Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT

Ann. I. H. Poincaré – AN 21 (2004) 97–120

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

A Liouville theorem for solutions of the Monge–Ampère equation with periodic data

Un théorème de Liouville pour les solutions de l'équation de Monge–Ampère avec données periodiques

L. Caffarelli [a,1], YanYan Li [b,*,2]

[a] *Department of Mathematics, The University of Texas, Austin, TX 78712, USA*
[b] *Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd., Piscataway, NJ 08854, USA*

Received 10 December 2002; accepted 8 January 2003

Abstract

A classical result of Jörgens, Calabi and Pogorelov states that any strictly convex smooth function u with $\det(D^2u) = \text{constant}$ in R^n must be a quadratic polynomial. We establish the following extension: any strictly convex smooth function u with $\det(D^2u)$ being 1-periodic in each variable must be the sum of a quadratic polynomial and a function which is 1-periodic in each variable. Given any positive periodic right-hand side, the existence and uniqueness of such solutions are well known.

Résumé

Selon un théorème classique de Jörgens, Calabi et Pogorelov, toute solution régulière et strictement convexe de l'équation $\det(D^2u) = \text{constante}$ dans R^n doit être égale à un polynôme quadratique. On démontre le résultat suivant : si u une fonction régulière et strictement convexe telle que $\det(D^2u)$ est 1-périodique par rapport à chaque variable, alors u est la somme d'un polynôme quadratique et d'une fonction 1-périodique par rapport à chaque variable. Étant donnée une fonction périodique et positive f, l'existence et l'unicité des solutions de $\det(D^2u) = f$ est un problème bien connu.

* Corresponding author.
E-mail address: yyli@math.rutgers.edu (Y.Y. Li).
[1] Partially supported by National Science Foundation Grant DMS-0140388 and G-37-X71-G4.
[2] Partially supported by National Science Foundation Grant DMS-0100819 and a Rutgers University Research Council Grant.

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2003.01.005

0. Introduction

Solutions of Monge–Ampère equations with periodic right-hand side appear in several contexts of geometry and applied mathematics: when lifting the equation from a Hessian manifold, in problems of optimal transportation, vorticity arrays, homogenization, etc. One question is the existence and uniqueness of periodic solutions. The basic converse question, from the point of view, for instance, of homogenization, is the classification of entire solutions: Let f be a positive periodic function and let u be an entire solution of $\det(D^2u) = f$, is u the sum of a quadratic polynomial and a periodic function? The answer is "yes" and this is the main purpose of the present work. Note that a particular case is the classical theorem of Jörgens, Calabi and Pogorelov [17,10,21] which asserts that classical convex solutions of

$$\det(D^2u) = 1, \quad \text{in } \mathbb{R}^n \tag{1}$$

must be quadratic polynomials. A simpler and more analytical proof, along the lines of affine geometry, of the theorem was later given by Cheng and Yau [12]. The first author extended the result for classical solutions to viscosity solutions [3]. Trudinger and Wang proved [22] that the only open convex subset Ω of $\mathbb{R}^n$ which admits a convex C^2 solution of $\det(D^2u) = 1$ in Ω with $\lim_{x\to\partial\Omega} u(x) = \infty$ is $\Omega = \mathbb{R}^n$. In an earlier paper [7], we proved that for any convex viscosity solution of $\det(D^2u) = 1$ outside a bounded subset of $\mathbb{R}^n$, $n \geqslant 3$, there exist an $n \times n$ real symmetric positive definite matrix A, a vector $b \in \mathbb{R}^n$ and a constant $c \in \mathbb{R}$ such that $\limsup_{|x|\to\infty} |x|^{n-2}(u - [\frac{1}{2}x'Ax + b\cdot x + c]) < \infty$. Existence of classical solutions to Dirichlet problem for Monge–Ampère equations was studied by Caffarelli, Nirerberg and Spruck in [8].

Description of our results: In the present paper we extend the theorem of Jörgens, Calabi and Pogorelov to

$$\det(D^2u) = f, \quad \text{in } \mathbb{R}^n, \tag{2}$$

where f is a positive periodic function.

Let $f \in C^0(\mathbb{R}^n)$ satisfy

$$f(x) > 0 \quad \forall x \in \mathbb{R}^n, \tag{3}$$

and, for some $a_1, \ldots, a_n > 0$,

$$f(x + a_i e_i) = f(x), \quad \forall x \in \mathbb{R}^n,\ 1 \leqslant i \leqslant n, \tag{4}$$

where $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$.

We are interested in convex solutions of (2), i.e., solutions u of (2) satisfying

$$(D^2u) > 0, \quad \text{in } \mathbb{R}^n. \tag{5}$$

We establish

Theorem 0.1. *Let $f \in C^\alpha(\mathbb{R}^n)$, $0 < \alpha < 1$, satisfy* (3) *and* (4), *and let $u \in C^2(\mathbb{R}^n)$ be a convex solution of* (2). *Then there exist $b \in \mathbb{R}^n$ and a symmetric positive definite $n \times n$ matrix A with $\det(A) = ⨍_{\prod_{1\leqslant i\leqslant n}[0,a_i]} f$, such that $v := u - [\frac{1}{2}x'Ax + b\cdot x]$ is a_i-periodic in ith variable, i.e.,*

$$v(x + a_i e_i) = v(x), \quad \forall x \in \mathbb{R}^n,\ 1 \leqslant i \leqslant n. \tag{6}$$

Some remarks:

Remark 0.1. The theorem of Jögens, Calabi, and Pogorelov is an easy consequence of the above theorem.

Remark 0.2. Let f be a bounded positive function in $C^\alpha(\mathbb{R}^n)$, $0 < \alpha < 1$, and let $u \in C^0(\mathbb{R}^n)$ be a convex viscosity of (2). Then $u \in C^{2,\alpha}(\mathbb{R}^n)$.

Remark 0.3. Because of the affine invariance, Theorem 0.1 still holds when the periodicity assumption is assumed in any n linearly independent directions, instead of in the $e_1, \dots, e_n$ directions.

Remark 0.4. By the affine invariance of the problem, we only need to establish Theorem 0.1 for $a_i = 1$ $\forall i$ and for f satisfying in addition

$$\int_{[0,1]^n} f = 1. \tag{7}$$

Remark 0.5. We believe that Theorem 0.1 holds for any convex viscosity solution u under the weaker hypothesis that $f \in L^\infty(\mathbb{R}^n)$, (4) holds a.e., and $0 < \inf_{\mathbb{R}^n} f \leqslant \sup_{\mathbb{R}^n} f < \infty$.

The existence and uniqueness (modulo constants) of solutions to periodic Monge–Ampère equations were studied by the second author.

Theorem 0.2 [20]. *Let $\mathbb{T}^n$ be a flat torus, $f \in C^\infty(\mathbb{T}^n)$ be a positive function, and let A be a symmetric positive definite $n \times n$ matrix satisfying*

$$\det(A) = \fint_{\mathbb{T}^n} f. \tag{8}$$

Then there exists a $v \in C^\infty(\mathbb{T}^n)$ satisfying

$$\det(A + D^2 v) = f, \quad \text{on } \mathbb{T}^n, \tag{9}$$

$$(A + D^2 v) > 0, \quad \text{on } \mathbb{T}^n. \tag{10}$$

Moreover, condition (8) *is necessary for the solvability of* (9)*, and solutions of* (9) *and* (10) *are unique up to addition of constants.*

Remark 0.6. If the smoothness assumption of f in Theorem 0.2 is weakened to $f \in C^{k,\alpha}(\mathbb{T}^n)$, $k \geqslant 0$, $0 < \alpha < 1$, there exists a solution $u \in C^{k+2,\alpha}(\mathbb{R}^n)$. For $k \geqslant 4$, the method in [20] is applicable; for $0 \leqslant k \leqslant 3$, this can be established by a smooth approximation of f based on the $C^{2,\alpha}$ theory of the first author in [2], together with the C^0 estimate of solutions in [20]. A different proof of Theorem 0.2 was given by the first author in [4]. Monge–Ampère equations on Hessian manifolds were studied in Cheng and Yau [11] and Caffarelli and Viaclovsky [9]. We plan to pursue some extensions of Theorem 0.1 in such a more general setting.

An auxiliary result: in our proof of Theorem 0.1, we need a homogenization type estimate. It states that a solution w of the Monge–Ampère equation with periodic right-hand side differs from the corresponding solution $\overline{w}$, with constant right-hand side, a power of the diameter of the lattice. Let $O \subset \mathbb{R}^n$ be a convex open subset satisfying

$$B_1 \subset O \subset B_n, \tag{11}$$

and let $\overline{w} \in C^0(\overline{O}) \cap C^\infty(O)$ denote the convex solution of

$$\begin{cases} \det(D^2 \overline{w}) = 1, & \text{in } O, \\ \overline{w} = 0, & \text{on } \partial O. \end{cases}$$

Let $\varepsilon_1, \dots, \varepsilon_n$ be n linearly independent vectors in $\mathbb{R}^n$, and let $g \in C^0(\mathbb{R}^n)$ be a positive function satisfying

$$g(x + \varepsilon_i) = g(x), \quad \forall x \in \mathbb{R}^n,\ 1 \leqslant i \leqslant n, \tag{12}$$

and

$$\fint_{\Omega_i} g = 1, \tag{13}$$

where $\Omega_i = \{x \in \mathbb{R}^n \mid x = \sum_i t_i \varepsilon_i,\ 0 \leqslant t_i \leqslant 1\}$ is the fundamental domain for the periodicity.

We consider

$$\begin{cases} \det(D^2 w) = g, & \text{in } O, \\ w = 0, & \text{on } \partial O. \end{cases} \tag{14}$$

We give an estimate to the L^∞ norm of $|w - \overline{w}|$ on $\overline{O}$:

Theorem 0.3. *Let $\varepsilon_1, \ldots, \varepsilon_n \in \mathbb{R}^n$ and $O \subset \mathbb{R}^n$ be as above, $g \in C^0(\mathbb{R}^n)$ be a positive function satisfying* (12) *and* (13)*, and let $w \in C^2(O) \cap C^0(\overline{O})$ be the convex viscosity solution of* (14). *Then for some constants $\beta, C > 0$, depending only on n and the upper bound of g, we have*

$$\|w - \overline{w}\|_{L^\infty(O)} \leqslant C \sum_i \|\varepsilon_i\|^\beta. \tag{15}$$

Remark 0.7. It is easy to see that we only need to establish Theorem 0.3 with an additional hypothesis that $g \in C^\infty(\mathbb{R}^n)$. The reason is that once we have estimate (15) with the constant C independent of the smoothness of g, we can approximate g by smooth g_j and obtain estimate (15) for w_j, the convex solution with respect to g_j, and then let j go to infinity. For the same reason, estimate (15) only requires the regularity of g be L^∞, while solution w is in the viscosity sense.

Remark 0.8. In view of a lemma of F. John (see, e.g., [14]), $B_1 \subset O \subset B_n$ can be replaced by $B_{r_1} \subset O \subset B_{r_2}$, $0 < r_1 \leqslant r_2 < \infty$, and then constants β and C in Theorem 0.3 depend also on r_1 and r_2.

Our paper is organized as follows. In Section 1, we establish Theorem 0.3. The first ingredient of our proof is the power deterioration of all derivatives of solutions to Monge–Ampère equations with constant right-hand side (Lemma 1.1), which we prove by modifying the Pogorelov estimates together with the $C^{2,\alpha}$ estimates of Evans and Krylov and the Schauder theory. We have found out recently, that step 1 of Lemma 1.1 is a particular case of a theorem of Chou and Wang in [13]. The second ingredient is the use of the periodic corrector (Theorem 0.2). In Section 2 we establish Theorem 0.1. The first step in our proof is to capture the quadratic behavior of the entire solution (Proposition 2.1). This follows from the general iteration scheme of the first author developed in [2], together with Theorem 0.3. Our second step is to establish an L^∞ bound for the second derivatives of the entire solution (Proposition 2.2). This is achieved by an application of the theory on the linearized Monge–Ampère operators developed by Caffarelli and Gutiérrez, with the help of the quadratic behavior of the solution obtained in the first step. More specifically, we make use of

Theorem 0.4 [6]. *Let O be a convex open subset of $\mathbb{R}^n$ satisfying $B_1 \subset O \subset B_n$, $n \geqslant 2$, and let $\phi \in C^2(\overline{O})$ be a convex function satisfying, for some constants λ and Λ,*

$$\begin{cases} 0 < \lambda \leqslant \det(D^2\phi) \leqslant \Lambda < \infty, & \text{in } O, \\ \phi = 0, & \text{on } \partial O. \end{cases}$$

Assume that $w \in C^2(O)$ satisfies

$$a_{ij} w_{ij} \geqslant 0, \quad w \geqslant 0, \quad \text{in } O,$$

where $a_{ij} = \det(D^2\phi)\phi^{ij}$ *is the linearization of the Monge–Ampère operator at* u. *Then, for any* $r > s > 0$,

$$\max_{x\in O,\ \mathrm{dist}(x,\partial O)>r} w \leqslant C \int_{x\in O,\ \mathrm{dist}(x,\partial O)>s} w,$$

where C *depends only on* n, λ, Λ, r *and* s.

Remark 0.9. This type of inequality is known in the literature as local maximum principle (see, e.g., p. 244 of [16]). In our case, it follows by noting that Theorems 1 and 4 in [6] are valid for supersolutions, and thus, the measure part of the proof of Lemma 4.1 in [6] applies to subsolutions. The details then follow exactly those of Theorem 4.8 in [5].

The third step in our proof of Theorem 0.1 is to capture $\sup_{\mathbb{R}^n} \Delta_e^2 u$ (Proposition 2.3), where $\Delta_e^2 u$ denotes the second incremental quotient of u and e is any period of u. By the first two steps, the second incremental quotient $\Delta_e^2 u$ is a subsolution for some uniformly elliptic operator. This step is then achieved by an appropriate use of the estimates of Krylov and Safonov on the second incremental quotient of u. To conclude the proof of Theorem 0.1, we make use of the periodic corrector and the Harnack inequality of Krylov and Safonov.

1. Proof of Theorem 0.3

In this section we prove Theorem 0.3. We first show that solutions with constant right-hand side deteriorate, together with all their derivatives, as a power of the distance to the boundary. This combines a modification of Pogorelov estimates [21] together with the the $C^{2,\alpha}$ interior estimates of Evans and Krylov [15,18] and Schauder estimates.

Lemma 1.1. *Let* $O \subset \mathbb{R}^n$ *be an open convex subset satisfying* $B_1 \subset O \subset B_n$, *and let* $u \in C^2(O) \cap C^0(\overline{O})$ *be a convex solution of*

$$\begin{cases} \det(D^2u) = 1, & \text{in } O,\\ u = 0, & \text{on } \partial O.\end{cases}$$

Then, for some positive constants C_k *and* β_k, *depending only on* n *and* k,

$$\left|D^k u(x)\right| \leqslant C_k\, \mathrm{dist}(x,\partial O)^{-\beta_k}, \quad x \in O,\ k = 1, 2, \ldots.$$

Proof. *Step* 1. Second derivative estimates. This is a modification of the original proof of Pogorelov [21]. For Reader's convenience, we include the proof. Using $\frac{1}{2}(|x|^2 - 1)$ and $\frac{1}{2}(|x|^2 - n^2)$ as comparison functions, we have, by the maximum principle, that $-\frac{n^2}{2} \leqslant \min_{\overline{O}} u \leqslant -\frac{1}{2}$.

We deduce from the above, using the convexity of u and the fact that $u = 0$ on ∂O, that

$$u(x) \leqslant -\frac{1}{4n}\,\mathrm{dist}(x,\partial O), \quad \forall x \in O. \tag{16}$$

By a barrier argument (see, e.g., Lemma 1 in [1] or Lemma 6.1 in [7]), we have, for any $0 < \alpha < 1$, that

$$u(x) \geqslant \begin{cases} -C\,\mathrm{dist}(x,\partial O)^{2/n}, & x \in O,\ n \geqslant 3,\\ -C\,\mathrm{dist}(x,\partial O)^{\alpha}, & x \in O,\ n = 2.\end{cases} \tag{17}$$

Here and in the following, C denotes various positive constants depending only on n when $n \geqslant 3$, and depends only on α when $n = 2$.

For $\delta > 0$, let

$$O' = \{x \in O \mid u(x) < -\delta\},$$

and let $w = u + \delta$. By (17),

$$\operatorname{dist}(O', \partial O) \geqslant \begin{cases} \delta^{n/2}/C, & n \geqslant 3, \\ \delta^{1/\alpha}/C, & n = 2. \end{cases}$$

It follows from the above and the convexity of u that

$$|\nabla w| = |\nabla u| \leqslant \begin{cases} C\delta^{(2-n)/2}, & \text{in } O',\ n \geqslant 3, \\ C\delta^{(\alpha-1)/\alpha} & \text{in } O',\ n = 2. \end{cases}$$

Also w is strictly convex and satisfies

$$\begin{cases} \det(D^2 w) = 1, & \text{in } O', \\ w = 0, & \text{on } \partial O'. \end{cases}$$

For simplicity, we will only treat the case $n \geqslant 3$ since the case $n = 2$ can be handled the same way.

Define $M > 0$ by

$$e^M = \max_{\overline{O'}} \left\{ |w| w_{11} e^{\delta^{n-2}|w_1|^2/2} \right\}.$$

By a translation of coordinates, the maximum is achieved at $0 \in O'$. Making the following affine transformation

$$\begin{cases} x_1' = x_1 + \sum_{i=2}^n \frac{w_{1i}(0)}{w_{11}(0)} x_i, \\ x_j' = x_j, \quad 2 \leqslant j \leqslant n, \end{cases}$$

and then rotating $x_2', \ldots, x_n'$ variables, we may assume without loss of generality that $(w_{ij}(0))$ diagonal.

Let

$$h := \log(-w) + \log w_{11} + \frac{\delta^{n-2} w_1^2}{2}.$$

Then h has a local maximum at 0 with $M = h(0)$. It follows that

$$h_i = \frac{w_i}{w} + \frac{w_{11i}}{w_{11}} + \delta^{n-2} w_1 w_{1i} = 0, \quad \text{at } 0,\ 1 \leqslant i \leqslant n, \tag{18}$$

and, at 0,

$$h_{ii} = \frac{w_{ii} w - w_i^2}{w^2} + \frac{w_{11ii} w_{11} - w_{11i}^2}{w_{11}^2} + \delta^{n-2} w_{1i}^2 + \delta^{n-2} w_1 w_{1ii} \leqslant 0, \quad 1 \leqslant i \leqslant n.$$

So at 0,

$$0 \geqslant \sum_i \frac{h_{ii}}{w_{ii}} = \frac{n}{w} - \sum_i \frac{w_i^2}{w^2 w_{ii}} + \frac{1}{w_{11}} \sum_i \frac{w_{11ii}}{w_{ii}} - \sum_i \frac{1}{w_{ii}} \left(\frac{w_{11i}}{w_{11}} \right)^2 + \delta^{n-2} w_{11} + \delta^{n-2} w_1 \sum_i \frac{w_{1ii}}{w_{ii}}.$$

It follows that at 0,

$$\delta^{n-2} |w| w_{11} \leqslant n + \sum_i \frac{w_i^2}{|w| w_{ii}} - \frac{|w|}{w_{11}} \sum_i \frac{w_{11ii}}{w_{ii}} + \frac{|w|}{w_{11}^2} \sum_i \frac{w_{11i}^2}{w_{ii}} - \delta^{n-2} |w| w_1 \sum_i \frac{w_{1ii}}{w_{ii}}. \tag{19}$$

Applying ∂_1 to the equation of w, we have

$$\sum_{i,j} w^{ij} w_{1ij} = 0,$$

where (w^{ij}) denotes the inverse matrix of (w_{ij}). In particular,

$$\sum_i \frac{w_{1ii}}{w_{ii}} = 0 \quad \text{at } 0. \tag{20}$$

Applying ∂_1 to (20), we have

$$\sum_i \frac{w_{11ii}}{w_{ii}} + \sum_{i,j} \partial_1\big(w^{ij}\big) w_{1ij} = 0, \quad \text{at } 0. \tag{21}$$

A calculation shows that at 0, we have

$$\partial_1\big(w^{ij}\big) = -\frac{w_{1ij}}{w_{ii} w_{jj}}, \quad 1 \leqslant i, j \leqslant n. \tag{22}$$

Putting (22) into (21), we have, at 0, that

$$\sum_i \frac{w_{11ii}}{w_{ii}} = -\sum_{i,j} \partial_1\big(w^{ij}\big) w_{1ij} = \sum_{i,j} \frac{w_{1ij}^2}{w_{ii} w_{jj}}. \tag{23}$$

Using (23) and (20), we deduce from (19) that, at 0,

$$\begin{aligned}
\delta^{n-2}|w| w_{11} &\leqslant n + \sum_i \frac{w_i^2}{|w| w_{ii}} - \frac{|w|}{w_{11}} \sum_{i,j} \frac{w_{1ij}^2}{w_{ii} w_{jj}} + \frac{|w|}{w_{11}^2} \sum_i \frac{w_{11i}^2}{w_{ii}} - \delta^{n-2}|w| w_1 \sum_i \frac{w_{1ii}}{w_{ii}} \\
&= n + \sum_i \frac{w_i^2}{|w| w_{ii}} - \frac{|w|}{w_{11}} \sum_{j \geqslant 2,\ i} \frac{w_{1ij}^2}{w_{ii} w_{jj}}.
\end{aligned} \tag{24}$$

By (18), we have, at 0, that

$$\left(\frac{w_i}{w}\right)^2 = \left(\frac{w_{11i}}{w_{11}} + \delta^{n-2} w_1 w_{1i}\right)^2, \quad 1 \leqslant i \leqslant n.$$

Write

$$\begin{aligned}
\sum_i \frac{w_i^2}{|w| w_{ii}} &= \frac{w_1^2}{|w| w_{11}} + |w| \sum_{j \geqslant 2} \left(\frac{w_j}{w}\right)^2 \frac{1}{w_{jj}} = \frac{w_1^2}{|w| w_{11}} + |w| \sum_{j \geqslant 2} \frac{1}{w_{jj}} \left(\frac{w_{11j}}{w_{11}}\right)^2 \\
&= \frac{w_1^2}{|w| w_{11}} + \frac{|w|}{w_{11}} \sum_{j \geqslant 2} \frac{w_{11j}^2}{w_{11} w_{jj}}.
\end{aligned}$$

Putting the above into (24), we have

$$\begin{aligned}
\delta^{n-2}|w| w_{11} &\leqslant n + \frac{w_1^2}{|w| w_{11}} + \frac{|w|}{w_{11}} \sum_{j \geqslant 2} \frac{w_{11j}^2}{w_{11} w_{jj}} - \frac{|w|}{w_{11}} \sum_{j \geqslant 2,\ i} \frac{w_{1ij}^2}{w_{ii} w_{jj}} \\
&= n + \frac{w_1^2}{|w| w_{11}} - \frac{|w|}{w_{11}} \sum_{i,j \geqslant 2} \frac{w_{1ij}^2}{w_{ii} w_{jj}} \leqslant n + \frac{w_1^2}{|w| w_{11}}.
\end{aligned}$$

It follows that

$$\delta^{n-2}|w| w_{11} \mathrm{e}^{\delta^{n-2} w_1^2/2} \leqslant n \mathrm{e}^{\delta^{n-2} w_1^2/2} + \frac{w_1^2 \mathrm{e}^{\delta^{n-2} w_1^2}}{|w| w_{11} \mathrm{e}^{\delta^{n-2} w_1^2/2}}.$$

Therefore

$$\delta^{n-2} \mathrm{e}^M \leqslant C + \frac{C}{\delta^{n-2} \mathrm{e}^M},$$

which implies

$$e^M \leqslant \frac{C}{\delta^{n-2}}.$$

Thus

$$|u_{11}| = |w_{11}| \leqslant \frac{C}{\delta^{n-1}}, \quad \text{in } O'',$$

where

$$O'' = \{x \in O' \mid w(x) < -\delta\} = \{x \in O \mid u(x) < -2\delta\}.$$

Since x_1 direction is chosen arbitrarily, we have

$$|D^2 u| \leqslant \frac{C}{\delta^{n-1}}, \quad \text{in } O''.$$

Now for any $x \in O$, set $\delta = |u(x)|/2$. We deduce from the above and (16) that

$$|D^2 u(x)| \leqslant C\delta^{1-n} \leqslant C \operatorname{dist}(x, \partial O)^{1-n}.$$

The second derivative estimate is established for $n \geqslant 3$. As mentioned earlier, the second derivative estimate for $n = 2$ case can be proved essentially the same way.

Step 2. Now we establish higher order derivative estimates, combining the estimates of Evans and Krylov with a normalization argument. For $x \in O$, let $d := \frac{1}{2}\operatorname{dist}(x, \partial O)$. Without loss of generality, $d < 1/2$. Set

$$v(y) = u(x+y) - u(x) - Du(x) \cdot y, \quad y \in B_d.$$

By the equation of u and the second derivative estimates of u, we have, for some positive constant C depending only on n,

$$\det\big(D^2 v(y)\big) = 1, \quad y \in B_d,$$

and

$$\frac{d^{\alpha_1}}{C} I \leqslant \big(D^2 v(y)\big) \leqslant \frac{C}{d^{\alpha_1}} I, \quad y \in B_d.$$

So for some positive constants α_2 and α_3 depending only on n, r_1 and r_2,

$$B_{d^{\alpha_2}} \subset \{y \mid v(y) < d^{\alpha_3}\} \subset B_{d/2}.$$

By a lemma of F. John, there exists some affine transformation $Ay = ay + b$ with $\det(a) = 1$ such that

$$B_R \subset A(\Omega) \subset B_{nR},$$

where $\Omega = \{y \mid v(y) < d^{\alpha_3}\}$. Since $|B_{d^{\alpha_2}}| \leqslant |\Omega| = |A(\Omega)| \leqslant |B_{d/2}|$, we have $d^{\alpha_2}/n \leqslant R \leqslant d/2$.

Let

$$w(z) = \frac{1}{R^2}\big(v\big(A^{-1}(Rz)\big) - d^{\alpha_3}\big), \quad z \in O := \frac{1}{R} A(\Omega).$$

Then w is a strict convex solution of

$$\det(D^2 w) = 1, \quad \text{in } O,$$

satisfying

$$w = 0, \quad \text{on } \partial O.$$

We also know that

$$w(\bar{z}) = \min_{\overline{O}} w,$$

where $\bar{z} = \frac{1}{R} A(0)$.

By the usual comparison argument and some barrier function argument, we know that

$$w(\bar{z}) \leqslant -C, \quad \text{and} \quad \operatorname{dist}(\bar{z}, \partial O) \geqslant \frac{1}{C},$$

where $C > 0$ is some number depending only on n. By the Pogorelov estimates, Evans and Krylov estimates and Schauder estimates, we have

$$\left|D^k w(\bar{z})\right| \leqslant C(k, n).$$

Since

$$D^2 w(\bar{z}) = \left(a^{-1}\right)^t D^2 v(0)\left(a^{-1}\right),$$

and since $(D^2 v(0)) \geqslant C^{-1} d^{\alpha_1} I$ and $(D^2 w(\bar{z})) \leqslant CI$, we have $(a^{-1})^t (a^{-1}) \leqslant Cd^{-\alpha_1} I$, i.e., $\|a^{-1}\| \leqslant Cd^{-\alpha_1/2}$. On the other hand, $\det(a) = 1$, so we have $\|a\| \leqslant Cd^{-(n-1)\alpha_1/2}$. The higher derivative estimates then follow from the above estimates of $|D^k w(\bar{z})|$. Lemma 1.1 is established. □

Now we prove the homogenization estimate: The main idea consists in showing that, if large, the maximum of the difference between w and $\overline{w}$ occurs far from the boundary, in the region where $\overline{w}$ is regular, and we may use there the periodic corrector plus a small quadratic polynomial to make out of w a super (sub) solution of the equation satisfied by $\overline{w}$.

Proof of Theorem 0.3. By Remark 0.7, we may assume without loss of generality that $g \in C^\infty(\mathbb{R}^n)$. Our proof makes use of Lemma 1.1 and Theorem 0.2. Throughout the proof, and unless otherwise stated, $\beta_i, \mu_i \in (0, 1)$ and $C_i > 1$ denote various positive constants depending only on n and the upper bound of g. Let

$$m = \max_{\overline{O}} |w - \overline{w}|.$$

By a barrier function argument,

$$-C_1 \operatorname{dist}(x, \partial O)^{\beta_1} \leqslant w, \quad \overline{w} \leqslant 0. \tag{25}$$

In particular $m \leqslant C_1$.

We will only treat the case

$$m = \max_{\overline{O}} (w - \overline{w}) > 0,$$

since the other case can be handled similarly.

Let $\bar{x} \in O$ be a maximum point of $w - \overline{w}$:

$$m = w(\bar{x}) - \overline{w}(\bar{x}).$$

By (25),

$$\operatorname{dist}(\bar{x}, \partial O) \geqslant \mu_1 m^{1/\beta_1}. \tag{26}$$

Let

$$u(x) = w(x) + \frac{m}{(6n)^2}|x - \bar{x}|^2.$$

Then

$$(u - \overline{w})(\bar{x}) = m.$$

On the other hand, since

$$|u - w| \leqslant \frac{m}{9}, \quad \text{on } \overline{O}, \tag{27}$$

we have

$$(u-\overline{w})\leqslant \frac{m}{9},\quad \text{on } \partial O.$$

So for some interior point $\tilde{x}\in O$,

$$(u-\overline{w})(\tilde{x})=\max_{\overline{O}}(u-\overline{w})\geqslant m. \tag{28}$$

By (28) and (27),

$$(w-\overline{w})(\tilde{x})\geqslant (u-\overline{w})(\tilde{x})-\frac{m}{9}\geqslant \frac{8m}{9}.$$

It follows, by (25), that

$$\operatorname{dist}(\tilde{x},\partial O)\geqslant \mu_1 m^{1/\beta_1}. \tag{29}$$

Here the values of μ_1 and β_1 are possibly smaller than previous values.

Let $\xi\in C^\infty(\mathbb{R}^n)$ be the unique solution of

$$\det\left(D^2\left[\frac{1}{2}x'D^2\overline{w}(\tilde{x})x+\xi(x)\right]\right)=g(x),\quad x\in\mathbb{R}^n,$$

satisfying

$$\left(D^2\left[\frac{1}{2}x'D^2\overline{w}(\tilde{x})x+\xi(x)\right]\right)>0,\quad x\in\mathbb{R}^n,$$
$$\xi(x+\varepsilon_i)=\xi(x),\quad x\in\mathbb{R}^n,\ 1\leqslant i\leqslant n,$$

and

$$\int_{\Omega_i}\xi=0.$$

The existence and uniqueness of ξ follows from Theorem 0.2.

Claim.

$$\|\xi\|_{L^\infty(\mathbb{R}^n)}\leqslant C_2 m^{-\beta_2}\sum_i\|\varepsilon_i\|^2. \tag{30}$$

Proof. Let

$$\varphi(x)=\frac{1}{2}x'D^2\overline{w}(\tilde{x})x+\xi(x),\quad x\in\mathbb{R}^n,$$

and for any fixed $y\in\mathbb{R}^n$ and $1\leqslant i\leqslant n$, let

$$h(t)=\xi(y+t\varepsilon_i),\quad t\in\mathbb{R}.$$

Since $(D^2\varphi)>0$ in $\mathbb{R}^n$, we have $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\varphi(y+t\varepsilon_i)>0$ for $t\in\mathbb{R}$. Consequently,

$$h''(t)\geqslant -\varepsilon_i'D^2\overline{w}(\tilde{x})\varepsilon_i\geqslant -\|\varepsilon_i\|^2\left\|D^2\overline{w}(\tilde{x})\right\|\geqslant -C_2\|\varepsilon_i\|^2m^{-\beta_3}.$$

Since h is a periodic function of period 1, we can argue as in [20]: let $\bar{t}\in[-1,0]$ be a point where $h'=0$. For all $0<t<s<1$, we have, by the above lower bound of h'', that

$$h(s)-h(t)=\int_t^s h'(\tau_1)\,\mathrm{d}\tau_1=\int_t^s\int_{\bar{t}}^{\tau_1}h''(\tau_2)\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1\geqslant -4C_2\|\varepsilon_i\|^2m^{-\beta_3}.$$

The above estimate, together with the fact that h is 1-periodic, implies that the oscillation of h is bounded by $4C_2\|\varepsilon_i\|^2 m^{-\beta_3}$. Since h is 1-periodic, the oscillation of h is bounded by $4C_2\|\varepsilon_i\|^2 m^{-\beta_3}$, and the estimate (30) follows easily. □

Since $\tilde{x}$ is an interior maximum point of $u - \overline{w}$, we have

$$\big(D^2(u-\overline{w})(\tilde{x})\big) \leqslant 0,$$

i.e.,

$$0 < \big(D^2 w(\tilde{x})\big) \leqslant \big(D^2\overline{w}(\tilde{x})\big) - \frac{2m}{(6n)^2} I. \tag{31}$$

Let

$$v(x) = \overline{w}(x) + \xi(x) - \frac{m}{(6n)^2}|x-\bar{x}|^2 + \frac{m}{(12n)^2}|x-\tilde{x}|^2.$$

Then

$$w(x) - v(x) = u(x) - \left(\overline{w}(x) + \xi(x) + \frac{m}{(12n)^2}|x-\tilde{x}|^2\right).$$

By (29) we can find β_3 and C_3 such that

$$B_{m^{\beta_3}/C_3}(\tilde{x}) \subset O,$$

and

$$\big|D^3\overline{w}(x)\big| \leqslant C_3 m^{-\beta_3}, \quad \forall x \in B_{m^{\beta_3}/C_3}(\tilde{x}).$$

Thus, we can find larger β_4 and C_4 such that

$$B_{m^{\beta_4}/C_4}(\tilde{x}) \subset B_{m^{\beta_3}/C_3}(\tilde{x}),$$

$$\begin{aligned}
\big(D^2 v(x)\big) &= \left(D^2\overline{w}(x) + D^2\xi(x) - \frac{6m}{(12n)^2} I\right) \\
&\leqslant \left(D^2\overline{w}(\tilde{x}) + n^2 C_3 m^{-\beta_3}|x-\tilde{x}| I + D^2\xi(x) - \frac{6m}{(12n)^2} I\right) \\
&\leqslant \left(D^2\overline{w}(\tilde{x}) + D^2\xi(x) + \frac{n^3 C_3}{C_4} m^{\beta_4-\beta_3} I - \frac{6m}{(12n)^2} I\right) \\
&< \big(D^2\overline{w}(\tilde{x}) + D^2\xi(x)\big), \quad \forall x \in B_{m^{\beta_4}/C_4}(\tilde{x}).
\end{aligned}$$

It follows that for every $x \in B_{m^{\beta_4}/C_4}(\tilde{x})$ with $(D^2 v(x)) \geqslant 0$, we have

$$\det\big(D^2 v(x)\big) < \det\big(D^2\overline{w}(\tilde{x}) + D^2\xi(x)\big) = g(x) = \det\big(D^2 w(x)\big). \tag{32}$$

Now

$$\begin{aligned}
(w-v)(\tilde{x}) &= u(\tilde{x}) - \overline{w}(\tilde{x}) - \xi(\tilde{x}) \\
&\geqslant (u-\overline{w})(\tilde{x}) - C_2 m^{-\beta_2} \sum_i \|\varepsilon_i\|^2.
\end{aligned}$$

Since $(u-\overline{w})(\tilde{x})$ is the maximum value of $u-\overline{w}$, we have, for all $x \in \partial B_{m^{\beta_4}/C_4}(\tilde{x})$, that

$$(w-v)(x) = (u-\overline{w})(x) - \xi(x) - \frac{m}{(12n)^2}|x-\tilde{x}|^2$$
$$\leqslant (u-\overline{w})(\tilde{x}) + C_2 m^{-\beta_2}\sum_i \|\varepsilon_i\|^2 - \frac{m^{1+2\beta_4}}{(12nC_4)^2}.$$

If

$$2C_2 m^{-\beta_2}\sum_i \|\varepsilon_i\|^2 \geqslant \frac{m^{1+2\beta_4}}{(12nC_4)^2},$$

we are done. Otherwise,

$$(w-v)(x) < (w-v)(\tilde{x}), \quad \forall x \in \partial B_{m^{\beta_4}/C_4}(\tilde{x}).$$

Let $x_1 \in B_{m^{\beta_4}/C_4}(\tilde{x})$ be an interior maximum point of $w-v$, then $(D^2 v(x_1)) \geqslant (D^2 w(x_1)) > 0$ and $\det(D^2 v(x_1)) \geqslant \det(D^2 w(x_1))$. This violates (32). Theorem 0.3 is established. □

2. Proof of Theorem 0.1

In this section we prove Theorem 0.1. We follow the three steps sketched in the introduction.

Step 1. Modulo an affine transformation, the behavior of u at infinity is $\frac{1}{2}|x|^2$:

Proposition 2.1. *There exist some $n \times n$ symmetric positive definite matrix A with* $\det(A) = 1$*, and some positive constants ε and C, such that*

$$\left|u(x) - \frac{1}{2}x'Ax\right| \leqslant C|x|^{2-\varepsilon}, \quad \forall |x| \geqslant 1. \tag{33}$$

We can always normalize u so that

$$u(0) = 0 \quad \text{and} \quad u \geqslant 0 \quad \text{in } \mathbb{R}^n. \tag{34}$$

By Lemma 2.1 in [7],

$$\liminf_{|x|\to\infty} \frac{u(x)}{|x|^{2/n}} > 0. \tag{35}$$

For $M > 0$, let

$$\Omega_M = \{x \in \mathbb{R}^n;\ u(x) < M\}.$$

By Propositions 2.4, 2.5 in [7],

$$C^{-1}M^{n/2} \leqslant |\Omega_M| \leqslant CM^{n/2} \quad \text{for all } M \geqslant 1 \tag{36}$$

for some positive constant C depending only on n, $\max_{\mathbb{R}^n} f$ and $\min_{\mathbb{R}^n} f$.

By a normalization lemma of John–Cordoba and Gallegos (see [14]), there exists some affine transformation

$$A_M(x) = a_M x + b_M,$$

with $\det(a_M) = 1$ such that

$$B_R \subset A_M(\Omega_M) \subset B_{nR}, \tag{37}$$

for some $R = R_M > 0$. It follows from (36) that

$$C^{-1}\sqrt{M} \leqslant R \leqslant C\sqrt{M}, \tag{38}$$

where $C \geqslant 1$ depends only on n, $\max_{\mathbb{R}^n} f$ and $\min_{\mathbb{R}^n} f$.

By Proposition 2.6 in [7],

$$2nR \geqslant \operatorname{dist}\big(a_M(\Omega_{M/2}), \partial a_M(\Omega_M)\big) \geqslant C^{-1}R, \tag{39}$$

and consequently

$$B_{R/C} \subset a_M(\Omega_M) \subset B_{2nR}, \tag{40}$$

where $C \geqslant 1$ depends only on n, $\max_{\mathbb{R}^n} f$ and $\min_{\mathbb{R}^n} f$.

For convenience, we make a normalization to unit size. Let

$$u_M(x) = \frac{1}{R^2} u\big(a_M^{-1}(Rx)\big), \quad x \in O_M := \frac{1}{R} a_M(\Omega_M).$$

By (40),

$$B_{1/C} \subset O_M \subset B_{2n}. \tag{41}$$

It is clear that

$$u_M(0) = \frac{1}{R^2} u(0) = 0,$$

$$\det\big(D^2 u_M(x)\big) = f\big(a_M^{-1}(Rx)\big), \quad x \in O_M, \tag{42}$$

and by (38),

$$u_M|_{\partial O_M} = \frac{M}{R^2} \in \big[C^{-1}, C\big].$$

Then, by the convexity of u_M,

$$0 \leqslant u_M \leqslant C \quad \text{in } O_M.$$

Let

$$E = \big\{k_1 e_1 + \cdots + k_n e_n;\ k_1, \ldots k_n \text{ are integers},\ k_1^2 + \cdots + k_n^2 > 0\big\},$$

and

$$E_j = \big\{e \in E;\ e = k_1 e_1 + \cdots + k_n e_n,\ |k_i| \leqslant j\big\}.$$

For $e \in E$, let

$$\tilde{e} = \frac{1}{R} a_M(e)$$

be the grids corresponding to e, for function u_M.

Lemma 2.1. *For some positive constants α and C, depending only on n, $\max_{\mathbb{R}^n} f$ and $\min_{\mathbb{R}^n} f$,*

$$\|\tilde{e}\| \leqslant C R^{-\alpha} \|e\|, \quad \forall e \in E.$$

Proof. For any $x \in \partial O_M$, we have, by [1],

$$u_M\left(\frac{1}{2}x\right) \geqslant \frac{1}{C} u_M(x),$$

from which we deduce

$$u\left(\frac{1}{2}y\right) \geqslant \frac{1}{C} u(y), \quad \forall |y| \geqslant 1.$$

Consequently, for some positive constants β, C depending only on n, $\max_{\mathbb{R}^n} f$ and $\min_{\mathbb{R}^n} f$,

$$u(y) \leqslant C|y|^\beta, \quad \forall |y| \geqslant 1. \tag{43}$$

For $\lambda e \in \partial\Omega_M$, we have, by (43),

$$M = u(\lambda e) \leqslant C|\lambda e|^\beta.$$

On the other hand, since $\frac{1}{R} a_M(\lambda e) \in \partial O_M \subset B_{2n}$, we have

$$|\lambda| \|\tilde{e}\| = \left| \frac{1}{R} a_M(\lambda e) \right| \leqslant 2n.$$

Lemma 2.1 follows from the above two inequalities. □

Let $\bar{\xi}$ be the convex solution of

$$\begin{cases} \det(D^2\bar{\xi}) = 1, & O_M, \\ \bar{\xi} = \frac{M}{R^2}, & \partial O_M. \end{cases}$$

Proof of Proposition 2.1. Given Theorem 0.3, the proof of Proposition 2.1 follows from the general iteration scheme of the first author in [2]. A proof can also be found in [7], see Propositions 3.1, 3.2 and (41) there; the only difference is that $\|u_M - \bar{\xi}\|_{L^\infty(O_M)} \leqslant CR^{-1}$ is known there instead of $\|u_M - \bar{\xi}\|_{L^\infty(O_M)} \leqslant CR^{-\alpha}$ which we have here. But the modification of the proof is very minor. □

One consequence of Proposition 2.1 is that for some positive constant C,

$$\|a_M\|, \|a_M^{-1}\| \leqslant C, \quad \forall M \geqslant 1. \tag{44}$$

Let

$$F(D^2u) = \det\bigl(D^2u\bigr)^{1/n},$$

and

$$F_{ij}(D^2u) = \frac{\partial F}{\partial u_{ij}}.$$

A consequence of the concavity of F is the following

Lemma 2.2. *Let f satisfy* (4) (*with $a_i = 1$*)*, and let u satisfy* (2)*. Then for every $e \in E$,*

$$F_{ij}\bigl(D^2u(x)\bigr)\partial_{ij}\bigl[u(x+e) + u(x-e) - 2u(x)\bigr] \geqslant 0 \quad \text{on } \mathbb{R}^n. \tag{45}$$

Remark 2.1. For (45) to hold, we only need that f is e-periodic and u satisfies (2).

Proof of Lemma 2.2. By the concavity of F, the equation of u, and the periodicity of f, we have

$$F\bigl(D^2w(x)\bigr) \geqslant \frac{1}{2}\bigl[F\bigl(D^2u(x+e)\bigr) + F\bigl(D^2u(x-e)\bigr)\bigr] = \frac{1}{2}\bigl[f(x+e) + f(x-e)\bigr] = f(x),$$

where $w(x) := \frac{1}{2}[u(x+e) + u(x-e)]$.

On the other hand, by the concavity of F and the equation of u,

$$F(D^2w) \leqslant F(D^2u) + F_{ij}(D^2u)\partial_{ij}(w-u) = f + F_{ij}(D^2u)\partial_{ij}(w-u).$$

The lemma follows immediately from the above two inequalities. □

Step 2. L^∞ estimate of the Hessian of u:

Proposition 2.2. *There exists some positive constant C such that*

$$\frac{I}{C} \leqslant \left(D^2u(x)\right) \leqslant CI, \quad \forall x \in \mathbb{R}^n. \tag{46}$$

For nonzero $e \in \mathbb{R}^n$, we introduce a notation for the second incremental quotient:

$$\Delta_e^2 u(x) = \frac{u(x+e) + u(x-e) - 2u(x)}{\|e\|^2},$$

where $\|e\|$ denotes the Euclidean norm of e.

The following lemma is a consequence of Theorem 0.4, a result of Caffarelli and Gutiérrez on the linearization of the Monge–Ampère operator.

Lemma 2.3. *For $r > 0$ and $e \in E$, there exists M_0, depending on u, r and $\|e\|$, such that for all $M \geqslant M_0$,*

$$\int\limits_{x \in O_M,\ \mathrm{dist}(x, \partial O_M) > r} \Delta_{\tilde{e}}^2 u_M \leqslant C, \tag{47}$$

and

$$0 < \Delta_{\tilde{e}}^2 u_M(x) \leqslant C, \quad \forall x \in O_M, \text{ and } \mathrm{dist}(x, \partial O_M) > r, \tag{48}$$

where C depends only on n, r, $\max_{\mathbb{R}^n} f$ and $\min_{\mathbb{R}^n} f$.

Remark 2.2. We emphasize that the constant C in Lemma 2.3 does not depend on $\|e\|$.

Proof of Lemma 2.3. Let $e \in E$, $\Delta_{\tilde{e}}^2 u_M$ is positive since u is strictly convex follows from the strict convexity of u. By Lemma 2.1, $|\tilde{e}| \to 0$ as $M \to \infty$. So, there exists M_0 such that for $M \geqslant M_0$, $|\tilde{e}| \leqslant r/8$. Let L be a line parallel to $\tilde{e}$, we have, by Lemma A.1 in Appendix A, that

$$\int\limits_{L \cap \{x \in O_M,\ \mathrm{dist}(x, \partial O_M) > r/4\}} \Delta_{\tilde{e}}^2 u_M \leqslant C, \quad \forall M \geqslant M_0,$$

where C depends on n, r, $\max_{\mathbb{R}^n} f$ and $\min_{\mathbb{R}^n} f$. Estimate (47) follows by integrating the above over all such lines.

To prove (48), we observe that u_M satisfies

$$0 < \min f \leqslant \det(D^2 u_M) = f\left(A_M^{-1}(Rx)\right) \leqslant \max f < \infty,$$

By Lemma 2.2, $w := \Delta_{\tilde{e}}^2 u_M$ satisfies

$$a_{ij}(x) w_{ij}(x) \geqslant 0, \quad x \in O_M \text{ and } \mathrm{dist}(x, \partial O_M) > r/2,$$

where a_{ij} is the linearization of the Monge–Ampère operator at u_M. Estimate (48) follows from (47) and Theorem 0.4, with r replaced by $r/2$. □

Lemma 2.4.

$$\gamma := \sup_{e \in E} \sup_{y \in \mathbb{R}^n} \Delta_e^2 u(y) < \infty. \tag{49}$$

Proof. For $e \in E$ and $y \in \mathbb{R}^n$, let $x = \frac{1}{R} a_M(y)$. Take M large so that $y \in \Omega_{M/2}$, we have, by (39),

$$\mathrm{dist}(x, \partial O_M) \geqslant \frac{1}{C}$$

for some constant C depending only on n, $\min_{\mathbb{R}^n}$ and $\max_{\mathbb{R}^n} f$. Then by (48) and (44),

$$\Delta_e^2 u(y) = \frac{\|a_M(e)\|^2}{\|e\|^2}\Delta_{\tilde{e}}^2 u_M(x) \leqslant C\|a_M\|^2 \leqslant C. \qquad \square$$

The following lemma is a consequence of Lemma 2.1 in [7].

Lemma 2.5. *For $\lambda > 0$ and $r \geqslant 2$, let $u \in C^2((-3,3)^{n-1} \times (-r,r))$ satisfy*

$$(D^2u) > 0, \quad \det(D^2u) \geqslant \lambda, \quad \textit{in } (-3,3)^{n-1} \times (-r,r),$$

and

$$0 \leqslant u \leqslant 1 \quad \textit{in } (-2,2)^n.$$

Then, for some positive constant $C = C(n) > 0$,

$$\max_{|s|\leqslant r} u(0',s)^n \geqslant \left(\frac{r\lambda}{C} - 1\right).$$

The next lemma is a consequence of the Pogorelov estimate.

Lemma 2.6. *Let $g \in C^4(\overline{B}_1)$ be a positive function, and let $v \in C^4(B_1) \cap C^0(\overline{B}_1)$ be a convex function satisfying*

$$\det(D^2v) = g, \quad \textit{on } B_1,$$

and

$$v(0) = 0.$$

We assume that

$$0 < \mu \leqslant v \leqslant \mu^{-1}, \quad \textit{on } \partial B_1.$$

Then for some $r_0 \in (0,1)$ and $C > 0$, depending only on n, μ, $\min_{\overline{B}_1} g$, and $\|g\|_{C^4(\overline{B}_1)}$, we have that

$$|D^2v| \leqslant C, \quad \textit{on } B_{r_0}.$$

Remark 2.3. In the above lemma, B_1 can be replaced by any bounded open subset Ω, then C, r_0 will depend on $\operatorname{dist}(0, \partial\Omega)$ and $\operatorname{diam}(\Omega)$.

Proof of Lemma 2.6. We only need to show that there exists some $\bar{r} > 0$, depending only on μ, such that

$$B_{2\bar{r}} \subset \{x \in B_1;\ v(x) < \mu/2\}. \tag{50}$$

Indeed let $v(\bar{x}) = \mu/2$, by the convexity of v,

$$v(x) \geqslant v(\bar{x}) + \nabla v(\bar{x})(x - \bar{x}), \quad \forall x \in \overline{B}_1. \tag{51}$$

In particular,

$$0 = v(0) \geqslant v(\bar{x}) - \nabla v(\bar{x})\bar{x},$$

i.e.,

$$\frac{\mu}{2} = v(\bar{x}) \leqslant \left|\nabla v(\bar{x})\right||\bar{x}|. \tag{52}$$

Taking $x \in \partial B_1$ such that $\nabla v(\bar{x})$ and $x - \bar{x}$ point the same direction, we have, by (51) and (52),

$$\mu^{-1} \geqslant v(x) \geqslant v(\bar{x}) + \left|\nabla v(\bar{x})\right||x - \bar{x}| \geqslant \frac{\mu}{2} + \left|\nabla v(\bar{x})\right|\left(1 - |\bar{x}|\right). \tag{53}$$

It follows from (52) and (53) that

$$\frac{\mu}{2} \leqslant \left|\nabla v(\bar{x})\right||\bar{x}| \leqslant \frac{\mu^{-1} - \mu/2}{1 - |\bar{x}|}|\bar{x}|.$$

Clearly $|\bar{x}| \geqslant 3\bar{r}$ for some $\bar{r}$ depending only on μ. □

Now we give the

Proof of Proposition 2.2. For $x \in \mathbb{R}^n$, let

$$\tilde{u}(z) = u(z + x) - \left[u(x) + \nabla u(x)z\right].$$

Then

$$\tilde{u}(0) = 0, \quad \tilde{u} \geqslant 0 \quad \text{in } \mathbb{R}^n.$$

Since

$$\sup_{e\in E}\sup_{z\in\mathbb{R}^n} \Delta_e^2 \tilde{u}(z) = \sup_{e\in E}\sup_{y\in\mathbb{R}^n} \Delta_e^2 u(y) \leqslant \gamma,$$

we have (using $\sup_{e\in E} \Delta_e^2 \tilde{u}(0) \leqslant \gamma$ and the convexity of $\tilde{u}$)

$$\sup_{B_r} \tilde{u} \leqslant C(n)\gamma r^2, \quad \forall 1 \leqslant r < \infty.$$

On the other hand, for $\bar{z} \in \partial B_r$, we have (using $\sup_{e\in E} \Delta_e^2 \tilde{u}(\bar{z}/2) \leqslant \gamma$)

$$\tilde{u}\left(\frac{\bar{z}}{2} + e\right) + \tilde{u}\left(\frac{\bar{z}}{2} - e\right) - 2\tilde{u}\left(\frac{\bar{z}}{2}\right) \leqslant \gamma\|e\|^2, \quad \forall e \in E.$$

It follows, by the convexity of $\tilde{u}$ and the fact that $\tilde{u}(0) = 0$, that

$$\tilde{u}(z) \leqslant 2\tilde{u}\left(\frac{\bar{z}}{2}\right) + C(n)\gamma \leqslant \tilde{u}(\bar{z}) + C(n)\gamma, \quad \forall z \in \frac{\bar{z}}{2} + (-2, 2)^n.$$

Applying Lemma 2.5 to $\tilde{u}(\bar{z}/2 + \cdot)/(\tilde{u}(\bar{z}) + C(n)\gamma)$ (modulo a rotation, i.e., think of $\bar{z}/|\bar{z}|$ as e_n), we have (recall that $\tilde{u}(0) = 0$)

$$\tilde{u}(\bar{z})^n = \max_{|s|\leqslant|\bar{z}|/2} \tilde{u}\left(\frac{\bar{z}}{2} + s\frac{\bar{z}}{|\bar{z}|}\right)^n \geqslant \left(\frac{r \min_{\mathbb{R}^n} f}{C(n)[\tilde{u}(\bar{z}) + \gamma]^n} - 1\right)\left(\tilde{u}(\bar{z}) + C(n)\gamma\right)^n.$$

If $\tilde{u}(\bar{z}) \leqslant \gamma$, then

$$\tilde{u}(\bar{z})^n \geqslant \gamma^n\left(\frac{r \min_{\mathbb{R}^n} f}{C(n)\gamma^n} - 1\right).$$

Fix some suitably large r, depending only on n, γ and $\min_{\mathbb{R}^n} f$, such that

$$\gamma^n\left(\frac{r \min_{\mathbb{R}^n} f}{C(n)\gamma^n} - 1\right) \geqslant 1,$$

we have $\tilde{u}(\bar{z}) \geqslant 1$. So, for such r, we have

$$\min_{\partial B_r} \tilde{u} \geqslant \min\{\gamma, 1\}.$$

Since

$$\det\bigl(D^2\tilde{u}(z)\bigr) = f\bigl(z + x - [x]\bigr),$$

where $[x]$ denotes the integer part of x. We have, by Lemma 2.6, that

$$\bigl|D^2u(x)\bigr| = \bigl|D^2\tilde{u}(0)\bigr| \leqslant C(r).$$

Since $0 < \min f \leqslant \det(D^2u) \leqslant \max f < \infty$, estimate (46) follows from the above. □

Step 3. To capture $\sup_{\mathbb{R}^n} \Delta_e^2 u$ for $e \in E$:

Proposition 2.3.

$$\sup_{\mathbb{R}^n} \Delta_e^2 u = \frac{e'Ae}{\|e\|^2}, \quad \forall e \in E. \tag{54}$$

First, two lemmas:
For $\lambda \geqslant 1$, let

$$u^\lambda(x) = \frac{u(\lambda x)}{\lambda^2}, \quad x \in \mathbb{R}^n.$$

We denote

$$Q(x) = \frac{1}{2}x'Ax.$$

Lemma 2.7. *For* $0 < \beta < 1$,

$$u^\lambda \to Q \quad \text{in } C^{1,\beta}_{\mathrm{loc}}\bigl(\mathbb{R}^n\bigr) \text{ as } \lambda \to \infty.$$

Proof. By Proposition 2.1,

$$u^\lambda \to Q \quad \text{in } C^0_{\mathrm{loc}}\bigl(\mathbb{R}^n\bigr) \text{ as } \lambda \to \infty.$$

On the other hand, by Proposition 2.2, we have, for some constant C independent of λ, that

$$\bigl|D^2u^\lambda\bigr| \leqslant C, \quad \text{on } \mathbb{R}^n.$$

Lemma 2.7 follows immediately. □

We will need a standard result for subsolutions of uniformly elliptic equations (see, e.g., Lemma 6.3 in [5]).

Lemma 2.8. *Let* $0 < \lambda \leqslant \Lambda < \infty$, *and*

$$\lambda I \leqslant \bigl(a_{ij}(x)\bigr) \leqslant \Lambda I, \quad \text{on } B_1.$$

Assume that

$$a_{ij}(x)v_{ij} \geqslant 0, \quad \text{on } B_1,$$

$$v \leqslant 1, \quad \text{on } B_1,$$

and, for some $\varepsilon, \mu > 0$,

$$\frac{|\{v \leqslant 1-\varepsilon\} \cap B_1|}{|B_1|} \geqslant \mu.$$

Then, for some $C = C(n, \lambda, \Lambda, \varepsilon, \mu) > 0$,

$$v \leqslant 1 - C^{-1}, \quad \textit{on } B_{1/2}.$$

Now we give the

Proof of Proposition 2.3. Let

$$\alpha = \sup_{\mathbb{R}^n} \Delta_e^2 u, \qquad \beta = \frac{e' A e}{\|e\|^2}.$$

By (49), $\alpha < \infty$. Let $\hat{e} = \lambda^{-1} e$, we know

$$0 < \Delta_{\hat{e}}^2 u^\lambda(x) \leqslant \alpha, \quad x \in \mathbb{R}^n. \tag{55}$$

It follows from Lemma A.2 and the Lebesgue dominated convergence theorem that

$$\lim_{\lambda\to\infty} \int_{B_1} \Delta_{\hat{e}}^2 u^\lambda = \int_{B_1} \beta = \beta |B_1|. \tag{56}$$

In particular, by (55) and (56), $\alpha \geqslant \beta$. We want to prove that $\alpha = \beta$. Suppose the contrary,

$$\alpha > \beta. \tag{57}$$

It follows from (56) that

$$\limsup_{\lambda\to\infty} \left(\frac{\alpha+\beta}{2} \left| \left\{ \Delta_{\hat{e}_k}^2 u^\lambda \geqslant \frac{\alpha+\beta}{2} \right\} \cap B_1 \right| \right) \leqslant \lim_{\lambda\to\infty} \int_{B_1} \Delta_{\hat{e}}^2 u^\lambda = \beta |B_1|.$$

Consequently, for large λ,

$$\frac{|\{\Delta_{\hat{e}}^2 u^\lambda \geqslant (\alpha+\beta)/2\} \cap B_1|}{|B_1|} \leqslant 1 - \mu,$$

where, by (57), $\mu = \frac{1}{2}(1 - \frac{2\beta}{\alpha+\beta}) > 0$. Or, equivalently, for large λ,

$$\frac{|\{\Delta_{\hat{e}}^2 u^\lambda \leqslant (\alpha+\beta)/2\} \cap B_1|}{|B_1|} \geqslant \mu.$$

Applying Lemma 2.8 to $v = \frac{2}{\alpha+\beta} \Delta_{\hat{e}}^2 u^\lambda$, we have, for some $C > 0$,

$$\sup_{B_{1/2}} \Delta_{\hat{e}}^2 u^\lambda \leqslant \alpha - C^{-1}, \quad \text{for large } \lambda.$$

It follows that

$$\alpha = \sup_{\mathbb{R}^n} \Delta_e^2 u = \lim_{\lambda\to\infty} \sup_{B_{1/2}} \Delta_{\hat{e}}^2 u^\lambda < \alpha,$$

a contradiction. □

We are about to complete the proof of Theorem 0.1: choose $b \in \mathbb{R}^n$ so that

$$w(e_k) = w(-e_k), \quad 1 \leqslant k \leqslant n,$$

where

$$w(x) = u(x) - Q(x) - b \cdot x.$$

Clearly, $w(0)=0$, and, by Proposition 2.3, $\Delta^2_{e_k} w \leqslant 0$ for $1 \leqslant k \leqslant n$. Then, by Lemma A.3,

$$w(je_k) \leqslant 0, \quad \forall 1 \leqslant k \leqslant n,\ \forall j = 0, \pm 1, \pm 2, \ldots. \tag{58}$$

Since

$$\sup_{x\in\mathbb{R}^n} \left|D^2 w(x)\right| < \infty,$$

it follows from (58) that

$$w(\lambda e_k) \leqslant C, \quad \forall 1 \leqslant k \leqslant n,\ \lambda \in \mathbb{R}. \tag{59}$$

By Theorem 0.2 and Remark 0.6, there exists a unique $g \in C^{2,\alpha}(\mathbb{R}^n)$ satisfying

$$\det\bigl(D^2(Q+g)\bigr) = f, \quad D^2(Q+g) > 0,$$

$$g(x+e_k) = g(x), \quad \forall 1 \leqslant k \leqslant n,\ x \in \mathbb{R}^n,$$

and

$$\int_{[0,1]^n} g = 0.$$

Set

$$h = w - g.$$

We will show that h is a constant on $\mathbb{R}^n$.

Since

$$\det\bigl(D^2(Q+g)\bigr) = f, \quad \frac{I}{C} \leqslant \bigl(D^2(Q+g)\bigr) \leqslant CI, \quad \text{in } \mathbb{R}^n,$$

and

$$\det\bigl(D^2(Q+w)\bigr) = f, \quad \frac{I}{C} \leqslant \bigl(D^2(Q+w)\bigr) = (D^2u) \leqslant CI, \quad \text{in } \mathbb{R}^n,$$

$h = (Q+w) - (Q+g)$ satisfies

$$a_{ij}(x)\partial_{ij} h = 0, \quad \text{in } \mathbb{R}^n,$$

where the coefficients $(a_{ij}(x))$ satisfies, for some constants $0 < \lambda \leqslant \Lambda < \infty$, that

$$\lambda I \leqslant \bigl(a_{ij}(x)\bigr) \leqslant \Lambda I, \quad \forall x \in \mathbb{R}^n.$$

So h is an entire solution to a uniformly elliptic equation.

Theorem 0.1 will follow from the following

Lemma 2.9.

$$\sup_{\mathbb{R}^n} h < \infty. \tag{60}$$

Proof. Let

$$M_i = \sup_{x\in[-i,i]^n} h(x), \quad i = 1, 2, \ldots.$$

Suppose the contrary of (60), we have

$$\lim_{i\to\infty} M_i = \infty. \tag{61}$$

Then we can show that for some constant C,

$$M_{2^i} \leqslant 4M_{2^{i-1}} + C, \quad \forall i = 1, 2, \ldots. \tag{62}$$

Indeed, let $(k_1, \ldots, k_n) \in [-m, m]^n$. Define

$$\varepsilon_j = \begin{cases} 1, & \text{if } k_j \text{ is odd,} \\ 0, & \text{if } k_j \text{ is even.} \end{cases}$$

Let

$$e = \left(\frac{k_1 \pm \varepsilon_1}{2}, \ldots, \frac{k_n \pm \varepsilon_n}{2} \right).$$

Since

$$\Delta_e^2 h(e) = \Delta_e^2 w(e) \leqslant 0,$$

we have

$$h(k_1 \pm \varepsilon_1, \ldots, k_n \pm \varepsilon_n) = h(2e) = h(2e) + h(0) \leqslant 2h(e) \leqslant 2M_{[(m+1)/2]+1}.$$

Since $\sup_{\mathbb{R}^n} |D^2 h| < \infty$, it follows from the above that for some C independent of i,

$$h(k_1, \ldots, k_n) \leqslant 2M_{[(m+1)/2]+1} + C.$$

It follows that

$$M_m \leqslant 2M_{[(m+1)/2]+1} + C.$$

Replacing m by $[(m+1)/2] + 1$ in the above, we have

$$M_{[(m+1)/2]+1} \leqslant 2M_{[(m+3)/4]+1} + C.$$

Taking $m = 2^i$ and using $[(2^i + 3)/4] + 1 \leqslant 2^{i-1}$ for $i \geqslant 3$, we have

$$M_{2^i} \leqslant 4M_{[(2^i+3)/4]+1} \leqslant 4M_{2^{i-1}} + 3C.$$

Estimate (62) is established.

Let

$$H_i(x) = \frac{h(2^i x)}{M_{2^i}}, \quad x \in [-1, 1]^n.$$

By (59),

$$H_i(\lambda e_k) \leqslant \frac{C}{M_{2^i}}, \quad 1 \leqslant k \leqslant n, \ i = 1, 2, \ldots. \tag{63}$$

By (62), for some positive constant C,

$$\max_{[-1/2,1/2]^n} H_i = \frac{M_{2^{i-1}}}{M_{2^i}} \geqslant \frac{M_{2^i} - C}{4M_{2^i}} \frac{1}{8}, \quad \text{for large } i. \tag{64}$$

We know that h satisfies an uniformly elliptic equation, so does H_i (with ellipticity constants independent of i).

By the definition,

$$H_i \leqslant 1 \quad \text{on } [-1, 1]^n,$$

and,

$$H_i(0) = \frac{h(0)}{M_{2^i}} \to 0, \quad \text{as } i \to \infty.$$

Since $1-H_i$ is non-negative, we have, by the Harnack inequality of Krylov and Safonov [19], that

$$\max_{[-8/9,8/9]^n}(1-H_i)\leqslant C\big(1-H_i(0)\big)=C\big(1+\mathrm{o}(1)\big)\leqslant 2C,$$

and there exists some $0<\alpha<1$ and H such that

$$H_i\to H \quad \text{in } C^\alpha\left(\left[-\frac{3}{4},\frac{3}{4}\right]^n\right) \text{ along a subsequence } i\to\infty.$$

By (64),

$$\max_{[-1/2,1/2]^n} H\geqslant\frac{1}{8}, \tag{65}$$

and by (63),

$$H(\lambda e_k)\leqslant 0,\quad 1\leqslant k\leqslant n,\ |\lambda|\leqslant\frac{3}{4}. \tag{66}$$

We also know that

$$H(0)=\lim_{i\to\infty}H_i(0)=0. \tag{67}$$

Since $\Delta_e^2 h\leqslant 0$ for $e\in E$, we have

$$\Delta^2_{2^{-i}e}H_i\leqslant 0,\quad \forall e\in E.$$

Since the convergence of H_i to H is uniform, we know from the above that H is *concave*.

Now H is a concave function satisfying (65), (66) and (67). Moreover H is the uniform limit of $\{H_i\}$, for which a uniform Harnack inequality holds. This leads to contradiction. Indeed, since H is concave and $H(0)=0$, let $l(x)$ be a linear function such that $l-H\geqslant 0$ in $(-\frac{3}{4},\frac{3}{4})^n$. Since $l-H$ is the uniform limit of $\{l-H_i\}$, the Harnack inequality applies to $l-H$ as well, thus $l-H\equiv 0$. By (66), $H\equiv l\equiv 0$ which contradicts to (65). Lemma 2.9 is established. □

Finally the

Proof of Theorem 0.1. Since h is an entire solution to a uniformly elliptic equation and, by Lemma 2.9, h is bounded from above, it then follows from the Harnack inequality that h is a constant. Theorem 0.1 is established. □

Appendix A

Lemma A.1. *Let $g\in C^2(-1,1)$ be a strictly convex function, and let $0<|h|\leqslant\varepsilon$. Then*

$$\Delta_h^2 g(x)>0,\quad \forall |x|\leqslant 1-2\varepsilon, \tag{A.1}$$

and

$$\int_{-1+2\varepsilon}^{1-2\varepsilon}\Delta_h^2 g\leqslant\frac{C}{\varepsilon}\operatorname{osc}_{(-1,1)}g, \tag{A.2}$$

where C is some universal constant, and $\operatorname{osc}_{(-1,1)}g:=\sup_{-1<s<t<1}|g(s)-g(t)|$.

Proof. For $-1<a<b<1$,

$$\int_a^b \Delta_h^2 g = \frac{1}{h^2}\int_a^b \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\big[g_i(x+sh)+g(x-sh)\big]\,\mathrm{d}s\,\mathrm{d}x$$
$$= \frac{1}{h}\int_0^1 \left[\int_{b-sh}^{b+sh} g'(y)\,\mathrm{d}y - \int_{a-sh}^{a+sh} g'(y)\,\mathrm{d}y\right]\mathrm{d}s. \tag{A.3}$$

By the convexity of g,

$$\max_{|x|\leqslant 1-\varepsilon}\big|g'(x)\big| \leqslant \frac{C}{\varepsilon}\mathrm{osc}_{(-1,1)}g.$$

Lemma A.1 follows easily from the above. □

Our next lemma is elementary.

Lemma A.2. *Let g_i converges to g in $C^1[-1,1]$, $g \in C^2(-1,1)$, and $|h_i| \to 0$. Then for all $-1 < a < b < 1$,*

$$\lim_{i\to\infty}\int_a^b \Delta_{h_i}^2 g_i = g'(b) - g'(a) = \int_a^b g''.$$

Proof. By (A.3),

$$\int_a^b \Delta_{h_i}^2 g_i = \frac{1}{h_i}\int_0^1 \left[\int_{b-sh_i}^{b+sh_i} g_i'(y)\,\mathrm{d}y - \int_{a-sh_i}^{a+sh_i} g_i'(y)\,\mathrm{d}y\right]\mathrm{d}s.$$

By the C^1 convergence of g_i to g,

$$\lim_{i\to\infty}\left(\frac{1}{h_i}\int_0^1 \int_{a-sh_i}^{a+sh_i} \big|g_i'(y) - g(y)\big|\,\mathrm{d}y\,\mathrm{d}s + \frac{1}{h_i}\int_0^1 \int_{b-sh_i}^{b+sh_i} \big|g_i'(y) - g(y)\big|\right) = 0.$$

It follows that

$$\lim_{i\to\infty}\int_a^b \Delta_{h_i}^2 g_i = \lim_{i\to\infty}\frac{1}{h_i}\int_0^1 \left[\int_{b-sh_i}^{b+sh_i} g'(y)\,\mathrm{d}y - \int_{a-sh_i}^{a+sh_i} g'(y)\,\mathrm{d}y\right]\mathrm{d}s = g'(b) - g'(a).$$

Lemma A.2 is established. □

Lemma A.3. *Let $g \in C^0(\mathbb{R})$, $g(0) = 0$, $g(1) = g(-1)$, and $\Delta_1^2 g(x) \leqslant 0$ for all $x \in \mathbb{R}$. Then*

$$g(m+1) \leqslant g(m), \quad g(-m-1) \leqslant g(-m), \quad \textit{for all non-negative integer } m. \tag{A.4}$$

Consequently, $g(m) \leqslant 0$ for all integer m.

Proof. Since the hypothesis is satisfied also by $g(-x)$, we only need to establish the first inequality in (A.4). For every integer k, we have

$$g(k+1) + g(k-1) - 2g(k) = \Delta_1^2 g(k) \leqslant 0. \tag{A.5}$$

Take $k = 0$ in (A.5), we have

$$2g(1) = g(1) + g(-1) \leqslant 2g(0) = 0.$$

So the first inequality in (A.4) holds for $m = 0$. We prove (A.4) by induction. Assuming that the first inequality in (A.4) holds for $m - 1$ for some $m \geqslant 1$, take $k = m$ in (A.5), we have

$$g(m+1) + g(m-1) \leqslant 2g(m) \leqslant g(m) + g(m-1).$$

So $g(m+1) \leqslant g(m)$, i.e., the first inequality in (A.4) holds for m. □

References

[1] L. Caffarelli, A localization property of viscosity solutions to the Monge–Ampère equation and their strict convexity, Ann. of Math. 13 (1990) 129–134.
[2] L. Caffarelli, Interior $W^{2,p}$ estimates for solutions of the Monge–Ampère equation, Ann. of Math. 131 (1990) 135–150.
[3] L. Caffarelli, Graduate Course at the Courant Institute, New York University, New York, 1995.
[4] L. Caffarelli, Monotonicity properties of optimal transportation and the FKG and related inequalities, Comm. Math. Phys. 214 (2000) 547–563.
[5] L. Caffarelli, X. Cabre, Fully Nonlinear Elliptic Equations, in: American Mathematical Society Colloquium Publications, vol. 43, American Mathematical Society, Providence, RI, 1995.
[6] L. Caffarelli, C. Gutiérrez, Properties of the solutions of the linearized Monge–Ampère equation, Amer. J. Math. 119 (1997) 423–465.
[7] L. Caffarelli, Y.Y. Li, An extension to a theorem of Jörgens, Calabi, and Pogorelov, Comm. Pure Appl. Math. 56 (2003) 549–583.
[8] L. Caffarelli, L. Nirenberg, J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations, I. Monge–Ampère equation, Comm. Pure Appl. Math. 37 (1984) 369–402.
[9] L. Caffarelli, J. Viaclovsky, On the regularity of solutions to Monge–Ampère equations on Hessian manifolds, Comm. Partial Differential Equations 26 (2001) 2339–2351.
[10] E. Calabi, Improper affine hypersurfaces of convex type and a generalization of a theorem by K. Jörgens, Michigan Math. J. 5 (1958).
[11] S.Y. Cheng, S.T. Yau, The real Monge–Ampère equation and affine flat structures, in: Proceedings of the Symposium on Differential Geometry and Differential Equations, vols. 1–3, Beijing, 1980, Science Press, Beijing, 1982, pp. 339–370.
[12] S.Y. Cheng, S.T. Yau, Complete affine hypersurfaces. I. The completeness of affine metrics, Comm. Pure Appl. Math. 39 (1986) 839–866.
[13] K.-S. Chou, X.-J. Wang, A variational theory of the Hessian equation, Comm. Pure Appl. Math. 54 (2001) 1029–1064.
[14] M. De Guzman, Differentiation of Integrals in $\mathbb{R}^n$, in: Lecture Notes, vol. 481, Springer-Verlag, Berlin, 1976.
[15] L.C. Evans, Classical solutions of fully nonlinear, convex, second-order elliptic equations, Comm. Pure Appl. Math. 35 (1982) 333–363.
[16] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, second ed., in: Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 224, Springer-Verlag, Berlin, 1983.
[17] K. Jörgens, Über die Lösungen der Differentialgleichung $rt - s^2 = 1$, Math. Ann. 127 (1954) 130–134.
[18] N.V. Krylov, Boundedly inhomogeneous elliptic and parabolic equation in a domain, Izv. Akad. Nauk SSSR 47 (1983) 75–108.
[19] N.V. Krylov, M.V. Safonov, An estimate of the probability that a diffusion process hits a set of positive measure, Dokl. Akad. Nauk. SSSR 245 (1979) 253–255; English translation in: Soviet Math. Dokl. 20 (1979) 253–255.
[20] Y.Y. Li, Some existence results of fully nonlinear elliptic equations of Monge–Ampère type, Comm. Pure Appl. Math. 43 (1990) 233–271.
[21] A.V. Pogorelov, On the improper affine hypersurfaces, Geom. Dedicata 1 (1972) 33–46.
[22] N. Trudinger, X. Wang, The Bernstein problem for affine maximal hypersurfaces, Invent. Math. 140 (2000) 399–422.