L^2)} \\ &\lesssim N^{\frac{1}{2}}\|P_{<1}w_1\|_{\dot{Z}^0}\|P_{<1}w_2\|_{\dot{Z}^0} \end{aligned} \tag{52}$$

due to the L^4 estimate (24). We may sum up all dyadic pieces for $N \lesssim 1$.

Let us turn to the proof of (50). The estimate for $I_1(P_{<1}w_1, P_{<1}v_2)$ is already covered by (52). Assume $N_1 \leqslant 1 \leqslant N_2$. By Theorem 2.8 and Proposition 2.10, we obtain

$$\begin{aligned} N^{-\frac{1}{2}}\left\|P_N I_1(P_{N_1}w_1, P_{N_2}v_2)\right\|_{U_S^2} &= N^{-\frac{1}{2}} \sup_{\|f\|_{V_S^2}=1}\left|\int_0^1\int_{\mathbb{R}^2} P_{N_1}w_1 P_{N_2}v_2 \partial_x P_N f\,dx\,dy\,dt\right| \\ &\lesssim N_1^{\frac{1}{4}-\varepsilon}\|P_{N_1}w_1\|_X N_2^{-\frac{1}{2}}\|P_{N_2}v_2\|_{U_S^2} \end{aligned}$$

where we applied (48) in the last step. Now, we sum up with respect to $N_1 \leqslant 1$. Finally, we perform the summation of the squared dyadic pieces with respect to $N \sim N_2$. □

4. Proof of the main results

In this section we present the proofs of the main results stated in Section 1. We follow the general approach via the contraction mapping principle, which is well known.

For regular functions, the Cauchy problem (1) on the time interval $(0, T)$ for $0 < T \leqslant \infty$ is equivalent to

$$u(t) = e^{tS}u_0 - \frac{1}{2}I_T(u, u)(t), \quad t \in (0, T). \tag{53}$$

This allows for a generalization to rough functions: Whenever we refer to a solution of (1) on $(0, T)$, the operator equation (53) is assumed to be satisfied.

Proof of Theorem 1.1. Let α_0 be as in (47). We then have $\alpha_0 e^{\cdot S}u_0 \in \dot{Z}^{-\frac{1}{2}}$, which implies that $e^{\cdot S}u_0 \in \dot{Z}^{-\frac{1}{2}}([0,\infty))$ for $u_0 \in \dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$ and

$$\left\|e^{\cdot S}u_0\right\|_{\dot{Z}^{-\frac{1}{2}}([0,\infty))} \leqslant \|u_0\|_{\dot{H}^{-\frac{1}{2},0}}.$$

Let

$$\dot{B}_\delta := \left\{u_0 \in \dot{H}^{-\frac{1}{2},0}\left(\mathbb{R}^2\right) \mid \|u_0\|_{\dot{H}^{-\frac{1}{2},0}} < \delta\right\}$$

for $\delta = (4C+4)^{-2}$, with the constant $C > 0$ from (45). Define

$$D_r := \left\{u \in \dot{Z}^{-\frac{1}{2}}\left([0,\infty)\right) \mid \|u\|_{\dot{Z}^{-\frac{1}{2}}([0,\infty))} \leqslant r\right\},$$

with $r = (4C+4)^{-1}$. Then, for $u_0 \in \dot{B}_\delta$ and $u \in D_r$,

$$\left\|e^{\cdot S}u_0 - \frac{1}{2}I_\infty(u, u)\right\|_{\dot{Z}^{-\frac{1}{2}}([0,\infty))} \leqslant \delta + Cr^2 \leqslant r,$$

due to (45) and Corollary 3.4. Similarly,

$$\begin{aligned}\left\|\frac{1}{2}I_\infty(u_1,u_1)-\frac{1}{2}I_\infty(u_2,u_2)\right\|_{\dot Z^{-\frac12}([0,\infty))} &\leqslant C\big(\|u_1\|_{\dot Z^{-\frac12}([0,\infty))}+\|u_2\|_{\dot Z^{-\frac12}([0,\infty))}\big)\|u_1-u_2\|_{\dot Z^{-\frac12}([0,\infty))}\\ &\leqslant \frac{1}{2}\|u_1-u_2\|_{\dot Z^{-\frac12}([0,\infty))},\end{aligned}$$

hence $\Phi: D_r \to D_r$, $u \mapsto e^{\cdot S}u_0 - \frac{1}{2}I_\infty(u,u)$ is a strict contraction. It therefore has a unique fixed point in D_r, which solves (53). By the implicit function theorem the map $F_+ : \dot B_\delta \to D_r$, $u_0 \mapsto u$ is analytic because the map $(u_0,u) \mapsto e^{\cdot S}u_0 - \frac{1}{2}I_\infty(u,u)$ is analytic. Due to the embedding $\dot Z^{-\frac12}([0,\infty)) \subset C([0,\infty), \dot H^{-\frac12}(\mathbb{R}^2))$ the regularity of the initial data persists under the time evolution. Concerning the results with respect to the negative time axis, we reverse the time $t \mapsto -t$ and apply the same arguments. □

Remark 4.1. Up to now, we only know that the solution u is unique in the subset $D_r \subset \dot Z^{-\frac12}([0,\infty))$. The proof of the uniqueness assertion in the larger space $Z^{-\frac12}([0,T])$ will follow from the results in the subsequent subsection.

Proof of Corollary 1.7. For initial data $u_0 \in \dot H^{-\frac12,0}(\mathbb{R}^2)$, $\|u_0\|_{\dot H^{-\frac12,0}} < \delta$, the solution u which was constructed above satisfies

$$u(t) = e^{tS}\left(u_0 - e^{-\cdot S}\frac{1}{2}I_\infty(u,u)\right)(t), \quad t \in (0,\infty).$$

The existence of the limit $u_0 - e^{-tS}\frac12 I_\infty(u,u)(t) \to u_+$ in $\dot H^{-\frac12,0}(\mathbb{R}^2)$ as $t \to \infty$ follows from Corollary 3.4. The analyticity of the map $V_+ : u_0 \mapsto u_+$ follows from the analyticity of F_+ shown above.

An obvious modification of the above proof also yields persistence of higher initial regularity, in particular if $u_0 \in L^2(\mathbb{R}^2;\mathbb{R})$, then $u(t) \in L^2(\mathbb{R}^2;\mathbb{R})$ for all t. It remains to show that $\|V_+(u_0)\|_{L^2} = \|u_0\|_{L^2}$. By approximation and a direct calculation for smooth, real valued solutions, we easily see that the L^2-norm is conserved. Due to the strong convergence in $\dot H^{-\frac12,0}(\mathbb{R}^2)$ we have weak convergence $e^{-tS}u(t) \rightharpoonup u_+$ in $L^2(\mathbb{R}^2)$ for $t \to \infty$, hence $\|u_+\|_{L^2} \leqslant \|u_0\|_{L^2}$. Let F_- be the flow map with respect to $(-\infty,0)$ according to Theorem 1.1, which is Lipschitz continuous. Because $\lim_{t\to\infty} e^{tS}u_+ - u(t) = 0$ in $\dot H^{-\frac12,0}(\mathbb{R}^2)$, it follows $\lim_{t\to\infty} F_-(-t, e^{tS}u_+) = u_0$ in $\dot H^{-\frac12,0}(\mathbb{R}^2)$. Moreover, due to the L^2 conservation $\|u_+\|_{L^2} = \|F_-(-t,e^{tS}u_+)\|_{L^2}$ we have weak convergence $F_-(-t,e^{tS}u_+) \rightharpoonup u_0$ in $L^2(\mathbb{R}^2)$. Altogether,

$$\|u_0\|_{L^2} \leqslant \lim_{t\to\infty}\big\|F_-\big(-t,e^{tS}u_+\big)\big\|_{L^2} = \|u_+\|_{L^2}.$$

The existence and analyticity of the local inverse W_+ follows from the inverse function theorem, because $V_+(0) = 0$ and by (45) and Corollary 3.4 we observe $DV_+(0) = Id$. □

Proof of Theorem 1.2. For some $\delta > 0$ and $R \geqslant \delta$ we define

$$B_{\delta,R} := \big\{u_0 \in H^{-\frac12,0}\big(\mathbb{R}^2\big) \bigm| u_0 = v_0 + w_0,\ \|v_0\|_{\dot H^{-\frac12,0}} < \delta,\ \|w_0\|_{L^2} < R\big\}.$$

Let $u_0 \in B_{\delta,R}$ with $u_0 = v_0 + w_0$. We have $\chi e^{\cdot S}w_0 \in X$ and $\chi e^{\cdot S}v_0 \in \dot Z^{-\frac12,0}$, which implies that $e^{\cdot S}u_0 \in Z^{-\frac12}([0,1])$ and

$$\big\|e^{\cdot S}u_0\big\|_{Z^{-\frac12}([0,1])} \lesssim \delta + R.$$

We start with the case $R = \delta = (4C+4)^{-2}$, with the constant $C > 0$ from (49). Define

$$D_r := \big\{u \in Z^{-\frac12}\big([0,1]\big) \bigm| \|u\|_{Z^{-\frac12}([0,1])} \leqslant r\big\},$$

with $r = (4C+4)^{-1}$. As above, we use (49) to verify that

$$\Phi: D_r \to D_r, \quad u \mapsto e^{\cdot S}u_0 - \frac{1}{2}I_1(u,u)$$

is a strict contraction, for $u_0 \in \dot{B}_{\delta,R}$. It therefore has a unique fixed point in D_r, which solves (53) on the interval (0, 1). By the implicit function theorem the map $B_{\delta,R} \to D_r$, $u_0 \mapsto u$ is analytic. We also have the embedding $Z^{-\frac{1}{2}}([0,1]) \subset C([0,1]; H^{-\frac{1}{2}}(\mathbb{R}^2))$. Now, we assume that $u_0 \in B_{\delta,R}$ for $R \geqslant \delta = (4C+4)^{-2}$. We define $u_{0,\lambda} = \lambda^2 u_0(\lambda\cdot, \lambda^2\cdot)$. For $\lambda = R^{-2}\delta^2$ we observe $u_{0,\lambda} \in B_{\delta,\delta}$. Therefore we find a solution $u_\lambda \in Z^{-\frac{1}{2},0}([0,1])$ on (0, 1) with $u_\lambda(0) = u_{0,\lambda}$. By rescaling (2) we find a solution $u \in Z^{-\frac{1}{2},0}([0, \delta^6 R^{-6}])$ on $(0, \delta^6 R^{-6})$ with $u(0) = u_0$. We notice that in (49), the left hand side is in the homogeneous space $\dot{Z}^{-\frac{1}{2},0}$, hence all of the above remains valid (or even becomes easier) if we exchange $Z^{-\frac{1}{2},0}([0,1])$ by the smaller space $\dot{Z}^{-\frac{1}{2},0}([0,1])$.

It remains to show the uniqueness claim. Assume that $u_1, u_2 \in Z^{-\frac{1}{2},0}([0,T])$ are two solutions such that $u_1(0) = u_2(0)$. Moreover, we assume that

$$T' := \sup\{0 \leqslant t \leqslant T \mid u_1(t) = u_2(t)\} < T.$$

By a translation in t it is enough to consider $T' = 0$. A combination of (45) and (48) yields the following: Decompose $u_j = v_j + w_j$, where $v_j \in X([0,T])$, $w_j \in \dot{Z}^{-\frac{1}{2},0}([0,T])$ and $w_j(0) = 0$. Then, there exists $C > 0$, such that for all small $0 < \tau \leqslant T'$

$$\begin{aligned}\|u_1 - u_2\|_{Z^{-\frac{1}{2},0}([0,\tau])} &= \left\| \frac{1}{2} I_\tau(u_1,u_1) - \frac{1}{2} I_\tau(u_2,u_2) \right\|_{Z^{-\frac{1}{2},0}([0,\tau])} \\ &\leqslant C\tau^{\frac{1}{4}-\varepsilon}\big(\|v_1\|_{X([0,\tau])} + \|v_2\|_{X([0,\tau])}\big)\|u_1 - u_2\|_{Z^{-\frac{1}{2},0}([0,\tau])} \\ &\quad + C\big(\|w_1\|_{\dot{Z}^{-\frac{1}{2},0}([0,\tau])} + \|w_2\|_{\dot{Z}^{-\frac{1}{2},0}([0,\tau])}\big)\|u_1 - u_2\|_{Z^{-\frac{1}{2},0}([0,\tau])}.\end{aligned}$$

We apply Proposition 2.24, part (ii) and obtain

$$\|u_1 - u_2\|_{Z^{-\frac{1}{2},0}([0,\tau])} \leqslant \frac{1}{2}\|u_1 - u_2\|_{Z^{-\frac{1}{2},0}([0,\tau])},$$

which contradicts the definition of T'. □

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