Riemannian metric of the averaged energy minimization problem in orbital transfer with low thrust

Métrique Riemannienne du problème de minimisation de l'énergie en transfert orbital à poussée faible

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Abstract

This article deals with the optimal transfer of a satellite between Keplerian orbits using low propulsion and is based on preliminary results of Epenoy et al. (1997) where the optimal trajectories of the energy minimization problem are approximated using averaging techniques. The averaged Hamiltonian system is explicitly computed. It is related to a Riemannian problem whose distance is an approximation of the value function. The extremal curves are analyzed, proving that the system remains integrable in the coplanar case. It is also checked that the metric associated with coplanar transfers towards a circular orbit is flat. Smoothness of small Riemannian spheres ensures global optimality of the computed extremals.
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Résumé

Cet article concerne le problème du transfert d'un satellite entre orbites elliptiques dans le cadre de la poussée faible et se fonde sur les résultats préliminaires d'Epenoy et al. (1997) où les trajectoires optimales du problème de minimisation de l'énergie sont approchées par moyennation. On fait un calcul explicite du Hamiltonien moyenné et on prouve qu'il s'agit du Hamiltonien associé à un problème Riemannien dont la distance approche la fonction valeur. On montre en analysant les extrémales que le système moyenné reste intégrable dans le cas coplanaire, et on vérifie que la métrique associée est plate dans le cas de transferts coplanaires vers des orbites circulaires. La régularité des sphères Riemanniennes de petit rayon garantit l'optimalité globale des extrémales calculées.
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1. Introduction

An important problem in astronautics is to transfer a satellite between elliptic orbits. Current research projects concern orbital transfer with electro-ionic propulsion where the thrust is very low. If we assume the mass constant, the Kepler equation describing the motion of the satellite can be normalized to

$$\ddot{q} = -\frac{q}{|q|^3} + u,$$

where $q = (q_1, q_2, q_3)$ is the position of the satellite and u is the thrust, $|u| \leqslant \varepsilon$, ε being the normalized maximum amplitude of the control. The thrust can be decomposed in a moving frame, the so-called *radial-orthoradial frame*. In this frame, $u = u_1 F_1 + u_2 F_2 + u_3 F_3$ with $F_1 = q/|q|$, $F_2 = F_3 \times F_1$, and $F_3 = q \times \dot{q}/|q \times \dot{q}|$. The state of the system is described by an angle, the *true longitude l*, and by five *equinoctial elements* corresponding to first integrals of the unperturbed motion, for instance $x = (P, e, h)$ where P is the *semi-latus rectum* of the osculating conic, $e = (e_x, e_y)$ the *eccentricity vector*, and $h = (h_x, h_y)$ the *inclination vector*. We restrict the problem to the *elliptic domain*, that is to the manifold X of elliptic trajectories of the Kepler equation,

$$X = \{P > 0, \ |e| < 1\}.$$

The system is smooth on $X \times \mathbf{S}^1$ and described by the following equations [7],

$$\dot{x} = \sum_{i=1}^{3} u_i F_i(l, x), \quad \dot{l} = g_0(l, x) + g_1(l, x, u), \tag{1.1}$$

where

$$F_1 = P^{1/2}\left(\sin l \frac{\partial}{\partial e_x} - \cos l \frac{\partial}{\partial e_y}\right),$$

$$F_2 = P^{1/2}\left[\frac{2P}{W}\frac{\partial}{\partial P} + \left(\cos l + \frac{e_x + \cos l}{W}\right)\frac{\partial}{\partial e_x} + \left(\sin l + \frac{e_y + \sin l}{W}\right)\frac{\partial}{\partial e_y}\right],$$

$$F_3 = \frac{P^{1/2}}{W}\left(-Z e_y \frac{\partial}{\partial e_x} + Z e_x \frac{\partial}{\partial e_y} + \frac{C \cos l}{2}\frac{\partial}{\partial h_x} + \frac{C \sin l}{2}\frac{\partial}{\partial h_y}\right),$$

with

$$W = 1 + e_x \cos l + e_y \sin l, \quad Z = h_x \sin l - h_y \cos l, \quad C = 1 + |h|^2$$

and

$$g_0 = \frac{W^2}{P^{3/2}}, \qquad g_1 = P^{1/2}\frac{Z}{W}u_3.$$

An important subproblem is to transfer the satellite between coplanar orbits. The corresponding subsystem is deduced by setting both the inclination h and the control u_3 to zero.

In orbital transfer, the system must be steered from an initial position represented by (l_0, x_0) to a terminal orbit defined by some x_f, taking into account physical cost functions such as the transfer time t_f, the energy $\int_0^{t_f} |u|^2 \, dt$, or the consumption $\int_0^{t_f} |u| \, dt$. The resulting criterions take the form

$$\int_0^{t_f} F_\varepsilon^0(l, x, v) \, dt \to \min,$$

where $v = u/\varepsilon$ is the renormalized control, $|v| \leqslant 1$. The analysis of the system is intricate and is achieved mainly using numerical simulations: see [7] for the time optimal case, and [13] for the minimization of the consumption. If we use low propulsion, we can observe on the numerical results that the evolution of the equinoctial parameters are essentially given by the averaged behaviour of the extremal solutions. This observation was the starting point of [10,12] where a preliminary analysis of the averaged energy minimization problem was performed (the constraint $|v| \leqslant 1$ being relaxed since it is automatically fulfilled for a big enough fixed transfer time). Using symbolic machine

computation, this led in the coplanar transfer case to an averaged system that can be mathematically evaluated by means of standard integral calculus and is integrable by quadratures if the system is transferred towards a circular orbit.

The aim of this article is to present the computations in the general case and to regard averaging as a means of approximating optimal control problems by sub-Riemannian ones. Such problems have the interesting property that L^2 or L^1 minimization and final time minimization share the same extremals. In the orbit transfer case, the problem we obtain is even Riemannian, in dimension three for coplanar trajectories, five otherwise, for additional control directions associated with Lie brackets of the original vector fields are generated when averaging with respect to the fast variable l. Another contribution is to prove that the coplanar Kepler equation remains integrable when it is perturbed by small controls and averaged. Analyzing the Riemannian metric associated with the problem, we even show that it is almost flat, in the sense that the metric is actually flat for the two-dimensional subsystem of transfers towards circular orbits, and that it is a deformation of the three-dimensional Euclidean metric otherwise. Besides, we also point out the effect of the geometry of the elliptic domain on existence of geodesics and completeness of the metric, and illustrate how it is responsible for the loss of compactness of large enough Riemannian spheres.

The organization of the article is the following. In Section 2, we introduce the category of optimal control problems analyzed further, and define the associated Riemannian structure. In Section 3, the averaged Hamiltonian is explicitly computed in the non-coplanar case, as well as the induced Riemannian metric. In Section 4, we prove that the system is integrable by quadratures in the coplanar case and compute flat coordinates for transfers towards circular orbits. In the concluding Section 5, we evaluate numerically Riemannian spheres for a coplanar transfer, allowing us to get global optimality results. We also discuss the merits of this study in the framework of continuation methods.

2. Energy minimization for affine control systems

Let X be an n-dimensional smooth manifold. An *affine control system* is a system of the form

$$\dot{x} = F_0(x) + \sum_{i=1}^m u_i F_i(x),$$

where the *drift*, F_0, and $F_1, \ldots, F_m$ are smooth vector fields. The set of admissible controls is the set of bounded measurable mappings valued in a prescribed subset U of $\mathbf{R}^m$. Let (x, u) be a pair solution of the differential equation on $[0, t_f]$, then the *energy* is the integral $\int_0^{t_f} |u|^2 \, dt$, $|u| = (u_1^2 + \cdots + u_m^2)^{1/2}$ being the Euclidean norm. The associated minimization problem is called the *energy minimization problem*.

2.1. Unconstrained extremals of the problem with drift

According to the maximum principle [16], optimal trajectories are extremal curves, solution of

$$\dot{x} = \frac{\partial H}{\partial p}(x, p, u), \qquad \dot{p} = -\frac{\partial H}{\partial x}(x, p, u),$$

and if U is equal to $\mathbf{R}^m$ (or to an open subset of $\mathbf{R}^m$), we deduce from the maximization condition that $\partial H/\partial u$ is zero where

$$H(x, p, u) = p^0 \sum_{i=1}^m u_i^2 + \left\langle p, F_0 + \sum_{i=1}^m u_i F_i \right\rangle.$$

The nonpositive constant p^0 can be normalized either to 0, in the so-called *abnormal* or *exceptional case*, or to $-1/2$ in the *normal case*. If we define the *Poincaré coordinates* $P_i = \langle p, F_i \rangle$, $i = 0, \ldots, m$, we get from the maximization condition that $u_i = P_i$ in the normal case. Plugging this into H defines the *true Hamiltonian* function, still denoted H, whose integral curves are the normal extremals:

$$H(x, p) = P_0 + \frac{1}{2} \sum_{i=1}^m P_i^2.$$

2.2. Sub-Riemannian problems

Consider the previous minimization problem with $F_0 = 0$, that is for the symmetric system

$$\dot{x} = \sum_{i=1}^{m} u_i F_i,$$

the final time t_f being fixed. Let $\mathcal{D}$ be the distribution spanned by the vector fields $F_1, \ldots, F_m$, and assume that its rank is constant and equal to m at each point. Then, according to Maupertuis principle, the energy minimization problem is equivalent to the minimization of the length, $\int_0^{t_f} |u| \, \mathrm{d}t$, among curves tangent to the distribution. The underlying geometry is called *sub-Riemannian* (*SR*). It is a generalization of Riemannian geometry for which m is equal to n.

Geometrically, the SR problem amounts to defining a metric on $\mathcal{D}$, the vector fields $F_1, \ldots, F_m$ forming an orthonormal frame. Using our system representation, our gauge group is made of diffeomorphisms, $x = \varphi(y)$, and feedbacks, $v = O(x)u$, where $O(x)$ is an orthogonal matrix preserving the control magnitude. Locally, we can represent the SR metric on the kernel of an appropriate vectorial one-form by $g = \sum_{i,j=1}^{m} g_{ij}(x) \, \mathrm{d}x_i \, \mathrm{d}x_j$ and the gauge group acts by changes of coordinates only.

According to the previous subsection, the normal extremals are integral solutions of the Hamiltonian $H = (1/2)(P_1^2 + \cdots + P_m^2)$. Thus, parameterizing the extremal curves by arclength amounts to a restriction to the level set $H = 1/2$. Let z be the pair (x, p) and let $z(t, z_0)$ be the normal extremal at time t contained in $H = 1/2$ and starting from z_0. If we fix x_0, the *exponential mapping* is the map

$$\exp_{x_0} : (t, p_0) \mapsto x(t, x_0, p_0).$$

If X is connected and if the Lie algebra generated by the distribution is of full rank, then, under mild assumptions, for each pair (x_0, x_f) there exists an optimal curve joining x_0 to x_f and the length of such a curve is the *SR distance* between these points. We note $S(x_0, r)$ the SR sphere with center x_0 and positive radius r. A *conjugate point* along a normal extremal is defined as follows. Let t_c be a positive time such that the exponential is not immersive for some p_0. Then t_c is a conjugate time on the extremal starting from (x_0, p_0), and its image $x(t_c)$ is a *conjugate point*. The *conjugate locus* $C(x_0)$ is the set of first conjugate points when all the normal extremals starting from x_0 are considered. The point where the extremal ceases to be minimizing is called the *cut point*, and such points form the *cut locus* $L(x_0)$.

2.3. Averaging for energy minimization

According to the introduction, the elliptic transfer is modelled by a system on $X \times \mathbf{S}^1$, see (1.1):

$$\dot{x} = \sum_{i=1}^{m} u_i F_i(l, x), \quad \dot{l} = g_0(l, x) + g_1(l, x, u).$$

For such a system, we assume that g_0 and g_1 are smooth, g_0 positive and g_1 linear in the control. We set as before $u = \varepsilon v$, $|v| \leqslant 1$, and consider the energy minimization problem

$$\int_0^{t_f} |u|^2 \, \mathrm{d}t = \varepsilon^2 \int_0^{t_f} |v|^2 \, \mathrm{d}t \to \min$$

for a fixed positive final time t_f. Since $\dot{l}$ is positive, we can reparameterize the trajectories by l,

$$\frac{\mathrm{d}x}{\mathrm{d}l} = \frac{\varepsilon}{g_0(l, x) + \varepsilon g_1(l, x, v)} \sum_{i=1}^{m} v_i F_i(l, x),$$

and the cost function becomes

$$\varepsilon^2 \int_{l_0}^{l_f} |v|^2 \frac{\mathrm{d}l}{g_0(l, x) + \varepsilon g_1(l, x, v)}.$$

In order to perform the analytic computation, we first relax the problem by dropping the bound on the control, $|v| \leqslant 1$. Indeed, the underlying idea is that, for a given positive ε, the constraint will be automatically fulfilled for a big enough final time. Practically, we also replace the problem with fixed time by fixed cumulated longitude.

By virtue of the maximum principle, optimal trajectories are extremals, that is integral curves of the following Hamiltonian:

$$H_\varepsilon(l, x, p, v) = \frac{\varepsilon}{g_0 + \varepsilon g_1} \left(p^0 \varepsilon |v|^2 + \sum_{i=1}^m v_i P_i \right),$$

where p^0 is nonpositive and $P_i = \langle p, F_i \rangle$, $i = 1, \ldots, m$. In the normal case, p^0 is negative and normalized not to $-1/2$ but to $-1/(2\varepsilon)$ for obvious homogeneity reasons. As a result, up to first order in ε, we have the approximation below:

$$H_\varepsilon(l, x, p, v) = \frac{\varepsilon}{g_0} \left(1 - \varepsilon \frac{g_1}{g_0} + \cdots \right) \left(-\frac{1}{2} |v|^2 + \sum_{i=1}^m v_i P_i \right)$$

$$= \frac{\varepsilon}{g_0} \left(-\frac{1}{2} |v|^2 + \sum_{i=1}^m v_i P_i \right) + \mathrm{o}(\varepsilon).$$

Neglecting the $\mathrm{o}(\varepsilon)$ term, the maximized first order approximation in the normal case is the true Hamiltonian

$$H(l, x, p) = \frac{1}{2g_0} \sum_{i=1}^m P_i^2,$$

where the multiplicative factor ε has been omitted for the sake of simplicity. We observe that, since g_0 is positive, H can be written as a sum of m squares, $H = (1/2) \sum_{i=1}^m (P_i / g_0^{1/2})^2$.

Lemma 1. *The function H is a nonnegative quadratic form in p with constant rank m denoted $w(l, x)$.*

Since H is 2π-periodic with respect to l, it makes sense to compute its average.

Definition 2. The *averaged Hamiltonian* is

$$\overline{H}(x, p) = \frac{1}{2\pi} \int_0^{2\pi} H(l, x, p) \, \mathrm{d}l.$$

The following lemma is crucial.

Lemma 3. *The averaged Hamiltonian also defines a nonnegative quadratic form in p denoted $\overline{w}(x)$. Moreover,*

$$\mathrm{Ker}\, \overline{w}(x) = \bigcap_{l \in \mathbf{S}^1} \mathrm{Ker}\, w(l, x).$$

Proof. We fix x and p. By construction, $H(l, x, p)$ is nonnegative for every l in $\mathbf{S}^1$, hence $\overline{H}(x, p)$ is also nonnegative, as well as the form $\overline{w}(x)$. Clearly, if p is in $\bigcap_{l \in \mathbf{S}^1} \mathrm{Ker}\, w(l, x)$, then $\overline{H}(x, p) = 0$. Conversely, if $\overline{H}(x, p) = 0$, then

$$\int_0^{2\pi} H(l, x, p) \, \mathrm{d}l = 0.$$

Since $H(l, x, p)$ is nonnegative, it has to be zero for each l. This proves the result. $\quad\square$

According to this lemma, the rank of $\overline{w}(x)$ is not less than m, and we can only expect it to increase. The geometric interpretation is clear: the extremal control is a dynamic feedback of the form $u(l, x, p)$, periodic with respect to l, the so-called *fast variable*. The oscillations due to this coordinate generate new control directions, namely Lie brackets of the original vector fields. This leads to the following concept.

2.4. Sub-Riemannian structure of the averaged problem

Definition 4. The averaged system is said to be *regular* if the rank of $\overline{w}(x)$ is constant.

In this case, there exists an orthogonal matrix $R(x)$ such that, if $P = R(x)p$ then $\overline{w}(x)$ is written as a sum of squares, $(1/2)\sum_{i=1}^{k}\lambda_i(x)P_i^2$, where $\lambda_1, \ldots, \lambda_k$ are the nonnegative eigenvalues of the symmetric matrix $S(x)$ such that $\overline{w}(x) = (1/2)\,{}^t p S(x) p$. Hence, we can write

$$\overline{w}(x) = \frac{1}{2}\sum_{i=1}^{k}\left(\lambda_i^{1/2} P_i\right)^2 = \frac{1}{2}\sum_{i=1}^{k}\langle p, F_i\rangle^2,$$

where the F_i's are smooth vector fields on X.

Proposition 5. *If the averaged system is regular of rank k, the averaged Hamiltonian $\overline{H}$ can be written as a sum of squares, $(1/2)\sum_{i=1}^{k} P_i^2$, $P_i = \langle p, F_i\rangle$, and is the Hamiltonian of the SR problem*

$$\dot{x} = \sum_{i=1}^{k} u_i F_i(x), \qquad \int_0^{t_f}\sum_{i=1}^{k} u_i^2\,dt \to \min$$

with k not less than m. If k is equal to n, then $\overline{H}$ is the Hamiltonian of a Riemannian problem.

3. Application to orbital transfer with low thrust

We use the representation of the introduction: the state is described by the equinoctial elements (P, e, h) and the control is decomposed in the radial-orthoradial frame (this choice is justified further). We start with the averaging of the coplanar transfer.

3.1. Averaged system for coplanar transfers

Applying the process previously discussed, we obtain the Hamiltonian $H = (1/2)(P_1^2 + P_2^2)$ with

$$P_1 = \frac{P^{5/4}}{W}(p_{e_x}\sin l - p_{e_y}\cos l),$$
$$P_2 = \frac{P^{5/4}}{W}\left[p_P\frac{2P}{W} + p_{e_x}\left(\cos l + \frac{e_x + \cos l}{W}\right) + p_{e_y}\left(\sin l + \frac{e_y + \sin l}{W}\right)\right].$$

As before, $W = 1 + e_x\cos l + e_y\sin l$, and the computation of the averaged has the complexity of integrating terms of the form $Q(\cos l, \sin l)/W^k$ where Q is a polynomial and k is an integer between two and four. Since

$$\int_0^{2\pi}\frac{Q(\cos l, \sin l)}{W^k}\,dl = \int_{\mathbf{S}^1}\frac{Q(z/2 + 1/(2z), z/(2i) - 1/(2iz))}{W^k}\frac{dz}{iz},$$

such integrals are evaluated by means of the integrand residues. Writing $e = e_x + ie_y$, we have $W = (\bar{e}z^2 + 2z + e)/(2z)$ and, for e not zero, there are two distinct poles,

$$z = \frac{-1 \pm (1 - |e|^2)^{1/2}}{\bar{e}}.$$

The product of these poles is the unit complex $e/\bar{e}$, so that only one of them belongs to the open unit disk, namely $z = [-1 + (1 - |e|^2)^{1/2}]/\bar{e}$. In contrast, when using the tangential-normal frame as in [10], W is replaced by $W(1 + 2e_x\cos l + 2e_y\sin l + |e|^2)$ and two poles among four are to be taken into account.

An inspection of the Hamiltonian shows that the following averages are required, for which we give the results:

$$\overline{1/W^2} = \delta^3,$$

$$\overline{\cos l/W^3} = -(3/2)e_x\delta^5, \qquad \overline{\sin l/W^3} = -(3/2)e_y\delta^5,$$

$$\overline{\cos^2 l/W^3} = (1/2)(\delta^3 + 3e_x^2\delta^5), \qquad \overline{\sin^2 l/W^3} = (1/2)(\delta^3 + 3e_y^2\delta^5),$$

$$\overline{\cos l \sin l/W^3} = (3/2)e_x e_y\delta^5,$$

$$\overline{1/W^4} = (1/2)(2 + 3|e|^2)\delta^7,$$

$$\overline{\cos l/W^4} = -(1/2)e_x(4 + |e|^2)\delta^7, \qquad \overline{\sin l/W^4} = -(1/2)e_y(4 + |e|^2)\delta^7,$$

$$\overline{\cos^2 l/W^4} = (1/2)(\delta^5 + 5e_x^2\delta^7), \qquad \overline{\sin^2 l/W^4} = (1/2)(\delta^5 + 5e_y^2\delta^7),$$

$$\overline{\cos l \sin l/W^4} = (5/2)e_x e_y\delta^7,$$

with $\delta = 1/(1 - |e|^2)^{1/2}$. Substituting these expressions, we obtain the averaged Hamiltonian

$$\overline{H}(x, p) = \frac{P^{5/2}}{4(1 - |e|^2)^{5/2}}\Big[4p_P^2 P^2\big(-3 + 5(1 - |e|^2)^{-1}\big) + p_{e_x}^2\big(5(1 - |e|^2) + e_y^2\big) + p_{e_y}^2\big(5(1 - |e|^2) + e_x^2\big)$$
$$- 20p_P p_{e_x} Pe_x - 20p_P p_{e_y} Pe_y - 2p_{e_x}p_{e_y}e_x e_y\Big].$$

3.2. Change of coordinates

At this point, we take advantage of the computation in [10] and make the following change of variables:

$$P = \frac{1 - \rho^2}{n^{2/3}}, \quad e_x = \rho\cos\theta, \quad e_y = \rho\sin\theta,$$

where n is the so-called *mean movement* [17]. This amounts to the Mathieu transformation $x = \varphi(y)$ and $p = q(\partial\varphi/\partial y)^{-1}$, where q is the new adjoint state variable. The next proposition summarizes the computation.

Proposition 6. *In coordinates* (n, ρ, θ), *the averaged Hamiltonian is*

$$\overline{H} = \frac{1}{4n^{5/3}}\left[18n^2 p_n^2 + 5(1 - \rho^2)p_\rho^2 + (5 - 4\rho^2)\frac{p_\theta^2}{\rho^2}\right], \tag{3.1}$$

and $\rho = 0$ *corresponds to a circular orbit for which the change of variables is singular.*

In particular, $\overline{H}$ is the Hamiltonian of a Riemannian problem in $\mathbf{R}^3$ defined by

$$ds^2 = \frac{1}{9n^{1/3}}\,dn^2 + \frac{2n^{5/3}}{5(1 - \rho^2)}\,d\rho^2 + \frac{2n^{5/3}}{5 - 4\rho^2}\rho^2\,d\theta^2,$$

and (n, ρ, θ) are *orthogonal coordinates*.

Up to a scalar, this result was obtained by symbolic computation in [10]. However, the complexity of the computation prevented the authors from tackling the general non-coplanar transfer problem.

3.3. Averaged system for non-coplanar transfers

The complete Hamiltonian is $H = (1/2)(P_1^2 + P_2^2 + P_3^2)$ with

$$P_3 = \frac{P^{5/4}}{W}\left(-Zp_{e_x}e_y + Zp_{e_y}e_x + \frac{C}{2}p_{h_x}\cos l + \frac{C}{2}p_{h_y}\sin l\right)$$

and P_1, P_2 unchanged. As previously, we use (n, ρ, θ) as coordinates, and we make a polar representation of h, $h_x = \sigma\cos\Omega$, $h_y = \sigma\sin\Omega$. The angle Ω is the so-called *longitude of the ascending node* [17]. We get

$$\overline{H} = \frac{1}{4n^{5/3}}\left[18n^2 p_n^2 + 5(1-\rho^2)p_\rho^2 + (5-4\rho^2)\frac{p_\theta^2}{\rho^2}\right] + \frac{1}{4n^{5/3}}\frac{(\sigma^2+1)^2}{4}\frac{1+4\rho^2}{1-\rho^2}\left(\cos\omega p_\sigma + \sin\omega\frac{p_{\theta\Omega}}{\sigma}\right)^2$$

$$+ \frac{1}{4n^{5/3}}\frac{(\sigma^2+1)^2}{4}\left(-\sin\omega p_\sigma + \cos\omega\frac{p_{\theta\Omega}}{\sigma}\right)^2,$$

where $\omega = \theta - \Omega$ is the *angle of the pericenter*, and where

$$p_{\theta\Omega} = \frac{2\sigma^2}{\sigma^2+1}p_\theta + p_\Omega.$$

This decomposition shows that $\overline{H}$ is the sum of five squares, $(1/2)\sum_{i=1}^{5} P_i^2$, where the $P_i = \langle p, F_i\rangle$ are five independent linear forms in p.

Theorem 7. *The averaged Hamiltonian of the non-coplanar transfer is associated with a five-dimensional Riemannian problem.*

More precisely, our computation asserts that $\overline{H} = \overline{H}_1 + \overline{H}_2$ where $\overline{H}_1$ is the averaged Hamiltonian (3.1) related to the coplanar transfer and where

$$\overline{H}_2 = \frac{1}{8n^{5/3}}\frac{(\sigma^2+1)^2}{2}\left[\frac{1+4\rho^2}{1-\rho^2}\left(\cos\omega p_\sigma + \sin\omega\frac{p_{\theta\Omega}}{\sigma}\right)^2 + \left(-\sin\omega p_\sigma + \cos\omega\frac{p_{\theta\Omega}}{\sigma}\right)^2\right]$$

comes from the averaged action of the vector field orthogonal to the osculating plane. We begin by studying this last term alone.

In this case, n and ρ are constant and we can absorb the effect of $1/(8n^{5/3})$. We set $K = [(1+4\rho^2)/(1-\rho^2)]^{1/2}$ for ρ in $]0, 1[$ (note that $K > 1$), the singular case $\rho = 0$ being excluded. Replacing the coordinate θ by $\omega = \theta - \Omega$, we have $\overline{H}_2 = (1/2)(P_4^2 + P_5^2)$ where $x = (\sigma, \omega, \Omega)$, and where P_4 and P_5 are the Hamiltonian lifts of the vector fields

$$F_4 = K(\sigma^2+1)\left[\cos\omega\frac{\partial}{\partial\sigma} + \frac{\sin\omega}{\sigma}\left(\frac{\sigma^2-1}{\sigma^2+1}\frac{\partial}{\partial\omega} + \frac{\partial}{\partial\Omega}\right)\right] \quad\text{and}$$

$$F_5 = (\sigma^2+1)\left[-\sin\omega\frac{\partial}{\partial\sigma} + \frac{\cos\omega}{\sigma}\left(\frac{\sigma^2-1}{\sigma^2+1}\frac{\partial}{\partial\omega} + \frac{\partial}{\partial\Omega}\right)\right].$$

Hence, $\overline{H}_2$ is induced by the SR problem

$$\dot{x} = u_1 F_4(x) + u_2 F_5(x), \qquad \int_0^{t_f}\left(|u_1|^2 + |u_2|^2\right)dt \to \min.$$

The vectors F_4, F_5 and $[F_4, F_5]$ readily form a frame, and $\mathscr{D} = \text{Span}\{F_4, F_5\}$ is a two-dimensional *contact distribution*. The induced geometry is well studied [6], and we have the proposition hereafter.

Proposition 8. *The averaged Hamiltonian $\overline{H}_2$ corresponds to the SR problem in dimension three defined by the contact distribution*

$$(\sigma^2+1)\,d\omega - (\sigma^2-1)\,d\Omega = 0.$$

The metric is

$$g = \frac{1}{(\sigma^2+1)^2}\left(\frac{\cos^2\omega}{K^2} + \sin^2\omega\right)d\sigma^2 + \frac{\sigma^2}{(\sigma^2-1)^2}\left(\frac{\sin^2\omega}{K^2} + \cos^2\omega\right)d\omega^2$$

$$- \frac{2\sigma\cos\omega\sin\omega}{(\sigma^2+1)(\sigma^2-1)}\left(1 - \frac{1}{K^2}\right)d\sigma\,d\omega.$$

In the sequel, we focus on the analysis of the extremal curves of the coplanar system.

4. Integrability of the averaged coplanar system

4.1. Geometric preliminaries

If we restrict our study to coplanar transfers, the variables (n, ρ, θ) form an orthogonal set of coordinates for the averaged Hamiltonian (3.1) that we shall renormalize to its half as in [10],

$$\overline{H} = \frac{1}{8n^{5/3}} \left[18n^2 p_n^2 + 5(1 - \rho^2) p_\rho^2 + (5 - 4\rho^2) \frac{p_\theta^2}{\rho^2} \right]. \tag{4.1}$$

The singularity of these coordinates at $\rho = 0$ comes from the polar transformation centered on circular orbits. In the coplanar case, Ω is zero and θ is equal to ω, the angle of the pericenter. We observe that, since the Hamiltonian does not depend on θ, the coordinate is *cyclic* in order that its dual variable p_θ is a first integral of the averaged motion. If we restrict further the system to the four-dimensional symplectic space $\{\theta = p_\theta = 0\}$, $\overline{H} = [1/(8n^{5/3})][18n^2 p_n^2 + 5(1 - \rho^2) p_\rho^2]$ is analytic and associated with a planar Riemannian metric. Geometrically, the condition $p_\theta = 0$ is the transversality condition for a transfer towards a circular orbit for which the angle of the pericenter is unprescribed. This is the case for the important problem of steering the system to a geostationary orbit.

In (4.1), the Hamiltonian is the half sum of three squares that are Hamiltonian lifts of the following vector fields:

$$F_1 = \frac{3}{\sqrt{2}} n^{1/6} \frac{\partial}{\partial n},$$

$$F_2 = \frac{\sqrt{5}}{2} \frac{(1 - \rho^2)^{1/2}}{n^{5/6}} \frac{\partial}{\partial \rho},$$

$$F_3 = \frac{1}{2} \frac{(5 - 4\rho^2)^{1/2}}{\rho} \frac{\partial}{\partial \theta},$$

where n is positive and ρ belongs to $[0, 1[$, ρ being zero for circular orbits and equal to one for parabolic orbits. These vector fields are analytic for ρ positive and the system $\dot{x} = \sum_{i=1}^{3} u_i F_i(x)$ is controllable by virtue of Chow's theorem. As a result, two points close enough in the elliptic domain are joined by an extremal minimizing curve of the associated Riemannian problem, $\int_0^{t_f} \sum_{i=1}^{3} |u_i|^2 \to \min$. Therefore, contrary to what is observed in lunar theory, there are no first integrals depending only upon state variables.

The Hamiltonian system is (for the sake of clarity, we use the time t instead of the cumulated longitude l to parameterize the extremals)

$$\dot{n} = \frac{9}{2} n^{1/3} p_n, \tag{4.2}$$

$$\dot{\rho} = \frac{5}{4} \frac{(1 - \rho^2)}{n^{5/3}} p_\rho, \tag{4.3}$$

$$\dot{\theta} = \frac{1}{4} \frac{(5 - 4\rho^2)}{\rho^2 n^{5/3}} p_\theta, \tag{4.4}$$

$$\dot{p}_n = -\frac{3}{4} \frac{p_n^2}{n^{2/3}} + \frac{25}{24} \frac{(1 - \rho^2)}{n^{8/3}} p_\rho^2 + \frac{5}{24} \frac{(5 - 4\rho^2)}{\rho^2 n^{8/3}} p_\theta^2, \tag{4.5}$$

$$\dot{p}_\rho = \frac{5}{4} \frac{(\rho p_\rho^2 + p_\theta^2/\rho^3)}{n^{5/3}}, \tag{4.6}$$

$$\dot{p}_\theta = 0. \tag{4.7}$$

4.2. Normal coordinates

We first normalize our coordinates.

Theorem 9. *The metric*

$$g = \frac{2}{9n^{1/3}} \, dn^2 + \frac{4n^{5/3}}{5(1 - \rho^2)} \, d\rho^2 + \frac{4n^{5/3}}{5 - 4\rho^2} \rho^2 \, d\theta^2 \tag{4.8}$$

is isomorphic to $g = \mathrm{d}r^2 + r^2(\mathrm{d}\varphi^2 + G(\varphi)\,\mathrm{d}\theta^2)$ in the elliptic domain.

Proof. The main step is to normalize the two-dimensional metric $g = 2\,\mathrm{d}n^2/(9n^{1/3}) + 4n^{5/3}\,\mathrm{d}\rho^2/[5(1-\rho^2)]$ associated with transfer towards circular orbits. We set

$$r = \frac{2^{3/2}}{5}n^{5/6}, \qquad \varphi = \frac{1}{c}\arcsin\rho \tag{4.9}$$

with $c = \sqrt{2/5}$, in order that g takes the so-called *polar form*, $g = \mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2$. The transformation (4.9) is well defined for $n > 0$ and φ in $]-\varphi_c, \varphi_c[$, $\varphi_c = \pi/(2c)$. Letting

$$G(\varphi) = \frac{25}{2}\frac{\sin^2(c\varphi)}{1 + 4\cos^2(c\varphi)}$$

gives the desired expression. $\square$

This normal form makes the analysis of transfers towards circular orbits straightforward.

4.3. Coplanar transfers to circular orbits

We first restrict the elliptic domain to $\theta = 0$ but relax it into

$$X_0 = \{n > 0,\ \rho \in\ ]-1, 1[\}.$$

In polar coordinates (4.9), the *two-dimensional elliptic subdomain* is $X_0 = \{r > 0,\ \varphi \in\ ]-\varphi_c, \varphi_c[\}$, and the relaxation allows to go through the previous singularity, $\rho = 0$. The metric is reduced to the polar form $g = \mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2$ that we can clearly rewrite $g = r^2[(\mathrm{d}r/r)^2 + \mathrm{d}\varphi^2]$. Defining $u = \ln r$, the metric becomes $g = \mathrm{e}^{2u}(\mathrm{d}u^2 + \mathrm{d}\varphi^2)$. It belongs to the category of *Liouville metrics* [3] of the form $\lambda(u, \varphi)(\mathrm{d}u^2 + \mathrm{d}\varphi^2)$, $\lambda(u, \varphi) = f(u) + g(\varphi)$. Such metrics are known to be integrable. Actually, in these coordinates, orthonormal vector fields are defined by $F_1 = \lambda^{-1/2}\,\partial/\partial u$, $F_2 = \lambda^{-1/2}\,\partial/\partial\varphi$, and the Hamiltonian is $H = [1/(2\lambda)](p_u^2 + p_\varphi^2)$ with $\lambda = \mathrm{e}^{2u}$. Consequently, φ is a cyclic coordinate so p_φ is a first integral equal to $\langle p, F\rangle$ with $F = \partial/\partial\varphi$, and integrability holds thanks to Liouville theorem. By Noether theorem, the metric is invariant under the action of F and the Lie derivative $\mathscr{L}_F g$ is zero. These computations are summarized as follows.

Lemma 10. *The metric g is a Liouville metric with a linear first integral and the geodesic flow can be integrated using elementary functions.*

More precisely, using the polar form of the metric, we set $x = r\sin\varphi$, $z = r\cos\varphi$, and write g as the *flat* metric in dimension two with zero Gauss curvature:

$$g = \mathrm{d}x^2 + \mathrm{d}z^2. \tag{4.10}$$

Proposition 11. *The geodesics of the averaged coplanar transfer towards circular orbits are straight lines in suitable coordinates, namely*

$$x = \frac{2^{3/2}}{5}n^{5/6}\sin\left(\frac{1}{c}\arcsin\rho\right) \quad and \quad z = \frac{2^{3/2}}{5}n^{5/6}\cos\left(\frac{1}{c}\arcsin\rho\right)$$

with $c = \sqrt{2/5}$.

4.4. Coplanar transfers to general orbits

The integrability properties of the extremal flow restricted to $\{\theta = p_\theta = 0\}$ was already obtained in [10] thanks to symbolic computation in the original geometric coordinates, n and ρ. We shall also make the integration in these variables and extend it further to the full system. The integration is a consequence of the previous decomposition of the metric into a polar part $g_1 = \mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2$, and a second metric $g_2 = \mathrm{d}\varphi^2 + G(\varphi)\,\mathrm{d}\theta^2$ where θ is a cyclic coordinate. We need a preliminary result.

Lemma 12. *The coordinate $v = n^{5/3}$ is a degree two polynomial of time,*

$$v = \frac{25}{4} C_0 t^2 + \dot{v}(0)t + v(0).$$

Proof. We first note that $(d/dt)(np_n) = \dot{n}p_n + n\dot{p}_n = (5/3)\overline{H}$, and the Hamiltonian is constant along extremal curves, $\overline{H} = C_0$. Since $v = n^{5/3}$, (4.2) implies that $\dot{v} = (15/2)np_n$. Then $\ddot{v} = (25/2)C_0$, whence the conclusion. $\quad\square$

This computation allows us to integrate the two-dimensional subsystem on $\{\theta = p_\theta = 0\}$, even without taking advantage of the flat coordinates (4.10). Indeed, a first integral linear in p is obtained by noting that

$$\frac{d\rho}{dp_\rho} = \frac{1 - \rho^2}{\rho p_\rho}.$$

Hence, separating the variables leads to

$$p_\rho(1 - \rho^2)^{1/2} = C_1, \tag{4.11}$$

associated with the isometry defined by $(1 - \rho^2)^{1/2}\partial/\partial\rho$. To compute ρ, we proceed as follows. We know that v is a polynomial of degree two, $v = (25/4)C_0 t^2 + \dot{v}(0)t + v(0)$, whose discriminant is $\Delta = -(125/8)[1 - \rho^2(0)]p_\rho^2(0)$. It is negative if $p_\rho(0)$ is not zero, the integration for $p_\rho(0) = 0$ being straightforward. In the former case we have

$$|\Delta|^{1/2} = \left(\frac{5}{2}\right)^{3/2} \left(1 - \rho^2(0)\right)^{1/2} |p_\rho(0)| > 0$$

and $d\rho/(1 - \rho^2) = (5/4)C_1 \, dt/v$. Introducing $w = \arcsin\rho$ we get

$$[w]_0^t = \sqrt{\frac{2}{5}}\, \text{sign}\, p_\rho(0)[\arctan T]_0^t$$

with $T = (2at + b)/|\Delta|^{1/2}$, $a = (25/4)C_0$ and $b = \dot{v}(0)$. This gives the parameterization of the geodesics of the two-dimensional Riemannian problem underlying the transfer towards circular orbits.

In order to integrate the full system, we observe that once v has been computed, we can integrate using a reparameterization. Indeed, let us introduce

$$\overline{H}' = 5(1 - \rho^2)p_\rho^2 + (5 - 4\rho^2)\frac{p_\theta^2}{\rho^2}.$$

The induced Hamiltonian system in the symplectic space $(\rho, \theta, p_\rho, p_\theta)$ will be integrated thanks to the change of time $dT = dt/(8n^{5/3})$. It is associated with the extremal flows of the metric $g_2 = d\varphi^2 + G(\varphi)\, d\theta^2$ that is still Liouville, but with nonzero Gaussian curvature:

$$-\frac{1}{G^{1/2}} \frac{d^2 G^{1/2}}{d\varphi^2} \neq 0.$$

Though the integration is easily performed putting the metric in standard form, $g_2 = G(\varphi)[(d\varphi/G^{1/2}(\varphi))^2 + d\theta^2]$, we shall use another method coming from mechanics. Since θ is cyclic, p_θ is constant, $p_\theta = C_2$, and $\overline{H}'$ is a Hamiltonian function of the two symplectic variables (ρ, p_ρ) depending upon the parameter C_2. The associated planar system is completely integrable and θ can be computed by quadratures. More precisely, we have:

$$\overline{H}' = 5(1 - \rho^2)p_\rho^2 + \frac{5 - 4\rho^2}{\rho^2}C_2^2 = C_3^2.$$

Using $p_\rho = \dot{\rho}/10(1 - \rho^2)$, we obtain

$$\frac{d\rho}{dT} = \pm 2\sqrt{5}\, \frac{(1 - \rho^2)^{1/2}}{\rho}\left[(C_3^2 + 4C_2^2)\rho^2 - 5C_2^2\right]^{1/2}.$$

Letting $w = 1 - \rho^2$, we eventually get $(dw/dT)^2 = R$, where R is the degree two polynomial $R = 80w[(C_3^2 - C_2^2) - (C_3^2 + 4C_2^2)w]$. Whence the last result of the section.

Theorem 13. *The averaged coplanar system is completely integrable by quadratures and the value function solution of the Hamilton–Jacobi equation for the minimization of the energy can be computed.*

Remark 14. The normal form of the metric and the integrability properties of the geodesic flows are important regarding optimality issues. Indeed, the conjugate and cut loci can be analytically computed and the geometric form reveals two curvatures: the zero curvature of the metric g_1, and the curvature of the metric g_2.

5. Computation of the Riemannian spheres and concluding remarks

5.1. Riemannian spheres associated with coplanar transfers to circular orbits

On the two-dimensional elliptic subdomain X_0, the metric is real analytic and the distance (as well as the energy value function) inherits the standard properties of analytic Riemannian metrics. On balls of small enough radius, it is a continuous subanalytic function, and cut points are either conjugate points or points where two minimizing extremal curves intersect. Moreover, the cut locus is the set of singularities of the sphere and is subanalytic. Here, since the metric is the standard Euclidean one, the cut and conjugate loci are empty, but the geometry of the domain is related to the existence of geodesics and the completeness of the metric. Using flat coordinates, the following is obvious.

Proposition 15. *Given two points of the two-dimensional elliptic subdomain, existence of an energy minimizing trajectory between these points holds if and only if the segment line joining them is included in the domain.*

The restrictions on existence arise from the fact that X_0 is not *geodesically convex*, that is not convex in flat coordinates (x, y) because $\varphi_c > \pi/2$ (see Fig. 1). To some extent, this non-convexity is related to the singularity $\rho = 0$ since convexity is regained by removing the relaxation in the definition of X_0 and prescribing ρ (resp. φ) to $[0, 1[$ (resp. $[0, \varphi_c[$). As illustrated by Fig. 1 (left), there are geodesics originating from a given point towards arbitrary targets if and only if X_0 is starshaped with respect to this point.

The geometry of the domain also has the consequence hereafter.

Proposition 16. *The metric is not complete and big enough spheres are not compact.*

The result is clear since contacts within finite time occur with the boundary, either with the origin or with $\{\varphi = \pm\varphi_c\}$, that is either with $\{n = 0\}$ (trajectories going to infinity) or $\{\rho = \pm 1\}$ (parabolic trajectories). The kind of the first contact, which is responsible for the loss of compactness of the associated sphere, depends on the initial point. See Fig. 1 (right) for a classification. For a fixed point in X_0, the incompleteness of geodesics—depending on the target—is illustrated by Fig. 2. Such geodesics as well as the corresponding spheres are presented Fig. 3 in (n, ρ), that (n, e_x) coordinates (for $\theta = 0$, e_x is indeed equal to ρ when relaxed to $]-1, 1[$).
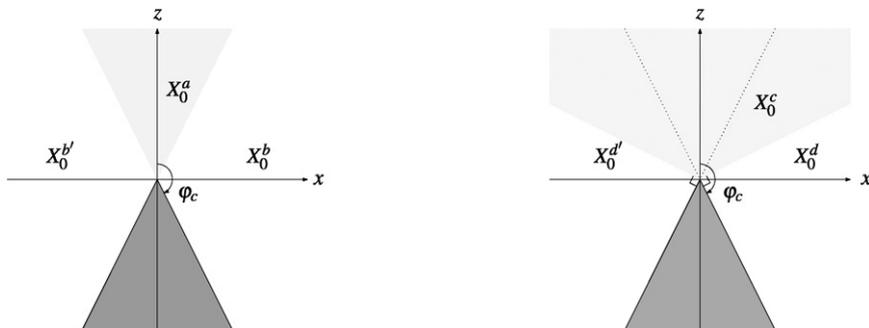

Fig. 1. Existence and completeness of geodesics. On the left, for points in X_0^a there are geodesics towards any target since X_0 is starshaped with respect to such points. Conversely, for points in X_0^b (resp. $X_0^{b'}$), existence of geodesics is lost for targets such that $\varphi \leqslant \varphi_0 - \pi$ (resp. $\varphi \geqslant \varphi_0 + \pi$). On the right, contacts with ∂X_0 are classified according to contact with $\{n = 0\}$ for initial points in X_0^c, or contact with $\{\rho = 1\}$ (resp. $\rho = -1$) for initial points in X_0^d (resp. $X_0^{d'}$).
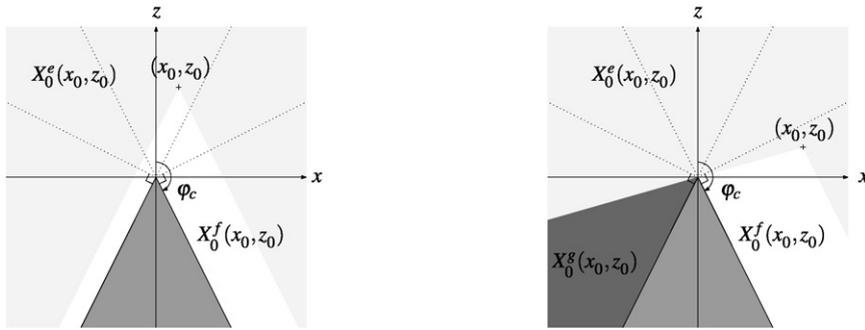
Fig. 2. Existence and completeness of geodesics. Given (x_0, z_0) in the two-dimensional elliptic subdomain, targets are classified into three categories. First, those in $X_0^e(x_0, z_0)$: geodesics exist and are complete. Second, those in $X_0^f(x_0, z_0)$ for which geodesics exist but are not complete because of contact with the boundary. Finally, those in $X_0^g(x_0, z_0)$ such that there are no geodesics at all (on the right).
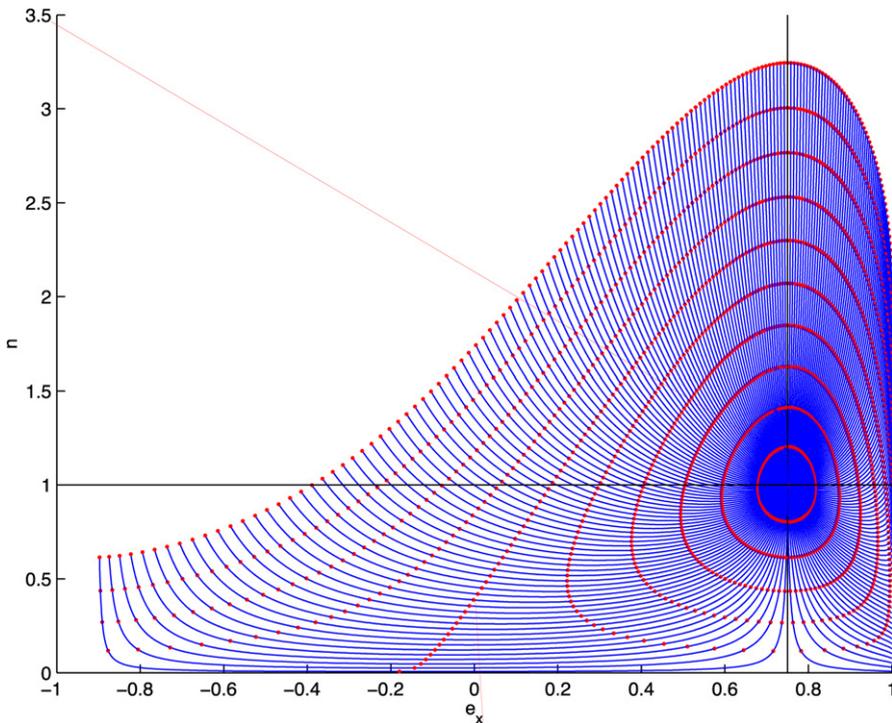

Fig. 3. Geodesics in (n, e_x) coordinates of the two-dimensional subsystem related to transfer towards circular orbits starting from $(n_0, \rho_0) = (1, 7.5e - 1)$. Dots indicate equi-criterion points, that is points on the same Riemannian spheres in $\{e_y = 0\}$ for t_f between $2e - 1$ and 2. We implicitly use the fact that because we stay on the same energy level, the radius of the Riemannian sphere is given by the final time. The given initial conditions are classified according to Fig. 1 and belong to $X_0^b \cap X_0^d$: there are no geodesics towards e_x too close to -1 (case of Fig. 2 (right)) and the loss of compactness of the sixth sphere is due to a contact with $\{\rho = 1\}$.

5.2. Riemannian spheres associated with arbitrary coplanar transfers

The three-dimensional case is more complex. In addition to the previous existence and completeness issues, the metric is not flat anymore and, because of the curvature, we can expect cut and conjugate points for large enough spheres. We shall restrict here to spheres of small radii and connect the metric to the three-dimensional Euclidean metric (in spherical coordinates) using the following homotopy:

$$g_\lambda = \mathrm{d}r^2 + r^2\left[\mathrm{d}\varphi^2 + \frac{\sin^2(c_\lambda \varphi)}{1 + 4\lambda \cos^2(c_\lambda \varphi)}\, \mathrm{d}\theta^2\right],$$
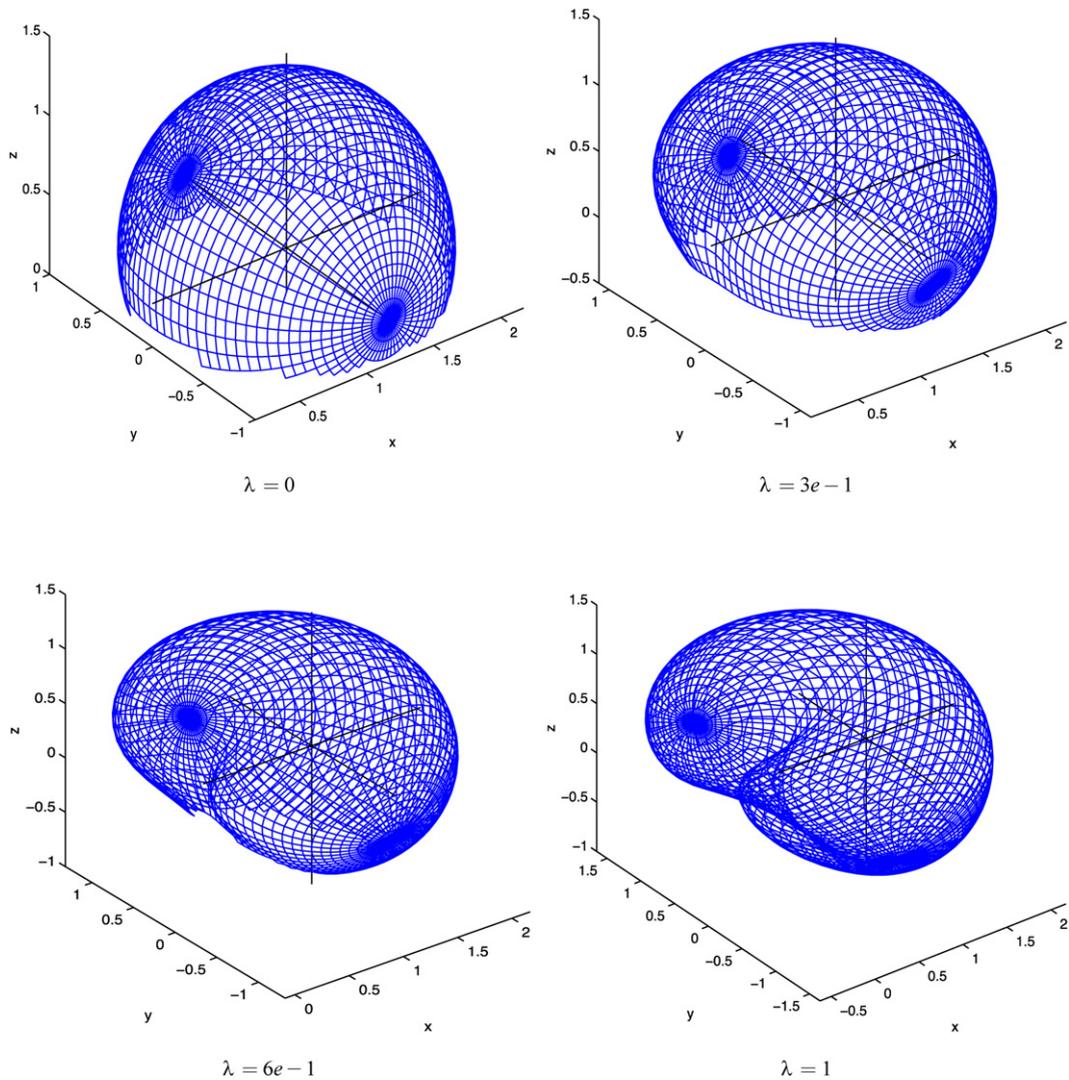
$\lambda = 0$

$\lambda = 3e - 1$

$\lambda = 6e - 1$

$\lambda = 1$

Fig. 4. Spheres in Cartesian coordinates (x, y, z) of the three-dimensional metric g_λ for λ between 0 and 1. The spheres are truncated in accordance with the domain definition, $\varphi \in]-\pi/(2c_\lambda), \pi/(2c_\lambda)[$. As before, the initial point is $(n_0, \rho_0, \theta_0) = (1, 7.5e - 1, 0)$ and the final time is equal to 1.

where $c_\lambda = (1 - \lambda)c_0 + \lambda c_1$ for $c_0 = 1$, $c_1 = \sqrt{2/5}$, λ in $[0, 1]$, and where we have slightly renormalized the metric (4.8) changing θ into $(5/\sqrt{2})\theta$ so as to get $g = \mathrm{d}r^2 + r^2(\mathrm{d}\varphi^2 + L(\varphi)\,\mathrm{d}\theta^2)$, with $L(\varphi) = \sin^2(c\varphi)/[1 + 4\cos^2(c\varphi)]$. The resulting continuous deformation of $\mathbf{S}^2$ is presented Fig. 4 in coordinates $x = r \sin\varphi \cos\theta$, $y = r \sin\varphi \sin\theta$, $z = r \cos\varphi$, as well as projections of the sphere for $\lambda = 1$ in Fig. 5. The absence of conjugate point is checked using the `cotcot`[1] algorithm presented in [5], and smoothness of small spheres ensures global optimality. Fig. 6 exhibits geodesics in dimension three together with two imbricated spheres in equinoctial coordinates for comparison with Fig. 1.

5.3. Approximation results

According to standard results in averaging for ordinary differential equations [11,14] and optimal control [9,15], the averaged system provides approximations of the true solutions. Indeed, the system is approximated by

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[1] The code is freely available at `www.n7.fr/apo/cotcot` so as to reproduce figures presented in the paper.
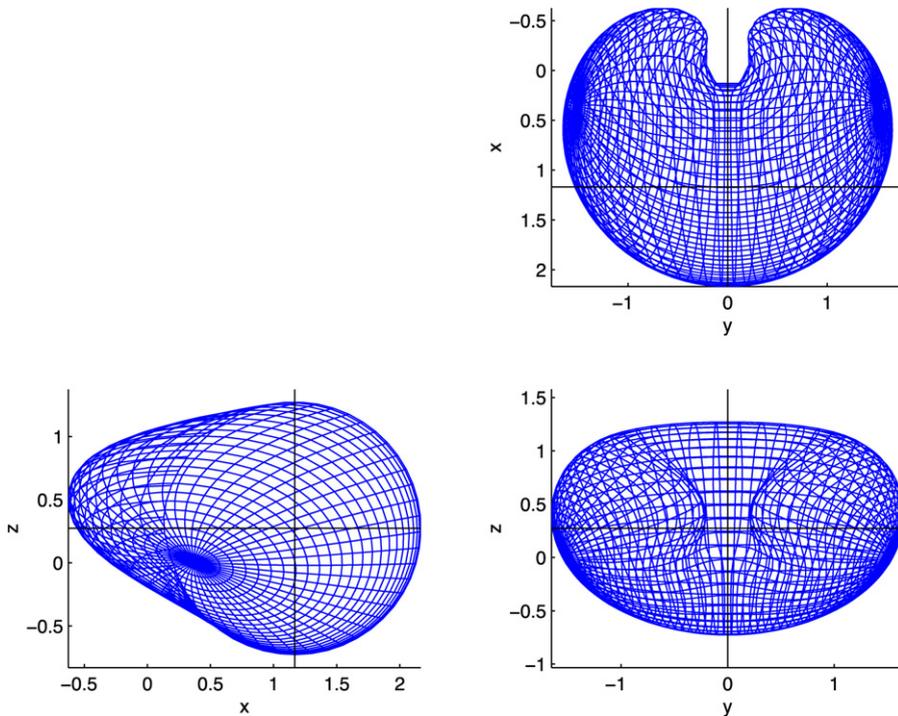
Fig. 5. Standard projections of the sphere in Cartesian coordinates (x, y, z) for $\lambda = 1$, same initial point and radius as in Fig. 4. The two-dimensional flat metric in $\{y = 0\}$ is observed in the xz-projection.

$$\frac{\mathrm{d}x}{\mathrm{d}l} = \varepsilon \sum_{i=1}^{3} u_i F_i(l, x) \quad \text{and} \quad \frac{\mathrm{d}c}{\mathrm{d}l} = \frac{\varepsilon^2}{g_0(l, x)} \sum_{i=1}^{3} u_i^2,$$

where c is the cost variable. We replace the cost c by εc and parameterize the trajectories by the *slow variable* $\tilde{l} = \varepsilon l$ to absorb ε. By virtue of the maximum principle, the extremals are integral curves of the Hamiltonian and the averaged system describes the evolution of the first term in the asymptotic expansion of the extremals in the two variables l and $\tilde{l}$ [15]: $x(\tilde{l}, l) = x_0(\tilde{l}) + \varepsilon x_1(\tilde{l}, l) + \cdots$, $p(\tilde{l}, l) = p_0(\tilde{l}) + \varepsilon p_1(\tilde{l}, l) + \cdots$. The extremal control is also developed as $u(\tilde{l}, l) = u_0(\tilde{l}, l) + \varepsilon u_1(\tilde{l}, l) + \cdots$. Herebefore, each function is periodic in l, and $z_0 = (x_0, p_0)$ is an extremal of the averaged Hamiltonian. The first order approximation u_0 of the control is obtained by dynamic feedback as a function of x_0 and p_0 and depends not only on $\tilde{l}$ but also on the fast variable l. From a classical result [2], the difference $z - z_0$ is uniformly of order $\mathrm{o}(\varepsilon)$ for a length l of order $1/\varepsilon$. The cost is then an approximation of the true cost up to order $\mathrm{o}(\varepsilon^2)$.

Approximating an energy minimization problem by a Riemannian one reveals practically important features. Since scaling the final fixed time $l_f \to l_f/\varepsilon$ readily results in a reverse scaling on the optimal control, $u \to \varepsilon u$, whatever the bound ε, it is thus possible to satisfy the constraint $|u| \leqslant \varepsilon$ by increasing the final time. Moreover, the product $l_f \times \varepsilon$ remains constant. Now, from the analysis of the previous paragraph, the same is also asymptotically true on the non-averaged energy minimization problem. Hence, an appropriate increase in l_f provides energy minimizing curves of the original problem such that the condition on the bound of the control is fulfilled, and $l_f \times \varepsilon$ tends to a constant as ε tends to zero. The same is valid for the original time of the real system too, since l and t are related by $\mathrm{d}l/\mathrm{d}t = g_0(l, x) + \mathrm{o}(\varepsilon)$. Finally, we assume that such energy minimizing curves are good approximations of minimum time ones by analogy with the Riemannian case where both problems share the same extremals. This is the starting point of the continuation approach discussed in next paragraph.

Remark 17. The previous arguments allow us to justify the asymptotic relation $\bar{t}_f \times \varepsilon \to$ constant as $\varepsilon \to 0$ that is used in practice to initialize the search for the minimum time $\bar{t}_f$ of the controlled Kepler equation [7]. This is an important step towards the answer to the conjecture in [8].
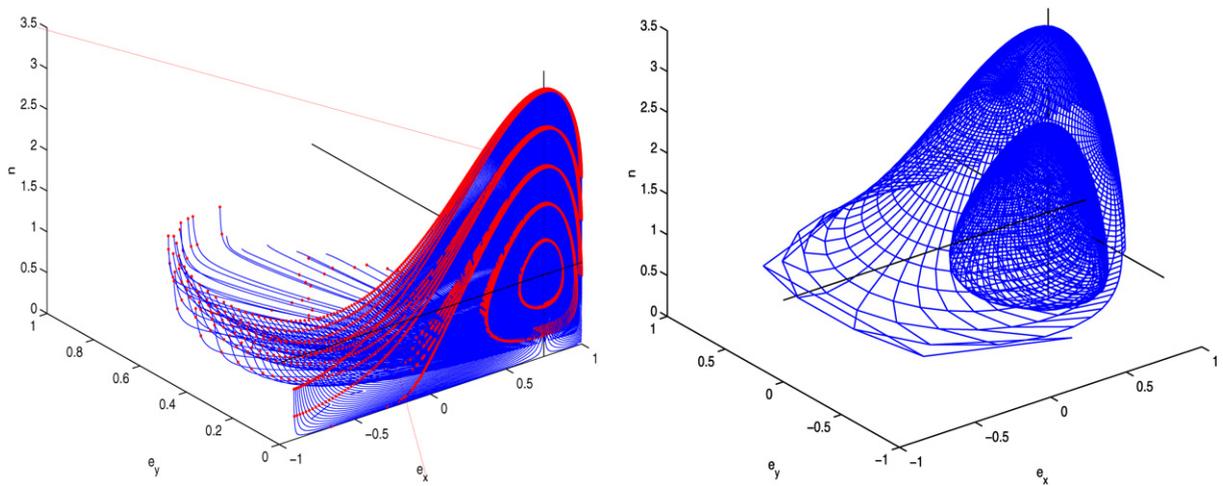
Fig. 6. Geodesics and spheres in dimension three and equinoctial coordinates (n, e_x, e_y), same initial condition as before. On the left, some geodesics up to $t_f = 2$. As previously, dots indicate points on the same spheres. The two-dimensional trajectories are observed in the plane $\{e_y = 0\}$ (compare with Fig. 3). On the right, two imbricated spheres for $t_f = 1$ and 2, respectively. The smaller one is compact and the same as in Fig. 5, but in different coordinates.

5.4. Averaging and continuation

The earliest motivation for averaging in orbit transfer is to solve the problem numerically, using a *shooting method*. Given an initial point (l_0, x_0) as well as a terminal orbit x_f, if t_f denotes the transfer time, the *shooting mapping* is defined on $\{p_{l_f} = 0\}$ by

$$S(p_0, t_f) = x(t_f, l_0, x_0, p_0) - x_f, \tag{5.1}$$

where $x(t, l_0, x_0, p_0)$ is the state component at time t of the extremal curve for initial conditions (l_0, x_0, p_0), that is the image of p_0 by the appropriate exponential mapping. As Eq. (5.1) is nonlinear, it is crucial to have a good initial guess not only of t_f, but also of p_0.

The physical problems are the minimization of the time and the minimization of the consumption, $\int_0^{t_f} |u| \, dt$. The thrust is bounded according to $|u| \leqslant \varepsilon$, and each shooting problem can be analyzed using *continuation methods* [1], which are mainly continuation on the thrust bound ε, or continuation on the cost, for instance by considering an homotopy connecting the energy problem to the consumption one:

$$\int_0^{t_f} \left[(1 - \lambda)|u|^2 + \lambda|u| \right] dt = (1 - \lambda)\|u\|_{\mathrm{L}^2}^2 + \lambda\|u\|_{\mathrm{L}^1}.$$

For each problem, we can replace for low thrust the Hamiltonian system by its averaged, and this amounts to an additional homotopy from the averaged to the initial problem. In this framework, the study allows to initialize the computation of the adjoint vector by the adjoint vector of the energy minimization averaged system. Furthermore, in the case of coplanar transfers, the system is integrable and the adjoint state can even be computed analytically.

Regarding the geometric analysis of the extremal flow, the role of the averaged system is clear. We define a Hamiltonian system close to the original one that keeps the properties of the system and has additional rigidity. For instance, a discrete symmetry group can be observed on time-minimizing extremals of the real system, see numerical experiments in [4]. By averaging, we add new symmetries that can be used to analyze the extremal curves. According to Noether theorem, these additional symmetries are obtained from symmetries of the Hamiltonian. Besides, the simpler averaged system can be much more easily used than the initial one to evaluate the cut locus or to compute periodic geodesics. This will be the objective of further studies to conclude the geometric analysis of the orbit transfer.

Acknowledgements

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