

## A RANS 3D model with unbounded eddy viscosities

### Sur un modèle de turbulence de type RANS 3D avec des viscosités turbulentes non bornées

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#### Abstract

We consider the Reynolds Averaged Navier–Stokes (RANS) model of order one  $(\mathbf{u}, p, k)$  set in  $\mathbb{R}^3$  which couples the Stokes Problem to the equation for the turbulent kinetic energy by  $k$ -dependent eddy viscosities in both equations and a quadratic term in the  $k$ -equation. We study the case where the velocity and the pressure satisfy periodic boundary conditions while the turbulent kinetic energy is defined on a cell with Dirichlet boundary conditions. The corresponding eddy viscosity in the fluid equation is extended to  $\mathbb{R}^3$  by periodicity. Our contribution is to prove that this system has a solution when the eddy viscosities are nondecreasing, smooth, unbounded functions of  $k$ , and the eddy viscosity in the fluid equation is a concave function.

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#### Résumé

On considère le modèle de turbulence moyenné d'ordre 1 issu des équations de Navier–Stokes (modèle RANS) satisfait par la vitesse moyenne  $u$ , la pression moyenne  $p$  et l'énergie cinétique turbulente  $k$  (ECT), le problème étant posé dans  $\mathbb{R}^3$ . On ne considère pas les termes de convection dans ce problème. Les équations pour la vitesse et la pression sont couplées avec l'équation pour l'ECT par des viscosités turbulentes fonctions de l'ECT et un terme quadratique dans le second membre de l'équation pour l'ECT. On considère le cas de conditions aux limites périodiques pour la vitesse et la pression, l'ECT étant définie dans une cellule avec des conditions de Dirichlet homogènes sur le bord et étendue à  $\mathbb{R}^3$  par périodicité. Les viscosités turbulentes correspondantes sont également étendues à  $\mathbb{R}^3$  par périodicité. Notre contribution dans ce travail est la preuve de l'existence d'une solution faible assez régulière à ce système, à savoir  $H^2$ , quand les viscosités turbulentes sont des fonctions croissantes de l'ECT, de classe  $C^2$ , non bornées et de plus la viscosité dans l'équation du fluide est une fonction concave.

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### 1. Introduction

#### 1.1. Position of the problem

We study problem (1.1)–(1.5) below set in  $\mathbb{R}^3$ . The unknowns are the vector field  $\mathbf{u}$  and the scalar functions  $k$  and  $p$ . The scalar  $k$  is defined on  $Q = [0, 1]^3$  with Dirichlet boundary conditions while  $\mathbf{u}$  and  $p$  are  $Q$ -periodic with zero mean value on  $Q$ ,

$$\nabla \cdot ([v_t(k, \ell)]^e \nabla \mathbf{u}) + \nabla p = \mathbf{f} \quad \text{in } \mathcal{D}'_{\text{per}}, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \mathcal{D}'_{\text{per}}, \tag{1.2}$$

$$-\nabla \cdot (\mu_t(k, \ell) \nabla k) = v_t(k, \ell) [|\nabla \mathbf{u}|^2]_Q - \frac{k\sqrt{k}}{\ell} \quad \text{in } \mathcal{D}'(Q), \tag{1.3}$$

$$(\mathbf{u}, p) \text{ } Q\text{-periodic, } \int_Q \mathbf{u} = \mathbf{0}, \quad \int_Q p = 0, \tag{1.4}$$

$$k|_{\partial Q} = 0, \quad k \geq 0 \text{ a.e. in } Q. \tag{1.5}$$

In the equations above,  $\nabla \cdot \mathbf{v} = \partial_i v^i$  ( $\mathbf{v} = (v^1, v^2, v^3)$ ) is the divergence operator. We use the following definitions: being given a scalar function  $h$  defined on  $Q$ ,  $[h]^e$  denotes its  $Q$ -periodic extension to  $\mathbb{R}^3$  and if  $h$  is a  $Q$ -periodic function,  $[h]_Q$  denotes its restriction to  $Q$ . The space  $\mathcal{D}'_{\text{per}}$  stands for the distributional space deduced from  $\mathcal{D}'(Q)$  by  $Q$  periodic reproduction.

The functions  $v_t$  and  $\mu_t$  are continuous on  $\mathbb{R}_+ \times \mathbb{R}_+$  and satisfy throughout the paper the growth conditions,  $\forall (k, \ell) \in \mathbb{R}_+ \times \mathbb{R}_+$ ,

$$0 < v \leq v_t(k, \ell), \quad v_t(k, \ell) \leq C_1(1 + \ell k^\alpha), \quad 0 < \alpha \leq 1/2, \tag{1.6}$$

$$0 < \mu \leq \mu_t(k, \ell), \quad \mu_t(k, \ell) \leq C_2(1 + \ell k^\gamma), \quad 0 < \gamma \leq 1/2. \tag{1.7}$$

Finally,  $\mathbf{f}$  is a  $H^1$   $Q$ -periodic field with zero mean value on  $Q$  and  $\ell$  is a nonnegative bounded function. We shall note in the remainder

$$\mathbf{F} = \left\{ \mathbf{f}: \mathbb{R}^3 \rightarrow \mathbb{R}^3, \text{ } Q\text{-periodic such that } \mathbf{f} \in (H^1_{\text{loc}}(\mathbb{R}^3))^3, \int_Q \mathbf{f} = \mathbf{0} \right\}. \tag{1.8}$$

#### 1.2. Physical meaning of the system

##### 1.2.1. General orientation

Systems of the form (1.1)–(1.3) play an important role in the modelization of turbulent flows. Indeed, they are the mathematical form of the RANS (Reynolds Averaged Navier–Stokes) model of order 1 used to simulate a stationary mean flow when the convection is neglected in front of the Reynolds stress. These systems are very often used in engineering or in geophysics, see for instance in [6,10,17,23,25,28] and [30], Chapter 4.

In such systems, the vector field  $\mathbf{u}$  stands for the statistical mean velocity,  $p$  is the mean pressure and  $k$  the turbulent kinetic energy (TKE). Roughly speaking, the TKE measures the variation around the average of the turbulent fields. The function  $v_t$  is the eddy viscosity,  $\mu_t$  is the eddy diffusion function and  $\ell$  is a local length scale.

The eddy viscosity and the eddy diffusion function involved in realistic models are defined by the following formula

$$v_t(k) = v + C_1 \ell \sqrt{k}, \tag{1.9}$$

$$\mu_t(k) = \mu + C_2 \ell \sqrt{k}, \tag{1.10}$$

where  $C_i$  are dimensionless constants.

The first term in the r.h.s of the  $k$ -equation (1.3),  $v_t(k, \ell) [|\nabla \mathbf{u}|^2]_Q$ , is the energy the large scales give to the small scales. This is a source of TKE. The second term,  $-\varepsilon = -(k\sqrt{k})/\ell$ , is the inverse cascade term which measures the energy rate returned by the small scales to the large scales.

*1.2.2. Physical realism of the model*

Physicists, like for instance Chen et al. [11], claim that the local length scale  $\ell$  is a constant when the turbulence is homogeneous and isotropic. Others, quoting Batchelor [2], claim that in this case there is no production of Turbulent Kinetic Energy, making useless any RANS model in such case. However, as shown in Mohammadi–Pironneau [30] (Hyp (H4) page 53), isotropy of the fluctuation is one of the main assumption to justify the derivation of the equation for the Turbulent Kinetic Energy.

In [26], we have used the same model to simulate a flow inside and outside a rigid fishing net. In this situation, the turbulence is neither homogeneous nor isotropic. In the numerical code, we have chosen  $\ell$  to be the size of the mesh. Therefore,  $\ell$  is not constant and varies with the position of the node. The numerical results obtained in [26] fit very well with the experimental data, which makes this simple turbulence model very accurate in this situation.

More sophisticated RANS models exist, in which an equation is written to compute  $\ell$ , see for instance [29]. Unfortunately, these models are still discussed in the case of geophysical flows, see the discussion in [15]. Indeed, the physical arguments to derive them are generally not convincing. Moreover, they are numerically unstable and very few mathematical results can be obtained on this class of 2 degree closure model, see also in [23], Section 4.5, Chapter 4 concerning also the well know  $(k, \varepsilon)$  model.

We also notice that in the case of very important industrial numerical applications, engineers firstly study the case where  $\ell$  is a constant in RANS models, as for instance in [28].

This bibliography shows how much these questions about turbulence modelization attract controversial reactions.

*1.3. Former works and what problem are we looking for*

The analogue of system (1.1)–(1.3) has already been studied in a bounded domain with homogeneous boundary conditions when  $v_t$  is a *bounded* function of  $k$ , and  $\ell$  is a constant. In this case we shall write  $v_t(k)$  in place of  $v_t(k, \ell)$ . The existence of a solution has been proved in this case (see [23], Chapter 6, Theorem 6.1.1, and [24]). Uniqueness questions are discussed in [9], where we prove that the solution is unique when the eddy viscosities are *smooth bounded functions close to a constant*.

We also mention that the problem of coupling two such systems with *bounded* eddy viscosities has been studied in [3,4] and [5], always for  $\ell$  constant.

*All the results mentioned above do not deal with the case where  $v_t = v_t(k)$  in the fluid equations is an unbounded function of  $k$ , like in the physical case described by formula (1.9). However, these former results are still valid when the eddy diffusion function in the  $k$ -equation satisfies the growth condition (1.7). Nevertheless, as far as we know, it remains an open problem to know if there exists a solution to these RANS equations when  $v_t = v_t(k)$  is an unbounded function of  $k$  in the 3D case. We are precisely studying this unbounded case in the present paper.*

**Remark 1.1.** As already said, all known existence results are obtained when  $\ell$  is a constant. Returning back to the case where  $\ell$  varies, is continuous satisfying  $0 < \ell_m \leq \ell(x) \leq \ell_M < \infty$  and  $v_t = v_t(k, \ell)$ , there is no doubt that when  $v_t$  is in  $L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$  and continuous with respect to the  $k$  variable, the existence of a solution can be obtained without changing the proofs.

*1.4. The main result*

This paper is mainly devoted to the case where  $\ell > 0$  is a constant. Therefore we note  $v_t = v_t(k)$  instead of  $v_t(k, \ell)$ . We aim to give a first answer to the question set by unbounded  $v_t = v_t(k)$  for the model introduced above. We prove an existence result when the velocity and the pressure satisfy periodic boundary conditions and when  $v_t$  is a smooth unbounded concave function having a bounded derivative.

The viscosities are subject to satisfy Properties 1.1 and 1.2 described below.

**Properties 1.1.** *The eddy viscosity  $v_t$  must satisfy the following properties.*

$$v_t \text{ is a } C^2\text{-class function on } \mathbb{R}_+, \tag{1.11}$$

$$v_t \text{ is nondecreasing, i.e. } v_t'(k) \geq 0, \quad k \geq 0, \tag{1.12}$$

$$v_t \text{ is concave, i.e. } v_t''(k) \leq 0, \quad k \geq 0, \quad (1.13)$$

$$v_t' \text{ is bounded.} \quad (1.14)$$

**Properties 1.2.** The eddy diffusion function  $\mu_t$  as for it must be such that

$$\mu_t \text{ is a } C^1\text{-class function on } \mathbb{R}_+, \quad (1.15)$$

$$\mu_t \text{ is a nondecreasing function on } \mathbb{R}^+, \quad (1.16)$$

$$\exists \theta > 0; \quad \forall k \geq 0, \quad C_3(\mu^{\frac{1}{\theta}} + k)^\theta \leq \mu_t(k). \quad (1.17)$$

Our main result is the following.

**Theorem 1.1.** Assume that  $\ell > 0$  is a constant and that Properties 1.1 and 1.2 hold. Let  $\mathbf{f} \in \mathbf{F}$ . There exists a constant  $\kappa = \kappa(\theta, \|\tilde{v}_t'\|_\infty)$  such that for every  $\ell > 0$  satisfying the condition

$$\ell v > \kappa(1 + v^{-\frac{3}{2(1+\theta)}} \|\mathbf{f}\|_Q \|\cdot\|_{(L^2(Q))^3}^{\frac{3}{1+\theta}}), \quad (1.18)$$

there exists

$$(\mathbf{u}, p, k) \in (H_{\text{loc}}^2(\mathbb{R}^3))^3 \times L_{\text{loc}}^2(\mathbb{R}^3) \times W_0^{1,6}(Q)$$

solution to problem (1.1)–(1.5).

### 1.5. Further comments, boundary conditions

We first note that the restrictive condition (1.18) is due to the term  $\varepsilon = k\sqrt{k}/\ell$  in the  $k$ -equation. We do not know how to remove this condition, except by neglecting  $\varepsilon$  in the  $k$ -equation which would be unrealistic.

One may wonder why dealing with periodic conditions in the fluid equations. This is simply because we shall consider in our proof of Theorem 1.1 the formal derivative of Eqs. (1.1) and (1.2), the fluid part of the system. Therefore, this makes it possible to study the gradients of the velocity and the pressure because they also satisfy periodic conditions. However, in the case of a domain in  $\mathbb{R}^3$ , we do not have any informations about the values taken by the gradient of the velocity at the boundary. Periodic conditions remove this difficulty.

We conjecture that the same result holds in a bounded domain in  $\mathbb{R}^3$  with homogeneous Dirichlet boundary conditions for  $\mathbf{u}$ , but we have the feeling that the proof will be hard and very technical to write.

Now the question arises to know why we do not study periodic conditions for  $k$  and why did we have consider this so strange situation. This is because such periodic conditions on  $k$  yields the compatibility condition

$$\int_Q v_t(k) |\nabla \mathbf{u}|^2 = \frac{1}{\ell} \int_Q k\sqrt{k}, \quad (1.19)$$

an unrealistic condition. Indeed, when one lets  $\ell$  go to infinity in (1.19), one would have zero as limit for  $\mathbf{u}$  unless  $k$  blows up in the space  $L^{3/2}$ , which is not the case thanks to the classical known estimates giving a uniform bound for  $k$  in each  $L^s$ ,  $s < 3$ . Therefore, this is possible if and only if  $\mathbf{f} = 0$ , where in this case  $\mathbf{u} = 0$ ,  $k = 0$  and  $p = 0$ .

This is why we had to consider  $k$  defined only inside a cell  $Q$  with homogeneous Dirichlet boundary conditions and then to take the periodic extension of the corresponding eddy viscosity in the fluid equation (1.1). Notice that this does not imply that the  $k$ -equation is satisfied in whole  $\mathbb{R}^3$ .

The physical consequence is that the TKE is a constant on the interface of the cells, describing homogeneous boundary layers there.

### 1.6. About the eddy viscosities properties

The question is how does Properties 1.1 and 1.2 fit with physical reality and what about the numerical reality when simulations are performed with codes using such models.

Actually, the growth hypotheses are well satisfied by realistic  $\nu_t$  and  $\mu_t$  which are nondecreasing functions, as well as  $\nu_t$  is a concave function, one of the main feature of our result. However the required regularities for  $\nu_t$  and  $\mu_t$  fail because of the behavior of the realistic viscosities near 0. Let us go into more details.

The eddy diffusion function  $\mu_t$  given by formula (1.10) is continuous and satisfies the growth condition (1.7), as well as it is a nondecreasing function satisfying the below growth condition (1.17) with  $\theta = 1/2$ . Therefore, these assumptions fit well with the physical reality in the case of  $\mu_t$ . As already said, the  $C^1$ -class condition is not satisfied because of the singularity at 0. Therefore, the function  $\mu_t$  given by formula (1.10) should be replaced by

$$\mu_t(k) = \mu + C_2 \ell \sqrt{\tau + k}, \quad k \geq 0, \tau > 0, \tag{1.20}$$

We conjecture that the  $C^1$ -class hypothesis can be removed and only a continuity hypothesis on  $\mu_t$  should be enough to conclude. However this remains an open problem.

Because of the same reason due to a lack of regularity near 0,  $\nu_t$  is not a  $C^2$ -class function with a bounded derivative when it is defined by the formula (1.9) even if the growth condition (1.6) is satisfied. However, when  $\nu_t$  is defined by the physical formula (1.9) it is a nondecreasing and concave function. From this point of view, we are glad to observe a good physical correspondence with our mathematical analysis. Therefore, as we did for  $\mu_t$ , formula (1.6) should be replaced by

$$\nu_t(k) = \nu + C_1 \ell \sqrt{\rho + k}, \quad k \geq 0, \rho > 0, \tag{1.21}$$

a function which satisfies Properties 1.1. It seems to us that this is more difficult to remove this  $C^2$ -class hypothesis on  $\nu_t$  than in the case of  $\mu_t$ .

The viscosities properties are involved because of the regularity considerations which are the key of the present work. Indeed, we shall show in the remainder how to construct a solution to our problem with a  $H^2$  velocity. As said before, we shall consider the formal derivative of Eqs. (1.1), (1.2). A bound on  $\nu_t'$  is crucial to obtain an a priori  $H^2$  estimate on  $\mathbf{u}$  as well as the concavity and the nondecreasing hypothesis on  $\nu_t$ .

Finally, what is the role played by the below growth condition (1.17)? Actually, the equation for  $k$  is naturally an equation “with a second hand side in  $L^1$ ” due to the production term  $\nu_t(k)|\nabla \mathbf{u}|^2$ . Thus the classical Boccardo–Gallouët’s inequality [7] yields  $k \in \bigcap_{p < 3/2} W^{1,p}$ . As said already, regularity is the key of our result. We shall construct a solution  $k$  continuous, especially bounded on  $\mathbb{R}^3$ . To do this, we need to increase the regularity of the terms  $\nu_t(k)|\nabla \mathbf{u}|^2$  and  $-k\sqrt{k}$  in the  $k$ -equation (1.3). The “below growth condition” (1.17) is one needed feature among others to derive such regularity.

*In conclusion, the existence of a solution to the system holds for eddy viscosities of the form (1.20) and (1.21) in the place of (1.9) and (1.10), because they do have derivative’s singularity at zero point. We stress that this encountered difficulty for small values of  $k$  in such models is well known by engineers who use truncations in their numerical codes when  $k$  is near 0 (see [28]). Therefore, there is a good correspondence between our mathematical analysis and the numerical reality. Physical formula are obtained by a dimensional analysis. Therefore (1.20) and (1.21) would be also physically accurate if one could find a physical meaning to the small quantities  $\tau$  and  $\rho$ , more than just formal cut-off.*

### 1.7. Additional bibliographical remarks

#### 1.7.1. Scalar systems, unbounded viscosities: renormalized and energy solutions

Let us mention that in [23], Chapter 5, Section 5.2, [12,13,18], and [19], one considers the following system set in an open bounded domain in  $\mathbb{R}^n$  with homogeneous boundary conditions for scalar quantities  $(u, k)$ , always for  $\ell$  constant,

$$-\nabla \cdot (\nu_t(k) \nabla \mathbf{u}) = f, \tag{1.22}$$

$$-\nabla \cdot (\mu_t(k) \nabla k) = \nu_t(k) |\nabla \mathbf{u}|^2. \tag{1.23}$$

In Refs. [18,12] and [13], the system (1.22), (1.23) is involved in heat conduction problems while in [19] and [23] Section 5.2, it is studied as a simplification of the RANS model to focus on the question raised by the quadratic term and remove for clarity the difficulties due to the pressure term, the incompressibility constrain and the dissipation term  $-\varepsilon$ . In [12] and [18], existence results are proved when  $\nu_t$  and  $\mu_t$  are bounded functions of  $k$ .

In [23] Section 5.2, one proves the existence of a renormalized solution to the scalar system (1.22), (1.23) when  $v_t$  and  $\mu_t$  are unbounded functions of  $k$  (but still satisfy a growth condition at infinity). The main result of Section 5.2 in [23], Theorem 5.3.1, has been obtained in collaboration with F. Murat. In [19] one proves the existence of an energy solution in the same unbounded case and when  $v_t$  is regularized near zero like in formula (1.21).

Notice that we have not been able to adapt to the RANS model (1.1)–(1.3) the techniques of [19] and [23], Section 5.2 when the eddy viscosities are unbounded. This is directly linked to the impossibility to give a renormalized sense to the Stokes and/or the Navier–Stokes equations in the spirit of Di Perna–Lions (see [16]) and Lions and Murat [27].

**Remark 1.2.** In Remark 1.1 we have said that the known existence results can be obtained when  $\ell$  varies, is non-negative, continuous bounded,  $v_t = v_t(k, \ell)$  is in  $L^\infty$  and continuous with respect to  $k$ . Unfortunately, we think that the proofs in [19] and [23] cannot be directly adapted to scalar systems in this case, which is an interesting open mathematical question.

*1.7.2. Scalar systems, unbounded viscosities: 2-dimensional case*

In [13] the authors prove the existence of a solution to the simplified scalar system (1.22), (1.23) when  $v_t$  and  $\mu_t$  are unbounded functions in the 2D case by proving that  $k \in L^\infty$ . The techniques of [13] can be adapted to RANS systems like (1.1)–(1.3) in the 2D case but it does not work in the 3D case under current consideration. Indeed, in the 2D case, Boccardo–Gallouët estimate [7] yields  $k \in \bigcap_{p < \infty} L^p$ . In [13], one fills the gap between  $k \in \bigcap_{p < \infty} L^p$  and  $L^\infty$  by a nice improvement of [7] to this case.

In our present 3D case, [7] yields  $k \in \bigcap_{p < 3} L^p$ . Then the gap to reach  $L^\infty$  is too large in the 3D case and will not be filled by this way. However, we also prove that  $k \in L^\infty$  by showing firstly that  $u \in H^2$ .

*1.8. Case  $\ell$  nonconstant*

We have also investigated the case where  $\ell$  is not a constant. Indeed, this case is very important from the physical point of view and for the applications as we said before. This case is very difficult and unfortunately, we are not able to prove a similar result as in Theorem 1.1. The reasons will be made clear until the end of the paper. The only result that we are able to prove here in this direction is the following.

**Theorem 1.2.** *Assume that*

$$v_t(k, \ell) = v_0 + C_1 \ell(k + \rho)^\alpha, \quad v_0 > 0, C_1 > 0, \rho > 0, 0 < \alpha < \frac{1}{2}, \tag{1.24}$$

$$\mu_t = \mu_0 \in \mathbb{R}_+^*, \tag{1.25}$$

$$\varepsilon(k, \ell) = \frac{k^{1+\theta}}{\ell(x)}, \quad 0 < \theta < 1/2, \tag{1.26}$$

$$\ell \text{ is a function of class } C^2, \quad -\Delta \ell \geq 0, \quad 0 < \ell_m \leq \ell(x) < \ell_M < \infty, \tag{1.27}$$

$$\left( v - \frac{\tilde{\Pi}}{\ell_m} - \|\nabla \ell\|_{L^\infty} \tilde{\Omega} \right) > 0 \tag{1.28}$$

where  $\kappa = \kappa(\rho, \ell_M)$ ,  $\tilde{\Pi} = \tilde{\Pi}(\mathbf{f}, \rho, \alpha, v, \theta)$ ,  $\tilde{\Omega} = \tilde{\Omega}(\mathbf{f}, \rho, \alpha, v, \theta)$ . Then there exists

$$(\mathbf{u}, p, k) \in (H_{\text{loc}}^2(\mathbb{R}^3))^3 \times L_{\text{loc}}^2(\mathbb{R}^3) \times W_0^{1,6}(Q)$$

solution to the problem

$$-\nabla \cdot ([v_t(k, \ell)]^e \nabla \mathbf{u}) + \nabla p = \mathbf{f} \quad \text{in } \mathcal{D}'_{\text{per}}, \tag{1.29}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \mathcal{D}'_{\text{per}}, \tag{1.30}$$

$$-\mu_0 \Delta k = v_t(k, \ell) [|\nabla \mathbf{u}|^2]_Q - \varepsilon(k, \ell) \quad \text{in } \mathcal{D}'(Q), \tag{1.31}$$

$$(\mathbf{u}, p) \text{ } Q\text{-periodic, } \int_Q \mathbf{u} = \mathbf{0}, \int_Q p = 0, \tag{1.32}$$

$$k|_{\partial Q} = 0, \quad k \geq 0 \text{ a.e. in } Q. \tag{1.33}$$

The constants  $\tilde{\Gamma}$  and  $\tilde{\mathcal{L}}$  in (1.28) depend on  $\mathbf{f}, \rho, \alpha, \nu$  and  $\theta$  and will be precised in Section 4. Of course, in the proof of Theorem 1.2, we shall use the fact that  $\nu_t$  is concave with respect to  $k$ , that is  $\partial^2 \nu_t / \partial k^2 \leq 0$ . We conjecture that it is possible to prove the analogue of Theorem 1.1, in particular when  $\alpha = 1/2$  and  $\mu_t = \mu_0 + C_2 \ell (k + \tau)^\alpha$ ,  $\varepsilon(k, \ell) = k\sqrt{k}/\ell(x)$ . This work is in progress. However, we do not know how to remove condition (1.28).

### 1.9. Organization of the paper

The continuation of this article is the following. Its main part is devoted to the case where  $\ell$  is a constant,  $\nu_t = \nu_t(k)$ . We first construct carefully smooth approximations to the RANS system (1.1)–(1.5) after changing the variable  $k$  into  $\tilde{k}$  thanks to the transformation

$$\tilde{k} = \int_0^k \mu_t(k') \, dk'. \tag{1.34}$$

Then one shows that the sequence of corresponding velocities is bounded in  $H^2$  by studying the formal derivative of the subsystem (1.1), (1.2). The periodic conditions play a role at this step, because one knows that the gradient of the velocity still verifies periodic conditions and therefore one can deduce estimates for it. A  $L^\infty$  bound for the sequence of TKE can be obtained. That makes it possible to reduce the problem to the case of bounded eddy viscosities and to pass to the limit in the equations as in former works.

In a last section, we prove Theorem 1.2.

## 2. Construction of approximations

### 2.1. Orientation

In this section as well as in Section 3,  $\ell > 0$  is a constant and one writes  $\nu_t(k)$  and  $\mu_t(k)$  instead of  $\nu_t(k, \ell)$  and  $\mu_t(k, \ell)$ . We start by some transformations of the  $k$ -equation. The backward term is first replaced by  $-k\sqrt{|k|}$  as well as  $\nu_t(k)$  and  $\mu_t(k)$  are replaced by  $\nu_t(|k|)$  and  $\mu_t(|k|)$ , as far as we do not have proved yet the positivity of  $k$ .

Next, we shall use the change of variable (1.34) mentioned above. That makes it possible to change the operator  $-\nabla \cdot (\mu_t \nabla)$  into the operator  $-\Delta$ . We shall obtain a new system that we shall study in the remainder, the system (2.7)–(2.11) below. The solutions of this system provide solutions to the RANS system (1.1)–(1.5) (see Proposition 2.1 below).

Next, we shall construct an approximated system to the new system (2.7)–(2.11) with a smooth bounded eddy viscosity in the fluid equation and a regularized r.h.s for the  $\tilde{k}$  equation as well as a regularization of the cube  $Q$ . We need to regularize  $Q$  by smooth approximated convex domains  $Q_\varepsilon$  again because of a regularity’s consideration,  $Q$  being not a domain having a  $C^1$  boundary.

The existence of a smooth solution to this approximated system will be proved by using Leray–Schauder fixed point Theorem (see [22]).

Throughout the paper, we shall assume that the eddy viscosity  $\nu_t$  satisfies Properties 1.1 and 1.2 as well as the growth conditions (1.6) and (1.7).

### 2.2. Transformation of the system

As far as we do not have any information on the  $k$ ’s sign, we shall first replace Eqs. (1.1)–(1.3) by the following ones,

$$-\nabla \cdot ([\nu_t(|k|)]^c \nabla \mathbf{u}) + \nabla p = \mathbf{f}, \tag{2.1}$$

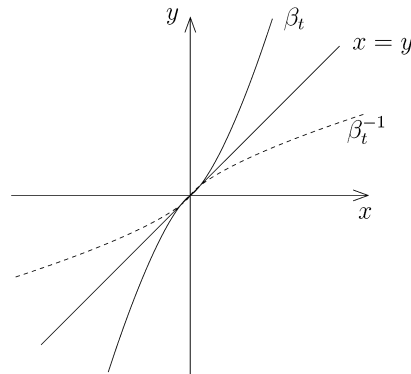


Fig. 1.

$$\nabla \cdot \mathbf{u} = 0, \tag{2.2}$$

$$-\nabla \cdot (\mu_t(|k|)\nabla k) = \nu_t(|k|)[|\nabla \mathbf{u}|^2]_Q - \frac{k\sqrt{|k|}}{\ell}. \tag{2.3}$$

To remove the operator  $-\nabla \cdot (\mu_t \nabla)$ , we introduce the odd function  $\beta_t$  defined on  $\mathbb{R}$  by

$$\begin{aligned} \forall k \geq 0, \quad \tilde{k} = \beta_t(k) &= \int_0^k \mu_t(k') dk', \\ \forall k \leq 0, \quad \tilde{k} = \beta_t(k) &= -\beta_t(-k). \end{aligned} \tag{2.4}$$

The function  $\beta_t$  is a  $C^2$ -class function because  $\mu_t$  is of class  $C^1$ . This is also an odd nondecreasing function, convex on  $\mathbb{R}^+$  because  $\mu_t$  is nondecreasing (see (1.16)). Thus the inverse function  $\beta_t^{-1}$  exists. It is a  $C^2$ -class odd function, concave on  $\mathbb{R}^+$  (see Fig. 1).

Let  $\mathcal{E}$  be the function defined by

$$k\sqrt{|k|} = \mathcal{E}(\tilde{k}) = \beta_t^{-1}(\tilde{k})\sqrt{|\beta_t^{-1}(\tilde{k})|} \tag{2.5}$$

and  $\tilde{\nu}_t$  the function defined by

$$\nu_t(|k|) = \tilde{\nu}_t(\tilde{k}) = \nu_t(|\beta_t^{-1}(\tilde{k})|). \tag{2.6}$$

Using the variable  $\tilde{k}$ , the system (1.1)–(1.5) becomes

$$-\nabla \cdot ([\tilde{\nu}_t(\tilde{k})]^e \nabla \mathbf{u}) + \nabla p = \mathbf{f} \quad \text{in } \mathcal{D}'_{\text{per}}, \tag{2.7}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \mathcal{D}'_{\text{per}}, \tag{2.8}$$

$$-\Delta \tilde{k} = \tilde{\nu}_t(\tilde{k})[|\nabla \mathbf{u}|^2]_Q - \frac{\mathcal{E}(\tilde{k})}{\ell} \quad \text{in } \mathcal{D}'(Q), \tag{2.9}$$

$$(\mathbf{u}, p) \text{ } Q\text{-periodic}, \quad \int_Q \mathbf{u} = 0, \quad \int_Q p = 0, \tag{2.10}$$

$$\tilde{k}|_{\partial Q} = 0, \quad \tilde{k} \geq 0 \text{ a.e. in } Q. \tag{2.11}$$

By the below growth condition (1.17), one has for  $k \geq 0$ ,

$$\tilde{k} = \beta_t(k) \geq \frac{C_3}{\theta + 1} (\mu^{\frac{1}{\theta}} + k)^{1+\theta} - \frac{C_3}{\theta + 1} \mu^{\frac{1+\theta}{\theta}},$$

which can be rewritten

$$|k| = |\beta_t^{-1}(\tilde{k})| \leq C_4(1 + |\tilde{k}|^{\frac{1}{1+\theta}}), \quad \theta > 0, \tag{2.12}$$



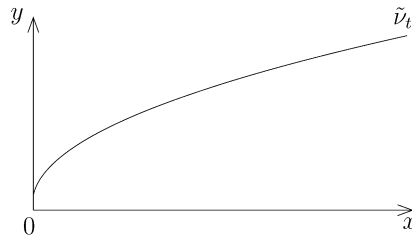


Fig. 2.

because  $\beta_t$  is odd. In the last inequality,  $C_4$  is a constant. Therefore the following holds:

$$\forall \tilde{k} \in \mathbb{R}, \quad 0 < v \leq \tilde{v}_t(\tilde{k}) \leq C_5(1 + |\tilde{k}|^{\frac{\alpha}{1+\theta}}), \quad 0 < \alpha \leq 1/2, \theta > 0, \tag{2.13}$$

$$\forall \tilde{k} \in \mathbb{R}, \quad |\mathcal{E}(\tilde{k})| \leq C_6(1 + |\tilde{k}|^{\frac{3}{2(1+\theta)}}), \quad \theta > 0. \tag{2.14}$$

The function  $v_t$  has a bounded derivative function as well as  $(\beta_t^{-1})'$ , because  $\beta_t^{-1}$  is concave on  $\mathbb{R}_+$ , odd, of class  $C^2$ . Moreover, because  $\beta_t^{-1}$  and  $v_t$  are non decreasing  $C^2$ -class functions, concave on  $\mathbb{R}^+$ ,  $\tilde{v}_t$  defined by (2.6) satisfies the same following properties  $v_t$  on  $\mathbb{R}^+$  (see Fig. 2).

**Properties 2.1.** *The function  $\tilde{v}_t$  satisfies*

$$\tilde{v}_t \text{ is a } C^2\text{-class function on } \mathbb{R}_+, \tag{2.15}$$

$$\tilde{v}_t \text{ is nondecreasing,} \tag{2.16}$$

$$\tilde{v}_t \text{ is concave,} \tag{2.17}$$

$$\tilde{v}'_t \text{ is bounded.} \tag{2.18}$$

**Proposition 2.1.** *Let  $(\mathbf{u}, p, \tilde{k})$  be a solution to the system (2.7)–(2.11) where*

$$\mathbf{u} \in (H^2_{\text{loc}}(\mathbb{R}^3))^3, \quad p \in L^2_{\text{loc}}(\mathbb{R}^3), \quad \tilde{k} \in W^{1,6}_0(Q).$$

*Let  $k = \beta_t^{-1}(\tilde{k})$ . Then  $k \in W^{1,6}_0(Q)$  and  $(\mathbf{u}, p, k)$  is a solution to the system (1.1)–(1.5).*

**Proof.** We mainly have to check the regularity of  $k$ . Notice that  $\tilde{k} \in W^{1,6}_0(Q) \subset C^0(Q)$  (the dimension is 3). Let  $n_0 = \|\tilde{k}\|_{L^\infty}$  and  $T_{n_0}$  be the truncature function at height  $n_0$ . It means

$$T_{n_0}(x) = x \quad \text{if } |x| \leq n_0, \quad T_{n_0}(x) = n_0 \frac{x}{|x|} \quad \text{if } n_0 \leq |x|, \tag{2.19}$$

by assuming that  $n_0 \neq 0$ . If it is not, then the result is obvious.

Because  $T_{n_0}(\tilde{k}) = \tilde{k}$ ,

$$k = \beta_t^{-1}(\tilde{k}) = \beta_t^{-1} \circ T_{n_0}(\tilde{k}). \tag{2.20}$$

The function  $\beta_t^{-1} \circ T_{n_0}$  is Lipschitz uniformly on  $\mathbb{R}$  and its derivative has a finite number of discontinuities. Thus thanks to a deep result due to G. Stampacchia (see [32], Lemma 1.2, page 17)  $k \in W^{1,6}_0(Q)$  and one has

$$\nabla k = (\beta_t^{-1} \circ T_{n_0})'(\tilde{k}) \nabla \tilde{k}.$$

The end of the proof is straightforward.  $\square$

*The remainder of the present section is devoted to prove the existence of a solution to the system (2.7)–(2.11), a solution which will be obtained by approximations.*

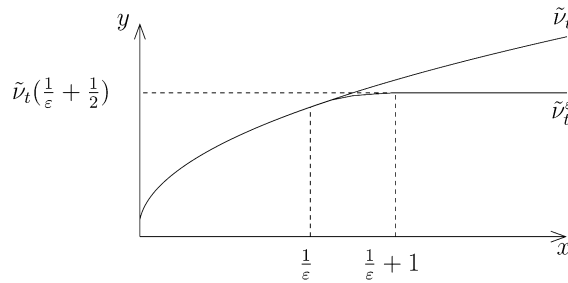


Fig. 3.

2.3. Construction of approximations

This subsection is divided into the following steps:

- constructing smooth bounded approximations to  $\tilde{\nu}_t$  and  $\mathcal{E}$ ,
- setting the approximated system and statement of the existence result, further comments,
- proving the existence result by fixed point theorem.

2.3.1. Smoothing  $\tilde{\nu}_t$  and  $\mathcal{E}$

One defines smooth bounded approximations to the eddy viscosity  $\tilde{\nu}_t$  in the fluid equation (2.7) and the function  $\mathcal{E}$  in the r.h.s of the  $\tilde{k}$ -equation (2.9), starting by  $\nu_t$ .

Let  $\varepsilon > 0$  and let us consider the function  $\tilde{\nu}_t^\varepsilon$  (see Fig. 3) be such that

$$\tilde{\nu}_t^\varepsilon \text{ is a } C^2\text{-class function on } \mathbb{R}_+, \tag{2.21}$$

$$\forall x \in [0, 1/\varepsilon], \quad \tilde{\nu}_t^\varepsilon(x) = \tilde{\nu}_t(x), \tag{2.22}$$

$$\forall x \geq 1 + 1/\varepsilon, \quad \tilde{\nu}_t^\varepsilon(x) = \tilde{\nu}_t(1/2 + 1/\varepsilon), \tag{2.23}$$

$$\forall x \in \mathbb{R}_+, \quad (\tilde{\nu}_t^\varepsilon)'(x) \leq \tilde{\nu}_t'(x), \tag{2.24}$$

$$\forall x \in \mathbb{R}, \quad \tilde{\nu}_t^\varepsilon(x) = \tilde{\nu}_t^\varepsilon(-x). \tag{2.25}$$

The existence of the sequence  $(\tilde{\nu}_t^\varepsilon)_{\varepsilon>0}$  is straightforward by Properties 2.1. Moreover, the following is satisfied by  $(\tilde{\nu}_t^\varepsilon)_{\varepsilon>0}$ . We note that  $\tilde{\nu}_t^\varepsilon = \tilde{\nu}_t$  on the range  $[0, 1/\varepsilon]$ .

**Properties 2.2.** Each  $\tilde{\nu}_t^\varepsilon$  is a  $C^2$ -class function such that

$$\tilde{\nu}_t^\varepsilon \text{ is nondecreasing,} \tag{2.26}$$

$$\tilde{\nu}_t^\varepsilon \text{ is concave,} \tag{2.27}$$

$$\forall \varepsilon > 0, \quad \tilde{\nu}_t^\varepsilon \text{ is a bounded function,} \tag{2.28}$$

$$((\tilde{\nu}_t^\varepsilon)')_{\varepsilon>0} \text{ is uniformly bounded with respect to } \varepsilon, \tag{2.29}$$

$$\forall \varepsilon > 0, \forall x \in \mathbb{R}, \quad 0 < \nu \leq \tilde{\nu}_t^\varepsilon(x). \tag{2.30}$$

Let us consider  $\mathcal{E}$  now.

**Definition 2.1.** Let  $\mathcal{E}_\varepsilon$  (see Fig. 4) be defined by

$$\forall x \in [0, 1/\varepsilon], \quad \mathcal{E}_\varepsilon(x) = \beta_t^{-1}(x) \sqrt{\sqrt{\beta_t^{-1}(x)^2 + \varepsilon^2} + \varepsilon^2}, \tag{2.31}$$

$$\forall x \geq 1 + 1/\varepsilon, \quad \mathcal{E}_\varepsilon(x) = \beta_t^{-1}(1/2 + 1/\varepsilon) \sqrt{\sqrt{\beta_t^{-1}(1/2 + 1/\varepsilon)^2 + \varepsilon^2} + \varepsilon^2}, \tag{2.32}$$

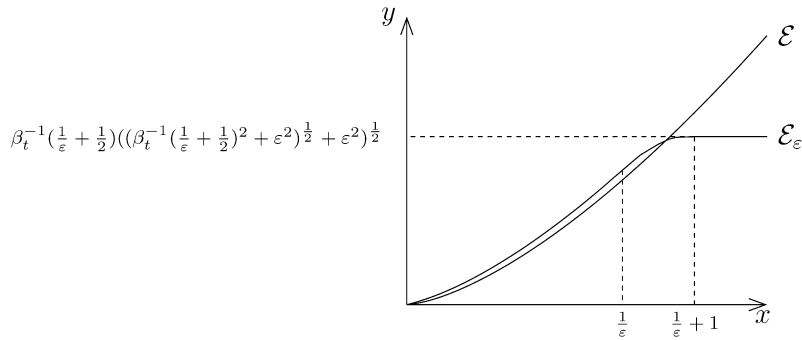


Fig. 4.

$\mathcal{E}_\varepsilon$  is a function of class  $C^2$  on  $\mathbb{R}$ , (2.33)

$\mathcal{E}_\varepsilon(-x) = -\mathcal{E}_\varepsilon(x)$ . (2.34)

The following lemma is straightforward.

**Lemma 2.1.** *The sequence  $(\mathcal{E}_\varepsilon)_{\varepsilon>0}$  converges to  $\mathcal{E}$  uniformly on the compact sets of  $\mathbb{R}$ . Moreover we have*

$$|\mathcal{E}_\varepsilon(\tilde{k})| \leq C_7(1 + |\tilde{k}|^{\frac{3}{2(1+\theta)}}), \quad \theta > 0 \tag{2.35}$$

thanks to the growth condition (2.14) where the constant  $C_7$  do not depend on  $\varepsilon$ .

For reasons that will be made clear in the remainder, we have to split  $\mathcal{E}_\varepsilon$  as a product of two functions  $\zeta_\varepsilon$  and  $\gamma_\varepsilon$ , where  $\zeta_\varepsilon$  is the function defined by the following

**Definition 2.2.** Let  $\zeta_\varepsilon$  be defined by

$$\forall x \in [0, 1/\varepsilon], \quad \zeta_\varepsilon(x) = \sqrt{\sqrt{\beta_t^{-1}(x)^2 + \varepsilon^2} + \varepsilon^2}, \tag{2.36}$$

$$\forall x \geq 1 + 1/\varepsilon, \quad \zeta_\varepsilon(x) = \sqrt{\sqrt{\beta_t^{-1}(1/2 + 1/\varepsilon)^2 + \varepsilon^2} + \varepsilon^2}, \tag{2.37}$$

$\zeta_\varepsilon$  is a function of class  $C^2$  on  $\mathbb{R}$ , (2.38)

$\zeta_\varepsilon(-x) = \zeta_\varepsilon(x)$ . (2.39)

**Definition 2.3.** Let  $\gamma_\varepsilon$  be the function defined by

$$\forall x \in [0, 1/\varepsilon], \quad \gamma_\varepsilon(x) = \beta_t^{-1}(x) \tag{2.40}$$

$$\forall x \geq 1 + 1/\varepsilon, \quad \gamma_\varepsilon(x) = \beta_t^{-1}(1/2 + 1/\varepsilon), \tag{2.41}$$

$\gamma_\varepsilon$  is a function of class  $C^2$  on  $\mathbb{R}$ , (2.42)

$\gamma_\varepsilon(-x) = -\gamma_\varepsilon(x)$ . (2.43)

Notice that one has  $\zeta_\varepsilon > 0$ ,  $\gamma_\varepsilon$  is odd and

$$\mathcal{E}_\varepsilon(x) = \gamma_\varepsilon(x)\zeta_\varepsilon(x). \tag{2.44}$$

Notice also that thanks to the growth condition (2.12) satisfied by  $\beta^{-1}$ , one has

$$|\gamma_\varepsilon(\tilde{k})| \leq D_1(1 + |\tilde{k}|^{\frac{1}{1+\theta}}), \quad \theta > 0, \tag{2.45}$$

$$|\zeta_\varepsilon(\tilde{k})| \leq D_2(1 + |\tilde{k}|^{\frac{1}{2(1+\theta)}}), \quad \theta > 0, \tag{2.46}$$

for  $D_1$  and  $D_2$  be constant.

2.3.2. *Some geometrical considerations*

The domain  $Q$  has not a smooth boundary. However, we want to deal with very smooth approximations  $\tilde{k}_\varepsilon$ , that means  $\tilde{k}_\varepsilon \in H^3$  at least. It is well known that high regularity not hold as well as if  $Q$  would have a  $C^\infty$  boundary. However,  $Q$  is a Lipschitz domain and there exists a cone  $C$  such that  $Q$  has the cone property determined by the cone  $C$ , as defined in [1], Chapter IV, Definition 4.3. Moreover, it is straightforward that the cone  $C$  can be chosen such that there exists a sequence  $(Q_\varepsilon)_{\varepsilon>0}$  of open bounded sets in  $\mathbb{R}^3$  satisfying

$$\partial Q_\varepsilon \text{ is } C^\infty, \tag{2.47}$$

$$\forall \varepsilon_1 < \varepsilon_2, \quad Q_{\varepsilon_2} \subset Q_{\varepsilon_1} \subset Q, \tag{2.48}$$

$$\bigcap_{\varepsilon>0} Q_\varepsilon = Q, \tag{2.49}$$

$$Q_\varepsilon \text{ is convex for each } \varepsilon, \tag{2.50}$$

$$Q_\varepsilon \text{ has the cone property determined by the cone } C \text{ for each } \varepsilon. \tag{2.51}$$

Now, combining Lemma 5.10 in [1], Chapter 5 and Theorem 1.4.3.4, Section 1.4, Chapter 1 in [21], one sees that for all exponent  $p$  and each  $q$  with  $q \leq p^*$ , there exists a constant  $K = K(p, q, C)$  and which does not depend on  $\varepsilon$  and such that one has

$$\forall u \in W_0^{1,p}(Q_\varepsilon), \quad \|u\|_{L^q(Q_\varepsilon)} \leq K \|u\|_{W_0^{1,p}(Q_\varepsilon)}. \tag{2.52}$$

Moreover, let us consider the elliptic problem

$$-\Delta u_\varepsilon = f|_{Q_\varepsilon} \quad \text{in } Q_\varepsilon, \tag{2.53}$$

$$u_\varepsilon = 0 \quad \text{on } \partial Q_\varepsilon, \tag{2.54}$$

where  $f \in L^2(Q)$  and  $f|_{Q_\varepsilon}$  stands for the restriction of  $f$  to  $Q_\varepsilon$ . Then, there exists a constant  $C_\varepsilon$  such that

$$\|u\|_{H^2(Q_\varepsilon)} \leq C_\varepsilon \|f\|_{L^2(Q)}.$$

Arguing as in [21], Theorem 3.2.1.2, page 147, one sees that the sequence  $(C_\varepsilon)_{\varepsilon>0}$  converges to  $C$  while  $(u_\varepsilon)_{\varepsilon>0}$  converges to  $u$  solution to

$$-\Delta u = f \quad \text{in } Q, \tag{2.55}$$

$$u = 0 \quad \text{on } \partial Q, \tag{2.56}$$

which satisfies

$$\|u\|_{H^2(Q)} \leq C \|f\|_{L^2(Q)}.$$

We indicate that we could use an other approach to treat this question of the regularity by using the results of [14], Chapter 8.

2.3.3. *Approximated system*

Now we are able to introduce the approximated system.

Let  $(\rho_\varepsilon)_{\varepsilon>0}$  be a sequence of molifiers. Let  $D$  be a given  $L^1$  function defined on  $Q$ . One denotes the extension of  $D$  by 0 outside  $Q$  still by  $D$ , to give a sense to  $D \star \rho_\varepsilon$ . The system that we consider is the system (2.57)–(2.61) below.

$$-\nabla \cdot ([\tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon)]^e \nabla \mathbf{u}_\varepsilon) + \nabla p_\varepsilon = \mathbf{f} \quad \text{in } \mathcal{D}'_{\text{per}}, \tag{2.57}$$

$$\nabla \cdot \mathbf{u}_\varepsilon = 0 \quad \text{in } \mathcal{D}'_{\text{per}}, \tag{2.58}$$

$$-\Delta \tilde{k}_\varepsilon = (\tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon)[|\nabla \mathbf{u}_\varepsilon|^2]_{Q_\varepsilon}) \star \rho_\varepsilon - \frac{\mathcal{E}_\varepsilon(\tilde{k}_\varepsilon)}{\ell} \quad \text{in } \mathcal{D}'(Q_\varepsilon), \tag{2.59}$$

$$(\mathbf{u}_\varepsilon, p_\varepsilon) \quad Q\text{-periodic}, \quad \int_Q \mathbf{u}_\varepsilon = 0, \quad \int_Q p_\varepsilon = 0, \tag{2.60}$$

$$\tilde{k}_\varepsilon|_{\partial Q_\varepsilon} = 0, \quad \tilde{k}_\varepsilon \geq 0, \quad \text{a.e. in } Q_\varepsilon. \tag{2.61}$$

In Eq. (2.57),  $[\tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon)]^e$  means what follows.

- In  $Q_\varepsilon$ ,  $[\tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon)]^e$  is equal to  $\tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon)$  where  $\tilde{k}_\varepsilon$  is computed by (2.59) and  $[|\nabla \mathbf{u}_\varepsilon|^2]_{Q_\varepsilon}$  stands for the restriction of  $|\nabla \mathbf{u}_\varepsilon|^2$  to  $Q_\varepsilon$ .
- In  $Q \setminus Q_\varepsilon$ ,  $[\tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon)]^e = [\tilde{v}_t^\varepsilon(0)]^e$  and then it is extended to  $\mathbb{R}^3$  by periodicity.

**Theorem 2.1.** *Let  $\varepsilon > 0$  be fixed. Assume that  $\mathbf{f} \in \mathbf{F}$ . The system (2.57)–(2.61) has a solution  $(\mathbf{u}_\varepsilon, p_\varepsilon, \tilde{k}_\varepsilon)$  such that*

$$\mathbf{u}_\varepsilon \in (H_{\text{loc}}^1(\mathbb{R}^3))^3, \quad p \in L_{\text{loc}}^2(\mathbb{R}^3), \tag{2.62}$$

$$\tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon)[|\nabla \mathbf{u}_\varepsilon|^2]_{Q_\varepsilon} \in L^1(Q_\varepsilon), \tag{2.63}$$

$$k_\varepsilon \in H^3(Q_\varepsilon) \cap H_0^1(Q_\varepsilon) \subset C^1(Q_\varepsilon). \tag{2.64}$$

Moreover, the following estimates hold, uniform in  $\varepsilon$ ,

$$\|[\mathbf{u}_\varepsilon]_Q\|_{(H^1(Q))^3} \leq \frac{C_1}{\nu^2} \int_Q |\mathbf{f}|^2, \tag{2.65}$$

$$\|\tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon)[|\nabla \mathbf{u}_\varepsilon|^2]_{Q_\varepsilon}\|_{L^1(Q_\varepsilon)} \leq \frac{C_1}{\nu} \int_Q |\mathbf{f}|^2, \tag{2.66}$$

$$\|\tilde{k}_\varepsilon\|_{W_0^{1,p}(Q_\varepsilon)} \leq \frac{C(p)C_1}{\nu} \int_Q |\mathbf{f}|^2, \quad \forall p < 3/2, \tag{2.67}$$

where  $C_1$  is the Poincaré’s constant on  $Q$  and  $C(p)$  does not depend on  $\varepsilon$  and satisfies  $\lim_{p \rightarrow 3/2^-} C(p) = \infty$ .

The proof of Theorem 2.1 is postponed until the end of this section. Let us first introduce the function spaces that we use, which are

$$\mathcal{V} = \left\{ \mathbf{v} = (v^1, v^2, v^3) \in (C^\infty(\mathbb{R}^3))^3, \text{ } Q\text{-periodic, } \int_Q \mathbf{v} = 0, \nabla \cdot \mathbf{v} = 0 \right\}, \tag{2.68}$$

$$\mathcal{V} = \left\{ \mathbf{v} = (v^1, v^2, v^3) \in (H_{\text{loc}}^1(\mathbb{R}^3))^3, \text{ } Q\text{-periodic, } \int_Q \mathbf{v} = 0, \nabla \cdot \mathbf{v} = 0 \right\}. \tag{2.69}$$

Light modifications of a classical result in [20], Corollary 2.5, Chapter 1, yield

$$\bar{\mathcal{V}} = \mathcal{V}. \tag{2.70}$$

The key point is to prove that the following variational problem has a solution.

$$\text{Find } (\mathbf{u}_\varepsilon, k_\varepsilon) \in V \times [H^3(Q_\varepsilon) \cap H_0^1(Q_\varepsilon)] \text{ be such that} \tag{2.71}$$

$$\forall \mathbf{v} \in V, \quad \int_Q \tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon) \nabla \mathbf{u}_\varepsilon : \nabla \mathbf{v} = \int_Q \mathbf{f} \cdot \mathbf{v}, \tag{2.72}$$

$$\forall w \in H^1(Q_\varepsilon), \quad \int_{Q_\varepsilon} \nabla \tilde{k}_\varepsilon \cdot \nabla w = \int_{Q_\varepsilon} (\tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon)[|\nabla \mathbf{u}_\varepsilon|^2]_{Q_\varepsilon}) \star \rho_\varepsilon \cdot w - \int_{Q_\varepsilon} \mathcal{E}_\varepsilon(\tilde{k}_\varepsilon) w. \tag{2.73}$$

**Remark 2.1.** Once problem (2.71)–(2.73) is solved, one knows by De Rham theorem (see [20], Theorem 2.3, Chapter 1) that there exists  $p_\varepsilon$  periodic with mean value 0 on  $Q$  and such that  $p_\varepsilon \in L_{\text{loc}}^2(\mathbb{R}^3)$  and such that  $(\mathbf{u}_\varepsilon, p_\varepsilon, k_\varepsilon)$  is a solution to problem (2.57)–(2.61).

It remains to prove the existence of a solution to problem (2.71)–(2.73). Before doing this, we prove the positivity of  $\tilde{k}_\varepsilon$ .

**Lemma 2.2.** *Let  $(\mathbf{u}_\varepsilon, k_\varepsilon)$  be any solution of the variational problem (2.71)–(2.73). Then  $\tilde{k}_\varepsilon \geq 0$  a.e. in  $\mathbb{R}^3$ .*

**Proof of Lemma 2.2.** Take the function  $w = -\tilde{k}_\varepsilon^- \in H^1(Q_\varepsilon)$  as test function in (2.73). Because  $(\tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon)[|\nabla \mathbf{u}_\varepsilon|^2]_{Q_\varepsilon}) \star \rho_\varepsilon \geq 0$  and  $\mathcal{E}_\varepsilon$  is odd and nonnegative on  $\mathbb{R}^+$ , one has

$$\int_{Q_\varepsilon} |\nabla \tilde{k}_\varepsilon^-| \leq 0.$$

Then  $\tilde{k}_\varepsilon^- = 0$  a.e. and therefore  $\tilde{k}_\varepsilon \geq 0$  a.e. in  $Q_\varepsilon$ . The proof of Lemma 2.2 is finished.  $\square$

### 2.3.4. Proof of Theorem 2.1

The proof of Theorem 2.1 is now reduced to proving that the variational problem (2.71)–(2.73) has a solution and to checking that the solution is regular as claimed in the statement. We shall use Leray–Schauder fixed point theorem (see [22]). The proof is divided into the four following steps,

- constructing a map  $\Gamma$  on an appropriate Sobolev space  $W_0^{1,p}(Q_\varepsilon)$ , the fixed points of which being solutions to the variational problem (2.71)–(2.73),
- obtaining estimates giving the existence of a ball  $B = B(0, R)$ , being such that  $\Gamma(B) \subset B$ ; notice that  $B$  is a convex compact set for the weak topology of  $B$ ,
- proving the  $\Gamma$ 's continuity for the weak topology of  $B$ ,
- checking the regularity of the solutions.

*Step 1 – Construction of the map  $\Gamma$ .* Let  $1 < p < 3/2$  close to  $3/2$  and to be fixed later. Being given  $q \in W_0^{1,p}(Q_\varepsilon)$ . Let us consider the variational problem

$$\text{Find } (\mathbf{u}, \tilde{k}) \in V \times W_0^{1,p}(Q_\varepsilon) \text{ be such that} \tag{2.74}$$

$$\forall \mathbf{v} \in V, \quad \int_Q \tilde{v}_t^\varepsilon(q) \nabla \mathbf{u} : \nabla \mathbf{v} = \int_Q \mathbf{f} \cdot \mathbf{v}, \tag{2.75}$$

$$\forall w \in W_0^{1,p'}(Q_\varepsilon), \quad \int_{Q_\varepsilon} \nabla \tilde{k} \cdot \nabla w = \int_{Q_\varepsilon} (\tilde{v}_t^\varepsilon(q)[|\nabla \mathbf{u}|^2]_{Q_\varepsilon}) \star \rho_\varepsilon \cdot w - \int_{Q_\varepsilon} \gamma_\varepsilon(\tilde{k}) \zeta_\varepsilon(q) w, \tag{2.76}$$

where the functions  $\gamma_\varepsilon$  and  $\zeta_\varepsilon$  are defined in Definitions 2.2 and 2.3 above and are such that  $\mathcal{E}_\varepsilon = \gamma_\varepsilon \zeta_\varepsilon$ .

Note first that  $\tilde{k}$  is not involved in (2.75). Indeed,  $\tilde{v}_t^\varepsilon(q) \in L^\infty(\mathbb{R}^3)$  (see (2.28)) and  $\tilde{v}_t^\varepsilon(q) \geq \nu > 0$  (see (2.30)). Therefore, there exists a unique  $\mathbf{u} \in V$  solution to problem (2.75) by Lax–Milgram theorem. In the remainder we shall denote by  $\mathbf{u}(q)$  this solution.

It is also easily seen that once the fluid problem (2.75) is solved, the TKE problem (2.76) has a unique solution  $\tilde{k} \in H^3(Q_\varepsilon) \cap H_0^1(Q_\varepsilon)$ . Indeed, thanks to the growth condition (2.35) one knows that  $\mathcal{E}_\varepsilon$  is continuous with a subcritical growth. Estimate (2.79) just below shows that  $\tilde{v}_t^\varepsilon(q)[|\nabla \mathbf{u}(q)|^2]_{Q_\varepsilon} \in L^1(Q_\varepsilon)$  and therefore  $(\tilde{v}_t^\varepsilon(q)[|\nabla \mathbf{u}(q)|^2]_{Q_\varepsilon}) \star \rho_\varepsilon \in C^\infty(Q_\varepsilon)$ . Moreover, it can be proved that  $\tilde{k} \in C^\infty(Q_\varepsilon)$ . To do this, one can use the results in [31] or iterate the result of Theorem IX.25 in [8], Chapter IX following the “bootstrapping” method.

The functional  $\Gamma$  is then defined by

$$\Gamma(q) = \tilde{k}, \tag{2.77}$$

where  $\tilde{k}$  is the unique solution to the problem (2.76) once (2.75) is solved. The functional  $\Gamma$  maps  $W_0^{1,p}(Q_\varepsilon)$  onto itself. Notice that by using a similar argument than in Lemma 2.2, it is easily proved that

$$\tilde{k} \geq 0, \quad \text{a.e. in } Q_\varepsilon. \tag{2.78}$$

The two next steps are devoted to prove that  $\Gamma$  has a fixed point  $\tilde{k}_\varepsilon$  by using Leray–Schauder theorem. Therefore, the couple  $(\mathbf{u}(\tilde{k}_\varepsilon), \tilde{k}_\varepsilon)$  will be a solution to the problem (2.71)–(2.73).

*Step 2 – Looking for a ball  $B = B(0, R)$  such that  $\Gamma(B) \subset B$ .* Let  $q \in W_0^{1,p}(Q_\varepsilon)$ . We begin with seeking for an estimate for  $\mathbf{u}(q)$ . Taking  $\mathbf{u}(q)$  as test function in (2.75) and using (2.30) yield, after classical computations,

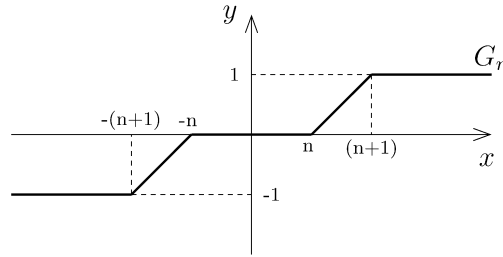


Fig. 5.

$$\int_Q \tilde{v}_t^\varepsilon(q) |\nabla \mathbf{u}(q)|^2 \leq \frac{C_1}{\nu} \int_Q |\mathbf{f}|^2, \tag{2.79}$$

$$\int_Q |\nabla \mathbf{u}(q)|^2 \leq \frac{C_1}{\nu^2} \int_Q |\mathbf{f}|^2, \tag{2.80}$$

where  $C_1$  is the Poincaré’s constant. By (2.80) and  $Q_\varepsilon \subset Q$ ,

$$\| \tilde{v}_t^\varepsilon(q) [|\nabla \mathbf{u}(q)|^2]_{Q_\varepsilon} \|_{L^1(Q_\varepsilon)} \leq \frac{C_1}{\nu} \int_Q |\mathbf{f}|^2. \tag{2.81}$$

Therefore, by Young’s inequality (see in [8], Theorem IV.30, Chapter IV),

$$\| (\tilde{v}_t^\varepsilon(q) [|\nabla \mathbf{u}(q)|^2]_{Q_\varepsilon}) \star \rho_\varepsilon \|_{L^1(Q_\varepsilon)} \leq \frac{C_1}{\nu} \int_Q |\mathbf{f}|^2. \tag{2.82}$$

Now we look for an estimate for  $\tilde{k} = \Gamma(q)$  in  $W_0^{1,p}$  (see (2.85) below). We aim to use Boccardo–Gallouët’s estimate (see [7]). Therefore, we have to prove that the gradient of  $\tilde{k}$  is uniformly bounded in  $L^2$  norm on the sets  $\{n \leq |\tilde{k}| \leq n + 1\}$ .

To do this, let us consider the odd function  $G_n$  piecewise linear, equal to 0 on  $[0, n]$  and 1 on  $[n + 1, \infty[$  (see Fig. 5).

Take  $G_n(\tilde{k})$  as test function in the  $\tilde{k}$ -equation (2.76). We note that  $\gamma_\varepsilon$  is an odd function, which is nonnegative on  $\mathbb{R}_+$  (see (2.43) and above) as well as  $G_n$ . We also note that  $\zeta_\varepsilon$  is everywhere nonnegative. Therefore, one has

$$0 \leq \int_Q \gamma_\varepsilon(\tilde{k}) \zeta_\varepsilon(q) G_n(\tilde{k}).$$

Notice that even if we already know that  $\tilde{k} \geq 0$ , one does not need any information on the sign of  $\tilde{k}$  to obtain the inequality above. Therefore, estimate (2.82) satisfied by the production term yields

$$\int_{n \leq |\tilde{k}| \leq n+1} |\nabla \tilde{k}|^2 \leq \frac{C_1}{\nu} \int_Q |\mathbf{f}|^2, \tag{2.83}$$

where one has used  $0 \leq G_n(\tilde{k}) \leq 1$ . Therefore, by Boccardo–Gallouët’s estimate [7], one knows that for all  $r \in [1, 3/2[$ , there exists a constant  $C_\varepsilon(r)$  which depends on  $r$ , where  $\lim_{r \rightarrow 3/2} C_\varepsilon(r) = \infty$  and such that

$$\| \tilde{k} \|_{W_0^{1,r}(Q_\varepsilon)} \leq \frac{C_\varepsilon(r) C_1}{\nu} \int_Q |\mathbf{f}|^2. \tag{2.84}$$

One recalls that only the Hölder and Sobolev’s inequalities are used for proving the Boccardo–Gallouët’s inequality. Therefore, the remarks in Subsection 2.3.2 and more precisely (2.52), make sure that there exists  $C(r)$  such that for each  $\varepsilon > 0$  one has  $C_\varepsilon(r) < C(r)$  and the inequality (2.84) becomes

$$\| \tilde{k} \|_{W_0^{1,r}(Q_\varepsilon)} \leq \frac{C(r) C_1}{\nu} \int_Q |\mathbf{f}|^2 = R_r. \tag{2.85}$$

Let  $B = B(O, R_p) \subset W_0^{1,p}(Q_\varepsilon)$  which is a convex set, compact for the weak topology of  $W_0^{1,p}(Q_\varepsilon)$ . Estimate (2.85) makes sure that  $\Gamma(B) \subset B$ .

*Step 3 –  $\Gamma$ 's continuity on  $B$ .* We need to prove that  $\Gamma$  is a continuous function on  $W_0^{1,p}(Q_\varepsilon)$  for the weak topology of  $B$ . Since  $B$  is bounded and the space  $W_0^{1,p}(Q_\varepsilon)$  is a separable space, the weak topology has a metric associated, following Theorem III.25 Chapter III in [8]. It means that there exists a distance  $d$  such that the weak topology on  $B$  is the topology induced by  $d$ . Therefore it is enough to prove that  $\Gamma$  is sequentially weak continuous.

Let us consider  $(q_n)_{n \in \mathbb{N}} \subset B^{\mathbb{N}}$  which converges in  $B$  to  $q$  for the weak topology of  $W_0^{1,p}(Q_\varepsilon)$  (here  $\varepsilon$  is fixed). We have to prove that

$$(\tilde{k}_n)_{n \in \mathbb{N}} = (\Gamma(q_n))_{n \in \mathbb{N}} \text{ converges weakly to } \tilde{k} = \Gamma(q).$$

We proceed in three substeps:

- extracting subsequences,
- passing to the limit in the fluid equation,
- passing to the limit in the  $\tilde{k}$  equation and concluding.

*Extracting subsequences.* In what follows, we shall extract a finite number of sequences. These subsequences will always be denoted by using the same notation. At the end of the procedure, we shall note that the whole sequence converges due to the uniqueness of the limit.

By Sobolev Embedding theorem, one can extract a subsequence from the sequence  $(q_n)_{n \in \mathbb{N}}$  which converges strongly to  $q$  in  $L^r(Q_\varepsilon)$  for all  $r \in [1, p^*]$ . Moreover, this sequence can be chosen such that it converges almost everywhere to  $q$  in  $Q_\varepsilon$ .

On the other side, one knows that the sequence  $(\tilde{k}_n)_{n \in \mathbb{N}}$  is bounded in  $W_0^{1,p}(Q_\varepsilon)$  (bound (2.85)). Therefore, one can extract again a subsequence such that  $(\tilde{k}_n)_{n \in \mathbb{N}}$  converges

- weakly to some  $\tilde{k} \in B$  in  $W_0^{1,p}(Q_\varepsilon)$ ,
- strongly to  $\tilde{k}$  in  $L^r(Q_\varepsilon)$  for all  $r \in [1, p^*]$ ,
- a.e. in  $Q_\varepsilon$ .

We have to show that  $\tilde{k} = \Gamma(q)$ .

*Passing to the limit in the fluid equation.* Now we study the sequence  $(\mathbf{u}(q_n))_{n \in \mathbb{N}}$ . We must prove that this sequence strongly converges in  $V$  to  $\mathbf{u}(q)$ .

Every  $q_n$  and  $q$  are extended by zero in  $Q \setminus Q_\varepsilon$ . Thanks to the bound (2.80), the sequence  $(\mathbf{u}(q_n))_{n \in \mathbb{N}}$  is bounded in  $V$ . Thus, up to a subsequence, it weakly converges in  $V$  to some  $\mathbf{u}$ , and  $([\mathbf{u}(q_n)]_Q)_{n \in \mathbb{N}}$  strongly converges in  $(L^2(Q))^3$  to  $[\mathbf{u}]_Q$ .

Let  $\mathbf{v} \in V$  be a test vector field. The function  $\tilde{v}_i^\varepsilon$  being continuous,  $(\tilde{v}_i^\varepsilon(q_n))_{n \in \mathbb{N}}$  converges almost everywhere to  $\tilde{v}_i^\varepsilon(q)$ . Thus,  $(\tilde{v}_i^\varepsilon(q_n)[\nabla \mathbf{v}]_Q)_{n \in \mathbb{N}}$  converges a.e. to  $\tilde{v}_i^\varepsilon(q)[\nabla \mathbf{v}]_Q$  and one has

$$|\tilde{v}_i^\varepsilon(q_n)[\nabla \mathbf{v}]_q| \leq \|\tilde{v}_i^\varepsilon\|_{L^\infty(\mathbb{R})} |[\nabla \mathbf{v}]_Q| \in L^2(Q).$$

Therefore, by Lebesgue's theorem,  $(\tilde{v}_i^\varepsilon(q_n)[\nabla \mathbf{v}]_Q)_{n \in \mathbb{N}}$  strongly converges to  $\tilde{v}_i^\varepsilon(q)[\nabla \mathbf{v}]_Q$  in  $[L^2(Q)]^{3 \times 3}$ , which yields

$$\lim_{n \rightarrow \infty} \int_Q \tilde{v}_i^\varepsilon(q_n) \nabla \mathbf{u}(q_n) : \nabla \mathbf{v} = \int_Q \tilde{v}_i^\varepsilon(q) \nabla \mathbf{u} : \nabla \mathbf{v} = \int_Q \mathbf{v} \cdot \mathbf{f}. \quad (2.86)$$

Therefore one has  $\mathbf{u} = \mathbf{u}(q)$ . Strong convergence is not proved yet. Using  $\mathbf{u}(q_n)$  as test vector field in the equation satisfied by  $\mathbf{u}(q_n)$  itself gives on one hand,

$$\int_Q \tilde{v}_i^\varepsilon(q_n) |\nabla \mathbf{u}(q_n)|^2 = \int_Q \mathbf{f} \cdot \mathbf{u}(q_n), \quad (2.87)$$



while on the other hand, taking  $\mathbf{u}(q) = \mathbf{v}$  as test vector field in (2.86) yields

$$\int_Q \tilde{v}_t^\varepsilon(q) |\nabla \mathbf{u}(q)|^2 = \int_Q \mathbf{f} \cdot \mathbf{u}(q). \tag{2.88}$$

The strong convergence of  $([\mathbf{u}(q_n)]_Q)_{n \in \mathbb{N}}$  in  $(L^2(Q))^3$  yields

$$\lim_{n \rightarrow \infty} \int_Q \mathbf{f} \cdot \mathbf{u}(q_n) = \int_Q \mathbf{f} \cdot \mathbf{u}(q).$$

Thus, (2.87) combined to (2.88) shows that

$$\lim_{n \rightarrow \infty} \int_Q \tilde{v}_t^\varepsilon(q_n) |\nabla \mathbf{u}(q_n)|^2 = \int_Q \tilde{v}_t^\varepsilon(q) |\nabla \mathbf{u}(q)|^2.$$

Therefore, by arguing as in [23] and [24] and thanks to the strict positivity of  $\tilde{v}_t^\varepsilon$  (see (2.30)) one deduces the strong convergence of  $(\mathbf{u}(q_n))_{n \in \mathbb{N}}$  to  $\mathbf{u}(q)$  in  $V$  and also the strong convergence in  $L^1(Q)$  (and  $L^1(Q_\varepsilon)$  of course) of  $(\tilde{v}_t^\varepsilon(q_n) |\nabla \mathbf{u}(q_n)|^2)_{n \in \mathbb{N}}$  to  $\tilde{v}_t^\varepsilon(q) |\nabla \mathbf{u}(q)|^2$ , up to a subsequence. This is satisfied by the whole sequence thanks to the uniqueness of the possible limit.

*Passing to the limit in the  $\tilde{k}$  equation.* Let  $\varphi \in \mathcal{D}(Q_\varepsilon)$ . The strong convergence in  $L^1(Q_\varepsilon)$  of  $(\tilde{v}_t^\varepsilon(q_n) |\nabla \mathbf{u}(q_n)|^2)_{n \in \mathbb{N}}$  to  $\tilde{v}_t^\varepsilon(q) |\nabla \mathbf{u}(q)|^2$  yields

$$\lim_{n \rightarrow \infty} \int_{Q_\varepsilon} \varphi(\tilde{v}_t^\varepsilon(q_n) |\nabla \mathbf{u}(q_n)|^2) \star \rho_\varepsilon = \int_{Q_\varepsilon} \varphi(\tilde{v}_t^\varepsilon(q) |\nabla \mathbf{u}(q)|^2) \star \rho_\varepsilon. \tag{2.89}$$

Because of the weak convergence in  $W_0^{1,p}(Q_\varepsilon)$  of the sequence  $(\tilde{k}_n)_{n \in \mathbb{N}}$  to  $\tilde{k}$  one has

$$\lim_{n \rightarrow \infty} \int_{Q_\varepsilon} \nabla \tilde{k}_n \cdot \nabla \varphi = \int_{Q_\varepsilon} \nabla \tilde{k} \cdot \nabla \varphi. \tag{2.90}$$

Since:

- the functions  $\gamma_\varepsilon$  and  $\zeta_\varepsilon$  satisfy conditions (2.45) and (2.46) and are continuous,
- when  $p$  is chosen close enough to  $3/2$ , such that  $p^* > 2$ , sequences  $(\tilde{k}_n)_{n \in \mathbb{N}}$  and  $(q_n)_{n \in \mathbb{N}}$  strongly converge to  $\tilde{k}$  and  $q$  in  $L^2(Q_\varepsilon)$  and *a.e.*,

one also has

$$\lim_{n \rightarrow \infty} \int_{Q_\varepsilon} \gamma_\varepsilon(\tilde{k}_n) \zeta_\varepsilon(q_n) \varphi = \int_{Q_\varepsilon} \gamma_\varepsilon(\tilde{k}) \zeta_\varepsilon(q) \varphi. \tag{2.91}$$

By putting together (2.89), (2.90) and (2.91), one sees that we were able to pass to the limit in each term in the  $\tilde{k}$  equation proving  $\Gamma(q) = \tilde{k}$ . As already mentioned we have extracted a finite number of subsequence, and the limit being unique, the whole sequence  $(\tilde{k}_n)_{n \in \mathbb{N}}$  converges to  $\tilde{k}$  and the weak continuity of  $\Gamma$  is proved.

**Summarize.** *The continuity of the functional  $\Gamma$  for the weak topology of  $B$  is proved. The ball  $B$  which is preserved by  $\Gamma$  is a compact set when  $W_0^{1,p}(Q_\varepsilon)$  is equipped by its weak topology. The set  $B$  is also a convex set. Consequently, the map  $\Gamma$  has a fixed point denoted by  $\tilde{k}_\varepsilon$  in the ball  $B$ . The couple*

$$(\mathbf{u}_\varepsilon, \tilde{k}_\varepsilon) = (\mathbf{u}(\tilde{k}_\varepsilon), \tilde{k}_\varepsilon) \in V \times W_0^{1,p}(Q_\varepsilon)$$

*is a solution to the variational problem (2.71)–(2.73).*

Step 4 – Check of the regularity. Estimates (2.65), (2.66) and (2.67) are deduced from (2.79), (2.80) and (2.85). In particular, recall that

$$\tilde{v}_r^\varepsilon(\tilde{k}_\varepsilon)[|\nabla \mathbf{u}_\varepsilon|^2]_{Q_\varepsilon} \in L^1(Q_\varepsilon), \quad \tilde{k}_\varepsilon \in \bigcap_{r < 3/2} W_0^{1,r}(Q_\varepsilon) \subset \bigcap_{s < 3} L^s(Q_\varepsilon). \tag{2.92}$$

Now we check the  $H^3$  regularity for  $\tilde{k}$ . Notice first that in the r.h.s of the  $\tilde{k}$ -equation,

$$(\tilde{v}_r^\varepsilon(\tilde{k}_\varepsilon)[|\nabla \mathbf{u}_\varepsilon|^2]_{Q_\varepsilon}) \star \rho_\varepsilon \in C^\infty(Q_\varepsilon). \tag{2.93}$$

On the other side, the growth condition (2.35) satisfied by  $\mathcal{E}_\varepsilon$  shows that

$$\int_{Q_\varepsilon} |\mathcal{E}_\varepsilon(\tilde{k}_\varepsilon)|^2 \leq E_1 \text{mes}(Q_\varepsilon) + E_2 \int_{Q_\varepsilon} |\tilde{k}_\varepsilon|^{\frac{3}{1+\theta}}, \tag{2.94}$$

for  $E_1$  and  $E_2$  two constants. This can be rephrased as

$$\|\mathcal{E}_\varepsilon(\tilde{k}_\varepsilon)\|_{L^2(Q_\varepsilon)} \leq E_3 \left[ 1 + \|\tilde{k}_\varepsilon\|_{L^{\frac{3}{3/(1+\theta)}}(Q_\varepsilon)} \right]. \tag{2.95}$$

Let  $p$  be such that  $3/(2 + \theta) = p < 3/2$ . Therefore  $3/(1 + \theta) = p^*$  and Sobolev’s inequality yields

$$\|\mathcal{E}_\varepsilon(\tilde{k}_\varepsilon)\|_{L^2(Q_\varepsilon)} \leq E_4 \left[ 1 + \|\tilde{k}_\varepsilon\|_{W_0^{1,p}(Q_\varepsilon)}^{\frac{3}{1+\theta}} \right]. \tag{2.96}$$

By the  $W_0^{1,p}$  estimate (2.85) satisfied by  $\tilde{k}_\varepsilon$ , one deduces

$$\|\mathcal{E}_\varepsilon(\tilde{k}_\varepsilon)\|_{L^2(Q_\varepsilon)} \leq E_5 \left[ 1 + v^{-\frac{3}{2(1+\theta)}} \|\mathbf{f}\|_{L^2(Q)}^{\frac{3}{(1+\theta)}} \right], \tag{2.97}$$

where the constant here only depends on  $\theta$  and  $Q$ . One may object that the constant should also depend on  $\varepsilon$ . In fact they do not: see the remarks at Subsection 2.3.2 above.

We have shown that the second term in the r.h.s of (2.73) is in  $L^2(Q_\varepsilon)$ , the first one being in  $C^\infty(Q_\varepsilon)$ . Therefore, it follows from the classical elliptic theory that  $\tilde{k}_\varepsilon \in H^2(Q_\varepsilon)$  because  $Q_\varepsilon$  is a smooth domain.

Now notice that  $\mathcal{E}_\varepsilon$  is a  $C^2$ -class function and that its derivative function is bounded. Therefore, thanks to the Stampacchia’s result ([32], Lemma 1.2, page 15),  $\mathcal{E}_\varepsilon(\tilde{k}_\varepsilon) \in H^1(Q_\varepsilon)$ . Then, Eq. (2.73) is an equation with a r.h.s in  $H^1$ . Therefore, thanks to the elliptic theory again,  $\tilde{k}_\varepsilon \in H^3(Q_\varepsilon) \cap H_0^1(Q_\varepsilon)$ .

Theorem 2.1 is now entirely proved.  $\square$

In order to prove the claimed result in the introduction, Theorem 1.1, we have to pass now to the limit in Eqs. (2.57)–(2.61) when  $\varepsilon$  tends to 0. This is the aim of the next section.

### 3. Proof of Theorem 1.1

#### 3.1. Orientation

Let us consider  $(\mathbf{u}_\varepsilon, p_\varepsilon, \tilde{k}_\varepsilon)$  a solution to the approximated system (2.57)–(2.61), the existence of which being guaranteed by Theorem 2.1. We aim to obtain a  $L^\infty$  estimate for the sequence  $(\tilde{k}_\varepsilon)_{\varepsilon > 0}$  which does not depend on  $\varepsilon$ . The strategy consists in studying the formal derivative of Eq. (2.57) to get a uniform  $H^2$  bound for  $(\mathbf{u}_\varepsilon)_{\varepsilon > 0}$  to obtain a uniform  $W^{1,6}$  bound on the sequence  $\tilde{k}_\varepsilon$ , which does not depend on  $\varepsilon$ .

Before doing that, we first prove a general helpful regularity result.

#### 3.2. General regularity theory

The general problem that we consider in this subsection is the following Stokes problem

$$-\nabla \cdot (a(x)\nabla \mathbf{u}) + \nabla p = \mathbf{f} \quad \text{in } \mathcal{D}'_{\text{per}}, \tag{3.1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \mathcal{D}'_{\text{per}}, \tag{3.2}$$

$$(\mathbf{u}, p) \quad Q\text{-periodic}, \quad \int_Q \mathbf{u} = 0, \quad \int_Q p = 0, \tag{3.3}$$

where  $\mathbf{f} \in \mathbf{F}$  and  $a = a(x)$  is a  $Q$ -periodic function at least in  $L^\infty_{\text{loc}}(\mathbb{R}^3)$ , to make sure that problem (3.1)–(3.3) has a solution  $(\mathbf{u}, p) \in V \times L^2_{\text{loc}}(\mathbb{R}^3)$ .

Even if the regularity of the solution of such a Stokes Problem with nonconstant coefficients has not been directly investigated (as far as we know), one may expect that:

*let  $a = a(x)$  be a continuous  $Q$ -periodic function, then  $\mathbf{u} \in H^2_{\text{loc}}(\mathbb{R}^3)$ .*

In the remainder, we shall prove a weaker result, but which presents the advantage that the proof is very simple and which is also a preparation to the next results we shall prove.

As usual in the paper,  $\mathbf{f} \in \mathbf{F}$ .

**Theorem 3.1.** *Assume that  $a = a(x)$  is  $Q$ -periodic and  $a \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^3)$ . Let  $(\mathbf{u}, p) \in V \times L^2_{\text{loc}}(\mathbb{R}^3)$  be a solution to problem (3.1)–(3.3). Then  $(\mathbf{u}, p) \in (H^2_{\text{loc}}(\mathbb{R}^3))^3 \times H^1_{\text{loc}}(\mathbb{R}^3)$ .*

**Proof.** Because  $a \in L^\infty_{\text{loc}}(\mathbb{R}^3)$  and is  $Q$ -periodic, the existence of a unique periodic solution  $\mathbf{u} \in V \times L^2_{\text{loc}}(\mathbb{R}^3)$  to problem (3.1)–(3.3) is a consequence of Lax–Milgram theorem combined to De Rham theorem and the Nečas estimates (see in [20], Chapter 1, §2).

*The result will be obtained by deriving Eqs. (3.1), (3.2).*

Let us consider the following Stokes problem where  $\mathbf{D}$  and  $\mathbf{P}$  are unknowns.

$$-\nabla \cdot (a(x)\nabla \mathbf{D}) + \nabla \mathbf{P} = \nabla \mathbf{f} + \nabla \cdot (\nabla a \otimes \nabla \mathbf{u}) \quad \text{in } \mathcal{D}'_{\text{per}}, \tag{3.4}$$

$$\nabla \cdot \mathbf{D} = 0 \quad \text{in } \mathcal{D}'_{\text{per}}, \tag{3.5}$$

$$(\mathbf{D}, \mathbf{P}) \quad Q\text{-periodic}, \quad \int_Q \mathbf{D} = 0, \quad \int_Q \mathbf{P} = 0. \tag{3.6}$$

In the problem above,  $\mathbf{D} = (d_r^i)_{1 \leq i, r \leq 3}$  is a second order tensor while  $\mathbf{P} = (P_r)_{1 \leq r \leq 3}$  is a first order tensor. Eqs. (3.4) and (3.5) can be rephrased as

$$-\partial_j (a(x)\partial_j d_r^i) + \partial_i P_r = \partial_r f^i + \partial_j (\partial_r a \partial_j u^i), \tag{3.7}$$

$$\partial_i d_r^i = 0, \tag{3.8}$$

by using the convention of the repeated indexes summation. Notice first that because  $\mathbf{f} \in \mathbf{F}$ ,  $\nabla \mathbf{f} \in L^2_{\text{loc}}$  and is a periodic field with a mean equal to zero. Next one has  $-\nabla \cdot (\nabla a \otimes \nabla \mathbf{u}) \in (V \otimes V)'$ , because  $a \in W^{1,\infty}_{\text{loc}}$  and  $\nabla \mathbf{u} \in L^2_{\text{loc}}$ , both being periodic. This term being also periodic with zero mean value, one knows thanks to Lax–Milgram theorem that there exists a unique  $\mathbf{D} \in V \otimes V$  such that

$$\forall \mathbf{E} = (e_r^i)_{1 \leq i, r \leq 3} \in V \otimes V, \tag{3.9}$$

$$\int_Q a(x)\partial_j d_r^i \partial_j e_r^i = \int_Q \partial_r f^i e_r^i - \int_Q \partial_r a \partial_j u^i e_r^i. \tag{3.10}$$

The existence of  $\mathbf{P} \in L^2_{\text{loc}}$  being such that  $(\mathbf{D}, \mathbf{P})$  is a solution to problem (3.4)–(3.6), is a consequence of De Rham’s theorem combined with the classical Nečas estimates.

*We prove now that  $\mathbf{D} = \nabla \mathbf{u}$ , which will show the  $(H^2_{\text{loc}})^3$  regularity of  $\mathbf{u}$ .*

Let  $\mathbf{E} = (e_r^i)_{1 \leq i, r \leq 3} \in V \otimes \mathcal{V}$ . Take  $\mathbf{v} = (\partial_r e_r^1, \partial_r e_r^2, \partial_r e_r^3) \in \mathcal{V}$  as vector test in (3.4). We have

$$\int_Q a(x)\partial_j u^i \partial_j \partial_r e_r^i = \int_Q f^i \partial_r e_r^i. \tag{3.11}$$

Integrating by part with respect to  $r$  and using the periodic boundary condition yields in the sense of the distributions

$$-\langle \partial_r (a(x) \partial_j u^i), \partial_j e_r^i \rangle = - \int_Q \partial_r f^i e_r^i. \tag{3.12}$$

Therefore

$$\langle a(x) \partial_j (\partial_r u^i), \partial_j e_r^i \rangle + \int_Q \partial_r a \partial_j u^i e_r^i = \int_Q \partial_r f^i e_r^i. \tag{3.13}$$

By the fact that  $\bar{V} = V$ , one deduces from (3.10) and uniqueness that  $\mathbf{D} = \nabla \mathbf{u}$ . It is easily seen that  $\nabla p = \mathbf{P}$  and Theorem 3.1 is proved.  $\square$

**Corollary 3.1.** *Let  $(\mathbf{u}_\varepsilon, p_\varepsilon, \tilde{k}_\varepsilon)$  be a solution to the system (2.57)–(2.61). Then one has  $(\mathbf{u}_\varepsilon, p_\varepsilon) \in (H_{\text{loc}}^2)^3 \times H_{\text{loc}}^1$ .*

**Proof.** One knows that  $\tilde{k}_\varepsilon$  is a  $C^1$ -class function already (see (2.64)). Because  $\tilde{v}_t^\varepsilon$  is a  $C^2$ -class function,  $[\tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon)]^e$  is at least in  $W_{\text{loc}}^{1,\infty}$ . Then the result is a consequence of Theorem 3.1.  $\square$

### 3.3. $H^2$ estimate for $\mathbf{u}_\varepsilon$ and $\tilde{k}_\varepsilon$

Recall that  $v_t$  satisfies Properties 1.1,  $\tilde{v}_t$  satisfies Properties 2.1 and  $\tilde{v}_t^\varepsilon$  has been built in order to satisfy Properties 2.2.

Throughout the rest of the paper,  $(\mathbf{u}_\varepsilon, p_\varepsilon, \tilde{k}_\varepsilon)$  is a solution to the approximated system (2.57)–(2.61). The aim of this part is to deduce a  $H^2$  estimate on  $\mathbf{u}_\varepsilon$ , uniform in  $\varepsilon$ . For doing this, one uses the equation deduced from the fluid equation by a formal derivation.

We consider the system (3.14)–(3.16) below. It is obtained after derivating in terms in Eqs. (2.57) and (2.58),

$$-\nabla \cdot ([\tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon)]^e \nabla \mathbf{D}) - \nabla \cdot ([(\tilde{v}_t^\varepsilon)'(\tilde{k}_\varepsilon) \nabla \tilde{k}_\varepsilon]^e \otimes \mathbf{D}) + \nabla \mathbf{P} = \nabla \mathbf{f} \quad \text{in } \mathcal{D}'_{\text{per}}, \tag{3.14}$$

$$\nabla \cdot \mathbf{D} = 0 \quad \text{in } \mathcal{D}'_{\text{per}}, \tag{3.15}$$

$$(\mathbf{D}, \mathbf{P}) \text{ } Q\text{-periodic,} \quad \int_Q \mathbf{D} = 0, \quad \int_Q \mathbf{P} = 0. \tag{3.16}$$

As above, the unknowns are the tensor  $\mathbf{D} = (d_r^i)_{1 \leq i, r \leq 3}$  and the vector field  $\mathbf{P} = (P_r)_{1 \leq i \leq 3}$ . By definition,

$$\nabla \cdot (\tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon) \nabla \mathbf{D}) = (\partial_j (\tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon) \partial_j d_r^i))_{1 \leq i, r \leq 3}, \quad \nabla \mathbf{P} = (\partial_i P_r)_{1 \leq i, r \leq 3},$$

and finally

$$\nabla \cdot ((\tilde{v}_t^\varepsilon)'(\tilde{k}_\varepsilon) \nabla \tilde{k}_\varepsilon \otimes \mathbf{D}) = (\partial_j ((\tilde{v}_t^\varepsilon)'(\tilde{k}_\varepsilon) \partial_r \tilde{k}_\varepsilon d_j^i))_{1 \leq i, r \leq 3}, \quad \nabla \cdot \mathbf{D} = (\partial_i d_r^i)_{1 \leq r \leq 3}.$$

The scalar  $\tilde{k}_\varepsilon$  is extended by 0 in  $Q \setminus Q_\varepsilon$  without changing the notation.

**Lemma 3.1.** *System (3.14)–(3.16) has a unique solution  $(\mathbf{D}_\varepsilon, \mathbf{P}_\varepsilon) \in V \otimes V \times (L_{\text{loc}}^2)^3$  and one has  $\mathbf{D}_\varepsilon = \nabla \mathbf{u}_\varepsilon$  a.e. in  $\mathbb{R}$ .*

**Proof.** Let us consider the variational problem

$$\text{Find } \mathbf{D} \in V \otimes V \text{ such that} \tag{3.17}$$

$$\forall \mathbf{E} \in V \otimes V, \quad \int_Q \tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon) \nabla \mathbf{D} : \nabla \mathbf{E} + \int_Q ((\tilde{v}_t^\varepsilon)'(\tilde{k}_\varepsilon) \nabla \tilde{k}_\varepsilon \otimes \mathbf{D}) : \nabla \mathbf{E} = \int_Q \nabla \mathbf{f} : \mathbf{E}. \tag{3.18}$$

Above “:” denotes the contracted tensor’s product, that means that  $\mathbf{A} : \mathbf{B} = a_i^j B_j^i$  for second order tensors,  $\mathbf{A} : \mathbf{B} = a_{ij}^k B_{ij}^k$  for three order, and so on. In particular, Eq. (3.18) means that  $\forall \mathbf{E} = (e_r^i)_{1 \leq i, r \leq 3}$  one has

$$\int_Q \tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon) \partial_j d_r^i \partial_j e_r^i + \int_Q ((\tilde{v}_t^\varepsilon)'(\tilde{k}_\varepsilon) \partial_r \tilde{k}_\varepsilon d_i^j \partial_j e_r^i) = \int_Q \partial_r f^i e_r^i, \tag{3.19}$$

where  $\mathbf{f} = (f^1, f^2, f^3)$ .

Thanks to the results above (in particular (2.64), (2.29) and (2.30)), one already knows that

$$\tilde{v}_t^\varepsilon(\tilde{k}^\varepsilon) \geq \nu > 0, \quad \tilde{v}_t^\varepsilon(\tilde{k}^\varepsilon) \in L^\infty(Q), \quad (\tilde{v}_t^\varepsilon)'(\tilde{k}_\varepsilon)\nabla\tilde{k}_\varepsilon \in L^\infty(Q).$$

Therefore, (3.17), (3.18) has a unique solution  $\mathbf{D}_\varepsilon$  thanks to Lax–Milgram theorem. The existence of  $\mathbf{P}_\varepsilon$  such that (3.14)–(3.16) is satisfied is a consequence of De Rham theorem (see also in [33], Proposition 1.1, Chapter 1).

One already knows that  $\mathbf{u}_\varepsilon \in (H_{\text{loc}}^2)^3$ , thanks to Corollary 3.1. Let  $\mathbf{E} = (e_r^i)_{1 \leq i, r \leq 3} \in \mathcal{V} \otimes \mathcal{V}$  (see the definition of  $\mathcal{V}$  in (2.68)). Let us take the vector field  $\mathbf{v} = -(\partial_r e_r^1, \partial_r e_r^2, \partial_r e_r^3)$  as test tensor in the variational fluid equation (2.72). This leads to

$$-\int_Q \tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon) \partial_j u_\varepsilon^i \partial_j \partial_r e_r^i = -\int_Q f^i \partial_r e_r^i. \tag{3.20}$$

By a part integration using the periodic conditions, a legal computation thanks to the regularity mentioned above, (3.20) becomes

$$\int_Q \tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon) \partial_j (\partial_r u_\varepsilon^i) \partial_j e_r^i + \int_Q (\tilde{v}_t^\varepsilon)'(\tilde{k}_\varepsilon) \partial_r \tilde{k}_\varepsilon \partial_j u_\varepsilon^i \partial_j e_r^i = \int_Q \partial_r f^i e_r^i. \tag{3.21}$$

We stress that in (3.21), each term makes sense. Comparing (3.21) to (3.19), one deduces that  $\mathbf{D}_\varepsilon = \nabla \mathbf{u}_\varepsilon$  by uniqueness of the solution to problem (3.17), (3.18) combined to the fact that  $\bar{\mathcal{V}} = \mathcal{V}$ . The proof is finished.  $\square$

In the following, we still note  $\mathbf{D}_\varepsilon = \nabla \mathbf{u}_\varepsilon$ .

**Theorem 3.2.** *Assume that  $\mathbf{f} \in \mathbf{F}$  (see (1.8)). There exists a constant  $\kappa = \kappa(\theta, \|v_t'\|_\infty)$  such that for every  $\ell$  satisfying the condition*

$$\ell \nu > \kappa \left(1 + \nu^{-\frac{3}{2(1+\theta)}} \|\mathbf{f}\|_Q \left\| \frac{3}{(L^2(Q))^3} \right\| \right), \tag{3.22}$$

there exists a constant  $\zeta = \zeta(\mathbf{f}, \nu, \ell, \theta, \|v_t'\|_\infty)$  being such that for all  $\varepsilon > 0$ ,

$$\|\mathbf{D}_\varepsilon\|_Q \left\|_{(H^1(Q))^{3 \times 3}} \leq \zeta. \tag{3.23}$$

**Corollary 3.2.** *When condition (3.22) is satisfied, the sequence  $(\mathbf{u}_\varepsilon)_{\varepsilon > 0}$  is bounded in  $(H_{\text{loc}}^2(\mathbb{R}^3))^3$ .*

**Corollary 3.3.** *Under the condition (3.22), there exists a constant  $\nu = \nu(\mathbf{f}, \nu, \ell, \theta, \|v_t'\|_\infty)$  being such that for all  $\varepsilon > 0$ , one has*

$$\|\tilde{k}_\varepsilon\|_{H^2(Q_\varepsilon)} \leq \nu. \tag{3.24}$$

**Proof of Theorem 3.2.** We take  $\mathbf{D}_\varepsilon = \nabla \mathbf{u}_\varepsilon$  as test tensor in (3.18). We first note that since  $d_j^i = \partial_j u^i$ , then  $\partial_j d_r^i = \partial_r d_j^i$ . Therefore, one has

$$\int_Q (\tilde{v}_t^\varepsilon)'(\tilde{k}_\varepsilon) \nabla k_\varepsilon \mathbf{D}_\varepsilon \otimes \nabla \mathbf{D}_\varepsilon = \int_Q (\tilde{v}_t^\varepsilon)'(\tilde{k}_\varepsilon) \partial_r \tilde{k}_\varepsilon d_j^i \partial_j d_r^i = \int_Q (\tilde{v}_t^\varepsilon)'(\tilde{k}_\varepsilon) \partial_r \tilde{k}_\varepsilon d_j^i \partial_r d_j^i, \tag{3.25}$$

which yields

$$\int_Q (\tilde{v}_t^\varepsilon)'(\tilde{k}_\varepsilon) \nabla k_\varepsilon \mathbf{D}_\varepsilon \otimes \nabla \mathbf{D}_\varepsilon = \frac{1}{2} \int_Q (\tilde{v}_t^\varepsilon)'(\tilde{k}_\varepsilon) \partial_r \tilde{k}_\varepsilon \partial_r (d_j^i)^2. \tag{3.26}$$

Integrating by parts yields,

$$\frac{1}{2} \int_Q (\tilde{v}_t^\varepsilon)'(\tilde{k}_\varepsilon) \partial_r \tilde{k}_\varepsilon \partial_r (d_j^i)^2 = -\frac{1}{2} \int_Q \partial_r [(\tilde{v}_t^\varepsilon)'(\tilde{k}_\varepsilon) \partial_r \tilde{k}_\varepsilon] (d_j^i)^2, \tag{3.27}$$

and

$$-\int_Q \partial_r [(\tilde{v}_t^\varepsilon)'(\tilde{k}_\varepsilon) \partial_r \tilde{k}_\varepsilon] (d_j^i)^2 = -\int_Q (\tilde{v}_t^\varepsilon)''(\tilde{k}_\varepsilon) (\partial_r \tilde{k}_\varepsilon)^2 (d_j^i)^2 + \int_Q (-\Delta \tilde{k}_\varepsilon) (\tilde{v}_t^\varepsilon)'(\tilde{k}_\varepsilon) (d_j^i)^2. \tag{3.28}$$

Therefore, taking  $\mathbf{D}_\varepsilon$  as test tensor in (3.18) yields

$$\int_Q \tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon) |\nabla \mathbf{D}_\varepsilon|^2 + \int_Q [-(\tilde{v}_t^\varepsilon)''(\tilde{k}_\varepsilon)] |\nabla \tilde{k}_\varepsilon|^2 |\mathbf{D}_\varepsilon|^2 + \int_Q (-\Delta \tilde{k}_\varepsilon) (\tilde{v}_t^\varepsilon)'(\tilde{k}_\varepsilon) |\mathbf{D}_\varepsilon|^2 = \int_Q \nabla \mathbf{f} : \mathbf{D}_\varepsilon. \tag{3.29}$$

The function  $\tilde{v}_t^\varepsilon$  being a concave function (it plays a role here, see (2.27) and (1.13)), one has  $-(\tilde{v}_t^\varepsilon)''(\tilde{k}_\varepsilon) \geq 0$  and Eq. (3.29) yields

$$\int_Q \tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon) |\nabla \mathbf{D}_\varepsilon|^2 + \int_Q (-\Delta \tilde{k}_\varepsilon) (\tilde{v}_t^\varepsilon)'(\tilde{k}_\varepsilon) |\mathbf{D}_\varepsilon|^2 \leq \int_Q \nabla \mathbf{f} : \mathbf{D}_\varepsilon. \tag{3.30}$$

By using the equation satisfied by  $\tilde{k}_\varepsilon$  (see (2.59)), inequality (3.30) becomes

$$\int_Q \tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon) |\nabla \mathbf{D}_\varepsilon|^2 + \int_Q (\tilde{v}_t^\varepsilon)'(\tilde{k}_\varepsilon) (\tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon) |\mathbf{D}|^2) \star \rho_\varepsilon \cdot |\mathbf{D}_\varepsilon|^2 \leq \frac{1}{\ell} \int_Q (\tilde{v}_t^\varepsilon)'(\tilde{k}_\varepsilon) |\mathbf{D}_\varepsilon|^2 \mathcal{E}_\varepsilon(\tilde{k}_\varepsilon) + \int_Q \nabla \mathbf{f} : \mathbf{D}_\varepsilon. \tag{3.31}$$

Now thanks to the fact that  $\tilde{v}_t^\varepsilon$  is a nondecreasing function (see (2.26)) and  $\tilde{v}_t^\varepsilon$  is nonnegative, (3.31) combined to (2.30) yields

$$\nu \int_Q |\nabla \mathbf{D}_\varepsilon|^2 \leq \frac{1}{\ell} \int_Q (\tilde{v}_t^\varepsilon)'(\tilde{k}_\varepsilon) |\mathbf{D}_\varepsilon|^2 \mathcal{E}_\varepsilon(\tilde{k}_\varepsilon) + \int_Q \nabla \mathbf{f} : \mathbf{D}_\varepsilon. \tag{3.32}$$

Bound (2.24) states that  $\|(\tilde{v}_t^\varepsilon)'\|_\infty \leq \|(v_t)'\|_\infty$  and recall that  $(v_t)'$  is bounded. Therefore (3.32) yields

$$\nu \int_Q |\nabla \mathbf{D}_\varepsilon|^2 \leq \frac{\|(v_t)'\|_\infty}{\ell} \int_Q |\mathbf{D}_\varepsilon|^2 \mathcal{E}_\varepsilon(\tilde{k}_\varepsilon) + \int_Q \nabla \mathbf{f} : \mathbf{D}_\varepsilon. \tag{3.33}$$

The first term in the r.h.s of (3.33) is considered in what follows. By Cauchy–Schwarz inequality, one has

$$\frac{\|(v_t)'\|_\infty}{\ell} \int_Q |\mathbf{D}_\varepsilon|^2 \mathcal{E}_\varepsilon(\tilde{k}_\varepsilon) \leq \frac{\|(v_t)'\|_\infty}{\ell} \|\mathcal{E}_\varepsilon\|_{L^2(Q_\varepsilon)} \|\mathbf{D}\|_{(L^4(Q_\varepsilon))^{3 \times 3}}^2, \tag{3.34}$$

and by Sobolev inequality,

$$\frac{\|(v_t)'\|_\infty}{\ell} \int_Q |\mathbf{D}_\varepsilon|^2 \mathcal{E}_\varepsilon(\tilde{k}_\varepsilon) \leq H \frac{\|(v_t)'\|_\infty}{\ell} \|\mathcal{E}_\varepsilon\|_{L^2(Q_\varepsilon)} \int_Q |\nabla \mathbf{D}_\varepsilon|^2. \tag{3.35}$$

When inserting in (3.35) estimate (2.97) for  $\mathcal{E}_\varepsilon$ , one deduces (where the constant  $\Upsilon$  below does not depend on  $\varepsilon$ , but on  $\theta$ )

$$\frac{\|(v_t)'\|_\infty}{\ell} \int_Q |\mathbf{D}_\varepsilon|^2 \mathcal{E}_\varepsilon(\tilde{k}_\varepsilon) \leq \Upsilon \frac{\|(v_t)'\|_\infty}{\ell} [1 + \nu^{-\frac{3}{2(1+\theta)}} \|\mathbf{f}\|_{L^2(Q)}^{\frac{3}{(1+\theta)}}] \int_Q |\nabla \mathbf{D}_\varepsilon|^2. \tag{3.36}$$

Let us write

$$\kappa = \Upsilon \|(v_t)'\|_\infty. \tag{3.37}$$

By reporting (3.36) inside (3.33), one obtains

$$\left( \nu - \frac{\kappa}{\ell} [1 + \nu^{-\frac{3}{2(1+\theta)}} \|\mathbf{f}\|_{L^2(Q)}^{\frac{3}{(1+\theta)}}] \right) \int_Q |\nabla \mathbf{D}_\varepsilon|^2 \leq \int_Q \nabla \mathbf{f} : \mathbf{D}_\varepsilon. \tag{3.38}$$

One knows that

$$\sigma = \sigma(\theta, \mathbf{f}, \ell, \nu) = \left( \nu - \frac{\kappa}{\ell} \left[ 1 + \nu^{-\frac{3}{2(1+\theta)}} \|\mathbf{f}\|_{L^2(Q)}^{\frac{3}{(1+\theta)}} \right] \right) > 0 \tag{3.39}$$

by the hypothesis (3.22) in Theorem 3.2. Therefore, (3.38) can be rewritten under the following form

$$\sigma \int_Q |\nabla \mathbf{D}_\varepsilon|^2 \leq \int_Q \mathbf{f} : \mathbf{D}_\varepsilon, \quad \sigma > 0. \tag{3.40}$$

By Cauchy–Schwarz inequality combined with Poincaré’s inequality one has

$$\sigma \int_Q |\nabla \mathbf{D}_\varepsilon|^2 \leq C_p \|\nabla \mathbf{f}\|_{(L^2(Q))^{3 \times 3}} \|\nabla \mathbf{D}\|_{(L^2(Q))^{3 \times 3 \times 3}}. \tag{3.41}$$

Finally, one uses Young inequality to derive the inequality

$$\|\nabla \mathbf{D}_\varepsilon\|_{(L^2(Q))^{3 \times 3 \times 3}} \leq \zeta = \sqrt{C_p} \frac{\|\nabla \mathbf{f}\|_{(L^2(Q))^{3 \times 3}}}{\sigma} \tag{3.42}$$

where  $\sigma$  is defined by (3.39). Inequality (3.42) can be rewritten under the form (3.23) and Theorem 3.2 is entirely proved.  $\square$

Corollary 3.2 is obvious because  $\mathbf{D}_\varepsilon = \nabla \mathbf{u}_\varepsilon$ . We are left to prove Corollary 3.3.

**Proof of Corollary 3.3.** Notice that the sequence of the  $Q$ -restrictions  $([\mathbf{u}_\varepsilon]_Q)_{\varepsilon>0}$  are bounded in the space  $(H^2(Q))^3$  and then in  $(W^{1,6}(Q))^3$ . Therefore, the sequence  $(\|\nabla \mathbf{u}_\varepsilon\|_{L^3(Q)}^2)_{\varepsilon>0}$  is bounded in  $L^3(Q)$ .

We also have  $\tilde{v}_t^\varepsilon \leq \tilde{v}_t$ . The scalar  $\tilde{k}_\varepsilon$  is extended by zero on  $Q \setminus Q_\varepsilon$ . Therefore the growth condition (2.13) yields

$$\|\tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon)\|_{L^6(Q)} \leq \rho \left[ 1 + \|\tilde{k}_\varepsilon\|_{L^{6\alpha/(1+\theta)}(Q)}^{\alpha/(1+\theta)} \right]. \tag{3.43}$$

Therefore when  $p = 3/(2 + \theta)$ , there exists a constant  $\mathcal{E} = \mathcal{E}(\nu, \theta, \mathbf{f})$  being such that for any  $\varepsilon > 0$ ,

$$\|\tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon)\|_{L^6(Q)} \leq \mathcal{E}. \tag{3.44}$$

To obtain this inequality we have used

- the  $W^{1,p}$  estimates (2.67) satisfied by  $\tilde{k}_\varepsilon$ ;
- the Sobolev inequality;
- the fact that  $\alpha \leq 1/2$ .

Consequently, for any  $\varepsilon > 0$  one has  $\tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon)|[\nabla \mathbf{u}_\varepsilon]_{Q_\varepsilon}|^2 \in L^2(Q_\varepsilon)$  and there exists a constant which does not depend on  $\varepsilon$  and denoted by  $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}(\theta, \nu, \mathbf{f})$  being such that

$$\|\tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon)|[\nabla \mathbf{u}_\varepsilon]_{Q_\varepsilon}|^2\|_{L^2(Q_\varepsilon)} \leq \tilde{\mathcal{E}}. \tag{3.45}$$

Moreover, thanks to Young inequality, one also has

$$\|[\tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon)|[\nabla \mathbf{u}_\varepsilon]_{Q_\varepsilon}|^2] \star \rho_\varepsilon\|_{L^2(Q_\varepsilon)} \leq \tilde{\mathcal{E}}. \tag{3.46}$$

We combine this last estimate with the  $L^2$  estimate (2.97) on  $\mathcal{E}_\varepsilon$  obtained in Subsection 2.3.4 when proving Theorem 2.1. Then, one observes that the  $\tilde{k}_\varepsilon$ ’s equation is an elliptic equation with a second hand side uniformly bounded in  $L^2$ , that means

$$-\Delta \tilde{k}_\varepsilon = \mathbf{K}_\varepsilon, \tag{3.47}$$

where

$$\|\mathbf{K}_\varepsilon\|_{L^2(Q_\varepsilon)} \leq \Phi = \Phi(\mathbf{f}, \nu, \theta, \|(v_t)'\|_\infty, \ell).$$

Thus, the  $H^2$  claimed estimates (3.24) satisfied by  $\tilde{k}_\varepsilon$  follows from the classical elliptic theory. Recall that here the constant  $\nu$  involved in (3.24) does not depend on  $\varepsilon$  thanks to the geometrical considerations of Subsection 2.3.2.  $\square$

**Before passing to the limit.** The space  $H^2(Q_\varepsilon) \cap H_0^1(Q_\varepsilon)$  is embedded in  $W_0^{1,6}(Q_\varepsilon)$ , and the constant can be chosen large enough, independent on  $\varepsilon$  (see Subsection 2.3.2). In what follows, we shall consider the function  $\tilde{k}_\varepsilon$  equal to itself in  $Q_\varepsilon$  and equal to 0 in  $Q \setminus Q_\varepsilon$  without any change of the notation. This new function lies in  $W_0^{1,6}(Q)$  and for this function, one has the estimate, uniform in  $\varepsilon$ ,

$$\|\tilde{k}_\varepsilon\|_{W_0^{1,6}(Q)} \leq \Theta = \Theta(\mathbf{f}, \nu, \theta, \|(v_t)'\|_\infty, \ell). \tag{3.48}$$

There is also an other consequence. The space  $W_0^{1,6}(Q)$  is embedded in  $C^0(Q)$  because we are in  $\mathbb{R}^3$ . Therefore, each  $\tilde{k}_\varepsilon$  remains a continuous function and the following estimate holds, uniform in  $\varepsilon$ ,

$$\|\tilde{k}_\varepsilon\|_{L^\infty(Q)} \leq \tilde{\Theta} = \tilde{\Theta}(\mathbf{f}, \nu, \theta, \|(v_t)'\|_\infty, \ell). \tag{3.49}$$

We are now ready to pass to the limit in the approximated system when  $\varepsilon$  tends to 0. Throughout the remainder, one assumes that (3.22) holds (also referred as (1.18)).

### 3.4. Passing to the limit: proof of Theorem 1.1

We now finish the proof of Theorem 1.1. To do this, we must pass to the limit in the approximated system (2.57)–(2.61) when  $\varepsilon$  tends to zero.

Notice that thanks

- to estimate (3.49);
- to the definition of  $\tilde{v}_t^\varepsilon$  which is equal to  $\tilde{v}_t$  on the range  $[0, 1/\varepsilon]$  (Subsection 2.3.1),

when  $\varepsilon$  is such that  $\varepsilon \leq (\tilde{\Theta})^{-1}$ , one has

$$\tilde{v}_t^\varepsilon(\tilde{k}_\varepsilon) = \tilde{v}_t(\tilde{k}_\varepsilon) = \tilde{v}_t \circ T_{\tilde{\Theta}}(\tilde{k}_\varepsilon) = \tilde{v}_t(\tilde{k}_\varepsilon), \tag{3.50}$$

and the function  $\tilde{v}_t$  involved above is a bounded nonnegative continuous function (recall that  $T_{\tilde{\Theta}}$  is the truncation function at height  $\tilde{\Theta}$  as defined by the sentence (2.19)).

One denotes by  $\tilde{\mathcal{E}}_\varepsilon$  the function defined by

$$\tilde{\mathcal{E}}_\varepsilon = \mathcal{E}_\varepsilon \circ T_{\tilde{\Theta}}. \tag{3.51}$$

On one hand, one has

$$\mathcal{E}_\varepsilon(\tilde{k}_\varepsilon) = \tilde{\mathcal{E}}_\varepsilon(\tilde{k}_\varepsilon), \tag{3.52}$$

on the other hand, the sequence  $(\tilde{\mathcal{E}}_\varepsilon)_{\varepsilon>0}$  is uniformly bounded in  $L^\infty(\mathbb{R})$  and uniformly converges to  $\tilde{\mathcal{E}} = \mathcal{E} \circ T_{\tilde{\Theta}}$ . The equations satisfied by  $(\mathbf{u}_\varepsilon, p_\varepsilon, \tilde{k}_\varepsilon)$  can be written under the form

$$-\nabla \cdot ([\tilde{v}_t(\tilde{k}_\varepsilon)]^\ell \nabla \mathbf{u}_\varepsilon) + \nabla p_\varepsilon = \mathbf{f} \quad \text{in } \mathcal{D}'_{\text{per}}, \tag{3.53}$$

$$\nabla \cdot \mathbf{u}_\varepsilon = 0 \quad \text{in } \mathcal{D}'_{\text{per}}, \tag{3.54}$$

$$-\Delta \tilde{k}_\varepsilon = (\tilde{v}_t(\tilde{k}_\varepsilon)[|\nabla \mathbf{u}_\varepsilon|^2]_{Q_\varepsilon}) \star \rho_\varepsilon - \frac{\tilde{\mathcal{E}}_\varepsilon(\tilde{k}_\varepsilon)}{\ell} \quad \text{in } \mathcal{D}'(Q_\varepsilon), \tag{3.55}$$

$$(\mathbf{u}_\varepsilon, p_\varepsilon) \text{ } Q\text{-periodic}, \quad \int_Q \mathbf{u}_\varepsilon = 0, \quad \int_Q p_\varepsilon = 0, \tag{3.56}$$

$$\tilde{k}_\varepsilon|_{\partial Q_\varepsilon} = 0, \quad \tilde{k}_\varepsilon \geq 0 \text{ a.e. in } Q_\varepsilon. \tag{3.57}$$

The problem is now a problem with a bounded eddy viscosity, as considered in many former works, see for instance [24]. Therefore, arguing as in [24] where in addition we also use the  $H^2$ -estimate for  $\mathbf{u}$  (see for instance (3.23)) and the  $W_0^{1,6}$ -estimate for  $\tilde{k}_\varepsilon$  (see (3.48)), one can extract from the sequence  $(\mathbf{u}_\varepsilon, \tilde{k}_\varepsilon)_{\varepsilon>0}$  a subsequence (still denoted by the same) which converges to some  $(\mathbf{u}, \tilde{k}) \in [V \cap H_{\text{loc}}^2(\mathbb{R}^3)] \times W_0^{1,6}(Q)$  and such that



- $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$  weakly converges to  $\mathbf{u}$  in  $H_{\text{loc}}^2(\mathbb{R}^3)$ , strongly in  $V$ .
- $(\tilde{k}_\varepsilon)_{\varepsilon>0}$  converges to  $\tilde{k} \geq 0$ 
  - weakly in  $W_0^{1,6}(Q)$ ,
  - uniformly.
- $((\tilde{v}_t(\tilde{k}_\varepsilon)[|\nabla \mathbf{u}_\varepsilon|^2]_{Q_\varepsilon}) \star \rho_\varepsilon)_{\varepsilon>0}$  strongly converges in  $L^1(Q)$  to  $\tilde{v}_t(\tilde{k})[|\nabla \mathbf{u}|^2]_Q$ .

Notice that in addition one has

$$\|\tilde{k}\|_{L^\infty(Q)} \leq \tilde{\Theta}. \tag{3.58}$$

The end of the proof follows now the scheme of former proofs written several times before. That is why we skip the details. Therefore, there exists  $p \in L_{\text{loc}}^2(\mathbb{R}^3)$  be such that

$$-\nabla \cdot ([\tilde{v}_t(\tilde{k})]^e \nabla \mathbf{u}) + \nabla p = \mathbf{f} \quad \text{in } \mathcal{D}'_{\text{per}}, \tag{3.59}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \mathcal{D}'_{\text{per}}, \tag{3.60}$$

$$-\Delta \tilde{k} = \tilde{v}_t(\tilde{k})[|\nabla \mathbf{u}|^2]_Q - \frac{\tilde{\mathcal{E}}(\tilde{k})}{\ell} \quad \text{in } \mathcal{D}'(Q), \tag{3.61}$$

$$(\mathbf{u}, p) \text{ } Q\text{-periodic, } \int_Q \mathbf{u} = 0, \int_Q p = 0, \tag{3.62}$$

$$\tilde{k}|_{\partial Q} = 0, \quad \tilde{k} \geq 0 \text{ a.e. in } Q. \tag{3.63}$$

Finally one has, by the  $L^\infty$  estimate (3.58) for  $\tilde{k}$  and  $\tilde{v}_t = \tilde{v}_t$ , combined to  $\tilde{\mathcal{E}} = \mathcal{E}$  on the range  $[0, \tilde{\Theta}]$ ,

$$\tilde{v}_t(\tilde{k}) = \tilde{v}_t(\tilde{k}), \quad \tilde{\mathcal{E}}(\tilde{k}) = \mathcal{E}(\tilde{k}).$$

Therefore,  $(\mathbf{u}, p, \tilde{k}) \in [V \cap H_{\text{loc}}^2(\mathbb{R}^3)] \times L_{\text{loc}}^2(\mathbb{R}^3) \times W_0^{1,6}(Q)$  is a solution to system (2.7)–(2.11), and thanks to Proposition 2.1,

$$(\mathbf{u}, p, \beta_t^{-1}(\tilde{k})) \in [V \cap H_{\text{loc}}^2(\mathbb{R}^3)] \times L_{\text{loc}}^2(\mathbb{R}^3) \times W_0^{1,6}(Q)$$

is a solution to problem (1.1)–(1.5). The proof of our existence result is now complete.

#### 4. Case where $\ell$ is not a constant

##### 4.1. Orientation and preliminary result

We prove in this section Theorem 1.2. In this case,  $v_t = v_t(k, \ell)$ . We do the following assumptions

$$v_t(k, \ell) = v_0 + C_1 \ell (k + \rho)^\alpha, \quad v_0 > 0, C_1 > 0, \rho > 0, 0 < \alpha < \frac{1}{2}, \tag{4.1}$$

$$\mu_t = \mu_0 \in \mathbb{R}_+^*, \tag{4.2}$$

$$\varepsilon(k) = \frac{k^{1+\theta}}{\ell(x)}, \quad 0 < \theta < 1/2, \tag{4.3}$$

$$\ell \text{ is a function of class } C^2, \quad -\Delta \ell \geq 0, \quad 0 < \ell_m \leq \ell(x) < \ell_M < \infty, \tag{4.4}$$

$$\left( v - \frac{\tilde{\Pi}}{\ell_m} - \|\nabla \ell\|_{L^\infty} \tilde{\Omega} \right) > 0. \tag{4.5}$$

The constants  $\tilde{\Pi}$  and  $\tilde{\Omega}$  depend on  $\mathbf{f}, \rho, \alpha, v$  and  $\theta$  and will be made clear in the remainder. Therefore, we are looking at the system:

$$-\nabla \cdot ([v_t(k, \ell)]^e \nabla \mathbf{u}) + \nabla p = \mathbf{f} \quad \text{in } \mathcal{D}'_{\text{per}}, \tag{4.6}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \mathcal{D}'_{\text{per}}, \tag{4.7}$$

$$-\mu_0 \Delta k = v_t(k, \ell) [|\nabla \mathbf{u}|^2]_Q - \frac{k^{1+\theta}}{\ell(x)} \quad \text{in } \mathcal{D}'(Q), \tag{4.8}$$

$$(\mathbf{u}, p) \text{ } Q\text{-periodic, } \int_Q \mathbf{u} = \mathbf{0}, \quad \int_Q p = 0, \tag{4.9}$$

$$k|_{\partial Q} = 0, \quad k \geq 0 \quad \text{a.e. in } Q. \tag{4.10}$$

We note that we cannot take  $\mu_t = \mu_t(k, \ell)$  and we assume that  $\mu_t$  is a constant. This is because of transformation (2.4) which aims to replace the variable  $k$  by  $\tilde{k}$  and the operator  $-\nabla \cdot (\mu_t \nabla k)$  by  $-\Delta \tilde{k}$ . Indeed, if one assumes for instance that  $\mu_t = \mu_t(k, \ell) = \mu_0 + C_2 \ell(k + \tau)^\alpha$ , the transformation (2.4) will induce in the  $\tilde{k}$ -equation several additional nonlinear terms that we cannot currently estimate. Without loss of generality, we shall assume that  $\mu_0 = 1$ . Therefore,  $\beta_t = Id$  and  $k = \tilde{k}$ ,  $\tilde{v}_t = v_t$ . The price to pay is also the fact that for a regularity reason, one cannot take  $\varepsilon(k) = k\sqrt{k}/\ell(x)$ , here replaced by  $k^{1+\theta}/\ell(x)$ , where  $0 < \theta < 1/2$ .

Moreover, as we shall see in the remainder, the fact that we have to restrict ourselves to the case  $\alpha < 1/2$  is due to the appearance of the term

$$\int_Q \frac{\partial^2 v_t}{\partial k \partial \ell}(k, \ell) \nabla \ell \cdot \nabla k |\mathbf{D}|^2$$

when one wants to obtain an estimate for the variable  $\mathbf{D}$ . Of course, there is no hope to invoke a sign argument. Therefore, we just can wait for a regularity argument. Notice that when  $v_t(k, \ell) = v_0 + C_1 \ell(k + \rho)^\alpha$ , then

$$\frac{\partial^2 v_t}{\partial k \partial \ell}(k, \ell) = \frac{C_1 \alpha}{(k + \rho)^{1-\alpha}},$$

and here  $k \geq 0$ . We prove the following lemma.

**Lemma 4.1.** *Assume that  $k \geq 0$  is a function in the space  $H^1_{\text{loc}}(Q)$  and that there exists a constant  $C$  such that*

$$\forall n \in \mathbb{N}, \quad \int_{n \leq k \leq n+1} |\nabla k|^2 \leq C. \tag{4.11}$$

Then for any  $0 < \alpha < 1/2$ ,

$$\frac{\nabla k}{(k + \rho)^{1-\alpha}} \in L^2(Q). \tag{4.12}$$

**Proof.** One has

$$\int_Q \frac{|\nabla k|^2}{(k + \rho)^{2-2\alpha}} = \sum_{n=0}^{\infty} \int_{n \leq k \leq n+1} \frac{|\nabla k|^2}{(k + \rho)^{2-2\alpha}}.$$

By using (4.11),

$$\int_{n \leq k \leq n+1} \frac{|\nabla k|^2}{(k + \rho)^{2-2\alpha}} \leq C \frac{1}{(n + \rho)^{2-2\alpha}},$$

hence

$$\int_Q \frac{|\nabla k|^2}{(k + \rho)^{2-2\alpha}} \leq C \sum_{n=0}^{\infty} \frac{1}{(n + \rho)^{2-2\alpha}} = CS(\rho) < \infty,$$

since  $\alpha < 1/2$ , and the lemma is proved.  $\square$

4.2. Proof of Theorem 1.2

As we have shown, the heart of the proof of Theorem 1.1 is estimate (3.42). The process of building approximations remains the same in the present case as well as passing to the limit. Indeed, we notice that  $v_t$  is continuous with respect to the  $k$ -variable, which is the required property to pass to the limit in the viscous term. Therefore, we just have to derive the same “à priori” estimate as (3.42), the remainder of the proof following the same scheme as in the case  $\ell$  constant. Recall that  $\mathbf{u}$  is a periodic solution to

$$-\nabla \cdot (v_t(k, \ell)\nabla\mathbf{u}) + \nabla p = \mathbf{f}, \tag{4.13}$$

where we have omitted here to quote the periodic extension of  $v_t$  and  $k$  for the simplicity. Moreover, we take grant that  $k \geq 0$ , as already proved in Lemma 2.2, the proof here being exactly the same (by replacing the term  $\varepsilon(k)$  by  $k|k|^\theta/\ell(x)$ ).

Deriving formally Eq. (4.13) yields, with  $\mathbf{D} = (d_j^i)_{1 \leq i, j \leq 3} = \nabla\mathbf{u}$  and  $\mathbf{P} = \nabla p$ ,

$$-\nabla \cdot (v_t(k, \ell)\nabla\mathbf{D}) - \nabla \cdot \left( \left[ \frac{\partial v_t}{\partial k}(k, \ell)\nabla k + \frac{\partial v_t}{\partial \ell}(k, \ell)\nabla \ell \right] \otimes \mathbf{D} \right) + \nabla\mathbf{P} = \nabla\mathbf{f}. \tag{4.14}$$

We take  $\mathbf{D}$  as test tensor in (4.14) and we integrate by parts, using as before the rule  $\partial_j d_r^i = \partial_r d_j^i$ . We obtain (we omit the details, the calculus being the same as in the previous section)

$$\int_Q v_t(k, \ell)|\nabla\mathbf{D}|^2 - \frac{1}{2} \int_Q \nabla \cdot \left[ \frac{\partial \tilde{v}_t}{\partial k}(k, \ell)\nabla k + \frac{\partial v_t}{\partial \ell}(k, \ell)\nabla \ell \right] |\mathbf{D}|^2 = \int_Q \nabla\mathbf{f} \cdot \mathbf{D}. \tag{4.15}$$

One has

$$\begin{aligned} \nabla \cdot \left[ \frac{\partial v_t}{\partial k}(k, \ell)\nabla k + \frac{\partial v_t}{\partial \ell}(k, \ell)\nabla \ell \right] &= \frac{\partial v_t}{\partial k}(k, \ell)\Delta k + \frac{\partial v_t}{\partial \ell}(k, \ell)\Delta \ell + \frac{\partial^2 v_t}{\partial k^2}(k, \ell)|\nabla k|^2 \\ &\quad + \frac{\partial^2 v_t}{\partial \ell^2}(k, \ell)|\nabla \ell|^2 + 2 \frac{\partial^2 v_t}{\partial k \partial \ell}(k, \ell)\nabla k \cdot \nabla \ell. \end{aligned} \tag{4.16}$$

Observe now that

$$\frac{\partial^2 v_t}{\partial \ell^2}(k, \ell) = 0, \quad \frac{\partial v_t}{\partial \ell}(k, \ell) = C_1(k + \rho)^\alpha \geq 0, \quad \frac{\partial^2 v_t}{\partial k \partial \ell}(k, \ell) = \frac{C_1\alpha}{(k + \rho)^{1-\alpha}}$$

and

$$\frac{\partial^2 v_t}{\partial k^2}(k, \ell) = -C_1 \frac{\alpha(1 - \alpha)\ell(x)}{(k + \rho)^{2-\alpha}} \leq 0.$$

Using the equation

$$-\Delta k = \tilde{v}_t |\nabla\mathbf{u}|^2 - \frac{k^{1+\theta}}{\ell(x)}$$

we obtain:

$$\begin{aligned} &\int_Q v_t(k, \ell)|\nabla\mathbf{D}|^2 + \frac{1}{2} \int_Q v_t(k, \ell)C_1\alpha\ell(x)(k + \rho)^{\alpha-1}|\mathbf{D}|^2|\nabla\mathbf{u}|^2 - \frac{1}{2} \int_Q \frac{k^{1+\theta}}{\ell(x)}|\mathbf{D}|^2 + \frac{1}{2} \int_Q C_1(k + \rho)^\alpha(-\Delta\ell)|\mathbf{D}|^2 \\ &\quad + \frac{1}{2} \int_Q \frac{1}{2}C_1\alpha(1 - \alpha)\ell(x)(k + \rho)^{\alpha-2}|\nabla k|^2|\mathbf{D}|^2 - C_1\alpha \int_Q \frac{\nabla k}{(k + \rho)^{1-\alpha}} \cdot \nabla \ell |\mathbf{D}|^2 \\ &= \int_Q \nabla\mathbf{f} \cdot \mathbf{D}. \end{aligned} \tag{4.17}$$

Thanks to the hypothesis  $-\Delta\ell \geq 0$  and the fact that  $\ell$  is a  $C^2$ -class function on  $Q$  and  $v_t \geq v$ , one has

$$v \int_Q |\nabla \mathbf{D}|^2 \leq \frac{1}{\ell_m} \int_Q k^{1+\theta} |\mathbf{D}|^2 + C_1 \alpha \|\nabla \ell\|_{L^\infty} \int_Q \frac{\nabla k}{(k+\rho)^{1-\alpha}} |\mathbf{D}|^2 + \int_Q \nabla \mathbf{f} \cdot \mathbf{D}. \quad (4.18)$$

One already knows by the analogue of (2.67) that  $k \in \bigcap_{p < 3/2} W_0^{1,p}(Q)$  and therefore  $k \in \bigcap_{p < 3} L^p(Q)$ . Because  $\theta < 1/2$ ,  $k^{1+\theta} \in L^2(Q)$  and

$$\|k^{1+\theta}\|_{L^2(Q)} \leq \Pi = \Pi(\|\mathbf{f}\|_{L^2(Q)}, \theta). \quad (4.19)$$

Consequently

$$\frac{1}{\ell_m} \int_Q k^{1+\theta} |\mathbf{D}|^2 \leq \frac{1}{\ell_m} \Pi \|\mathbf{D}\|_{L^4(Q)}^2 \leq \frac{1}{\ell_m} \Pi C_s \int_Q |\nabla \mathbf{D}|^2 = \frac{1}{\ell_m} \tilde{\Pi} \int_Q |\nabla \mathbf{D}|^2, \quad (4.20)$$

where  $C_s$  is the Sobolev constant. On the other hand, thanks to (2.83) and Lemma 4.1 one has

$$\int_Q \frac{\nabla k}{(k+\rho)^{1-\alpha}} |\mathbf{D}|^2 \leq \left\| \frac{\nabla k}{(k+\rho)^{1-\alpha}} \right\|_{L^2(Q)} \|\mathbf{D}\|_{L^4(Q)}^2 \leq \frac{C_p^2 \|\mathbf{f}\|_{L^2(Q)}}{v} S(\rho) \|\mathbf{D}\|_{L^4(Q)}^2$$

which yields combined to Sobolev inequality to an inequality under the form

$$\int_Q \frac{\nabla k}{(k+\rho)^{1-\alpha}} |\mathbf{D}|^2 \leq \Omega \int_Q |\nabla \mathbf{D}|^2, \quad (4.21)$$

where  $\Omega = \Omega(\rho, v, \mathbf{f})$ . Therefore (4.18) yields, by using (4.20) and (4.21)

$$\left( v - \frac{\tilde{\Pi}}{\ell_m} - \|\nabla \ell\|_{L^\infty} \tilde{\Omega} \right) \int_Q |\nabla \mathbf{D}|^2 \leq \int_Q \nabla \mathbf{f} \cdot \mathbf{D}, \quad (4.22)$$

where  $\tilde{\Omega} = C_1 \alpha \Omega$ . By using hypothesis (4.5) the end of the proof is obvious.

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