

# Instantaneous self-fulfilling of long-term prophecies on the probabilistic distribution of financial asset values

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Received 1 December 2005; accepted 21 December 2005

Available online 1 December 2006

## Abstract

Our goal here is to present various examples of situations where a “large” investor (i.e. an investor whose “size” challenges the liquidity or the depth of the market) sees his long-term guesses on some important financial parameters instantaneously confirmed by the market dynamics as a consequence of his trading strategy, itself based upon his guesses. These examples are worked out in the context of a model (i.e. a quantitative framework) which attempts to provide a rigorous basis for the qualitative intuitions of many practitioners. Our results may be viewed as some kind of reverse Black–Scholes paradigm where modifications of option prices affect today’s real volatility.

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## 1. Introduction

Our goal in this paper is to show through a natural quantitative model some bootstrap effects in financial markets with limited *liquidity* that we call self-fulfilling prophecies. Indeed, we shall argue that if an investor (typically a large investor or a collection of investors...), because of some belief (or conviction) about the future behavior of an asset, decides to act according to it, then he will affect the market and the asset dynamics in such a way that the belief will have much more chance to become true or at least partially true...

Such a phenomenon, as stated previously, is not surprising in a market with a limited liquidity: in such a market, if a large investor “changes its belief” at a date  $T_0$  on some asset price at a future date  $T_1$  by, for instance, increasing his own probability that this asset price will go up at time  $T_1$ , then he will buy the asset, therefore pushing up the price, hence increasing indeed the (real) probability of a higher asset price at time  $T_1$ .

Let us immediately mention that our goal here is to illustrate similar phenomena on more complex market parameters such as, e.g., volatility or correlations. We are perfectly aware that the above qualitative argument on prices may be challenged in many ways specially regarding the potential information analysis and reactions of other market players. However, we do believe that the above scenario actually takes place in the markets, at least during some transition periods, provided we concentrate upon a large investor (or a group of investors with similar behaviors...)

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when other market players are not aware or informed of these dynamical changes. Let us point out, by the way, that the preceding phenomenon is both good and bad news for the investor. Indeed, on the one hand, his conviction will be confirmed by the upward trend of the asset price while, on the other hand, since prices will go up because of his buying orders, the investor will have to buy, at least for some part of his investment strategies, at higher prices. . .

Although the effects we discuss in this paper share with the preceding simple example the same basic property of transforming subjective opinions into reality, we wish to show here much more subtle mechanisms. More precisely, we shall show that changes in subjective parameters of a large investor not only modify the probability distribution of asset prices at the end of the period (time  $T_1$  above), but also change some dynamical parameters such as, e.g., volatility on the whole time interval  $((T_0, T_1)$  above) with the same type of qualitative features as in the above simple example. For instance, if the belief of a large investor changes at time  $T_0$  towards an increased variance of an asset price at time  $T_1$  and if he decides to buy European options (with a maturity greater or equal to  $T_1$ ) with convex pay-offs, say for example call or “put-call”, in order to profit from the increased variance, then we will show that the volatility of the asset will increase (over the whole time interval  $(T_0, T_1)$ ) thus confirming in particular the growth of the variance at time  $T_1$ ! In particular, the long-term investor’s guess (concerning time  $T_1$ ) is immediately (at any time  $t > T_0$ ) confirmed since the volatility immediately increases. This is why we emphasized this observation through the word “instantaneous” in the title of this article. However, we do not claim that our theory accurately reproduces the dynamics over the whole period  $(T_0, T_1)$  (and thus, in particular, at time  $T_1$ ). An increased volatility of the asset will change the global information pattern and thus will induce further effects due to other market players. We believe that such “secondary reactions” are interesting and important issues but we do not wish to address them here since our goal here is to focus on the “primary effect” for the sake of clarity. This is the first example we consider and treat in details in Section 2 below.

We shall present in Sections 3 and 4 two more examples of similar phenomena. In Section 3, we consider an agent worrying about the increased probability of a crossing of some barrier (or similarly about an increase of the probability of default. . .) who invests (substantially) in European options. . . Then, exactly as in the preceding example, the volatility of the asset will increase thus confirming instantaneously the investor’s fears. . . . And our last example, developed in Section 4, concerns the correlation between two assets.

Our results may be also viewed as some kind of reverse Black–Scholes paradigm. We show here that, in a market with limited liquidity (or depth), changes in the real volatility can be driven by changes in the implied volatility (see for example Section 2).

At this stage, we need to indicate the mathematical model we use in order to exhibit the market effects described above. It is in fact the quantitative framework we developed in two previous papers [8,9] in order to investigate market behaviors in a context of limited liquidity, i.e. more precisely a market in which a large investor shifts the market prices proportionally to his buy/sell volume. With such a simple and natural framework, we have been able, through a utility maximization approach, to quantify the impact of investing in a European option on the volatility structure (see [8]). Note that this impact affects both the real and the risk-neutral probabilities on spot dynamics. And we have also derived, within this framework, a model for the formation of the volatility structure out of “infinitely many small” impacts caused by “infinitely many small” market players. And we refer to [9] for more details) on that theory which is somewhat inspired from Mean Field (or self-consistent) theories in Statistical Physics.

Let us also point out that we are not aware of any previous work discussing the phenomena shown in this paper. However, the quantitative framework we used in [8,9] and that we recall below is somewhat reminiscent of some phenomenological models proposed by R. Frey and A. Strumme [6], H. Föllmer [5]. At a more technical level, the examples in the following section are derived from our results in [8], which, in addition to the trading impact on prices model, rely upon a utility maximization approach to incomplete markets problems similar to the one developed in, e.g., G. Constantinides and Th. Zariphopoulou [3], or M. Musiela and Th. Zariphopoulou [11]. And the resulting utility maximization problems turn out to be novel stochastic control problems that require an extension (see J.-M. Lasry and P.-L. Lions [7] for more details) of the classical modern stochastic control theory (see, for instance, W.H. Fleming and M.H. Soner [4], M. Bardi and I. Capuzzo-Dolcetta [1]).

Next, we wish to emphasize that our theoretical framework falls within the general class of incomplete markets models and thus should not be thought to apply to all situations! Nevertheless, we believe (or, at least, we hope. . .) that both our framework and the type of effects we derive rigorously from it do capture at least quantitatively some of the behaviors observed in real markets and thus shed some light on their complex mechanisms.

Finally, although our arguments are mathematically funded and thus apply to rather general dynamics, we shall present our three examples in the simplest possible situation of pure Gaussian (i.e. Brownian) dynamics for the asset price with a linear buy/sell volume impact law i.e.

$$dS_t = \sigma dW_t + b dt + k d\alpha_t, \tag{1}$$

where  $S_t$  denotes the asset price at time  $t$ ,  $W_t$  is a Brownian motion,  $\sigma$  is the (extrinsic) volatility taken to be constant,  $b$  is the trend also taken to be constant,  $k$  is a positive parameter corresponding to the impact factor,  $\alpha_t$  is the position (resp. the aggregate position) of a large investor (resp. of a collection of investors sharing the same beliefs and behaviors...), and thus  $d\alpha_t$  corresponds to the instantaneous trading volume due to the investor or to his investing strategies. In fact, we make a further simplification by assuming that the trend  $b$  vanishes, although one can check, using the results of [8], that the presence of  $b$  does not modify the arguments made in the following sections!

### 2. Example 1: Variance and volatility

As explained in the Introduction, we consider a large investor who believes at time  $T_0$  that the variance of the probability distribution of an asset price will be larger at time  $T_1 > T_0$  than expected. We have to make a bit more precise what we mean by an expected variance. Several understandings are possible and the simplest one is the variance implied by the current (at time  $T_0$ ) market prices of a European option such as, for instance, a call (or a put, or a “call–put”...). More precisely, since we consider only Gaussian distributions to simplify the presentation as much as possible, according to Black–Scholes theory [2,10], the market price at time  $T_0$  of a European option (with maturity  $T_1$ ) whose pay-off is given by a scalar function  $\Phi$  is simply given by

$$\pi_m(S) = \int \Phi(S + \sqrt{V_m}S') \exp\left(-\frac{S'^2}{2}\right) (2\pi)^{-1/2} dS', \tag{2}$$

where  $S$  is the asset price at time  $T_0$ ,  $V_m$  corresponds to the variance and in the case of Brownian dynamics i.e. (1) (with  $\alpha_t \equiv 0$ ),  $V_m = \sigma^2(T_1 - T_0)$ . For a call (resp. a put, a call–put) of strike  $K$ ,  $\Phi$  is given by  $(S - K)_+$  (resp.  $(K - S)_+$ ,  $|S - K|$ ) and  $\pi_m$  by

$$\pi_m = V_m \exp\left(-\frac{(K - S)^2}{2V_m}\right) (2\pi V_m)^{-1/2} - (K - S)N\left(\frac{K - S}{\sqrt{V_m}}\right), \tag{3}$$

where  $N(x) = \int_x^{+\infty} (e^{-y^2/2}/\sqrt{2\pi}) dy$ .

In particular, if the investor believes that the variance will be  $V_i > V_m$ , this leads to a value (from the investor’s viewpoint) of the option given by

$$\pi_i(S) = \int \Phi(S + \sqrt{V_i}S') \exp\left(-\frac{S'^2}{2}\right) (2\pi)^{-1/2} dS'. \tag{4}$$

And as is well-known, this value  $\pi_i$  is strictly larger than the market price  $\pi_m$  as soon as the pay-off is *convex* (and not linear...!) as is the case for a call (or a put, or a call–put...). Notice indeed that we have (denoting by  $\pi$  the quantity given by (2) with  $V$  in place of  $V_m$ )

$$\frac{\partial \pi}{\partial V} = \frac{1}{2\sqrt{V}} \int \Phi'(S + \sqrt{V}S') S' \frac{e^{-(S')^2/2}}{\sqrt{2\pi}} dS' = \frac{1}{2} \int \Phi''(S + \sqrt{V}S') \frac{e^{-(S')^2/2}}{\sqrt{2\pi}} dS' > 0.$$

In other words, the investor believes that the market *underprices* all European options with convex pay-offs and maturity  $T_1$  (or greater or equal to  $T_1$ ...). This is why it is natural to assume that he will (or may) invest substantially into such options that is buying substantial amounts of, say calls (or puts, or call–puts) with maturity  $T \geq T_1$ . A more quantitative argument for the decision of investing into options with convex pay-offs is given in Section 3 below. We do not present it here in order to keep our presentation of this first example as simple as possible.

Next, the seller(s) of these options will have to hedge them (or the investor may wish to replicate the options himself, and the argument below is unchanged). One could also describe this market situation by saying that there is a much larger flow of option buyers than of option sellers, and thus the excess demand of options is matched, through delta hedging, by sales from trading desks. Then, because of the assumption of limited liquidity together with the

large volume of trading involved, the hedging will have an impact on the dynamics of the asset price. At this stage, we need to assume that the seller(s) either does not take into account that impact (Case B for “blind”), or incorporates it into the definition of his trading strategy (Case A for “aware”).

We begin with Case B: in that case, we assume that the hedging strategy is the one deduced from Black–Scholes theory [2,10] with a single dynamical model (Brownian dynamics) for the evolution of the market price given by

$$dS_t = \sigma dW_t. \tag{5}$$

Recall that  $V_m = \sigma^2(T_1 - T_0)$ . In other words, the position of the trader is given by

$$\alpha_t = Q \frac{\partial \pi}{\partial S}(S_t + \sigma(W_t - W_{T_0}), t) \quad \text{for } t \geq T_0, \tag{6}$$

with  $\pi(S, t) = E[\Phi(S + \sigma(W_t - W_{T_0}))]$  for  $t \geq T_0$ , and where  $Q$  denotes the (positive) volume of calls (or puts, or call–puts), bought and sold. We finally assume that the impact on price dynamics is represented by the model (1) and we recover the following evolution of the asset price

$$dS_t = \sigma dW_t + k d\alpha_t = \sigma dW_t + kQ \frac{\partial^2 \pi}{\partial S^2} \sigma dW_t + kQ \left( \frac{\partial \pi}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 \pi}{\partial S^2} \right) dt$$

by Itô’s formula. As is well-known (Black–Scholes equations [2,10]), the last term vanishes and we obtain finally

$$dS_t = \sigma(1 + kQ\Gamma) dW_t \tag{7}$$

where  $\Gamma = \partial^2 \pi / \partial S^2$  is the “gamma” of the option.

A variant of the above hedging strategy consists in keeping the (Black–Scholes) delta-hedging strategy but depending on the current observation of the asset price namely

$$\alpha_t = Q \frac{\partial \pi}{\partial S}(S_t, t)$$

with the same Black–Scholes price  $\pi$  as before or, even better, the Black–Scholes price corresponding to the volatility we are computing (i.e.  $\sigma = \sigma(S)$  as determined below). The resulting volatility then becomes  $\sigma(1 - kQ\Gamma)^{-1}$  and the rest of our analysis below is easily adapted to this more complex situation.

In particular, at time  $T_1$ , the variance of the distribution of the asset price whose evolution is given by (7) turns out to be

$$\bar{V} = \sigma^2 E \int_{T_0}^{T_1} (1 + kQ\Gamma)^2 dt \tag{8}$$

which is indeed larger than  $V_m$  since  $\Gamma > 0$

$$\bar{V} > \sigma^2(T_1 - T_0) = V_m,$$

thus confirming the investor’s initial belief!

We now investigate Case A: in that case, we assume that the seller is aware of the impact on the asset price dynamics and will try to define his hedging strategy accordingly. Once more, we assume that the trading impact is represented by (1) i.e.

$$dS_t = \sigma dW_t + k d\alpha_t. \tag{9}$$

This becomes an incomplete market problem that we solved completely in a previous work [8] through a utility maximization approach. And we showed in [8] that, for any utility function, the optimal hedging strategy is still given by (6)! Therefore, everything we said in case B above remains valid and we recover once again formula (8) for the variance of the distribution of the asset price at time  $T_1$ .

In conclusion, in both cases, the investor’s belief is indeed (instantaneously) confirmed as a consequence of the belief of a natural investment caused by it and the impact on the asset price by trading! As we mentioned above, this is indeed some kind of reverse Black–Scholes paradigm in limited liquidity markets. In our limited liquidity model, the excess demand on European options yields an increased implied volatility and leads the trading desks to sell and hedge options on the spot market, modifying thus immediately the real volatility.

Once more, as we emphasized in the Introduction, for the sake of clarity we do not wish to consider here further market reactions due to other agents adjusting their trading strategies to the new situation created.

### 3. Example 2: Barriers, defaults and volatility

We consider here the situation of an agent having invested in a (up and out) barrier option and, in order to simplify the presentation, we consider an option with maturity  $T_1$  whose pay-off vanishes if  $S_t$  exceeds some fixed level  $B$  for some  $t \leq T_1$ , and is equal to 1 otherwise. Furthermore, to simplify further notation, we simply take  $T_0 = 0$ . And we consider a situation where the investor decides at time  $T_0 = 0$  to raise his estimate of the probability that  $S_t$  will cross the barrier at level  $B$  before  $T_1$ , assuming of course that  $S = S_{T_0} < B$  i.e. raises his estimate of  $P(\max_{0 \leq t \leq T_1} S_t > B) = P_B$ .

Once more, we assume that the market prices such options according to the Gaussian (Brownian) dynamics (5) and thus the Black–Scholes price of the option  $\pi_m$  is given by

$$\pi_m = P\left(\max_{0 \leq t \leq T_1} S_t \leq B\right) = P\left(\max_{0 \leq t \leq T_1} W_t \leq \frac{B - S}{\sigma_m}\right) \tag{10}$$

(a quantity that can be made quite explicit... ) writing  $\sigma_m$  in place of  $\sigma$  to emphasize the fact that  $\sigma_m$  is the volatility implied by market prices. Note that  $P_B = 1 - \pi$ . In particular, raising the estimate of  $P_B$  means in fact that the investor believes that the volatility of the asset price is raised from  $\sigma_m$  to a higher value  $\sigma_i$  (and thus that the market, once more, misprices the option). At this stage, the investor may simply “get rid” of the options but we assume he will not do so. First of all, he may not be able to do it, but, in addition, we present now a rather convincing quantitative argument that shows that the optimal investment strategy in such a situation is to buy a European option with maturity  $T_1$  and a strictly convex pay-off. This is the argument we mentioned in the preceding section and thus also applies to the first example and whose discussion we postponed. Of course, the simpler argument presented in Section 2 also remains valid here, indicating the potential profit that can be made through investing in options with convex pay-offs!

We indeed argue that the optimal pay-off or profile of an investment in the asset can be deduced from a utility maximization approach. We thus consider an arbitrary investment profile  $f(S_{T_1})$  where  $f$  is an arbitrary scalar function and  $S_{T_1} = S + \sigma_i W_{T_1}$ . The current market price (Black–Scholes price) is given by

$$p(f) = E[f(S + \sigma_m W_{T_1})]. \tag{11}$$

We may then consider the following maximization problem

$$\sup_f E[U[f(S + \sigma_i W_{T_1}) - p(f)]], \tag{12}$$

where  $U$  is a utility function. In order to make the calculations that follow as simple as possible, we choose an exponential utility function  $U(z) = 1 - e^{-\lambda z}$  where  $\lambda > 0$  represents the absolute risk aversion (we could consider as well  $U(z) = z^\theta$  where  $\theta \in (0, 1)$  or  $U(z) = \log z$  for  $z \geq 0$ , examples that correspond to a constant relative risk aversion). The above concave optimization problem is easily shown to have a unique solution determined by the associated Euler–Lagrange equation namely

$$U'(f(x) - p(f)) e^{-x^2/(2\sigma_i^2 T_1)} = E[U'(f(S + \sigma_i W_{T_1}) - p(f))] e^{-x^2/(2\sigma_m^2 T_1)}$$

and thus we find for some  $C \in \mathbb{R}$

$$f(x) = C + \frac{a}{2\lambda} x^2, \quad a = \frac{1}{T_1} \left( \frac{1}{\sigma_m^2} - \frac{1}{\sigma_i^2} \right). \tag{13}$$

The constant  $C$  can be computed explicitly but we do not bother to do so here since its precise value is irrelevant. Let us also mention that in the case of a power utility function  $U(z) = z^\theta$  with  $\theta \in (0, 1)$  (resp. a logarithmic utility function  $U(z) = \log z$ ), we derive the following optimal pay-off for some (explicit) positive constant  $c$

$$f(x) = p + c e^{\frac{x^2}{2(1-\theta)T_1} \left( \frac{1}{\sigma_m^2} - \frac{1}{\sigma_i^2} \right)}$$

(resp.  $f(x) = p + c e^{\frac{x^2}{2\pi} \left( \frac{1}{\sigma_m^2} - \frac{1}{\sigma_i^2} \right)}$ ).

At this stage, we have shown that, in order to maximize the utility, the investor should buy a European option with maturity  $T_1$  and a strictly convex pay-off (actually, a quadratic pay-off in the case of an exponential utility). And we may then apply the rest of the argument introduced in Section 2 which leads to an increased volatility given by

$$\bar{\sigma} = \sigma(1 + k\Gamma) \tag{14}$$

where  $\sigma = \sigma_m$ ,  $\Gamma = \frac{\partial^2}{\partial S^2} E[f(S + \sigma(W_{T_1} - W_t))]$ .

In particular, if we take the optimal choice  $f$  given by (13), we finally obtain a constant volatility (greater than  $\sigma$ )

$$\bar{\sigma} = \sigma \left( 1 + \frac{k}{\lambda T_1} \left( \frac{1}{\sigma^2} - \frac{1}{\sigma_i^2} \right) \right). \tag{15}$$

And, indeed, the probability of  $S_t$  crossing the barrier level  $B$  before any time  $T \in (0, T_1)$  increases, confirming thus instantaneously the original fears of the investor!

We conclude the discussion of this example by observing that, as is well-known, a possible interpretation of the discussion above (replacing simply an upper barrier by a lower barrier) may be given in terms of default probability in the context of Merton’s firm model. . . .

#### 4. Correlations

Our last example concerns the correlations between assets. Once more, to simplify the presentation as much as possible, we only consider the case of two assets whose prices are given by  $X_t$  and  $Y_t$  and postulate Brownian dynamics

$$dX_t = \alpha dW_t, \quad dY_t = \beta dB_t, \tag{16}$$

where  $\alpha > 0$ ,  $\beta > 0$ ;  $W_t$ ,  $B_t$  are two correlated Brownian motions with a constant correlation  $\rho \in [-1, +1]$  ( $E[W_t B_t] = \rho t$  for all  $t \geq 0$  . . .).

At time  $T_0 = 0$  (in order to simplify notation), a (large) investor modifies his estimate of the correlation between  $X_t$  and  $Y_t$  at time  $T_1 (> 0)$  believing that a correlation parameter  $\rho_i$  different from  $\rho$  should be more accurate.

We claim that it is then natural for the investor to buy some European option with maturity  $T \geq T_1$  and with a pay-off  $\Phi = \Phi(x, y)$  where  $\Phi$  satisfies

$$\begin{cases} \text{sign}(\rho_i - \rho) \cdot \frac{\partial^2 \Phi}{\partial x \partial y} \geq 0 & \text{for all } x, y, \\ \frac{\partial^2 \Phi}{\partial x \partial y} \neq 0. \end{cases} \tag{17}$$

A simple argument for such an investment strategy follows the one made in Section 2, once we observe that the Black–Scholes price of such an option is strictly increasing in  $\rho$  provided we have precisely

$$\frac{\partial^2 \Phi}{\partial x \partial y} \geq 0 \quad \text{for all } x, y, \quad \frac{\partial^2 \Phi}{\partial x \partial y} \neq 0.$$

Indeed, the Black–Scholes price  $\pi$  solves the following equation

$$\begin{cases} \frac{\partial \pi}{\partial t} + \frac{\alpha^2}{2} \frac{\partial^2 \pi}{\partial x^2} + \alpha \beta \rho \frac{\partial^2 \pi}{\partial x \partial y} + \beta^2 \frac{\partial^2 \pi}{\partial y^2} = 0, \\ \pi|_{t=T_1} = \Phi. \end{cases}$$

Hence,  $\Gamma_c = \partial^2 \pi / \partial x \partial y$  solves the same equation with a non-trivial nonnegative terminal condition ( $\Gamma_c|_{t=T_1} = \partial^2 \Phi / \partial x \partial y$ ). And we conclude by the maximum principle observing that  $\partial \pi / \partial \rho$  solves

$$\begin{cases} \frac{\partial}{\partial t} \left( \frac{\partial \pi}{\partial \rho} \right) + \frac{\alpha^2}{2} \frac{\partial^2}{\partial x^2} \left( \frac{\partial \pi}{\partial \rho} \right) + \alpha \beta \rho \frac{\partial^2}{\partial x \partial y} \left( \frac{\partial \pi}{\partial \rho} \right) + \beta^2 \frac{\partial^2}{\partial y^2} \left( \frac{\partial \pi}{\partial \rho} \right) = -\alpha \beta \Gamma_c, \\ \frac{\partial \pi}{\partial \rho} \Big|_{t=T_c} = 0 \end{cases}$$

since  $\Gamma_c(x, y, t) > 0$  for all  $x, y, t < T_1$ . A direct (and more elementary) proof of this fact may be obtained by the examination of  $\pi = E[\Phi(X_t, Y_t)]$  but we chose to present the above proof since it remains valid for rather general models where  $\alpha = \alpha(x)$ ,  $\beta = \beta(y)$  and  $\rho = \rho(x, y)$ . . . .

Let us mention by the way that  $\partial^2 \Phi / \partial x \partial y \equiv 0$  simply means that  $\Phi = \Phi_1(x) + \Phi_2(y)$  in which case the correlation parameter has no influence on the option price.

A more elaborate argument in the spirit of the one made in Section 3 leads to a specific pay-off  $\Phi$ . Indeed, the utility maximization problem becomes

$$\sup_{\Phi} E_i [U(\Phi(X_{T_1}, Y_{T_1}) - p(\Phi))],$$

where  $F_i$  denotes the expectation for the probability  $P_i$  under which the correlation is given by  $\rho_i$ , and  $p(\Phi) = E[\Phi(X_{T_1}, Y_{T_1})]$  where  $E$  now corresponds to the correlation equal to  $\rho$ . Once more, we choose an exponential utility function  $U(z) = 1 - e^{-\lambda z}$  ( $\lambda > 0$ ). And, by a computation somewhat similar to the one made in Section 3, we obtain the following optimal pay-off

$$\Phi(x, y) = C + \frac{1}{2\lambda T_1} [Q(x, y) - Q_i(x, y)] \tag{18}$$

for some (irrelevant) constant  $C$ , where  $Q$  (resp.  $Q_i$ ) is the quadratic form correspond to the matrix

$$A = \frac{1}{1 - \rho^2} \begin{pmatrix} \frac{1}{\alpha^2} & -\frac{\rho}{\alpha\beta} \\ -\frac{\rho}{\alpha\beta} & \frac{1}{\beta^2} \end{pmatrix} \quad \left( \text{resp. } A_i = \frac{1}{1 - \rho_i^2} \begin{pmatrix} \frac{1}{\alpha^2} & -\frac{\rho_i}{\alpha\beta} \\ -\frac{\rho_i}{\alpha\beta} & \frac{1}{\beta^2} \end{pmatrix} \right).$$

And we claim that (17) holds. Indeed, we have easily

$$\frac{\partial^2 \Phi}{\partial x \partial y} = (2T_1 \lambda \alpha \beta)^{-1} \left( \frac{\rho_i}{1 - \rho_i^2} - \frac{\rho}{1 - \rho^2} \right)$$

and we conclude since the function  $x/(1 - x^2)$  is obviously strictly increasing on  $(-1, +1)$ .

For all the reasons detailed above, we assume that the investor buys an option with maturity  $T_1$  and with the pay-off given by  $QX_{T_1}Y_{T_1}$  (for some  $Q > 0$  if  $\rho_i > \rho$  and  $Q < 0$  if  $\rho_i < \rho$ ) and we fix ideas by choosing for example  $\rho_i > \rho$  and thus  $Q > 0$ . Let us observe indeed that the difference between this pay-off and  $\Phi(X_{T_1}, Y_{T_1})$  where  $\Phi$  is given by (18) is the sum of two split quadratic pay-offs in  $X_{T_1}$  and  $Y_{T_1}$  respectively, corresponding to the sum of options on each individual asset (options that obviously are not interesting from a correlation viewpoint...). Similarly, since  $xy = \frac{1}{2}(x + y)^2 - \frac{x^2}{2} - \frac{y^2}{2}$ , up to irrelevant options on each individual asset the above option is really an option on the basket composed by the two assets (and thus can be replicated by basket options...).

Next, we need to define an impact model. Once more, in order to keep the presentation as simple as possible, we consider a completely separated model

$$dX_t = \alpha dW_t + k_1 da_t, \quad dY_t = \beta dB_t + k_2 db_t, \tag{19}$$

where  $k_1, k_2 > 0$ ,  $a_t$  corresponds to the position in the asset  $X_t$  and  $b_t$  to the one in the asset  $Y_t$ .

Then, exactly as we did in Section 2, we are led to the choice

$$a_t = \frac{\partial \pi}{\partial x}(x + \alpha W_t, y + \beta B_t, t), \quad b_t = \frac{\partial \pi}{\partial y}(x + \alpha W_t, y + \beta B_t, t), \tag{20}$$

where  $\pi = QE[(x + \alpha(W_{T_1} - W_t))(y + \beta(B_{T_1} - B_t))] = Qxy + Q\alpha\beta\rho(T_1 - t)$ . Hence (20) yields

$$a_t = Q(y + \beta B_t), \quad b_t = Q(x + \alpha W_t).$$

And (19) then becomes

$$dX_t = \alpha dW_t + k_1 Q\beta dB_t, \quad dY_t = \beta dB_t + k_2 Q\alpha dW_t. \tag{21}$$

We may now conclude the discussion of this example since the initial correlation between the two assets  $(X_t, Y_t)$  was given by  $\rho t$  ( $= E[X_t Y_t] E[X_t^2]^{-1/2} E[Y_t^2]^{-1/2}$ ), while (21) yields a new correlation given by

$$t \{ \alpha\beta\rho + k_1 k_2 Q^2 \alpha\beta\rho + k_2 Q\alpha^2 + k_1 Q\beta^2 \} (\alpha^2 + k_1^2 Q^2 \beta^2 + 2\rho k_1 Q\alpha\beta)^{-1/2} (\beta^2 + k_2^2 Q^2 \alpha^2 + 2\rho k_2 Q\alpha\beta)^{-1/2}.$$

And a straightforward but tedious computation (that we leave to the reader...) indeed yields, using the positivity of  $\alpha, \beta, Q, k_1, k_2$ , the fact that the new correlation is indeed larger (for all  $t > 0$  i.e. instantaneously) than the original one.

It is worth pointing out that, in a market with limited liquidity, our argument shows that basket options or options on an index (in particular, an index of a specific industrial sector) modify the correlations structure (actually, increase correlations in the case of put or call options).

**References**

- [1] M. Bardi, I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations*, Birkhäuser, Boston, 1997.
- [2] F. Black, M. Scholes, The pricing of options and corporate liabilities, *J. Political Economy* 81 (1973) 637–659.
- [3] G. Constantinides, Th. Zariphopoulou, Bounds on prices of contingent claims in an intertemporal economy with proportional transaction costs and general preferences, *Finance Stoch.* 3 (1999) 345–369.
- [4] W.H. Fleming, M.H. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer, Berlin, 1993.
- [5] H. Föllmer, Stock price fluctuation as a diffusion in a random environment, in: S.D. Howison, F.P. Kelly, P. Wilmott (Eds.), *Mathematical Models in Finance*, Chapman–Hall, London, 1995.
- [6] R. Frey, A. Stremme, Portfolio insurance and volatility, Department of Economics, Univ. of Bonn (Discussion paper B-256).
- [7] J.-M. Lasry, P.-L. Lions, Une classe nouvelle de problèmes singuliers de contrôle stochastique, *C. R. Acad. Sci. Paris, Ser. I* 331 (2000) 879–885.
- [8] J.-M. Lasry, P.-L. Lions, Large investor trading impacts on volatility, Preprint.
- [9] J.-M. Lasry, P.-L. Lions, Towards a self-consistent theory of volatility, Preprint.
- [10] R. Merton, Theory of rational option pricing, *Bull. J. Econom. Manag. Sci.* 4 (1973) 141–183.
- [11] M. Musiela, Th. Zariphopoulou, Indifference prices and related measures, Preprint.