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Error estimates on homogenization of free boundary velocities in periodic media ☆

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Abstract

In this paper we consider a free boundary problem which describes contact angle dynamics on inhomogeneous surface. We obtain an estimate on convergence rate of the free boundaries to the homogenization limit in periodic media. The method presented here also applies to more general class of free boundary problems with oscillating boundary velocities.

Keywords: Homogenization; Free boundaries; Error estimates; Viscosity solutions

1. Introduction

Consider a bounded domain Ω in $\mathbb{R}^n$ containing $K = B_1(0)$. Let $\Omega_0 = \Omega - K$ and $\Gamma_0 = \partial\Omega$, and let u_0 satisfy

$$-\Delta u_0 = 0 \quad \text{in } \Omega_0, \qquad u_0 = 1 \quad \text{on } K, \quad \text{and} \quad u_0 = 0 \quad \text{on } \Gamma_0.$$

(See Fig. 1.)

Let us define $e_i \in \mathbb{R}^n$, $i = 1, \dots, n$, such that

$$e_1 = (1, 0, \dots, 0),\ e_2 = (0, 1, 0, \dots, 0), \dots, \text{ and } e_n = (0, \dots, 0, 1),$$

and consider a Lipschitz continuous function

$$g : \mathbb{R}^n \to [m, M], \quad g(x + e_i) = g(x) \quad \text{for } i = 1, \dots, n$$

with Lipschitz constant L. For simplicity in the analysis we will work with $m = 1$, $M = 2$ and $L = 10$, but the method in this paper applies to general $m, M > 0$ and L.

In this paper we consider the behavior, as $\epsilon \to 0$, of the viscosity solutions $u^\epsilon \geqslant 0$ of the following problem

$$(\mathrm{P})_\epsilon \quad \begin{cases} -\Delta u^\epsilon = 0 & \text{in } \{u^\epsilon > 0\}, \\ u^\epsilon_t = |Du^\epsilon|(|Du^\epsilon| - g(x/\epsilon)) & \text{on } \partial\{u^\epsilon > 0\} \end{cases}$$

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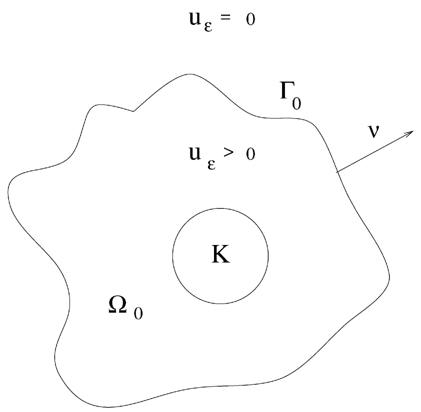

Fig. 1. Initial setting of the problem.

in $Q = (\mathbb{R}^n - K) \times (0, \infty)$ with initial data u_0 and smooth boundary data $f(x,t) > 0$ on $\partial K \times [0, \infty)$. Here Du denotes the spatial derivative of u.

We refer to $\Gamma_t(u^\epsilon) := \partial\{u^\epsilon(\cdot,t) > 0\} - \partial K$ as the *free boundary* of u^ϵ and to $\Omega_t(u^\epsilon) := \{u^\epsilon(\cdot,t) > 0\}$ as the *positive phase* of u^ϵ at time t. Note that if u^ϵ is smooth up to the free boundary, then the free boundary moves with outward normal velocity $V = u_t^\epsilon/|Du^\epsilon|$, and therefore the second equation in $(\mathrm{P})_\epsilon$ implies that

$$V = \left|Du^\epsilon\right| - g\left(\frac{x}{\epsilon}\right) = Du^\epsilon \cdot (-\nu) - g\left(\frac{x}{\epsilon}\right)$$

where $\nu = \nu_{(x,t)}$ denotes the outward normal vector at $x \in \Gamma_t(u)$ with respect to $\Omega_t(u)$.

A weak notion of solution is necessary since, due to the collision, neck-pinching or shrinking of free boundary parts, smooth solutions cease to exist in finite time even with smooth initial data and smooth velocity (see Remark 2). For the definition of viscosity solutions we refer to Section 2.

$(\mathrm{P})_\epsilon$ is a simplified model to describe contact line dynamics of liquid droplets on an irregular surface (see [2]). Here $u(x,t)$ denotes the height of the droplet. Heterogeneities on the surface, represented by $g(\frac{x}{\epsilon})$ in $(\mathrm{P})_\epsilon$, result in contact lines with a fine scale structure that may lead to pinning of the interface and hysteresis of the overall fluid shape.

For literature on homogenization of nonlinear PDEs and free boundary problems, we refer to [1] and [6].

Below we recall the main result obtained in [6].

Theorem 1.1. *(Theorem 0.1, [6].) Let u^ϵ be a viscosity solution of $(\mathrm{P})_\epsilon$ with initial data u_0 and boundary data f. Then there exists a continuous function*

$$r(q) = \mathbb{R}^n - \{0\} \to [-2, \infty), \quad r \text{ increases in } |q|,$$

such that the following holds:

(a) *If u_{ϵ_k} locally uniformly converges to u as $\epsilon_k \to 0$, then u is a viscosity solution of*

$$(\mathrm{P}) \quad \begin{cases} -\Delta u = 0 & \text{in } \{u > 0\}, \\ u_t = |Du| r(Du) & \text{on } \partial\{u > 0\} \end{cases}$$

in Q with initial data u_0 and boundary data f on ∂K.

(b) *If u is the unique viscosity solution of* (P) *in Q with initial data u_0 and boundary data f on ∂K, then the whole sequence $\{u_\epsilon\}$ locally uniformly converges to u.*

Uniqueness of u holds if the initial data satisfies one of the following (see Theorem 2.8 and the remark below):

(A) $\Omega = \Omega_0 \cup K$ is star-shaped with respect to a small ball $B_r(0)$;

(B) Γ_0 is locally Lipschitz and $|Du_0| > 2$ on Γ_0;
(C) Γ_0 is locally Lipschitz and $|Du_0| < 1$ on Γ_0;
where Du_0 on Γ_0 is taken as the limit from Ω_0.

(In case of (A), $\Omega_t(u)$ stays star-shaped with respect to $B_r(0)$ for $t > 0$. In case of (B) u strictly increases in time, and in case of (C) u strictly decreases in time for all times.)

The goal of this paper is to refine the analysis performed in [6] to provide a quantitative estimate on the distance between $\Omega_t(u^\epsilon)$ and $\Omega_t(u)$ at each time. The main result (Corollary 4.2) can be summarized as below:

$$\text{For sufficiently small } \epsilon > 0,\ \Omega_t(u^\epsilon) \text{ stays in } O(\epsilon^{1/70})\text{-neighborhood of } \Omega_t(u) \text{ for } 0 \leqslant t \leqslant \epsilon^{-1/300} \text{ if one of conditions (A)–(C) holds for the initial data.} \tag{1.1}$$

Such estimate is, to the best of author's knowledge, new for homogenization of free boundary problems. Below we sketch an outline of the paper. In Section 2 we recall the notion of viscosity solutions and their properties. In particular *comparison principle* (Theorem 2.6) is used frequently in the paper. In Section 3 we improve existing results obtained in [6] to derive Proposition 3.5 and Corollary 3.6. In Section 4 we state the main result (Theorem 4.1) and prove it with the help of Corollary 3.6 and Proposition 4.3. In Section 5 we prove Proposition 4.3, and thus finishing the proof of Theorem 4.1. We finish with Section 6, the corresponding result are stated for expanding free boundary problem $(\mathrm{P2})^\epsilon$: for this problem (1.1) holds for general initial data.

Remark 1. The analysis presented here and in [5,6] can be generalized to free boundary problems of the type

$$\begin{cases} (u_t) - \Delta u = 0 & \text{in } \{u > 0\}, \\ V = G(Du, \frac{x}{\epsilon}) & \text{on } \partial\{u > 0\} \end{cases}$$

where $G(p, y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is (i) Lipschitz continuous, (ii) strictly increasing with respect to $|p|$ and (iii) satisfies

$$b|p|\frac{\partial G}{\partial |p|} - aG \geqslant \left|\frac{\partial G}{\partial y}\right|$$

for some constants a and $b > 0$. For example, in $(\mathrm{P})^\epsilon$ we have

$$G(p, y) = |p| - g(y) \quad \text{and} \quad a = b = \frac{\operatorname{Lip} g}{\inf g}.$$

In $(\mathrm{P2})^\epsilon$ given in Section 6 we have

$$G(p, y) = g(y)|p| \quad \text{and} \quad a = 0, \quad b = \frac{\operatorname{Lip} g}{\inf g}.$$

2. Notations and viscosity solutions

We begin by recalling existence and uniqueness of viscosity solutions obtained in [6] for a general class of free boundary problem, including both (P) and $(\mathrm{P})^\epsilon$.

Let us consider a continuous function

$$F(q, y) : \big(\mathbb{R}^n - \{0\}\big) \times \mathbb{R}^n \to [-2, \infty)$$

such that

(a) F increases in $|q|$, $|q| - 2 \leqslant F(q, y, \nu) \leqslant |q| - 1$.
(b) $F(q, y + e_k) = F(q, y)$ for $k = 1, \ldots, n$.
(c) $|F(q, y_1) - F(q, y_2)| \leqslant L|y_1 - y_2|$ for $y_1, y_2 \in \mathbb{R}^n$.

Let $\Sigma \subset \mathbb{R}^n \times [0, \infty)$ be a space–time domain with smooth boundary, and consider the free boundary problem

$$(\tilde{\mathrm{P}})_\epsilon \quad \begin{cases} -\Delta u^\epsilon = 0 & \text{in } \{u^\epsilon > 0\}, \\ u^\epsilon_t - |Du^\epsilon| F(Du^\epsilon, \frac{x}{\epsilon}) = 0 & \text{on } \partial\{u^\epsilon > 0\} \end{cases}$$

in Σ with appropriate boundary data.

Let $\Sigma(s) := \Sigma \cap \{t = s\}$. For a nonnegative real valued function $u(x,t)$ defined for $(x,t) \in \Sigma$, define

$$\Omega(u) = \{(x,t) \in \Sigma\colon\ u(x,t) > 0\}, \qquad \Omega_t(u) = \{x\colon\ (x,t) \in \Sigma\colon\ u(x,t) > 0\};$$
$$\Gamma(u) = \partial\Omega(u) - \partial\Sigma, \qquad \Gamma_t(u) = \partial\Omega_t(u) - \partial\Sigma(t).$$

Below we define viscosity solutions of $(\tilde{\mathrm{P}})_\epsilon$.

Definition 2.1. A nonnegative, upper semi-continuous function u defined in Σ is a *viscosity subsolution* of $(\tilde{\mathrm{P}})_\epsilon$ if

(a) for each $a < T < b$ the set $\overline{\Omega(u)} \cap \{t \leqslant T\} \cap \Sigma$ is bounded; and
(b) for every $\phi \in C^{2,1}(\Sigma)$ such that $u - \phi$ has a local maximum in $\overline{\Omega(u)} \cap \{t \leqslant t_0\} \cap \Sigma$ at (x_0, t_0),
   (i) if $u(x_0, t_0) > 0$, then $-\Delta\phi(x_0, t_0) \leqslant 0$.
   (ii) if $(x_0, t_0) \in \Gamma(u)$, $|D\phi|(x_0, t_0) \neq 0$ and $-\Delta\phi(x_0, t_0) > 0$,

then

$$\left(\phi_t - |D\phi| F\left(D\phi, \frac{x_0}{\epsilon}\right)\right)(x_0, t_0) \leqslant 0.$$

Note that, because u is only upper semi-continuous, there may be points of $\Gamma(u)$ at which u is positive.

Definition 2.2. A nonnegative, lower semi-continuous function v defined in Σ is a *viscosity supersolution* of $(\tilde{\mathrm{P}})_\epsilon$ if for every $\phi \in C^{2,1}(\Sigma)$ such that $v - \phi$ has a local minimum in $\Sigma \cap \{t \leqslant t_0\}$ at (x_0, t_0), then

(i) if $v(x_0, t_0) > 0$, then $-\Delta\phi(x_0, t_0) \geqslant 0$.
(ii) if $(x_0, t_0) \in \Gamma(v)$, $|D\phi|(x_0, t_0) \neq 0$ and $-\Delta\phi(x_0, t_0) < 0$,

then

$$\left(\phi_t - |D\phi| F\left(D\phi, \frac{x_0}{\epsilon}\right)\right)(x_0, t_0) \geqslant 0.$$

Let K, Ω_0, Γ_0, f, u_0 and Q be as given in the introduction.

Definition 2.3. u is a viscosity subsolution of $(\tilde{\mathrm{P}})_\epsilon$ in Q with initial data u_0 and fixed boundary data $f > 0$ if

(a) u is a viscosity subsolution of $(\tilde{\mathrm{P}})_\epsilon$ in Q,
(b) u is upper semicontinuous in $\bar{Q}$, $u = u_0$ at $t = 0$ and $u \leqslant f$ on ∂K.
(c) $\overline{\Omega(u)} \cap \{t = 0\} = \overline{\Omega(u_0)}$.

Definition 2.4. u is a viscosity supersolution of $(\tilde{\mathrm{P}})_\epsilon$ in Q with initial data u_0 and boundary data f if u is a viscosity supersolution in Q, lower semicontinuous in $\bar{Q}$ with $u = u_0$ at $t = 0$ and $u \geqslant f$ on ∂K.

For a nonnegative real valued function $u(x,t)$ in $\Sigma \subset \mathbb{R}^n \times [0, \infty)$ we define

$$u^*(x,t) := \limsup_{(\xi,s)\in\Sigma\to(x,t)} u(\xi, s),$$

and

$$u_*(x,t) := \liminf_{(\xi,s)\in\Sigma\to(x,t)} u(\xi, s).$$

Note that u^* is upper semicontinuous and u_* is lower semicontinuous.

Definition 2.5. u is a viscosity solution of $(\tilde{\mathrm{P}})_\epsilon$ (in Q with initial data u_0 and boundary data f) if u is a viscosity supersolution and u^* is a viscosity subsolution of $(\tilde{\mathrm{P}})_\epsilon$ (in Q with initial data u_0 and boundary data f).

We say that a pair of functions $u_0, v_0 : \bar{D} \to [0,\infty)$ are (*strictly*) *separated* (denoted by $u_0 \prec v_0$) in $D \subset \mathbb{R}^n$ if

(i) the support of u_0, $\mathrm{supp}(u_0) = \overline{\{u_0 > 0\}}$ restricted in $\bar{D}$ is compact and
(ii) $u_0(x) < v_0(x)$ in $\mathrm{supp}(u_0) \cap \bar{D}$.

Theorem 2.6 *(Comparison principle, Theorem 1.7, [6]). Let h_1, h_2 be respectively viscosity sub- and supersolutions of* $(\tilde{\mathrm{P}})^\epsilon$ *in Σ. If $h_1 \prec h_2$ on the parabolic boundary of Σ, then $h_1(\cdot,t) \prec h_2(\cdot,t)$ in Σ.*

Theorem 2.7. *(Theorem 1.8, [6].) Suppose one of the conditions* (A)–(C) *holds for u_0. Then there exists a unique solution of* (P) *in Q with initial data u_0 and boundary data* 1.

Lemma 2.8. *(Lemma 1.9, [6].)*

(a) *Let u be a supersolution of* (P) *or* $(\mathrm{P})^\epsilon$ *in Q with fixed boundary data* 1. *Then $\Gamma(u)$ does not "jump inward" in time: for any point $x_0 \in \Gamma_{t_0}(u)$ with $t_0 > 0$ there exists a sequence of points $(x_n, t_n) \in \{u = 0\}$ such that $t_n < t_0$ and $(x_n, t_n) \to (x_0, t_0)$.*
(b) *Let u is a subsolution of* (P) *or* $(\mathrm{P})^\epsilon$ *in Q with fixed boundary data* 1. *Then $\Gamma(u)$ does not "jump outward" in time: for any point $x_0 \in \Gamma_{t_0}(u)$ with $t_0 > 0$ there exists a sequence of points $(x_n, t_n) \in \bar{\Omega}_t(u)$ such that $t_n < t_0$ and $(x_n, t_n) \to (x_0, t_0)$.*

Proof. 1. To prove (a), suppose that $x_0 \in \Gamma_{t_0}(u)$. If (a) fails for x_0, then $B_r(x_0) \subset \Omega_t(u)$ for $t_0 - r \leqslant t < t_0$ for some $r > 0$. On the other hand there exists $y_0 \in B_{r/2}(x_0)$ such that $u(y_0, t_0) > 2c_0 > 0$ for some $c_0 > 0$. Since u is lower semicontinuous, $u \geqslant c_0 > 0$ in $B_\delta(y_0) \times [t_0 - \delta, t_0]$ for some $0 < \delta < r/2$. Consider a barrier function $\phi(x,t)$ in

$$\Sigma := \big(\mathbb{R}^n - B_\delta(y_0)\big) \times [t_0 - \delta/2, t_0]$$

such that

$$\begin{cases} -\Delta\phi(\cdot,t) = 0 & \text{in } B_{r-2(t-t_0+\delta/2)}(x_0) - B_\delta(x_0), \\ \phi(\cdot,t) = 0 & \text{on } \partial B_{r-2(t-t_0+\delta/2)}(x_0), \\ \phi(\cdot,t) = c_0 & \text{on } \partial B_\delta(x_0). \end{cases}$$

Note that

$$\frac{\phi_t}{|D\phi|} = V = -2 < |D\phi| - 2 \leqslant r(D\phi) \quad \text{on } \Gamma(\phi).$$

Hence ϕ is a subsolution of both (P) and $(\mathrm{P})^\epsilon$ in Σ. It follows from Theorem 2.6 that $\phi \leqslant u$ in Σ, but this means that $u(\cdot, t_0) > 0$ in $B_{r/2}(x_0)$, contradicting the fact that $x_0 \in \Gamma_{t_0}(u)$.

2. The argument to prove (b) proceeds similarly. Suppose $x_0 \in \Gamma_{t_0}(u)$ and $B_r(x_0) \cap \bar{\Omega}_t(u) = \emptyset$ for $t_0 - \delta \leqslant t < t_0$. We may choose $r < \delta$. Let $r(t) := (t_0 - t)/(2r^2) + r/2$. Consider a barrier function $\phi(x,t)$ in

$$\Sigma := B_{2r}(x_0) \times \big[t_0 - r^4, t_0\big]$$

such that

$$\begin{cases} -\Delta\phi(\cdot,t) = 0 & \text{in } B_{2r}(x_0) - B_{r(t)}(x_0), \\ \phi(\cdot,t) = 0 & \text{on } \partial B_{r(t)}(x_0), \\ \phi(\cdot,t) = 1 & \text{on } \partial B_{2r}(x_0). \end{cases}$$

Note that in Σ we have $|D\phi| \leqslant C/r$ with a dimensional constant C. Hence if r is chosen sufficiently small, then

$$\frac{\phi_t}{|D\phi|} = V = -r'(t) = \frac{1}{2r^2} \geqslant |D\phi| \geqslant r(D\phi) \quad \text{on } \Gamma(\phi),$$

and thus ϕ is a supersolution of both (P) and $(P)^\epsilon$ in Σ. Again Theorem 2.6 yields that $u \leqslant \phi$ in Σ, but this means that $u(\cdot, t_0) \equiv 0$ in $B_{r/2}(x_0)$, contradicting the fact that $x_0 \in \Gamma_{t_0}(u)$. □

Remark 2. Note that above lemma does not guarantee the continuity of the free boundary in time. In fact free boundary parts may instantly disappear, for example in $n = 1$ if we superpose two radially symmetric functions (see the introduction in [4]). For $n > 1$ discontinuity of the free boundary also happens when the free boundary contains a slit in the middle of its positive phase: in this case the slit instantly disappears and at this time the discontinuity of the solution occurs as well. The discontinuity of the free boundary also happens if a portion of the positive phase gets disconnected by a neck pinching and instantly disappears. Hence the definition of the viscosity solution with semi-continuous sub and supersolutions are indeed necessary for $(\tilde{P})_\epsilon$.

For $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, let us denote the space and space–time balls by

$$B_r(x) := \{y \in \mathbb{R}^n\colon |y - x| \leqslant r\}$$

and

$$B_r^{(n+1)}(x, t) := \{(y, s) \in \mathbb{R}^n \times \mathbb{R}\colon |(y, s) - (x, t)| \leqslant r\}.$$

The following lemma will be used frequently in our analysis. The proof is parallel to that of Lemma 3.5 in [3].

Lemma 2.9.

(a) *If u is a viscosity subsolution of $(\tilde{P})_\epsilon$ in Q, then the sup-convolution*

$$\tilde{u}(x, t) := \sup_{y \in B_{m\epsilon - \delta t}(x)} u(y, t)$$

is a viscosity subsolution of $(\tilde{P})_\epsilon$ in

$$Q_{c,\delta} := \bigcup_{\{0 \leqslant t \leqslant m\epsilon/\delta\}} \big(\big(\mathbb{R}^n - (1 + m\epsilon - \delta t)K\big) \times t\big)$$

with $F(Du, \frac{x}{\epsilon})$ replaced by $F(Du, \frac{x}{\epsilon}) + Lm - \delta$.

(b) *If u is a supersolution of $(\tilde{P})_\epsilon$ in Q then the inf-convolution*

$$\tilde{u}(x, t) = \inf_{y \in B_{m\epsilon - \delta t}(x)} u(y, t)$$

is a viscosity supersolution of $(\tilde{P})_\epsilon$ in $Q_{c,\delta}$ with $F(Du, \frac{x}{\epsilon})$ replaced by $F(Du, \frac{x}{\epsilon}) - Lm + \delta$.

(a), (b) also holds with $B_{m\epsilon - \delta t}(x)$ replaced with space–time balls $B^{(n+1)}_{m\epsilon - \delta t}(x)$.

3. Properties of free boundaries in obstacle problems

3.1. Introduction of the obstacle problem and statement of previous results

First we recall some of the results obtained in [6]. These results address solutions of "obstacle problems" which we introduce below. For given nonzero vector $q \in \mathbb{R}^n$ and $r \in [-2, \infty)$, we denote $\nu = \frac{q}{|q|}$ and define

$$P_{q,r}(x, t) := |q|(rt - x \cdot \nu)_+, \qquad l_{q,r}(t) = \{x \in \mathbb{R}^n\colon rt = x \cdot \nu\}.$$

Note that the free boundary of $P_{q,r}$, $\Gamma_t(P_{q,r}) := l_{q,r}(t)$, propagates with normal velocity r with its outward normal direction ν.

Next we construct a domain with which the obstacle problems will be defined. In $e_1 - e_n$ plane, consider a vector $\mu = e_n + \sqrt{3}e_1$. Let l to be the line which is parallel to μ and passes through $3e_1$. Rotate l with respect to e_n-axis and

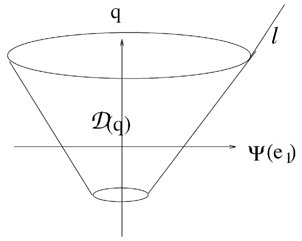

Fig. 2. The spatial domain for test functions.

define $\mathcal{D}$ to be the region bounded by the rotated image and $\{x\colon -1 \leqslant x \cdot e_n \leqslant r\}$ (see Fig. 2). For any nonzero vector $q \in \mathbb{R}^n$, let us define $\mathcal{D}(q) := \Psi(\mathcal{D})$, where Ψ is a rotation in $\mathbb{R}^n$ which maps e_n to $q/|q|$. Let us define

$$\mathcal{O} = \bigcup_{0 \leqslant t \leqslant 1} \big((1+3t)\mathcal{D}(q) \times \{t\}\big).$$

Let us define the space–time domain $Q_1 := \mathcal{D}(q) \times [0,1]$ for $r \geqslant 0$, and $Q_1 := \mathcal{O}$ for $r < 0$.

Next we define the maximal subsolution below $P_{q,r}$ and minimal supersolution above $P_{q,r}$ in Q_1:

$$\bar{u}_{\epsilon;q,r} := \big(\sup\big\{u\colon \text{ a subsolution of (P)}^\epsilon \text{ in } Q_1 \text{ with } u \leqslant P_{q,r}\big\}\big)^*,$$
$$\underline{u}_{\epsilon;q,r} := \big(\inf\big\{v\colon \text{ a supersolution of (P)}^\epsilon \text{ in } Q_1 \text{ with } u \geqslant P_{q,r}\big\}\big)_*.$$

Remark 3. Note that then $\bar{u}_{\epsilon;q,r}(\cdot,t)$ and $\underline{u}_{\epsilon;q,r}(\cdot,t)$ are both harmonic in their positive phases. The main reason for defining a rather complicated domain Q_1 is to guarantee that the free boundary of $\underline{u}_{\epsilon;q,r}$ and $\bar{u}_{\epsilon;q,r}$ does not detach too fast from $P_{q,r}$ as it gets away from the lateral boundary of Q_1 (see Lemma 2.4 in [6]).

Below we recall properties of $\bar{u}_{\epsilon;q,r}$ and $\underline{u}_{\epsilon;q,r}$ which we need later in the paper.

Lemma 3.1. *(Lemma 2.5, [6].)*

(a) $\bar{u}_{\epsilon;q,r}$ *is a subsolution of* (P)$_\epsilon$ *in* Q_1 *with* $\bar{u}_{\epsilon;q,r} \leqslant P_{q,r}$ *in* $\bar{Q}_1$ *and* $\bar{u}_{\epsilon;q,r} = P_{q,r}$ *on the parabolic boundary of* Q_1. *Moreover* $(\bar{u}_{\epsilon;q,r})_*$ *is a solution of* (P)$_\epsilon$ *away from* $\Gamma(\bar{u}_{\epsilon;q,r}) \cap l_{q,r}$.
(b) $\underline{u}_{\epsilon;q,r}$ *is a supersolution of* (P)$_\epsilon$ *in* Q_1 *with* $\underline{u}_{\epsilon;q,r} \geqslant P_{q,r}$ *in* $\bar{Q}_1$ *and* $\underline{u}_{\epsilon;q,r} = P_{q,r}$ *on the parabolic boundary of* Q_1. *Moreover* $\underline{u}_{\epsilon;q,r}$ *is a solution of* (P)$_\epsilon$ *away from* $\Gamma(\underline{u}_{\epsilon;q,r}) \cap l_{q,r}$.
(c) $\bar{u}_{\epsilon;q,r}$ *decreases in time if* $r<0$. $\underline{u}_{\epsilon;q,r}$ *increases in time if* $r>0$.

Lemma 3.2. *(Corollary 2.6, [6].) For any given nonzero vector* $q \in \mathbb{R}^n$, $\nu = \frac{q}{|q|}$ *and for any* $a \in [0,1]$, *there is* $\eta \in \mathbb{R}^n$ *such that* $a\nu + \eta \in \epsilon\mathbb{Z}^n$, $\eta \cdot \nu \geqslant \frac{1}{2}|\eta|$ *and* $\epsilon \leqslant |\eta| < 3\epsilon$. *For this* η *the following holds*:

(a) *For* $r>0$

$$\bar{u}_{\epsilon;q,r}(x + a\nu + \eta, t+\tau) \leqslant \bar{u}_{\epsilon;q,r}(x,t) \tag{3.1}$$

for $0 \leqslant \tau \leqslant r^{-1}(a + \eta\cdot\nu)$ *and*

$$\underline{u}_{\epsilon;q,r}(x + a\nu + \eta, t+\tau) \geqslant \underline{u}_{\epsilon;q,r}(x,t) \quad \text{in } Q_1 \tag{3.2}$$

for $\tau \geqslant r^{-1}(a + \eta\cdot\nu)$.
(b) *For* $r<0$ *the above inequalities are true with* ν, η *and* r *replaced by* $-\nu$, $-\eta$ *and* $|r|$, *and the range of* τ *for* $\bar{u}_{\epsilon;q,r}$ *and* $\underline{u}_{\epsilon;q,r}$ *interchanged.*

For a nonzero vector $q \in \mathbb{R}^n$ we set $\nu = \frac{q}{|q|}$ and define the *contact sets*

$$\underline{A}_{\epsilon;q,r} := \big(\Gamma(\underline{u}_{\epsilon;q,r}) \cap l_{q,r}\big) \cap \left(B_{1/2}\left(\frac{1}{2}r\nu\right) \times [1/2, 1]\right)$$

and

$$\bar{A}_{\epsilon;q,r} := \left(\Gamma(\bar{u}_{\epsilon;q,r}) \cap l_{q,r}\right) \cap \left(B_{1/2}\left(\frac{1}{2}r\nu\right) \times [1/2, 1]\right).$$

As the speed r of the obstacle $P_{q,r}$ increases, the *contact set from above* ($\underline{A}_{\epsilon;q,r}$) increases, and the *contact set from below* ($\bar{A}_{\epsilon;q,r}$) decreases. The free boundary speed $r(q)$ in the homogenization limit turns out to be the unique speed with which both contact sets are (in the limiting sense) nonempty:

Lemma 3.3. *(Lemma 3.12, [6].)*

$$r(q) = \inf\{r\colon\ \underline{A}_{\epsilon;q,r} \neq \emptyset \text{ for } \epsilon \leqslant \epsilon_0 \text{ with some } \epsilon_0 > 0\}$$

$$= \sup\{r\colon\ \bar{A}_{\epsilon;q,r} \neq \emptyset \text{ for } \epsilon \leqslant \epsilon_0 \text{ with some } \epsilon_0 > 0\}.$$

Moreover $\underline{A}_{\epsilon;q,r(q)}$ *and* $\bar{A}_{\epsilon;q,r(q)}$ *are both nonempty for any* $0 < \epsilon < 1/10$.

Remark 4. From scaling arguments it follows that if $\underline{A}_{\epsilon_0;q,r}$ ($\bar{A}_{\epsilon_0;q,r}$) is nonempty, then so is $\underline{A}_{\epsilon;q,r}(\bar{A}_{\epsilon;q,r})$ for $\epsilon \geqslant \epsilon_0$.

3.2. Improved estimates

For $A, B \subset Q_1$, let us define

$$d(A, B) = \inf\{d(x, y)\colon\ x \in A,\ y \in B\}.$$

In [6] we showed that $\Gamma(\bar{u}_{\epsilon;q,r})$ and $\Gamma(\underline{u}_{\epsilon;q,r})$, with $r = r(q)$ given in (2.1), are at most $M\epsilon$-away from $l_{q,r}(t)$ where M depends on several parameters, including the size of q (see Propositions 2.8 and 2.9, [6]). This *flatness constant* M is then used in the main proposition (Propositions 3.8 and 3.11 in [6]) to measure the free boundary detachment from the obstacle, when the speed of the obstacle is not the correct one for the homogenization limit. For the purpose of our investigation, it is necessary to refine the estimate on M such that the size of M it only depends on one perturbation parameter γ. This is what we will carry out below:

Lemma 3.4. *Let* $q \in \mathcal{R}^n - \{0\}$ *and* $r = r(q)$. *Then there exist dimensional constants* $0 < \gamma(n) < 1 < C(n)$ *such that for* $0 < \gamma < \gamma(n)$ *the following is true*:

(a) *If* $r_1 = (1-\gamma)r$ *and* $q_1 = (1-\gamma)q$, *then*

$$d\left(\Gamma(\underline{u}_{\epsilon;q_1,r_1}), l_{q_1,r_1}\right) < \frac{C(n)\epsilon}{\gamma}.$$

(b) *If* $r_2 = (1+\gamma)r$ *and* $q_2 = (1+\gamma)q$, *then*

$$d\left(\Gamma(\bar{u}_{\epsilon;q_2,r_2}), l_{q_2,r_2}\right) < \frac{C(n)\epsilon}{\gamma}.$$

Proof. The general idea for the proof of, for example (a), is the following: since $\underline{A}_{\epsilon;q,r}$ is nonempty and the free boundary velocity of $\Gamma(\underline{u}_{\epsilon;q,r})$ is increasing with respect to $|D\underline{u}_{\epsilon;q,r}|$, the size of $\underline{u}_{\epsilon;q,r}$ near $l_{q,r}$ should stay small: otherwise $\Gamma(\underline{u}_{\epsilon;q,r})$ will completely detach from $l_{q,r}$. Now suppose part of $\Gamma(\underline{u}_{\epsilon;q,r})$ is trying to get away from $l_{q,r}$. Since u is already small near $l_{q,r}$ and is harmonic in its positive set, $|D\underline{u}_{\epsilon;q,r}|$ is very small near the far away part of $\Gamma(\underline{u}_{\epsilon;q,r})$. This and the free boundary motion law forces $\Gamma(\underline{u}_{\epsilon;q,r})$ recede, putting it closer to $l_{q,r}$. This heuristic argument suggests that $\Gamma(\underline{u}_{\epsilon;q,r})$ cannot be too far away from $l_{q,r}$ to begin with. Unfortunately the rigorous proof of above reasoning is rather complicated, and we will divide the proof into several steps. Observe that by scaling law

$$r\left((1-\gamma)q\right) \leqslant (1-\gamma)r(q) \quad \text{and} \quad r\left((1+\gamma)q\right) \geqslant (1+\gamma)r(q),$$

and thus both $\underline{A}_{\epsilon;q_1,r_1}$ and $\bar{A}_{\epsilon;q_2,r_2}$ are nonempty for $0 < \epsilon < 1/2$. Also observe that it is enough to prove the lemma for $r^{-1}\epsilon \leqslant t \leqslant 1$.

1. Let $\nu := \frac{q}{|q|}$. We first prove (a) in the case $r \leqslant 0$. We begin by claiming that

$$\underline{u}_{\epsilon;q_1,r_1}(\cdot,t) \leqslant C\epsilon \quad \text{on } D := \{x\colon 0 \leqslant x\cdot\nu \geqslant rt - 2\epsilon\}. \tag{3.3}$$

Suppose our claim fails with $r < 0$. Then $\underline{u}_{\epsilon;q_1,r_1}(x_0,t) > C\epsilon$ for some $x_0 \in D$. By lower semicontinuity, we then have $\underline{u}_{\epsilon;q_1,r_1}(\cdot,t) \geqslant C\epsilon$ in a small ball $B_\delta(x_0)$, $\delta > 0$.

Choose a lattice vector $\xi \in \epsilon\mathcal{Z}^n$ such that $|\xi - (\xi\cdot\nu)\nu| \leqslant 2\epsilon$ and $\xi\cdot\nu = -10\epsilon$. Due to Lemma 3.2, we have

$$\underline{u}_{\epsilon;q_1,r_1}(x+\xi,t) \geqslant \underline{u}_{\epsilon;q,r}(x,t_0) \quad \text{in } B_{1/2}(0)\times[t_0,t_0+5\epsilon].$$

Hence

$$\underline{u}_{\epsilon;q_1,r_1} \geqslant C\epsilon \quad \text{in } B_\delta(y_0)\times[t_0,t_0+5\epsilon],\ y_0 = x_0+\xi.$$

Next let $r(t) := 4(t-t_0)+\delta/2$, $C_1 := c(n)C$ where $c(n)$ is a small dimensional constant to be determined, and construct a barrier function $\phi(x,t)$ solving

$$\begin{cases} -\Delta\phi(\cdot,t) = 0 & \text{in } B_{2r(t)}(y_0) - B_{r(t)}(y_0),\\ \phi = C_1\epsilon & \text{in } B_{r(t)}(y_0)\times[t_0,t_0+5\epsilon],\\ \phi(\cdot,t) = 0 & \text{in } \mathbb{R}^n - B_{2r(t)}(y_0). \end{cases}$$

If C is sufficiently large such that $|D\phi| > 6$ on $\Gamma(\phi)$ for $t_0 \leqslant t \leqslant t_0+5\epsilon$, then

$$\frac{\phi_t}{|D\phi|} = r'(t) = 4 \leqslant |D\phi| - 2.$$

Hence ϕ is a subsolution of $(\mathrm{P})_\epsilon$ in

$$\Sigma := \bigcup_{t_0\leqslant t\leqslant t_0+5\epsilon} \big(\mathbb{R}^n - B_{r(t)}(y_0)\big)\times t.$$

2. In the following paragraph we show that

$$\phi \leqslant \underline{u}_{\epsilon;q_1,r_1} \quad \text{in } \Sigma. \tag{3.4}$$

Proof of (3.4). By construction $\phi \leqslant \underline{u}_{\epsilon;q_1,r_1}$ in $\Sigma\cap\{t=t_0\}$. Next observe that, if $\underline{u}_{\epsilon;q_1,r_1}(\cdot,t)$ is positive in $B_{\frac{3}{2}r(t)}(y_0)$, by interior Harnack inequality for harmonic functions applied to $\underline{u}_{\epsilon;q,r}(\cdot,t)$ in $B_{\frac{3}{2}r(t)}(y_0)$ yields that

$$\underline{u}_{\epsilon;q_1,r_1}(\cdot,t) \geqslant C_1\epsilon = \phi \quad \text{in } B_{r(t)}(y_0), \tag{3.5}$$

where $C_1 = c(n)C$ with $c(n)$ a dimensional constant.

On the other hand, suppose that (3.5) holds for $t_0 \leqslant t < s$ for some $t_0 \leqslant s \leqslant t_0+5\epsilon$. Then we claim that

$$\underline{u}_{\epsilon;q_1,r_1} > 0 \quad \text{in } \bigcup_{t_0\leqslant t\leqslant s} B_{2r(t)}(y_0)\times\{t\}.$$

To see this, begin by applying Theorem 2.6 to ϕ and $\underline{u}_{\epsilon;q_1,r_1}$ in Σ to yield $\phi \leqslant \underline{u}_{\epsilon;q_1,r_1}$ in $\Sigma\cap\{t_0\leqslant t<s\}$. As a consequence $B_{2r(t)}(y_0) \subset \Omega_t(\underline{u}_{\epsilon;q_1,r_1})$ for $t<s$. Now Lemma 2.8 and the continuity of $r(t)$ yields that

$$B_{\frac{3}{2}r(t)}(y_0) \subset \Omega_t(\underline{u}_{\epsilon;q_1,r_1}) \quad \text{for } s\leqslant t\leqslant t+\delta_0 \text{ for some } \delta_0 > 0.$$

Thus (3.5) holds for $t_0 \leqslant t \leqslant s+\delta_0$. This argument states that (3.5) holds for all times $t_0 \leqslant t \leqslant t_0+5\epsilon$, and as a consequence $\phi \leqslant \bar{u}_{\epsilon;q_1,r_1}$ in Σ. □

(3.4) states, in particular,

$$\underline{u}_{\epsilon;q_1,r_1}(x,t_0+5\epsilon) > 0 \quad \text{in } B_{20\epsilon}(y_0) \supset B_{8\epsilon}(x_0).$$

Observe that, by definition of $\underline{u}_{\epsilon;q_1,r_1}$,

$$\frac{1}{2}\underline{u}_{\epsilon;q_1,r_1}(2x,2t) \leqslant \underline{u}_{\epsilon/2;q_1,r_1}(x-\eta,t+\tau) \quad \text{in } \frac{1}{2}Q_1 + (\eta,-\tau) \tag{3.6}$$

when $\tau > 0$ and $\eta \in \epsilon\mathbb{Z}^n$ satisfies $|\eta| \leqslant \frac{1}{2}$ and $\eta \cdot \nu \geqslant |r_1|\tau$. In particular it follows that

$$\underline{A}_{\epsilon/2;q_1,r_1} = \emptyset,$$

contradicting the fact that $r_1 \geqslant r(q_1)$. We have shown (3.3).

3. So far we have shown that u is small near $l_{q,r}$. The next step is to show that $|Du|$ is small on free boundary parts far away from $l_{q,r}$. To do this we need to regularize the free boundary in some sense: this is done via sup-convolution as follows. Define

$$v(x,t) := \sup_{y \in B_{\gamma\epsilon/80}(x)} (1-\gamma)^{-1} \underline{u}_{2\epsilon;q_1,r_1}\left(y + \frac{\gamma\epsilon}{20}\nu, (1-\gamma)^{-1}t\right).$$

We claim that

$$v(x,t) \leqslant 2\underline{u}_{\epsilon;q,r}(x/2, t/2). \tag{3.7}$$

Thanks to Lemma 2.9, v is a subsolution of $(\mathrm{P})^\epsilon$ away from $l_{q,r}$ with $v \leqslant P_{q,r}$. From these facts (3.7) seem plausible. However we need to go around the technical difficulty arising at $l_{q,r}$, so a slightly different route is taken.

Let us choose $y \in B_{\gamma\epsilon/80}(0)$ and let $\xi = y - \frac{\gamma\epsilon}{20}\nu$. Then

$$w(x,t) := 2(1-\gamma)\underline{u}_{\epsilon;q,r}\left(\frac{(x+\xi)}{2}, \frac{(1-\gamma)t}{2}\right)$$

is a supersolution of $(\mathrm{P})_{2\epsilon}$. This is because w is harmonic in its positive set and w satisfies the free boundary motion law

$$\begin{aligned} V_{x,t} = \frac{w_t}{|Dw|}(x,t) &\geqslant (1-\gamma)\left(|D\underline{u}_{\epsilon;q,r}|\left(\frac{(x+\xi)}{2}, \frac{(1-\gamma)t}{2}\right) - g\left(\frac{x+\xi}{\epsilon}\right)\right) \\ &\geqslant |Dw|(x,t) - (1-\gamma)\left(g\left(\frac{x}{2\epsilon}\right) + \frac{5}{8}\gamma\right) \\ &\geqslant |Dw|(x,t) - g\left(\frac{x}{2\epsilon}\right). \end{aligned}$$

(Here the second inequality is due to the fact that Lip $g \leqslant 10$ and $g \geqslant 1$.)

Moreover

$$w(x,t) \geqslant 2(1-\gamma)P_{q,r}\left(\frac{(x+\xi)}{2}, (1-\gamma)(t)\right) \geqslant P_{q_1,r_1} \quad \text{in } Q_1.$$

Since $\underline{u}_{2\epsilon;q_1,r_1}$ is the smallest supersolution of $(\mathrm{P})_{2\epsilon}$ which stays above P_{q_1,r_1}, it follows that $\underline{u}_{\epsilon;q_1,r_1} \leqslant w$ and thus (3.7) is proved.

4. Pick $t_0 > 0$. Let x_0 be the furthest point of $\Gamma_{t_0}(v)$ from $l_{q_1,r_1}(t_0)$ in $Q_1 \cap \{t = t_0\}$. We may assume that

$$d_0 := d\big(x_0, l_{q,r}(t_0)\big) > \frac{C(n)}{\gamma},$$

where $C(n)$ is a large dimensional constant, to be determined. Due to the barrier argument in the proof of Lemma 2.4 in [6], if $\gamma \leqslant (10C(n))^{-1}$, then (x_0, t_0) is more than 10ϵ away from the lateral boundary of Q_1.

Due to (3.7), (3.6) and due to the fact that $\underline{A}_{\epsilon;q,r} \neq \emptyset$ for $0 < \epsilon < 1/2$, for any ϵ neighborhood of a point in

$$S = \big\{x\colon\ d_0 - 20\epsilon \leqslant d\big(x, l_{q,r}(t_0 - 10\epsilon)\big) \leqslant d_0\big\}$$

there exists z_0 in the zero set of $\underline{u}_{\epsilon;q,r}(\cdot, t_0)$, and therefore in the zero set of $v(\cdot, t_0)$. Choose z_0 such that $d(z_0, x_0) \in (4\epsilon, 6\epsilon)$.

By definition of v,

$$\underline{u}_{\epsilon,q_1,r_1}\big(\cdot, (1-\gamma)^{-1}(t_0 - 10\epsilon)\big) = 0 \quad \text{in } B_{\gamma\epsilon/80}(\tilde{z}_0), \tag{3.8}$$

where $\tilde{z}_0 := z_0 - \frac{\gamma\epsilon}{20}\nu$.

On the other hand, recall that $\underline{u}^*_{\epsilon;q_1,r_1}$ is a subsolution of $(\mathrm{P})^\epsilon$, and in particular a subharmonic function in x-variable, away from $l_{q_1,r_1}(t)$. Moreover $\underline{u}_{\epsilon;q_1,r_1}(\cdot,t_0)$ vanishes in $\{x\colon x\cdot\nu \geqslant d_0 + r_1 t_0\}$, and $\underline{u}^*_{\epsilon;q_1,r_1}(\cdot,t_0) \leqslant C\epsilon$ on $l_{q_1,r_1}(t)$ by (3.3). Consequently in the domain $Q_1 \cap \{x\colon x\cdot\nu \geqslant r_1 t\} \cap \{t = t_0\}$

$$\underline{u}_{\epsilon;q_1,r_1}\big(x,(1-\gamma)^{-1}t_0\big) \leqslant \frac{C\epsilon}{d_0}\big(d_0 - d\big(x, l_{q_1,r_1}(t_0)\big)\big)_+.$$

Thanks to Lemma 3.2, in the domain $Q_1 \cap \{x\colon x\cdot\nu \geqslant r_1 t + 3\epsilon\} \cap \{t \leqslant t_0\}$.

$$\underline{u}_{\epsilon;q_1,r_1}\big(x,(1-\gamma)^{-1}t\big) \leqslant \frac{C\epsilon}{d_0}\big(d_0 + 3\epsilon - d(x, l_{q_1,r_1})(t)\big)_+.$$

In particular

$$\underline{u}_{\epsilon;q_1,r_1}(\cdot,t) \leqslant \frac{24C\gamma}{C(n)}\epsilon \quad \text{in } S \times [t_0 - 2\epsilon, t_0]. \tag{3.9}$$

Note that $B_{10\epsilon}(\tilde{z}_0)$ is a subset of S. Now let us consider a barrier $\phi(x,t)$ defined in $\Sigma := B_{10\epsilon}(\tilde{z_0}) \times [t_1, t_0]$, $t_1 := (1-\gamma)^{-1}(t_0 - 10\epsilon)$ such that

$$\begin{cases} -\Delta\phi(\cdot,t) = 0 & \text{in } B_{10\epsilon}(\tilde{z_0} - B_{r(t)}(\tilde{z_0}),\ r(t) = \frac{\gamma\epsilon}{80} + (t - t_1), \\ \phi(\cdot,t) = \frac{24C\gamma}{C(n)}\epsilon & \text{on } \partial B_{10\epsilon}(\tilde{z_0}), \\ \phi(\cdot,t) = 0 & \text{on } \partial B_{r(t)}(\tilde{z}_0). \end{cases}$$

If $C(n)$ is chosen sufficiently large, then ϕ is a subsolution of $(\mathrm{P})_\epsilon$ in Σ. Eqs. (3.8) and (3.9) would then yield that $\underline{u}_{\epsilon;q_1,r_1}(\cdot,t_0) \equiv 0$ in $B_{8\epsilon}(\tilde{z}_0)$. But this is a contradiction to the fact that $x_0 \in \Gamma_t(v)$, since from our choice of $\tilde{z}_0$ it follows that $v(\cdot,t_0) = 0$ in $B_{2\epsilon}(x_0)$. We have thus shown that (a) holds for $r \leqslant 0$.

6. Next we prove (a) for $r \geqslant 0$. If $0 \leqslant r \leqslant 2$ then parallel argument as above applies to yield (a), thus let us consider the case $r \geqslant 2$. Here arguing as in the proof of (3.3) yields that

$$\underline{u}_{\epsilon;q_1,r_1}(\cdot,t) \leqslant Cr\epsilon \quad \text{on } \big\{x\colon 0 \leqslant d\big(x, l_{q_1,r_1}(t)\big) \leqslant 2\epsilon\big\}, \tag{3.10}$$

where C is the same dimensional constant as in (3.3).

Let x_0 be the furthest point in $\Gamma(\underline{u}_{\epsilon;q_1,r_1})$ from $l_{q_1,r_1}(t_0)$, with

$$d_0 = d\big(x_0, l_{q_1,r_1}(t_0)\big) \geqslant \frac{\epsilon}{\gamma}.$$

Equipped with (3.10), we can argue as in step 5 to yield

$$\underline{u}_{\epsilon;q_1,r_1}(x,t) \leqslant \frac{Cr\epsilon}{d_0}\big(d_0 + 3\epsilon - d\big(x, l_{q_1,r_1}(t)\big)\big)_+ \quad \text{in } \{x\colon x\cdot\nu \geqslant r_1\} \times \{t \leqslant t_0\}.$$

We are now ready to yield a contradiction. Our barrier this time is

$$h(x,t) := Cr\gamma\left(d_0 + 3\epsilon - d(x, l_{q_1,r_1})\left(t_0 - \frac{10\epsilon}{r}\right) + C(r+2)\gamma\left(t - t_0 + \frac{10\epsilon}{r}\right)\right)_+.$$

$h(x,t)$ is then a planar supersolution of $(\mathrm{P})^\epsilon$ in

$$\Sigma := Q_1 \cap \{x\colon x\cdot\nu \geqslant r_1 t\} \cap \left\{t_0 - \frac{10\epsilon}{r} \leqslant t \leqslant t_0\right\}.$$

Hence Theorem 2.6 applied to $\underline{u}_{\epsilon;q_1,r_1}$ and h yields that $\underline{u}_{\epsilon;q_1,r_1} \leqslant h$ in Σ.

If $\gamma \leqslant (4C)^{-1}$, then the positive set of h does not reach x_0 by time t_0: precisely

$$\Omega_{t_0}(h) \subset \big\{x\colon d(x, l_{q_1,r_1})(t_0) < d_0 - 2\epsilon\big\}.$$

Hence we reach a contradiction.

7. As for the proof of (b), the case for $r \leqslant 0$ is shown in the proof of Proposition 2.9 (a) in [6]: the argument is indeed similar to the proof of (a) for $r \leqslant 0$, with simplifications due to the fact that the corresponding sub-convolution v is also a subsolution of $(P)^\epsilon$ in Q_1. For $0 \leqslant r \leqslant 2$ a stronger version of (b) is Proposition 2.8(b) in [6]. Thus it remains to consider the case $r \geqslant 2$. First observe that, if $x_0 \in \Gamma_t(\bar{u}_{2\epsilon;q_2,r_2})$ with $d(x_0, l_{q_2,r_2}(t)) > \epsilon$ then for a dimensional constant C

$$\bar{u}_{2\epsilon;q_2,r_2}(\cdot,t) < Cr\epsilon \quad \text{in } B_{2\epsilon}(x_0 - 3\epsilon\nu). \tag{3.11}$$

If not a barrier argument as in step 2 using Lemma 3.2(a) yields that $x_0 \in \Omega_t(\bar{u}_{2\epsilon;q_2,r_2})$, a contradiction.

Pick $t_0 > 0$. Suppose y_0 is the furthest point of $\Gamma_{t_0}(\bar{u}_{2\epsilon;q_2,r_2})$ from $l_{q,r}(t_0)$ in Q_1 with

$$d_0 = d\big(x_0, l_{q_2,r_2}(t_0)\big) \geqslant \frac{\epsilon}{\gamma}.$$

As in (3.6) we have

$$\frac{1}{2}\bar{u}_{2\epsilon;q_2,r_2}(2x,2t) \geqslant \bar{u}_{\epsilon;q_2,r_2}(x+\eta,t+\tau) \quad \text{in } \frac{1}{2}Q_1 + (\eta,-\tau) \tag{3.12}$$

when $\tau > 0$ and $\eta \in \epsilon\mathbb{Z}^n$ satisfies $|\eta| \leqslant \frac{1}{2}$ and $\eta\cdot\nu \geqslant r\tau$. It then follows from (3.11) and (3.12) that

$$\bar{u}_{\epsilon;q_2,r_2}(\cdot,t_0) \leqslant Cr\epsilon \quad \text{on } B_{3/4}(t_0\nu) \cap \big(l_{q,r}(t_0) - (d_0+3\epsilon)\nu\big). \tag{3.13}$$

(3.13) and the fact that $\bar{u}_{\epsilon;q_2,r_2}(\cdot,t_0)$ is subharmonic yields that

$$\bar{u}_{\epsilon;q_2,r_2}(\cdot,t_0) \leqslant Cr\gamma\epsilon \quad \text{in } B_{2/3}(t_0\nu) \cap \{x\colon x\cdot\nu \geqslant r_2t_0 - 5\epsilon\}.$$

Above equation and Lemma 3.2 says that for $t \geqslant t_0$

$$\bar{u}_{\epsilon;q,r}(\cdot,t_0) \leqslant Cr\gamma\epsilon \quad \text{in } B_{4/7}(t_0\nu) \cap \{x\colon x\cdot\nu \geqslant r_2t - 3\epsilon\}. \tag{3.14}$$

Now a barrier argument similar to that in step 6 would yield that

$$\bar{u}_{\epsilon;q,r}\left(\cdot, t_0 + \frac{1}{r}\epsilon\right) \equiv 0 \quad \text{on } l_{q_2,r_2}\left(t_0 + \frac{1}{r_2}\epsilon\right),$$

contradicting the fact that $\bar{A}_{\epsilon;q,r} \neq \emptyset$ for $0 < \epsilon < \frac{1}{2}$. □

Replacing the flatness constant M in Propositions 2.8 and 2.9 in [6] with $\frac{C(n)}{\gamma}$ in Lemma 3.4, Propositions 3.8 and 3.11 in [6] now reads as below.

Proposition 3.5. *(Propositions 3.8 and 3.11 in [6].) There exists dimensional constant $C_1 > 0$ such that for any nonzero vector $q \in \mathbb{R}^n$ and for $r = r(q) \neq 0$ the following is true*:

Let us fix $0 < \gamma \ll 1$ and $0 < \epsilon < \epsilon_0 = \frac{r\gamma^{11}}{n}$.

(a) *For $r_1 \geqslant (1-\gamma)r$ and $q_1 \leqslant (1-\gamma)q$,*

$$d\big(\Gamma_t(\bar{u}_{\epsilon;q_1,r_1}), l_{q_1,r_1}(t) \cap B_{1/4}(0)\big) > \frac{C_1\epsilon}{\gamma}$$

for $t \geqslant \frac{C_1\epsilon}{|r|\gamma^3}$.

(b) *For $r_2 \leqslant (1+\gamma)r$ and $q_2 \geqslant (1+\gamma)q$,*

$$d\big(\Gamma_t(\underline{u}_{\epsilon;q_2,r_2}), l_{q_2,r_2}(t) \cap B_{1/4}(0)\big) > \frac{C_1\epsilon}{\gamma}$$

for $t \geqslant \frac{C_1\epsilon}{|r|\gamma^3}$.

Remark 5. Note that by scaling argument it follows that $(1-a)r((1+a)q)$ increases in a.

Proposition 3.5 states that if the obstacle speed r_1 (r_2) is too fast (slow) compared to the size of q_1 (q_2), then the maximal subsolution (minimal supersolution) of $(\mathrm{P})_\epsilon$ stays away from the obstacle. We will use the following variation of Proposition 3.5 in our analysis in Section 4 (see Proposition 4.3).

Corollary 3.6. *Let $0 < \epsilon < c(n)$ and C_1 be the constant given in Proposition 3.5. Let u^ϵ solve $(\mathrm{P})^\epsilon$ in $\Sigma := 2B_{\epsilon^{1/2}}(0) \times [-\alpha_\epsilon, 0]$, where*

$$\alpha_\epsilon := \min\left[\frac{\epsilon^{4/5}}{|r|}, \epsilon^{3/5}\right].$$

(a) *If $(u^\epsilon)^* \leqslant P_{q_0,r_0}$ in Σ and if*

$$r_0 \geqslant \left(1-\epsilon^{1/25}\right) r\left(\left(1+\epsilon^{1/25}\right) q_0\right) + 2\epsilon^{1/25}$$

then

$$d\left(\Gamma_0\left(\left(u^\epsilon\right)^*\right), l_{q_0,r_0}(0) \cap B_{\epsilon^{1/2}/4}(0)\right) > C_1 \epsilon^{24/25}.$$

(b) *If $u^\epsilon \geqslant P_{q_0,r_0}$ in Σ and if*

$$r_0 \leqslant \left(1+\epsilon^{1/25}\right) r\left(\left(1-\epsilon^{1/25}\right) q_0\right) - 2\epsilon^{1/25},$$

then

$$d\left(\Gamma_0\left(u^\epsilon\right), l_{q_0,r_0}(0) \cap B_{\epsilon^{1/2}/4}(0)\right) > C_1 \epsilon^{24/25}.$$

Proof. We only prove (a), since parallel arguments hold for (b).

Choose $\xi \in \epsilon\mathbb{Z}^n$ such that $|\xi - r\alpha_\epsilon \nu| \leqslant 2\epsilon$, $(\xi - r\alpha_\epsilon)\cdot \nu \leqslant 0$. $\nu = q_0/|q_0|$. Define

$$\tilde{u}^\epsilon(x,t) := e^{-1/2} u^\epsilon\left(\epsilon^{1/2}(x-\xi), \epsilon^{1/2}(t-\alpha_\epsilon)\right).$$

Then $(\tilde{u}^\epsilon)^*$ is a subsolution of $(\mathrm{P})^{\epsilon^{1/2}}$ in $\tilde{\Sigma} := B_{10}(0) \times [0, \alpha_\epsilon \epsilon^{-1/2}]$ with $(\tilde{u}^\epsilon)^* \leqslant P_{q_0,r_0}$. Note that $\mathcal{O} \cap \{0 \leqslant t \leqslant \alpha_\epsilon\}$ is contained in $\tilde{\Sigma}$. Hence by definition of $\bar{u}$ as the maximal subsolution above $P_{q,r}$ in $\mathcal{O}$ we obtain

$$\left(\tilde{u}^\epsilon\right)^* \leqslant \bar{u}_{\epsilon^{1/2};q_0,r_0} \quad \text{in } \tilde{\Sigma}.$$

Therefore if $|r((1+\epsilon^{1/25})q_0)| > \epsilon^{1/25}$, then (a) follows from Proposition 3.5 with ϵ replaced by $\epsilon^{1/2}$ and $\gamma = \epsilon^{1/25}$.

If $|r((1+\epsilon^{1/25})q_0)| \leqslant \epsilon^{1/25}$, then by our hypothesis in (a) it follows that $|r_0| \geqslant \epsilon^{1/25}$ and one can apply Proposition 3.5 with q_0 replaced by $\tilde{q} = \alpha q_0$ with which

$$r_0 = \left(1-\epsilon^{1/25}\right) r\left(\left(1+\epsilon^{1/25}\right)\tilde{q}\right).$$

Since $r(q)$ increases in $|q|$, we have $\alpha > 1$. It follows that $u^\epsilon \leqslant P_{\tilde{q},r_0}$ in Σ. Thus one can apply Proposition 3.2 with ϵ replaced by $\epsilon^{1/2}$ and $\gamma = \epsilon^{1/25}$ and use the fact that

$$\left(\tilde{u}^\epsilon\right)^* \leqslant \bar{u}_{\epsilon^{1/2};\tilde{q},r_0} \quad \text{in } \tilde{\Sigma}$$

to derive the conclusion. □

Below we sketch a formal argument to prove (1.1). Suppose u^ϵ and u respectively solve $(\mathrm{P})^\epsilon$ and (P) with same initial data u_0. Suppose we can perturb u to construct a new function w_1 which satisfies the following:

(i) $d(\Gamma_t(w_1), \Gamma_t(u)) < \epsilon^{1/70}$ for $t \geqslant 0$.
(ii) w_1 satisfies (P) with $r(Du)$ replaced by

$$\left(1-\epsilon^{1/25}\right) r\left(\left(1+\epsilon^{1/25}\right) Dw_1\right) + \epsilon^{1/25}. \tag{3.15}$$

(iii) $u^\epsilon(\cdot,0) \prec w_1(\cdot,0)$ and $u^\epsilon \leqslant w_1$ for $x \in K$.

Now assume that $\Gamma(u^\epsilon)$ touches $\Gamma(w_1)$ for the first time at $P_0 = (x_0, t_0)$. Then $t_0 > 0$ and $u^\epsilon \leqslant w_1$ in $Q \cap \{t \leqslant t_0\}$. Let

$$q_0 = Dw_1(P_0), \qquad r_0 = \frac{(w_1)_t}{|Dw_1|}(P_0). \tag{3.16}$$

Note that, due to (3.15),

$$r_0 \geqslant \bigl(1 - \epsilon^{1/25}\bigr) r\bigl(\bigl(1 + \epsilon^{1/25}\bigr) q_0\bigr) + \epsilon^{1/25}. \tag{3.17}$$

Let ξ be a space–time translate of P_{q_0,r_0} such that $l_{q_0,r_0} + \xi$ touches P_0. If one can show that $u^\epsilon \leqslant P_{q_0,r_0} + \xi$ in $\epsilon^{1/2}$-neighborhood of P_0, then a contradiction would follow due to Corollary 3.6, yielding $u^\epsilon \leqslant w_1$. A parallel argument applies to constructing a perturbation function w_2 which will bound u^ϵ from below. Once we obtain $w_2 \leqslant u^\epsilon \leqslant w_1$ with

$$d\bigl(\Gamma_t(w_k), \Gamma_t\bigl(u^\epsilon\bigr)\bigr) \leqslant \epsilon^{1/70} \quad \text{for } t \geqslant 0,\ k = 1, 2,$$

(1.1) follows.

In Sections 4, 5 we show a rigorous version of above formal argument to prove (1.1). The challenge is to find correct perturbations w_1, w_2 of u and to find q_0 and r_0 for which (3.17) is satisfied and $u^\epsilon \leqslant P_{q_0,r_0} + \xi$ in $\epsilon^{1/2}$-neighborhood of P_0. (Note that (3.16) would not apply to nonsmooth w_1.)

4. Statement of main result

Let u be a solution of (P) in Q with initial data u_0, and fix $t_0 > \epsilon^{1/30}$ and $\epsilon > 0$. In the domain

$$Q_\epsilon := \bigl(\mathbb{R}^n - K_\epsilon\bigr) \times \bigl[\epsilon^{1/30}, \epsilon^{-1/300}\bigr], \quad K_\epsilon := \bigl(1 + \epsilon^{1/70} + 2\epsilon^{1/30}\bigr) K$$

we define

$$u_1(x,t) := u\bigl(\bigl(1 + \epsilon^{1/70}\bigr)^{-1} x, \bigl(1 + \epsilon^{1/70}\bigr)^{-1}\bigl(1 - \epsilon^{1/60}\bigr) t + t_0\bigr), \tag{4.1}$$

and the inf-convolutions

$$v_1(x,t) := \inf_{y \in B_{\epsilon^{1/30} - \epsilon^{1/27} t}(x)} u_1(y,t), \tag{4.2}$$

and

$$w_1(x,t) := \inf_{(y,s) \in B^{(n+1)}_{\epsilon^{1/30}}(x,t)} v_1(y,s). \tag{4.3}$$

Then w_1 is a viscosity supersolution of

$$\begin{cases} -\Delta w_1 = 0 & \text{in } \{w_1 > 0\}, \\ V = (1 - \epsilon^{1/60}) r((1 + \epsilon^{1/70}) Dw_1) + \epsilon^{1/27} & \text{on } \Gamma(w_1) \end{cases}$$

in Q_ϵ.

The *convoluted* functions v_1 and w_1 is introduced to improve the free boundary regularity of u_1: any free boundary point $(x_0, t_0) \in \Gamma(w_1)$ has both an exterior space–time ball and an exterior space ball, lying in the zero set of w_1 and touching (x_0, t_0) (or x_0) on their boundaries.

Similarly in the domain

$$\tilde{Q}_\epsilon := \bigl(\mathbb{R}^n - K\bigr) \times \bigl[\epsilon^{1/30}, \epsilon^{-1/300}\bigr], \quad \tilde{K}_\epsilon = \bigl(1 + 2\epsilon^{1/30}\bigr) K$$

we define

$$u_2(x,t) := u^*\bigl(\bigl(1 - \epsilon^{1/70}\bigr)^{-1} x, \bigl(1 - \epsilon^{1/70}\bigr)^{-1}\bigl(1 + \epsilon^{1/60}\bigr) t + t_1\bigr), \tag{4.4}$$

and

$$v_2(x,t) := \sup_{y \in B_{\epsilon^{1/30}-\epsilon^{1/27}t}(x)} u_2(y,t),$$
$$w_2(x,t) := \sup_{(y,s) \in B^{(n+1)}_{\epsilon^{1/30}}(x,t)} v_2(y,s).$$

Then w_2 is a viscosity subsolution of

$$\begin{cases} -\Delta w_2 = 0 & \text{in } \Omega(w_2), \\ V = (1+\epsilon^{1/60}) r((1-\epsilon^{1/70}) Dw_2) - \epsilon^{1/27} & \text{on } \Gamma(w_2) \end{cases}$$

in $\tilde{Q}_\epsilon$, with interior ball properties at the free boundary.

Suppose that there exist constants $\epsilon^{1/30} \leqslant t_0, t_1 < \infty$, respectively given in (4.1) and (4.4), and $\tau > 0$ such that the corresponding w_2 and w_1 satisfy

(H1) $w_2(x,0) \prec u^\epsilon(x,\tau) \prec w_1(x,0)$.

and for all $t \geqslant 0$

(H2) $u^\epsilon(x,t+\tau) < w_1(x,t)$ for $x \in K_\epsilon$, $w_2(x,t) < u^\epsilon(x,t+\tau)$ for $x \in K$.

Theorem 4.1. *Suppose u and u^ϵ satisfies* (H1), (H2) *with some t_0, t_1 and τ. Then*

$$w_2(x,t) \leqslant u^\epsilon(x,t+\tau) \leqslant w_1(x,t) \quad \textit{in } Q_\epsilon.$$

Suppose $\Omega(u_0) \subset B_R(0)$. From a barrier argument with radially symmetric solutions of (P), using the fact that $r(|Du|) \in [|Du|-2, |Du|-1]$, it follows that

$$B_{R_1}(0) \subset \Omega_t(u) \subset B_{R_2}(0) \quad \text{for } t \geqslant 0, \tag{4.5}$$

where R_i depends on n and u_0. In particular R_2 is given as the maximum of a dimensional constant and R.

Corollary 4.2. *Suppose u solves* (P) *and u^ϵ solves* (P)$^\epsilon$, *with initial data u_0. Also suppose $\Omega(u_0) \subset B_R(0)$ and one of the conditions* (A)–(C) *holds. Then for any $T > 0$, there exist positive constants $\epsilon_0 = \epsilon(n, u_0, T)$ and $C_0 = C(n,R)$ such that for* $0 < \epsilon < \epsilon_0$

$$d\big((x,t), \Gamma\big(u^\epsilon\big)\big) \leqslant C_0 \epsilon^{1/70} \quad \textit{for } (x,t) \in \Gamma(u) \cap [0,T]. \tag{4.6}$$

Proof. 1. First suppose that (A) holds. Since Ω is star-shaped with respect to $B_r(0)$, it follows that for $0 < \epsilon < \epsilon_0 = \epsilon_0(r)$ and for $t_0 = \tau = t_1 = \epsilon^{1/30}$

$$\Omega_0(w_2) \Subset \Omega_\tau\big(u^\epsilon\big) \Subset \Omega_0(w_1). \tag{4.7}$$

Due to (4.5) and barrier arguments with radially symmetric harmonic functions it follows that

$$|Du|(\cdot,t) \sim C(n,u_0) \quad \text{for } x \in K. \tag{4.8}$$

Therefore, for sufficiently small ϵ depending on n and u_0, (H2) holds. In particular maximum principle for harmonic functions yield (H1) due to (4.7) and (H2). Hence if ϵ is chosen sufficiently small that $T \leqslant \epsilon^{-1/300}$ then Theorem 4.1 yields (4.6) with

$$C_0 = C(n) \sup_{(x,t) \in \Omega(u)} |x|.$$

Due (4.5), $C_0 = C(n,R)$.

2. Next suppose that (B) holds. Then the free boundary velocity is strictly positive at $t = 0$. Since Γ_0 is locally Lipschitz, by a barrier argument one can check that there exists $\epsilon^{1/30} = t_0 < \tau, t_1 = O(\epsilon^{1/70})$ satisfying

$$\Omega_0(w_2) \subset \Omega_\tau(u) \subset \Omega_0(w_1)$$

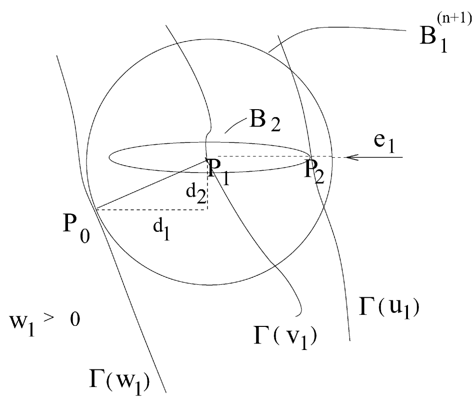

Fig. 3.

if $\epsilon > 0$ is sufficiently small depending on u_0. The rest of argument is the same as in the case of (A). Parallel argument applies to the case (C), for which the free boundary velocity is strictly negative at $t = 0$. □

Proof of Theorem 4.1. Suppose our theorem is false. Then either $(u^\epsilon)^*$ crosses w_1 from below or u^ϵ crosses w_2 from above in finite time. Suppose the former, that is

$$0 < t_0 = \sup\{t\colon\ \Omega_t\big((u^\epsilon)^*\big) \prec \Omega_t(w_1)\} < \infty.$$

For simplicity we denote $(u^\epsilon)^*$ by u^ϵ in the rest of the proof.

Suppose $\bar{\Omega}_{t_0}(u^\epsilon)$ is a compact subset of $\Omega(w_1) - K_\epsilon$. Since $u^\epsilon < w_1$ on K_ϵ and $(u^\epsilon - w_1)(\cdot, t_0)$ is subharmonic in $\Omega_{t_0}(u^\epsilon) - K_\epsilon$, it follows from the maximum principle for harmonic functions that $u^\epsilon(\cdot, t) < w_1(\cdot, t)$ in $\Omega_{t_0}(u^\epsilon)$, and thus $u^\epsilon(\cdot, t_0) \prec w_1(\cdot, t_0)$. Due to the lower semicontinuity of $w_1 - u^\epsilon$, then for a small time period after t_0 the supports of u^ϵ and w_1 stays strictly ordered and thus $u^\epsilon(\cdot, t) \prec w_1(\cdot, t)$, contradicting the definition of t_0.

On the other hand suppose $u^\epsilon(x_0, t_0) > 0$ at some $x_0 \in \Gamma_{t_0}(w_1)$. By construction, there exists a space–time ball $B^{(n+1)}$ of radius $\epsilon^{1/30}$ such that

$$\mathcal{E} := \{(x,t)\colon\ |x - y| \leqslant \epsilon^{1/30/2} \text{ for some } (y,t) \in B^{(n+1)}\}$$

lies in the zero set of w_1 and touches (x_0, t_0) on its boundary (see Fig. 3). A barrier argument based on this set, similar to the one given in the proof of Lemma 2.8(b), leads to a contradiction.

From above discussion we conclude that at $t = t_0$ we have $\Omega_{t_0}(u^\epsilon) \subset \Omega_{t_0}(w_1)$, $u^\epsilon = 0$ on $\Gamma_{t_0}(w_1)$, and there exists $P_0 := (p_0, t_0)$ such that $p_0 = \Gamma_{t_0}(u^\epsilon) \cap \Gamma_{t_0}(w_1)$. In particular due to (H2) $u^\epsilon \leqslant w_1$ for $t \leqslant t_0$.

Next we investigate the geometry of $\Gamma(w_1)$ at the contact point P_0. By definition of w_1, the set $\Omega(w_1)$ lies outside

$$B_1^{(n+1)} := B_{\epsilon^{1/30}}^{(n+1)}(P_1) \tag{4.9}$$

with $P_1 = (p_1, t_1) \in \Gamma(v_1)$, touching $\Gamma(w_1)$ at P_0 (see Fig. 2). On the other hand $\Omega(u_1)$ has an interior space ball $B_2 := B_{\epsilon^{1/30} - \epsilon^{1/6} t_1}(P_1)$ touching $\Gamma(u_1)$ at $P_2 = (p_2, t_1)$. We rotate the coordinates such that

$$P_0 - P_1 = (d_1 e_1, -d_2) \in \mathbb{R}^n \times \mathbb{R}, \quad \text{where } d_1 \geqslant 0 \text{ and } e_1 = (1, 0, \dots, 0).$$

$P_1 - P_2$ is then also parallel to e_1. Observe that, if $\Gamma(w_1)$ were smooth, d_2/d_1 equals the (outward) normal velocity of $\Gamma(w_1)$ at P_0. Barrier arguments with radially symmetric barrier in $2B_1^{(n+1)} - B_1^{(n+1)}$, as in the proof of Theorem 2.2 in [4], yields that

$$d_1 \neq 0 \quad \text{and} \quad \frac{d_2}{d_1} \geqslant -2.$$

(Formally speaking $d_1 \neq 0$ since otherwise $\Gamma(w_1)$ would have infinite normal velocity at P_0: but this is impossible because $|Dw_1|$ stays finite on $\Gamma(w_1)$ due to the exterior ball property.)

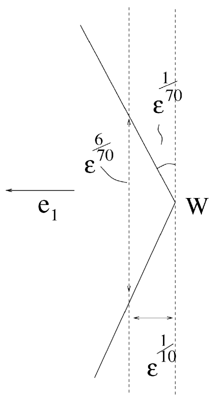

Fig. 4.

Let us define

$$r_0 = \frac{d_2}{d_1} \in [-2, \infty) \quad \text{and} \quad q_0 = m e_1,$$

where

$$m = \min_{x \in W,\ x_1 = \epsilon^{1/10}} \frac{u_1(x + p_2, t_1)}{\epsilon^{1/10}}$$

and

$$W = \big\{x\colon\ x_1 := x \cdot e_1 \geqslant 0,\ |x - x_1 e_1| \leqslant \big(1 - \epsilon^{1/70}\big)|x|\big\}$$

(see Fig. 4).

We will prove, in the next section, the following proposition:

Proposition 4.3. *For* $0 < \epsilon < c(n)$ *let* $q_1 = (1 + \epsilon^{1/50}) q_0$. *Then*

$$\big(u^\epsilon\big)^* \leqslant P_{q_1, r_0} + P_0 + \epsilon^{29/30} e_1 \quad \text{in } B_{\epsilon^{1/2}}(x_0) \times (t_0 - \alpha_\epsilon, t_0),$$

where α_ϵ *is as given in Corollary* 3.6 *and*

$$r_0 \geqslant \big(1 - \epsilon^{1/60}\big) r\big(\big(1 + \epsilon^{1/60}\big) q_1\big) + 2\epsilon^{1/25}.$$

If above proposition is true, then due to Corollary 3.6 and Remark 5

$$d\big(\Gamma_{t_0}\big(\big(u^\epsilon\big)^*\big), x_0\big) > C_1 \epsilon^{24/25} - \epsilon^{29/30},$$

where C_1 is a dimensional constant. Hence for $0 < \epsilon < c(n)$,

$$d\big(\Gamma_{t_0}\big(\big(u^\epsilon\big)^*\big), x_0\big) > \frac{C_1}{2} \epsilon^{24/25},$$

which contradicts the fact that $x_0 \in \Gamma_{t_0}((u^\epsilon)^*)$.

Parallel argument holds for the case u^ϵ crossing w_2 from above. □

5. Proof of Proposition 4.3

It remains to show Proposition 4.3. We begin with the following lemma.

Lemma 5.1.

$$r_0 > \big(1 - \epsilon^{1/60}\big) r\big(\big(1 + \epsilon^{1/65}\big) q_0\big) + \epsilon^{1/27}$$

for $0 \leqslant \epsilon \leqslant c(n)$.

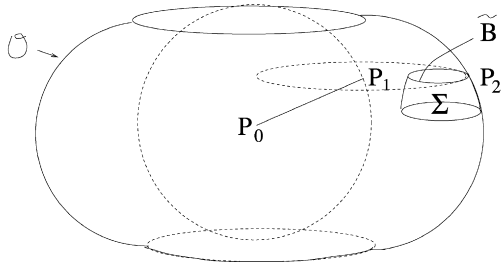

Fig. 5.

Proof. Recall that u_1 satisfies the free boundary motion law

$$V \geqslant \left(1-\epsilon^{1/60}\right) r\left(\left(1+\epsilon^{1/70}\right) D u_1\right) \quad \text{on } \Gamma(u_1)$$

in the viscosity sense. As mentioned in the previous section, $\Omega_{t_1}(u_1)$ has an interior space ball $B_{\epsilon^{1/30}-\epsilon^{1/27}t_1}(P_1)$ touching $p_2 \in \Gamma_{t_1}(u_1)$. Therefore one can also find a space ball $\tilde{B}$ of radius $\epsilon^{1/13}$ in $\Omega_{t_1}(u_1)$ touching p_2. In fact from (4.2), (4.3)

$$\mathcal{O} \subset \Omega(u_1),$$

where $\mathcal{O}$ is a "flat" space–time ball-like set given by

$$\mathcal{O} := \left\{(x,t)\colon\ |x-y| \leqslant \epsilon^{1/30} - \epsilon^{1/27} t \text{ for some } y \in B_1^{(n+1)}\right\},$$

where $B_1^{(n+1)}$ is as given in (4.9) (see Fig. 5).

Let

$$\mathcal{C}(t) = a(t)\tilde{B},$$

where $a(t) = \sup\{s\colon\ s\tilde{B} \times \{t\} \subset \mathcal{O}\}$ and

$$\Sigma = \bigcup_{t_1-\delta \leqslant t \leqslant t_1} \left(\mathcal{C}(t) - \frac{1}{2}\mathcal{C}(t)\right) \times \{t\},$$

where δ is small and to be determined. We now construct $\phi(x,t)$ in Σ as follows:

$$\begin{cases} -\Delta\phi(\cdot,t) = 0 & \text{in } \mathcal{C}(t) - (1-\epsilon^{1/10})\mathcal{C}(t), \\ \phi(\cdot,t) = (1-\epsilon)m\epsilon^{1/10} > 0 & \text{on } (1-\epsilon^{1/10})\partial\mathcal{C}(t), \\ \phi(\cdot,t) = 0 & \text{on } \partial\mathcal{C}(t). \end{cases}$$

Then we have

$$|D\phi|(P_2) \geqslant \left(1 - C\epsilon^{1/10-1/13}\right)m, \qquad \frac{\phi_t}{|D\phi|}(P_2) = r_0 - \epsilon^{1/27}.$$

Note that

$$\mathcal{S} = \left\{(x+p_2, t_1)\colon\ x_1 = \epsilon^{1/10}\right\} \cap \tilde{B}$$

is a set of width $\epsilon^{1/10}$ in e_1-direction and of width

$$C\epsilon^{1/20+1/26} \leqslant \epsilon^{6/70} \quad \text{for } 0 < \epsilon < c(n)$$

in other directions, and $\mathcal{S} \subset W + p_2$. Hence

$$\phi(\cdot,t) = (1-\epsilon)m\epsilon^{1/10} < u_1(\cdot,t) \quad \text{on } \mathcal{S} \times [t_1-\delta, t_1],$$

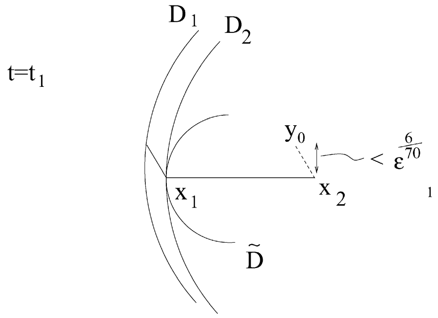

Fig. 6.

if δ is chosen sufficiently small, first at $t = t_1$ by definition of m, and then for other times by lower semi-continuity of u. Moreover Σ is a subset of $\Omega(u_1)$ by construction. Therefore by maximum principle of harmonic functions

$$\phi \leqslant u_1 \quad \text{in } \{x + p_2\colon x_1 \leqslant \epsilon^{1/10}\} \cap \tilde{B} \times [t_1 - \delta, t_1],$$

and in particular $u_1 - \phi$ has a local minimum zero at P_2.

Using the definition of viscosity supersolution, if ϵ is sufficiently small,

$$\begin{aligned} r_0 = \frac{\phi_t}{|D\phi|}(P_2) + \epsilon^{1/27} &\geqslant (1 - \epsilon^{1/60}) r\big((1 + \epsilon^{1/70})|D\phi|(P_2)\big) + \epsilon^{1/27} \\ &\geqslant (1 - \epsilon^{1/60}) r\big((1 + \epsilon^{1/70})(1 - \epsilon^{3/130}) q_0\big) + \epsilon^{1/27} \\ &\geqslant (1 - \epsilon^{1/60}) r\big((1 + \epsilon^{1/65}) q_0\big) + \epsilon^{1/27}. \qquad \square \end{aligned}$$

Our next goal is to construct a barrier which bounds w_1 from above and lies below (a perturbation of) $P_{q_0,r_0} + P_0$. Such barrier will be constructed by small increments, starting from investigation of u_1 at p_2.

By definition of m, there exists $y_0 \in W \cap \{x\colon x_1 = \epsilon^{1/10}\} + p_2$ such that

$$u_1(y_0, t_1) = m\epsilon^{1/10}.$$

By definition of v_1 we then have

$$v_1(x, t_1) \leqslant m\epsilon^{1/10} \quad \text{in } D_1 := B_{\epsilon^{1/30} - \epsilon^{1/27} t_1}(y_0). \tag{5.1}$$

Recall that $\Omega_{t_1}(v_1)$ has an exterior ball $B_{\epsilon^{1/30} - \epsilon^{1/27} t_1}(p_2)$ touching $p_1 \in \Gamma_{t_1}(v_1)$. Thus $\Omega_{t_1}(v_1)$ also has an exterior spatial ball $\tilde{D} = B_{\epsilon^{1/30}/4}(\tilde{x})$ touching p_1.

Since $y_0 - x_2 = \epsilon^{1/10} e_1 + \mu$ with $\mu \cdot e_1 = 0$, $|\mu| \leqslant \epsilon^{6/70}$, a straightforward calculation yields that

$$\partial D_1 \text{ is outside } \big(1 + 4\epsilon^{1/15} - \epsilon^{2/15}\big)\tilde{D}. \tag{5.2}$$

(See Fig. 6.)

Let $h(x)$ be the harmonic function in the ring domain

$$\Pi := \big(1 + 4\epsilon^{1/15} - \epsilon^{2/15}\big)\tilde{D} - \tilde{D}$$

with boundary data

$$h = m\epsilon^{1/10} \quad \text{on } \big(1 + 4\epsilon^{1/15} - \epsilon^{2/15}\big)\partial\tilde{D}, \qquad h = 0 \quad \text{on } \partial\tilde{D}.$$

Then $|Dh| = m(1 + C\epsilon^{1/15})$ on $\partial\tilde{D}$: in fact from the explicit formula for radially symmetric harmonic functions it follows that $|Dh| \leqslant m(1 + C\epsilon^{1/15})$ in Π.

Due to (5.1) and (5.2), for $0 < \epsilon < c(n)$ $v_1(\cdot, t_1) \leqslant m\epsilon^{1/10} \leqslant h$ on the outer boundary of Π, and thus

$$v_1(\cdot, t_1) \leqslant h \quad \text{on } \Pi. \tag{5.3}$$

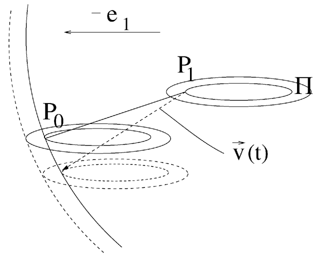

Fig. 7.

Next we construct a barrier for w_1, using the information gathered from above. Let us construct the space–time ring domain

$$\mathcal{C} = \bigcup_{t_0-\alpha_\epsilon \leqslant t \leqslant t_0} \big(\Pi + a(t)e_1\big) \times \{t\}$$

where

$$a(t) > 0, \vec{v}(t) := \big(a(t)e_1, t - t_1\big) \in \partial B^{(n+1)}_{\epsilon^{1/30}}(0).$$

In particular $a(t) \in C^2$, $a(t_0) = d_1$ and $a'(t_0) = -r_0$ (see Fig. 7).

Now define $\varphi(x,t) = h(x - a(t)e_n)$ in $\mathcal{C}$. Then by definition of w_1 and (5.3)

$$w_1(x,t) \leqslant v_1\big(x - a(t)e_n, t_1\big) \leqslant \varphi(x,t) \quad \text{in } \mathcal{C}. \tag{5.4}$$

Finally we bound φ from above by $P_{q_0,r_0}(x,t) + P_0$. Note that $\Gamma_t(\varphi)$ is a sphere of radius $\epsilon^{1/30}/4$. This fact and the twice differentiability of $a(t)$ yields that, in $B_{\epsilon^{1/2}}(x_0) \times [t_0 - \epsilon^{1/2}, t_0]$, $\Gamma_t(\varphi)$ is in $\epsilon^{1-1/30}$-neighborhood of its space–time tangent plane at (x_0, t_0), which is $l_{q_0,r_0}(t) + P_0$. Since $|Dh| \leqslant m(1 + C\epsilon^{1/5})$ in Π, so is $|D\varphi|$ in $\mathcal{C}$. Therefore

$$\varphi \leqslant \big(1 + \epsilon^{1/50}\big) P_{q_0,r_0} + P_0 + \epsilon^{29/30} e_1 \quad \text{in } B_{\epsilon^{1/2}}(x_0) \times \big[t_0 - \epsilon^{1/2}, t_0\big]. \tag{5.5}$$

Recall that we have $(u^\epsilon)^* \leqslant w_1$ for $t \leqslant t_0$. This and (5.4), (5.5) proves our proposition.

6. Remarks on an expanding free boundary problem

As stated in Corollary 3.6, for problem $(P)^\epsilon$ and (P) our error estimate is only obtained for the class of initial data (A)–(C). This is because uniqueness does not hold for solutions of (P) with general initial data.

Below we show that stronger result holds for problems with expanding free boundaries.

Let u_0, Ω, K, g and Γ_0 the same as in the introduction, and let $u(x,t)$ solve

$$(\mathrm{P2})^\epsilon \quad \begin{cases} -\Delta u^\epsilon(\cdot,t) = 0 & \text{in } \Omega_t(u) - K, \\ V = g(\frac{x}{\epsilon})|Du^\epsilon| & \text{on } \Gamma(u), \\ u^\epsilon = 1 & \text{on } K, \end{cases}$$

in $Q = (\mathbb{R}^n - K) \times (0,\infty)$ with initial data u_0. The following result was recently shown in [5] and [7]:

Theorem 6.1. *(See [5,7].) Let u^ϵ be a viscosity solution of* $(\mathrm{P2})_\epsilon$ *with initial data u_0. In addition suppose that Γ_0 is C^1. Then u^ϵ locally uniformly converges to the unique viscosity solution of*

$$(\mathrm{P2}) \quad \begin{cases} -\Delta u(\cdot,t) = 0 & \text{in } \Omega_t(u) - K, \\ V = (\langle \frac{1}{g} \rangle)^{-1}|Du| & \text{on } \Gamma(u), \\ u = 1 & \text{on } K \end{cases}$$

in Q with initial data u_0. Here $\langle h \rangle$ denotes the average of h, i.e., $\int_{[0,1]^n} h(x)\,dx$.

Parallel analysis as in Sections 3–5, yields the following:

Proposition 6.2. *Proposition* 3.5 *holds for* u *and* u^ϵ, *respectively solving* (P2) *and* $(\mathrm{P2})^\epsilon$.

Corollary 6.3. *If* Γ_0 *is* C^1, *then for sufficiently small* $\epsilon > 0$ *depending on* Γ_0

$$d\big((x,t), \Gamma(u)\big) \leqslant \epsilon^{1/90} \quad \text{for } (x,t) \in \Gamma\big(u^\epsilon\big).$$

Proof. Since Γ_0 is C^1 and u_0 is harmonic in Ω_0 with $u_0 = 1$ on K, one can conclude that

$$\frac{u(-de_n, 0)}{d} \in \big[d^{1/8}, d^{-1/8}\big] \quad \text{for small } d > 0.$$

Hence by a barrier argument, one can check that for sufficiently small $t > 0$ the set $\Gamma_t(u)$ lies outside $t^{9/8}$-neighborhood and inside $t^{7/8}$-neighborhood of $\Omega_0(u)$.

It follows that for sufficiently small $\epsilon > 0$, (H1) and (H2) in Proposition 3.5 is satisfied with $t_0 = \epsilon^{1/30}$, $\tau = \epsilon^{1/80}$ and $t_1 = 2\epsilon^{1/80}$. □

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