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Increasing radial solutions for Neumann problems without growth restrictions

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Abstract

We study the existence of positive increasing radial solutions for superlinear Neumann problems in the ball. We do not impose any growth condition on the nonlinearity at infinity and our assumptions allow for interactions with the spectrum. In our approach we use both topological and variational arguments, and we overcome the lack of compactness by considering the cone of nonnegative, nondecreasing radial functions of $H^1(B)$.

Résumé

Nous étudions l'existence de solutions radiales positives croissantes de problèmes de Neumann super linéaires dans la boule. Nous n'imposons aucune restriction de croissance sur la non linéarité à l'infini et nos hypothèses permettent également une interaction avec le spectre. Notre approche combinne des arguments topologiques et variationnels. Nous contournons le manque de compacité en travaillant dans le cône des fonctions radiales, positives et croissantes de $H^1(B)$.

Keywords: Supercritical problems; Krasnosel'skiĭ fixed point; Invariant cone; Gradient flow

1. Introduction

In this paper we are mainly concerned with the semilinear Neumann problem

$$\begin{cases} -\Delta u + u = a\big(|x|\big) f(u) & \text{in } B, \\ u > 0 & \text{in } B, \\ \partial_\nu u = 0 & \text{on } \partial B, \end{cases} \tag{1.1}$$

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where B is the unit ball in $\mathbb{R}^N$, $N \geqslant 2$. We study the existence of radial solutions of (1.1) under suitable assumptions on a and f. The problem has been studied extensively in the case where $f(u) = u^p$ with some $p > 1$ and $a \equiv 1$. Note that in this case there always exists the constant solution $u \equiv 1$ of (1.1). This already shows that the solvability of (1.1) depends in a quite different way on the data than in the case of Dirichlet boundary conditions, in which nontrivial solutions only exist in the subcritical range

$$p < \frac{N+2}{N-2} \quad \text{if } N \geqslant 3 \tag{1.2}$$

as a consequence of Pohozaev's identity, see [14]. Note that the subcriticality assumption (1.2) ensures that the problem (1.1) with $f(u) = u^p$ is accessible by variational methods, i.e., the (formal) energy functional corresponding to (1.1) is well defined in $H^1(B)$. Moreover, due to the compact embedding $H^1(B) \hookrightarrow L^{p+1}(B)$, the existence of a solution to (1.1) follows in a standard way through the mountain pass theorem [2] if a is a positive continuous function on $\overline{B}$. In the critical and supercritical case, namely when (1.2) does not hold, most of the available results on the existence of positive solutions are devoted to perturbative cases where either a small diffusion constant is added in front of $-\Delta$, see [1, Chapters 9 and 10] and the references therein or a slightly supercritical exponent is considered, see e.g. [8]. The present paper deals with the nonperturbative problem and is therefore more closely related to the recent works [5,16]. In [5], the authors considered the Neumann problem for the Hénon equation $-\Delta u + u = |x|^\alpha u^p$, and they apply a shooting method to prove that this problem admits a positive and radially increasing solution for every $p > 1$ and $\alpha > 0$. Very recently, Serra and Tilli [16] showed the existence of the same type of solutions for problem (1.1), provided that a is an increasing function such that $a(r) > 0$ a.e. in B and $f \in C^1([0,\infty))$ satisfies

$$f(0) = f'(0) = 0, \qquad f'(t)t - f(t) > 0 \quad \text{and} \quad f(t)t \geqslant \mu F(t) := \int_0^t f(s)\,ds, \tag{1.3}$$

for $t \in (0,\infty)$, with some constant $\mu > 2$. These assumptions, which hold for $f(u) = u^p$, $p > 1$, play a crucial role in the approach of Serra and Tilli, who minimize the energy functional corresponding to (1.1) among nonnegative, radial and radially nondecreasing functions within the associated Nehari manifold. Reducing to nonnegative and nondecreasing radial trial functions in $H^1(B)$ gives rise to boundedness and compactness properties even for supercritically growing nonlinearities. It is not obvious that restrictions of this type still lead to a solution of (1.1), but Serra and Tilli could prove this with the help of assumptions (1.3).

The purpose of the present paper is twofold. First, we generalize the results of Serra and Tilli to a wider class of functions f by means of a new approach based on topological fixed point theory and invariance properties of the cone of nonnegative, nondecreasing radial functions in $H^1(B)$. In particular, we give a rather short proof of the existence of an increasing radial solution of (1.1). More precisely, we first establish a priori estimates on the solutions of (1.1) in this cone and then apply a suitable version of Krasnosel'skiĭ's fixed point theorem (see [12]). The second aim of this paper is related to the case a constant, say $a \equiv 1$, where any fixed point of f gives rise to a constant solution of (1.1). In this case we will be concerned with the existence of *nonconstant* increasing solutions. To state our main results, we now list our assumptions on a and f:

(a) $a \in C^1([0,1],\mathbb{R})$ is nondecreasing and $a_0 := a(0) > 0$;
$(f1)$ $f \in C^1([0,+\infty),\mathbb{R})$, $f(0) = 0$ and $f'(0) = \lim_{s\to 0^+} \frac{f(s)}{s} = 0$;
$(f2)$ f is nondecreasing;
$(f3)$ $\liminf_{s\to+\infty} \frac{f(s)}{s} > \frac{1}{a_0}$.

In particular, these assumptions on f allow the nonlinearity to have supercritical growth as well as resonant growth, i.e. $\lim_{s\to+\infty} f(s)/s = \lambda$ with $\lambda > 1$ being a Neumann eigenvalue of the operator $-\Delta + 1$ in B, and they are much weaker than (1.3). In particular, f may have multiple positive fixed points and the quotient $f(s)/s$ may oscillate between values in an interval of the form $[c,\infty)$ with $c > 1/a_0$ for large s, whereas (1.3) forces this quotient to be strictly increasing. Our first existence result for (1.1) is the following.

Theorem 1.1. *Assume* (a), $(f1)$, $(f2)$, $(f3)$ *and suppose moreover that* $a(|x|)$ *is nonconstant. Then there exists at least one nonconstant nondecreasing radial solution of* (1.1).

The existence of solutions for such general nonlinearities f underscore the difference between Dirichlet and Neumann boundary conditions for supercritical elliptic problems, see also the related recent papers [7,9,15]. In contrast to the method of Serra and Tilli in [16], our approach based on topological fixed point theory does not require the (formal) variational structure of problem (1.1) and therefore applies to the more general problem

$$\begin{cases} -\Delta u + b\big(|x|\big)x\cdot\nabla u + u = a\big(|x|\big)f(u) & \text{in } B,\\ u>0 & \text{in } B,\\ \partial_\nu u = 0 & \text{on } \partial B, \end{cases} \tag{1.4}$$

provided that the following assumption holds:

(b) $b\in C([0,1],\mathbb{R})$ is nonpositive, and $\frac{d}{dr}(b(r)r) > -1-\frac{N-1}{r^2}$ in $(0,1)$.

Theorem 1.2. *Assume* (a), (b), $(f1)$, $(f2)$, $(f3)$ *and suppose moreover that* $a(|x|)$ *is nonconstant. Then there exists at least one nonconstant nondecreasing radial solution of* (1.4).

In case a is a constant function, say $a\equiv 1$, assumptions $(f1)$–$(f3)$ imply the existence of $u_0>0$ such that $f(u_0)=u_0$, so that u_0 is a constant solution of (1.1). Moreover, there exist nonlinearities satisfying $(f1)$–$(f3)$ (with $a_0=1$) and such that the problem

$$\begin{cases} -\Delta u + u = f(u) & \text{in } B,\\ u>0 & \text{in } B,\\ \partial_\nu u = 0 & \text{on } \partial B \end{cases} \tag{1.5}$$

only admits this constant solution (see Proposition 4.1 below, where we adapt an argument of [6]). We need the following additional assumption:

$(f4)$ there exists $u_0>0$ such that $f(u_0)=u_0$ and $f'(u_0)>\lambda_2^{rad}$.

Here $\lambda_2^{rad}>1$ is the second radial eigenvalue of $-\Delta+1$ in the unit ball with Neumann boundary conditions. We prove the following result.

Theorem 1.3. *Assume* $(f1)$–$(f4)$ *with* $a\equiv 1$. *Then there exists at least one nonconstant increasing radial solution of* (1.5).

To our knowledge, this is the first existence result for nonconstant solutions of (1.5) under assumptions $(f1)$–$(f4)$ and even under the more restrictive conditions (1.3) and $(f4)$. An inspection of the proof of Theorem 1.3 shows that we find nonconstant solutions of (1.5) in every order interval of the form $[u_-,u_+]$, where u_- and u_+ are ordered fixed points of f with the property that there exists another fixed point $u_0\in(u_-,u_+)$ such that $f'(u_0)>\lambda_2^{rad}$.

We note that the topological fixed point method does not give sufficient information to detect a nonconstant solution of (1.5), moreover it seems impossible to use the spectral assumption $(f4)$ within a shooting approach to derive Theorem 1.3. Therefore we use a variational approach, but this leads to several difficulties. First, the (formal) energy functional associated with (1.5) is not well defined and of class C^1 in $H^1(B)$ under the sole assumptions $(f1)$–$(f4)$. We overcome this problem by truncating the nonlinearity f and by recovering the original problem by means of a priori estimates on the solutions. Then we construct a suitable convex subset $\mathcal{C}_*$ of the cone of nonnegative, nondecreasing radial functions in $H^1(B)$ such that u_0 is the only constant solution of (1.5) in $\mathcal{C}_*$, and we show that this set is positively invariant under the corresponding gradient flow. Then we set up a variational principle of mountain pass type within $\mathcal{C}_*$, and – using assumption $(f4)$ – we show that the corresponding critical point is different from u_0. Within this last step, a further problem occurs; the set $\mathcal{C}_*$ has empty interior in the H^1-topology, and even though one could prove that $\mathcal{C}_*\cap X$ has interior points in the topology of the smaller space $X=\mathcal{C}^2(\overline{B})\subset H^1(\Omega)$, the constant solution u_0 is still a boundary point of $\mathcal{C}_*\cap X$. Therefore it does not seem possible to use standard Morse theory (i.e. the calculation of critical groups) to distinguish critical points obtained via deformations in $\mathcal{C}_*$ from the constant solution u_0. In particular, this prevents us from using the techniques in [17], where the authors prove an abstract mountain pass theorem in order intervals.

The paper is organized as follows. In Section 2 we introduce the cone of radial, nonnegative, nondecreasing functions and its properties. In Section 3 we obtain a priori estimates on the solutions of (1.1) in the cone, which allows to prove Theorem 1.1 by applying a suitable fixed point theorem in the cone. In Section 4 we fix $a(|x|) = 1$ and provide the proof of Theorem 1.3.

We close the introduction with an open problem. Our construction of the nonconstant solution u of (1.5) provided in Theorem 1.3 implies that u intersects the constant solution u_0. This raises the question whether it is possible to construct radial solutions with a given number of intersections with u_0 provided that $f'(u_0)$ is sufficiently large. More precisely, we conjecture that there exists a radial solution with k intersections with u_0 provided that $f'(u_0) > \lambda_k^{rad}$.

2. The cone of nonnegative, nondecreasing, radial functions

We will look for solutions to (1.1) and (1.4) in the space of radial H^1 functions in the ball, that we denote by $H^1_{rad}(B)$. If $u \in H^1_{rad}(B)$ then we can assume it is continuous in $(0, 1]$ and the following set is well defined

$$\mathcal{C} = \big\{u \in H^1_{rad}(B):\ u \geqslant 0 \text{ and } u(r) \leqslant u(s) \text{ for every } 0 < r \leqslant s \leqslant 1\big\}.$$

Observe that if $u \in \mathcal{C}$, then $u \in C(\overline{B})$, and in particular it is a bounded function. In fact, since u is nondecreasing, we can assume continuity also at the origin by setting $u(0) = \lim_{r\to 0^+} u(r)$. Moreover, u is differentiable almost everywhere and $u'(r) \geqslant 0$ where it is defined.

It is easy to see that $\mathcal{C}$ is a closed convex cone in $H^1(B)$, that is

(i) if $u \in \mathcal{C}$ and $\lambda > 0$ then $\lambda u \in \mathcal{C}$;
(ii) if $u, v \in \mathcal{C}$ then $u + v \in \mathcal{C}$;
(iii) if $u, -u \in \mathcal{C}$ then $u \equiv 0$;
(iv) $\mathcal{C}$ is closed for the topology of H^1.

We will refer to $\mathcal{C}$ as the cone of nonnegative, nondecreasing functions. Notice also that it is weakly closed in H^1 and as already mentioned, it has empty interior in the H^1-topology.

As observed by Serra and Tilli in [16], $\mathcal{C}$ is a good set when dealing with supercritical equations because of the a priori bound stated in the following lemma.

Lemma 2.1. *There exists a constant C only depending on the dimension N such that*

$$\|u\|_{L^\infty(B)} \leqslant C\|u\|_{W^{1,1}(B)} \quad \text{for all } u \in \mathcal{C}.$$

Proof. For every $u \in \mathcal{C}$ we have $\|u\|_{L^\infty(B)} = \|u\|_{L^\infty(B\setminus B_{1/2})}$. Since u is radial and the space $W^{1,1}((1/2, 1))$ is continuously embedded in $L^\infty((1/2, 1))$, we deduce that there exists $C > 0$, only depending on the dimension N, such that

$$\|u\|_{L^\infty(B)} = \|u\|_{L^\infty(B\setminus B_{1/2})} \leqslant C\|u\|_{W^{1,1}(B\setminus B_{1/2})} \leqslant C\|u\|_{W^{1,1}(B)}. \qquad \square$$

Remark 2.2. Lemma 2.1 implies that the embedding $\mathcal{C} \subset L^\infty(\Omega)$ is bounded when $\mathcal{C}$ is considered with the metric induced by the $H^1(B)$-norm. However, this embedding is not continuous if $N \geqslant 3$, since the sequence $(u_n)_n \subset \mathcal{C}$ defined by $u_n(x) = |x|^{1/n}$ satisfies $\|u_n - 1\|_{H^1(B)} \to 0$ as $n \to +\infty$ and $\|u_n - 1\|_{L^\infty} \geqslant 1$ for all n. Nevertheless we have the following continuity property.

Lemma 2.3. *Let $g : [0, \infty) \to \mathbb{R}$ be continuous, and let $(u_n)_n \subset \mathcal{C}$ be a sequence with $u_n \rightharpoonup u$ weakly in $H^1(B)$. Then for every $p \in [1, \infty)$ we have*

$$g \circ u_n \to g \circ u \quad \text{in } L^p(B) \text{ as } n \to \infty.$$

Proof. Let $p \in [1, \infty)$. Suppose by contradiction that – passing to a subsequence – we have

$$\liminf_{n\to\infty} \int_B \big|g(u_n) - g(u)\big|^p\, dx > 0. \tag{2.6}$$

Since $u_n \to u$ in $L^2(B)$, we may pass to a subsequence such that $u_n \to u$ a.e. in B. Moreover, by Lemma 2.1 we have $u \in L^\infty(B)$ and $\sup_{n\in\mathbb{N}} \|u_n\|_{L^\infty(B)} < \infty$, hence also

$$\sup_{n\in\mathbb{N}} \big\| g(u_n) - g(u) \big\|_{L^\infty(B)} < \infty.$$

We now infer from Lebesgue's theorem that

$$\lim_{n\to\infty} \int_B \big| g(u_n) - g(u) \big|^p \, dx = 0$$

but this contradicts (2.6). The claim follows. □

3. Existence of solutions via a topological method

In this section we will prove Theorem 1.2, and we note that Theorem 1.1 immediately follows from Theorem 1.2. Throughout this section we assume conditions (a), (b) and $(f1)$–$(f3)$. We first recall well-known properties of the linear differential operator $\mathcal{L} := -\Delta + b(|x|)x \cdot \nabla + Id$.

Lemma 3.1. *Let*

$$\mathcal{H}(B) := \big\{ v \in H^2(B) \colon \partial_r v \in H_0^1(B) \big\},$$

where ∂_r denotes the derivative in direction $x/|x|$. For every $w \in L^2(B)$, the equation $\mathcal{L}v = w$ admits a unique solution $v \in \mathcal{H}(B)$, and $\|v\|_{H^2(B)} \leqslant C\|w\|_{L^2(B)}$ with a constant $C > 0$ independent of w. Moreover, if $w \in L^p(B)$ for some $p \in (2,\infty)$, then $v \in W^{2,p}(B)$. Also, if $w \in H^1(B)$, then $v \in H^3(B)$.

Proof. The assertions are true by standard elliptic regularity if $b \equiv 0$. Moreover, since the first order term in $\mathcal{L}$ defines a compact perturbation, $\mathcal{L}$ is a Fredholm operator of index zero when considered as a map between the spaces $\mathcal{H}(B) \to L^2(B)$, $\mathcal{H}(B) \cap W^{2,p}(B) \to L^p(B)$ and $\mathcal{H}(B) \cap H^3(B) \to H^1(B)$, respectively. Therefore it remains to prove the following:

$$\text{the equation } \mathcal{L}v = 0 \text{ only admits the trivial solution in } \mathcal{H}(B). \tag{3.7}$$

To prove this, let $v \in \mathcal{H}(B)$ solve $\mathcal{L}v = 0$, i.e. $-\Delta v + v = b(|x|)x \cdot \nabla v$. Since the map $x \mapsto b(|x|)$ is Lipschitz in B as a consequence of assumption (b), it follows from standard elliptic regularity that $v \in C^{2,\alpha}(\overline{B})$ for some $\alpha > 0$. Moreover, by the strong maximum principle, v neither may attain a positive maximum nor a negative minimum in B. Since moreover $\partial_r v = 0$ on ∂B, the Hopf Lemma implies that v cannot attain a positive maximum nor a negative minimum on ∂B. Therefore $v \equiv 0$, as claimed in (3.7). □

We will prove Theorem 1.2 by applying a suitable fixed point theorem to the operator $T : \mathcal{C} \to H^1(B)$ defined as

$$T(u) = v \quad \text{with} \quad \begin{cases} -\Delta v + b\big(|x|\big)x \cdot \nabla v + v = a\big(|x|\big) f(u) & \text{in } B, \\ \partial_\nu v = 0 & \text{on } \partial B. \end{cases} \tag{3.8}$$

Notice that the function $x \mapsto a(|x|) f(u(x))$ is contained in $\mathcal{C}$ whenever $u \in \mathcal{C}$, since $u \in L^\infty(B)$ by Lemma 2.1. The first step is of course to prove that $T(\mathcal{C}) \subseteq \mathcal{C}$.

Lemma 3.2. *Let $w \in \mathcal{C}$; then the equation*

$$\begin{cases} -\Delta v + b\big(|x|\big)x \cdot \nabla v + v = w & \text{in } B, \\ \partial_\nu v = 0 & \text{on } \partial B, \end{cases}$$

admits a unique solution $v = T(w)$, which belongs to $\mathcal{C}$.

Proof. Since $w \in \mathcal{C} \subset H^1(B) \cap L^\infty(B)$, it follows from Lemma 3.1 that there exists a unique solution v in $\mathcal{H}(B) \cap H^3(B) \cap W^{2,p}(B)$ (for every $p < \infty$). Hence $v \in C^{1,\alpha}(\overline{B})$ and $\partial_\nu v = 0$ on ∂B. Since the solution is radial (because it is unique), we may write the equation for v in polar coordinates as

$$-v'' + \left(b(r)r - \frac{N-1}{r}\right)v' + v = w, \qquad v'(0) = v'(1) = 0,$$

where v' denotes the derivative with respect to $r = |x|$. Note that, as a function of r, we have $z := v' \in H^2_{loc}(0,1)$, so differentiation yields

$$\left(b(r)r - \frac{N-1}{r}\right)z' + \left(\big[b(r)r\big]' + \frac{N-1}{r^2} + 1\right)z = z'' + w'.$$

We point out that the left-hand side of this equation is continuous in $(0,1)$ (since $H^2_{loc}(0,1) \subset C^1(0,1)$), and thus the continuity of the right-hand side follows. Now suppose by contradiction that z attains a negative local minimum at a point $r_0 \in (0,1)$, then at this point we have $z'(r_0) = 0$ and

$$\left(\big[b(r)r\big]' + \frac{N-1}{r^2} + 1\right)z\bigg|_{r_0} < 0$$

by assumption (b). Therefore, by continuity, there exists a neighborhood U of r_0 in $(0,1)$ with

$$\left(b(r)r - \frac{N-1}{r}\right)z' + \left(\big[b(r)r\big]' + \frac{N-1}{r^2} + 1\right)z < 0 \quad \text{in } U.$$

Since $w' \geqslant 0$ in $(0,1)$, it then follows that $z'' < 0$ a.e. in U, which yields that z' is strictly decreasing in U. This however contradicts our assumption that z attains a negative minimum at r_0. Since moreover $z(0) = z(1) = 0$, we conclude that $v' = z \geqslant 0$ in $(0,1)$, so that $v \in \mathcal{C}$. □

Corollary 3.3. *The operator T defined by (3.8) satisfies $T(\mathcal{C}) \subseteq \mathcal{C}$.*

Proof. Observe that if $u \in \mathcal{C}$, the assumptions on $a(r)$ and f imply that $a(r)f(u) \in \mathcal{C}$. Henceforth, the conclusion follows from Lemma 3.2. □

In order to apply a fixed point theorem in the cone, we need a priori estimates on the solutions of (1.1) and on the solutions of a family of auxiliary problems depending on some parameters $\lambda \geqslant 0$ and $0 < \mu \leqslant 1$.

Lemma 3.4. *There exists a constant $\bar{\lambda}$ such that the following problem*

$$\begin{cases} -\Delta u + b(r)x \cdot \nabla u + u = a(r)f(u) + \lambda & \text{in } B, \\ u \geqslant 0 & \text{in } B, \\ \partial_\nu u = 0 & \text{on } \partial B, \end{cases} \tag{3.9}$$

does not admit any solution in $\mathcal{C}$ for $\lambda > \bar{\lambda}$. Moreover, there exists a constant K_1 such that every solution u of (3.9) with $0 \leqslant \lambda \leqslant \bar{\lambda}$ satisfies $\|u\|_{L^1(B)} \leqslant K_1$.

Proof. By assumption $(f3)$ there exist $M, \delta > 0$ such that

$$\frac{f(s)}{s} \geqslant \frac{1+\delta}{a_0} \quad \text{for every } s \geqslant M, \tag{3.10}$$

where $a_0 = a(0)$. Let $u \in \mathcal{C}$ be a solution of (3.9). Since $b(r)x \cdot \nabla u(x) \leqslant 0$ by assumption (b), integrating the equation in (3.9) in B yields

$$\begin{aligned} \int_B u\,dx &\geqslant \int_B \big[u + b(r)x \cdot \nabla u(x)\big]\,dx = \int_{\{u \leqslant M\}} a(r)f(u)\,dx + \int_{\{u > M\}} a(r)f(u)\,dx + \lambda|B| \\ &\geqslant \int_{\{u>M\}} a(r)\frac{1+\delta}{a_0}u\,dx + \lambda|B| \geqslant (1+\delta)\int_{\{u>M\}} u\,dx + \lambda|B|. \end{aligned}$$

Therefore

$$M|B| \geqslant \int\limits_{\{u\leqslant M\}} u\,dx \geqslant \delta \int\limits_{\{u>M\}} u\,dx + \lambda|B|$$

and the lemma is proved. □

From now on, we fix $\bar{\lambda}$ as in the previous lemma.

Lemma 3.5. *Assume $0 \leqslant \lambda \leqslant \bar{\lambda}$. There exist two constants K_∞, K_2 such that if $u \in \mathcal{C}$ solves (3.9), then*

$$\|u\|_{L^\infty(B)} \leqslant K_\infty \quad \text{and} \quad \|u\|_{H^1(B)} \leqslant K_2.$$

Proof. Let $u \in \mathcal{C}$ be a solution of (3.9). In radial coordinates, the equation for u can be written in the form

$$\big(r^{N-1}u'\big)' = r^{N-1}\big(u(r) + b(r)ru'(r) - a(r)f\big(u(r)\big) - \lambda\big) \leqslant r^{N-1}u(r).$$

Therefore

$$u'(r) \leqslant \frac{1}{r^{N-1}} \int\limits_0^r u(t)t^{N-1}\,dt \leqslant \frac{1}{r^{N-1}|\partial B|} \int\limits_B u\,dx \leqslant \frac{K_1}{r^{N-1}|\partial B|},$$

with K_1 defined in the previous lemma. Since $u' \geqslant 0$, we deduce from the previous inequality that $\|u\|_{W^{1,1}(B)} \leqslant 2K_1$, so that Lemma 2.1 gives the first estimate. As for the estimate of the H^1-norm, we multiply the equation in (3.9) by u and integrating in the ball yields

$$\int\limits_B \big(|\nabla u|^2 + u^2\big)\,dx = \int\limits_B \big[a(r)f(u)u - b(r)x\cdot\nabla u\big]u\,dx + \bar{\lambda}\int\limits_B u\,dx.$$

Since u is a priori bounded in $W^{1,1}(B)$ and $L^\infty(B)$, the right-hand side is a priori bounded as well, and the a priori bound in $H^1(B)$ follows. □

Remark 3.6. An inspection of the proofs of Lemmas 3.4 and 3.5 shows the following. First, it is possible to choose

$$\bar{\lambda} := \min\big\{s \geqslant 0:\ f(t) \geqslant t \text{ for } t \geqslant s\big\}$$

in Lemma 3.4. Moreover, the a priori bounds in these lemmas only depend on some properties of f and not on the nonlinearity itself. More precisely, if $M > 0$ and $\delta > 0$ are fixed, then K_1, K_2 and K_∞ can be chosen independently for all nonnegative nonlinearities f satisfying (3.10). This will be important in Section 4 where we work with a truncated problem.

Lemma 3.7. *There exists a constant k_2 such that for every $0 < \mu < 1$ and for every solution $u \not\equiv 0$ of*

$$\begin{cases} -\Delta u + b(r)x\cdot\nabla u + u = \mu a(r)f(u) & \text{in } B,\\ u \geqslant 0 & \text{in } B,\\ \partial_\nu u = 0 & \text{on } \partial B, \end{cases} \tag{3.11}$$

we have $\|u\|_{H^1(B)} \geqslant k_2$.

Proof. By Lemma 3.1, there exists a constant $C > 0$ such that

$$\big\|-\Delta u + b(r)x\cdot\nabla u + u\big\|_{L^2(B)} \geqslant C\|u\|_{L^2(B)} \quad \text{for all } u \in \mathcal{H}(B). \tag{3.12}$$

Assume by contradiction the existence of $u_n \not\equiv 0$, solutions of (3.9) with $0 < \mu_n < 1$, such that $\|u_n\|_{H^1(B)} \to 0$ as $n \to +\infty$. Then $\|u_n\|_{L^\infty(B)} \to 0$ by Lemma 2.1. By assumption $(f1)$ we have

$$\frac{f(u_n(x))}{u_n(x)} \leqslant \frac{1}{n} \quad \text{for all } x \in B,$$

for n sufficiently large, and it then follows from (3.12) that

$$C^2\|u_n\|^2_{L^2(B)} \leqslant \mu_n^2 \int_B \big[a(r) f(u_n)\big]^2 \leqslant \left(\frac{\mu_n a(1)}{n}\right)^2 \int_B u_n^2\,dx = \left(\frac{\mu_n a(1)}{n}\right)^2 \|u_n\|^2_{L^2(B)}.$$

Since $u_n \not\equiv 0$ for every n, this yields a contradiction for n large. □

We now turn to the proof of Theorem 1.1. We are in a position to apply a generalization of a fixed point theorem by Krasnosel'skiĭ (see [10,11]) to the operator T defined by (3.8) in the cone $\mathcal{C}$. This theorem is proved by Benjamin in [4, Appendix 1], but we refer to Kwong [12] where the approach is more elementary. We also quote [3] and [13].

Proof of Theorem 1.1. Let us check the assumptions of the fixed point theorem in [12] (expansive form):

(i) $T : \mathcal{C} \to \mathcal{C}$ by Corollary 3.3;
(ii) T is completely continuous on $\mathcal{C}$. Indeed let $\{u_n\} \subset \mathcal{C}$ be a sequence bounded in $H^1(B)$. By Lemma 2.1 it is bounded in $L^\infty(B)$, hence $\{v_n = T(u_n)\}$ is bounded in $H^2(B)$ by Lemma 3.1. Therefore, by the compactness of the embedding $H^2(B) \hookrightarrow H^1(B)$, a subsequence of $(v_n)_n$ converges in the H^1-norm;
(iii) For every $\lambda \geqslant 0$, for every $u \in \mathcal{C}$ with $\|u\|_{H^1(B)} = 2K_2$ (defined in Lemma 3.5) we have $u - T(u) \neq \lambda$. In fact notice that $u - T(u) = \lambda$ if and only if u solves Eq. (3.9), hence this property is a consequence of Lemma 3.5;
(iv) For every $0 < \mu < 1$, for every $u \in \mathcal{C}$ with $\|u\|_{H^1(B)} = k_2/2$ (defined in Lemma 3.7) we have $\mu T(u) \neq u$. In fact we have $\mu T(u) = u$ if and only if u solves Eq. (3.11), hence property (iv) is a consequence of Lemma 3.7.

We then conclude that there exists a fixed point of T in $\mathcal{C}$. Such a fixed point is of course a nonconstant solution of (1.1) since a is nonconstant. Moreover it is strictly positive and strictly increasing by the maximum principle. This completes the proof. □

4. Existence of solutions via a variational method

In the case where a is a constant function, say $a \equiv 1$, the following proposition and remark show that (1.5) may only admit the constant solution $u \equiv u_0$ in $H^1_{rad}(B)$. The argument is adapted from [6] where it is shown that if $f(u) = u^p$ and p is close to 1, $u_0 \equiv 1$ is the unique solution of (1.5).

Recall that $\lambda_2^{rad} > 1$ is the second radial eigenvalue of $-\Delta + 1$ in the unit ball with Neumann boundary conditions. Fix $\delta \in (0, \lambda_2^{rad})$ and let $M > 0$. By Lemma 3.5 and Remark 3.6, there exists $K_\infty > 0$ such that, if f satisfies $(f1)$–$(f3)$ and (3.10) with these values of M, δ and $a_0 \equiv 1$, then every solution $u \in \mathcal{C}$ of (1.5) satisfies $\|u\|_{L^\infty(B)} \leqslant K_\infty$.

Proposition 4.1. *Let $\delta \in (0, \lambda_2^{rad})$ and $M > 0$. Assume f satisfies $(f1)$–$(f3)$ and (3.10) with $a_0 = 1$. If $f'(s) < \lambda_2^{rad}$ for every $s \in [0, K_\infty]$, then (1.5) only admits constant solutions in $H^1_{rad}(B)$.*

Proof. Let $u \in H^1_{rad}(B)$ be a solution of (1.5). We can write $u = v + \lambda$ for some $\lambda \in \mathbb{R}$ and $v \in H^1_{rad}(B)$ satisfying

$$\int_B v\,dx = 0 \quad \text{and} \quad \lambda_2^{rad} \int_B v^2\,dx \leqslant \int_B \big(|\nabla v|^2 + v^2\big)\,dx.$$

Multiplying (1.5) by v and integrating by parts, we obtain

$$\begin{aligned}\lambda_2^{rad} \int_B v^2\,dx &\leqslant \int_B \big(|\nabla v|^2 + v^2\big)\,dx = \int_B f(v+\lambda)v\,dx \\ &= \int_B \big[f(v+\lambda) - f(\lambda)\big]v\,dx = \int_B f'(\lambda + cv)v^2\,dx,\end{aligned}$$

with some function $c = c(x)$ satisfying $0 \leqslant c \leqslant 1$ in B. Now, since $\|u\|_{L^\infty(B)} \leqslant K_\infty$, we also have $\|\lambda + cv\|_{L^\infty(B)} \leqslant K_\infty$, hence $f'(\lambda + cv) < \lambda_2^{rad}$ by assumption. This yields $v = 0$. □

Remark 4.2. If, in addition to the assumptions of Proposition 4.1, f only has one positive fixed point, then this fixed point is the only radial solution of (1.5). This is true e.g. if f is given as $f(u) = g(u)u$ with a strictly increasing C^1-function $g : [0, \infty) \to \mathbb{R}$ satisfying $g(0) = 0$ and $\lim_{t\to\infty} g(t) \in (1, \lambda_2^{rad})$.

In the remainder of this section we will prove Theorem 1.3. For this reason in the following we will assume that $a(r) \equiv 1$ and we always assume $(f1)$–$(f4)$ (with $a_0 = 1$). As we already mentioned in the introduction, we shall find a solution of (1.5) by a minimax technique. This will allow us to prove that it is nonconstant through an energy comparison. The first step is to consider a truncated problem which can be cast into a variational setting in $H^1(B)$. We will then recover the original problem through the a priori bounds on the solutions proved in the previous section.

Lemma 4.3. *There exist $p > 1$ satisfying $p < \frac{N+2}{N-2}$ if $N \geqslant 3$ and a function $\tilde{f}$ satisfying $(f1)$–$(f4)$ and*

$$\lim_{s\to\infty} \frac{\tilde{f}(s)}{s^p} = 1, \tag{4.13}$$

such that if $u \in \mathcal{C}$ solves $-\Delta u + u = \tilde{f}(u)$ in B with $\partial_\nu u = 0$ on ∂B, then u solves (1.5).

Proof. Fix $M, \delta > 0$ such that (3.10) holds for f with $a_0 = 1$, i.e.

$$f(s) \geqslant (1 + \delta)s \quad \text{for } s \geqslant M. \tag{4.14}$$

By Remark 3.6, there exists $K_\infty > 0$ such that, for any nonnegative nonlinearity $\tilde{f} : [0, \infty) \to \mathbb{R}$ satisfying $\tilde{f}(s) \geqslant (1 + \delta)s$ for $s \geqslant M$ and any solution $u \in \mathcal{C}$ of the problem

$$-\Delta u + u = \tilde{f}(u) \quad \text{in } B, \qquad \partial_\nu u = 0 \quad \text{on } \partial B \tag{4.15}$$

we have $\|u\|_{L^\infty(B)} \leqslant K_\infty$. Now fix $s_0 > \max\{K_\infty, M\}$, and fix $p > 1$ with $p < \frac{N+2}{N-2}$ if $N \geqslant 3$. To define the truncated function $\tilde{f} \in C^1([0, \infty))$ we distinguish the following cases.

Case 1: $f(s_0) = (1 + \delta)s_0$. Then it follows from (4.14) that $f(s)$ touches the line $(1 + \delta)s$ from above at s_0, so that the two curves are tangent at s_0. Therefore $f'(s_0) = 1 + \delta$ and we set

$$\tilde{f}(s) = \begin{cases} f(s) & \text{for } 0 \leqslant s \leqslant s_0; \\ f(s_0) + f'(s_0)(s - s_0) + (s - s_0)^p & \text{for } s > s_0. \end{cases}$$

Then $\tilde{f} \in C^1([0, \infty))$ satisfies (4.13), and it also satisfies (4.14), so that every solution of (4.15) is also a solution of (1.5) by the choice of K_∞ and s_0.

Case 2: $f(s_0) > (1 + \delta)s_0$. Then we may first modify f in a right neighborhood $(s_0, s_0 + \varepsilon)$ of s_0, in such a way that $f(s) \geqslant (1 + \delta)s$ for $s \leqslant s_0 + \varepsilon$ and $f'(s_0 + \varepsilon) = 1 + \delta$. Then we define $\tilde{f}$ as in Case 1 with s_0 replaced by $s_0 + \varepsilon$. □

In the following, we may also assume that $\tilde{f}$ is defined on the whole real line by setting $\tilde{f} \equiv 0$ on $(-\infty, 0]$. It then follows by standard arguments from the subcritical growth assumption (4.13) that the functional $I : H^1_{rad}(B) \to \mathbb{R}$ defined by

$$u \mapsto I(u) = \int_B \left(\frac{|\nabla u|^2 + u^2}{2} - \tilde{F}(u) \right) dx,$$

where $\tilde{F}(s) := \int_0^s \tilde{f}(t)\, dt$ is well defined and of class C^2 in $H^1(B)$. Moreover, critical points of I are radial solutions of (1.5). We look for critical points of I by applying a mountain pass type argument in a suitable subset of $\mathcal{C}$, which is based on invariance properties of the corresponding flow.

Since the truncated nonlinearity $\tilde{f}$ has a subcritical growth at infinity, the Palais–Smale condition holds. We include a proof for completeness though this is a standard fact.

Lemma 4.4. *The action functional I satisfies the Palais–Smale condition.*

Proof. Let $(u_n)_n \subset H^1_{rad}(B)$ be a sequence with $I'(u_n) \to 0$ and such that $I(u_n)$ remains bounded. It easily follows from (4.13) there exist $R_0 > 0$ and $\mu \in (2, p+1)$ such that $\tilde{f}(s)s \geqslant \mu \tilde{F}(s)$ for $s \geqslant R_0$. Hence we have

$$I(u_n) - \frac{1}{\mu} I'(u_n)u_n \geqslant \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2_{H^1(B)} + \int\limits_{\{u_n \leqslant R_0\}} \left(\frac{\tilde{f}(u_n)u_n}{\mu} - \tilde{F}(u_n)\right) dx.$$

Since $\mu > 2$, the H^1-norm of the sequence $\{u_n\}$ is bounded, hence $u_n \rightharpoonup u$ weakly in $H^1(B)$ after passing to a subsequence, where u also is a critical point of I. Using the subcritical growth of f given by (4.13) and the compact embedding $H^1(B) \hookrightarrow L^p(B)$, it is then easy to see that $\tilde{f}(u_n) \to \tilde{f}(u)$ strongly in the dual space $[H^1(B)]'$ of $H^1(B)$, and therefore – regarding $-\Delta + Id$ as an isomorphism $H^1(B) \to [H^1(B)]'$ – we have

$$u_n = [-\Delta + Id]^{-1}\tilde{f}(u_n) \to [-\Delta + Id]^{-1}\tilde{f}(u) = u \quad \text{in } H^1(B),$$

as required. □

By assumption $(f4)$, we may now fix $u_0 \in (0, \infty)$ with $f(u_0) = u_0$ and $f'(u_0) > \lambda_2^{rad}$. Moreover, since $u_0 < K_\infty$, it follows from the proof of Lemma 4.3 that $\tilde{f}(u_0) = f(u_0) = u_0$ and $\tilde{f}'(u_0) = f'(u_0) > \lambda_2^{rad}$. Since $\lambda_2^{rad} > 1$, u_0 is an isolated fixed point of $\tilde{f}$, so we can define

$$u_- := \sup\{t \in [0, u_0):\ \tilde{f}(t) = t\}$$

and

$$u_+ := \inf\{t > u_0:\ \tilde{f}(t) = t\}.$$

We point out that $u_+ = \infty$ is possible. Next, we define the convex set

$$\mathcal{C}_* := \{u \in \mathcal{C}:\ u_- \leqslant u \leqslant u_+ \text{ a.e. in } B\}.$$

Clearly, $\mathcal{C}_*$ is closed and convex. Moreover we have

Lemma 4.5. *Fix $c \in \mathbb{R}$ and assume that there exist $\varepsilon, \delta > 0$ such that $\|\nabla I(u)\|_{H^1(B)} \geqslant \delta$ for every $u \in \mathcal{C}_*$ with $|I(u) - c| \leqslant 2\varepsilon$. Then there exists $\eta : \mathcal{C}_* \to \mathcal{C}_*$ continuous with respect to the H^1-topology which satisfies the following properties*

(i) $I(\eta(u)) \leqslant I(u)$ *for every* $u \in \mathcal{C}_*$;
(ii) $I(\eta(u)) \leqslant c - \varepsilon$ *if* $|I(u) - c| < \varepsilon$;
(iii) $\eta(u) = u$ *if* $|I(u) - c| > 2\varepsilon$.

Proof. We first show that the operator T defined in (3.8) – with $a(r) \equiv 1$, $b(r) \equiv 0$ and $\tilde{f}$ in place of f – satisfies

$$T(\mathcal{C}_*) \subset \mathcal{C}_*. \tag{4.16}$$

Let $w \in \mathcal{C}_*$ and denote by $v \in H^1(B)$ the unique solution of

$$\begin{cases} -\Delta v + v = \tilde{f}(w) & \text{in } B, \\ \partial_\nu v = 0 & \text{on } \partial B. \end{cases}$$

Then $v \in \mathcal{C}$ by Lemma 3.2, so we only have to prove that $u_- \leqslant v \leqslant u_+$ a.e. in B. Note that $h = v - u_-$ satisfies

$$-\Delta h + h = \tilde{f}(w) - u_- \geqslant 0 \quad \text{in } B \quad \text{and} \quad \partial_\nu h = 0 \quad \text{on } \partial B.$$

Here we used the fact that $\tilde{f}$ is nondecreasing and $\tilde{f}(u_-) = u_-$. Multiplying this equation with h^- and integrating by parts, we obtain $\|h^-\|^2_{H^1} \leqslant 0$ and therefore $h^- \equiv 0$, i.e. $v \geqslant u_-$ a.e. in B. Very similarly, if $u_+ < \infty$, we show that $v \leqslant u_+$ a.e. in B. Hence we conclude that $v \in \mathcal{C}_*$ and (4.16) follows.

Next, we take a smooth cut-off function $\chi : \mathbb{R} \to [0, 1]$ such that $\chi(s) = 1$ if $|s - c| < \varepsilon$ and $\chi(s) = 0$ if $|s - c| > 2\varepsilon$. For $u \in H^1(B)$ consider the following Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\eta(t,u) = -\chi\big(I\big(\eta(t,u)\big)\big)\dfrac{\nabla I(\eta(t,u))}{\|\nabla I(\eta(t,u))\|_{H^1(B)}} & t>0, \\ \partial_\nu \eta(t,u)(x) = 0 & t>0,\ x\in\partial B, \\ \eta(0,u) = u. \end{cases} \tag{4.17}$$

Since $I\in C^2(H^1(B),\mathbb{R})$, the normalized gradient vector field appearing in (4.17) is locally Lipschitz continuous and globally bounded, hence there exists a unique solution $\eta(\cdot,u)\in C^1([0,+\infty),H^1(B))$. We set

$$\eta(u) := \eta\left(\frac{2\varepsilon}{\delta},u\right). \tag{4.18}$$

Properties (i), (ii) and (iii) are standard, so it remains to prove that $\eta(\mathcal{C}_*)\subset\mathcal{C}_*$. To this aim we consider the approximation of the flow line $t\mapsto\eta(t,u)$ given by the Euler polygonal. The first segment of the polygonal is given by the expression

$$\bar\eta(t,u) = u - \frac{t}{\lambda}\nabla I(u) = u - \frac{t}{\lambda}\big(u - T(u)\big), \quad t\in(0,1),$$

where $\lambda = \frac{\chi(I(\eta(t,u)))}{\|\nabla I(\eta(t,u))\|_{H^1(B)}}$ and T is the operator defined in (3.8) (with $a(r)=1$). By writing

$$\bar\eta(t,u) = \left(1-\frac{t}{\lambda}\right)u + \frac{t}{\lambda}T(u), \quad t\in(0,1),$$

we see that it is contained in $\mathcal{C}_*$ by (4.16) and the convexity $\mathcal{C}_*$. Finally, since the vector field in (4.17) is locally Lipschitz, the Euler polygonals are known to converge in $H^1(B)$ to the flow line $t\mapsto\eta(t,u)$, which therefore must be contained in $\mathcal{C}_*$. □

Lemma 4.6. *Let $\tau>0$ be such that $\tau<\min\{u_0-u_-,u_+-u_0\}$. Then there exists $\alpha>0$ such that*

(i) *$I(u)\geqslant I(u_-)+\alpha$ for every $u\in\mathcal{C}_*$ with $\|u-u_-\|_{L^\infty(B)}=\tau$;*
(ii) *if $u_+<\infty$, then $I(u)\geqslant I(u_+)+\alpha$ for every $u\in\mathcal{C}_*$ with $\|u-u_+\|_{L^\infty(B)}=\tau$.*

Proof. Suppose by contradiction that there exists a sequence $(w_n)_n\subset\mathcal{C}$ of increasing nonnegative functions such that $\|w_n\|_{L^\infty(B)} = w_n(1) = \tau$ for all n and $\limsup_{n\to\infty}[I(u_-+w_n)-I(u_-)]\leqslant 0$. Since

$$\begin{aligned} I(u_-+w_n)-I(u_-) &= \frac12\int_B\big(|\nabla w_n|^2+|w_n|^2+2u_-w_n\big)\,dx - \int_B\big(\tilde F(u_-+w_n)-\tilde F(u_-)\big)\,dx \\ &= \frac12\int_B|\nabla w_n|^2\,dx + \int_B\int_0^1\big(u_-+tw_n-\tilde f(u_-+tw_n)\big)w_n\,dt\,dx \end{aligned}$$

and

$$s-\tilde f(s)>0 \quad \text{for } s\in(u_-,u_0), \tag{4.19}$$

we then conclude that $\|\nabla w_n\|_{L^2(B)}\to 0$ as $n\to\infty$. Hence the sequence w_n converges to the constant solution $w\equiv\tau$ in the H^1-norm. By Lemma 2.3 we therefore conclude that

$$\begin{aligned} 0\geqslant \lim_{n\to\infty}\big[I(u_-+w_n)-I(u_-)\big] &= \lim_{n\to\infty}\int_B\int_0^1\big(u_-+tw_n-\tilde f(u_-+tw_n)\big)w_n\,dx \\ &= \int_B\int_0^1\big(u_-+t\tau-\tilde f(u_-+t\tau)\big)\tau\,dt\,dx. \end{aligned}$$

This however contradicts (4.19). Hence there exists $\alpha_1>0$ such that (i) holds.

In a similar way, now using the fact that $s - \tilde{f}(s) < 0$ for $s \in (u_0, u_+)$, we find $\alpha_2 > 0$ such that (ii) holds if $u_+ < \infty$. The claim then follows with $\alpha := \min\{\alpha_1, \alpha_2\}$. □

In the following, we first consider the case

$$u_+ < \infty.$$

Moreover, we fix τ and α as in Lemma 4.6, and we define

$$U_\pm := \left\{ u \in \mathcal{C}_* \colon\ I(u) < I(u_\pm) + \frac{\alpha}{2},\ \|u - u_\pm\|_{L^\infty(B)} < \tau \right\}. \tag{4.20}$$

Then we have:

Proposition 4.7. *Let*

$$\Gamma = \left\{ \gamma \in C\big([0,1], \mathcal{C}_*\big) \colon\ \gamma(0) \in U_-,\ \gamma(1) \in U_+ \right\}$$

and

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I\big(\gamma(t)\big).$$

Then $c \geqslant \max\{I(u_-), I(u_+)\} + \alpha$ and c is a critical level for I. More precisely, there exists a critical point $u \in \mathcal{C}_*$ of I with $I(u) = c$.

Proof. It follows immediately from Lemma 4.6 that $c > \max\{I(u_-), I(u_+)\} + \alpha$. Moreover, Γ is nonempty, since the path of constant functions

$$t \mapsto (1-t)u_- + t u_+ \tag{4.21}$$

is contained in Γ. Consequently, $c < \infty$. Assume by contradiction that there does not exist a critical point $u \in \mathcal{C}_*$ of I with $I(u) = c$. By Lemma 4.4, this implies the existence of $\varepsilon, \delta > 0$ such that $\|\nabla I(u)\|_{H^1(B)} \geqslant \delta$ for all $u \in \mathcal{C}_*$ satisfying $|I(u) - c| \leqslant 2\varepsilon$. Without loss of generality, we may assume that $4\varepsilon < \alpha$. Correspondingly, let η be the deformation defined in Lemma 4.5, and let $\gamma \in \Gamma$ be such that $\max_{t\in[0,1]} I(\gamma(t)) \leqslant c + \varepsilon$.

Defining $\bar{\gamma} : [0,1] \to \mathcal{C}_*$ by $\bar{\gamma}(t) = \eta(\gamma(t))$, we then have $\bar{\gamma}(0) = \gamma(0)$ and $\bar{\gamma}(1) = \gamma(1)$ because of Lemma 4.5(iii) and the fact that $I(u_\pm) < c - 2\varepsilon$ by our choice of ε and α. Hence $\bar{\gamma} \in \Gamma$. However, by Lemma 4.5(i) and (ii) we have

$$\max_{t\in[0,1]} I\big(\bar{\gamma}(t)\big) \leqslant c - \varepsilon,$$

contradicting the definition of c. The claim then follows. □

In order to show that the critical value c in Proposition 4.7 does not yield a constant solution of (1.5), it suffices to show that $c < I(u_0)$. To show this, we will now make use of the assumption $\tilde{f}'(u_0) > \lambda_2^{rad}$. The strategy is to find a curve $\gamma \in \Gamma$ such that $\max_{t\in[0,1]} I(\gamma(t)) < I(u_0)$. This is achieved by suitably perturbing the constant path defined in (4.21) around u_0, moving in the direction of the eigenfunction associated to λ_2^{rad}. We will need a series of lemmas. Let us start with some simple properties of the eigenfunction associated to λ_2^{rad}.

Lemma 4.8. *Let v be an eigenfunction associated to λ_2^{rad}, that is*

$$\begin{cases} -\Delta v + v = \lambda_2^{rad} v & \text{in } B, \\ \partial_\nu v = 0 & \text{on } \partial B, \\ v \text{ radial}. \end{cases}$$

Then v is unique up to a multiplicative factor and we can choose it increasing. Moreover, $\int_B v\,dx = 0$.

Proof. By writing the equation for v in radial coordinates we see that it satisfies a Sturm–Liouville problem. Hence v is unique up to a multiplicative factor, it is monotone and has exactly one zero. By taking $-v$ if necessary, we can assume it is increasing. We refer to [6] for the explicit form of the eigenfunctions. By integrating the equation for v we deduce $(\lambda_2^{rad} - 1)\int_B v\,dx = 0$, and therefore $\int_B v\,dx = 0$. □

In the following v will always denote a positive eigenfunction associated to λ_2^{rad}.

Lemma 4.9. *Consider the function*

$$\psi : \mathbb{R}^2 \to \mathbb{R}, \qquad \psi(s,t) = I'\big(t(u_0 + sv)\big)(u_0 + sv).$$

There exist $\varepsilon_1, \varepsilon_2 > 0$ and a C^1-function $g : (-\varepsilon_1, \varepsilon_1) \to (1-\varepsilon_2, 1+\varepsilon_2)$ such that for $(s,t) \in U := (-\varepsilon_1, \varepsilon_1) \times (1-\varepsilon_2, 1+\varepsilon_2)$ we have $\psi(s,t) = 0$ if and only if $t = g(s)$.

Moreover:

(i) $g(0) = 1$, $g'(0) = 0$;
(ii) $I(g(s)(u_0 + sv)) < I(u_0)$ *for* $s \in (-\varepsilon_1, \varepsilon_1)$.

Proof. Since I is a C^2-functional, ψ is of class C^1 with $\psi(0,1) = 0$,

$$\frac{\partial}{\partial t}\bigg|_{(0,1)} \psi(s,t) = I''(u_0)(u_0, u_0) = \int_B \big[1 - \tilde{f}'(u_0)\big]u_0^2\,dx < \big(1 - \lambda_2^{rad}\big)|B|u_0^2 < 0$$

and

$$\frac{\partial}{\partial s}\bigg|_{(0,1)} \psi(s,t) = I'(u_0)v + I''(u_0)(u_0, v) = \big[1 - \tilde{f}'(u_0)\big]u_0 \int_B v\,dx = 0.$$

Thus the existence of $\varepsilon_1, \varepsilon_2$ and g, as well as property (i), follow from the implicit function theorem. To prove (ii), we write $g(s) = 1 + o(s)$, so that

$$g(s)(u_0 + sv) - u_0 = \big(g(s) - 1\big)u_0 + g(s)sv = sv + o(s)$$

and therefore, by Taylor expansion,

$$\begin{aligned} I\big(g(s)(u_0 + sv)\big) - I(u_0) &= \frac{1}{2}I''(u_0)\big[sv + o(s), sv + o(s)\big] + o\big(s^2\big) = \frac{s^2}{2}I''(u_0)(v,v) + o\big(s^2\big) \\ &= \frac{s^2}{2}\int_B \big(|\nabla v|^2 + v^2 - \tilde{f}'(u_0)v^2\big)\,dx + o\big(s^2\big). \end{aligned}$$

Since

$$\int_B \big(|\nabla v|^2 + v^2 - \tilde{f}'(u_0)v^2\big)\,dx < \int_B \big(|\nabla v|^2 + v^2 - \lambda_2^{rad}v^2\big)\,dx = 0,$$

property (ii) holds after making $\varepsilon_1, \varepsilon_2$ smaller if necessary. □

Lemma 4.10. *Let τ be given as in Lemma 4.6, and fix $t_-, t_+ > 0$ such that*

$$t_-u_0 \in U_-, \qquad t_+u_0 \in U_+ \quad \text{and} \quad u_- < t_-u_0 < u_0 < t_+u_0 < u_+, \tag{4.22}$$

where $U_\pm$ are defined in (4.20). For $s \geqslant 0$ define

$$\gamma_s : [t_-, t_+] \to H^1(B), \qquad \gamma_s(t) = t(u_0 + sv). \tag{4.23}$$

Then there exists $s > 0$ such that $\gamma_s(t_\pm) \in U_\pm$, $\gamma_s(t) \in \mathcal{C}_*$ for $t_- \leqslant t \leqslant t_+$ and

$$\max_{t_- \leqslant t \leqslant t_+} I\big(\gamma_s(t)\big) < I(u_0). \tag{4.24}$$

Proof. We first observe that the function $t \mapsto I(\gamma_0(t))$ has a unique maximum point at 1, since

$$\frac{d}{dt} I\big(\gamma_0(t)\big) = I'(tu_0)u_0 = |B|\big(tu_0 - \tilde{f}(tu_0)\big)u_0$$

and $tu_0 - \tilde{f}(tu_0) > 0$ in $[t_-, 1)$ while $tu_0 - \tilde{f}(tu_0) < 0$ in $(1, t_+]$. Consider the neighborhood U of $(s,t) = (0,1)$ found in Lemma 4.9. By continuity, there exists $s_0 > 0$ such that

$$I\big(\gamma_s(t)\big) < I(u_0) \quad \text{for every } (s,t) \in [-s_0, s_0] \times [t_-, t_+] \setminus U.$$

On the other hand, if (s,t) in U is such that t is the global maximum of the function γ_s, then

$$0 = \frac{d}{dt} I\big(\gamma_s(t)\big) = I'\big(t(u_0 + sv)\big)(u_0 + sv)$$

and therefore $t = g(s)$ and $I(t(u_0 + sv)) < I(u_0)$ by Lemma 4.9. Hence (4.24) follows. By (4.22) and since v is an increasing function, we may choose $s \in (0, s_0)$ so small such that

$$\gamma_s(t_-) = t_-(u_0 + sv) \in U_- \quad \text{and} \quad \gamma_s(t_+) = t_+(u_0 + sv) \in U_+.$$

By convexity, we then also have $\gamma_s(t) \in \mathcal{C}_*$ for all $t \in [t_-, t_+]$. □

End of the proof of Theorem 1.3 in the case $u_+ < \infty$. Proposition 4.7 provides in $\mathcal{C}$ a mountain pass type critical point of I which, by Lemma 4.3, is a solution of (1.5). As emphasized before, it only remains to prove that $c < I(u_0)$, which implies that the critical point found in Proposition 4.7 is not constant. To this end, we note that Lemma 4.10 implies that – after an affine transformation of the independent variable – the path γ_s defined in (4.23) belongs to Γ and satisfies $\max_t I(\gamma_s(t)) < I(u_0)$ for some $s > 0$. Hence $c < I(u_0)$, as claimed. □

Now we consider the case

$$u_+ = \infty.$$

We then fix τ and α as in Lemma 4.6(i), and we keep the definition of U_- from (4.20). In addition, we now set

$$U_+ := \big\{u \in \mathcal{C}_*\colon\ u \geqslant u_0,\ I(u) \leqslant I(u_-)\big\}. \tag{4.25}$$

Then we have

Proposition 4.11. *Let Γ and c be defined as in Lemma 4.7 (with U_+ now defined as in (4.25)). Then $c > I(u_-) + \alpha$, and there exists a critical point $u \in \mathcal{C}_*$ of I with $I(u) = c$.*

Proof. It follows from Lemma 4.6(i) that $c > I(u_-) + \alpha$. Moreover, considering again $M, \delta > 0$ such that (3.10) holds, we find that, for $t > M$,

$$\begin{aligned} I(t \cdot 1) &= |B|\left(\frac{t^2}{2} - \tilde{F}(t)\right) = |B|\left(\frac{t^2}{2} - \int_0^t \tilde{f}(s)\,ds\right) \\ &\leqslant |B|\left(\frac{t^2}{2} - \int_0^M \tilde{f}(s)\,ds - (1+\delta)\int_M^t s\,ds\right) \\ &= \frac{|B|}{2}\left(t^2 - 2\int_0^M \tilde{f}(s)\,ds - (1+\delta)(t-M)^2\right) \to -\infty \end{aligned}$$

as $t \to \infty$. Hence, for $\Lambda > 0$ sufficiently large, the path $[0,1] \to \mathcal{C}_*$, $t \mapsto u_- + \Lambda t$ of constant functions is contained in Γ. Consequently, $c < \infty$. Assume by contradiction that there does not exist a critical point $u \in \mathcal{C}_*$ of I with $I(u) = c$. By Lemma 4.4, this implies the existence of $\varepsilon, \delta > 0$ such that $\|\nabla I(u)\|_{H^1(B)} \geqslant \delta$ for all $u \in \mathcal{C}_*$ satisfying

$|I(u) - c| \leqslant 2\varepsilon$. Without loss of generality, we may assume that $4\varepsilon < \alpha$. Correspondingly, let η be the deformation defined in Lemma 4.5, and let $\gamma \in \Gamma$ be such that $\max_{t\in[0,1]} I(\gamma(t)) \leqslant c + \varepsilon$.

Defining $\bar\gamma : [0,1] \to \mathcal{C}_*$ by $\bar\gamma(t) = \eta(\gamma(t))$, we then have $\bar\gamma(0) = \gamma(0)$ and $\bar\gamma(1) = \gamma(1)$ because of Lemma 4.5(iii) and the fact that $I(u_\pm) < c - 2\varepsilon$ by our choice of ε and α. Hence $\bar\gamma \in \Gamma$. However, by Lemma 4.5(i) and (ii) we have

$$\max_{t\in[0,1]} I\big(\bar\gamma(t)\big) \leqslant c - \varepsilon,$$

contradicting the definition of c. The claim then follows. □

Again we need to show $c < I(u_0)$ for the critical value c in Proposition 4.7.

Lemma 4.12. *Let τ be given as in Lemma 4.6, and fix t_-, t_+ such that*

$$t_- u_0 \in U_-, \qquad t_+ u_0 \in U_+ \quad \text{and} \quad u_- < t_- u_0 < u_0 < t_+ u_0 < \infty,$$

where U_- is defined in (4.20) and U_+ is defined in (4.25). For $s \geqslant 0$ define

$$\gamma_s : [t_-, t_+] \to H^1(B), \qquad \gamma_s(t) = t(u_0 + sv). \tag{4.26}$$

Then there exists $s > 0$ such that $\gamma_s(t_\pm) \in U_\pm$, $\gamma_s(t) \in \mathcal{C}_*$ for $t_- \leqslant t \leqslant t_+$ and $\max_{t_-\leqslant t\leqslant t_+} I(\gamma_s(t)) < I(u_0)$.

Proof. The proof is exactly the same as the one of Lemma 4.10 (using Lemma 4.9). The only difference here is the new definition of U_+. □

End of the proof of Theorem 1.3 in the case $u_+ = \infty$. Lemma 4.12 implies that – after an affine transformation of the independent variable – the path γ_s defined in (4.26) belongs to Γ and satisfies $\max_t I(\gamma_s(t)) < I(u_0)$, so that $c < I(u_0)$. Hence the mountain pass type critical point of I in $\mathcal{C}_*$ provided by Proposition 4.7 is not constant. □

We conclude with the remark that the method presented in this section also applies to obtain decreasing solutions in the subcritical regime assuming for instance the standard Ambrosetti–Rabinowitz condition.

Remark 4.13. Let $a(|x|) \in C(\overline{B})$ be nonincreasing and strictly positive. Let f satisfy $(f1)$, $(f2)$ and assume moreover that

$$\text{there exist } C > 0,\ 2 < p < 2^*\ \text{such that } \big|f(s)\big| \leqslant C|s|^{p-1},$$
$$\text{there exist } R_0 > 0,\ \mu > 2\ \text{such that } f(s)s \geqslant \mu F(s)\ \text{for every } s \geqslant R_0.$$

Then the following hold

(i) if $a(|x|)$ is nonconstant, there exists at least one decreasing radial solution of (1.1);
(ii) if $a(|x|) = 1$ and $(f4)$ holds then (1.5) admits both an increasing and a decreasing radial solution, which are not constant.

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