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Minimization of the zeroth Neumann eigenvalues with integrable potentials

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Abstract

For an integrable potential q on the unit interval, let $\lambda_0(q)$ be the zeroth Neumann eigenvalue of the Sturm–Liouville operator with the potential q. In this paper we will solve the minimization problem $\tilde{\mathbf{L}}_1(r) = \inf_q \lambda_0(q)$, where potentials q have mean value zero and L^1 norm r. The final result is $\tilde{\mathbf{L}}_1(r) = -r^2/4$. The approach is a combination of variational method and limiting process, with the help of continuity results of solutions and eigenvalues of linear equations in potentials and in measures with weak topologies. These extremal values can yield optimal estimates on the zeroth Neumann eigenvalues.

Résumé

Soit $\lambda_0(q)$ la zéro-ème valeur propre de Neumann de l'opérateur de Sturm–Liouville pour un potentiel intégrable q de l'intervalle $[0, 1]$. Dans cet article nous résolvons le problème de minimisation $\tilde{\mathbf{L}}_1(r) = \inf_q \lambda_0(q)$ pour les potentiels q de valeur moyenne zéro et de norme L^1 égale à r. Le résultat est $\tilde{\mathbf{L}}_1(r) = -r^2/4$. L'approche est une combinaison de méthode variationnelle et de procédé de limite, utilisant des résultats de continuité des solutions et des valeurs propres d'équations linéaires en les potentiels et les mesures dans des topologies faibles. Ces valeurs extrémales peuvent donner des estimations optimales sur les zéro-èmes valeurs propres de Neumann.

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1. Introduction and main results

Due to numerous applications of eigenvalues, extremal problems for eigenvalues are important in many problems in applied sciences. For example, the minimal values of the principal Neumann eigenvalues with sign-changing weights

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are crucial in population dynamics [2,4,10,20]. Mathematically, these are interesting variational problems. For example, in a classical paper [9], Krein has applied the Pontrayjin's Maximum Principle [25, §§48.6–48.8] to find all extremal values of the weighted Dirichlet eigenvalues $\lambda_m(w)$, where the weights w belong to the following class

$$\int_{[0,1]} w(t)\,\mathrm{d}t = r \quad \text{and} \quad 0 \leqslant w(t) \leqslant h < \infty \quad \text{for a.e. } t \in [0,1].$$

After that, several interesting extremal problems for eigenvalues with potentials or weights have been solved successfully by using different approaches [1,7,8,12,18,24].

The present extremal problems for eigenvalues are motivated by recent papers [4,15,22,29,30]. We are concerned with Sturm–Liouville operators. Given a potential $q \in \mathcal{L}^p := L^p(I,\mathbb{R})$, the Lebesgue space of the unit interval $I := [0,1]$ with the L^p norm $\|\cdot\|_p$, where $1 \leqslant p \leqslant \infty$, consider the eigenvalue problem

$$\ddot{y} + \big(\lambda + q(t)\big)y = 0. \tag{1.1}$$

With the Dirichlet boundary condition

$$y(0) = y(1) = 0, \tag{1.2}$$

eigenvalues of problem (1.1) are denoted by $\{\lambda_m^D(q)\}_{m\in\mathbb{N}}$, while, with the Neumann boundary condition

$$\dot{y}(0) = \dot{y}(1) = 0, \tag{1.3}$$

eigenvalues of problem (1.1) are denoted by $\{\lambda_m^N(q)\}_{m\in\mathbb{Z}^+}$, where $\mathbb{Z}^+ := \{0,1,2,\ldots\}$. See [25,26]. Denote the L^p balls and spheres in $\mathcal{L}^p$ by

$$B_p[r] := \big\{q \in \mathcal{L}^p\colon\ \|q\|_p \leqslant r\big\}, \qquad S_p[r] := \big\{q \in \mathcal{L}^p\colon\ \|q\|_p = r\big\}.$$

Consider the following extremal values for $\lambda_m^\sigma(q)$, where $\sigma = D$ or N,

$$\mathbf{L}_{m,p}^\sigma(r) := \inf\big\{\lambda_m^\sigma(q)\colon\ q \in B_p[r]\big\}, \qquad \mathbf{M}_{m,p}^\sigma(r) := \sup\big\{\lambda_m^\sigma(q)\colon\ q \in B_p[r]\big\}. \tag{1.4}$$

Due to basic properties of eigenvalues, the infimum and supremum in balls of (1.4) are the same as those on the corresponding spheres, i.e.

$$\mathbf{L}_{m,p}^\sigma(r) = \inf\big\{\lambda_m^\sigma(q)\colon\ q \in S_p[r]\big\}, \qquad \mathbf{M}_{m,p}^\sigma(r) = \sup\big\{\lambda_m^\sigma(q)\colon\ q \in S_p[r]\big\}.$$

Though $B_p[r]$ and $S_p[r]$ are infinite-dimensional, authors of [22,29] have applied many different theories to show that all of these extremal values are finite. Moreover, for the most revelent case $p = 1$, all of these extremal values have been constructed explicitly. For example, for the zeroth Neumann eigenvalues $\lambda_0^N(q)$, one has

$$\mathbf{L}_{0,1}^N(r) = \mathbf{Z}_0^{-1}(r), \qquad \mathbf{M}_{0,1}^N(r) = \lambda_0^N(-r) = r, \tag{1.5}$$

where $\mathbf{Z}_0 : (-\infty,0] \to [0,\infty)$ is defined by

$$\mathbf{Z}_0(x) = \sqrt{-x}\tanh\sqrt{-x}, \quad x \in (-\infty,0]. \tag{1.6}$$

See [29, §6]. Moreover, when $r > 0$, $\mathbf{L}_{0,1}^N(r)$ cannot be attained by any potential $q \in B_1[r]$. The obtention of $\mathbf{L}_{0,1}^N(r)$ is not trivial, because L^1 balls have no compactness even in the weak topology. To obtain explicit result of $\mathbf{L}_{0,1}^N(r)$, Zhang and his coauthors have developed several ideas different from the preceding works, including (i) the continuity of eigenvalues in potentials with weak topologies of $\mathcal{L}^p$ [15,23,28], (ii) the continuous Fréchet differentiability of eigenvalues in potentials with the L^p norms [17,23,25], (iii) the variational construction to the L^p case for $p \in (1,\infty)$ [22,29], and (iv) the limiting analysis of the L^p results as $p \downarrow 1$ [22,29]. These results can yield some optimal estimates on $\lambda_0^N(q)$.

A basic property on eigenvalues is

$$\lambda_m^\sigma(q) = -\bar{q} + \lambda_m^\sigma(\tilde{q}), \tag{1.7}$$

where $\bar{q} := \int_I q \in \mathbb{R}$ is the mean value of q and $\tilde{q} := q - \bar{q}$ has the zero mean value. Because of (1.7), it is natural to study the following further extremal problems on eigenvalues $\lambda_m^\sigma(q)$,

$$\tilde{\mathbf{L}}_{m,p}^\sigma(r) := \inf\{\lambda_m^\sigma(q)\colon q \in \tilde{B}_p[r]\}, \qquad \tilde{\mathbf{M}}_{m,p}^\sigma(r) := \sup\{\lambda_m^\sigma(q)\colon q \in \tilde{B}_p[r]\}. \tag{1.8}$$

Here

$$\tilde{\mathcal{L}}^p := \{q \in \mathcal{L}^p\colon \bar{q} = 0\}, \qquad \tilde{B}_p[r] := B_p[r] \cap \tilde{\mathcal{L}}^p, \qquad \tilde{S}_p[r] := S_p[r] \cap \tilde{\mathcal{L}}^p.$$

As $\tilde{B}_p[r] \subset B_p[r]$, extremal values of (1.8) are well defined. Moreover, one has always

$$-\infty < \mathbf{L}_{m,p}^\sigma(r) \leqslant \tilde{\mathbf{L}}_{m,p}^\sigma(r) \leqslant \tilde{\mathbf{M}}_{m,p}^\sigma(r) \leqslant \mathbf{M}_{m,p}^\sigma(r) < +\infty.$$

Combining with (1.7), the solutions of extremal problems (1.8) can yield optimal estimates on eigenvalues $\lambda_m^\sigma(q)$ which are different from those obtained from extremal values (1.4). For detailed discussions, see Section 5.1.

As an initial step toward the complete solutions of problems (1.8), in this paper we will follow the scheme of [22,29] to solve extremal problems (1.8) for the zeroth Neumann eigenvalues $\lambda_0^N(q)$, written $\lambda_0(q)$ as well. For simplicity, $\tilde{\mathbf{L}}_{0,p}^N(r)$ and $\tilde{\mathbf{M}}_{0,p}^N(r)$ of (1.8) are written as $\tilde{\mathbf{L}}_p(r)$ and $\tilde{\mathbf{M}}_p(r)$ respectively. That is,

$$\tilde{\mathbf{L}}_p(r) := \inf\{\lambda_0(q)\colon q \in \tilde{B}_p[r]\}, \qquad \tilde{\mathbf{M}}_p(r) := \sup\{\lambda_0(q)\colon q \in \tilde{B}_p[r]\}, \tag{1.9}$$

where $p \in [1,\infty]$ and $r \in [0,\infty)$.

Recall that

$$\lambda_0(q) \leqslant -\bar{q} \quad \forall q \in \mathcal{L}^1, \tag{1.10}$$

where the equality holds when and only when q is constant. See, for example, [27, Lemma 3.3] and Remark 2.4 as well. In particular, one has $\lambda_0(q) \leqslant 0 = \lambda_0(0)$ for all $q \in \tilde{B}_p[r]$. Hence the supremum of $\lambda_0(q)$ in the ball $\tilde{B}_p[r]$ is

$$\tilde{\mathbf{M}}_p(r) = \max\{\lambda_0(q)\colon q \in \tilde{B}_p[r]\} = \lambda_0(0) = 0 \quad \forall r \in [0,\infty),\ p \in [1,\infty]. \tag{1.11}$$

Note that the infimum $\tilde{\mathbf{L}}_p(r)$ of $\lambda_0(q)$ is taken over the ball $\tilde{B}_p[r]$. However, we will prove in Lemma 2.5 that it coincides with the infimum on the sphere $\tilde{S}_p[r]$, i.e.

$$\tilde{\mathbf{L}}_p(r) \equiv \inf\{\lambda_0(q)\colon q \in \tilde{S}_p[r]\}. \tag{1.12}$$

The final answer to the most interesting case $p = 1$ is surprisingly simple.

Theorem 1.1. *For any $r \geqslant 0$, one has*

$$\tilde{\mathbf{L}}_1(r) = \inf_{q \in \tilde{B}_1[r]} \lambda_0(q) = \inf_{q \in \tilde{S}_1[r]} \lambda_0(q) = -r^2/4. \tag{1.13}$$

Moreover, when $r > 0$, $\tilde{\mathbf{L}}_1(r)$ cannot be attained by any potential in $\tilde{B}_1[r]$.

If eigenvalues of *measure differential equations* (MDE) [14] are used, we have the following characterization on 'minimizers' of $\tilde{\mathbf{L}}_1(r)$.

Theorem 1.2. *For any $r \geqslant 0$, one has*

$$\tilde{\mathbf{L}}_1(r) = \lambda_0(\pm\nu_r), \quad \text{where } \nu_r := (r/2)(\delta_1 - \delta_0). \tag{1.14}$$

Here δ_a is the unit Dirac measure located at a and $\lambda_0(\mu)$ denotes the zeroth Neumann eigenvalue for MDE with the measure μ.

For the proof of Theorem 1.1, we will follow the scheme of [22,29]. Roughly speaking, the L^1 problems can be approximated by the L^p problems as $p \downarrow 1$, while the L^p problems can be solved using variational method. Due to an additional parameter caused by the constraint $\bar{q} = 0$, the critical equations for $\tilde{\mathbf{L}}_p(r)$, $p \in (1,\infty)$, are more complicated than those in [22,29]. The analysis for critical equations of $\tilde{\mathbf{L}}_p(r)$ will be more tricky with the help of deep results on MDE.

The paper is organized as follows. In Section 2, we will establish some topological relation between $\tilde{\mathcal{L}}^p$ spheres, balls and $\tilde{\mathcal{L}}^1$ spheres, balls. Then we will prove in Lemma 2.2 that how the infimum in $\tilde{B}_1[r]$ can be approximated by the minimal values in $\tilde{B}_p[r]$, $p \in (1,\infty)$. For convenience, we will briefly review some deep results on MDE in [14], including continuous dependence of solutions and eigenvalues of MDE on measures with the weak* topology. For the measure ν_r in (1.14), all Neumann eigenvalues will be found in Example 2.8. In Section 3, we will derive the critical equation for the problem $\tilde{\mathbf{L}}_p(r)$, $p \in (1,\infty)$. The dynamics and quantitative properties of the critical equation will be given. In Section 4 we will concentrate on the limiting analysis for the critical problems $\tilde{\mathbf{L}}_p(r)$ as $p \downarrow 1$. Here we will make use of the results on MDE, especially from the point of view of the weak* topology of measures. Theorem 1.1 will be proved after we find all limits in Section 4.1. For the proof of Theorem 1.2, see Remark 4.10. In Section 5, we will first give some application of Theorem 1.1 to optimal estimates of the zeroth Neumann eigenvalues. Due to the relation between the zeroth Neumann eigenvalues and the zeroth periodic eigenvalues [26], the corresponding extremal problems for the zeroth periodic eigenvalues in $\tilde{B}_1[r]$ will be solved with help of the scaling technique.

Since the L^1 space has no local compactness even in its weak topology and the space of measures is locally compact in its weak* topology, it is quite natural to use MDE in the limiting analysis of problems $\tilde{\mathbf{L}}_p(r)$ as $p \downarrow 1$. In fact, with the help of MDE, our analysis on the limiting case of critical equations is very concise, especially compared with that in [22,29]. In the limiting process, we will have a natural understanding for the minimal potentials $q_{p,r}$ in $\tilde{\mathcal{L}}^p$ balls when p changes from ∞ to 1. In particular, it will be proved that $q_{p,r}$, as measures, has the limiting measures $\pm\nu_r$ in the weak* topology of measures. This gives a natural explanation to Theorems 1.1 and 1.2. We think that the approach of this paper is also useful for other extremal problems in L^1 spaces.

2. Weak topologies and eigenvalues of MDE

2.1. Eigenvalues with potentials in the L^p topologies

A basic topological fact on $\tilde{\mathcal{L}}^p$ spheres (and $\tilde{\mathcal{L}}^p$ balls) is as follows.

Lemma 2.1. *For any $q \in \tilde{S}_1[r]$, there exists $q_p \in \tilde{S}_p[r]$ such that* $\lim_{p\downarrow 1} \|q_p - q\|_1 = 0$. *Hence one has*

$$\tilde{S}_1[r] \subset \operatorname{closure}_{(\tilde{\mathcal{L}}^1, \|\cdot\|_1)} \bigcup_{p\in(1,\infty)} \tilde{S}_p[r].$$

Proof. The proof is a refinement of [29, Lemma 2.1]. One can assume that $r \in (0,\infty)$. Define a family of functions

$$f_p(t) = r^{1/p^*} \big|q(t)\big|^{1/p} \cdot \operatorname{sign}\big(q(t)\big), \quad p \in (1,\infty).$$

Then $f_p \in S_p[r]$. By the Lebesgue dominated convergence theorem, one has $\lim_{p\downarrow 1} \|f_p - q\|_1 = 0$. In particular, $|\bar{f}_p| = |\bar{f}_p - \bar{q}| \leqslant \|f_p - q\|_1 \to 0$. Thus

$$\|f_p - \bar{f}_p\|_p \geqslant \|f_p\|_p - \|\bar{f}_p\|_p = r - |\bar{f}_p| \to r > 0.$$

Hence

$$q_p := r \frac{f_p - \bar{f}_p}{\|f_p - \bar{f}_p\|_p} \in \tilde{S}_p[r]$$

is well defined when p is close to 1. Moreover,

$$q_p = \frac{r}{\|f_p - \bar{f}_p\|_p} f_p - \frac{r}{\|f_p - \bar{f}_p\|_p} \bar{f}_p \to q$$

in $\|\cdot\|_1$. The proof is complete. □

From the topological fact of Lemma 2.1 and continuity of eigenvalues in L^1 potentials, we have the following limiting relation.

Lemma 2.2. *For any $r \in (0, \infty)$, one has the following limiting equality*

$$\tilde{\mathbf{L}}_1(r) = \lim_{p \downarrow 1} \tilde{\mathbf{L}}_p(r) = \inf_{p \in (1,\infty)} \tilde{\mathbf{L}}_p(r) \in (-\infty, 0). \tag{2.1}$$

Proof. The proof is similar to [29, Lemma 2.2]. By the Hölder inequality, one has

$$\tilde{B}_1[r] \supset \tilde{B}_{p_1}(r) \supset \tilde{B}_{p_2}(r) \quad \forall 1 \leqslant p_1 \leqslant p_2 \leqslant \infty.$$

Thus, as r is fixed, $\tilde{\mathbf{L}}_p(r)$ is increasing in $p \in (1, \infty)$ and has the lower bound $\tilde{\mathbf{L}}_1(r) > -\infty$. Therefore one has the second equality of (2.1). Moreover, one has

$$\tilde{\mathbf{L}}_1(r) \leqslant \lim_{p \downarrow 1} \tilde{\mathbf{L}}_p(r). \tag{2.2}$$

On the other hand, given $q \in \tilde{B}_1[r]$ with $\hat{r} := \|q\|_1 \in (0, r]$, from Lemma 2.1 one has $q_p \in \tilde{S}_p[\hat{r}] \subset \tilde{B}_p[r]$ such that $\|q_p - q\|_1 \to 0$. Thus $\lambda_0(q_p) \geqslant \tilde{\mathbf{L}}_p(r)$ for all $p \in (1, \infty)$. By the continuity of $\lambda_0(q)$ in $q \in (\mathcal{L}^1, \|\cdot\|_1)$, one has

$$\lambda_0(q) = \lim_{p \downarrow 1} \lambda_0(q_p) \geqslant \lim_{p \downarrow 1} \tilde{\mathbf{L}}_p(r).$$

Taking the infimum over $q \in \tilde{B}_1[r]$, we get

$$\tilde{\mathbf{L}}_1(r) \geqslant \lim_{p \downarrow 1} \tilde{\mathbf{L}}_p(r). \tag{2.3}$$

Now the first equality of (2.1) follows from (2.2) and (2.3). □

Let $p \in [1, \infty]$. Consider $\lambda_0(q)$ as a nonlinear functional of potentials $q \in (\mathcal{L}^p, \|\cdot\|_p)$, $1 \leqslant p \leqslant \infty$, it is continuously Fréchet differentiable [17,23,25]. The Fréchet derivative is

$$\partial_q \lambda_0(q) = -W^2, \tag{2.4}$$

where W is an eigenfunction associated with $\lambda_0(q)$ and satisfies the normalization condition

$$\|W\|_2 = \left(\int_I W^2(t)\,\mathrm{d}t \right)^{1/2} = 1.$$

Result (2.4) is understood as a bounded linear functional of $(\mathcal{L}^p, \|\cdot\|_p)$ defined by

$$\mathcal{L}^p \ni h \to -\int_I W^2(t) h(t)\,\mathrm{d}t \in \mathbb{R}.$$

Using the derivatives of eigenvalues, we can obtain the following results.

Lemma 2.3. *Let $q \in \mathcal{L}^p$, $p \in [1, \infty]$. Then $\Lambda(\tau) := \lambda_0(\tau q)$ is continuously differentiable in $\tau \in \mathbb{R}$. Moreover, one has*

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \Lambda(\tau) = -\int_I q(t) W^2(t; \tau q)\,\mathrm{d}t, \quad \forall \tau \in \mathbb{R}, \tag{2.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\Lambda(\tau)}{\tau} = -\frac{1}{\tau^2} \int_I \dot{W}^2(t; \tau q)\,\mathrm{d}t, \quad \forall \tau \neq 0. \tag{2.6}$$

Here $W(\cdot; \tau q)$ is the normalized eigenfunction associated with $\lambda_0(\tau q)$.

Proof. Formula (2.5) follows from (2.4) immediately. For the zeroth periodic eigenvalues, formula (2.6) is given in [29, Lemma 2.11]. One sees that the proof there applies also to the Neumann eigenvalues $\lambda_0(q)$. □

Remark 2.4. (i) Both (2.5) and (2.6) hold for all Dirichlet and Neumann eigenvalues of Sturm–Liouville operators.

(ii) Let $q \in \mathcal{L}^p$ be non-constant. Then $W(t; \tau q)$ is also non-constant for any $\tau \neq 0$. Formula (2.6) shows that $\Lambda(\tau)/\tau$ is strictly decreasing in $\tau \in (0, \infty)$. In particular, one has

$$\lambda_0(q) = \frac{\Lambda(1)}{1} < \lim_{\tau \downarrow 0} \frac{\Lambda(\tau)}{\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\Lambda(\tau)\Big|_{\tau=0} = -\int_I q(t)\,\mathrm{d}t,$$

following from (2.5) because $W^2(t; 0 \cdot q) \equiv 1$. This gives another proof for inequality (1.10).

Lemma 2.5. *One has relation* (1.12) *for the infimum on* $\tilde{B}_p[r]$ *and on* $\tilde{S}_p[r]$.

Proof. Since $\tilde{S}_p[r] \subset \tilde{B}_p[r]$, we have

$$\inf_{q \in \tilde{B}_p[r]} \lambda_0(q) \leqslant \inf_{q \in \tilde{S}_p[r]} \lambda_0(q).$$

On the other hand, let $0 \neq q \in \tilde{B}_p[r]$. Denote $\hat{\tau} = r/\|q\|_p \geqslant 1$ and $\hat{q} = \hat{\tau} q \in \tilde{S}_p[r]$. It follows from (2.6) that

$$\lambda_0(q) = \frac{\Lambda(1)}{1} \geqslant \frac{\Lambda(\hat{\tau})}{\hat{\tau}} = \frac{1}{\hat{\tau}}\lambda_0(\hat{q}) \geqslant \lambda_0(\hat{q}),$$

because $1/\hat{\tau} \leqslant 1$ and $\lambda_0(\hat{q}) < 0$. Thus

$$\inf_{q \in \tilde{B}_p[r]} \lambda_0(q) \geqslant \inf_{q \in \tilde{S}_p[r]} \lambda_0(q).$$

These give (1.12). □

2.2. *Weak* topology for measures and eigenvalues of MDE*

Let us recall from [14] some results on measure differential equations, which are a special class of the so-called generalized ordinary differential equations [16,21].

Let $I = [0, 1]$ be the unit interval. Denote by $\mathcal{C} := \mathcal{C}(I, \mathbb{R})$ the Banach space of continuous functions of I with the supremum norm $\|\cdot\|_\infty$. The space $\mathcal{M}_0 := \mathcal{M}_0(I, \mathbb{R})$ of measures on I is the dual space of $(\mathcal{C}, \|\cdot\|_\infty)$. By the Riesz representation theorem [3], $\mathcal{M}_0$ can be characterized as

$$\mathcal{M}_0 = \big\{\mu : I \to \mathbb{R}\text{: } \mu(0+) = 0,\ \mu(t+) = \mu(t)\ \forall t \in (0, 1),\ \|\mu\|_{\mathbf{V}} < +\infty\big\}.$$

Here $\mu(t+) := \lim_{s \downarrow t} \mu(s)$ is the right-limit, while $\|\cdot\|_{\mathbf{V}}$ is the total variation defined by

$$\|\mu\|_{\mathbf{V}} := \sup\left\{\sum_{i=0}^{n-1}\big|\mu(t_{i+1}) - \mu(t_i)\big|\text{: } 0 = t_0 < t_1 < \cdots < t_n = 1,\ n \in \mathbb{N}\right\}.$$

In the space $\mathcal{M}_0$ of measures, $\|\cdot\|_{\mathbf{V}}$ is a norm and $(\mathcal{M}_0, \|\cdot\|_{\mathbf{V}})$ is a Banach space. Note that any potential $q \in \mathcal{L}^p$ defines an (absolutely continuous) measure $\mu_q \in \mathcal{M}_0$ by

$$\mu_q(t) := \int_{[0,t]} q(s)\,\mathrm{d}s, \quad t \in I.$$

It is well known that

$$\|\mu_q\|_{\mathbf{V}} = \|q\|_1. \tag{2.7}$$

That is, $(\mathcal{L}^1, \|\cdot\|_1)$ is isometrically embedded into $(\mathcal{M}_0, \|\cdot\|_{\mathbf{V}})$. Another topology in $\mathcal{M}_0$ is that of the weak* convergence, denoted by w^* [5,13]. Precisely, $\mu_n \to \mu_0$ in $(\mathcal{M}_0, w^*)$ iff

$$\int_I f(t)\,\mathrm{d}\mu_n(t) \to \int_I f(t)\,\mathrm{d}\mu_0(t) \quad \forall f \in \mathcal{C}.$$

By the Banach–Alaoglu theorem [13], $(\mathcal{M}_0, w^*)$ is locally compact and locally sequentially compact.

Given a real measure $\mu \in \mathcal{M}_0$, the second-order, scalar, linear MDE with the measure μ is written as

$$\mathrm{d}\dot{y} + y\,\mathrm{d}\mu(t) = 0, \quad t \in I. \tag{2.8}$$

With the initial value $(y(0), \dot{y}(0)) = (y_0, z_0) \in \mathbb{R}^2$, the solution $y(t)$, $t \in I$, of Eq. (2.8) and its generalized right-derivative (or its velocity) $\dot{y}(t)$, $t \in I$, are determined by the following integral system

$$y(t) = y_0 + \int_{[0,t]} \dot{y}(s)\,\mathrm{d}s, \qquad t \in [0, 1], \tag{2.9}$$

$$\dot{y}(t) = \begin{cases} z_0 & \text{for } t = 0, \\ z_0 - \int_{[0,t]} y(s)\,\mathrm{d}\mu(s) & \text{for } t \in (0, 1]. \end{cases} \tag{2.10}$$

Here y and $\dot{y}$ are respectively continuous and of bounded variation on I. The integrals in (2.9) and (2.10) are respectively the Lebesgue integral and the Riemann–Stieltjes integral [3]. Then $(y(t), \dot{y}(t))$ is uniquely determined on I. To emphasize their dependence on y_0, z_0 and μ, let us write $(y(t), \dot{y}(t))$ as $(y(t; y_0, z_0, \mu), \dot{y}(t; y_0, z_0, \mu))$, $t \in I$. Some deep results in [14] are as follows.

Theorem 2.6. *(See [14].) The following solution mapping*

$$\mathbb{R}^2 \times \left(\mathcal{M}_0, w^*\right) \to (\mathcal{C}, \|\cdot\|_\infty), \quad (y_0, z_0, \mu) \to y(\cdot; y_0, z_0, \mu)$$

is continuous. Meanwhile, the following functional is also continuous

$$\mathbb{R}^2 \times \left(\mathcal{M}_0, w^*\right) \to \mathbb{R}, \quad (y_0, z_0, \mu) \to \dot{y}(1; y_0, z_0, \mu).$$

The eigenvalue problem for MDE (2.8) is

$$\mathrm{d}\dot{y} + \lambda y\,\mathrm{d}t + y\,\mathrm{d}\mu(t) = 0, \quad t \in I. \tag{2.11}$$

Some basic results in [14] are as follows. With the Dirichlet boundary condition (1.2), problem (2.11) has a sequence of eigenvalues

$$\lambda_1^D(\mu) < \lambda_2^D(\mu) < \cdots < \lambda_m^D(\mu) < \cdots, \quad \lambda_m^D(\mu) \to +\infty,$$

while, with the Neumann boundary condition

$$\dot{y}(0) = \dot{y}(1) = 0, \tag{2.12}$$

problem (2.11) has a sequence of eigenvalues

$$\lambda_0^N(\mu) < \lambda_1^N(\mu) < \cdots < \lambda_m^N(\mu) < \cdots, \quad \lambda_m^N(\mu) \to +\infty.$$

Eigenvalues of MDE are extensions of eigenvalues of problem (1.1), i.e. $\lambda_m^\sigma(q) = \lambda_m^\sigma(\mu_q)$ for $q \in \mathcal{L}^1$. It is standard to prove that eigenvalues $\lambda_m^\sigma(\mu)$ of (2.11) are continuously Fréchet differentiable in measures $\mu \in (\mathcal{M}_0, \|\cdot\|_{\mathbf{V}})$. Based on results of Theorem 2.6, $\lambda_m^\sigma(\mu)$ possess the following continuous dependence on $\mu \in \mathcal{M}_0$.

Theorem 2.7. *(See [14].) Let $m \in \mathbb{N}$ for $\sigma = D$ or $m \in \mathbb{Z}^+$ for $\sigma = N$. As a nonlinear functional, the following is continuous*

$$\left(\mathcal{M}_0, w^*\right) \to \mathbb{R}, \quad \mu \to \lambda_m^\sigma(\mu).$$

Recall that, in the Lebesgue space $\mathcal{L}^p$, $p \in [1, \infty]$, the weak topology w_p is defined as $q_n \to q_0$ in $(\mathcal{L}^p, w_p)$ iff

$$\int_I f(t)q_n(t)\,\mathrm{d}t \to \int_I f(t)q_0(t)\,\mathrm{d}t \quad \forall f \in \mathcal{L}^{p^*}, \ p^* := \frac{p}{p-1} \in [1, \infty].$$

By the definition of weak topologies, $(\mathcal{L}^p, w_p)$ is continuously embedded into $(\mathcal{M}_0, w^*)$. Hence Theorem 2.7 has generalized the continuity results for eigenvalues of Sturm–Liouville operators in potentials/weights with weak topologies [15,19,23,28].

Example 2.8. Let ν_r be the measure given in (1.14). Precisely,

$$\nu_r(t) = \begin{cases} r/2 & \text{for } t = 0, \\ 0 & \text{for } t \in (0,1), \\ r/2 & \text{for } t = 1. \end{cases} \tag{2.13}$$

We are going to find all eigenvalues of

$$\mathrm{d}\overset{\bullet}{y} + \lambda y\,\mathrm{d}t + y\,\mathrm{d}\nu_r(t) = 0, \quad t \in [0,1], \tag{2.14}$$

with the Neumann boundary condition (2.12). For this purpose, we need only to consider the fundamental solution $\varphi_1(t)$ of (2.14) satisfying $(y(0), \overset{\bullet}{y}(0)) = (1,0)$. At the initial time $t = 0$, one has $(\varphi_1(0), \overset{\bullet}{\varphi}_1(0)) = (1,0)$. Following definition (2.9)–(2.10) for solutions of MDE, for $t \in (0,1]$, one has the following system

$$\varphi_1(t) = 1 + \int_{[0,t]} \overset{\bullet}{\varphi}_1(s)\,\mathrm{d}s, \tag{2.15}$$

$$\overset{\bullet}{\varphi}_1(t) = -\int_{[0,t]} \lambda\varphi_1(s)\,\mathrm{d}t - \int_{[0,t]} \varphi_1(s)\,\mathrm{d}\nu_r(s). \tag{2.16}$$

For $t \in (0,1)$, as

$$\int_{[0,t]} \varphi_1(s)\,\mathrm{d}\nu_r(s) = -\frac{r}{2}\varphi_1(0) = -\frac{r}{2},$$

system (2.15)–(2.16) for $(\varphi_1(t), \overset{\bullet}{\varphi}_1(t))$ is reduced to the classical ODE

$$\ddot{y} + \lambda y = 0,$$

with the initial condition $(y(0+), \dot{y}(0+)) = (1, r/2)$, which is caused by the jump of $\nu_r(t)$ at $t = 0$. The solution is, by setting $\omega = \sqrt{\lambda}$,

$$\begin{aligned} \varphi_1(t) &= y(t) = \cos\omega t + \frac{r}{2}\frac{\sin\omega t}{\omega}, \quad t \in (0,1), \\ \overset{\bullet}{\varphi}_1(t) &= \dot{y}(t) = -\omega\sin\omega t + \frac{r}{2}\cos\omega t, \quad t \in (0,1). \end{aligned} \tag{2.17}$$

As solutions of MDE are continuous in t, formula (2.17) is also true at $t = 0, 1$. At $t = 1$, the velocity $\overset{\bullet}{\varphi}_1(1)$ is obtained from the integral equality (2.16)

$$\begin{aligned} \overset{\bullet}{\varphi}_1(1) &= -\int_{[0,1]} \lambda\varphi_1(s)\,\mathrm{d}s - \int_{[0,1]} \varphi_1(s)\,\mathrm{d}\nu_r(s) \\ &= -\int_{[0,1]} \lambda\varphi_1(s)\,\mathrm{d}s - \frac{r}{2}\big(\varphi_1(1) - \varphi_1(0)\big) \\ &= -\big(\lambda + r^2/4\big)\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}. \end{aligned} \tag{2.18}$$

See (2.17) for $\varphi_1(t)$.

Now the Neumann eigenvalues of ν_r are obtained by solving $\overset{\bullet}{\varphi}_1(1) = 0$. Using (2.18), we have

$$\lambda_0(\nu_r) = -r^2/4, \qquad \lambda_m(\nu_r) = (m\pi)^2 \quad \text{for } m \in \mathbb{N}. \tag{2.19}$$

That is, $\lambda_m(\nu_r)$ differ from the classical eigenvalues $\lambda_m(0) = (m\pi)^2$ only by the zeroth Neumann eigenvalue. Moreover, (2.19) shows that $\lambda_m(-\nu_r) = \lambda_m(\nu_{-r}) = \lambda_m(\nu_r)$.

3. Minimal eigenvalues in $\tilde{B}_p[r]$ with $1 < p < \infty$

Besides balls $B_p[r]$, $\tilde{B}_p[r]$ and spheres $S_p[r]$, $\tilde{S}_p[r]$ in the Lebesgue spaces $(\mathcal{L}^p, \|\cdot\|_p)$, let us introduce the following balls and spheres in the measure space $(\mathcal{M}_0, \|\cdot\|_{\mathbf{V}})$

$$B_0[r] := \{\mu \in \mathcal{M}_0:\ \|\mu\|_{\mathbf{V}} \leqslant r\},$$
$$S_0[r] := \{\mu \in \mathcal{M}_0:\ \|\mu\|_{\mathbf{V}} = r\},$$
$$\tilde{B}_0[r] := \left\{\mu \in B_0[r]:\ \int_I \mathrm{d}\mu = \mu(1) - \mu(0) = 0\right\},$$
$$\tilde{S}_0[r] := \{\mu \in S_0[r]:\ \mu(1) - \mu(0) = 0\}.$$

By the Hölder inequality and equality (2.7), one has, for $1 \leqslant p \leqslant \infty$,

$$B_p[r] \subset B_1[r] \subset B_0[r], \qquad \tilde{B}_p[r] \subset \tilde{B}_1[r] \subset \tilde{B}_0[r].$$

Since balls $B_0[r]$ and $\tilde{B}_0[r]$ are sequentially compact in $(\mathcal{M}_0, w^*)$, Theorem 2.7 implies that $\lambda_m^\sigma(\cdot)$ are bounded in $B_0[r]$. In particular, all extremal values of (1.4) and (1.8) are finite and well defined, including the case $p = 1$ we are interested in. Topologically, $\tilde{\mathcal{L}}^p \subset (\mathcal{L}^p, \|\cdot\|_p)$ is a closed subspace. From definition of weak topologies, $\tilde{\mathcal{L}}^p$ is also closed in $(\mathcal{L}^p, w_p)$.

In the following, we always assume that $r \in (0, \infty)$ and $p \in (1, \infty)$. In this case $\tilde{B}_p[r]$ is sequentially compact in $(\mathcal{L}^p, w_p)$. By Theorem 2.7, one has some $q_{p,r} \in \tilde{B}_p[r]$, called a minimizer of $\tilde{\mathbf{L}}_p(r)$, such that $\lambda_0(q_{p,r}) = \tilde{\mathbf{L}}_p(r)$. The aim of this section is to give a characterization for $q_{p,r}$ and $\tilde{\mathbf{L}}_p(r)$.

3.1. Critical equations and the dynamics

Note that $\tilde{B}_p[r] \subset \mathcal{L}^p$ is a domain of codimension one with boundary $\tilde{S}_p[r]$. Geometrically, the boundary $\tilde{S}_p[r]$ is differentiable because $p \in (1, \infty)$.

By Lemma 2.5, minimizers $q_{p,r}$ are on $\tilde{S}_p[r]$. This can be proved in another way.

Lemma 3.1. *One has $q_{p,r} \in \tilde{S}_p[r]$. That is, minimizers $q_{p,r}$ for $\tilde{\mathbf{L}}_p(r)$ must be on the corresponding spheres.*

Proof. Suppose that $q_{p,r}$ is in the interior of the ball $\tilde{B}_p[r]$. Then $q_{p,r}$ is a minimizer of the following minimization problem

$$\operatorname{Min} \lambda_0(q) \quad \text{subject to} \quad \|q\|_p < r \text{ and } \bar{q} = 0.$$

Note that the Fréchet derivative of the linear functional $\mathcal{L}^p \ni q \to \bar{q} \in \mathbb{R}$ is

$$\partial_q \bar{q} = 1, \tag{3.1}$$

understood as in (2.4). By the Lagrangian multiplier method, $q_{p,r}$ satisfies $-W^2 = c_1$, where W is a normalized eigenfunction associated with $\lambda_0(q_{p,r})$ and c_1 is a constant. This implies that $W(t)$ is constant and therefore, the potential $q_{p,r}$ is also constant. Since $\bar{q}_{p,r} = 0$, we obtain $q_{p,r} = 0$ and $\tilde{\mathbf{L}}_p(r) = \lambda_0(q_{p,r}) = \lambda_0(0) = 0$. This is impossible because we have known that $\tilde{\mathbf{L}}_p(r) < 0$. □

Remark 3.2. The proof of Lemma 3.1 shows that, as a nonlinear functional in $\tilde{\mathcal{L}}^p$, $\lambda_0(q)$ has $q = 0$ as its unique critical point. From inequality (1.10), $q = 0$ is the global maximizer of $\lambda_0(\cdot)$ in $\tilde{\mathcal{L}}^p$.

Due to Lemma 3.1, $\tilde{\mathbf{L}}_p(r)$ can be reduced to the following constraint minimization problem in the space $\mathcal{L}^p$,

$$\tilde{\mathbf{L}}_p(r) = \operatorname{Min} \lambda_0(q) \quad \text{subject to} \quad \bar{q} = 0 \text{ and } \|q\|_p = r. \tag{3.2}$$

The nonlinear functional $\mathcal{L}^p \ni q \to \|q\|_p \in \mathbb{R}$ is continuously Fréchet differentiable, with the Fréchet derivative

$$\partial_q \|q\|_p = \|q\|_p^{1-p} \phi_p(q). \tag{3.3}$$

Here the mapping $\phi_p : \mathbb{R} \to \mathbb{R}$ is defined by $\phi_p(s) := |s|^{p-2}s$.

Applying the Lagrangian multiplier method to problem (3.2), it follows from formulas (2.4), (3.1) and (3.3) that minimizers $q := q_{p,r}$ of problem (3.2) satisfy for some constants c_i,

$$W^2 = c_0\phi_p(q) + c_1, \tag{3.4}$$

where W is a normalized eigenfunction associated with $\lambda_0(q)$, i.e. $W(t)$ satisfies

$$\ddot{W} + (\ell + q)W = 0, \quad \ell := \lambda_0(q), \tag{3.5}$$

boundary condition (1.3) and normalization condition $\|W\|_2 = 1$. Since W is the zeroth Neumann eigenfunction, let us assume that $W(t) > 0$ for all $t \in I$.

Lemma 3.3. *The Lagrangian multipliers c_0 and c_1 in* (3.4) *are positive.*

Proof. By (3.5), we have

$$\int_I W\ddot{W}\,\mathrm{d}t + \ell \int_I W^2\,\mathrm{d}t + \int_I qW^2\,\mathrm{d}t = 0. \tag{3.6}$$

Since W satisfies (1.3), the first term of (3.6) is $-\int_I \dot{W}^2\,\mathrm{d}t$. By (3.4), one has

$$\int_I qW^2\,\mathrm{d}t = \int_I q\big(c_0\phi_p(q) + c_1\big)\,\mathrm{d}t = c_0 \int_I |q|^p\,\mathrm{d}t,$$

because $\phi_p(s)s = |s|^p$ and $\bar{q} = 0$. Substituting into (3.6), we obtain the following equality

$$c_0 \int_I |q|^p\,\mathrm{d}t = \int_I \dot{W}^2\,\mathrm{d}t - \ell \int_I W^2\,\mathrm{d}t. \tag{3.7}$$

Since $\ell = \lambda_0(q) = \tilde{\mathbf{L}}_p(r) < 0$, this shows that $c_0 > 0$.

On the other hand, it follows from Eq. (3.4) that

$$q = \phi_{p^*}\big((W^2 - c_1)/c_0\big) = \phi_{p^*}\big(W^2 - c_1\big)/\phi_{p^*}(c_0).$$

As $\bar{q} = 0$, we have $c_1 = W^2(t_1) > 0$ for some t_1. □

In the following we use the idea in [22,29] to give a reduction for Eqs. (3.4) and (3.5). Since $c_i > 0$, let us introduce

$$m = c_1/c_0 > 0, \qquad y(t) = W(t)/\sqrt{c_0}.$$

Then $y(t)$ is also a positive, non-constant eigenfunction associated with $\lambda_0(q) = \ell$. From (3.4), one has

$$q = \phi_{p^*}\big(y^2 - m\big). \tag{3.8}$$

Inserting (3.8) into the equation

$$\ddot{y} + \ell y + q(t)y = 0 \tag{3.9}$$

for the eigenfunction $y(t)$, we obtain the following autonomous Schrödinger equation for $y(t)$,

$$\ddot{y} + \ell y + \phi_{p^*}\big(y^2 - m\big)y = 0. \tag{3.10}$$

As for the minimization problem (3.2), it follows from (3.8) and condition $\bar{q} = 0$ that $y(t)$ satisfies

$$\int_I \phi_{p^*}\big(y^2(t) - m\big)\,\mathrm{d}t = 0. \tag{3.11}$$

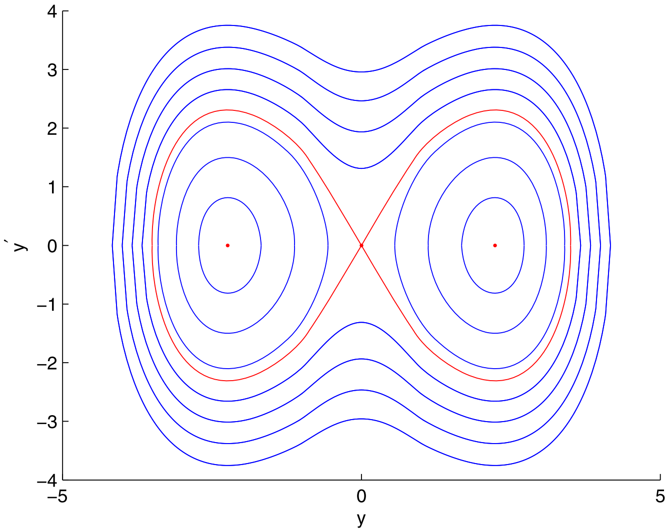

Fig. 1. Phase portrait of critical Eq. (3.10).

Moreover, (3.8) implies that $|q(t)|^p = |y^2(t) - m|^{p(p^*-1)} = |y^2(t) - m|^{p^*}$. Thus condition $\|q\|_p = r$ is transformed into

$$\int_I \left|y^2(t) - m\right|^{p^*} \mathrm{d}t = r^p. \tag{3.12}$$

Definition 3.4. Eq. (3.10) is called the *critical equation* of problem (3.2), while system of Eqs. (3.10)–(3.12) is called the *critical system* of problem (3.2). Here $\ell \in (-\infty, 0)$ is a parameter.

For the minimization problem of $\lambda_0(q)$ in $B_p[r]$, the critical system is composed of Eqs. (3.10) and (3.12) where m is taken as 0. See [29]. For the present minimization problem of $\lambda_0(q)$ in $\tilde{B}_p[r]$, we have an additional parameter $m > 0$ and an additional constraint (3.11) on solutions y, which are caused by the additional constraint $\bar{q} = 0$ on potentials. Critical equations like (3.10) are typical in many eigenvalues minimization problems and in Sobolev inequalities [6].

The dynamics of critical equation (3.10) is as follows. Eq. (3.10) is invariant under transformation $y \to -y$. Since $m > 0$ and $\ell < 0$, Eq. (3.10) has three equilibria

$$y_0 = 0 \quad \text{and} \quad y_\pm = y_\pm(m, \ell) := \pm\sqrt{m + |\ell|^{p-1}}.$$

The corresponding linearized equations are

$$\ddot{y} + \alpha_0 y = 0, \quad \alpha_0 := \ell - \phi_{p^*}(m) < 0,$$
$$\ddot{y} + \alpha_\pm y = 0, \quad \alpha_\pm := 2\big(p^* - 1\big)|\ell|^{2-p}\big(m - \phi_p(\ell)\big) > 0.$$

Hence y_0 is hyperbolic and $y_\pm$ are elliptic. Emanating from the hyperbolic equilibrium $(0, 0)$, Eq. (3.10) has two homoclinic orbits. Inside homoclinic orbits, Eq. (3.10) has two families of non-constant periodic solutions surrounding equilibria $(y_\pm, 0)$ which are strictly positive or strictly negative respectively. Outside these homoclinic orbits, the phase portrait is filled by a family of sign-changing periodic orbits. See Fig. 1.

3.2. Construction of minimal potentials

We are going to construct minimizers of problem (3.2), which are determined by critical system (3.10)–(3.12). Note that $y(t)$, $t \in [0, 1]$, is a positive, non-constant solution of Eq. (3.10) satisfying the Neumann condition (1.3). By Eq. (3.10), $y(t)$ satisfies the following conservation law

$$\dot{y}^2(t) + F_p\big(y(t)\big) \equiv F_p\big(y(0)\big), \tag{3.13}$$

where

$$F_p(x) := \ell x^2 + \frac{1}{p^*}\big|x^2 - m\big|^{p^*}, \quad x \in \mathbb{R}. \tag{3.14}$$

Let us introduce

$$a := y(0) > 0, \qquad b := y(1) > 0. \tag{3.15}$$

Due to (1.3) and (3.13), a and b are correlated by

$$F_p(a) = F_p(b). \tag{3.16}$$

As a solution of autonomous equation (3.10), $y(t)$, $t \in [0, 1]$, can be extended to the whole line $\mathbb{R}$, still denoted by $y(t)$. Note that Eq. (3.10) is invariant under the reflection of time $t \to -t$. As $\dot{y}(0) = 0$, one sees that

$$y(-t) \equiv y(t), \quad t \in \mathbb{R}. \tag{3.17}$$

Furthermore, we assert that $y(t)$ satisfies

$$y(t+2) \equiv y(t), \quad t \in \mathbb{R}. \tag{3.18}$$

To see this, let us notice that $y_1(t) := y(t+1)$ and $y_2(t) := y(1-t)$, $t \in \mathbb{R}$, are solutions of Eq. (3.10). Due to (1.3) and (3.15), one has $(y_i(0), \dot{y}_i(0)) = (b, 0)$, $i = 1, 2$. Hence

$$y(t+1) \equiv y(1-t) \equiv y(t-1),$$

where the last equality follows from (3.17). Thus we have equality (3.18). Since $y(t) > 0$ for $t \in [0, 1]$, it follows from (3.17) and (3.18) that $y(t)$ is a positive, non-constant 2-periodic solution of Eq. (3.10).

Since $q(t)$, defined by (3.8), is a minimizer of problem (3.2), we have the following important observation on $y(t)$.

Lemma 3.5. *The solution $y(t)$ of system (3.10)–(3.12) has the minimal period 2.*

Proof. The proof is similar to that of [29, Lemma 2.12]. In fact, as we know that $y(t)$ is non-constant, the present proof is relatively easier. □

Remark 3.6. From Lemma 3.5, we have necessarily $a \neq b$, where a and b are as in (3.15). Otherwise, $b = a$ would imply that $y(t+1) \equiv y(t)$, because both $y(t)$ and $y(t+1)$ are solutions of (3.10) with the same initial value $(a, 0)$ at $t = 0$. This means that $y(t)$ is 1-periodic, which is impossible.

Since $a \neq b$, in the following we consider the case

$$a = y(0) < y(1) = b. \tag{3.19}$$

In fact, for the case $a > b$, instead of $y(t)$, one can consider the solution $\hat{y}(t) := y(1-t)$.

Lemma 3.7. *Suppose that $y(t)$ satisfies (3.19). Then $y(t)$ is strictly increasing on* $[0, 1]$.

Proof. Since we have assumed (3.19), it suffices to prove that $\dot{y}(t) \neq 0$ for all $t \in (0, 1)$. Otherwise, if $\dot{y}(\tau) = 0$ for some $\tau \in (0, 1)$, arguing as above, $y(t)$ has 2τ as its period, a contradiction to Lemma 3.5. □

Now we are able to construct the minimizers $q_{p,r}$ and the minimal eigenvalues $\lambda_0(q_{p,r}) = \tilde{\mathbf{L}}_p(r)$, where $r \in (0, \infty)$ and $p \in (1, \infty)$. Note that Eq. (3.10) contains two parameters $m > 0$ and $\ell < 0$. Given $a \in (0, y_+(m, \ell))$, it follows from the conservation law (3.13) that the minimal period of the solution $y(t; a)$ of Eq. (3.10) satisfying $(y(0), \dot{y}(0)) = (a, 0)$ is $2\mathbf{T}_p(a)$, where

$$\mathbf{T}_p(a) = \int_a^b \frac{\mathrm{d}x}{\sqrt{F_p(a) - F_p(x)}},$$

with $b \in (y_+(m,\ell), \infty)$ being determined by Eq. (3.16). Solving

$$\mathbf{T}_p(a) = 1,$$

one can obtain $a = A_p(m,\ell)$. This gives the solution $y = y(t; m,\ell)$ of Eq. (3.10). Finally, by inserting $y(t; m,\ell)$ into (3.11) and (3.12), we obtain a system for (m,ℓ)

$$\int_I \phi_{p^*}\bigl(y^2(t; m,\ell) - m\bigr)\,\mathrm{d}t = 0,$$

$$\int_I \bigl|y^2(t; m,\ell) - m\bigr|^{p^*}\,\mathrm{d}t = r^p.$$

The solution for ℓ gives $\tilde{\mathbf{L}}_p(r)$, while $q_{p,r}$ is given by Eq. (3.8). Note that even for the case $p = 2$, $q_{p,r}$ and $\tilde{\mathbf{L}}_p(r)$ cannot be expressed using elementary functions. This is similar to the problems in [22,29]. On the other hand, it is possible to give an expression for $\tilde{\mathbf{L}}_p(r)$ using singular integrals as in [22,29].

We end this section by deriving some equalities on solutions $y(t)$ of Eq. (3.10) which are used in (3.8). In order to emphasize the dependence of the objects $y(t)$, a, b etc. on the exponent $p \in (1,\infty)$, we write $\ell_p := \tilde{\mathbf{L}}_p(r)$ and

$$y(t) = y_p(t), \qquad q(t) = q_p(t), \qquad a = a_p, \qquad m = m_p, \qquad b = b_p.$$

Since the periodic orbit $(y_p(t), \dot{y}_p(t))$ surrounds the equilibrium $(y_+, 0)$, we have $a_p < y_+ < b_p$. In particular, one has the following inequality

$$b_p^2 > m_p - \phi_p(\ell_p). \tag{3.20}$$

Moreover, condition (3.11) implies that $m_p = y_p^2(t_p)$ for some $t_p \in I$. Hence we have another inequality

$$a_p^2 < m_p. \tag{3.21}$$

Note that a_p^2 and b_p^2 are respectively the minimum and the maximum of $y_p^2(t)$, while m_p determined by (3.11) is called the p^*-th mean value of $y_p^2(t)$ in literature.

Lemma 3.8. *One has the following equalities*

$$-\ell_p \int_I y_p^2\,\mathrm{d}t + \int_I \dot{y}_p^2\,\mathrm{d}t = r^p, \tag{3.22}$$

$$\ell_p \int_I y_p^2\,\mathrm{d}t + \int_I \dot{y}_p^2\,\mathrm{d}t = F_p(a_p) - \frac{1}{p^*} r^p, \tag{3.23}$$

$$\int_I \frac{\dot{y}_p^2}{y_p^2}\,\mathrm{d}t = -\ell_p. \tag{3.24}$$

Proof. Equality (3.22) is just (3.7), if the condition $\|q\|_p = r$ is used. By (3.13) and (3.14),

$$\begin{aligned} F_p(a_p) &= \int_I \dot{y}_p^2\,\mathrm{d}t + \int_I F_p(y_p)\,\mathrm{d}t \\ &= \int_I \dot{y}_p^2\,\mathrm{d}t + \ell_p \int_I y_p^2\,\mathrm{d}t + \frac{1}{p^*}\int_I \bigl|y_p^2 - m_p\bigr|^{p^*}\,\mathrm{d}t. \end{aligned}$$

With condition (3.12) for the last term, we obtain (3.23). Since the zeroth eigenfunction $y_p(t)$ is positive, we can rewrite Eq. (3.10) as

$$-\ell_p - q_p(t) = \frac{\ddot{y}_p}{y_p}.$$

Integrating it over I, we obtain

$$-\ell_p = -\ell_p - \bar{q}_p = \int_I \frac{\ddot{y}_p}{y_p}\,\mathrm{d}t = \frac{\dot{y}_p}{y_p}\bigg|_0^1 - \int_I \dot{y}_p\,\mathrm{d}\frac{1}{y_p} = \int_I \frac{\dot{y}_p^2}{y_p^2}\,\mathrm{d}t,$$

where the Neumann condition (1.3) is used. This gives (3.24). □

4. Infimum of $\lambda_0(q)$ in $\tilde{B}_1[r]$

4.1. Limiting approach

The following approach is a combination of the limiting technique in [22,29] and results on MDE in [14]. Our task is to show that, as $p \downarrow 1$, all objects for minimization problem $\tilde{\mathbf{L}}_p(r)$, including y_p, m_p, ℓ_p, a_p and b_p, will have limits in appropriate sense, with the limits being denoted by y_0, m_0, ℓ_0 etc. To this end, we use the following argument. For any sequence p_n of exponents such that $p_n \downarrow 1$, we will prove that these objects have some subsequences which are convergent. Finally we will prove that these limits can be explicitly expressed using r only and are independent of the choice of sequences. This ensures that the convergence of $\lim_{p\downarrow 1} y_p$ etc. However, for simplicity, we still write the limit process as $\lim_{p\downarrow 1}$, which is understood as finding subsequences from given sequences $p_n \downarrow 1$.

At first let us restate result (2.1) as

$$\lim_{p\downarrow 1} \ell_p = \ell_0 := \tilde{\mathbf{L}}_1(r) \in (-\infty, 0). \tag{4.1}$$

This is important in the following analysis. For example, let $p_0 \in (1, \infty)$ be fixed. Then (3.22) and (4.1) imply that

$$\{y_p\colon\ 1 < p \leqslant p_0\} \subset \mathcal{H}^1 := \mathcal{H}^1(I, \mathbb{R}) \quad \text{is bounded.} \tag{4.2}$$

Consequently, we can assert that, as $p \downarrow 1$,

$$y_p \to y_0 \text{ in } \big(\mathcal{H}^1, w\big), \qquad y_p \to y_0 \quad \text{in } (\mathcal{C}, \|\cdot\|_\infty), \tag{4.3}$$

following simply from the embedding theorem. As explained before, (4.3) means that for any $p_n \downarrow 1$, one has a subsequence $p_{n'} \downarrow 1$ such that $y_{p_{n'}} \to y_0$ for some y_0, which is presumably assumed to depend on sequences. By (4.3), we conclude

$$\begin{cases} a_p = y_p(0) \to a_0 := y_0(0), \\ b_p = y_p(1) \to b_0 := y_0(1), \\ m_p = y_p^2(t_p) \to m_0. \end{cases} \tag{4.4}$$

Lemma 4.1. *We assert that $a_0 > 0$, $m_0 \geqslant a_0^2$ and $b_0 \geqslant \sqrt{m_0 + 1}$.*

Proof. The second and the third result follow simply from (3.21) and (3.20) respectively.

In order to prove that $a_0 > 0$, we will apply MDE to simplify the argument. The potential q_p induces a measure

$$Q_p(t) = \int_{[0,t]} q_p(s)\,\mathrm{d}s \in B_0[r] \subset \mathcal{M}_0.$$

By compactness of $B_0[r]$, we can assume that

$$Q_p \to \nu_* \quad \text{in } \big(\mathcal{M}_0, w^*\big). \tag{4.5}$$

Eq. (3.10) for y_p is the same as the linear equation (3.9), which can be understood as an MDE with the measure $\ell_p\mu_0 + Q_p$, where μ_0 is the Lebesgue measure of I. By (4.1) and (4.5), we have the convergence $\ell_p\mu_0 + Q_p \to \ell_0\mu_0 + \nu_*$ in w^*.

Note that the initial value of y_p is $(a_p, 0)$. If $a_p \to a_0 = 0$, it follows from Theorem 2.6 that $y_p \to y_0^*$ in $(\mathcal{C}, \|\cdot\|_\infty)$, where y_0^* is the solution of the limiting MDE with the measure $\ell_0\mu_0+\nu_*$ and with initial value $(0,0)$. Hence $y_0^*(t) \equiv 0$. That is, we would have $y_p \to 0$ in $(\mathcal{C}, \|\cdot\|_\infty)$. It then follows from (4.1) and (4.4) that

$$m_p = y_p^2(t_p) \to 0,$$
$$F_p(y_p) = \ell_p y_p^2 + \frac{1}{p^*}\left|y_p^2 - m_p\right|^{p^*} \to 0 \quad \text{in } \left(\mathcal{C}, \|\cdot\|_\infty\right),$$
$$F_p(a_p) = F_p\big(y_p(0)\big) \to 0.$$

Now conservation law (3.13) shows that $\dot{y}_p \to 0$ in $(\mathcal{C}, \|\cdot\|_\infty)$. These imply that the limiting case of (3.22) is the equality $0 = r$, which is impossible. □

We remark that the argument using MDE has actually given another proof for the second convergence of (4.3). Moreover, by Lemma 3.7 and convergence results (4.3), the fact $a_0 > 0$ implies that

$$y_0(t) = \lim_{p\downarrow 1} y_p(t) \geqslant \lim_{p\downarrow 1} y_p(0) = a_0 > 0, \quad t \in I.$$

As a consequence of (3.23) and (4.2), as $p \downarrow 1$, $\{F_p(a_p)\} \subset \mathbb{R}$ is bounded. Thus we can assume that

$$F_p(a_p) \to E_0 \in \mathbb{R}. \tag{4.6}$$

Suggested by (3.22)–(3.24), let us introduce

$$h_0(t) := E_0 - \ell_0 y_0^2(t). \tag{4.7}$$

We have the following convergence result.

Lemma 4.2. *One has $h_0(t) \geqslant 0$ for all t. Moreover,*

$$\dot{y}_p^2 \to h_0 \quad \text{in } \left(\mathcal{L}^1, \|\cdot\|_1\right). \tag{4.8}$$

Proof. Let us introduce

$$h_p(t) := F_p(a_p) - \ell_p y_p^2(t), \qquad g_p(t) := -\left|y_p^2(t) - m_p\right|^{p^*}/p^*.$$

The conservation law (3.13) can be rewritten as

$$\dot{y}_p^2 - h_0 = (h_p - h_0) + g_p.$$

By (4.3) and (4.6), we can obtain

$$h_p \to h_0 \quad \text{in } \left(\mathcal{C}, \|\cdot\|_\infty\right). \tag{4.9}$$

On the other hand, by using (3.8), one has $g_p = -(y_p^2 - m_p)q_p/p^*$. Thus

$$\|g_p\|_1 \leqslant \|q_p\|_1 \left\|y_p^2 - m_p\right\|_\infty/p^* \leqslant r\left\|y_p^2 - m_p\right\|_\infty/p^* \leqslant r\left(\|y_p\|_\infty^2 + \|y_p\|_\infty^2\right)/p^* \to 0,$$

i.e.

$$g_p \to 0 \quad \text{in } \left(\mathcal{L}^1, \|\cdot\|_1\right). \tag{4.10}$$

Now (4.8) follows simply from (4.9) and (4.10).

Finally, (4.8) shows that $h_0(t) \geqslant 0$ for a.e. t. As $h_0(t)$ is continuous, $h_0(t) \geqslant 0$ for all t. □

Now we can use convergence results (4.1) and (4.8) on y_p and $\dot{y}_p$ to obtain the following results.

Lemma 4.3. *We have*

$$E_0 = 0, \tag{4.11}$$
$$a_0^2 + 1 = m_0 = b_0^2 - 1. \tag{4.12}$$

Proof. Let $p \downarrow 1$ in (3.24). By using (4.3), (4.7) and (4.8), we get

$$\int_I \frac{\dot{y}_p^2}{y_p^2}\,\mathrm{d}t \to \int_I \frac{h_0}{y_0^2}\,\mathrm{d}t = E_0 \int_I \frac{1}{y_0^2}\,\mathrm{d}t - \ell_0.$$

One has necessarily $E_0 = 0$.

Next, result (4.11) is the same as

$$\left(m_p - a_p^2\right)^{p^*}/p^* = F_p(a_p) - \ell_p a_p^2 \to -\ell_0 a_0^2 \in (0, \infty). \tag{4.13}$$

As $p \downarrow 1$, one has $p^* \uparrow +\infty$. Hence (4.13) implies that $m_p - a_p^2 \to 1$. This proves the first equality of (4.12). On the other hand, it follows from $F_p(b_p) = F_p(a_p)$ that

$$\left(b_p^2 - m_p\right)^{p^*}/p^* = F_p(b_p) - \ell_p b_p^2 = F_p(a_p) - \ell_p b_p^2 \to -\ell_0 b_0^2 \in (0, \infty),$$

because we have known that $E_0 = 0$. Similarly, this convergence shows that $b_p^2 - m_p \to 1$, which gives the second equality of (4.12). □

We will use the following elementary inequality.

Lemma 4.4. *There holds the following inequality*

$$D_h(\varepsilon) := (\sqrt{h+\varepsilon} - \sqrt{h}\,)^2 \leqslant |\varepsilon|, \quad \forall h \geqslant 0,\ \varepsilon \geqslant -h. \tag{4.14}$$

Proof. Explicitly,

$$D_h(\varepsilon) = 2h + \varepsilon - 2\sqrt{h^2 + h\varepsilon}.$$

When $h = 0$, one has $D_0(\varepsilon) = \varepsilon$, $\varepsilon \geqslant 0$, and (4.14) is evident.

When $h > 0$ and $\varepsilon \in [0, \infty)$, one has (4.14), because $D_h(\varepsilon) = \varepsilon + 2(h - \sqrt{h^2 + h\varepsilon}) \leqslant \varepsilon$.

When $h > 0$ and $\varepsilon \in [-h, 0]$, one has

$$D_h(\varepsilon) \leqslant |\varepsilon| \quad \Longleftrightarrow \quad h - |\varepsilon| \leqslant \sqrt{h^2 - h|\varepsilon|} \quad \Longleftrightarrow \quad \varepsilon^2 \leqslant h|\varepsilon|.$$

The last is obvious because $\varepsilon \in [-h, 0]$. □

Notice from (4.7) and (4.11) that $h_0 = -\ell_0 y_0^2$.

Lemma 4.5. *We assert that $y_0(t)$ satisfies*

$$y_0(t) = a_0 + \int_0^t \sqrt{h_0(s)}\,\mathrm{d}s = a_0 + \sqrt{-\ell_0} \int_0^t y_0(s)\,\mathrm{d}s, \quad t \in I. \tag{4.15}$$

Proof. By conservative law (3.13), we have, for $t \in I$,

$$\dot{y}_p(t) = \sqrt{h_p(t) + g_p(t)}.$$

Thus

$$y_p(t) - a_p = \int_0^t \sqrt{h_0(s) + \varepsilon_p(s)}\,\mathrm{d}s = \int_0^t \sqrt{h_0(s)}\,\mathrm{d}s + I_p(t), \tag{4.16}$$

where

$$\varepsilon_p(t) := h_p(t) - h_0(t) + g_p(t) \geqslant -h_0(t),$$
$$I_p(t) := \int_0^t \left(\sqrt{h_0(s) + \varepsilon_p(s)} - \sqrt{h_0(s)}\right) \mathrm{d}s.$$

By the Cauchy inequality, we have for $t \in I$,

$$\begin{aligned}\left|I_p(t)\right|^2 &\leqslant t \int_0^t \left(\sqrt{h_0(s) + \varepsilon_p(s)} - \sqrt{h_0(s)}\right)^2 \mathrm{d}s \\ &\leqslant \int_I \left(\sqrt{h_0(s) + \varepsilon_p(s)} - \sqrt{h_0(s)}\right)^2 \mathrm{d}s \\ &\leqslant \int_I \left|\varepsilon_p(s)\right| \mathrm{d}s,\end{aligned}$$

where the last inequality follows from (4.14). By (4.9) and (4.10), we can obtain

$$\|I_p\|_\infty^2 \leqslant \|\varepsilon_p\|_1 \leqslant \|h_p - h_0\|_1 + \|g_p\|_1 \to 0.$$

By letting $p \downarrow 1$ in (4.16), we obtain (4.15). □

Proof of Theorem 1.1. Now we can deduce the infimum $\tilde{\mathbf{L}}_1(r)$ by finding ℓ_0. Denote

$$\omega_0 := \sqrt{-\ell_0} > 0.$$

From (4.15), we know that $y_0 \in \mathcal{C}^1(I, \mathbb{R})$ satisfies

$$\dot{y}_0 = \omega_0 y_0, \quad t \in I.$$

Since $y_0(0) = a_0$, we get

$$y_0(t) = a_0 e^{\omega_0 t}, \quad t \in I.$$

By (4.12), we have

$$2 = b_0^2 - a_0^2 = \left(e^{2\omega_0} - 1\right) a_0^2. \tag{4.17}$$

Equalities (3.22) and (3.23) can be rewritten as

$$-2\ell_p \int_I y_p^2 = \left(1 + \frac{1}{p^*}\right) r^p - F_p(a_p), \qquad 2 \int_I \dot{y}_p^2 = \frac{r^p}{p} + F_p(a_p).$$

Both of the limiting equalities are

$$r = 2\omega_0^2 \int_I y_0^2 = 2\omega_0^2 \int_I a_0^2 e^{2\omega_0 t} \, \mathrm{d}t = \omega_0 \left(e^{2\omega_0} - 1\right) a_0^2 = 2\omega_0.$$

See (4.17). Hence

$$\omega_0 = r/2, \qquad \ell_0 = -\omega_0^2 = -r^2/4.$$

The other limits are

$$a_0 = \left(\frac{2}{e^{2\omega_0} - 1}\right)^{1/2} = \left(\frac{2}{e^r - 1}\right)^{1/2},$$
$$b_0 = \left(a_0^2 + 2\right)^{1/2} = \left(\frac{2e^r}{e^r - 1}\right)^{1/2},$$
$$m_0 = \left(a_0^2 + 1\right)^{1/2} = \left(\frac{e^r + 1}{e^r - 1}\right)^{1/2},$$
$$y_0(t) = a_0 e^{\omega_0 t} = \left(\frac{2}{e^r - 1}\right)^{1/2} e^{rt/2}, \quad t \in [0, 1]. \tag{4.18}$$

All of these limits are determined by r and are independent of the choices of sequences of exponents. As explained at the beginning, as $p \downarrow 1$, these objects have limits given above. In particular, $\tilde{\mathbf{L}}_1(r) = \ell_0 = -r^2/4$, proving (1.13).

When $r > 0$, the fact that $\tilde{\mathbf{L}}_1(r)$ cannot be attained by any potential from $\tilde{B}_1[r]$ will be proved in the next subsection after we find the minimal measures. □

We can give the precise meaning for the convergence of solutions y_p.

Lemma 4.6. *As $p \downarrow 1$, one has $y_p \to y_0$ in the Sobolev space $(W^{1,2}(I), \|\cdot\|_{W^{1,2}})$.*

Proof. Note that the limiting solution $y_0(t)$ in (4.18) is smooth on I. Moreover, one has

$$h_0(t) = -\ell_0 y_0^2(t) = \frac{r^2}{2(e^r - 1)} e^{rt} = \dot{y}_0^2(t), \quad t \in I.$$

Hence convergence result (4.8) can be stated as

$$\dot{y}_p^2 \to \dot{y}_0^2 \quad \text{in } \left(\mathcal{L}^1, \|\cdot\|_1\right). \tag{4.19}$$

As $p \downarrow 1$, one has

$$\|\dot{y}_p - \dot{y}_0\|_1 = \int_I \frac{|\dot{y}_p^2(t) - \dot{y}_0^2(t)|}{|\dot{y}_p(t) + \dot{y}_0(t)|} \, \mathrm{d}t \to 0, \tag{4.20}$$

because $|\dot{y}_p(t) + \dot{y}_0(t)|$ has a uniform positive lower bound. Moreover, by using (4.19) and (4.20),

$$\begin{aligned}\|\dot{y}_p - \dot{y}_0\|_2^2 &= \int_I (\dot{y}_p - \dot{y}_0)^2 \, \mathrm{d}t = \int_I \dot{y}_p^2 \, \mathrm{d}t + \int_I \dot{y}_0^2 \, \mathrm{d}t - 2\int_I \dot{y}_p \dot{y}_0 \, \mathrm{d}t \\ &= \int_I \left(\dot{y}_p^2 - \dot{y}_0^2\right) \mathrm{d}t - 2\int_I (\dot{y}_p - \dot{y}_0)\dot{y}_0 \, \mathrm{d}t \\ &\to 0,\end{aligned}$$

i.e. $\dot{y}_p \to \dot{y}_0$ in $(\mathcal{L}^2, \|\cdot\|_2)$. Combining with the second result of (4.3), the lemma is proved. □

Remark 4.7. As $p \downarrow 1$, one has $y_p \not\to y_0$ in $(C^1(I), \|\cdot\|_{C^1})$, because $\dot{y}_p(0) = \dot{y}_p(1) = 0$ for all $p \in (1, \infty)$ and $\dot{y}_0(1) > \dot{y}_0(0) > 0$. See (1.3) and (4.18) respectively.

4.2. Minimal measures

Given $r \in (0, \infty)$, we have from (2.1)

$$\tilde{\mathbf{L}}_1(r) = \lim_{p \downarrow 1} \lambda_0(q_{p,r}), \tag{4.21}$$

where $q_{p,r}$ is the minimal potential of $\lambda_0(q)$ in $\tilde{B}_p[r]$, $p \in (1, \infty)$. Associated with $\lambda_0(q_{p,r})$ is the eigenfunction $y_{p,r}(t)$. The limiting eigenfunction $y_0(t)$ of $y_{p,r}(t)$ is given by (4.18). Let $Q_{p,r}$ be the measure induced by the potential $q_{p,r} \in \tilde{S}_p[r]$

$$Q_{p,r}(t) := \int_{[0,t]} q_{p,r}(s)\,\mathrm{d}s \in \tilde{B}_0[r] \subset \mathcal{M}_0.$$

We will work out the limiting measures of $Q_{p,r}$ in the space $(\mathcal{M}_0, w^*)$ of measures. To this end, let us recall the Alexandroff theorem for weak* convergence of measures [5, p. 316]. Since any $\mu \in \mathcal{M}_0$ is regular in the sense there, we have the following characterization.

Lemma 4.8. *Let μ_n, $\mu_0 \in \mathcal{M}_0$. Then $\mu_n \to \mu_0$ in $(\mathcal{M}_0, w^*)$ iff* (i) *$\{\|\mu_n\|_{\mathbf{V}}\}_{n\in\mathbb{N}}$ is bounded, and* (ii) *for any open subset B of the space I satisfying $\mu_0(B) = \mu_0(\bar{B})$, one has $\mu_n(B) \to \mu_0(B)$.*

Since $\mu \in \mathcal{M}_0$ is normalized as $\mu(0+) = 0$, the unit Dirac measure $\delta_a \in \mathcal{M}_0$ at the point $a \in [0, 1]$ is, for $a \in (0, 1]$,

$$\delta_a(t) = \begin{cases} 0 & \text{for } t \in [0, a), \\ 1 & \text{for } t \in [a, 1], \end{cases}$$

while, for $a = 0$,

$$\delta_0(t) = \begin{cases} -1 & \text{for } t = 0, \\ 0 & \text{for } t \in (0, 1]. \end{cases}$$

Considered as in the dual space of $(\mathcal{C}, \|\cdot\|_\infty)$, δ_a is the following linear functional

$$\delta_a(f) = f(a) \quad \forall f \in \mathcal{C}.$$

Lemma 4.9. *In the weak* topology, as $p \downarrow 1$, the induced measure $Q_{p,r}$ of $q_{p,r}$ tends to $\nu_r = (r/2)(\delta_1 - \delta_0)$.*

Proof. Let us consider the family of minimal potentials $q_{p,r} = q_p$ in the proof of Theorem 1.1. Note that y_0 of (4.18) satisfies

$$\left|y_0^2(t) - m_0\right| < 1 \quad \text{on } (0, 1), \qquad y_0^2(0) - m_0 = -1, \qquad y_0^2(1) - m_0 = +1.$$

Since $m_p \to m_0$ in $\mathbb{R}$ and $y_p \to y_0$ in $(\mathcal{C}, \|\cdot\|_\infty)$, for any $\varepsilon \in (0, 1/2)$, there exists $p = p_\varepsilon > 1$ such that

$$\Delta_\varepsilon := \sup_{t\in[\varepsilon,1-\varepsilon],\ p\in(1,p_\varepsilon]} \left|y_p^2(t) - m_p\right| < 1.$$

Therefore, by recalling that $q_p(t) = \phi_{p^*}(y_p^2(t) - m_p)$, we have

$$\sup_{t\in[\varepsilon,1-\varepsilon]} \left|q_p(t)\right| \leqslant \Delta_\varepsilon^{p^*-1} \quad \forall p \in (1, p_\varepsilon].$$

By the Alexandroff theorem, one sees that any limiting measure μ of $Q_{p,r}$ must be constant on $(-1, 1)$. Since $\mu(0+) = 0$, μ takes the form

$$\mu(t) = \begin{cases} m_0 & \text{for } t = 0, \\ 0 & \text{for } t \in (0, 1), \\ m_1 & \text{for } t = 1. \end{cases}$$

It follows from $\bar{q}_{p,r} = 0$ that μ satisfies

$$0 = \int_I \mathrm{d}\mu(t) = m_1 - m_0.$$

That is, $m_1 = m_0$ and $\mu = \nu_{2m_0}$, where the family of measures $\nu_{r'}$, $r' \in \mathbb{R}$, is as in (2.13). Since $q_p(t) < 0$ for $t > 0$ small, one sees that $m_0 \geqslant 0$. By Theorem 2.7 and result (4.21), one has $\tilde{\mathbf{L}}_1(r) = \lambda_0(\nu_{2m_0})$. Using results (1.13) and (2.19), this is $-r^2/4 = -m_0^2$. As $m_0 \geqslant 0$, one has $m_0 = r/2$ and $\mu = \nu_r$. Since $\mu = \nu_r$ is independent of the choice of sequences of exponents, we conclude that $Q_{p,r} \to \nu_r \in \tilde{S}_0[r] \subset \tilde{B}_0[r]$ in $(\mathcal{M}_0, w^*)$. □

Remark 4.10. For $p \in (1, \infty)$, $\tilde{\mathbf{L}}_p(r)$ has another minimal potential $\hat{q}_{p,r}(t) \equiv q_{p,r}(1-t)$. The corresponding measure $\hat{Q}_{p,r}$ will tend to $-\nu_r$ in $(\mathcal{M}_0, w^*)$ as $p \downarrow 1$.

Result (1.14) of Theorem 1.2 can be obtained from (4.21), based on Theorem 2.7 and Lemma 4.9.

Now let us complete the proof of Theorem 1.1. Let $r > 0$. We need to show that $\tilde{\mathbf{L}}_1(r)$ cannot be realized by any $q \in \tilde{B}_1[r]$. Otherwise, assume that there exists some $q \in \tilde{B}_1[r]$ such that

$$\lambda_0(q) = \tilde{\mathbf{L}}_1(r) = -r^2/4. \tag{4.22}$$

By Lemma 2.5, one can assume that $q \in \tilde{S}_1[r]$. Let us take a normalized eigenfunction W associated with $\lambda_0(q)$. From (1.1), (1.3) and (4.22), one has

$$\int_I \left(\dot{W}^2 - q(t)W^2\right) \mathrm{d}t = -r^2/4. \tag{4.23}$$

Since $W(t)$ is the zeroth Neumann eigenfunction, one may assume that $W(t)$ is positive and strictly increasing in $t \in [0, 1]$. As $q \in \tilde{S}_1[r]$, by denoting

$$q_\pm(t) := \max\left\{\pm q(t), 0\right\} \geqslant 0,$$

one has $\|q_\pm\|_1 = r/2$. Thus

$$\begin{aligned}\int_I q(t)W^2(t)\,\mathrm{d}t &= \int_I q_+(t)W^2(t)\,\mathrm{d}t - \int_I q_-(t)W^2(t)\,\mathrm{d}t \\ &< (r/2)\left(W^2(1) - W^2(0)\right) \\ &= \int_I W^2(t)\,\mathrm{d}\nu_r(t).\end{aligned}$$

Combining with (4.23), we obtain

$$\lambda_0(\nu_r) \leqslant \int_I \dot{W}^2(t)\,\mathrm{d}t - \int_I W^2(t)\,\mathrm{d}\nu_r(t) < \int_I \dot{W}^2(t)\,\mathrm{d}t - \int_I q(t)W^2(t)\,\mathrm{d}t = -r^2/4,$$

a contradiction with results (1.13) and (1.14). □

5. Applications and further problems

5.1. Estimates of the zeroth Neumann eigenvalues

We will give some application of Theorem 1.1 to estimates of the zeroth Neumann eigenvalues.

Let $\mathbf{Z}_0(x)$ be as in (1.6). By using result (1.5) obtained in [29], one has the following lower bound for $\lambda_0(q)$,

$$\lambda_0(q) \geqslant \mathbf{Z}_0^{-1}\left(\|q\|_1\right) \quad \forall q \in \mathcal{L}^1. \tag{5.1}$$

Moreover, the equality holds iff $q = 0$.

Note that $q \in \mathcal{L}^1$ can be decomposed into $q = \bar{q} + \tilde{q}$, where $\tilde{q} := q - \bar{q} \in \tilde{\mathcal{L}}^1$. Denote $\tilde{q}_+(t) := \max\{\tilde{q}(t), 0\}$. Then $\tilde{q} \leqslant \tilde{q}_+$ and $\|\tilde{q}_+\|_1 = \|\tilde{q}\|_1/2$. By the monotonicity of eigenvalues in potentials, result (1.7) and inequality (5.1), one has another lower bound

$$\lambda_0(q) = -\bar{q} + \lambda_0(\tilde{q}) \geqslant -\bar{q} + \lambda_0(\tilde{q}_+) \geqslant -\bar{q} + \mathbf{Z}_0^{-1}\left(\|\tilde{q}\|_1/2\right) \quad \forall q \in \mathcal{L}^1. \tag{5.2}$$

Moreover, the equality holds iff $\tilde{q} = 0$. However, (5.2) is not optimal, because the negative part of q has been neglected.

From (1.10) and Theorem 1.1, we can obtain the following optimal estimates on the zeroth Neumann eigenvalues $\lambda_0(q)$.

Theorem 5.1. *There hold*

$$-\bar q - \|\tilde q\|_1^2/4 \leqslant \lambda_0(q) \leqslant -\bar q \quad \forall q \in \mathcal{L}^1. \tag{5.3}$$

Moreover, equalities of (5.3) *hold iff q is constant, i.e.* $\tilde q = 0$.

The lower bound in (5.3) is new in literature. Moreover, it is easy to verify that the lower bound in (5.3) does improve (5.2) when $\tilde q \neq 0$.

5.2. Scaling results and the zeroth periodic eigenvalues

So far we have been dealing with extremal values of $\lambda_0(q)$ with potentials on the unit interval $I = [0, 1]$. These results can be easily extended to the zeroth Neumann eigenvalues with L^1 potentials on any finite interval J, by exploiting the scaling technique for eigenvalues as in [29, §2]. We only state the results.

Let $J \subset \mathbb{R}$ be a finite interval of the length $|J|$. For $q \in L^p(J)$, the mane value is $\bar q := \frac{1}{|J|}\int_J q$. The zeroth Neumann eigenvalue of (1.1) on the interval J is still denoted by $\lambda_0(q)$.

Theorem 5.2. *Let $p \in [1, \infty]$ and $r \in [0, \infty)$. By denoting $\hat r := |J|^{2-1/p} r$, one has*

$$\inf\bigl\{\lambda_0(q)\colon q \in L^p(J),\ \bar q = 0,\ \|q\|_{L^p(J)} \leqslant r\bigr\} \equiv |J|^{-2}\tilde{\mathbf{L}}_p(\hat r),$$
$$\sup\bigl\{\lambda_0(q)\colon q \in L^p(J),\ \bar q = 0,\ \|q\|_{L^p(J)} \leqslant r\bigr\} \equiv |J|^{-2}\tilde{\mathbf{M}}_p(\hat r),$$

where $\tilde{\mathbf{L}}_p(r)$ and $\tilde{\mathbf{M}}_p(r)$ are as in (1.9). *In particular, it follows from Theorem* 1.1 *and* (1.11) *that*

$$\inf\bigl\{\lambda_0(q)\colon q \in L^1(J),\ \bar q = 0,\ \|q\|_{L^1(J)} \leqslant r\bigr\} \equiv -r^2/4,$$
$$\sup\bigl\{\lambda_0(q)\colon q \in L^p(J),\ \bar q = 0,\ \|q\|_{L^p(J)} \leqslant r\bigr\} \equiv 0 \quad \forall p \in [1, \infty].$$

Next let us consider the zeroth periodic eigenvalues of problem (1.1). Given $T > 0$, for $q \in L^p(\mathbb{S}_T) \cong L^p([0, T])$, $\mathbb{S}_T = \mathbb{R}/T\mathbb{Z}$, denote by $\hat\lambda_0(q)$ the zeroth periodic eigenvalue of (1.1) with the periodic boundary condition

$$y(T) - y(0) = \dot y(T) - \dot y(0) = 0.$$

See [11,26]. A relation between the zeroth periodic eigenvalues $\hat\lambda_0(\cdot)$ of period T and the zeroth Neumann eigenvalues $\lambda_0(\cdot)$ on $[0, T]$ is

$$\hat\lambda_0(q) = \max_{s\in[0,T]} \lambda_0(q_s), \tag{5.4}$$

where $q_s(\cdot) := q(s + \cdot)$ denote translations of potentials. See, for example, [26, Theorem 4.3]. In particular, one has

$$\lambda_0(q) \leqslant \hat\lambda_0(q) \leqslant -\bar q \quad \forall q \in L^p(\mathbb{S}_T).$$

We conclude

$$\sup\bigl\{\hat\lambda_0(q)\colon q \in L^p(\mathbb{S}_T),\ \bar q = 0,\ \|q\|_{L^p(\mathbb{S}_T)} \leqslant r\bigr\} \equiv \hat\lambda_0(0) = 0, \quad p \in [1, \infty],\ r \in [0, \infty).$$

As for the corresponding infimum for $\hat\lambda_0(q)$, by relation (5.4) and scaling results in Theorem 5.2 for Neumann eigenvalues, we have the following results.

Theorem 5.3. *Let $T > 0$, $p \in [1, \infty]$ and $r \in [0, \infty)$. By denoting $\hat r := T^{2-1/p} r$, one has*

$$\inf\bigl\{\hat\lambda_0(q)\colon q \in L^p(\mathbb{S}_T),\ \bar q = 0,\ \|q\|_{L^p(\mathbb{S}_T)} \leqslant r\bigr\} \equiv 4T^{-2}\tilde{\mathbf{L}}_p(\hat r/4),$$

where $\tilde{\mathbf{L}}_p(r)$ is as in (1.9). *In particular, it follows from Theorem* 1.1 *that*

$$\inf\bigl\{\hat\lambda_0(q)\colon q \in L^1(\mathbb{S}_T),\ \bar q = 0,\ \|q\|_{L^1(\mathbb{S}_T)} \leqslant r\bigr\} \equiv -r^2/16. \tag{5.5}$$

For the detailed proof of this theorem, we refer to [29, §6]. Like the infimum for the zeroth Neumann eigenvalues, (5.5) cannot be attained by any potential q when $r > 0$. If the period T is taken as 1, it follows from Theorem 1.2 that the 'minimizer' of (5.5) is, after some translation of t, the following singular measure

$$\hat{\nu}_r := (r/4)(-\delta_0 + 2\delta_{1/2} - \delta_1) \in \tilde{S}_0[r] \subset \tilde{B}_0[r]. \tag{5.6}$$

5.3. *Further extremal problems on eigenvalues*

Let us give a summary on extremal problems of eigenvalues of Sturm–Liouville operators with integrable potentials.

In the L^1 ball $B_1[r]$ of potentials, the infimum and supremum of all Dirichlet and Neumann eigenvalues of any order have been found in [22,29]. By exploiting the relation between periodic/anti-periodic eigenvalues and Dirichlet/Neumann eigenvalues in [26], partial results on periodic/anti-periodic eigenvalues have been obtained as well. A common feature is that the supremum can be attained by potentials on spheres $S_1[r]$, while the infimum on $B_1[r]$ cannot be attained by any potential. In fact, by using eigenvalues for measure differential equations [14], one has, for example,

$$\inf\{\lambda_0(q)\colon q \in B_1[r]\} = \lambda_0(r\delta_0) = \lambda_0(r\delta_1).$$

That is, by considering $q(t)$ as the density of strings and $\|q\|_1$ the total mass of strings, the zeroth Neumann eigenvalue will reach its minimum when the mass is located at one of the end-points of the interval [0, 1]. For the zeroth periodic eigenvalues $\hat{\lambda}_0(q)$, one has from [29] that

$$\inf\{\hat{\lambda}_0(q)\colon q \in B_1[r]\} = \hat{\lambda}_0(r\delta_{1/2}).$$

That is, the zeroth periodic eigenvalue will reach its minimum when the mass is located at the middle of the interval [0, 1]. Higher order Neumann and Dirichlet eigenvalues will reach its minimum when the mass is located at some points on the interval [0, 1]. These 'minimal' measures can be found from [14,22].

In the L^1 ball $\tilde{B}_1[r]$ of potentials of the zero mean value, in this paper we have only solved the extremal problems for the zeroth Neumann and periodic eigenvalues, with the 'minimal' measures as in (1.14) and (5.6) respectively. Result (1.14) means that the zeroth Neumann eigenvalue will reach its minimum when the mass is located at both end-points of [0, 1] in an asymmetric way.

We end the paper with the following open problem: For $m \in \mathbb{N}$, find the extremal values $\tilde{\mathbf{L}}^{\sigma}_{m,1}(r)$ and $\tilde{\mathbf{M}}^{\sigma}_{m,1}(r)$, defined by (1.8), for eigenvalues of L^1 potentials with zero mean value, and their minimial/maximal measures/potentials.

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