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Ann. I. H. Poincaré – AN 21 (2004) 1–23

ANNALES DE L'INSTITUT HENRI POINCARÉ
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Uniqueness of motion by mean curvature perturbed by stochastic noise

Unicité du mouvement par courbure moyenne perturbé par un bruit stochastique

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Received 13 May 2002; accepted 19 November 2002

Abstract

We present some uniqueness (non-fattening) results for the motion by mean curvature perturbed by stochastic noise. It is well known that for special initial data, the deterministic motion has multiple solutions, i.e., it develops interior. Our result for a particular evolution of curves in $\mathbb{R}^2$ illustrates that stochastic perturbations can select a unique solution in a natural way. The noise we use is white in time and constant in space. The results are formulated both almost surely and in probability law.

Résumé

Nous présentons des résultats d'unicité pour le mouvement par courbure moyenne, perturbé par un bruit stochastique. Il est bien connu que pour certaines conditions initiales, le mouvement a plusieurs solutions, i.e. il acquiert un intérieur. Notre résultat pour l'évolution de courbes spécifiques dans $\mathbb{R}^2$ illustre le fait que les perturbations stochastiques peuvent sélectionner une unique solution de manière naturelle. Le bruit utilisé ici est blanc dans le temps et constant dans l'espace. Nous donnons nos résultats en termes presque sûrs ainsi qu'en loi de probabilité.

1. Introduction

The study of the motion by mean curvature (MMC) of curves and surfaces before and after the development of singularities has a long history. It involves interesting mathematical theories coming from nonlinear partial

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[1] Partially supported by the NSF.

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2002.11.001

differential equations, geometric measure theory, singular perturbations and asymptotic analysis. It also has many applications in materials science and image processing. We refer to [1,25–27] for surveys on the mathematical studies and physical interpretation of MMC.

This paper investigates the issue of the non-uniqueness of, or, equivalently, the development of interior for solutions of MMC. It is well known that multiple solutions can arise from some initial data. When this happen, the notion of solutions used so far cannot uniquely predict the evolution of the surfaces. Some additional information is needed to select a unique solution. This further manifests itself in the fact that different approximation schemes can produce different solutions. As we also have in mind the physical applications of MMC in the study of materials interfacial motions, the phenomena of non-uniqueness would mean that some underlying physical processes might not be captured by the present model.

In this work we explore the use of noise to select a unique solution. The incorporation of stochastic perturbations has been widely considered in the physics community. The noise can come from thermal fluctuations, impurities or the atomistic processes describing the surface motions. The mathematical theory for the study of noise naturally involves nonlinear stochastic partial differential equations. Compared to its deterministic counterpart, this is largely an open area, which only recently has begun to be investigated in the work of P.-L. Lions and one of the authors (see [20–23]). The results presented here are among the first which describe quantitatively the effects of noise in terms of the statistics of the solutions.

A time dependent hypersurface $\Gamma(t)$ in $\mathbb{R}^N$ is said to evolve by MMC for $t \geqslant 0$, if the outward normal velocity v_n at every point of the surface equals the mean curvature κ

$$v_n = -\kappa, \tag{1.1}$$

with the sign convention for κ chosen so that a sphere shrinks.

There are several, basically equivalent, methods to construct sets moving by mean curvature. Classical partial differential equation and differential geometry techniques have been used for the study of smooth flows. But this approach requires special treatment to continue the solution when singularities and topological changes to the surfaces occur. We refer to [2] for a survey of this approach and related questions.

The first global in time (weak) solution for MMC was constructed in [9] using the theory of varifolds. This method, which also works for higher co-dimensional curves and surfaces, is so far only applicable in the isotropic case. In addition, there is a high degree of non-uniqueness in the solutions.

Another approach, which has been used widely to study a large number of applications – see [25] for a general survey – is the level-set formulation. This method makes use of a continuous function u such that its zero- (or any c-) level set defined as

$$\Gamma(t) = \left\{x \in \mathbb{R}^N\colon\ u(x,t) = 0\right\}$$

evolves by MMC. This function would then be a solution of the following degenerate parabolic equation

$$u_t = |Du| \operatorname{div}\left(\frac{Du}{|Du|}\right) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+. \tag{1.2}$$

Equations like (1.2) admit unique globally defined uniformly continuous solutions in the viscosity sense (see [10, 12,15,17]). Furthermore the invariance properties of (1.2) yield that $\Gamma(t)$ is uniquely defined as a function of the initial set $\Gamma_0 = \{x\colon\ u(x,0) = 0\}$. The afortiomentioned non-uniqueness phenomena are, however, reflected in the fact that $\Gamma(t)$ can develop non-empty interior, a fact referred to as fattening. In this case, $\Gamma(t)$ may not represent geometrically a hypersurface.

We say that u is has no-interior at time t and level-0 if

$$\partial\left\{x\colon\ u(x,t) > 0\right\} = \left\{x\colon\ u(x,t) = 0\right\}. \tag{1.3}$$

If the above fails, we say that u fattens or it develops interior. When this occurs, there are at least two solutions of MMC (see Theorem 2.1 of [5]), namely

$$\partial\Gamma^*(t) = \left\{x\colon\ u(x,t) \geqslant 0\right\} \quad \text{and} \quad \partial\Gamma_*(t) = \left\{x\colon\ u(x,t) > 0\right\},$$

which by definition differ from each other at time t. In fact $\Gamma^*(t)$ and $\Gamma_*(t)$ are respectively the maximum and minimum solutions. Although any other solution is trapped in between them, its location is not easily prescribed. Such a non-uniqueness phenomenon also corresponds to the fact that the solutions of MMC do not depend continuously on the initial data (in the L^1 topology). Different approximations of the initial curve or surfaces can lead to solutions which do not stay close to each other. Explicit examples of fattening are given in [3–5,7,12, 18,24], etc. General sufficient conditions for u not to develop fattening are given in [5].

The non-uniqueness issue described above also appears in the study of the convergence of a number of macroscopic and microscopic models in phase transitions (see [25] for a general overview). A canonical example is the study of the convergence of the phase field equation for MMC. In this approach, a phase-ordering parameter φ evolves according to the Allen–Cahn equation

$$\varphi_t - \Delta\varphi + \varepsilon^{-2}\varphi\big(\varphi^2 - 1\big) = 0, \tag{1.4}$$

where $\varepsilon > 0$ is a small parameter. Given reasonable initial data, as $\varepsilon \to 0$, φ approximates a sharp interface and the zero level set of φ converges to a solution of MMC. This convergence was established past singularities in the viscosity sense in [13] and later by [16] using varifolds. The result of [13] have been extended to more general anisotropic motions (see [25] and [6]). However, when the solution to MMC is not unique, it is not clear which solution φ will converge to.

Given the above approaches to solve MMC, we observe the following hierarchy of solutions:

$$\begin{pmatrix}\text{smooth}\\ \text{flows}\end{pmatrix} \subseteq \begin{pmatrix}\text{limits of}\\ \text{Allen–Cahn (1.4)}\end{pmatrix} \subseteq \begin{pmatrix}\text{Brakke}\\ \text{flows [9]}\end{pmatrix} \subseteq \begin{pmatrix}\text{zero level-}\\ \text{set of (1.2)}\end{pmatrix}.$$

It would be interesting to understand this hierarchy more quantitatively. This motivates us to search for a selection principle for the solutions of MMC.

In this work, we incorporate noise to MMC through the level set formulation. There are so far relatively few mathematical results which can handle in a general setting the stochastic perturbations for geometric motions. One of the main difficulties is how to combine the nonlinear, usually smoothing, effect of the surface evolutions and the roughening effect of the noise. In [28,29], variational minimization and stochastic calculus are combined in the framework of geometric measure theory to construct global solutions for stochastic MMC and dendritic crystal growth with Gibbs–Thomson condition. Funaki [14] considers a stochastic perturbation of (1.4) and shows that, as $\varepsilon \to 0$, φ converges to a front moving with normal velocity equal to mean curvature plus white noise, as long as the flow remains smooth and convex. This result is proven in [21] to hold for the generalized stochastic flow globally in time, i.e. past singularities.

A new theory for "stochastic" viscosity solutions for fully nonlinear second-order PDEs, which include the geometric PDEs such as (1.2) arising in the level set method, has recently been put forward in [20–23] by Lions and one of the authors. This theory applies to equations of the form

$$du + F\big(D^2u, Du, x, t\big)\,dt = \sum_{i=1}^{m} H^i(Du) \circ dW_t^i, \tag{1.5}$$

where $\{W^i(t)\colon\ i = 1, \ldots, m\}$ is a collection of independent Wiener processes and $\circ$ denotes the Stratonovich stochastic differential, and yields the existence, uniqueness and stability properties of the solutions. We describe briefly in Section 7 the results on stochastic PDEs, which are relevant to our analysis.

We apply the machinery developed for (1.5) to study stochastic MMC and to show that for a particular choice of an initial surface, for which there is non-uniqueness for the MMC, the stochastic motion converges to a unique deterministic MMC. Our results are formulated both almost surely and in probability. Similar results were also obtained independently in [11].

This paper is organized as follows. In Section 2 we summarize most of the notation used in the paper. The main results are stated in Section 3. The definitions of generalized flows and some technical lemmas are presented in

Section 4. The proofs of the theorems are presented in Sections 5 and 6. Section 7 summarizes the facts about stochastic viscosity solutions, which are used in the paper.

2. Notation

We summarize here most of the notation which are used often in the paper.

Let $\mathcal{O}$ be the collection of open subsets of $\mathbb{R}^N$. For any element U of $\mathcal{O}$, $\mathbf{1}_U(x)$ is the characteristic function of U which equals one for $x \in U$ and zero otherwise. ∂U, $\operatorname{Int} U$, $\overline{U}$ and U^c refer to the topological boundary, interior, closure and complement of U. For a given topological space X, $UC(X)$ and $BUC(X)$ refer to the spaces of uniformly continuous and bounded uniformly continuous real valued functions respectively. A sequence $\{f_n\}_{n\geqslant 1}$ defined on a locally compact topological space X is said to converge to f in $C(X)$, if it converges uniformly on compact subsets of X. Given a family of function $\{f_\varepsilon\}_{\varepsilon\geqslant 0}$, the symbols f^* and f_* denote to the upper- and lower semi-continuous limits of the family, i.e.,

$$f^*(x) = \limsup_{z\to x,\ \varepsilon\to 0} f_\varepsilon(z) \quad \text{and} \quad f_*(x) = \liminf_{z\to x,\ \varepsilon\to 0} f_\varepsilon(z).$$

For any continuous function f, we write

$$f^+(t) = \sup_{0\leqslant s\leqslant t} f(s) \quad \text{and} \quad f^-(t) = -\inf_{0\leqslant s\leqslant t} f(s).$$

We also denote by $B_r(p)$ and B_r the open balls of radius r centered at p and the origin $(0,0)$ respectively. In addition, $\mathcal{B}(t)$ denotes some general time varying balls $B_{R(t)}$ with radius $R(t) > 0$. When we use this notation, the exact values of the $R(t)$'s are not too important – they can be different even when $\mathcal{B}(t)$ appears in consecutive mathematical expressions. Finally ω denotes a realization of the Brownian motion.

3. The main results

Consider the pair of two touching balls $U_0 = B_1(p_1) \cup B_1(p_2)$, where $p_1 = (-1,0)$ and $p_2 = (1,0)$. The boundary ∂U_0 has the shape of a "figure-∞". As shown in [12] there are at least two generalized flows $U^*(t)$ and $U_*(t)$ for (1.1) starting from U_0. Using the language of the generalized front propagation, $U_*(t)$ and $U^*(t)$ are precisely given by

$$U^*(t) = \operatorname{Int}\bigl\{x\colon\ u(x,t) \geqslant 0\bigr\} \quad \text{and} \quad U_*(t) = \bigl\{x\colon\ u(x,t) > 0\bigr\}, \tag{3.1}$$

where $u \in BUC(\mathbb{R}^2 \times [0,\infty))$ is the unique viscosity solution to (1.2) with initial condition such that

$$U_0 = \bigl\{x\colon\ u(x,0) > 0\bigr\}, \quad \partial U_0 = \bigl\{x\colon\ u(x,0) = 0\bigr\}, \quad \text{and} \quad \overline{U}_0^c = \bigl\{x\colon\ u(x,0) < 0\bigr\}.$$

The fact that $U^*(t)$ is not equal to $U_*(t)$ is exactly due to the failure of the no-interior condition (1.3).

Geometrically, the first flow $\{U^*(t)\}_{t\geqslant 0}$ is characterized by the property that it contains the origin $(0,0)$ in its interior for small positive $t > 0$, i.e., there is some $\mathcal{B}(t)$ such that $\mathcal{B}(t) \subseteq U^*(t)$. (See Fig. 1.) In this sense, we say that the figure-∞ opens vertically at the origin.

The second flow $\{U_*(t)\}_{t\geqslant 0}$ is the union of two disjoint balls:

$$U_*(t) = B_{R(t)}(p_1) \cup B_{R(t)}(p_2),$$

where the radius $R(t) = \sqrt{R(0)^2 - 2t}$ satisfies the ordinary differential equation:

$$\frac{dR(t)}{dt} = -\frac{1}{R(t)}, \quad R(0) = 1. \tag{3.2}$$

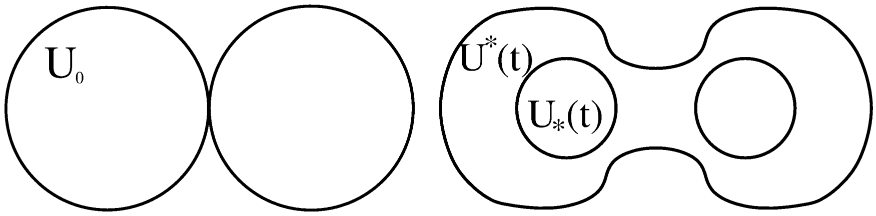

Fig. 1. An example of multiple solutions arising from two touching circles.

We say in this case that the figure-∞ opens horizontally.

The situation changes completely, if we consider, for $\varepsilon > 0$, the generalized flow with normal velocity

$$v_n = -\kappa + \varepsilon \dot{W}(t,\omega),$$

or the stochastic level set evolution

$$du = |Du|\,\mathrm{div}\left(\frac{Du}{|Du|}\right)dt + \varepsilon|Du| \circ dW(t,\omega). \tag{3.3}$$

The following result is obtained in the paper:

Theorem 3.1 (Almost sure convergence). *Let $U^\varepsilon(t,\omega)$ be a generalized flow to* (3.3) *starting from U_0. Then, for almost every ω* (*with respect to the Wiener measure*), *as $\varepsilon \to 0$, $U^\varepsilon(t,\omega)$ converges to $U^*(t)$ in the sense that, for all $t > 0$ and $x \in U^*(t) \cup \overline{U^*(t)}^c$,* $\lim_{\varepsilon\to 0} \mathbf{1}_{U^\varepsilon(t,\omega)}(x) = \mathbf{1}_{U^*(t)}(x)$.

Our second result is motivated by the following example of [7]. Consider the initial datum to be two separated balls $V_0 = B_1(q_1) \bigcup B_1(q_2)$ where $q_1 = (-2,0)$ and $q_2 = (2,0)$, and consider the generalized flow to

$$v_n = -\kappa + g(t),$$

or the level-set evolution

$$u_t = |Du|\left(\mathrm{div}\left(\frac{Du}{|Du|}\right) + g(t)\right), \tag{3.4}$$

where $g(t)$ is a time varying function chosen so that the two balls enlarge initially, because the value of $g(t)$ is large enough to offset the shrinking effect due to the curvature term. However, $g(t)$ decreases in t. Its precise form is chosen to make the two balls touch at some time t_*. At this time, the value of $g(t_*)$ equals exactly the curvature so that the two touching balls begin to separate. There are (at least) two generalized flows for $t > t_*$. The first one, denoted by $V^*(t)$, is similar to $U^*(t)$ – the figure-∞ opens vertically. The other one, denoted by $V_*(t)$, is similar to $U_*(t)$ – the figure-∞ opens horizontally. Similarly to (3.1), $V^*(t)$ and $V_*(t)$ can be defined as

$$V^*(t) = \mathrm{Int}\{x\colon\ v(x,t) \geqslant 0\} \quad \text{and} \quad V_*(t) = \{x\colon\ v(x,t) > 0\},$$

where $v \in BUC(\mathbb{R}^2 \times \mathbb{R}_+)$ is the unique viscosity solution to

$$v_n = -\kappa + g(t),$$

with initial condition satisfying

$$V_0 = \{x\colon\ v(x,0) > 0\}, \quad \partial V_0 = \{x\colon\ v(x,0) = 0\}, \quad \text{and} \quad \overline{V}_0^c = \{x\colon\ v(x,0) < 0\}.$$

The geometric intuition behind this example is that an external force is added to (1.2) to drive the curve into a configuration (such as two touching balls) so that non-unique solutions can arise. After this moment, the external force is gradually turned off.

For the stochastic version we consider the generalized flow to the motion law

$$v_n = -\kappa + g(t) + \varepsilon \dot{W}(t,\omega),$$

or the level-set equation

$$du = |Du|\left(\operatorname{div}\left(\frac{Du}{|Du|}\right) + g(t)\right) dt + \varepsilon |Du| \circ dW(t,\omega), \tag{3.5}$$

and study the behavior of the solution as $\varepsilon \to 0$.

The following result holds.

Theorem 3.2 (Uniqueness in Probability Law). *Let $V^\varepsilon(t,\omega)$ be a generalized flow to* (3.5) *starting from V_0. Then*

$$\begin{aligned} &P\left\{\omega\colon \lim_{\varepsilon\to 0} \mathbf{1}_{V^\varepsilon(t,\omega)}(x) = \mathbf{1}_{V^*(t)}(x),\ t \geqslant 0 \textit{ and } x \in V^*(t) \cup \overline{V^*(t)}^c\right\} \\ &\quad = P\left\{\omega\colon \lim_{\varepsilon\to 0} \mathbf{1}_{V^\varepsilon(t,\omega)}(x) = \mathbf{1}_{V_*(t)}(x),\ t \geqslant 0 \textit{ and } x \in V_*(t) \cup \overline{V_*(t)}^c\right\} = \frac{1}{2}, \end{aligned} \tag{3.6}$$

where P is the underlying Wiener measure.

Few remarks are in order. The four generalized flows U^*, U_*, V^* and V_* are all stable solutions. Theorem 3.1 indicates that, in the limit of vanishing white noise, one of the stable solutions is always selected. The heuristic reason behind this is that under the white noise perturbation, $U^\varepsilon(t,\omega)$ will almost surely open vertically for small positive time and for all $\varepsilon \neq 0$. Once this has happened, the boundary of $U^\varepsilon(t,\omega)$ near the origin has high curvatures, which prevent the $U^\varepsilon(t,\omega)$ from going back to the "closed" figure-∞ shape.

In Theorem 3.2, we have in essence constructed a unique probability measure on the space of generalized flows for (3.3). The number $1/2$ is actually the probability of the two balls touching each other at some time under the combined effect of $g(t)$ and dW_t. It is also related to the probability of reaching a certain level by some diffusion process. Once the two balls touch, we can invoke Theorem 3.1. Otherwise, the two balls will just evolve separately from each other. This explains that our second result cannot be improved to hold in the almost sure sense.

4. The generalized flows and level-set formulations and some technical lemmas

We briefly summarize here the definitions of generalized flows and level-set formulations of front propagation and state some basic facts. All the definitions and statements below are supposed to hold a.s. in ω. Hence, whenever there is no confusion, for notational simplicity we suppress the ω dependence.

To this end, we assume that $F \in C(\mathcal{S}^N \times \mathbb{R}^N \backslash \{0\} \times \mathbb{R}^N \times \mathbb{R}_+)$ is degenerate elliptic, i.e.,

$$\text{if } X \geqslant Y, \quad \text{then } F(X,p,x,t) \leqslant F(Y,p,x,t), \tag{4.1}$$

geometric, i.e., for all $\lambda > 0$, $\mu \in R$,

$$F(\lambda X + \mu p \otimes p, \lambda p, x, t) = \lambda F(X,p,x,t), \tag{4.2}$$

and it satisfies

$$-\infty < F_*(O,0,x,t) = F^*(O,0,x,t) < +\infty. \tag{4.3}$$

Furthermore assume that $H \in C^{0,1}(\mathbb{R}^N)$ is positively homogeneous of degree one, i.e., it satisfies for all $\lambda > 0$, $p \in \mathbb{R}^N$,

$$H(\lambda p) = \lambda H(p). \tag{4.4}$$

The next definition is an extension to the random case of the definition put forward in [6] to study front propagation in anisotropic environments.

Definition 4.1. A family $\{S_t(\omega)\}_{t\in(a,b)}$ of open subsets of $\mathbb{R}^N$ is a generalized super-flow (resp. sub-flow) with normal velocity $-(F\,dt + H\,dW)$ if and only if for all $x_0 \in \mathbb{R}^N$, $t \in (a,b)$, $r > 0$, $\alpha > 0$ and all smooth functions $\varphi : \mathbb{R}^N \to R$ such that $\{x \in \mathbb{R}^N\colon \varphi(x) \geqslant 0\} \subset S_t(\omega) \cap B_r(x_0)$, (resp. $\{x \in \mathbb{R}^N\colon \varphi(x) \leqslant 0\} \subset \overline{S_t(\omega)}^c \cap \overline{B_r(x_0)})$ with $|D\varphi| \neq 0$ on $\{x \in \mathbb{R}^N\colon \varphi(x) = 0\}$, there exists $h_0 > 0$ depending only on α and φ through its C^4-norm in $\overline{B_r(x_0)}$ such that, for all $h \in (0, h_0)$,

$$\bigl\{x \in \mathbb{R}^N\colon \varphi(x) - h\bigl[F^*\bigl(D^2\varphi(x), D\varphi(x), x, t\bigr) + H\bigl(D\phi(x)\bigr)(W_{t+h} - W_t)\bigr] + \alpha > 0\bigr\} \cap \overline{B_r(x_0)} \subset S_{t+h}(\omega),$$

(resp.

$$\bigl\{x \in \mathbb{R}^N\colon \varphi(x) - h\bigl[F_*\bigl(D^2\varphi(x), D\varphi(x), x, t\bigr) - H\bigl(D\phi(x)\bigr)(W_{t+h} - W_t)\bigr] - \alpha < 0\bigr\} \cap \overline{B_r(x_0)} \subset \overline{S_{t+h}}^c(\omega)).$$

A family $\{S_t(\omega)\}_{t\in(a,b)}$ of open subsets of $\mathbb{R}^N$ is called a generalized flow with normal velocity $-(F\,dt + H\,dW)$ if it is both a sub- and super-flow.

We next formulate set evolutions in terms of the level set equations. Let $\mathcal{E}$ be the collection of triplets (Γ, D^+, D^-) of mutually disjoint subsets of $\mathbb{R}^N$ such that Γ is closed and $D^\pm$ are open and $\mathbb{R}^N = \Gamma \cup D^+ \cup D^-$. For any $(\Gamma_0 \cup D_0^+ \cup D_0^-) \in \mathcal{E}$, choose $u_0 \in BUC(\mathbb{R}^N)$ such that

$$\Gamma_0 = \bigl\{x\colon u_0(x) = 0\bigr\}, \qquad D_0^+ = \bigl\{x\colon u_0(x) > 0\bigr\}, \qquad D_0^- = \bigl\{x\colon u_0(x) < 0\bigr\},$$

and let, a.s in ω, $u(\cdot,\cdot,\omega) \in BUC(\mathbb{R}^N \times \mathbb{R}_+)$ be the unique solution of the initial value problem

$$\begin{cases} \text{(i)}\, du + F(D^2u, Du, x, t)\,dt + H(Du)\circ dW = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}_+, \\ \text{(ii)}\, u = u_0 & \text{on } \mathbb{R}^N \times \{0\}. \end{cases} \tag{4.5}$$

We set

$$\begin{aligned} &\Gamma_t(\omega) = \{x\colon u(x,t,\omega) = 0\}, \\ &D_t^+(\omega) = \{x\colon u(x,t,\omega) > 0\}, \quad \text{and} \\ &D_t^-(\omega) = \{x\colon u(x,t,\omega) < 0\}. \end{aligned}$$

It turns out (see, for example, [10,12,17,21]) that $(\Gamma_t(\omega), D_t^+(\omega), D_t^-(\omega))$ is independent of the particular choice of the initial datum u_o and only depends on (Γ_0, D_0^+, D_0^-). Geometrically, these sets represent the boundary, inside and outside respectively of some evolving set starting at (Γ_0, D_0^+, D_0^-).

Definition 4.2. For $t \geqslant 0$ and a.s. in ω, let $E_t(\omega)\colon \mathcal{E} \to \mathcal{E}$ be the map

$$E_t\bigl(\Gamma_0, D_0^+, D_0^-\bigr)(\omega) = \bigl(\Gamma_t(\omega), D_t^+(\omega), D_t^-(\omega)\bigr).$$

The collection $\{E_t\}_{t\geqslant 0}(\omega)$ is called the generalized level-set evolution with normal velocity $-(F\,dt + H\,dW)$. Given $(\Gamma_0, D_0^+, D_0^-) \in \mathcal{E}$, the collection of closed sets $\{\Gamma_t(\omega)\}_{t\geqslant 0}$ is called the generalized level-set front propagation of Γ_0 with normal velocity $-(F\,dt + H\,dW)$.

It follows (see [5,6,10,12,15,17,21,23]) that the map $\{E_t\}_{t\geqslant 0}(\omega)$ is well-defined and it satisfies, a.s. in ω, the semigroup properties: $E_0 = \mathrm{id}_{\mathcal{E}}$ and $E_{t+s} = E_t \circ E_s$ for all $t, s \geqslant 0$.

We summarize in the next proposition the relationships between the two definitions above and the connection to the issue of fattening. Its proof, when $H = 0$, can be found in [6]. The stochastic case is discussed in [21].

Proposition 4.3. *The following hold, a.s. in ω:*

(i) *A family* $\{S_t\}_{t\in[0,T]}$ *of open subsets of* $\mathbb{R}^N$ *is a generalized flow* (*resp. super- or sub-flow*) *with normal velocity* $-(F\,dt + H\,dW)$ *if and only if the function* $u(x,t) = \mathbf{1}_{S_t}(x) - \mathbf{1}_{\bar{S}_t^c}(x)$ *is a viscosity solution* (*resp. super- or sub-solution*) *of* (4.5)(i).

(ii) *Let* $\{S_t\}_{t\in[0,T]}$ *be a generalized flow and* $(\Gamma_t, D_t^+, D_t^-)_{t\in[0,T]}$ *be the generalized level-set evolution of* (Γ_0, D_0^+, D_0^-) *with normal velocity* $-(F\,dt + H\,dW)$ *such that* $D_0^+ = S_0$ *and* $D_0^- = \bar{S}_0^c$. *Then, for all* $t > 0$, $D_t^+ \subset S_t \subset D_t^+ \cap \Gamma_t$. *If the no-interior condition* (1.3) *holds, then* $S_t = D_t^+$.

The relationship between a given outward normal velocity of a set and F and H is the following: If $v_n = G(Dn, n, x, t) + K(n)\dot{W}$ denotes the outward normal velocity of Γ_t, then the corresponding F and H are given by

$$F(X, p, x, t) = -|p|G\left(-\left(I - \frac{p\otimes p}{|p|^2}\right)X, -\frac{p}{|p|}, x, t\right),$$

and

$$H(p) = K\left(-\frac{p}{|p|}\right)|p|.$$

For example, if $v_n = -\kappa + g(t) + \varepsilon\dot{W}$, then (4.5)(i) has the form

$$du = \operatorname{tr}\left(\left(I - \frac{p\otimes p}{|p|^2}\right)X\right)dt - g(t)|p|\,dt - \varepsilon|p| \circ dW.$$

Given this correspondence between v_n and F and H, in this paper, the phrases “generalized flow to (the motion law) v_n”, “to $-(F\,dt + H\,dW)$” and “to $du + F(D^2u, Du, x, t)\,dt + H(Du)\,d\dot{W} = 0$” all have the same meaning.

One of the most important properties of viscosity solutions is their comparison principle. Below we state this principle for the equation

$$du + F\big(D^2u, Du, x, t\big)dt = \varepsilon|Du| \circ dW_t. \tag{4.6}$$

For the proof we refer to [17], among others, for the determistic case, and to [21] for the stochastic case.

Proposition 4.4. (i) *Let* $u \in UC(\mathbb{R}^N \times \mathbb{R}_+)$ *be a sub-solution and* v *be a discontinuous super-solution of* (4.6). *If* $u(\cdot,0) \leqslant v(\cdot,0)$ *on* $\mathbb{R}^N$, *then* $u(\cdot,t) \leqslant v(\cdot,t)$ *on* $\mathbb{R}^N \times \mathbb{R}_+$. *A similar result holds if* u *is a discontinuous sub-solution and* $v \in UC(\mathbb{R}^N \times \mathbb{R}_+)$ *is a super-solution.*

(ii) *Let* u *and* v *be respectively an upper-semicontinuous sub-solution and a lower-semicontinuous super-solution of* (4.6) *in* $Q \times \mathbb{R}_+$, *where* Q *is a bounded subset of* $\mathbb{R}^N$. *If* $u \leqslant v$ *on* $(Q \times \{0\}) \cup (\partial Q \times \mathbb{R}_+)$, *then* $u \leqslant v$ *on* $Q \times \mathbb{R}_+$.

The next two propositions are needed in the paper to compare generalized flows.

Proposition 4.5. *Let* $A(t)$ *and* $B(t)$ *be respectively generalized sub-flow and super-flow of* (4.6). *If* $A(0) \subseteq B(0)$ *and* $\operatorname{dist}(\partial A(0), \partial B(0)) > 0$, *then* $A(t) \subseteq B(t)$ *for* $t \geqslant 0$.

Proof. Fix $\eta > 0$, define $v(x,t) = \mathbf{1}_{B(t)} - \mathbf{1}_{\overline{B(t)^c}}$, and let $u \in UC(\mathbb{R}^N \times \mathbb{R}_+)$ be the solution of (4.6) with initial datum

$$u_\eta(x,0) = \begin{cases} \min\left(\frac{\operatorname{dist}(x,\partial A(0))}{\eta}, 1\right) & \text{for } x \in A_0, \\ \max\left(-\frac{\operatorname{dist}(x,\partial A(0))}{\eta}, -1\right) & \text{for } x \in A_0^c. \end{cases}$$

Since the assumptions imply, for sufficiently small η, that $u(x,0) \leqslant v_*(x,0)$, it follows from Proposition 4.4 that $u \leqslant v_*$ in $\mathbb{R}^N \times \mathbb{R}_+$. Proposition 4.3 then yields that $A(t) \subseteq \{u(x,t) \geqslant 0\} \subseteq \{v_* = 1\} = B(t)$, and hence the claim. □

Proposition 4.6. *Let $\{G_t(\omega)\}_{t\geqslant 0}$ and $\{H_t(\omega)\}_{t\geqslant 0}$ be two generalized flows in $\mathbb{R}^2$ to the motion law* (4.6). *Let $Q=\{|x|\leqslant a(\omega),\, y\geqslant 0\}$ and $\partial Q = A^-\cup A^0\cup A^+$, where, for a positive number $a(\omega)$, $A^-=\{x=-a(\omega),\, y\geqslant 0\}$, $A^0=\{|x|\leqslant a(\omega),\, y=0\}$ and $A^+=\{x=a(\omega),\, y\geqslant 0\}$. Assume that there exist two positive constants $T(\omega)$ and $b(\omega)$ such that, for $t\in(0,T(\omega))$, the following conditions hold*:

$$\begin{cases} \text{(i) } G_0(\omega)\cap Q\subseteq H_0(\omega)\cap Q,\\ \text{(ii) } \operatorname{dist}((\partial G_0(\omega))\cap Q, (\partial H_0(\omega))\cap Q)>0,\\ \text{(iii) } G_t(\omega)\cap(A^-\cup A^+)\subseteq H_t(\omega)\cap(A^-\cup A^+),\\ \text{(iv) } \operatorname{dist}((\partial G_t(\omega))\cap(A^-\cup A^+), (\partial H_t(\omega))\cap(A^-\cup A^+))>b(\omega). \end{cases} \tag{4.7}$$

If either $A^0\subseteq G_t(\omega)$ or $A^0\subseteq H_t(\omega)$, then

$$G_t(\omega)\cap Q\subseteq H_t(\omega)\cap Q. \tag{4.8}$$

We omit the proof of Proposition 4.6, since it is based on Proposition 4.4 and it is similar to the one of Proposition 4.5.

We conclude this section with another proposition which gives a quantitative estimate, in terms of the difference of their normal velocities, of how far interfaces move away from each other. This estimate is a new one and, we believe, of independent interest. Since the results holds a.s. in ω, once again we suppress this explicit dependence.

Proposition 4.7. *Suppose E_0 and F_0 are two open subsets of $\mathbb{R}^N$ such that $E_0\subseteq F_0$ and $\operatorname{dist}(\partial E_0,\partial F_0)>0$. Let $E(t)$ and $F(t)$ be respectively the generalized flows to $v_n=-\kappa$ starting from $E(0)=E_0$ and $v_n=-\kappa+\varepsilon\dot W(t)$ starting from $F(0)=F_0$. Then, for all $t>0$ such that $\varepsilon W^-(t)<\operatorname{dist}(\partial E_0,\partial F_0)$,*

$$E(t)\subseteq F(t) \quad \textit{and} \quad \operatorname{dist}\big(\partial E(t),\partial F(t)\big)\geqslant \operatorname{dist}(\partial E_0,\partial F_0)+\varepsilon W(t). \tag{4.9}$$

Proof. 1. We only present here the key steps of the proof and refer to [5] for the justification of some of the arguments.

2. Consider the functions

$$u(x,t)=\begin{cases}0 & \text{if } x\in\overline{E}(t),\\ -\infty & \text{if } x\in\overline{E(t)}^c\end{cases} \quad\text{and}\quad v(x,t)=\begin{cases}+\infty & \text{if } x\in F(t),\\ 0 & \text{if } x\in F(t)^c.\end{cases}$$

It follows (see the proof of Theorem 2.1 of [5]) that u and v are respectively sub- and super-solutions of (1.1) and (3.3).

3. Next define the function

$$\rho(t)=\sup_{(x,y)\in\mathbb{R}^2}\big\{u(x,t)-v(x,t)-|x-y|\big\}.$$

It is obvious that

$$\rho(t)=-\operatorname{dist}\big(\partial E(t),\partial F(t)\big).$$

Moreover, standard arguments from the theory of viscosity solutions yield that d satisfies

$$d\rho\leqslant -\varepsilon\, dW.$$

Upon integrating we have

$$-\rho(t)\leqslant -\rho(0)-\varepsilon W(t),$$

and hence the claim. □

5. The proof of Theorem 3.1

The proof of Theorem 3.1 consists of three parts, which we state below in the next three propositions. All the results below hold for almost all ω with respect to the underlying Wiener measure. Finally, $\{U^\varepsilon(t,\omega)\}_{t\geqslant 0}$ refers to a generalized flow to (3.3) starting from U_0.

Proposition 5.1 (The Initial Opening). *For each $\varepsilon > 0$, there exist a time $T_1^\varepsilon(\omega) > 0$ and collection of balls $\{\mathcal{B}(t,\omega)\}_{t\geqslant 0}$ such that, for $0 < t < T_1^\varepsilon(\omega)$, $\mathcal{B}^\varepsilon(t,\omega) \subseteq U^\varepsilon(t,\omega)$. In addition, $T_1^\varepsilon(\omega) \geqslant h_3(\omega)\varepsilon^{2+\gamma}$, for some positive numbers $h_3(\omega)$ and γ.*

Proposition 5.2 (The Uniform Opening). *There exist a time $T^*(\omega) > 0$ and collection of balls $\{\mathcal{B}(t,\omega)\}_{t\geqslant 0}$, independent of ε, such that, for $t \in [0, T^*(\omega)]$, $\mathcal{B}(t,\omega) \subseteq U^\varepsilon(t,\omega)$.*

Proposition 5.3 (The Limiting Motion). *For all $t > 0$ and $x \in U^*(t) \cup \overline{U^*(t)}^c$, $\lim_{\varepsilon\to 0} \mathbf{1}_{U^\varepsilon(t,\omega)}(x) = \mathbf{1}_{U^*(t)}(x)$.*

The main step in the proofs of Propositions 5.1 and 5.2 is the construction of a time dependent open subset $D^\varepsilon(t,\omega) \subseteq \mathbb{R}^2$ such that, for $t \in [0, T^*(\omega)]$ and some $\mathcal{B}(t,\omega)$,

$$\mathcal{B}(t,\omega) \subseteq D^\varepsilon(t,\omega) \subseteq U^\varepsilon(t,\omega).$$

In order to make this construction more transparent, we replace the Brownian motion $\{W(t,\omega)\colon t \geqslant 0\}$ by a smooth function $\{W^\nu(t,\omega)\colon t \geqslant 0\}$ such that $W^\nu(0,\omega) = 0$ and $\lim_{\nu\to 0} W^\nu(t,\omega) = W(t,\omega)$ locally uniformly in t and a.s in ω. In view of the results of [20] and [21], our conclusions follow by letting $\nu \to 0$.

To simplify the presentation, below we suppress the dependence on ω, ν, ε in all the time dependent functions such as $W^\nu(t,\omega)$, $U^\varepsilon(t,\omega)$, $D^\varepsilon(t,\omega)$ and $\mathcal{B}(t,\omega)$, but we keep the explicit dependence on ω and ε in all of the constants.

First we introduce the following definition which will be useful for the proof of Proposition 5.1 below.

Definition 5.4. For fixed $R > 1$ and $0 \leqslant r < \sqrt{R^2-1}$, define $I(R)$ and $J(R,r)$ to be the open subsets of $\mathbb{R}^2$ (see Fig. 2)

$$I(R) = B_R(p_1) \cup B_R(p_2) \quad \text{and} \quad J(R,r) = \left(I(R)^c + \overline{B_r}\right)^c,$$

where $U + V = \{x + y\colon x \in U,\ y \in U\}$ for any $U, V \subset \mathbb{R}^2$.

The set $I(R)$ is the union of two balls centered at $(\pm 1, 0)$ with radius $R > 1$. Its boundary $\partial I(R)$, which consists of two circular arcs, is piece-wise smooth with corners located at $(0, \pm\sqrt{R^2-1})$ on the y-axis. The set $J(R,r)$

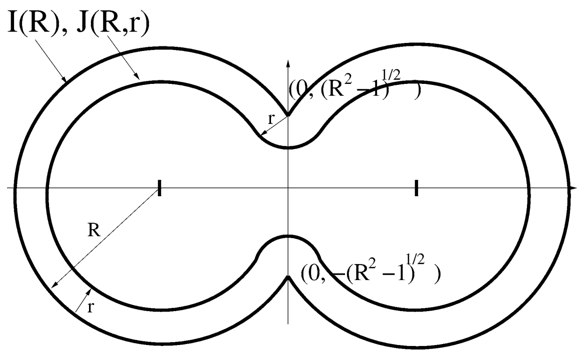

Fig. 2. Definitions of $I(R)$ and $J(R,r)$.

is the result of moving $\partial I(R)$ inward by distance r. Due to the two corners of $\partial I(R)$, $\partial J(R,r)$ consists of four circular arcs with centers at $(\pm 1, 0)$ and $(0, \pm\sqrt{R^2-1})$. The radii of these arcs are respectively R and r. The condition $r < \sqrt{R^2-1}$ ensures that $\partial J(R,r)$ is a connected curve and $(0,0)$ is in the interior of $J(R,r)$.

For future reference we note the following properties of $J(R,r)$:

$$\begin{cases} J(R,0) = I(R), & \\ J(R,r) + B_s = I(R-r+s) & \text{for all } r \leqslant s, \\ J(R,r) + B_s = J(R,r-s) & \text{for all } 0 \leqslant s \leqslant r, \\ (J(R,r)^c + \overline{B_s})^c = J(R,r+s) & \text{for all } r+s < \sqrt{\mathbb{R}^2-1}. \end{cases} \tag{5.1}$$

We now begin with the

Proof of Proposition 5.1. 1. Let $B_{R(t)}(p_1)$ and $B_{R(t)}(p_2)$ be the evolutions with normal velocity (3.3) of the two balls $B_1(p_1)$ and $B_1(p_2)$ comprising the initial set U_0. It is immediate that the radius $R(t)$ of the two balls satisfies the stochastic differential equation

$$dR(t) = -R(t)^{-1}\,dt + \varepsilon\,dW_t.$$

Let $X(t) = R(t) - 1$ be the x-coordinate of the right most point of the balls centered at p_1 (see Fig. 3). Then $X(t)$ solves

$$dX(t) = -\frac{1}{X(t)+1}\,dt + \varepsilon\,dW_t \quad \text{with } X(0) = 0,$$

i.e., $X(t)$ satisfies

$$X(t) = -\int_0^t \frac{ds}{X(s)+1} + \varepsilon W_t. \tag{5.2}$$

2. Since $W(t)$ is continuous and $W(0) = 0$, there exists a time interval $[0, \Lambda(\omega)]$, independent of ε, such that for all $\varepsilon > 0$,

$$\sup_{0\leqslant t\leqslant \Lambda(\omega)} \|X(t)\| \leqslant \frac{1}{2}.$$

Hence (5.2) implies that, for $t \in [0, \Lambda(\omega)]$,

$$X(t) \geqslant -2t + \varepsilon W(t). \tag{5.3}$$

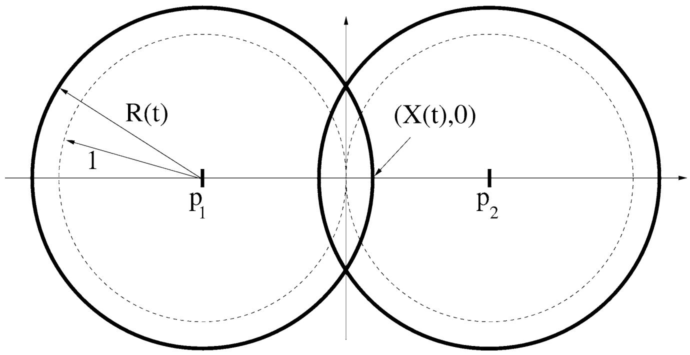

Fig. 3. Illustration of two circles crossing each other.

The comparison principle for (3.3) yields

$$B_{1+X(t)}(p_1) \cup B_{1+X(t)}(p_2) \subseteq U(t).$$

Therefore, as soon as $X(t) > 0$, the two balls will overlap with each other forcing $U(t)$ to open vertically. We show below that there exists a short time interval $[0, T_1^\varepsilon(\omega)]$ such that:

(i) there are many times t's in $[0, T_1^\varepsilon(\omega)]$ at which $X(t) > 0$, and
(ii) the set $U(t)$ remains opened vertically during the whole time interval $[0, T_1^\varepsilon(\omega)]$.

Lemma 5.5. *For $\delta \in (0, 1/2)$, let $\gamma = 4\delta/(1-2\delta)$. Then there exist positive constants $h_1(\omega)$, $h_2(\omega)$, and $h_3(\omega)$ such that, if $t \in [0, T_1^\varepsilon(\omega)]$, where $T_1^\varepsilon(\omega) = h_3(\omega)\varepsilon^{2+\gamma}$, then*

$$\varepsilon h_1(\omega) t^{1/2+\delta} \leqslant X^+(t), \qquad X^-(t) \leqslant \varepsilon h_2(\omega) t^{1/2-\delta}. \tag{5.4}$$

Indeed the estimate $X^+(t) > \varepsilon h_1(\omega) t^{1/2+\delta}$ implies that there are many times t's in the interval $[0, T_1^\varepsilon(\omega)]$ at which $X(t) > 0$. This in turn yields (i) above.

3. Proof of Lemma 5.5. 3.1. The classical continuity properties of the Brownian motion (see, for example, Theorem 2.9.23 in [19] and Theorem VIII.6 in [8]) yield the existence of positive constants $h_1(\omega), h_2(\omega)$ such that

$$h_1(\omega) t^{1/2+\delta} \leqslant W^+(t), \qquad W^-(t) \leqslant h_2(\omega) t^{1/2-\delta}. \tag{5.5}$$

3.2. It follows from (5.3) that, for all $s \in [0, t]$,

$$X^+(t) \geqslant \varepsilon W(s) - 2s \geqslant \varepsilon W(s) - 2t.$$

Hence

$$X^+(t) \geqslant \varepsilon W^+(t) - 2t \geqslant \varepsilon h_1(\omega) t^{1/2+\delta} - 2t = t^{1/2+\delta}\big(\varepsilon h_1(\omega) - 2t^{1/2-\delta}\big).$$

If t is restricted so that $4t^{1/2-\delta} < \varepsilon h_1(\omega)$, then $2X^+(t) \geqslant (2^{-1}\varepsilon h_1(\omega)) t^{1/2+\delta}$. Redefining $h_1(\omega)$ to be $2h_1(\omega)$ gives the left-hand side inequality of (5.5). All the other inequalities are proven similarly.

3.3. Finally, for $\gamma = 4\delta/(1-2\delta)$, set

$$T_1^\varepsilon(\omega) = \left(\frac{h_1(\omega)}{4}\right)^{2/(1-2\delta)} \varepsilon^{2/(1-2\delta)} = h_3(\omega)\varepsilon^{2+\gamma}. \qquad \square$$

4. To prove (ii) above we argue as follows: Since $X(t)$ is a smooth function of time, we may assume without loss of generality, that initially $X(t)$ is increasing and positive. If not, we can always reset the origin of time to be the first time this is true.

Next, for $i \geqslant 1$, we define, in the time interval $[0, T_1^\varepsilon(\omega)]$, two (finite) sequences of times $\{t_i\}$ and $\{s_i\}$ such that (see Fig. 4), for $i > 1$:

$$s_1 = 0 \quad \text{and} \quad s_i < t_i < s_{i+1},$$
$$X^+(t) = X(t) \quad \text{for } t \in [s_i, t_i],$$
$$X(t) < X(t_i) = X(s_{i+1}) \quad \text{for } t \in [t_i, s_{i+1}], \quad \text{and}$$
$$X(t)|_{[s_i,t_i]} \leqslant X(t)|_{[s_{i+1},t_{i+1}]}.$$

5. We define now the set $D(t)$ as follows:

$$\begin{cases} \text{(i) } D(t) = I(1 + X(t)) & \text{for } t \in [s_i, t_i]|_{\{i\geqslant 1\}}, \quad \text{and} \\ \text{(ii) } D(t) = J(1 + X(t_i), X(t_i) - X(t)) & \text{for } t \in [t_i, s_{i+1}]|_{\{i\geqslant 1\}}. \end{cases} \tag{5.6}$$

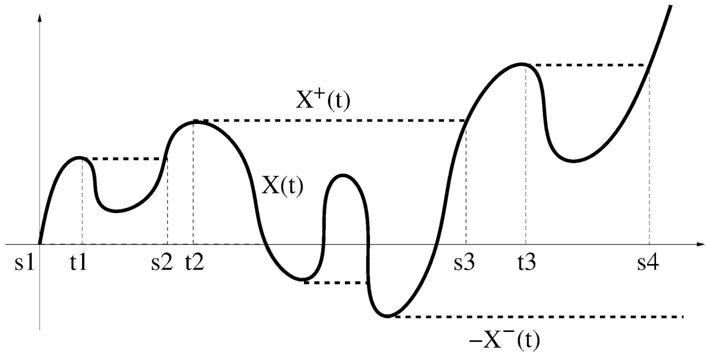

Fig. 4. Definition of s_i, t_i and $X^{\pm}(t)$.

The above definition together with Lemma 5.5 clearly imply that, for $0 < t < T_1^{\varepsilon}(\omega)$, there exists a $\mathcal{B}(t) \subseteq D(t)$, such that

$$\mathcal{B}(s_i) = B_{R_i} \quad \text{with } R_i \geqslant \left(\varepsilon s_i^{1/2+\delta}\right)^{1/2} \gg X(s_i).$$

We show below that

$$D(t) \subseteq U(t) \quad \text{for all } t \in \left[0, T_1^{\varepsilon}(\omega)\right]. \tag{5.7}$$

6. First we observe that $D(t)$ is in fact the generalized flow to the motion law

$$v_n = \frac{dX(t)}{dt}, \quad D(0) = U_0. \tag{5.8}$$

Indeed in each of the time intervals $[s_i, t_i]_{\{i \geqslant 1\}}$, $X(t)$ is increasing. Since

$$X(u)|_{u \in [s_i, t_i]} < X(v)|_{v \in [s_{i+1}, t_{i+1}]},$$

the solution to (5.8) is given exactly by (5.6)(i). In the remaining intervals $[t_i, s_{i+1}]_{\{i \geqslant 1\}}$, $X(t) \leqslant X(t_i) = X(s_{i+1})$. It then follows from (5.1) that the solution of (5.8) is given by (5.6)(ii). The maximum principle then yields that

$$D(t) \subseteq U(t) \quad \text{for } t \in [s_i, t_i].$$

7. To prove the same inclusion for $t \in [t_i, s_{i+1}]$ we need the following two lemmas, whose proof will be presented after the end of the ongoing one.

Lemma 5.6. *Fix* $R > 0$, $0 \leqslant r < \sqrt{R^2 - 1}$ *and let* $t \to \alpha(t)$ *be a smooth function such that* $\alpha(0) = 0$. *Then, for all* t *in any connected time interval containing* 0 *and such that* $0 \leqslant r - \alpha(t) < \sqrt{R^2 - 1}$, $\{J(R, r - \alpha(t))\}_{t \geqslant 0}$ *is a generalized flow to*

$$v_n = \frac{d\alpha}{dt} \quad \text{or} \quad u_t = |Du|\frac{da}{dt}. \tag{5.9}$$

Lemma 5.7. *Let* $\beta(t)$ *be the solution of*

$$\frac{d\beta}{dt} = -\left(R - r + \beta(t)\right)^{-1}, \quad \beta(0) = 0. \tag{5.10}$$

Then, for all t *such that* $0 \leqslant r - \beta(t) < \sqrt{R^2 - 1}$, $\{J(R, r - \beta(t))\}_{t \geqslant 0}$ *is a generalized sub-flow to the motion by mean curvature* (1.1) *or* (1.2).

8. To prove that $D(t) \subseteq U(t)$ for $t \in [t_i, s_{i+1}]$, we discretize $[t_i, s_{i+1}]$ as $\bigcup_j [t_i + j\Delta t, t_i + (j+1)\Delta t]$, where $0 \leqslant j \leqslant (s_{i+1} - t_i)(\Delta t)^{-1}$ and we make use of the above two lemmas in the following way.

Consider the following Euler approximation scheme of (5.7)

$$X_0 = X(t_i), \quad \text{and for } k \geqslant 0,$$
$$X_{2k+1} = X_{2k} + \varepsilon\big(W\big(t_i + (2k+1)\Delta t\big) - W(t_i + 2k\Delta t)\big),$$
$$X_{2k+2} = X_{2k+1} + Z(X_{2k+1}, \Delta t)$$

where $Z(a,t)$ is the solution at time t of

$$\frac{dZ}{dt}(a,t) = -\frac{1}{1+a+Z(a,t)}, \quad Z(0) = 0.$$

Let $X^{\Delta t}(t)$ be the linear interpolation of $\{X_j\}$ satisfying $X^{\Delta t}(t_i + j\Delta t) = X_j$. Then, as $\Delta t \to 0$, $X^{\Delta t}(t)$ converges to $X(t)$, uniformly on any finite time interval. This in turn leads to

$$\mathbf{1}_{D(t)}(x) = \lim_{\Delta t \to 0} \mathbf{1}_{J(1+X(t_i), X(t_i) - X^{\Delta t}(t))}(x), \quad \text{for } x \notin \partial J(1 + X(t_i), X(t_i) - X^{\Delta t}(t)). \tag{5.11}$$

Note that, for $t \in [t_i, s_{i+1}]$, Lemma 5.5 and the fact that $X(t) < X(t_i) = X(s_{i+1})$ yield that the right-hand side of the above equality is always defined for Δt sufficiently small.

9. Let $u \in BUC(\mathbb{R}^2 \times \mathbb{R}_+)$ be the unique solution to (3.3) with initial datum $u(\cdot,0) = g \in BUC(\mathbb{R}^2)$ such that

$$g \geqslant -1, \quad \{g > 0\} = U(t_i), \quad \{g = 0\} = \partial U(t_i), \quad \text{and} \quad \{g < 0\} = \overline{U(t_i)}^c.$$

For $0 < t < \Delta t$, we define the function $u^{\Delta t}$ by

$$u^{\Delta t}\big(x, k(\Delta t) + t\big) = \begin{cases} \mathcal{W}(t_i + k\Delta t, t) u^{\Delta t}(x, k\Delta t)(x) & \text{for } k \text{ even}, \\ \mathcal{K}(t) u^{\Delta t}(x, k\Delta t)(x) & \text{for } k \text{ odd}, \end{cases}$$

where $\mathcal{W}(s,t)f$ and $\mathcal{K}(t)f$ are respectively the solutions of

$$v_t = |Dv| \dot{W}(s+t) \quad \text{and} \quad v_t = |Dv| \operatorname{div}\left(\frac{Du}{|Du|}\right), \quad \text{with } v(0) = f.$$

The classical Trotter product formula (see Theorem 7.1) yields that, as $\Delta t \to 0$, $u^{\Delta t}$ converges in $C(\mathbb{R}^2 \times [0,T])$ to u.

Next, for $\eta > 0$, we consider the functions

$$p^{\Delta t}(x,t) = \eta\big(\mathbf{1}_{J(1+X(t_i), X(t_i) - X^{\Delta t}(t+\eta))} - \mathbf{1}_{\overline{J(1+X(t_i), X(t_i) - X^{\Delta t}(t+\eta))^c}}\big)$$

and

$$p(x,t) = \eta\big(\mathbf{1}_{D(t_i + t + \eta)} - \mathbf{1}_{\overline{D(t_i + t + \eta)^c}}\big).$$

Since, for t close to t_i, $X(t) \leqslant X(t_i)$, we can always find η small enough such that $p(\cdot,0) \leqslant g(\cdot)$ on $\mathbb{R}^2$. It then follows from Lemmas 5.6 and 5.7, Propositions 4.3 and 4.4 that, for $(x,t) \in \mathbb{R}^2 \times [0, \Delta t]$,

$$p^{\Delta t}(x, 2k\Delta t + t) \leqslant u^{\Delta t}(x, 2k\Delta t + t) \quad \text{and} \quad p^{\Delta t}(x, (2k+1)\Delta t + t) \leqslant u^{\Delta t}\big(x, (2k+1)\Delta t + t\big).$$

The convergences of $p^{\Delta t}$ to p and $u^{\Delta t}$ to u lead to

$$p_* \leqslant u \quad \text{on } \mathbb{R}^2 \times [t_i, s_{i+1} - \eta].$$

It follows that

$$D(t + \eta) \subseteq \big\{x \colon u(x,t) > 0\big\}.$$

But, in view of Proposition 4.3, we also have

$$\big\{x \colon u(x,t) > 0\big\} \subseteq U(t) \subseteq \big\{x \colon u(x,t) \geqslant 0\big\}.$$

Hence, for $t \in [t_1, s_{i+1} - \eta]$,

$$D(t+\eta) \subseteq U(t).$$

Finally, as $\eta \to 0$, the continuity of $t \to D(t)$ in t, gives the result. □

We continue with the proofs of Lemmas 5.6 and 5.7.

Proof of Lemma 5.6. Let $J(t)$ evolve according to (5.8) with initial datum $J(R,r)$. If α is nondecreasing in $[0,\eta]$, then $J(t) = U_0 + B_{\alpha(t)}$. If α is nonincreasing on $[0,\eta]$, then $J(t) = (U_0^c + \overline{B_{\alpha(t)}})^c$. Dividing the time interval into segments of monotonicity of $\alpha(t)$ and using (5.1) we conclude. □

The geometric reason behind Lemma 5.7 is that the radii of the circular arcs centered at p_1 and p_2 are shrinking according to MMC, while the radii of the circular arcs centered at $(0, \pm\sqrt{R^2-1})$ are expanding so that their normal velocity is opposite to the one coming from MMC. The rigorous argument is presented below in

Proof of Lemma 5.7. Let the function h be defined by

$$J\bigl(R, r - \beta(t)\bigr) \cap \bigl\{x \in \mathbb{R}^2\colon |x| \leqslant 1\bigr\} = \bigl\{(x,y)\colon -1 \leqslant x \leqslant 1, -h(x,t) < y < h(x,t)\bigr\}.$$

Since

$$h_t \leqslant \frac{h_{xx}}{1 + h_x^2},$$

and $J(R, r-\beta(t))$ is the graph, in $\{x \in \mathbb{R}^2\colon |x| \leqslant 1\}$, of h, it follows from the theory of MMC that $J(R, r-\beta(t))$ is a generalized sub-flow for MMC. The conclusion now follows. □

Proposition 5.2 asserts that $U(t)$ opens vertically for a time interval $(0, T^*(\omega)]$, which is independent of ε. The intuitive reason is that once $U(t)$ opens vertically, as it follows from Proposition 5.1, the part of $\partial U(t)$ near the origin has very high curvatures which pull $\partial U(t)$ further away in the vertical direction. Thus even under the effect of white noise perturbations, $U(t)$ can never go back to the figure-∞ shape.

To prove Proposition 5.2, we will construct two comparison sets and use Propositions 4.6 and 4.7 to extend the time interval during which $U(t)$ opens vertically, first from $[0, \mathrm{O}(\varepsilon^{2+\gamma})]$ to $[\mathrm{O}(\varepsilon^{2+\gamma}), \mathrm{O}(\varepsilon^{2-\alpha})]$ and then to $[\mathrm{O}(\varepsilon^{2-\alpha}), \mathrm{O}(1)]$. These two steps are made precise in the following two lemmas.

Lemma 5.8. *For any given constant $\alpha_1 \in (0, 1/2)$, there exist positive constants $\varepsilon_0(\omega)$ and $h_4(\omega)$ and collection of balls $\{\mathcal{B}(t)\}_{t \geqslant 0}$ such that, for all $\varepsilon \in (0, \varepsilon_0(\omega))$ and $t \in [0, h_4(\omega)\varepsilon^{2-\alpha_1}]$, $\mathcal{B}(t) \subseteq U(t)$.*

Proof. 1. Fix a constant $\beta \in (0, 1/2)$ to be specified later and apply Proposition 4.6 to the sets

$$F_0 = \bigl\{(x,y) \in \mathbb{R}^2\colon y < |x|\bigr\} \quad \text{and} \quad E_0 = \bigl\{(x,y) \in \mathbb{R}^2\colon y < |x| - \varepsilon^{2-\beta}\bigr\},$$

which satisfy

$$E_0 \subseteq F_0 \quad \text{and} \quad \operatorname{dist}(\partial E_0, \partial F_0) = \varepsilon^{2-\beta} > 0.$$

Proposition 4.6 yields that, if $|\varepsilon W^-(t)| \leqslant \varepsilon^{2-\beta}$, then $\operatorname{dist}(\partial E(t), \partial F(t)) \geqslant \operatorname{dist}(\partial E_0, \partial F_0) + \varepsilon W(t)$. It follows from (5.5) that, for all $t \in [0, h_4(\omega)\varepsilon^{2-\alpha_1}]$ and any $\delta \in (0, 1/2)$, we have

$$\varepsilon W^-(t) \leqslant \varepsilon h_2(\omega)\bigl(h_4(\omega)\varepsilon^{2-\alpha_1}\bigr)^{1/2-\delta} \leqslant \mathrm{O}(\omega)\varepsilon^{2-\alpha_1/2-(\alpha_1-2)\delta}.$$

If β and δ are chosen so that

$$0 < \alpha_1 + 2\delta(2-\alpha_1) < 2\beta, \tag{5.12}$$

then, for ε sufficiently small, $|\varepsilon W^-(t)| \leqslant \mathrm{O}(\omega)\varepsilon^{2-\beta}$. This also implies that $\mathrm{dist}(\partial E(t), \partial F(t)) > 0$ during this whole interval.

2. Since $E(t)$ is the solution to the mean curvature flow (1.1) and E_0 is a cone, it follows that $E(t)$ is a self-similar evolving shape of the form

$$E(t) = \{(x, y) \in \mathbb{R}^2\colon\ y < \sqrt{t} f(x/\sqrt{t}) - \varepsilon^{2-\beta}\}, \tag{5.13}$$

where $f(\cdot) : R \to \mathbb{R}_+$ is an even, convex and positive function.

At time $t = T_1^\varepsilon(\omega) = h_3(\omega)\varepsilon^{2+\gamma}$, the y-coordinate of the boundary $\partial E(t)$ at $x = 0$ satisfies, for sufficiently small ε,

$$\sqrt{T_1^\varepsilon(\omega)} f(0) - \varepsilon^{2-\beta} = \mathrm{O}(\omega)\varepsilon^{1+\gamma/2} - \varepsilon^{2-\beta} > 0. \tag{5.14}$$

In view of Lemma 5.5, the number γ can be taken to be as small as possible by choosing small enough δ. Hence

$$\{(x, y) \in \mathbb{R}^2\colon\ y < 0\} \subseteq E(t) \subseteq F(t) \qquad \text{for } t \in [T_1^\varepsilon(\omega), h_4(\omega)\varepsilon^{2-\alpha_1}]. \tag{5.15}$$

3. We now apply Proposition 4.6 with $H(t) = U(t + \eta)$, and $G(t) = F(t)$. It is clear that (4.7)(i) and (4.7)(ii) hold, if $0 < \eta$ is small enough and $a = 1/2$.

Next consider the set

$$A = B_{(\sqrt{3}-1)/4}\left(-\frac{1}{2}, \frac{\sqrt{3}+1}{4}\right) \cup B_{(\sqrt{3}-1)/4}\left(\frac{1}{2}, \frac{\sqrt{3}+1}{4}\right) \subseteq H_0 \backslash G_0.$$

If A evolves according to (3.3), it will not disappear before a time $T(\omega)$, which is independent of ε. Hence (4.7)(iii) and (4.7)(iv) also hold for $0 \leqslant t \leqslant T(\omega)$ and $b = (\sqrt{3} - 1)/8$.

4. For $t \in [0, T_1^\varepsilon(\omega)]$, Proposition 5.1 yields $\Gamma^0 \subseteq H(t)$. Hence Proposition 4.6 leads to (4.8) for this same time interval. Moreover, for $t \in [T_1^\varepsilon(\omega), h_4(\omega)\varepsilon^{2-\alpha_1}]$, in view of (5.14) and (5.15), we also have $\Gamma^0 \subseteq G(t)$. Then Proposition 4.6 again implies (4.8).

Combining all the above, we obtain, for $t \in [0, h_4(\omega)\varepsilon^{2-\alpha_1} - \eta]$, that $\mathcal{B}(t) \subseteq U(t + \eta)$. Letting $\eta \to 0$ concludes the proof. □

The next lemma is needed to conclude the proof of Proposition 5.2.

Lemma 5.9. *There exist $T^*(\omega) > 0$, $h_5(\omega) > 0$, $0 < \alpha_2 < 1/2$ and balls $\{\mathcal{B}(t)\}_{t\geqslant 0}$ such that, for $t \in [h_5(\omega)\varepsilon^{2-\alpha_2}, T^*(\omega)]$, $\mathcal{B}(t) \subseteq U(t)$.*

Proof. 1. Consider the sets $H(t) = U(t + \eta)$ and $G(t) = \overline{B_{R(t)}(0, 1)}^c$, where $R(t)$ solves (3.2) and η is a small positive number. It is easy to check that all the assumptions of Proposition 4.6 hold.

2. The lower y-coordinate of $\partial G(t)$ at $x = 0$ solves

$$\dot{y} = [1 - y(t)]^{-1} + \varepsilon \dot{W}(t), \quad y(0) = 0,$$

or in integral form

$$y(t) = \int_0^t \frac{ds}{1 - y(s)} + \varepsilon W(t).$$

For ε small enough, there exists $T^*(\omega)$ such that $|y(t)| \leqslant 1/2$ for $t \in [0 \leqslant T^*(\omega)]$. It follows that $y(t) \geqslant 2t + \varepsilon W(t)$. In addition, (5.5) yields that $|W(t)| \leqslant h_2(\omega)t^{1/2-\delta}$ in this same interval. Therefore,

$$t > \varepsilon|W(t)| \qquad \text{for } t > \varepsilon h_2(\omega)t^{1/2-\delta}.$$

This condition for t is equivalent to $t > h_5(\omega)\varepsilon^{2/(1+2\delta)} = h_5(\omega)\varepsilon^{2-\alpha_2}$ for some constant $h_5(\omega)$ and $\alpha_2 = 4\delta/(1+2\delta)$. Hence, for $t \in [h_5(\omega)\varepsilon^{2-\alpha_2}, T^*(\omega)]$, we have $y > t > 0$. □

We continue with the

Proof of Proposition 5.2. The conclusion follows by applying Proposition 4.6. Note that, in view of (5.12), we may assume that $0 < \alpha_2 < \alpha_1$. Then, for $t \in [0, h_4(\omega)\varepsilon^{2-\alpha_1}]$, Lemma 5.8 yields $\Gamma^0 \subseteq H(t)$. This leads to (4.8). Next, for $t \in [h_5(\omega)\varepsilon^{2-\alpha_2}, T^*(\omega)]$, we have $\Gamma^0 \subseteq G(t)$. Once more Proposition 4.6 implies (4.8). Therefore $G(t) \cap Q \subseteq H(t) \cap Q$ which implies that $\mathcal{B}(t) \subseteq U(t+\eta)$. Letting $\eta \to 0$ completes the proof. □

To prove Proposition 5.3, we need the following:

Theorem 5.10. *Let $\{K(t)\}_{t\geqslant 0}$ be a generalized flow to* (1.1) *starting from U_0. For any $r > 0$, let $r(t) = \sqrt{r^2 - 2t}$ and $T_r = (7/32)r^2$. If there exists $r_0 > 0$ such that, for all $r < r_0$ and $t \in [0, T_r]$, $B_{r-r(t)} \subseteq K(t)$ holds, then $K(t) = U^*(t)$.*

The idea of proof is to squeeze $K(t)$ between $U^*(t)$ and the following a re-scaled version of $U^*(t)$

$$\rho U^*\left(\frac{t-\delta}{\rho^2}\right) \subseteq K(t) \subseteq U^*(t), \quad 0 < \rho < 1,\ \delta > 0,$$

where, for any subset A of $\mathbb{R}^N$, $\rho A = \{\rho x\colon\ x \in A\}$.

Letting $\delta \to 0$ and then $\rho \to 1$ gives the desired result. The right-hand side inclusion is automatic by Proposition 4.3, while the left-hand side one requires an improved rate of opening for $K(t)$, which can be obtained from the hypotheses.

Proof of Theorem 5.10. 1. Let $\delta = T_r = (7/32)r^2$. The radius of the ball $B_{r-r(t)}$ at T_r is equal to $r/4$, or equivalently, $\sqrt{2\delta/7}$. Hence the rate of vertical opening is given, for t small, by $(2t/7)^{1/2}$. Below we improve this rate to be $Mt^{1/2}$, where M can be as large as possible.

2. The maximum principle and the hypotheses yield that, for all $0 < r < r_0$ and $t \in [0, T_r]$,

$$B_{\sqrt{1-2t}}(p_1) \cup B_{r-r(t)} \cup B_{\sqrt{1-2t}}(p_2) \subseteq K(t).$$

It follows that, for $t = \delta$, we have

$$B_{\sqrt{1-2\delta}}(p_1) \cup B_{\sqrt{2\delta/7}} \cup B_{\sqrt{1-2\delta}}(p_2) \subseteq K(\delta). \tag{5.16}$$

Consider the intersection point $\bar{A} = (\bar{x}, \bar{y})$ between $\partial B_{\sqrt{1-2\delta}}(p_2)$ and $\partial B_{\sqrt{2\delta/7}}$ which, obviously (see Fig. 5), satisfies

$$\bar{x}^2 + \bar{y}^2 = \frac{2\delta}{7} \quad \text{and} \quad (\bar{x}-1)^2 + \bar{y}^2 = 1 - 2\delta.$$

It follows that

$$\bar{x} = \frac{8\delta}{7} \quad \text{and} \quad \bar{y} = \sqrt{\frac{2\delta}{7} - \left(\frac{8\delta}{7}\right)^2},$$

and, hence, as $\delta \to 0$,

$$\frac{\bar{y}}{\bar{x}} \to \infty. \tag{5.17}$$

3. Let $G(t)$ be the generalized flow to (1.1) starting from $G(0)=\{y<|x|\tan\alpha\}$, where $\alpha\in(0,\frac{\pi}{2})$. Similarly to (5.13), $\partial G(t)$ has the self-similar shape

$$G(t)=\left\{(x,y)\in\mathbb{R}^2\colon\ y<\sqrt{t}\,f_\alpha(x/\sqrt{t})\right\},$$

where $f_\alpha(\cdot):R\to R^+$ is an even convex function. Hence the y-coordinate of $\partial G(\delta)$ at $x=0$ at time $t=\delta$ is given by $\sqrt{\delta}\,f_\alpha(0)$.

We apply once again Proposition 4.6 for the motion law (1.1) with $G(t)$ given above and $H(t)$ equal to $K(t+\delta)$. In view of (5.17), we have that, for all $M>0$, there exist $\delta_0,a,b>0$ and $\tan\alpha>M$ such that the hypotheses of Proposition 4.6 hold for $T=\delta<\delta_0$. Moreover, we may assume that $f_\alpha(0)>M$. Hence the following improved version of (5.16) holds:

$$B_{\sqrt{1-2\delta}}(p_1)\cup B_{M\delta^{1/2}}\cup B_{\sqrt{1-2\delta}}(p_2)\subseteq K(2\delta).$$

Considering again the intersection point $A^*=(x^*,y^*)$ between $\partial B_{\sqrt{1-2\delta}}(p_2)$ and $\partial B_{M\delta^{1/2}}$ (see Fig. 6), we find that it is of the form

$$x^*=\left(\frac{M^2+2}{2}\right)\delta\quad\text{and}\quad y^*=\sqrt{M^2\delta-x^{*2}}\approx M\delta^{1/2}\quad\text{as }\delta\to 0.$$

Next we choose $\rho=M^2/(M^2+2)$ – observe that $\rho\to 1$ as $M\to\infty$. Then, for M large enough, we have (see Fig. 6.):

$$B_\rho(-\rho,0)\cup B_\rho(\rho,0)\subseteq B_{\sqrt{1-2\delta}}(p_1)\cup B_{M\delta^{1/2}}\cup B_{\sqrt{1-2\delta}}(p_2)\subseteq K(2\delta).$$

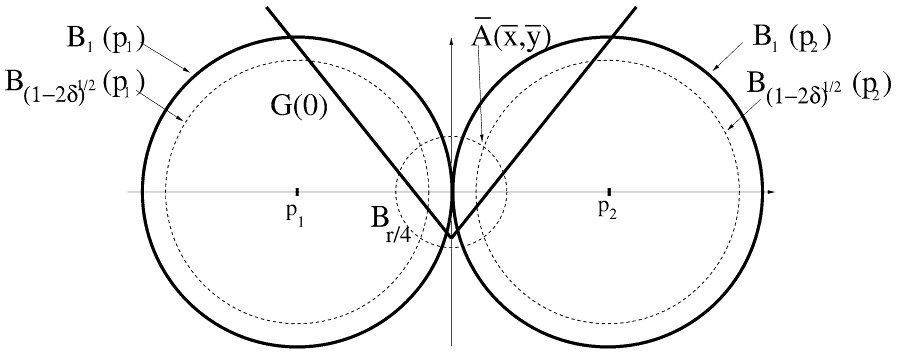

Fig. 5. Construction of a sub-solution using a wedge.

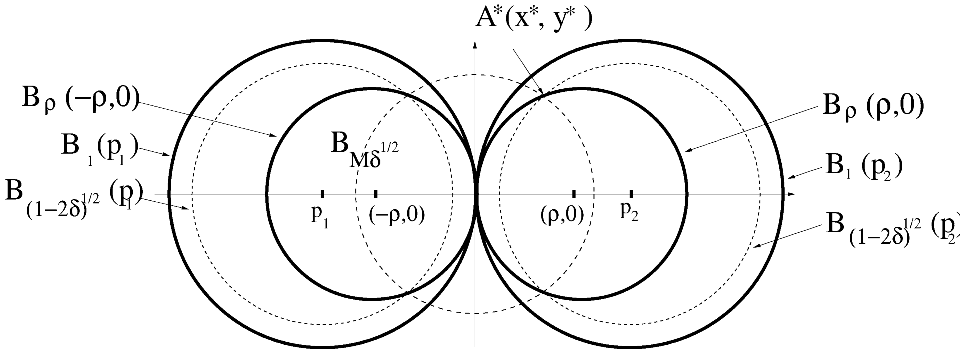

Fig. 6. Construction of a sub-solution using a re-scaled version of the initial data.

Since, $B_\rho(-\rho,0)\cup B_\rho(\rho,0)=\rho(B_1(p_1)\cup B_1(p_2))=\rho U_0$, it follows

$$\rho U_0\subseteq K(2\delta). \tag{5.18}$$

4. Define $v(x,t)=\mathbf{1}_{K(t+2\delta)}-\mathbf{1}_{\overline{K(t+2\delta)^c}}$, and, for $\eta>0$, let $u_\eta\in BUC(\mathbb{R}^2\times\mathbb{R}_+)$ be the solution to (1.2) with initial datum

$$u_\eta(x,0)=\begin{cases}\min(\frac{\operatorname{dist}(x,\partial U_0)}{\eta},1) & \text{for } x\in U_0,\\ \max(-\frac{\operatorname{dist}(x,\partial U_0)}{\eta},-1) & \text{for } x\in U_0^c.\end{cases}$$

To make use of (5.18), we also consider the solution u_η^ρ of (1.2) with initial datum $u_\eta^\rho(x,0)=u_\eta(x\rho^{-1},0)$. Note that, for all $\eta>0$ and $0<\rho<1$,

$$\{x\colon u_\eta(x,0)>0\}=U_0,\quad \{x\colon u_\eta(x,0)=0\}=\partial U_0,\quad \{x\colon u_\eta(x,0)<0\}=\overline{U}_0^c,\quad\text{and}$$
$$\{x\colon u_\eta^\rho(x,0)>0\}=\rho U_0,\quad \{x\colon u_\eta^\rho(x,0)=0\}=\rho\partial U_0,\quad \{x\colon u_\eta^\rho(x,0)<0\}=\rho\overline{U}_0^c.$$

The fact that $\operatorname{dist}(\partial(\rho U_0),\partial K(2\delta))>0$, yields the existence $\eta>0$ such that $u_\eta^\rho(\cdot,0)\leqslant v_*(\cdot,0)$ on $\mathbb{R}^2$. Since v is (viscosity) solution to (1.2), Proposition 4.4 implies that $u_\eta^\rho\leqslant v_*$ in $\mathbb{R}^2\times\mathbb{R}_+$, and, hence, for $t\geqslant\delta$,

$$\{x\colon u_\eta^\rho(x,t)\geqslant 0\}\subseteq K(t+2\delta)\quad\text{or equivalently}\quad \{x\colon u_\eta^\rho(x,t-\delta)\geqslant 0\}\subseteq K(t).$$

In addition the geometric properties of (1.2) also imply that

$$\{x\colon u_\eta^\rho(x,t-2\delta)\geqslant 0\}=\{x\colon u_1^\rho(x,t-2\delta)\geqslant 0\}.$$

Finally, in view of the uniqueness of the viscosity solution of (1.2), we have $u_1^\rho(x,t)=u_1(\rho^{-1}x,\rho^{-2}t)$. Hence the inclusion above can be written, for $t\geqslant\delta$, as

$$\left\{x\colon u_1\left(\frac{x}{\rho},\frac{t-2\delta}{\rho^2}\right)\geqslant 0\right\}\subseteq K(t).$$

5. Let p and $a>0$ be such that $p\in\operatorname{Int}\{x\colon u_1(x,t)\geqslant 0\}$ and $B_a(p)\subset\{x\colon u_1(x,t)\geqslant 0\}$. Using this ball as a comparison set, we deduce the existence of sufficiently small $0<\delta_1,\delta_2$ such that $B_{\delta_1}(p)\subset\{x\colon u_1(x,s)\geqslant 0\}$ for $s\in[t,t+\delta_2]$. Then the previous inclusion yields that $p\in K(t)$. This implies that $U^*(t)=\operatorname{Int}\{x\colon u_1(x,t)\geqslant 0\}\subseteq K(t)$. Finally the inclusion $K(t)\subseteq U^*(t)$ (see [6]) concludes the proof. □

We continue with the

Proof of Proposition 5.3. 1. Let u and $u^\varepsilon\in BUC(\mathbb{R}^2\times\mathbb{R}_+)$ be respectively the solutions to (1.1) and (3.3) with initial data such that

$$\{x\colon u(x,0)>0\}=\{x\colon u^\varepsilon(x,0)>0\}=U_0,$$
$$\{x\colon u(x,0)=0\}=\{x\colon u^\varepsilon(x,0)=0\}=\partial U_0,$$
$$\{x\colon u(x,0)<0\}=\{x\colon u^\varepsilon(x,0)<0\}=\overline{U}_0.$$

Proposition 4.3 yields

$$\{x\colon u^\varepsilon(x,t)>0\}\subseteq U^\varepsilon(t)\subseteq\{x\colon u^\varepsilon(x,t)\geqslant 0\}.$$

Moreover, in view of the stability properties of the stochastic viscosity solutions (Theorem 7.1), u^ε converges in $C(\mathbb{R}^N,\mathbb{R}_+)$ to u. Hence we have:

$$\mathbf{1}_{\{u(t)>0\}}\leqslant\liminf_{\varepsilon\to 0}\mathbf{1}_{U^\varepsilon(t)}\leqslant\limsup_{\varepsilon\to 0}\mathbf{1}_{U^\varepsilon(t)}\leqslant\mathbf{1}_{\{u(t)\geqslant 0\}}.$$

2. Arguing similarly to the proof of Lemma 5.9 yields that, for all sufficiently small $r > 0$ and $t \in [r^{2-\alpha}\varepsilon^{2-\alpha}h_4(\omega), T_r^\varepsilon(\omega)]$,

$$\overline{B_{r^\varepsilon(t)}(0,r)}^c \cap \left\{(x,y)\in\mathbb{R}^2\colon\ |x|<r/4, |y|<r\right\} \subseteq U(t)\cap\left\{(x,y)\in\mathbb{R}^2\colon\ |x|<r/4, |y|<r\right\},$$

where $r^\varepsilon(t)$ solves

$$dr^\varepsilon = -(r^\varepsilon)^{-1}\,dt + \varepsilon\,dW_t \quad \text{with } r^\varepsilon(0)=r \text{ and } r^\varepsilon\big(T_r^\varepsilon(\omega)\big) = 3r/4.$$

It is easy to see (no stochastic calculus needed) that, as $\varepsilon \to 0$, $r^\varepsilon(t) \to r(t) = \sqrt{r^2-2t}$ locally uniformly in t, $r(t)$ being the solution to $\dot r = -r^{-1}$. Moreover $T_r^\varepsilon \to T_r = 7r/32$.

We also have that

$$B_{r-r^\varepsilon(t)} \subseteq \overline{B_{r^\varepsilon(t)}(0,r)}^c \cap \left\{(x,y)\in\mathbb{R}^2\colon\ |x|<r/4, |y|<r\right\}.$$

Hence it follows, for $t\in[0,T_r^\varepsilon]$ and $\delta>0$ sufficiently small, that

$$B_{r-r^\varepsilon(t)} \subseteq U^\varepsilon(t+\delta).$$

Theorem 5.10 and Proposition 4.7 yield the existence of $\rho\in(0,1)$ and sufficiently small ε such that

$$\mathbf{1}_{\{x\colon u_1(x/\rho,(t-2\delta)/\rho^2)\geqslant 0\}} \leqslant \mathbf{1}_{\{x\colon u^\varepsilon(x,t)>0\}} \leqslant \mathbf{1}_{U^\varepsilon(t)} \leqslant \mathbf{1}_{\{x\colon u^\varepsilon(x,t)\geqslant 0\}}.$$

Letting $\varepsilon\to 0$, we obtain

$$\begin{aligned}\mathbf{1}_{\{x\colon u_1(x/\rho,(t-2\delta)/\rho^2)\geqslant 0\}} &\leqslant \liminf_{\varepsilon\to 0}\mathbf{1}_{\{x\colon u^\varepsilon(x,t)>0\}} \leqslant \liminf_{\varepsilon\to 0}\mathbf{1}_{U^\varepsilon(t)}\\ &\leqslant \limsup_{\varepsilon\to 0}\mathbf{1}_{U^\varepsilon(t)} \leqslant \limsup_{\varepsilon\to 0}\mathbf{1}_{\{x\colon u^\varepsilon(x,t)\geqslant 0\}} \leqslant \mathbf{1}_{\{x\colon u(x,t)\geqslant 0\}},\end{aligned}$$

which again gives

$$\mathbf{1}_{\mathrm{Int}\{x\colon u(x,t)\geqslant 0\}} \leqslant \liminf_{\varepsilon\to 0}\mathbf{1}_{U^\varepsilon(t)} \leqslant \limsup_{\varepsilon\to 0}\mathbf{1}_{U^\varepsilon(t)} \leqslant \mathbf{1}_{\{x\colon u(x,t)\geqslant 0\}}. \qquad \square$$

6. The proof of Theorem 3.2

The proof is based on the analysis of the stochastic perturbation of an ordinary differential equation. The idea is already given in the remarks following the statement of the theorem.

Proof of Theorem 3.2. 1. Consider first the deterministic motion law (3.4), which can be completely characterized by the motion of the two circles $B_{R(t)}(q_1)$ and $B_{R(t)}(q_2)$, and let T_e be their extinction time. If the function G is defined by

$$G(t,X) = -\frac{1}{X+2} + g(t),$$

then the x-coordinate of the right most point of $\partial B_{R(t)}(q_1)$, which is given by $X(t)=R(t)-2$, satisfies

$$dX(t) = G\big(t,X(t)\big)\,dt, \quad X(0)=-1, \quad \text{and} \quad X(T_e)=-2.$$

For simplicity, we define $X(t)\equiv -2$ for $t>T_e$. We assume that g and X are smooth functions of t, and that there exists a unique time $t_*\in[0,T_e]$ such that $X(t_*)=0$, i.e., $X(t)<0$ for $t\in[0,t_*)$.

2. Next consider the stochastic motion law (3.5). The the x-coordinate of the right most point of the evolving set, which is denoted by X_ε, solves the initial value problem

$$dX_\varepsilon(t) = G\big(t,X_\varepsilon(t)\big)\,dt + \varepsilon\,dW_t, \quad X_\varepsilon(0)=-1,$$

which also has has a unique solution.

Let $t_*^\varepsilon(\omega) = \inf\{t\colon\ X_\varepsilon(t) = 0\}$. The two balls $B_{R(t)}(q_1)$ and $B_{R(t)}(q_2)$ touch each other if and only if $t_*^\varepsilon(\omega) < \infty$. If this happens, we can invoke Theorem 3.1 to conclude that $V^\varepsilon(t,\omega)$ opens vertically for $t > t_*^\varepsilon(\omega)$. Otherwise, the two balls will never touch. In this case $V^\varepsilon(t,\omega)$ is just the union of two separated balls, which evolve independently of each other.

3. Define the sets

$$A = \bigl\{\omega\colon \text{ there exists } \varepsilon_0 \text{ such that for } 0 < \varepsilon < \varepsilon_0,\ X_\varepsilon(t) = 0 \text{ for some } t\bigr\}, \quad \text{and}$$
$$B = \bigl\{\omega\colon \text{ there exists } \varepsilon_0 \text{ such that for } 0 < \varepsilon < \varepsilon_0,\ X_\varepsilon(t) < 0 \text{ for all } t\bigr\}.$$

It follows that the claim we are trying to prove is equivalent to

$$P(A) = P(B) = \frac{1}{2}. \tag{6.1}$$

4. To prove the above equality, we write X_ε as $X_\varepsilon(t,\omega) = X(t) + \varepsilon Y_\varepsilon(t,\omega)$. Expanding $G(t, X_\varepsilon)$ around X, we find that Y^ε satisfies the differential equation

$$dY_\varepsilon = G_X\bigl(t, X(t)\bigr)Y_\varepsilon\,dt + \frac{\varepsilon}{2}G_{XX}(t,\xi){Y_\varepsilon}^2\,dt + dW_t,$$

where ξ is such that

$$\bigl|X(t) - \xi(t)\bigr| \leqslant \bigl|X(t) - Y_\varepsilon(t)\bigr|.$$

Since the coefficient of dW_t is a constant, deterministic theory is sufficient for the analysis of the above equation. It can be shown, in particular, that, a.s. in ω, as $\varepsilon \to 0$, $Y_\varepsilon(t)$ converges uniformly in $[0, T_e]$, to $Z(t)$ which solves the following linear stochastic differential equation:

$$dZ = G_X(t, X)Z\,dt + dW.$$

But Z is a (Hölder) continuous Gaussian processes with Hölder exponent $\gamma < \frac{1}{2}$. Therefore

$$P\bigl\{\omega\colon\ Z(t_*,\omega) > 0\bigr\} = P\bigl\{\omega\colon\ Z(t_*,\omega) < 0\bigr\} = \frac{1}{2}.$$

The claim (6.1) will follow as soon as we establish the inclusion

$$\bigl\{\omega\colon\ Z(t_*) > 0\bigr\} \subseteq A \quad \text{and} \quad \bigl\{\omega\colon\ Z(t_*) < 0\bigr\} \subseteq B.$$

5. To conclude we remark that

(i) $A = \{\omega\colon$ there exists ε_0 such that for $0 < \varepsilon < \varepsilon_0,\ \varepsilon Y_\varepsilon(t) = -X(t)$ for some $t\}$,
(ii) $B = \{\omega\colon$ there exists ε_0 such that for $0 < \varepsilon < \varepsilon_0,\ \varepsilon Y_\varepsilon(t) < -X(t)$ for all $t > 0\}$,
(iii) $X_\varepsilon(t_*) = 0,\ -X_\varepsilon(t) \geqslant 0$, and,
(iv) $\lim_{\varepsilon\to 0} -\varepsilon^{-1}X_\varepsilon = +\infty$, uniformly on $[0, T_e]\backslash(T_e - \eta, T_e + \eta)$ and all $\eta > 0$.

If $Z(t_*) > 0$, in view of the uniform convergence of Y_ε to Z, we have $Y_\varepsilon(t_*) > 0$ for ε small enough. Hence $\varepsilon Y_\varepsilon(t_*) \geqslant -X_\varepsilon(t_*) = 0$, which implies (i) above. If $Z(t_*) < 0$, the continuity of Z in t and the earlier remark yield that, for $t \in [0, T_e]$ and ε small enough, $-X_\varepsilon(t) > \varepsilon(Z(t) + \delta)$, where $2\delta = -Z(t_*)$. The uniform convergence of Y^ε to Z again leads to $-\varepsilon^{-1}X_\varepsilon(t) > Y_\varepsilon(t)$ for $t \in [0, T_e]$ and ε small enough. This implies (ii). The proof is now complete. □

7. Some of the basic properties of stochastic viscosity solutions

The notion of stochastic viscosity solutions for fully nonlinear, second-order, possibly degenerate, stochastic partial differential equations such as (3.3) or (3.5) was introduced by P.-L. Lions and one of the authors in [20–23]. Instead of repeating the definition, which is a bit cumbersome, we summarize in the next theorem, which is stated without proof, some of the key properties of the stochastic viscosity solutions of

$$\begin{cases} du + F(D^2u, Du, x, t)\,dt = \varepsilon|Du| \circ dW_t & \text{in } \mathbb{R}^N \times \mathbb{R}_+, \\ u(\cdot\,, 0) = u_0(\cdot) & \text{on } \mathbb{R}^N \end{cases} \tag{7.1}$$

and

$$\begin{cases} du = \varepsilon|Dv| \circ dW_t & \text{in } \mathbb{R}^N \times \mathbb{R}_+, \\ u(\cdot\,, 0) = u_0(\cdot) & \text{on } \mathbb{R}^N. \end{cases} \tag{7.2}$$

We have:

Theorem 7.1. *The following hold a.s. in ω:*

1. *There exists a unique solution to* (7.1) *and* (7.2).

2. *Let $\{\xi_\alpha(t)\}_{\alpha>0}$ and $\{\eta_\beta(t)\}_{\beta>0}$ be two families of smooth functions such that as α and $\beta \to 0$, ξ_α and η_β converge to W uniformly on compact in t and a.s. in ω. Let $\{u_\alpha\}_{\alpha>0}$ and $\{v_\beta\}_{\beta>0}$ in BUC$(\mathbb{R}^N \times \mathbb{R}_+)$ be the solutions to* (7.1) (*resp.* (7.2)) *with W replaced by ξ_α and η_β respectively. If $\lim_{\alpha,\beta\to 0} \|u_\alpha(\cdot\,, 0) - v_\beta(\cdot\,, 0)\|_{C(\mathbb{R}^N)} = 0$, then, for all $T > 0$, $\lim_{\alpha,\beta\to 0} \|u_\alpha - v_\beta\|_{C(\mathbb{R}^N\times[0,T])} = 0$. In particular, any smooth approximations of W produce solutions converging to the unique function stochastic viscosity solution of* (7.1) (*resp.* (7.2)).

3. *As $\varepsilon \to 0$, the solution u^ε of* (7.1) *converges in $C(\mathbb{R}^N \times \mathbb{R}_+)$ to the solution u of* (7.1) *with $\varepsilon = 0$.*

4. *Let S^F and S^W be respectively the solution operators of* (7.1) *for $\varepsilon = 0$ and* (7.2). *Then the function*

$$u^\Delta(\cdot\,, t) = S^W\big(t - [t/\Delta t]\big)\left(\prod_{i=1}^{[t/\Delta t]} \big[S^F(\Delta t)S^W(\Delta t)\big]\right)\varphi(\cdot)$$

converges in $C(\mathbb{R}^N \times \mathbb{R}_+)$ and a.s. in ω, as $\Delta t \to 0$, to the solution u of (7.1) *with $u(\cdot\,, 0) = \varphi(\cdot)$. Here $[x]$ denotes the largest integers less than or equal to x.*

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