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Approximation and relaxation of perimeter in the Wiener space

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Abstract

We characterize the relaxation of the perimeter in an infinite dimensional Wiener space, with respect to the weak L^2-topology. We also show that the rescaled Allen–Cahn functionals approximate this relaxed functional in the sense of Γ-convergence.

Keywords: Variational problems; Gamma-convergence; Infinite dimensional analysis

1. Introduction

Extending the variational methods and the geometric measure theory from the Euclidean to the Wiener space has recently attracted a lot of attention. In particular, the theory of functions of bounded variation in infinite dimensional spaces started with the works by Fukushima and Hino [21,22]. Since then, the fine properties of BV functions and sets of finite perimeter have been investigated in [4,5,3,1,11]. We point out that this theory is closely related to older works by M. Ledoux and P. Malliavin [25,26].

In the Euclidean setting it is well known that the perimeter can be approximated by means of more regular functionals of the form

$$\int \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{W(u)}{\varepsilon} \right) dx$$

when ε tends to zero, in the sense of Γ-convergence with respect to the strong L^1-topology [28,27]. An important ingredient in this proof is the compact embedding of BV in L^1.

A natural question is whether a similar approximation property holds in the infinite dimensional case. The main goal of this paper is answering to this question by computing the Γ-limit, as $\varepsilon \to 0$, of the Allen–Cahn-type functionals (see Section 2 for precise definitions)

$$F_\varepsilon(u) = \int_X \left(\frac{\varepsilon}{2} |\nabla_H u|_H^2 + \frac{W(u)}{\varepsilon} \right) d\gamma.$$

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In a Wiener space with a Hilbert structure there are two possible definitions of gradient, and consequently two different notions of Sobolev spaces, functions of bounded variation and perimeters [4,1]. In one definition the compact embedding of $BV_\gamma(X)$ in $L^1_\gamma(X)$ still holds [4, Th. 5.3] and the Γ-limit of F_ε is, as expected, the perimeter up to a multiplicative constant. We do not reproduce here the proof of this fact, since it is very similar to the Euclidean one. A more interesting situation arises when we consider the other definition of gradient, which gives rise to a more invariant notion of perimeter and is therefore commonly used in the literature [21,22,4]. In this case, the compact embedding of $BV_\gamma(X)$ in $L^1_\gamma(X)$ does not hold anymore. In particular sequences with uniformly bounded F_ε-energy are not generally compact in the (strong) L^1_γ-topology, even though they are bounded in $L^2_\gamma(X)$, and hence compact with respect to the weak $L^2_\gamma(X)$-topology. This suggests that the right topology for considering the Γ-convergence should rather be the weak $L^2_\gamma(X)$-topology.

A major difference with the finite dimensional case is the fact that the perimeter function defined by

$$F(u) = \begin{cases} P_\gamma(E) & \text{if } u = \chi_E \\ +\infty & \text{otherwise} \end{cases}$$

is no longer lower semicontinuous in this topology, and therefore cannot be the Γ-limit of the functionals F_ε. The problem is that the sets of finite perimeter are not closed under weak convergence of the characteristic functions. However, it is possible to compute the relaxation $\overline{F}$ of F (Theorem 4.6), which reads:

$$\overline{F}(u) = \begin{cases} \int_X \sqrt{\mathcal{U}^2(u) + |D_\gamma u|^2}\, d\gamma & \text{if } 0 \leqslant u \leqslant 1 \\ +\infty & \text{otherwise.} \end{cases}$$

Such functional is quite familiar to people studying log-Sobolev and isoperimetric inequalities in Wiener spaces [6,7,10].

Our main result is to show that the Γ-limit of F_ε, with respect to the weak $L^2_\gamma(X)$-topology, is a multiple of $\overline{F}$ (Theorem 5.3). The proof relies on the interplay between symmetrization, semicontinuity and isoperimetry.

The plan of the paper is the following. In Section 2 we recall some basic facts about Wiener spaces and functions of bounded variation. In Section 3 we give the main properties of the Ehrhard symmetrizations. We also prove a Pólya–Szegö inequality and a Bernstein-type result in the Wiener space (Propositions 3.12 and 3.5), which we believe to be interesting in themselves. In Section 4, we use the Ehrhard symmetrization to compute the relaxation of the perimeter (Theorem 4.6). Finally, in Section 5 we compute the Γ-limit of the functionals F_ε (Theorem 5.3) and discuss some consequences of this result.

2. Wiener space and functions of bounded variation

A clear and comprehensive reference on the Wiener space is the book by Bogachev [8] (see also [26]). We follow here closely the notation of [4]. Let X be a separable Banach space and let X^* be its dual. We say that X is a Wiener space if it is endowed with a non-degenerate centered Gaussian probability measure γ. That amounts to say that γ is a probability measure for which $x^*\sharp\gamma$ is a centered Gaussian measure on $\mathbb{R}$ for every $x^* \in X^*$. The non-degeneracy hypothesis means that γ is not concentrated on any proper subspace of X.

As a consequence of Fernique's Theorem [8, Th. 2.8.5], for every $x^* \in X^*$, the function $R^*x^*(x) = \langle x^*, x\rangle$ is in $L^2_\gamma(X) = L^2(X,\gamma)$. Let $\mathcal{H}$ be the closure of R^*X^* in $L^2_\gamma(X)$; the space $\mathcal{H}$ is usually called the reproducing kernel of γ. Let R, the operator from $\mathcal{H}$ to X, be the adjoint of R^* that is, for $\hat{h} \in \mathcal{H}$,

$$R\hat{h} = \int_X x\hat{h}(x)\, d\gamma$$

where the integral is to be intended in the Bochner sense. It can be seen that R is a compact and injective operator. We will let $Q = RR^*$. We denote by H the space $R\mathcal{H}$. This space is called the Cameron–Martin space. It is a separable Hilbert space with the scalar product given by

$$[h_1, h_2]_H = \langle \hat{h}_1, \hat{h}_2\rangle_{L^2_\gamma(X)}$$

if $h_i = R\hat{h}_i$. We will denote by $|\cdot|_H$ the norm in H. The space H is a dense subspace of X, with compact embedding, and $\gamma(H) = 0$ if X is of infinite dimension.

For $x_1^*, \dots, x_m^* \in X^*$ we denote by $\Pi_{x_1^*,\dots,x_m^*}$ the projection from X to $\mathbb{R}^m$ given by

$$\Pi_{x_1^*,\dots,x_m^*}(x) = \big(\langle x_1^*, x\rangle, \dots, \langle x_m^*, x\rangle\big).$$

We will also denote it by Π_m when specifying the points x_i^* is unnecessary. Two elements x_1^* and x_2^* of X^* will be called orthonormal if the corresponding $h_i = Qx_i^*$ are orthonormal in H. We will fix in the following an orthonormal basis of H given by $h_i = Qx_i^*$.

We also denote by $H_m = \operatorname{span}(h_1, \dots, h_m) \simeq \mathbb{R}^m$ and $X_m^\perp = \operatorname{Ker}(\Pi_m) = \overline{H_m^\perp}^X$, so that $X = \mathbb{R}^m \oplus X_m^\perp$. The map Π_m induces the decomposition $\gamma = \gamma_m \otimes \gamma_m^\perp$, with γ_m, $\gamma_m^\perp$ Gaussian measures on $\mathbb{R}^m$, $X_m^\perp$ respectively.

Proposition 2.1. *(See [8].) Let $\hat{h}_1, \dots, \hat{h}_m$ be in $\mathcal{H}$ then the image measure of γ under the map*

$$\Pi_{\hat{h}_1,\dots,\hat{h}_m}(x) = \big(\hat{h}_1(x), \dots, \hat{h}_m(x)\big)$$

is a Gaussian in $\mathbb{R}^m$. If the $\hat{h}_i$ are orthonormal, then such measure is the standard Gaussian measure on $\mathbb{R}^m$.

Given $u \in L^2_\gamma(X)$, we will consider the canonical cylindrical approximation $\mathbb{E}_m$ given by

$$\mathbb{E}_m u(x) = \int_{X_m^\perp} u\big(\Pi_m(x), y\big)\, d\gamma_m^\perp(y).$$

Notice that $\mathbb{E}_m u$ is a cylindrical functions depending only on the first m variables, and $\mathbb{E}_m u$ converges to u in $L^2_\gamma(X)$.

We will denote by $\mathcal{FC}^1_b(X)$ the space of cylindrical $\mathcal{C}^1$ bounded functions that is the functions of the form $v(\Pi_m(x))$ with v a $\mathcal{C}^1$ bounded function from $\mathbb{R}^m$ to $\mathbb{R}$. We denote by $\mathcal{FC}^1_b(X, H)$ the space generated by all functions of the form Φh, with $\Phi \in \mathcal{FC}^1_b(X)$ and $h \in H$.

We now give the definitions of gradients, Sobolev spaces functions of bounded variation. Given $u : X \to \mathbb{R}$ and $h = R\hat{h} \in H$, we define

$$\frac{\partial u}{\partial h}(x) = \lim_{t\to 0} \frac{u(x+th) - u(x)}{t}$$

whenever the limit exists, and

$$\partial_h^* u = \frac{\partial u}{\partial h} - \hat{h} u.$$

We define $\nabla_H u : X \to H$, the gradient of u by

$$\nabla_H u = \sum_{i=1}^{+\infty} \frac{\partial u}{\partial h_i} h_i$$

and the divergence of $\Phi : X \to H$ by

$$\operatorname{div}_\gamma \Phi = \sum_{i=1}^{+\infty} \partial_{h_i}^* [\Phi, h_i]_H.$$

The operator $\operatorname{div}_\gamma$ is the adjoint of the gradient in $L^2_\gamma(X)$ so that for every $u \in \mathcal{FC}^1_b(X)$ and every $\Phi \in \mathcal{FC}^1_b(X, H)$, the following integration by parts holds:

$$\int_X u \operatorname{div}_\gamma \Phi \, d\gamma = -\int_X [\nabla_H u, \Phi]_H \, d\gamma. \tag{1}$$

The ∇_H operator is thus closable in $L^2_\gamma(X)$ and we will denote by $H^1_\gamma(X)$ its closure in $L^2_\gamma(X)$. From this, formula (1) still holds for $u \in H^1_\gamma(X)$ and $\Phi \in \mathcal{FC}^1_b(X, H)$.

Following [21,4], given $u \in L^1_\gamma(X)$ we say that $u \in BV_\gamma(X)$ if

$$\int_X |D_\gamma u|_H = \sup\left\{ \int_X u \operatorname{div}_\gamma \Phi \, d\gamma;\ \Phi \in \mathcal{FC}^1_b(X, H),\ |\Phi|_H \leqslant 1\ \forall x \in X \right\} < +\infty.$$

We will also denote by $|D_\gamma u|(X)$ the total variation of u. If $u = \chi_E$ is the characteristic function of a set E we will denote $P_\gamma(E)$ its total variation and say that E is of finite perimeter if $P_\gamma(E)$ is finite. As shown in [4] we have the following properties of $BV_\gamma(X)$ functions.

Theorem 2.2. *Let $u \in BV_\gamma(X)$ then the following properties hold*:

- *$D_\gamma u$ is a countably additive measure on X with finite total variation and values in H (we will note the space of these measures by $\mathcal{M}(X, H)$), such that for every $\Phi \in \mathcal{FC}^1_b(X)$ we have*:
$$\int_X u \partial^*_{h_j} \Phi \, d\gamma = -\int_X \Phi \, d\mu_j \quad \forall j \in \mathbb{N}$$
where $\mu_j = [h_j, D_\gamma u]_H$.
- $|D_\gamma u|(X) = \inf \underline{\lim}\{\int_X |\nabla_H u_j|_H \, d\gamma \colon u_j \in H^1_\gamma(X),\ u_j \to u \text{ in } L^1_\gamma(X)\}$.

Proposition 2.3. *Let $u = v(\Pi_m)$ be a cylindrical function then $u \in BV_\gamma(X)$ if and only if $v \in BV_{\gamma_m}(\mathbb{R}^m)$. We then have*
$$\int_X |D_\gamma u|_H = \int_{\mathbb{R}^m} |D_{\gamma_m} v|.$$

Proposition 2.4. *[Coarea formula [4].] If $u \in BV_\gamma(X)$ then for every Borel set $B \subset X$,*
$$|D_\gamma u|(B) = \int_{\mathbb{R}} P_\gamma\big(\{u > t\}, B\big) \, dt. \tag{2}$$

In Proposition 3.12, we will need the following extension of Proposition 2.4.

Lemma 2.5. *For every function $u \in BV_\gamma(X)$ and every non-negative Borel function g,*
$$\int_X g(x) \, d|D_\gamma u|(x) = \int_{\mathbb{R}} \left(\int_X g(x) \, d|D_\gamma \chi_{E_t}|(x) \right) dt \tag{3}$$
where $E_t := \{u > t\}$.

Proof. The proof of this lemma mimic the standard proof in the Euclidean case [14, Th. 2.2]. By [20, Ch. 1, Th. 7] we can write g as
$$g = \sum_{i=1}^{+\infty} \frac{1}{i} \chi_{A_i}$$
where the $A_i \subset X$ are Borel sets. Using the coarea formula (2), we then get
$$\begin{aligned}
\int_X g(x) d|D_\gamma u|(x) &= \sum_{i=1}^{+\infty} \frac{1}{i} |D_\gamma u|(A_i) = \sum_{i=1}^{+\infty} \frac{1}{i} \int_{\mathbb{R}} |D_\gamma \chi_{E_t}|(A_i) \, dt \\
&= \int_{\mathbb{R}} \left(\int_X \sum_{i=1}^{+\infty} \frac{1}{i} \chi_{A_i} \, d|D_\gamma \chi_{E_t}|(x) \right) dt \\
&= \int_{\mathbb{R}} \int_X g(x) \, d|D_\gamma \chi_{E_t}|(x) \, dt. \qquad \square
\end{aligned}$$

In [4] it is also shown that sets with finite Gaussian perimeter can be approximated by smooth cylindrical sets.

Proposition 2.6. *Let $E \subset X$ be a set of finite Gaussian perimeter then there exists smooth sets $E_m \subset \mathbb{R}^m$ such that $\Pi_m^{-1}(E_m)$ converges in $L^1_\gamma(X)$ to E and $P_\gamma(\Pi_m^{-1}(E_m)) = P_{\gamma_m}(E_m)$ converges to $P_\gamma(E)$ when m tends to infinity.*

Note that, for half-spaces, the perimeter can be exactly computed [4, Cor. 3.11].

Proposition 2.7. *Let $h = R\hat{h} \in H$ and $c \in \mathbb{R}$ then the half-space*

$$E = \{x \in X\colon \hat{h}(x) \leqslant c\}$$

has perimeter

$$P_\gamma(E) = \frac{1}{\sqrt{2\pi}} e^{-\frac{c^2}{2|h|_H^2}}.$$

3. The Ehrhard symmetrization

The Ehrhard symmetrization has been introduced by Ehrhard in [18] for studying the isoperimetric inequality in a Gaussian setting. We recall the definition and the main properties of such symmetrization.

Definition 3.1. We define the functions Φ and α by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt \quad \text{and} \quad \alpha(x) = \Phi^{-1}(x);$$

we then let $\mathcal{U}(x) = \Phi' \circ \alpha(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\alpha^2(x)}{2}}$.

Notice that $\Phi(t)$ is the volume of the half-space $\{\hat{h}(x) < t\}$ and that $\mathcal{U}(x)$ is the perimeter of a half-space of volume x.

Lemma 3.2. *Let $\hat{h}_1, \hat{h}_2 \in \mathcal{H}$, with $|h_1|_H = |h_2|_H = 1$, and suppose that there exist $C_1, C_2 \in \mathbb{R}$ such that*

$$\{\hat{h}_1 < C_1\} \subset \{\hat{h}_2 < C_2\}.$$

Then $\hat{h}_1 = \hat{h}_2$.

Proof. Assume by contradiction $\hat{h}_1 \neq \hat{h}_2$ then,

$$\gamma\big(\{\hat{h}_1(x) < C_1\} \cap \{\hat{h}_2(x) \geqslant C_2\}\big) = \Pi_{\hat{h}_1,\hat{h}_2}\sharp\gamma\big(\{(x,y) \in \mathbb{R}^2\,,\ x < C_1 \text{ and } y \geqslant C_2\}\big)$$

which is positive since γ is non-degenerate. □

We now define the Ehrhard symmetrization.

Definition 3.3. Let $E \subset X$ and let $m \in \mathbb{N}$. The Ehrhard symmetral of E along the first m variables is defined as (see Fig. 1):

$$E^* := \begin{cases} \{(x, x_m, x_m^\perp) \in \mathbb{R}^{m-1} \times \mathbb{R} \times X_m^\perp\colon\ x_m < \alpha(\mathbb{E}_{m-1}\chi_E(x))\} & \text{if } m > 1 \\ \{x \in X\colon\ \langle x_1^*, x\rangle < \alpha(\gamma(E))\} & \text{if } m = 1. \end{cases}$$

The interest of this symmetrization is that it decreases the Gaussian perimeter, while keeping the volume fixed.

Proposition 3.4. *Let E be a set of finite perimeter and E^* be an Ehrhard symmetral of E, then*

$$\gamma(E^*) = \gamma(E), \tag{4}$$

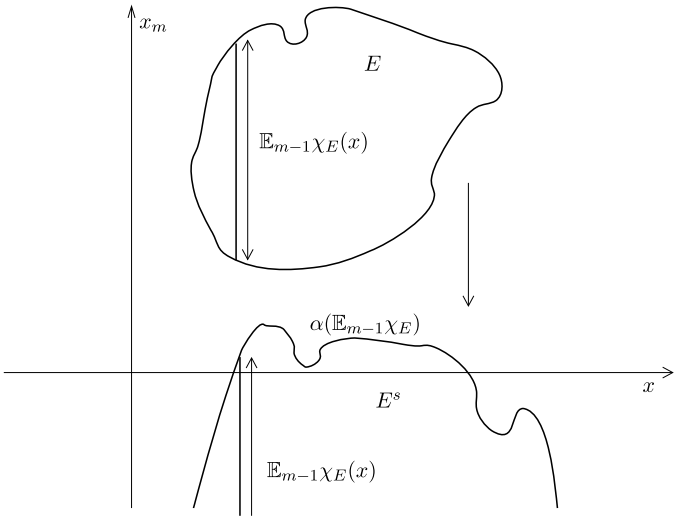

Fig. 1. The Ehrhard symmetrization.

$\mathbb{E}_{m-1}\chi_{E^*} = \mathbb{E}_{m-1}\chi_E$ *and*

$$P_\gamma\big(E^*\big) \leqslant P_\gamma(E). \tag{5}$$

In particular, we have the isoperimetric inequality

$$P_\gamma(E) \geqslant \mathcal{U}\big(\gamma(E)\big),$$

with equality if and only if E is a half-space.

For the proof we refer to [7,10], and to [4, Remark 4.7] for the extension to infinite dimensions.
We can also prove a stronger result which is a kind of Bernstein Theorem in this setting.

Proposition 3.5. *The half-spaces are the only local minimizers of the Gaussian perimeter with volume constraint.*

Proof. Let $E \subset X$ be a local minimizer of the (Gaussian) perimeter and let $v = \gamma(E)$. This means that, for every $R > 0$ and every set F of finite perimeter, with $\gamma(F) = v$ and $E \Delta F \subset B_R$ (where B_R denotes the ball of radius R centered at 0), we have

$$P_\gamma(E) \leqslant P_\gamma(F).$$

If E is not a half-space then, by Proposition 3.4, there exists $\eta > 0$ such that

$$P_\gamma(E) \geqslant \mathcal{U}(v) + \eta.$$

Let α_R be such that

$$\gamma(E \cap B_R) = \gamma\big(\big\{\big\langle x_1^*, x\big\rangle < \alpha_R\big\} \cap B_R\big).$$

We have that α_R tends to $\alpha(v)$ when R goes to infinity and $P_\gamma(\{\langle x_1^*, x\rangle < \alpha_R\})$ tends to $P_\gamma(\{\langle x_1^*, x\rangle < \alpha(v)\})$. Letting

$$F_R = \big(\big\{\big\langle x_1^*, x\big\rangle < \alpha_R\big\} \cap B_R\big) \cup \big(E \cap B_R^c\big)$$

we get

$$\begin{aligned}
\mathcal{U}(v) + \eta \leqslant P_\gamma(E) \leqslant P_\gamma(F_R) &\leqslant P_\gamma\big(\big\{\big\langle x_1^*, x\big\rangle < \alpha_R\big\} \cap B_R\big) + P_\gamma\big(E \cap B_R^c\big) \\
&\leqslant P_\gamma\big(\big\{\big\langle x_1^*, x\big\rangle < \alpha_R\big\}\big) + P_\gamma(B_R) + P_\gamma\big(E \cap B_R^c\big) \\
&\leqslant P_\gamma\big(\big\{\big\langle x_1^*, x\big\rangle < \alpha(v)\big\}\big) + \varepsilon(R) \\
&= \mathcal{U}(v) + \varepsilon(R),
\end{aligned}$$

where we used various time the inequality (see [23])

$$P_\gamma(E \cup F) + P_\gamma(E \cap F) \leqslant P_\gamma(E) + P_\gamma(F)$$

and where $\varepsilon(R)$ is a function which goes to zero when R goes to infinity. We thus found a contradiction. □

Remark 3.6. In the Euclidean setting, half-spaces are the only local minimizers of the perimeter only in dimension lower than 8 (see [23]). Notice also that if we drop the volume constraint, half-spaces are no longer local minimizers for the Gaussian perimeter, since there are no nonempty local minimizers.

In the sequel we will also need another transformation which from a finite dimensional function gives an Ehrhard symmetric set whose sections have volume prescribed by the original function. More precisely:

Definition 3.7. Given a measurable function $v : \mathbb{R}^m \to [0,1]$, we define its Ehrhard set $ES_m(v) \subset X$ by

$$ES_m(v) := \left\{\left(x, x_{m+1}, x_{m+1}^\perp\right) \in \mathbb{R}^m \times \mathbb{R} \times X_{m+1}^\perp \colon\ x_{m+1} < \alpha\left(v(x)\right)\right\}.$$

Given a measurable cylindrical function $u : X \to [0,1]$ depending only on the first m variables, that is, $u = v \circ \Pi_m$ for some $v : \mathbb{R}^m \to [0,1]$, we set

$$ES_m(u) := ES_m(v).$$

The link between Ehrhard sets and Ehrhard symmetrization is the following:

Proposition 3.8. *Let E be a set of finite perimeter and E^* be its Ehrhard symmetrization with respect to the first $(m+1)$ variables, then*

$$E^* = ES_m\left(\mathbb{E}_m(\chi_E)\right).$$

In the next proposition we compute the perimeter of Ehrhard sets. It slightly extends a result in [15].

Proposition 3.9. *Let $u \in BV_{\gamma_m}(\mathbb{R}^m)$ with $0 \leqslant u \leqslant 1$, then*

$$P_\gamma\left(ES_m(u)\right) = \int_{\mathbb{R}^m} \sqrt{\mathcal{U}(u)^2 + |D_{\gamma_m} u|^2}\, d\gamma_m$$

where

$$\int_{\mathbb{R}^m} \sqrt{\mathcal{U}(u)^2 + |D_{\gamma_m} u|^2}\, d\gamma_m = \int_{\mathbb{R}^m} \sqrt{\mathcal{U}(u)^2 + |\nabla u|^2}\, d\gamma_m + \left|D_\gamma^s u\right|(X)$$

and $D_\gamma u = \nabla u \gamma + D_\gamma^s u$ is the Radon–Nikodym decomposition of $D_\gamma u$.

Proof. By [15, Th. 4.3] the result holds for $u \in H^1_{\gamma_m}(\mathbb{R}^m)$. We will show by approximation that the same holds for $u \in BV_{\gamma_m}(\mathbb{R}^m)$.

Let $E = ES_m(u)$, then we can find sets E_n such that $\gamma(E_n \Delta E) \to 0$ and $P_\gamma(E_n) \to P_\gamma(E)$ as $n \to +\infty$, and all the E_n have smooth boundary and are Ehrhard symmetric. Thus, for every $n \in \mathbb{N}$, there exists a smooth function u_n such that $0 \leqslant u_n \leqslant 1$, $E_n = ES_m(u_n)$, $u_n \to u$ in $L^1_{\gamma_m}(\mathbb{R}^m)$, and

$$P_\gamma(E_n) = \int_{\mathbb{R}^m} \sqrt{\mathcal{U}(u_n)^2 + |D_{\gamma_m} u_n|^2}\, d\gamma_m.$$

Since, by Proposition 4.4, the functional $\int_{\mathbb{R}^m} \sqrt{\mathcal{U}(u)^2 + |D_{\gamma_m} u|^2}\, d\gamma_m$ is lower semicontinuous in $L^1_{\gamma_m}(\mathbb{R}^m)$, we get

$$\begin{aligned} P_\gamma(E) &= \lim_{n\to\infty} P_\gamma(E_n) \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^m} \sqrt{\mathcal{U}(u_n)^2 + |D_{\gamma_m} u_n|^2}\, d\gamma_m \\ &\geqslant \int_{\mathbb{R}^m} \sqrt{\mathcal{U}(u)^2 + |D_{\gamma_m} u|^2}\, d\gamma_m. \end{aligned}$$

The other inequality follows as in [15]. Let $\widetilde{E} = \Pi_{m+1}(E) \subset \mathbb{R}^{m+1}$ and observe that $\gamma_{m+1}(\widetilde{E}) = \gamma(E)$ and $P_{\gamma_{m+1}}(\widetilde{E}) = P_\gamma(E)$. By Vol'pert Theorem [2, Th. 3.108] there exists a set $B \subset \mathbb{R}^m$ such that for every $x \in B$, $\nu^{\widetilde{E}}_{m+1}(x, \alpha(u_E(x)))$ exists and is not equal to zero, where $\nu^{\widetilde{E}}_{m+1}$ denotes the last coordinate of the unit external normal to $\partial^* \widetilde{E}$. By [15, Lemma 4.4], γ_m-almost every $x \in B$ is a point of approximate differentiability for u. By Lemmas 4.5 and 4.6 of [15] we then have

$$\begin{aligned} P_{\gamma_{m+1}}(\widetilde{E}) &= P_{\gamma_{m+1}}(\widetilde{E}, B \times \mathbb{R}) + P_{\gamma_{m+1}}\big(\widetilde{E}, B^c \times \mathbb{R}\big) \\ &\leqslant \int_B \sqrt{\mathcal{U}(u)^2 + |\nabla u|^2}\, d\gamma_m + \int_{B^c} |D_{\gamma_m} u| + \int_{B^c} \mathcal{U}(u)\, d\gamma_m. \end{aligned}$$

As $\gamma_m(B^c) = 0$, we find that

$$\int_B \sqrt{\mathcal{U}(u)^2 + |\nabla u|^2}\, d\gamma_m + \int_{B^c} |D_{\gamma_m} u| = \int_{\mathbb{R}^m} \sqrt{\mathcal{U}^2(u) + |\nabla u|^2}\, d\gamma_m + \big|D^s_{\gamma_m} u\big|\big(\mathbb{R}^m\big)$$

and thus $P_\gamma(E) = P_{\gamma_{m+1}}(\widetilde{E}) \leqslant \int_{\mathbb{R}^m} \sqrt{\mathcal{U}(u)^2 + |D_{\gamma_m} u|^2}\, d\gamma_m$. □

The last transformation that we consider is the analog of the Schwartz symmetrization in the Gaussian setting, and was first introduced by Ehrhard in [19].

Definition 3.10. Let $u \in X \to \mathbb{R}$ be a measurable function and let $m \in \mathbb{N}$ be fixed. We define the m-dimensional Ehrhard symmetrization u^* of u as follows:

- for all $t \in \mathbb{R}$ we let E^*_t be the Ehrhard symmetrization of $E_t := \{u > t\}$ with respect to the first m variables;
- we let $u^*(x) := \inf\{t\colon\ x \in E^*_t\}$.

As (4) implies $\gamma(\{u^* > t\}) = \gamma(\{u > t\})$ for all $t \in \mathbb{R}$, from the Layer Cake formula it follows that, if $u \in L^2_\gamma(X)$, then $u^* \in L^2_\gamma(X)$ and

$$\int_X |u^*|^2\, d\gamma = \int_X |u|^2\, d\gamma. \tag{6}$$

Indeed, we have

$$\begin{aligned} \int_X |u|^2 d\gamma &= 2\int_0^{+\infty} t\,\gamma\big(\{u > t\}\big)\, dt - 2\int_{-\infty}^0 t\gamma\big(\{u < t\}\big)\, dt \\ &= 2\int_0^{+\infty} t\gamma\big(\{u^* > t\}\big)\, dt - 2\int_{-\infty}^0 t\gamma\big(\{u^* < t\}\big)\, dt \\ &= \int_X |u^*|^2\, d\gamma. \end{aligned}$$

Lemma 3.11. *Let $u, v : X \to [0, +\infty)$ belonging to $L^2_\gamma(X)$, then*

$$\left\|u^* - v^*\right\|_{L^2_\gamma(X)} \leqslant \|u - v\|_{L^2_\gamma(X)}. \tag{7}$$

Proof. The proof is a straightforward adaptation of the analogous proof for the Schwartz symmetrization [24, Th. 3.4]. Recalling (6) with $p = 2$, we have only to show that

$$\int_X uv\,d\gamma \leqslant \int_X u^* v^*\,d\gamma. \tag{8}$$

Again by the Layer Cake formula we have

$$\int_X uv\,d\gamma = \int_0^{+\infty}\int_0^{+\infty}\int_X \chi_{\{u>t\}}(x)\chi_{\{v>s\}}(x)\,d\gamma(x)\,dt\,ds.$$

Thus (8) would follow from the same inequality for sets, that is,

$$\gamma(A \cap B) \leqslant \gamma\big(A^* \cap B^*\big). \tag{9}$$

Let $x_m \in \mathbb{R}^m$ and assume that

$$\int_{X_m^\perp} \chi_A(x_m + y)\,d\gamma_m^\perp(y) \geqslant \int_{X_m^\perp} \chi_B(x_m + y)\,d\gamma_m^\perp(y)$$

then by definition of the Ehrhard symmetrization we have

$$B^* \cap \big(x_m + X_m^\perp\big) \subset A^* \cap \big(x_m + X_m^\perp\big)$$

and therefore

$$\begin{aligned}\int_{X_m^\perp} \chi_{A^*}(x_m + y)\chi_{B^*}(x_m + y)\,d\gamma_m^\perp(y) &= \int_{X_m^\perp} \chi_{A^*}(x_m + y)\,d\gamma_m^\perp(y)\\ &= \int_{X_m^\perp} \chi_A(x_m + y)\,d\gamma_m^\perp(y)\\ &\geqslant \int_{X_m^\perp} \chi_A(x_m + y)\chi_B(x_m + y)\,d\gamma_m^\perp(y).\end{aligned}$$

This inequality also holds if $\int_{X_m^\perp} \chi_B(x_m + y)\,d\gamma_m^\perp(y) \geqslant \int_{X_m^\perp} \chi_A(x_m + y)\,d\gamma_m^\perp(y)$ so that finally

$$\begin{aligned}\gamma\big(A^* \cap B^*\big) &= \int_{X_m}\int_{X_m^\perp} \chi_{A^*}(x + y)\chi_{B^*}(x + y)\,d\gamma_m^\perp(y)\,d\gamma_m(x)\\ &\geqslant \int_{X_m}\int_{X_m^\perp} \chi_A(x + y)\chi_B(x + y)\,d\gamma_m^\perp(y)\,d\gamma_m(x)\\ &= \gamma(A \cap B)\end{aligned}$$

which gives (9). □

As for the Schwartz symmetrization, a Pólya–Szegö principle holds for the Ehrhard symmetrization.

Proposition 3.12. *Let $u \in H^1_\gamma(X)$, let $m \in \mathbb{N}$ and let u^* be the m-dimensional Ehrhard symmetrization of u. Then $u^* \in H^1_\gamma$ and*

$$\int_X \left|\nabla_H u^*\right|_H^2 d\gamma \leqslant \int_X |\nabla_H u|_H^2 \, d\gamma. \tag{10}$$

Moreover, if $m = 1$ and equality holds in (10), then

$$u = \tilde{u}\big(\hat{h}(x)\big) \quad \text{for some } \hat{h} \in \mathcal{H},$$

and $\hat{h}$ can be chosen to be a unitary vector.

Proof. In [19, Th. 3.1], inequality (10) is proven for Lipschitz functions, in finite dimensions. We extend it by approximation to Sobolev functions.

We can assume $u \geqslant 0$, since we have $(u^\pm)^* = (u^*)^\pm$, where $u^\pm$, $(u^*)^\pm$ denote the positive and negative part of u and u^*, respectively.

Let $u_n \in \mathcal{FC}^1_b(X)$ be positive functions converging to u in $H^1_\gamma(X)$, then by (7), u_n^* converges to u^* in $L^2_\gamma(X)$ and thus by the lower semicontinuity of the $H^1_\gamma(X)$ norm we have

$$\int_X \left|\nabla_H u^*\right|_H^2 \leqslant \varliminf_{n\to\infty} \int_X \left|\nabla_H u_n^*\right|_H^2 \leqslant \varliminf_{n\to\infty} \int_X |\nabla_H u_n|_H^2 = \int_X |\nabla_H u|_H^2.$$

We now turn to the equality case for one-dimensional symmetrizations. For this we closely follow [10] and give an alternative proof of (10), based on ideas of Brothers and Ziemer [9] for the Schwartz symmetrization.

Let $u \in H^1_\gamma(X)$ and $\mu(t) = \gamma(\{u > t\}) = \gamma(\{u^* > t\})$. By the coarea formula (3), for all $t \in \mathbb{R}$ we have

$$\mu(t) = \gamma\big(\{u > t\} \cap \{\nabla_H u = 0\}\big) + \int_t^{+\infty} \left(\int_{\{\nabla_H u \neq 0\}} \frac{1}{|\nabla_H u|_H} \, d|D_\gamma \chi_{E_\tau}| \right) d\tau.$$

Hence

$$-\mu'(t) \geqslant \int_{\{\nabla_H u \neq 0\}} \frac{1}{|\nabla_H u|_H} \, d|D_\gamma \chi_{E_t}| \quad \text{for a.e. } t \in \mathbb{R}. \tag{11}$$

Since u^* is a function depending only on one variable, arguing as in [14] we get

$$\frac{d}{dt}\gamma\big(\{u^* > t\} \cap \{\nabla_H u^* = 0\}\big) = 0 \quad \text{for a.e. } t \in \mathbb{R}.$$

As u^* is monotone we have that $|\nabla_H u^*|_H$ is constant on $\{u^* = t\} \cap \{\nabla_H u^* \neq 0\}$. Observe also that, being u^* one-dimensional, $\{u^* = t\}$ has a well defined meaning. We thus find:

$$-\mu'(t) = \frac{P_\gamma(\{u^* > t\})}{|\nabla_H u^*|_{\{u^* = t\}}} \quad \text{for a.e. } t \in \mathbb{R},$$

which implies, recalling (11),

$$\frac{P_\gamma(\{u^* > t\})}{|\nabla_H u^*|_{\{u^* = t\}}} \geqslant \int_{\{\nabla_H u \neq 0\}} \frac{1}{|\nabla_H u|_H} \, d|D_\gamma \chi_{E_t}| \quad \text{for a.e. } t \in \mathbb{R}. \tag{12}$$

Let us note that as in [10, Lemma 4.2], using (3) with $g = \chi_{\{\nabla_H u = 0\}}$ we find

$$\int_X \chi_{\{\nabla_H u = 0\}} |\nabla_H u|_H \, d\gamma = 0 = \int_{\mathbb{R}} \int_X \chi_{\{\nabla_H u = 0\}} \, d|D_\gamma \chi_{E_t}|(x) \, dt$$

and thus for almost every $t \in \mathbb{R}$,

$$\int_X \chi_{\{\nabla_H u = 0\}} \, d|D_\gamma \chi_{E_t}|(x) = 0.$$

This shows that for almost every $t \in \mathbb{R}$, $\nabla_H u(x) \neq 0$ for $|D_\gamma \chi_{E_t}|$-almost every $x \in X$ and thus

$$\int_{\{\nabla_H u \neq 0\}} \frac{1}{|\nabla_H u|_H}\, d|D_\gamma \chi_{E_t}|(x) = \int_X \frac{1}{|\nabla_H u|_H}\, d|D_\gamma \chi_{E_t}|(x) \quad \text{for a.e. } t \in \mathbb{R}. \tag{13}$$

By (3), (5), (12) and (13), we eventually get

$$\begin{aligned}
\int_X \left|\nabla_H u^*\right|^2 d\gamma &= \int_{\mathbb{R}} \left|\nabla_H u^*\right|_{\{u^*=t\}} P_\gamma\left(\{u^* > t\}\right) dt \\
&= \int_{\mathbb{R}} \frac{P_\gamma(\{u^* > t\})^2}{\left(\frac{P_\gamma(\{u^*>t\})}{|\nabla_H u^*|_{\{u^*=t\}}}\right)}\, dt \\
&\leqslant \int_{\mathbb{R}} \frac{P_\gamma(\{u > t\})^2}{\int_X \frac{1}{|\nabla_H u|_H} d|D_\gamma \chi_{E_t}|(x)}\, dt \\
&\leqslant \int_{\mathbb{R}} \int_X |\nabla_H u|_H\, d|D_\gamma \chi_{E_t}|(x)\, dt \\
&= \int_X |\nabla_H u|_H^2\, d\gamma.
\end{aligned}$$

As a consequence, if equality holds in (10), then equality holds in the Gaussian isoperimetric inequality, that is,

$$P_\gamma(u > t) = P_\gamma\left(u^* > t\right) \quad \text{for a.e. } t \in \mathbb{R}.$$

This implies that almost every level-set of u is a half-space, i.e. for almost every $t \in \mathbb{R}$ there exists $\hat{h}_t \in \mathcal{H}$ such that $\{u > t\} = \{\hat{h}_t < \alpha(\mu(t))\}$, and without loss of generality we can assume that $|h_t|_H = 1$. Such half-spaces being nested, by Lemma 3.2 we have that $\hat{h}_t$ does not depend on t and thus $u(x) = v(\hat{h}(x))$. □

Remark 3.13. We notice that the fact that equality in (10) implies that u is one-dimensional is a specific feature of the Gaussian setting, and the analogous statement does not hold for the Schwartz symmetrization in the Euclidean case [9]. Indeed, this property is a consequence of the fact that Gaussian measures, differently from the Lebesgue measure, are not invariant under translations.

4. Relaxation of perimeter

In this section we compute the relaxation of the perimeter functional

$$F(u) := \begin{cases} P_\gamma(E) & \text{if } u = \chi_E \\ +\infty & \text{otherwise} \end{cases}$$

with respect to the weak $L^2_\gamma(X)$-topology. The fact that F is not lower semicontinuous can be easily checked by taking the sequence $E_n = \{\langle x_n^*, x\rangle < 0\}$. Indeed, the characteristic functions of these sets weakly converge to the constant function $1/2$, which is not a characteristic function, while the perimeter of E_n is constantly equal to $1/\sqrt{2\pi}$.

We will show that the relaxation of F is equal to

$$\overline{F}(u) := \begin{cases} \int_X \sqrt{\mathcal{U}^2(u) + |D_\gamma u|^2}\, d\gamma & \text{if } 0 \leqslant u \leqslant 1 \ \gamma\text{-a.e.} \\ +\infty & \text{otherwise} \end{cases}$$

where

$$\int_X \sqrt{\mathcal{U}^2(u) + |D_\gamma u|^2}\, d\gamma = \int_X \sqrt{\mathcal{U}^2(u) + |\nabla_H u|_H^2}\, d\gamma + \left|D_\gamma^s u\right|(X)$$

with $D_\gamma u = \nabla_H u\, d\gamma + D^s_\gamma u$. Observe that the functional $\overline{F}$ already appears in the seminal work of Bakry and Ledoux [6] and in the earlier work of Bobkov [7] in the context of log-Sobolev inequalities. This functional has been also studied in [10]. See also [4, Remark 4.3] where it appears in a setting closer to ours.

Let us first recall the definition of the lower semicontinuous envelope of a function (see [16] for more details).

Definition 4.1. Let X be a topological vector space. For every function $F : X \to \overline{\mathbb{R}}$, its lower semicontinuous envelope (or relaxed function) is the greatest lower semicontinuous function that lies below F.

When X is a metric space, the following characterization holds.

Proposition 4.2. *Let X be a metric space. For every function $F : X \to \overline{\mathbb{R}}$, the relaxed function $\overline{F}$ is given by*

$$\overline{F}(x) = \inf\Big\{ \varliminf_{n\to\infty} F(x_n) : x_n \to x \Big\} \quad x \in X.$$

We now show a representation formula for $\overline{F}$ which is reminiscent of the definition of the total variation and of the nonparametric area functional (see [23]). We start with a preliminary result.

Lemma 4.3. *Let $g \in L^\infty(X)$ with $g \geqslant 0$, let $\mu \in \mathcal{M}(X, H)$, and define*

$$\tilde{f}(g,\mu) := \sqrt{g^2 + |h|_H^2}\, d\gamma + |\mu^s|,$$

where $\mu = h\gamma + \mu^s$. There holds

$$\tilde{f}(g,\mu)(X) = \sup_{\substack{\Phi \in L^1_\mu(X,H) \\ \xi \in L^1_\mu(X)}} \left\{ \int_X [\Phi, d\mu]_H + \int_X g\,\xi\, d\gamma :\ |\Phi|_H^2 + |\xi|^2 \leqslant 1 \text{ a.e. in } X \right\}. \tag{14}$$

Proof. The proof is adapted from [17].

Notice first that, for $(\lambda, p) \in \mathbb{R} \times H$, the function $f(\lambda, p) := \sqrt{\lambda^2 + |p|_H^2}$ defines a norm on the product space $\mathbb{R} \times H$. Moreover, if we let $f_\lambda(p) := \sqrt{\lambda^2 + |p|_H^2}$, then the convex conjugate of f_λ is $f^*_\lambda(\Phi) = -\lambda\sqrt{1 - |\Phi|_H^2}$. We divide the proof into three steps.

Step 1. Let

$$M(g,\mu) = \sup_{\Phi \in L^1_\mu(X,H)} \left\{ \int_X [\Phi, d\mu]_H + \int_X g\sqrt{1 - |\Phi|_H^2}\, d\gamma :\ |\Phi|_H \leqslant 1 \text{ a.e. in } X \right\}.$$

We will show that

$$M(g, h\gamma) = \int_X f(g,h)\, d\gamma. \tag{15}$$

By definition of convex conjugate, it is readily checked that $M(g, h\gamma) \leqslant \int_X f(g,h)\, d\gamma$. We thus turn to the other inequality. By definition of the Bochner integral, for every $\delta > 0$, there exist $h_i \in H$ and $A_i \subset X$ with A_i disjoints Borel sets and $i \in [1, m]$ such that if we set

$$\theta = \sum_{i=1}^m \chi_{A_i} h_i$$

then $|\theta - h|_{L^1_\gamma} \leqslant \delta$. Analogously there exists $\eta_i \in X$ such that setting

$$\tilde{g} = \sum_{i=1}^m \chi_{A_i} \eta_i$$

we have $|\tilde g - g|_{L^1_\gamma} \leqslant \delta$. By the observation at the beginning of the proof and the triangle inequality we get

$$\big|f(\tilde g,\theta) - f(g,h)\big| \leqslant \big|f(\tilde g - g, \theta - h)\big| \leqslant |\tilde g - g| + |\theta - h|_H.$$

For every i, by definition of convex conjugate, there exists $\xi_i \in H$ with $|\xi_i|_H \leqslant 1$ such that

$$f(\eta_i, h_i) \leqslant [\xi_i, h_i]_H + \eta_i\sqrt{1 - |\xi_i|_H^2} + \delta.$$

From this, setting $\Phi = \sum_{i=1}^m \chi_{A_i}\xi_i$ we have

$$\begin{aligned}
\int_X f(g,h)\,d\gamma &\leqslant \int_X f(\tilde g,\theta)\,d\gamma + 2\delta \\
&= \sum_{i=1}^m \int_{A_i} f(\eta_i,h_i)\,d\gamma + 2\delta \\
&\leqslant \sum_{i=1}^m \int_{A_i} [\xi_i,h_i]_H + \eta_i\sqrt{1-|\xi_i|_H^2}\,d\gamma + 3\delta \\
&= \int_X [\Phi,h]_H + \tilde g\sqrt{1-|\Phi|_H^2}\,d\gamma + 3\delta.
\end{aligned}$$

Since $|\tilde g\sqrt{1-|\Phi|_H^2} - g\sqrt{1-|\Phi|_H^2}| \leqslant |\tilde g - g|$ we find

$$\int_X f(g,h)\,d\gamma \leqslant \int_X \Phi\cdot h - g\sqrt{1-|\Phi|_H^2}\,d\gamma + 4\delta \leqslant M(g,h\gamma) + 4\delta.$$

Since δ is arbitrary we have $M(g,h\gamma) = \int_X f(g,h)\,d\gamma$.

Step 2. The proof proceeds exactly as in [17] and we only sketch it. Recalling (15), it remains to show that

$$M\big(g,h\gamma+\mu^s\big) = M(g,h\gamma) + \big|\mu^s\big|(X).$$

One inequality is easily obtained, since

$$\begin{aligned}
M\big(g,h\gamma+\mu^s\big) &= \sup_\Phi \int_X [\Phi,h]_H\,d\gamma + \int_X \Phi\cdot d\mu^s + \int_X g(x)\sqrt{1-|\Phi|_H^2}\,d\gamma \\
&\leqslant \left(\sup_\Phi \int_X [\Phi,h]_H\,d\gamma + \int_X g(x)\sqrt{1-|\Phi|_H^2}\,d\gamma\right) + \int_X \big|d\mu^s\big| \\
&= M(g,h\gamma) + \big|\mu^s\big|(X).
\end{aligned}$$

For the opposite inequality, let $\delta > 0$ be fixed then there exist Φ_1 and Φ_2 such that

$$\begin{aligned}
M(g,h\gamma) &\leqslant \int_X [\Phi_1,h]_H\,d\gamma + \int_X g(x)\sqrt{1-|\Phi_1|_H^2}\,d\gamma + \delta \\
\big|\mu^s\big|(X) &\leqslant \int_X \big[\Phi_2, d\mu^s\big]_H + \delta.
\end{aligned}$$

Taking Φ equal to Φ_2 on a sufficiently small neighborhood of the support of μ^s and equal to Φ_1 outside this neighborhood, we get

$$M(g,h\gamma) + \big|\mu^s\big|(X) \leqslant \int_X [\Phi,h]_H\,d\gamma + \int_X g(x)\sqrt{1-|\Phi|_H^2}\,d\gamma + \int_X \big[\Phi,d\mu^s\big]_H + C\delta \leqslant M\big(g,h\gamma+\mu^s\big) + C\delta$$

which gives the opposite inequality.

Step 3. In order to conclude the proof, it is enough to notice that for every $\Phi \in L^1_\mu(X,H)$, with $|\Phi|_H \leqslant 1$, we have

$$\sup_{\xi\in L^1_\mu(X)}\left\{\int_X [\Phi, d\mu]_H + \int_X g\xi\, d\gamma\colon\ |\Phi|_H^2 + |\xi|^2 \leqslant 1 \text{ a.e. in } X\right\}$$
$$= \int_X [\Phi, d\mu]_H + \int_X g\sqrt{1-|\Phi|_H^2}\, d\gamma. \qquad \square$$

Proposition 4.4. *Let $u \in BV_\gamma(X)$ then*

$$\overline{F}(u) = \sup_{\substack{\Phi\in\mathcal{FC}^1_b(X,H)\\ \xi\in\mathcal{FC}^1_b(X)}}\left\{\int_X \big(u\,\mathrm{div}_\gamma\,\Phi + \mathcal{U}(u)\xi\big)\,d\gamma\colon\ \big|\Phi(x)\big|_H^2 + \big|\xi(x)\big|^2 \leqslant 1\ \forall x\in X\right\}. \tag{16}$$

Proof. We apply Lemma 4.3 with $\mu = Du$ and $g = \mathcal{U}(u)$. Since μ is tight [4], the space $\mathcal{FC}^1_b(X,H)$ is dense in $L^1_\mu(X,H)$ so that we can restrict the supremum in (16) to smooth cylindrical functions Φ, ξ. $\square$

Remark 4.5. Since $\mathcal{U}$ is concave, the duality formula (16) is not sufficient to prove that $\overline{F}$ is lower semicontinuous for the weak $L^2_\gamma(X)$-topology. It shows however the lower-semicontinuity of $\overline{F}$ in the strong $L^2_\gamma(X)$-topology.

We now prove that $\overline{F}$ is the lower semicontinuous envelope of F.

Theorem 4.6. *$\overline{F}$ is the relaxation of F in the weak $L^2_\gamma(X)$-topology.*

Proof. Let us first notice that F takes finite values only on functions of the closed unit ball of $L^2_\gamma(X)$ which is metrizable for the weak convergence. Therefore the relaxation and the sequential relaxation in the weak topology of $L^2_\gamma(X)$ coincide.

Let χ_{E_n} be a sequence of sets weakly converging in $L^2_\gamma(X)$ to $u \in BV_\gamma(X)$, with uniformly bounded perimeter. We shall show that

$$\varliminf_{n\to\infty} P_\gamma(E_n) \geqslant \overline{F}(u).$$

Notice that, by weak convergence, we necessarily have $0 \leqslant u \leqslant 1$ a.e. on X.

For all $n \geqslant 1$ and $k \geqslant 2$, we let E_n^k be the Ehrhard symmetral of E_n with respect to the first k variables. Recalling the notation of Section 3, we have

$$P_\gamma\big(E_n^{k+1}\big) \leqslant P_\gamma(E_n) \quad \text{and} \quad E_n^{k+1} = ES_k(\mathbb{E}_k\chi_{E_n}).$$

As $\int_X |D_\gamma \mathbb{E}_k(\chi_{E_n})|_H \leqslant P_\gamma(E_n)$ and $\mathbb{E}_k(\chi_{E_n})$ depends only on the first k variables, by the compact embedding of $BV_{\gamma_k}(\mathbb{R}^k)$ into $L^1_{\gamma_k}(\mathbb{R}^k)$ we can extract a subsequence from $\mathbb{E}_k(\chi_{E_n})$ which converges strongly to $u^k := \mathbb{E}_k(u)$. From this we get that $E_n^{k+1} = ES_k(\mathbb{E}_k\chi_{E_n})$ tends strongly to $E^{k+1} := ES_k(u^k)$. By the lower semicontinuity of the perimeter we then have

$$\varliminf_{n\to\infty} P_\gamma(E_n) \geqslant \varliminf_{n\to\infty} P_\gamma\big(E_n^{k+1}\big) \geqslant P_\gamma\big(E^{k+1}\big).$$

For every $\varphi \in \mathcal{FC}^1_b(X)$, with φ depending only of the $j \leqslant k$ first variables, there holds

$$\int_X \chi_{E_{k+1}}(x)\varphi(x)\,d\gamma(x) = \int_X u_k(x)\varphi(x)\,d\gamma(x) = \int_X u(x)\varphi(x)\,d\gamma(x),$$

which implies that the sequence $\chi_{E_{k+1}}$ tends weakly to u. In order to conclude the proof it remains to show that

$$\varliminf_{k\to\infty} P_\gamma\big(E^{k+1}\big) = \overline{F}(u).$$

Notice that, by Proposition 3.9, there holds

$$P_\gamma\big(E^{k+1}\big) = \overline{F}\big(u^k\big).$$

For every $\Phi \in \mathcal{FC}^1_b(X, H)$ and $\xi \in \mathcal{FC}^1_b(X)$, depending on the first k variables and such that the range of Φ is included in H_k, by Proposition 4.4, we have

$$\int_X \big(u^k \operatorname{div}_\gamma \Phi + \mathcal{U}\big(u^k\big)\xi\big)\, d\gamma = \int_X \big(u \operatorname{div}_\gamma \Phi + \mathcal{U}(u)\xi\big)\, d\gamma \leqslant \overline{F}(u).$$

Taking the supremum in Φ, ξ and recalling (16), we then get

$$\overline{F}\big(u^k\big) \leqslant \overline{F}(u) \quad \text{for all } k.$$

Repeating the same argument with u^{k+1} instead of u, we obtain that $\overline{F}(u^k)$ is nondecreasing in k. Therefore there exists $\ell \geqslant 0$ such that

$$\lim_{k\to\infty} \overline{F}\big(u^k\big) = \lim_{k\to\infty} P_\gamma\big(E^{k+1}\big) = \ell \leqslant \overline{F}(u).$$

Assume by contradiction that $\ell < \overline{F}(u)$. Then there exists $\delta > 0$ such that $\overline{F}(u^k) \leqslant \overline{F}(u) - \delta$ for all k, hence there exist $N \in \mathbb{N}$, $\Phi \in \mathcal{FC}^1_b(X, H)$ and $\xi \in \mathcal{FC}^1_b(X)$, depending only on the first N variables, such that

$$\int_X \big(u^k \operatorname{div}_\gamma \Phi + \mathcal{U}\big(u^k\big)\xi\big)\, d\gamma \leqslant \overline{F}\big(u^k\big) \leqslant \overline{F}(u) - \delta \leqslant \int_X \big(u \operatorname{div}_\gamma \Phi + \mathcal{U}(u)\xi\big)\, d\gamma - \frac{\delta}{2},$$

but for $k > N$ we have

$$\int_X \big(u^k \operatorname{div}_\gamma \Phi + \mathcal{U}\big(u^k\big)\xi\big)\, d\gamma = \int_X \big(u \operatorname{div}_\gamma \Phi + \mathcal{U}(u)\xi\big)\, d\gamma$$

which leads to a contradiction. □

Remark 4.7. Theorem 4.6 provides an example of a nonconvex functional, namely $\overline{F}$, which is lower semicontinuous for the weak $L^2_\gamma(X)$-topology. We also know that semicontinuity does not hold for general functional of the form

$$J(u) = \int_X f(u, D_\gamma u)\, d\gamma$$

since if we take for instance $f(u, p) := \sqrt{g^2(u) + |p|^2}$ with g such that $g(1/2) > \mathcal{U}(1/2)$ and $g(0) = g(1) = 0$, then, letting $u_n := \{\langle x_n^*, x\rangle < 0\}$, we have $u_n \rightharpoonup u = 1/2$ weakly in $L^2_\gamma(X)$, so that

$$J(u) = g\left(\frac{1}{2}\right) > \mathcal{U}\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2\pi}} = \lim_{n\to\infty} J(u_n).$$

One could wonder what are the right hypotheses for a functional of this form to be lower semicontinuous with respect to the weak topology.

5. Γ-limit for the Modica–Mortola functional

Let us briefly recall the definition of Γ-convergence. We refer to [16] for a comprehensive treatment of the subject.

Definition 5.1. Let X be a topological space, and let $F_n : X \to \overline{\mathbb{R}}$ be a sequence of functions. The Γ-lower limit and the Γ-upper limit of the sequence F_n is defined as

$$\Big(\Gamma - \varliminf_{n\to\infty} F_n\Big)(x) = \sup_{U\in\mathcal{N}(x)} \varliminf_{n\to\infty} \inf_{y\in U} F_n(y)$$

$$\Big(\Gamma - \varlimsup_{n\to\infty} F_n\Big)(x) = \sup_{U\in\mathcal{N}(x)} \varlimsup_{n\to\infty} \inf_{y\in U} F_n(y)$$

where $\mathcal{N}(x)$ denotes the set of all open neighborhoods of x in X. When the Γ-lower limit and the Γ-upper limit coincide, we say that the sequence F_n Γ-converges.

As for the relaxation, if X is a metric space we have a sequential characterization of the Γ-convergence.

Theorem 5.2. *Let X be a metric space. A sequence of functions F_n Γ-converges to $F : X \to \overline{\mathbb{R}}$ if and only if the following two conditions hold*:

- *for every sequence x_n converging to x, it holds $\underline{\lim}_{n\to\infty} F_n(x_n) \geqslant F(x)$*
- *for every $x \in X$ there exists a sequence x_n converging to x with $\overline{\lim}_{n\to\infty} F_n(x_n) \leqslant F(x)$.*

Let now $W \in C^1(\mathbb{R})$ be a double-well potential with minima in $\{0, 1\}$, that is, $W(t) \geqslant 0$ for all $t \in \mathbb{R}$, and $W(t) = 0$ iff $t \in \{0, 1\}$. A typical example of such potential is $W(t) = t^2(t-1)^2$.

For any $\varepsilon > 0$ we define the functionals $F_\varepsilon : L^2_\gamma(X) \to [0, +\infty]$ as

$$F_\varepsilon(u) := \begin{cases} \int_X (\frac{\varepsilon}{2}|\nabla_H u|^2_H + \frac{W(u)}{\varepsilon})\, d\gamma & \text{if } u \in H^1_\gamma(X) \\ +\infty & \text{if } u \in L^2_\gamma(X) \setminus H^1_\gamma(X). \end{cases}$$

We are ready to prove our main Γ-convergence result.

Theorem 5.3. *When ε tends to zero the functionals F_ε Γ-converge, in the weak topology of $L^2_\gamma(X)$, to the functional $c_W \overline{F}$, where $c_W = \int_0^1 \sqrt{2W(t)}\, dt$.*

Proof. Notice first that the Γ-limit does not change if we restrict the domain of F_ε to the functions $u \in H^1_\gamma(X)$ such that $0 \leqslant u \leqslant 1$. This follows from the following two facts:

- for all $u \in H^1_\gamma(X)$, letting $\tilde{u} = \min(\max(u, 0), 1)$, we have $F_\varepsilon(\tilde{u}) \leqslant F_\varepsilon(u)$;
- $F_\varepsilon(u) \geqslant \int_X \frac{W(u)}{\varepsilon} d\gamma$ for all $u \in H^1_\gamma(X)$, which implies that the Γ-limit is concentrated on the functions $u \in L^2_\gamma(X)$ such that $u(x) \in \{0, 1\}$ for a.e. $x \in X$.

Since the restricted domain is contained in the unit ball of $L^2_\gamma(X)$, which is metrizable for the weak $L^2_\gamma(X)$-topology, by Theorem 5.2 the Γ-limit and the sequential Γ-limit of F_ε coincide.

We now compute the Γ-liminf of F_ε.

Let $u_\varepsilon \in H^1_\gamma(X)$ be such that $0 \leqslant u_\varepsilon \leqslant 1$ and $F_\varepsilon(u_\varepsilon) \leqslant C$ for some $C > 0$, then $\|u_\varepsilon\|_{L^2_\gamma(X)} \leqslant 1$. As a consequence, there exists a weakly converging subsequence, still denoted by u_ε. Letting u be its weak limit, from $0 \leqslant u_\varepsilon \leqslant 1$ we get $0 \leqslant u \leqslant 1$. Using the coarea formula (2), we obtain the estimate

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &= \int_X \left(\frac{\varepsilon}{2}|\nabla_H u|^2_H + \frac{W(u)}{\varepsilon}\right) d\gamma \\ &\geqslant \int_X \sqrt{2W(u_\varepsilon)}\, |\nabla_H u|_H\, d\gamma \\ &= \int_0^1 \sqrt{2W(t)}\, P_\gamma\big(\{u_\varepsilon > t\}\big)\, dt. \end{aligned}$$

Fix now $\delta > 0$. From the fact that $\gamma(\{\delta \leqslant u_\varepsilon \leqslant 1-\delta\}) \to 0$ as $\varepsilon \to 0$, it follows that, for every sequence $t_\varepsilon \in [\delta, 1-\delta]$, then functions $\chi_{\{u_\varepsilon > t_\varepsilon\}}$ tend weakly to u in $L^2_\gamma(X)$. For every $\varepsilon > 0$ let us choose $t_\varepsilon \in [\delta, 1-\delta]$ such that

$$\int_\delta^{1-\delta} \sqrt{2W(t)}\, P_\gamma\big(\{u_\varepsilon > t\}\big)\, dt \geqslant \left(\int_\delta^{1-\delta} \sqrt{2W(t)}\, dt\right) P_\gamma\big(\{u_\varepsilon > t_\varepsilon\}\big).$$

Then, by Theorem 4.6 we have

$$\underline{\lim}_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geqslant \underline{\lim}_{\varepsilon\to 0}\left(\int_\delta^{1-\delta} \sqrt{2W(t)}\,dt\right) P_\gamma\big(\{u_\varepsilon > t_\varepsilon\}\big) \geqslant \left(\int_\delta^{1-\delta} \sqrt{2W(t)}\,dt\right)\overline{F}(u).$$

Since δ is arbitrary we get the Γ-liminf inequality.

The Γ-limsup is done similarly to the (Euclidean) finite dimensional case [28,27]. Since $\overline{F}$ is the relaxation of F in the weak $L^2_\gamma(X)$-topology and since we can approximate sets of finite perimeter by smooth cylindrical sets by Proposition 2.6, for every $u \in BV_\gamma(X)$ with $0 \leqslant u \leqslant 1$ there exists a sequence E_n of smooth cylindrical sets with χ_{E_n} converging weakly to u and such that $P_\gamma(E_n)$ tends to $\overline{F}(u)$. This shows that we can restrict ourselves to smooth cylindrical sets for computing the Γ-limsup of F_ε.

Let $m \in \mathbb{N}$ and $E = \Pi_m^{-1}(E_m)$, where $E_m \subset \mathbb{R}^m$ is a smooth set with finite Gaussian perimeter, and let

$$d^H(x, E) := d\big(\Pi_m(x), E_m\big)$$

where $d(x, E_m)$ is the usual distance function from E_m in $\mathbb{R}^m$. Notice that

$$d^H(x, E) = \min\big\{|x - y|_H;\ y \in E,\ x - y \in H\big\},$$

moreover d^H is differentiable almost everywhere with $|\nabla_H d^H(x, E)|_H = 1$.

Let $\delta > 0$, $\alpha_\delta := \max\{W(t):\ t \in [0, \delta] \cup [1 - \delta, 1]\}$ and define $W_\delta, H_\delta : [0, 1] \to \mathbb{R}$ as

$$W_\delta(t) := \begin{cases} \alpha_\delta & \text{if } 0 \leqslant t \leqslant \delta \\ W(t) & \text{if } \delta \leqslant t \leqslant 1 - \delta \\ \alpha_\delta & \text{if } 1 - \delta \leqslant t \leqslant 1. \end{cases}$$

$$H_\delta(t) := \int_0^t \frac{1}{\sqrt{2W_\delta(s)}}\,ds.$$

Finally let η_δ be the usual truncated one-dimensional transition profile defined as

$$\eta_\delta(t) := \begin{cases} 0 & \text{if } t \leqslant 0 \\ H_\delta^{-1}(t) & \text{if } 0 \leqslant t \leqslant H_\delta(1) \\ 1 & \text{if } t > H_\delta(1). \end{cases}$$

Observe that η_δ is a Lipschitz function which verifies $\frac{\eta_\delta'^2}{2} = W_\delta(\eta_\delta)$. We then set

$$u_\varepsilon(x) := \eta_\delta\left(\frac{d^H(x, E)}{\varepsilon}\right).$$

We finally have

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &= \int_X \left(\frac{\varepsilon}{2}|\nabla_H u_\varepsilon|_H^2 + \frac{W(u_\varepsilon)}{\varepsilon}\right) d\gamma \\ &\leqslant \int_X \left(\frac{\varepsilon}{2}|\nabla_H u_\varepsilon|_H^2 + \frac{W_\delta(u_\varepsilon)}{\varepsilon}\right) d\gamma \\ &= \int_X \frac{\varepsilon}{2}\eta_\delta'^2\left(\frac{d(\Pi_m(x))}{\varepsilon}\right)\left(\frac{|\nabla_H d(\Pi_m(x))|}{\varepsilon}\right)^2 + \frac{1}{\varepsilon}W_\delta\left(\frac{\eta_\delta(d(\Pi_m(x)))}{\varepsilon}\right) d\gamma \\ &= \int_{\mathbb{R}^m}\left[\frac{1}{2}\eta_\delta'^2\left(\frac{d}{\varepsilon}\right) + W_\delta\left(\eta_\delta\left(\frac{d}{\varepsilon}\right)\right)\right)\frac{|\nabla d|}{\varepsilon}\,d\gamma_m \\ &= \int_0^{H_\delta(1)} \left(\frac{\eta_\delta'^2(t)}{2} + W_\delta\big(\eta_\delta(t)\big)\right) P_{\gamma_m}\big(\{d > \varepsilon t\}\big)\,dt. \end{aligned}$$

The proof is completed since for every $t \in [0, H_\delta(1)]$, $P_{\gamma_m}(\{d > \varepsilon t\})$ tends to $P_{\gamma_m}(E_m)$ as $\varepsilon \to 0$, and

$$\int_0^{H_\delta(1)} \left(\frac{\eta_\delta'^2(t)}{2} + W_\delta\big(\eta_\delta(t)\big) \right) dt = \int_0^1 \sqrt{2W_\delta(t)}\, dt.$$

Thus we have

$$\overline{\lim_{\varepsilon\to 0}} F_\varepsilon(u_\varepsilon) \leqslant \left(\int_0^1 \sqrt{2W_\delta(t)}\, dt \right) P_{\gamma_m}(E_m),$$

which gives the desired inequality letting $\delta \to 0$. □

Remark 5.4. As in the Euclidean case, a similar result can be proven for the volume constrained problems. In this case, the proof of the Γ-liminf is exactly the same as in Theorem 5.3, and the Γ-limsup is also very similar. The only difference comes from the fact that we have to adapt the recovery sequence to have the right volume, and this can be done as in [27] by slightly translating η_δ.

We now state some simple implications of the Γ-convergence result.

Proposition 5.5. *Let $m \in [0, 1]$ and u_ε be a minimizer of*

$$\min_{\int_X u\, d\gamma = m} \int_X \left(\frac{\varepsilon}{2} |\nabla_H u|_H^2 + \frac{W(u)}{\varepsilon} \right) d\gamma \tag{17}$$

then $u_\varepsilon = v_\varepsilon(\hat{h}_\varepsilon(x))$ for some $\hat{h}_\varepsilon \in \mathcal{H}$ with $|h_\varepsilon|_H = 1$ and some v_ε minimizer of the one-dimensional problem

$$\min_{\int_{\mathbb{R}} v\, d\gamma_1 = m} \int_{\mathbb{R}} \frac{\varepsilon}{2} v'^2\, d\gamma + \int_{\mathbb{R}} \frac{W(v)}{\varepsilon}\, d\gamma_1. \tag{18}$$

in particular, v_ε (strongly) converges to the characteristic function of a half-line.

Proof. For every $u \in H^1_\gamma(X)$, by Proposition 3.12, we have $\int_X u^*\, d\gamma = \int_X u\, d\gamma$ and $F_\varepsilon(u^*) \leqslant F_\varepsilon(u)$, with equality only if u is of the form $u(x) = v(\hat{h}(x))$ for some $\hat{h} \in \mathcal{H}$ with $|h|_H = 1$. Using that $\hat{h}$ is the limit in $L^2_\gamma(X)$ of linear functions of the form $R^* x_i^*$, it is readily seen that $\nabla_H \hat{h} = h$, and thus we get

$$F_\varepsilon(u) = \int_X \left(\frac{\varepsilon}{2} v'\big(\hat{h}(x)\big)^2 + \frac{W(v(\hat{h}(x)))}{\varepsilon} \right) d\gamma = \int_{\mathbb{R}} \left(\frac{\varepsilon}{2} v'^2\, d\gamma + \int_{\mathbb{R}} \frac{W(v)}{\varepsilon} \right) d\gamma_1.$$

Therefore problem (17) reduces to the one-dimensional problem (18).

Using the compact embedding of $H^1_{\gamma_1}(\mathbb{R})$ in $L^2_{\gamma_1}(\mathbb{R})$ (see [4, Th. 4.10]) and the direct method of the calculus of variations, we get that (18) has a minimizer. Moreover, by the Γ-convergence of the one-dimensional functionals in the strong $L^2_{\gamma_1}(\mathbb{R})$-topology towards the a multiple of the perimeter (which can be obtained exactly as in the classical Modica–Mortola Theorem since compact embedding of $BV_{\gamma_1}(\mathbb{R})$ in $L^1_{\gamma_1}(\mathbb{R})$ holds), we find that every sequence of minimizers v_ε of (18) has a subsequence strongly converging towards the characteristic of the half-line of measure m. □

We finally give another convergence result for the prescribed curvature problem in case of uniqueness of minimizers.

Proposition 5.6. *Let $g \in L^2_\gamma(X)$, then the following assertions are equivalent*:

- *the functional*

$$F_g(E) = P_\gamma(E) + \int_E g\,d\gamma \tag{19}$$

has a unique minimizer in the class of sets of finite perimeter;
- *the functional*

$$\overline{F}_g(u) = \overline{F}(u) + \int_X ug\,d\gamma \tag{20}$$

has a unique minimizer in $BV_\gamma(X)$.

Moreover, when this holds the two minimizers coincides. Finally, if u_ε *is a sequence in* $H^1_\gamma(X)$ *satisfying*

$$\sup_\varepsilon \left(F_\varepsilon(u_\varepsilon) + \int_X u_\varepsilon g\,d\gamma \right) \leqslant C$$

for some $C > 0$, *then* u_ε *has a subsequence strongly converging to* χ_E *in* $L^2_\gamma(X)$, *where* E *is the common minimizer of* (19) *and* (20).

Proof. We first notice that the problem (19) always has a solution. Indeed, arguing as in [12], if E_n is a minimizing sequence for (19), it has a subsequence weakly converging to some $u \in BV_\gamma(X)$. By the lower semicontinuity of the total variation and the coarea formula we then have

$$\inf_E \left(P_\gamma(E) + \int_E g\,d\gamma \right) \geqslant \int_X |D_\gamma u|_H + \int_X ug\,d\gamma = \int_0^1 \left(P_\gamma\big(\{u > t\}\big) + \int_{\{u>t\}} g(x)\,d\gamma(x) \right) dt$$

and thus the sets $\{u > t\}$ minimize F_g for almost every t. As $\overline{F}$ is the relaxation of the perimeter we have that the minimum values in (19) and (20) are the same and thus any minimizer of F_g is also a minimizer of $\overline{F}_g$. This shows that if uniqueness does not hold in (19) then it does not hold in (20), too. Now, if u is a minimizer of $\overline{F}_g$, applying the coarea formula once again we get

$$\inf_E F_g(E) = \overline{F}_g(u) \geqslant \int_X |D_\gamma u|_H + \int_X ug\,d\gamma = \int_0^1 \left(P_\gamma\big(\{u > t\}\big) + \int_{\{u>t\}} g(x)\,d\gamma(x) \right) dt.$$

As above, this implies that $\{u > t\}$ solves (19) for almost every t. Therefore, if the minimizer of $\overline{F}_g$ is not a characteristic function, then uniqueness does not hold neither in (19) nor in (20). This proves the first part of the proposition.

The second statement easily follows from Theorem 5.3. Indeed, as the functionals $F_\varepsilon(u) + \int_X ug\,d\gamma$ Γ-converge to $\overline{F}_g$ in the weak $L^2_\gamma(X)$-topology, for every sequence u_ε bounded in energy, there exists a subsequence weakly converging to χ_E (where E is the unique minimizer of (19) and (20)). However, by the lower semicontinuity of the norm,

$$m^{\frac{1}{2}} \geqslant \varliminf_{\varepsilon\to 0} \|u_\varepsilon\|_{L^2_\gamma(X)} \geqslant \|\chi_E\|_{L^2_\gamma(X)} = m^{\frac{1}{2}}.$$

Thus $\|u_\varepsilon\|_{L^2_\gamma(X)}$ converges to $\|\chi_E\|_{L^2_\gamma(X)}$, which implies the strong convergence of u_ε. □

Remark 5.7. In [13], we provide an example of functionals for which uniqueness of minimizers holds, namely

$$P_\gamma(E) + \int_X (g - \lambda)\,d\gamma$$

where $g : X \to \mathbb{R}$ is convex and $\lambda \in (0, +\infty)$ is large enough.

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