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# Regularity criteria for the generalized viscous MHD equations

# Critères de régularité pour les équations MHD généralisées avec viscosité

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**Abstract**

In this paper, we consider regularity criteria for solutions to the 3D generalized MHD equations with fractional dissipative term $-(-\Delta)^{\alpha}u$ for the velocity field and $-(-\Delta)^{\beta}b$ for the magnetic field. For the case $\alpha=\beta$, it is proved that if the velocity field belongs to $L^{p,q}$ with $2\alpha/p+3/q\leqslant 2\alpha-1$ or the gradient of velocity field belongs to $L^{p,q}$ with $2\alpha/p+3/q\leqslant 3\alpha-1$ on $[0,T]$, then the solution remains smooth on $[0,T]$. The significance is that there is no restriction on the magnetic field. Moreover, the norms $\|u\|_{L^{p,q}}$ and $\|\Lambda^{\alpha}u\|_{L^{p,q}}$ are scaling dimension zero for $2\alpha/p+3/q=2\alpha-1$ and $2\alpha/p+3/q=3\alpha-1$ respectively. For $1\leqslant\beta\leqslant\alpha$, we find that the minimum sum of $\alpha$ and $\beta$ to guarantee the global existence of smooth solutions is $5/2$. Furthermore, we show that the weak solution actually is strong if the corresponding vorticity field $\omega=\nabla\times u$ satisfies certain condition in the high vorticity region.


**Résumé**

Dans ce papier nous considérons des critères de régularité pour les solutions des équations MHD en 3D généralisées avec un terme de dissipation fractionnel $-(-\Delta)^{\alpha}u$ pour le champ de vitesse et un terme $-(-\Delta)^{\beta}b$ pour le champ magnétique. Pour le cas $\alpha=\beta$, il est démontré que si le champ de vitesse est dans $L^{p,q}$ avec $2\alpha/p+3/q\leqslant 2\alpha-1$ ou le gradient de la vitesse est dans $L^{p,q}$ avec $2\alpha/p+3/q\leqslant 3\alpha-1$ sur $[0,T]$, alors la solution reste régulière sur $[0,T]$. Il est important de noter qu'il n'y a pas de restriction sur le champ magnétique. En plus, les normes $\|u\|_{L^{p,q}}$ et $\|\Lambda^{\alpha}u\|_{L^{p,q}}$ ont une dimension sous changement d'échelle égale à zéro pour $2\alpha/p+3/q=2\alpha-1$ et pour $2\alpha/p+3/q=3\alpha-1$ respectivement. Pour $1\leqslant\beta\leqslant\alpha$, nous trouvons que la somme minimale de $\alpha$ et $\beta$ qui garantit l'existence globale de solutions régulières est $5/2$. En plus nous montrons que les solutions faibles sont des solutions fortes si le champ de vorticité correspondant $\omega=\nabla\times u$ satisfait une certaine condition dans la région de vorticité.


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## 1. Introduction

In this paper, we consider the following 3D generalized viscous MHD equations

$$\begin{cases} \partial_t u + u\cdot\nabla u - b\cdot\nabla b + \nabla P = -(-\Delta)^\alpha u, \\ \partial_t b + u\cdot\nabla b - b\cdot\nabla u = -(-\Delta)^\beta b, \\ \operatorname{div} u = \operatorname{div} b = 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x), \end{cases} \tag{1.1}$$

where $u \in \mathbb{R}^3$ is the velocity field, $b \in \mathbb{R}^3$ is the magnetic field, $P(x,t)$ is a scalar pressure, and $u_0(x), b_0(x)$ with $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$ in the sense of distribution are the initial velocity and magnetic fields. $\alpha, \beta \geqslant 1$ are the parameters, and the operator $(-\Delta)^\gamma$ $(\gamma > 0)$ is defined by [13]

$$\widehat{(-\Delta)^\gamma} f(\xi) = |\xi|^{2\gamma}\hat{f},$$

where $\hat{f}$ denotes the Fourier transform of $f$. As usual, we write $(-\Delta)^{1/2}$ as $\Lambda$.

This system is of interest for various reasons. For example, it includes some known equations, say Navier–Stokes equation ($\alpha = \beta = 1$, $b = 0$) and standard MHD equations ($\alpha = \beta = 1$). Moreover, it has similar scaling properties and energy estimate as the Navier–Stokes and MHD equations. Heuristically, solutions of (1.1) should converge to that of Navier–Stokes and MHD equations as $\alpha, \beta \to 1$. We believe that the regularity studies of system (1.1) can improve the understanding of the Navier–Stokes and MHD equations.

For this dissipative system, it is easy to prove (see [11] for $\alpha = \beta = 1$) that problem (1.1) is local well-posed for any given initial datum $u_0, b_0 \in H^s(\mathbb{R}^3)$, $s \geqslant 3$. Moreover, just as what for other mechanical equations, say Navier–Stokes and MHD equations, it is proved by Wu [15] that Eq. (1.1) has a weak solution for any given $u_0, b_0 \in L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. But whether the unique local solution can exist globally or the weak solution is regular and unique is an outstanding challenge problem, just as the situation for Navier–Stokes and MHD equations. So a lot of literatures are devoted to find regularity criteria or prove partial regularity for these equations, such as [1–4,7,9,12,14,17–20,22] for Navier–Stokes equations, and [5,10,16,21] for MHD equations.

The paper is organized as follows. In Section 2, we impose conditions on the velocity field, and consider 2 cases. One is $\alpha = \beta$, the other is $\alpha > \beta$. In Section 3, we prove that the weak solution actually is strong if the corresponding vorticity field $\omega = \nabla \times u$ somehow is Hölder continuous with exponent $\frac{5}{2} - 2\alpha$.

Before going to next section, we write down the definition of weak solutions to (1.1).

**Definition 1.1.** A measurable vector pair $(u, b)$ is called a weak solution to generalized MHD equations (1.1), if $(u, b)$ satisfies the following properties

(i) $u \in L^\infty([0,T]; L^2) \cap L^2([0,T]; H^\alpha)$, $b \in L^\infty([0,T]; L^2) \cap L^2([0,T]; H^\beta)$.
(ii) $(u, b)$ verifies (1.1) in the sense of distribution; that is

$$\int_0^T\!\!\int_{\mathbb{R}^3} \left(\frac{\partial\phi}{\partial t} + (u\cdot\nabla)\phi\right) u \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^3} u_0\phi(x,0)\,\mathrm{d}x = \int_0^T\!\!\int_{\mathbb{R}^3} \left(u\Lambda^{2\alpha}\phi + b\cdot\nabla\phi\cdot b\right)\mathrm{d}x\,\mathrm{d}t,$$

$$\int_0^T\!\!\int_{\mathbb{R}^3} \left(\frac{\partial\phi}{\partial t} + (u\cdot\nabla)\phi\right) b \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^3} b_0\phi(x,0)\,\mathrm{d}x = \int_0^T\!\!\int_{\mathbb{R}^3} \left(b\Lambda^{2\beta}\phi + b\cdot\nabla\phi\cdot u\right)\mathrm{d}x\,\mathrm{d}t$$

for all $\phi \in C_0^\infty(\mathbb{R}^3 \times [0,T))$ with $\operatorname{div}\phi = 0$, and

$$\int_0^T\!\!\int_{\mathbb{R}^3} u\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t = 0, \qquad \int_0^T\!\!\int_{\mathbb{R}^3} b\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t = 0$$

for every $\phi \in C_0^\infty(\mathbb{R}^3 \times [0,T))$.

(iii) The energy inequality; that is

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\Lambda^\alpha u(s)\|_{L^2}^2\,\mathrm{d}s \leqslant \|u_0\|_{L^2}^2,$$

$$\|b(t)\|_{L^2}^2 + 2\int_0^t \|\Lambda^\beta b(s)\|_{L^2}^2\,\mathrm{d}s \leqslant \|b_0\|_{L^2}^2,$$

for $0 \leqslant t \leqslant T$.

Here, the space $H^s(\mathbb{R}^3)$, $s \in \mathbb{R}$, consists of functions $f$ satisfying

$$\|f\|_{H^s}^2 = \int_{\mathbb{R}^3} \left(1+|\xi|^2\right)^s \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi < \infty.$$

**Remark.** The definition of weak solution for (1.1) is similar to that for the Navier–Stokes equations. In [15], Wu defined the weak solutions without (iii), the energy inequality. However, from the existence proof (by Galerkin method), we can find the existence of weak solutions possessing the energy inequality.

## 2. Regularity criteria in terms of the velocity field

The first result is a regularity criterion similar to Serrin's [12] regularity class for the Navier–Stokes equations. To this end, we introduce the space $L^{\alpha,\gamma}$

$$\|u\|_{L^{p,q}} = \begin{cases} \left(\int_0^t \|u(\cdot,\tau)\|_{L^q}^p\,\mathrm{d}\tau\right)^{1/p} & \text{if } 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}_{0<\tau<t} \|u(\cdot,\tau)\|_{L^q} & \text{if } p = \infty, \end{cases}$$

where

$$\|u(\cdot,\tau)\|_{L^q} = \begin{cases} \left(\int_{\mathbb{R}^3} |u(x,\tau)|^q\,\mathrm{d}x\right)^{1/q} & \text{if } 1 \leqslant q < \infty, \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}^3} |u(x,\tau)| & \text{if } q = \infty. \end{cases}$$

We say $v \in L^{p,q}$ if $\|v\|_{L^{p,q}} < \infty$.

Note that if $\alpha = \beta$ and $(u(x,t), b(x,t))$ is a solution to (1.1), then $(u_\lambda, b_\lambda)$ with any $\lambda > 0$ is also a solution, where $u_\lambda(x,t) = \lambda^{2\alpha-1}u(\lambda x, \lambda^{2\alpha}t)$ and $b_\lambda(x,t) = \lambda^{2\alpha-1}b(\lambda x, \lambda^{2\alpha}t)$. Motivated by the work of Caffarelli, Kohn and Nirenberg [4] for the Navier–Stokes equations, we say that the norm $\|u\|_{L^{p,q}}$ is scaling dimension zero for $2\alpha/p + 3/q = 2\alpha - 1$ in the sense that $\|u_\lambda\|_{L^{p,q}} = \|u\|_{L^{p,q}}$ holds for all $\lambda > 0$ if and only if $2\alpha/p + 3/q = 2\alpha - 1$.

The first regularity criterion reads

**Theorem 2.1.** *Let $1 \leqslant \alpha = \beta \leqslant \frac{3}{2}$. Assume that the initial velocity and magnetic fields $u_0, b_0 \in H^3(\mathbb{R}^3)$. If*

$$u(x,t) \in L^{p,q}, \quad \text{with } \frac{2\alpha}{p} + \frac{3}{q} \leqslant 2\alpha - 1,\ \frac{3}{2\alpha-1} < q \leqslant \infty, \tag{2.1}$$

*then the solution remains smooth on $(0, T]$.*

Assume that Theorem 2.1 is true for a moment, just from the energy inequality for the weak solutions, then we have a corollary as follows.

**Corollary 2.2.** *Assume that the initial velocity and magnetic fields $u_0, b_0 \in H^3(\mathbb{R}^3)$. If $\alpha = \beta \geqslant \frac{5}{4}$, then all the global weak solutions to* (1.1) *are actually strong and unique.*

**Proof.** Due to the Gagliardo–Nirenberg inequality, for any weak solution defined by Definition 1.1, we have

$$\|u\|_{L^{4\alpha,3}} \leqslant C\|u\|_{L^{\infty,2}}^{1-1/(2\alpha)}\|\Lambda^\alpha u\|_{L^{2,2}}^{1/(2\alpha)}.$$

On the other hand,

$$\frac{2\alpha}{4\alpha}+\frac{3}{3}=\frac{3}{2}\leqslant 2\alpha-1$$

is true, provided that $\alpha \geqslant \frac{5}{4}$. Hence Corollary 2.2 follows from the result of Theorem 2.1 directly. □

**Remark 2.1.** The result of Corollary 2.2 was also showed by Wu [15] via a different approach.

Now we go to the proof of Theorem 2.1. Suppose $f \in H^2$, due to the fact that

$$\frac{\partial^2 f}{\partial x_i \partial x_j}=\partial_i\partial_j f=-R_iR_j\Delta f,$$

where $R_i$ is the Riesz transform, $\widehat{R_i g}(\xi)=-\mathrm{i}(\xi_i/|\xi|)\hat{g}(\xi)$ [13], and the boundedness of the operator $R_i : L^p \to L^p$, $1<p<\infty$, we have

$$\|\partial_i\partial_j f\|_{L^p} \leqslant C\|\Delta f\|_{L^p}, \quad 1<p<\infty. \tag{2.2}$$

In order to prove Theorem 2.1, first we show

$$u, b \in L^\infty\big(0,T;H^1\big)\cap L^2\big(0,T;H^{\alpha+1}\big), \tag{2.3}$$

if (2.1) holds.

Multiplying the first equation of (1.1) by $\Delta u$, after integration by parts and by taking the divergence free property into account, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2+\|\Lambda^{\alpha+1}u\|_{L^2}^2=-\int_{\mathbb{R}^3}\partial_i u_k\cdot\partial_k u_j\cdot\partial_i u_j\,\mathrm{d}x+\int_{\mathbb{R}^3}\partial_i b_k\cdot\partial_k b_j\cdot\partial_i u_j\,\mathrm{d}x \\ -\int_{\mathbb{R}^3} b_k\cdot\partial_i\partial_k u_j\cdot\partial_i b_j\,\mathrm{d}x. \tag{2.4}$$

Similarly, multiplying the second one by $\Delta b$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla b\|_{L^2}^2+\|\Lambda^{\alpha+1}b\|_{L^2}^2=-\int_{\mathbb{R}^3}\partial_i u_k\cdot\partial_k b_j\cdot\partial_i b_j\,\mathrm{d}x+\int_{\mathbb{R}^3}\partial_i b_k\cdot\partial_k u_j\cdot\partial_i b_j\,\mathrm{d}x \\ +\int_{\mathbb{R}^3} b_k\cdot\partial_k\partial_i u_j\cdot\partial_i b_j\,\mathrm{d}x. \tag{2.5}$$

Combining (2.4) and (2.5) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2\big)+\big\|\Lambda^{\alpha+1}u\big\|_{L^2}^2+\big\|\Lambda^{\alpha+1}b\big\|_{L^2}^2=-\int_{\mathbb{R}^3}\partial_i u_k\cdot\partial_k u_j\cdot\partial_i u_j\,\mathrm{d}x+\int_{\mathbb{R}^3}\partial_i b_k\cdot\partial_k b_j\cdot\partial_i u_j\,\mathrm{d}x \\ -\int_{\mathbb{R}^3}\partial_i u_k\cdot\partial_k b_j\cdot\partial_i b_j\,\mathrm{d}x+\int_{\mathbb{R}^3}\partial_i b_k\cdot\partial_k u_j\cdot\partial_i b_j\,\mathrm{d}x \\ =I+II+III+IV. \tag{2.6}$$

Then we estimate the above terms one by one. First we do the estimates for $q<\infty$.

$$
\begin{aligned}
|I| &= \left| \int_{\mathbb{R}^3} u_k (\partial_{ik} u_j \cdot \partial_i u_j + \partial_k u_j \cdot \partial_{ii} u_j) \right| \\
&\leqslant C \|u\|_{L^q} \|\nabla u\|_{L^s} \|\Delta u\|_{L^\gamma} \\
&\leqslant C \|u\|_{L^q} \|\nabla u\|_{L^2}^{\theta} \left\| \Lambda^{\alpha+1} u \right\|_{L^2}^{1-\theta} \|\nabla u\|_{L^2}^{\delta} \left\| \Lambda^{\alpha+1} u \right\|_{L^2}^{1-\delta} \\
&\leqslant \frac{1}{2} \left\| \Lambda^{\alpha+1} u \right\|_{L^2}^2 + C \|u\|_{L^q}^{2/(\theta+\delta)} \|\nabla u\|_{L^2}^2
\end{aligned}
\tag{2.7}
$$

where we used the Hölder inequality, Young inequality and Gagliardo–Nirenberg inequality (a combination of the interpolation and Sobolev inequalities) for the fractional Sobolev spaces [13] (see also [8], p. 89). The constants $1 < s, \gamma < \infty$ and $0 \leqslant \theta, \delta \leqslant 1$ satisfy

$$
\begin{cases}
\dfrac{1}{s} + \dfrac{1}{\gamma} + \dfrac{1}{q} = 1, \\
\dfrac{1}{s} - \dfrac{1}{3} = \theta \left( \dfrac{1}{2} - \dfrac{1}{3} \right) + (1-\theta) \left( \dfrac{1}{2} - \dfrac{\alpha+1}{3} \right), \\
\dfrac{1}{\gamma} - \dfrac{2}{3} = \delta \left( \dfrac{1}{2} - \dfrac{1}{3} \right) + (1-\delta) \left( \dfrac{1}{2} - \dfrac{\alpha+1}{3} \right).
\end{cases}
\tag{2.8}
$$

System (2.8) has 4 unknowns but 3 equations, so there are infinite many solutions. Note that $\alpha \leqslant \frac{3}{2}$ implies $q > \frac{3}{2}$, then one solution to (2.8) can be written as

$$
s = \frac{6q}{2q-3}, \quad \gamma = \frac{6q}{4q-3}, \quad \theta = \delta = \frac{1}{2} \left( \frac{1}{\alpha} + \frac{3}{\alpha q} \right) < 1. \tag{2.9}
$$

Moreover for any solution to (2.8), we have

$$
\frac{2}{\theta+\delta} = \frac{2\alpha}{2\alpha - 1 - 3/q} \leqslant p.
$$

Similarly, one can obtain

$$
\begin{aligned}
|II| + |III| + |IV| &\leqslant C \|u\|_{L^q} \|\nabla b\|_{L^s} \|\Delta b\|_{L^\gamma} \\
&\leqslant C \|u\|_{L^q} \|\nabla b\|_{L^2}^{\theta} \left\| \Lambda^{\alpha+1} b \right\|_{L^2}^{1-\theta} \|\nabla b\|_{L^2}^{\delta} \left\| \Lambda^{\alpha+1} b \right\|_{L^2}^{1-\delta} \\
&\leqslant \frac{1}{2} \left\| \Lambda^{\alpha+1} b \right\|_{L^2}^2 + C \|u\|_{L^q}^{2\alpha/(2\alpha-1-3/q)} \|\nabla b\|_{L^2}^2,
\end{aligned}
\tag{2.10}
$$

where $s, \gamma, \theta$ and $\delta$ are constants given by (2.9).

Putting (2.7) and (2.10) into (2.6), we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t} \left( \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \right) + \left\| \Lambda^{\alpha+1} u \right\|_{L^2}^2 + \left\| \Lambda^{\alpha+1} b \right\|_{L^2}^2 \leqslant C \|u\|_{L^q}^{2\alpha/(2\alpha-1-3/q)} \left( \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \right). \tag{2.11}
$$

Therefore, by applying the standard Gronwall inequality on (2.11), one has

$$
\begin{aligned}
&\sup_{0 \leqslant t \leqslant T} \left( \left\| \nabla u(\cdot,t) \right\|_{L^2}^2 + \left\| \nabla b(\cdot,t) \right\|_{L^2}^2 \right) + \int_0^T \left( \left\| \Lambda^{\alpha+1} u(\cdot,t) \right\|_{L^2}^2 + \left\| \Lambda^{\alpha+1} b(\cdot,t) \right\|_{L^2}^2 \right) \mathrm{d}t \\
&\quad \leqslant \left( \|\nabla u_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2 \right) \exp\left( \int_0^T C \left\| u(\cdot,t) \right\|_{L^q}^{2\alpha/(2\alpha-1-3/q)} \mathrm{d}t \right).
\end{aligned}
$$

Thanks to the boundedness of $u$ in $L^{p,q}$-norm, the above inequality implies (2.3) for $p < \infty$.

If $q = \infty$, we can use the Hölder, Gagliardo–Nirenberg and Young inequalities to obtain

$$
\begin{aligned}
|I|+|II|+|III|+|IV| &\leqslant C\|u\|_{L^\infty}\big(\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}+\|\nabla b\|_{L^2}\|\Delta b\|_{L^2}\big)\\
&\leqslant C\|u\|_{L^\infty}\big(\|\nabla u\|_{L^2}^{(2\alpha-1)/\alpha}\big\|\Lambda^{\alpha+1}u\big\|_{L^2}^{1/\alpha}+\|\nabla b\|_{L^2}^{(2\alpha-1)/\alpha}\big\|\Lambda^{\alpha+1}b\big\|_{L^2}^{1/\alpha}\big)\\
&\leqslant \frac{1}{2}\big(\big\|\Lambda^{\alpha+1}u\big\|_{L^2}^2+\big\|\Lambda^{\alpha+1}b\big\|_{L^2}^2\big)+C\|u\|_{L^\infty}^{2\alpha/(2\alpha-1)}\big(\|\nabla u\|_{L^2}^2+\|\nabla u\|_{L^2}^2\big).
\end{aligned}\tag{2.12}
$$

In this case, it is obvious that $\frac{2\alpha}{2\alpha-1}\leqslant p$. Finally, by combining (2.6) and (2.12), then we get (2.3).

After we have (2.3), the estimates for higher order derivatives can be obtained by an inductive procedure.

This completes the proof of Theorem 2.1.

Under the above scaling, it is easy to check that $\|\Lambda^\alpha u\|_{L^{p,q}}$ is scaling dimension zero; that is $\|\Lambda^\alpha u\|_{L^{p,q}}=\|\Lambda^\alpha u_\lambda\|_{L^{p,q}}$, if and only if $\frac{2\alpha}{p}+\frac{3}{q}=3\alpha-1$. The second regularity criterion concerned with $\Lambda^\alpha u$ reads

**Theorem 2.3.** *Let $1\leqslant\alpha=\beta\leqslant\frac{5}{4}$ and assume that the initial velocity and magnetic fields $u_0,b_0\in H^3(\mathbb{R}^3)$. If on $[0,T]$, $\Lambda^\alpha u$ satisfies*

$$
\Lambda^\alpha u\in L^{p,q},\quad \textit{with } \frac{2\alpha}{p}+\frac{3}{q}\leqslant 3\alpha-1,\ 1\leqslant p<\infty,\ \frac{3}{3\alpha-1}<q<\frac{3}{\alpha-1},\tag{2.13}
$$

*then the solution remains smooth on* $[0,T]$.

**Remark 2.2.** When $\alpha=\beta=\frac{5}{4}$, a direct consequence of Theorem 2.3 is Corollary 2.2. In this case, the weak solution defined by Definition 1.1 satisfies (2.13) with $p=q=2$.

**Proof.** We begin our proof from (2.6).

$$
\begin{aligned}
|I| &\leqslant \|\nabla u\|_{L^3}^3\leqslant C\|\nabla u\|_{L^2}^{3\theta_1}\big\|\Lambda^\alpha u\big\|_{L^q}^{3\theta_2}\big\|\Lambda^{\alpha+1}u\big\|_{L^2}^{3\theta_3}\\
&\leqslant \frac{1}{2}\big\|\Lambda^{\alpha+1}u\big\|_{L^2}^2+C\big\|\Lambda^\alpha u\big\|_{L^q}^{2\theta_2/\theta_1}\|\nabla u\|_{L^2}^2,
\end{aligned}\tag{2.14}
$$

where we used the Gagliardo–Nirenberg and Young inequalities. The constants satisfy

$$
\begin{cases}
\theta_1+\theta_2+\theta_3=1,\\
0=\theta_1\left(\frac{1}{2}-\frac{1}{3}\right)+\theta_2\left(\frac{1}{q}-\frac{\alpha}{3}\right)+\theta_3\left(\frac{1}{2}-\frac{\alpha+1}{3}\right),\\
3\theta_1=2-3\theta_3.
\end{cases}\tag{2.15}
$$

System (2.15) can be solved uniquely as

$$
\theta_1=1-\frac{1}{3\alpha}\left(1+\frac{3}{q}\right),\quad \theta_2=\frac{1}{3},\quad \theta_3=\frac{1}{3\alpha}\left(1-\alpha+\frac{3}{q}\right).
$$

Consequently, we have

$$
\frac{2\theta_2}{\theta_1}=\frac{2\alpha}{3\alpha-1-3/q}\leqslant p.
$$

On the other hand, by the Hölder, Gagliardo–Nirenberg and Young inequalities, we obtain

$$
\begin{aligned}
|II|+|III|+|IV| &\leqslant C\|\nabla u\|_{L^\gamma}\|\nabla b\|_{L^{2\gamma/(\gamma-1)}}^2\\
&\leqslant C\|u\|_{L^2}^{1-\theta}\big\|\Lambda^\alpha u\big\|_{L^q}^{\theta}\|\nabla b\|_{L^2}^{2-2\delta}\big\|\Lambda^{\alpha+1}b\big\|_{L^2}^{2\delta}\\
&\leqslant \frac{1}{2}\big\|\Lambda^{\alpha+1}b\big\|_{L^2}^2+C\|u\|_{L^2}^{(1-\theta)/(1-\delta)}\big\|\Lambda^\alpha u\big\|_{L^q}^{\theta/(1-\delta)}\|\nabla b\|_{L^2}^2\\
&= \frac{1}{2}\big\|\Lambda^{\alpha+1}b\big\|_{L^2}^2+C\|u\|_{L^2}^{(1-\theta)/(1-\delta)}\big\|\Lambda^\alpha u\big\|_{L^q}^{2\alpha/(3\alpha-1-3/q)}\|\nabla b\|_{L^2}^2.
\end{aligned}\tag{2.16}
$$

The constants $\gamma,\theta$ and $\delta$ satisfy

$$\begin{cases} \frac{1}{\gamma} - \frac{1}{3} = (1-\theta)\frac{1}{2} + \theta\left(\frac{1}{q} - \frac{\alpha}{3}\right), \\ \frac{\gamma-1}{2\gamma} - \frac{1}{3} = (1-\delta)\left(\frac{1}{2} - \frac{1}{3}\right) + \delta\left(\frac{1}{2} - \frac{\alpha+1}{3}\right), \\ \frac{\theta}{1-\delta} = \frac{2\alpha}{3\alpha - 1 - 3/q}. \end{cases} \tag{2.17}$$

First, formally system (2.17) can be solved with

$$\gamma = \frac{(5-4\alpha)3q}{(5-4\alpha)(q(1-\alpha)+3)}, \quad \delta = \frac{3}{2\alpha\gamma}, \quad \theta = \frac{5/6 - 1/\gamma}{1/2 + \alpha/3 - 1/q}.$$

If $1 \leqslant \alpha < \frac{5}{4}$, it is obvious that the above solution can be reduced to

$$\gamma = \frac{3q}{q(1-\alpha)+3}, \quad \delta = \frac{3}{2\alpha\gamma} = \frac{q(1-\alpha)+3}{2\alpha q}, \quad \theta = \frac{5/6 - 1/\gamma}{1/2 + \alpha/3 - 1/q} = 1. \tag{2.18}$$

When $\alpha = \frac{5}{4}$, we can define

$$\gamma = \lim_{\alpha \to 5/4} \frac{(5-4\alpha)3q}{(5-4\alpha)(q(1-\alpha)+3)} = \frac{3q}{q(1-\alpha)+3}$$

to get the same solution as (2.18).

Hence in (2.16), the following inequality was used

$$\|\nabla u\|_{L^\gamma} \leqslant C\left\|\Lambda^{\alpha-1}\nabla u\right\|_{L^q} \leqslant C\left\|\Lambda^{\alpha} u\right\|_{L^q}.$$

Thanks to the energy inequality, we have

$$|II| + |III| + |IV| \leqslant \frac{1}{2}\left\|\Lambda^{\alpha+1} b\right\|_{L^2}^2 + C\left\|\Lambda^{\alpha} u\right\|_{L^q}^{2\alpha/(3\alpha-1-3/q)} \|\nabla b\|_{L^2}^2. \tag{2.19}$$

Combining (2.6), (2.14) and (2.19), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2\right) + \left\|\Lambda^{\alpha+1} u\right\|_{L^2}^2 + \left\|\Lambda^{\alpha+1} b\right\|_{L^2}^2 \leqslant C\left\|\Lambda^{\alpha} u\right\|_{L^q}^{2\alpha/(3\alpha-1-3/q)}\left(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2\right). \tag{2.20}$$

Note that

$$\frac{2\alpha}{3\alpha - 1 - 3/q} \leqslant p, \quad \text{if } \frac{2\alpha}{p} + \frac{3}{q} \leqslant 3\alpha - 1,$$

then the bounds of $\|u, b\|_{L^\infty(0,T;H^1)}$ and $\|u, b\|_{L^2(0,T;H^{\alpha+1})}$ follow from (2.20) and the Gronwall inequality.

The proof is finished. □

**Remark 2.3.** When $\alpha = \beta = 1$, Theorem 2.3 still holds under the condition that $\nabla u \in L^{1,\infty}$ (see [21]). In fact, in this case, (2.16) simply reduced to

$$|II| + |III| + |IV| \leqslant C\|\nabla u\|_{L^\infty}\|\nabla b\|_{L^2}^2.$$

**Remark 2.4.** The regularity can be proved under the condition that the velocity field $u$ belongs to $L^{\infty,3/(2\alpha-1)}$ or $\Lambda^\alpha u$ belongs to $L^{\infty,3/(3\alpha-1)}$, provided that $\|u\|_{L^{\infty,3/(2\alpha-1)}}$ or $\|\Lambda^\alpha\|_{L^{\infty,3/(3\alpha-1)}}$ is sufficient small. The proof can be given by the similar argument used in [21].

Theorems 2.1 and 2.3 establish regularity criteria in terms of the velocity field (see also the corresponding results for MHD in [21]). It seems that the velocity field contributes more than the magnetic field for the effects of regularization. Now we want to investigate the regularity criterion on the parameter $\alpha$ and $\beta$ for $\alpha \neq \beta$. Our theorem reads

**Theorem 2.4.** *Let $1 \leqslant \beta \leqslant \frac{5}{4} \leqslant \alpha < \frac{5}{2}$. Assume that the initial velocity and magnetic field $u_0, b_0 \in H^3(\mathbb{R}^3)$. If*

$$\alpha \geqslant \frac{5}{2} - \beta, \tag{2.21}$$

*then the global weak solutions are actually strong and unique.*

**Proof.** We only need to prove (2.3) under the condition (2.21).

From (2.6) with $\alpha \neq \beta$, we do the following estimates. Taking $q = 2$ in (2.14), the first term $I$ can be bounded as

$$|I| \leqslant \frac{1}{2}\left\|\Lambda^{\alpha+1}u\right\|_{L^2}^2 + C\left\|\Lambda^{\alpha}u\right\|_{L^2}^{4\alpha/(6\alpha-5)}\|\nabla u\|_{L^2}^2. \tag{2.22}$$

By the Hölder, Gagliardo–Nirenberg and Young inequalities as well as the energy inequality for the weak solution, we have

$$\begin{aligned}
|II| + |III| + |IV| &\leqslant C\|\nabla u\|_{L^{a_1}}\|\nabla b\|_{L^{a_2}}^2 \\
&\leqslant C\|u\|_{L^2}^{1-\theta_1}\left\|\Lambda^{\alpha}u\right\|_{L^2}^{\theta_1}\left\|\Lambda^{\beta+1}b\right\|_{L^2}^{2\theta_2}\|\nabla b\|_{L^2}^{2(1-\theta_2)} \\
&\leqslant \frac{1}{2}\left\|\Lambda^{\beta+1}b\right\|_{L^2}^2 + C\|u\|_{L^2}^{(1-\theta_1)/(1-\theta_2)}\left\|\Lambda^{\alpha}u\right\|_{L^2}^{\theta_1/(1-\theta_2)}\|\nabla b\|_{L^2}^2 \\
&\leqslant \frac{1}{2}\left\|\Lambda^{\beta+1}b\right\|_{L^2}^2 + C\left\|\Lambda^{\alpha}u\right\|_{L^2}^{\theta_1/1-\theta_2}\|\nabla b\|_{L^2}^2.
\end{aligned} \tag{2.23}$$

The constants satisfy

$$\begin{cases}
\dfrac{1}{a_1} + \dfrac{2}{a_2} = 1, \\
\dfrac{1}{a_1} - \dfrac{1}{3} = (1-\theta_1)\dfrac{1}{2} + \theta_1\left(\dfrac{1}{2} - \dfrac{\alpha}{3}\right), \\
\dfrac{1}{a_2} - \dfrac{1}{3} = (1-\theta_2)\left(\dfrac{1}{2} - \dfrac{1}{3}\right) + \theta_2\left(\dfrac{1}{2} - \dfrac{\beta+1}{3}\right),
\end{cases} \tag{2.24}$$

where $0 \leqslant \theta_1, \theta_2 \leqslant 1$. If (2.24) is solvable with

$$\frac{\theta_1}{1-\theta_2} \leqslant 2,$$

by putting (2.22) and (2.23) into (2.6), and due to the energy inequality, we get the a priori estimate (2.3). The higher derivative estimates can be obtained similarly. Consequently, the weak solution actually is smooth.

The problem is to solve (2.24) under the condition (2.21). Letting $1/a_1 = b_1$, then (2.24) can be solved as

$$\theta_1 = \left(\frac{5}{6} - b_1\right)\frac{3}{\alpha}, \qquad \theta_2 = \frac{3b_1}{2\beta}. \tag{2.25}$$

Then due to the condition $\theta_1/(1-\theta_2) \leqslant 2$, we have

$$\frac{(5/6 - b_1)3\beta/\alpha}{2\beta - 3b_1} \leqslant 1.$$

Hence the restriction for $\alpha$ is

$$\alpha \geqslant \frac{(5-6b_1)\beta}{4\beta - 6b_1}. \tag{2.26}$$

Now, let

$$f(x) = \frac{(5-6x)\beta}{4\beta - 6x},$$

then $f'(x) \geqslant 0$ for $1 \leqslant \beta \leqslant \frac{5}{4}$.

From the condition $0 \leqslant \theta_1, \theta_2 \leqslant 1$, thanks to (2.25), we get

$$\frac{5}{6} - \frac{\alpha}{3} \leqslant b_1 \leqslant \frac{5}{6}.$$

Therefore from (2.26), one obtains

$$\alpha \geqslant f\left(\frac{5}{6} - \frac{\alpha}{3}\right).$$

Direct computation yields

$$\alpha \geqslant \frac{5}{2} - \beta.$$

Actually, in this case, (2.24) can be solved as

$$\theta_1 = 1, \quad a_1 = \frac{6}{5 - 2\alpha}, \quad \theta_2 = \frac{5 - 2\alpha}{4\beta}, \quad a_2 = \frac{12}{1 + 2\alpha}.$$

Hence in (2.23), the following Sobolev inequality is used

$$\|\nabla u\|_{L^{a_1}} \leqslant C \left\| \Lambda^{\alpha-1} \nabla u \right\|_{L^2} = C \left\| \Lambda^{\alpha} u \right\|_{L^2}.$$

The proof is complete. □

**Remark 2.5.** Corollary 2.2 and Theorem 2.4 show that if the indexes $\alpha$ and $\beta$ satisfy $1 \leqslant \beta \leqslant \alpha$ and $\alpha + \beta \geqslant \frac{5}{2}$, then the weak solution to (1.1) is regular and unique. It should be very interesting if one can prove the global existence of smooth solutions when $\alpha + \beta < \frac{5}{2}$.

## 3. Regularity criterion in terms of the vorticity field

In the study of fluid mechanics, the vorticity field is always an important and interesting issue. For example, for 3-D Navier–Stokes equations, Constantin and Fefferman [7] proved that if the direction of the vorticity field is somehow Lipschitz continuous in the high vorticity region, then the weak solution actually is a (unique) strong solution. Recently, there are several interesting results [2,3,19] which improved the result of Constantin and Fefferman by relaxing the condition on the vorticity field or by combining the regularity conditions on the direction of vorticity field and the vorticity field itself.

In this section, we want to find some sufficient condition imposed on the vorticity field to guarantee the regularity of weak solutions.

Taking curl operator on the system (1.1), one has

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u - b \cdot \nabla j + j \cdot \nabla b = -(-\Delta)^{\alpha} \omega, \\ \partial_t j + u \cdot \nabla j - j \cdot \nabla u - b \cdot \nabla \omega + \omega \cdot \nabla b = -(-\Delta)^{\beta} j + 2F(b, u), \\ \omega = \nabla \times u, \quad j = \nabla \times b, \\ \operatorname{div} u = \operatorname{div} b = 0, \\ \omega(x, t = 0) = \omega_0(x), \quad j(x, t = 0) = j_0(x), \end{cases} \tag{3.1}$$

where

$$F(b, u) = \begin{pmatrix} \partial_2 b \cdot \partial_3 u - \partial_3 b \cdot \partial_2 u \\ \partial_3 b \cdot \partial_1 u - \partial_1 b \cdot \partial_3 u \\ \partial_1 b \cdot \partial_2 u - \partial_2 b \cdot \partial_1 u \end{pmatrix}.$$

Our main theorem in this section reads

**Theorem 3.1.** *Let $1 \leqslant \alpha = \beta < \frac{5}{4}$ and assume that the initial velocity and magnetic fields $u_0, b_0 \in H^3(\mathbb{R}^3)$. There exist constants $C$, $\rho$ and $K$ independent of $t$, such that the vorticity field $\omega = \nabla \times u$ satisfies*

$$\left|\omega(x + y, t) - \omega(x, t)\right| \leqslant C \left|w(x + y, t)\right| |y|^{\delta}, \quad \text{for } |y| \leqslant \rho \text{ and } \left|\omega(x, t)\right| \geqslant K, \tag{3.2}$$

*on $[0, T]$, with $\delta = \frac{5}{2} - 2\alpha$. Then the corresponding weak solution $(u, b)$ to (1.1) is strong and hence smooth.*

**Remark 3.1.** For $\alpha=\beta=1$, we recover the result of He and Xin [10]. Even though, the detailed and crucial estimates established here are different from theirs.

**Proof.** First let us recall some basic facts as used by Constantin and Fefferman. Thanks to Biot–Savart law (see [6] for example), the velocity and magnetic fields can be expressed in terms of the vorticity and electrical fields respectively,

$$u(x,t)=-\frac{1}{4\pi}\int\limits_{\mathbb{R}^3}\nabla\left(\frac{1}{|y|}\right)\times\omega(x+y,t)\,\mathrm{d}y,$$

$$b(x,t)=-\frac{1}{4\pi}\int\limits_{\mathbb{R}^3}\nabla\left(\frac{1}{|y|}\right)\times j(x+y,t)\,\mathrm{d}y.$$

The strain matrix $S(x,t)$ in terms of $\omega(x,t)$ is given by

$$\begin{aligned}S(x,t)=S[\omega](x,t)&\equiv\frac{1}{2}\big(\nabla u(x,t)+\big(\nabla u(x,t)\big)^{\mathrm{T}}\big)\\&=\frac{3}{4\pi}P.V.\int\limits_{\mathbb{R}^3}M\big(\hat{y},\omega(x+y,t)\big)\frac{\mathrm{d}y}{|y|^3},\end{aligned}$$

where

$$\hat{y}=\frac{y}{|y|},\quad M(\hat{y},\omega)=\frac{1}{2}\big[\hat{y}\otimes(\hat{y}\times\omega)+(\hat{y}\times\omega)\otimes\hat{y}\big],$$

with $(a\otimes b)_{ij}=a_ib_j$. Moreover, we have

$$w(x,t)=\frac{1}{4\pi}P.V.\int\limits_{\mathbb{R}^3}\sigma(\hat{y})w(x+y,t)\frac{\mathrm{d}y}{|y|^3},$$

where $\sigma(\hat{y})=3\hat{y}\otimes\hat{y}-I$, with $I$ denoting the identity matrix.

From Calderón–Zygmund inequality [13], we have

$$\big\|S(x,t)\big\|_{L^p}\leqslant C\big\|\omega(x,t)\big\|_{L^p},\qquad\big\|T(x,t)\big\|_{L^p}\leqslant C\big\|j(x,t)\big\|_{L^p},\tag{3.3}$$

for any $1<p<\infty$, where $C$ is a constant depending only on $p$.

Now we need to establish a priori estimates. To this end, multiply the first equation of (3.1) by $\omega(x,t)$ and the second by $j(x,t)$ respectively, and take integral on the whole space, then we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\big\|\omega(t)\big\|_{L^2}^2+\big\|j(t)\big\|_{L^2}^2\big)+\big\|\Lambda^\alpha\omega(t)\big\|_{L^2}^2+\big\|\Lambda^\alpha j(t)\big\|_{L^2}^2\\&=\int\limits_{\mathbb{R}^3}\big(\omega(x,t)\cdot\nabla u(x,t)\cdot\omega(x,t)+j(x,t)\cdot\nabla u(x,t)\cdot j(x,t)\\&\quad-j(x,t)\cdot\nabla b(x,t)\cdot\omega(x,t)-\omega(x,t)\cdot\nabla b(x,t)\cdot j(x,t)\\&\quad+2F(b,u)(x,t)\cdot j(x,t)\big)\,\mathrm{d}x\equiv I_1+I_2+I_3+I_4+I_5.\end{aligned}\tag{3.4}$$

We do estimates one by one as follows.

Let $K$ be the number in Theorem 3.1. We split $\omega(x,t)$ as

$$\omega(x,t)=\chi\left(\frac{|\omega(x,t)|}{K}\right)\omega(x,t)+\left(1-\chi\left(\frac{|\omega(x,t)|}{K}\right)\right)\omega(x,t)=\omega_1(x,t)+\omega_2(x,t),$$

and split $S(x,t)$ as

$$S(x,t)=\chi\left(\frac{|\omega(x,t)|}{K}\right)S(x,t)+\left(1-\chi\left(\frac{|\omega(x,t)|}{K}\right)\right)S(x,t)=S_1(x,t)+S_2(x,t),$$

where the smooth bump function $\chi(\lambda) \in [0,1]$, is identically equal to one for $0 \leqslant \lambda \leqslant 1$ and identically equal to zero for $\lambda \geqslant 2$ or $\lambda \leqslant -1$. Hence it is obvious that

$$\|\omega_i(x,t)\|_{L^p} \leqslant \|\omega(x,t)\|_{L^p}, \qquad \|S_i(x,t)\|_{L^p} \leqslant \|S(x,t)\|_{L^p}, \tag{3.5}$$

for any $i = 1, 2$ and $1 < p < \infty$.

Now, $I_1$ is decomposed into

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^3} \omega(x,t)\cdot\nabla u(x,t)\cdot\omega(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^3} \big(S(x,t)\omega(x,t)\big)\cdot\omega(x,t)\,\mathrm{d}x \\ &= \sum_{i=1}^{2}\int_{\mathbb{R}^3}\left(\sum_{k=1}^{2}\big(S_i(x,t)\omega_1(x,t)\big)\cdot\omega_k(x,t) + \big(S_i(x,t)\omega_2(x,t)\big)\cdot\omega_1(x,t)\right)\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3}\big(S_1(x,t)\omega_2(x,t)\big)\cdot\omega_2(x,t)\,\mathrm{d}x + \int_{\mathbb{R}^3}\big(S_2(x,t)\omega_2(x,t)\big)\cdot\omega_2(x,t)\,\mathrm{d}x \\ &\equiv I_1^{(1)} + I_1^{(2)} + I_1^{(3)}. \end{aligned}$$

Now we do the estimates one by one.

$$\begin{aligned} \big|I_1^{(1)}\big| &\leqslant \left|\sum_{i=1}^{2}\int_{\mathbb{R}^3}\sum_{k=1}^{2}\big(S_i(x,t)\omega_1(x,t)\big)\cdot\omega_k(x,t)\,\mathrm{d}x\right| + \left|\sum_{i=1}^{2}\int_{\mathbb{R}^3}\big(S_i(x,t)\omega_2(x,t)\big)\cdot\omega_1(x,t)\,\mathrm{d}x\right| \\ &\leqslant K\sum_{i=1}^{2}\sum_{k=1}^{2}\|S_i(t)\|_{L^2}\|\omega_k(t)\|_{L^2} + K\sum_{i=1}^{2}\|S_i(t)\|_{L^2}\|\omega_2(t)\|_{L^2} \\ &\leqslant C\|\omega(t)\|_{L^2}^2, \end{aligned} \tag{3.6}$$

where we used (3.3) and (3.5).

By using the Hölder, Gagliardo–Nirenberg and Young inequalities and taking the boundedness of $\omega_1$ into account, we have

$$\begin{aligned} \big|I_1^{(2)}\big| &\leqslant \|\omega_1(t)\|_{L^4}\|\omega(t)\|_{L^4}\|\omega(t)\|_{L^2} \\ &\leqslant C\|\omega_1(t)\|_{L^4}\|\omega(t)\|_{L^2}^{(4\alpha-3)/(4\alpha)}\|\Lambda^\alpha\omega(t)\|_{L^2}^{3/(4\alpha)}\|\omega(t)\|_{L^2} \\ &\leqslant \frac{1}{12}\|\Lambda^\alpha\omega(t)\|_{L^2}^2 + C\|\omega_1(t)\|_{L^4}^{8\alpha/(8\alpha-3)}\|\omega(t)\|_{L^2}^2 \\ &\leqslant \frac{1}{12}\|\Lambda^\alpha\omega(t)\|_{L^2}^2 + C\|\omega(t)\|_{L^2}^{4\alpha/(8\alpha-3)}\|\omega(t)\|_{L^2}^2, \end{aligned} \tag{3.7}$$

where we used the following inequality

$$\|\omega_1(t)\|_{L^4} \leqslant \|\omega_1(t)\|_{L^\infty}^{1/2}\|\omega_1(t)\|_{L^2}^{1/2} \leqslant K^{1/2}\|\omega_1(t)\|_{L^2}^{1/2}.$$

Therefore the constant $C$ only depends on the given number $K$.

It follows from the definition of $S(x)$ that

$$\begin{aligned} \big|S_2(x,t)\big| &= \frac{3}{4\pi}\left(1-\chi\left(\frac{|\omega(x,t)|}{K}\right)\right)P.V.\int_{\mathbb{R}^3} M\big(\hat{y},\omega(x+y,t)\big)\frac{\mathrm{d}y}{|y|^3} \\ &\leqslant \left|\frac{3}{4\pi}\left(1-\chi\left(\frac{|\omega(x,t)|}{K}\right)\right)P.V.\int_{|y|\geqslant\rho} M\big(\hat{y},\omega(x+y,t)\big)\frac{\mathrm{d}y}{|y|^3}\right| \\ &\quad + \left|\frac{3}{4\pi}\left(1-\chi\left(\frac{|\omega(x,t)|}{K}\right)\right)P.V.\int_{|y|\leqslant\rho} M\big(\hat{y},\omega(x+y,t)\big)\frac{\mathrm{d}y}{|y|^3}\right| \end{aligned}$$

$$
\begin{aligned}
&= \left| \frac{3}{4\pi} \left(1 - \chi\left(\frac{|\omega(x,t)|}{K}\right)\right) P.V. \int\limits_{|y| \geqslant \rho} M\big(\hat{y}, \omega(x+y,t)\big) \frac{\mathrm{d}y}{|y|^3} \right| \\
&\quad + \left| \frac{3}{4\pi} P.V. \int\limits_{|y| \leqslant \rho} \left(1 - \chi\left(\frac{|\omega(x,t)|}{K}\right)\right) M\big(\hat{y}, \omega(x+y,t) - \omega(x,t)\big) \frac{\mathrm{d}y}{|y|^3} \right| \\
&\leqslant C\big(\rho^{-\delta} + K\big) \int\limits_{\mathbb{R}^3} \big|\omega(x+y,t)\big| \frac{\mathrm{d}y}{|y|^{3-\delta}},
\end{aligned}
$$

since the mean on the unit sphere of $M(\hat{y}, \cdot)$ is zero.

Therefore $I_1^{(3)}$ can be estimated as

$$
\begin{aligned}
\big|I_1^{(3)}\big| &\leqslant \big\|\omega(t)\big\|_{L^2} \big\|\omega(t)\big\|_{L^a} \big\|S_2(t)\big\|_{L^\gamma} \\
&\leqslant C\big(\rho^{-\delta} + K\big) \big\|\omega(t)\big\|_{L^2} \big\|\omega(t)\big\|_{L^a} \big\|\omega(t)\big\|_{L^p} \\
&\leqslant C\big(\rho^{-\delta} + K\big) \big\|\omega(t)\big\|_{L^2} \big\|\omega(t)\big\|_{L^2}^{2-\theta_1-\theta_2} \big\|\Lambda^\alpha \omega(t)\big\|_{L^2}^{\theta_1+\theta_2} \\
&\leqslant \frac{1}{12} \big\|\Lambda^\alpha \omega(t)\big\|_{L^2}^2 + C\big(\rho^{-\delta} + K\big)^{2/(2-\theta_1-\theta_2)} \big\|\omega(t)\big\|_{L^2}^{2/(2-\theta_1-\theta_2)} \big\|\omega(t)\big\|_{L^2}^2,
\end{aligned} \tag{3.8}
$$

where we used the Hölder, Hardy–Littlewood–Sobolev, Gagliardo–Nirenberg and Young inequalities, where the parameters $a, \gamma, p, \theta_1$ and $\theta_2$ satisfy

$$
\begin{cases}
\dfrac{1}{2} + \dfrac{1}{a} + \dfrac{1}{\gamma} = 1, \quad \dfrac{1}{\gamma} = \dfrac{1}{p} - \dfrac{\delta}{3}, \\
\dfrac{1}{a} = (1-\theta_1)\dfrac{1}{2} + \theta_1\left(\dfrac{1}{2} - \dfrac{\alpha}{3}\right), \quad \dfrac{1}{p} = (1-\theta_2)\dfrac{1}{2} + \theta_2\left(\dfrac{1}{2} - \dfrac{\alpha}{3}\right).
\end{cases} \tag{3.9}
$$

For any solution to the system (3.9), we have

$$
\frac{2}{2 - \theta_1 - \theta_2} = 2\alpha,
$$

provided that $\delta = \frac{5}{2} - 2\alpha$.

So consequently, it follows from (3.8) that

$$
\big|I_1^{(3)}\big| \leqslant \frac{1}{12} \big\|\Lambda^\alpha \omega(t)\big\|_{L^2}^2 + C\big(\rho^{-\delta} + K\big)^{2\alpha} \big\|\omega(t)\big\|_{L^2}^{2\alpha} \big\|\omega(t)\big\|_{L^2}^2. \tag{3.10}
$$

The question is that whether (3.9) can be solved. Since there are 5 unknowns and 4 equations, actually there are infinite many solutions. For example, we have the following solutions

$$
\theta_1 = \theta_2 = \frac{2\alpha - 1}{2\alpha}, \quad a = p = \frac{3}{2-\alpha}, \quad \gamma = \frac{6}{2\alpha - 1}. \tag{3.11}
$$

Similarly, we split $I_2$ as

$$
\begin{aligned}
I_2 &= \int\limits_{\mathbb{R}^3} S(x,t) j(x,t) \cdot j(x,t)\,\mathrm{d}x \\
&= \int\limits_{\mathbb{R}^3} S_1(x,t) j(x,t) \cdot j(x,t)\,\mathrm{d}x + \int\limits_{\mathbb{R}^3} S_2(x,t) j(x,t) \cdot j(x,t)\,\mathrm{d}x = I_2^{(1)} + I_2^{(2)}.
\end{aligned}
$$

Just as $I_1^{(2)}$, we have

$$\begin{aligned}
\left|I_2^{(1)}\right| &\leqslant \left\|S_1(t)\right\|_{L^4}\left\|j(t)\right\|_{L^4}\left\|j(t)\right\|_{L^2} \\
&\leqslant C\left\|\omega_1(t)\right\|_{L^4}\left\|j(t)\right\|_{L^4}\left\|j(t)\right\|_{L^2} \\
&\leqslant \frac{1}{12}\left\|\Lambda^\alpha j(t)\right\|_{L^2}^2 + C\left\|\omega_1(t)\right\|_{L^4}^{8\alpha/(8\alpha-3)}\left\|j(t)\right\|_{L^2}^2 \\
&\leqslant \frac{1}{12}\left\|\Lambda^\alpha j(t)\right\|_{L^2}^2 + C\left\|\omega(t)\right\|_{L^2}^{4\alpha/(8\alpha-3)}\left\|j(t)\right\|_{L^2}^2.
\end{aligned} \tag{3.12}$$

Similar to the estimates for $I_1^{(3)}$,

$$\begin{aligned}
\left|I_2^{(2)}\right| &\leqslant \left\|j(t)\right\|_{L^2}\left\|j(t)\right\|_{L^a}\left\|S_2(t)\right\|_{L^\gamma} \\
&\leqslant C\left(\rho^{-\delta}+K\right)\left\|j(t)\right\|_{L^2}\left\|j(t)\right\|_{L^a}\left\|\omega(t)\right\|_{L^p} \\
&\leqslant C\left(\rho^{-\delta}+K\right)\left\|j(t)\right\|_{L^2}\left\|j(t)\right\|_{L^a}^2 + C\left(\rho^{-\delta}+K\right)\left\|j(t)\right\|_{L^2}\left\|\omega(t)\right\|_{L^p}^2 \\
&\leqslant C\left(\rho^{-\delta}+K\right)\left\|j(t)\right\|_{L^2}\left\|j(t)\right\|_{L^2}^{2-2\theta_1}\left\|\Lambda^\alpha j(t)\right\|_{L^2}^{2\theta_1} + C\left(\rho^{-\delta}+K\right)\left\|j(t)\right\|_{L^2}\left\|\omega(t)\right\|_{L^2}^{2-2\theta_1}\left\|\Lambda^\alpha\omega(t)\right\|_{L^2}^{2\theta_1} \\
&\leqslant \frac{1}{12}\left(\left\|\Lambda^\alpha j(t)\right\|_{L^2}^2 + \left\|\Lambda^\alpha\omega(t)\right\|_{L^2}^2\right) + C\left(\rho^{-\delta}+K\right)^{2\alpha}\left\|j(t)\right\|_{L^2}^{2\alpha}\left(\left\|j(t)\right\|_{L^2}^2 + \left\|\omega(t)\right\|_{L^2}^2\right),
\end{aligned} \tag{3.13}$$

where $a, \gamma, p$ and $\theta_1$ are given by (3.11).

$$\begin{aligned}
I_3 &= \int_{\mathbb{R}^3} j(x,t)\cdot\nabla b(x,t)\cdot\omega(x,t)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} j(x,t)\cdot\nabla b(x,t)\cdot\omega_1(x,t)\,\mathrm{d}x + \int_{\mathbb{R}^3} j(x,t)\cdot\nabla b(x,t)\cdot\omega_2(x,t)\,\mathrm{d}x = I_3^{(1)} + I_3^{(2)}.
\end{aligned}$$

$I_3^{(1)}$ is trivial,

$$\begin{aligned}
\left|I_3^{(1)}\right| &\leqslant \left\|j(t)\right\|_{L^2}\left\|\nabla b(t)\right\|_{L^4}\left\|\omega_1(t)\right\|_{L^4} \\
&\leqslant C\left\|j(t)\right\|_{L^2}\left\|j(t)\right\|_{L^4}\left\|\omega_1(t)\right\|_{L^4} \\
&\leqslant \frac{1}{12}\left\|\Lambda^\alpha j(t)\right\|_{L^2}^2 + C\left\|\omega(t)\right\|_{L^2}^{4\alpha/(8\alpha-3)}\left\|j(t)\right\|_{L^2}^2,
\end{aligned} \tag{3.14}$$

where we used the following inequality

$$\|\nabla f\|_{L^p} \leqslant C\|\nabla\times f\|_{L^p}, \quad \text{for any } f\in W^{1,p} \text{ with } \operatorname{div} f = 0.$$

From the expression of $\omega(x,t)$, we obtain that

$$\begin{aligned}
\left|\omega_2(x,t)\right| &= \left|\frac{1}{4\pi}\left(1-\chi\left(\frac{|\omega(x,t)|}{K}\right)\right)P.V.\int_{\mathbb{R}^3}\sigma(\hat{y})\omega(x+y,t)\frac{\mathrm{d}y}{|y|^3}\right| \\
&\leqslant \left|\frac{1}{4\pi}\left(1-\chi\left(\frac{|\omega(x,t)|}{K}\right)\right)P.V.\int_{|y|\geqslant\rho}\sigma(\hat{y})\omega(x+y,t)\frac{\mathrm{d}y}{|y|^3}\right| \\
&\quad + \left|\frac{1}{4\pi}\left(1-\chi\left(\frac{|\omega(x,t)|}{K}\right)\right)P.V.\int_{|y|\leqslant\rho}\sigma(\hat{y})\omega(x+y,t)\frac{\mathrm{d}y}{|y|^3}\right| \\
&= \left|\frac{1}{4\pi}\left(1-\chi\left(\frac{|\omega(x,t)|}{K}\right)\right)P.V.\int_{|y|\geqslant\rho}\sigma(\hat{y})\omega(x+y,t)\frac{\mathrm{d}y}{|y|^3}\right| \\
&\quad + \left|\frac{1}{4\pi}P.V.\int_{|y|\leqslant\rho}\left(1-\chi\left(\frac{|\omega(x,t)|}{K}\right)\right)\sigma(\hat{y})\left(\omega(x+y,t)-\omega(x,t)\right)\frac{\mathrm{d}y}{|y|^3}\right| \\
&\leqslant C\left(\rho^{-\delta}+K\right)\int_{\mathbb{R}^3}\left|\omega(x+y,t)\right|\frac{\mathrm{d}y}{|y|^{3-\delta}},
\end{aligned}$$

since the mean on the unit sphere of $\sigma(\hat{y})$ is zero.

Similar to $I_2^{(2)}$, $I_3^{(2)}$ can be estimated as

$$\begin{aligned}
\left|I_3^{(2)}\right| &\leqslant \|j(t)\|_{L^2}\|j(t)\|_{L^a}\|\omega_2(t)\|_{L^\gamma} \\
&\leqslant C\left(\rho^{-\delta}+K\right)\|j(t)\|_{L^2}\|j(t)\|_{L^a}\|\omega(t)\|_{L^p} \\
&\leqslant \frac{1}{12}\left(\|\Lambda^\alpha j(t)\|_{L^2}^2+\|\Lambda^\alpha\omega(t)\|_{L^2}^2\right)+C\left(\rho^{-\delta}+K\right)^{2\alpha}\|j(t)\|_{L^2}^{2\alpha}\left(\|j(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2\right),
\end{aligned} \tag{3.15}$$

where $a, \gamma$ and $p$ are given by (3.11). $I_4$ can be estimated exactly as $I_3$,

$$\begin{aligned}
|I_4| \leqslant{}& \frac{1}{12}\left(\|\Lambda^\alpha j(t)\|_{L^2}^2+\|\Lambda^\alpha\omega(t)\|_{L^2}^2\right)+C\|\omega(t)\|_{L^2}^{(4\alpha)/(8\alpha-3)}\|j(t)\|_{L^2}^2 \\
&+C\left(\rho^{-\delta}+K\right)^{2\alpha}\|j(t)\|_{L^2}^{2\alpha}\left(\|j(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2\right).
\end{aligned} \tag{3.16}$$

Now we pay our attention to the last term $I_5$. First

$$\begin{aligned}
|I_5| &\leqslant C\|j(t)\|_{L^2}\|j(t)\|_{L^a}\|\omega(t)\|_{L^\gamma} \\
&\leqslant C\|j(t)\|_{L^2}\|j(t)\|_{L^a}\|\omega_1(t)\|_{L^\gamma}+\|j(t)\|_{L^2}\|j(t)\|_{L^a}\|\omega_2(t)\|_{L^\gamma} \\
&= I_5^{(1)}+I_5^{(2)},
\end{aligned} \tag{3.17}$$

where the constants $a$ and $\gamma$ are given by (3.11).

The estimate of $I_5^{(2)}$ is similar to that of $I_3^{(2)}$.

$$I_5^{(2)} \leqslant \frac{1}{24}\left(\|\Lambda^\alpha j(t)\|_{L^2}^2+\|\Lambda^\alpha\omega(t)\|_{L^2}^2\right)+C\left(\rho^{-\delta}+K\right)^{2\alpha}\|j(t)\|_{L^2}^{2\alpha}\left(\|j(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2\right). \tag{3.18}$$

$I_5^{(1)}$ is the crucial term, but can be treated as

$$\begin{aligned}
I_5^{(1)} &\leqslant C\|j(t)\|_{L^2}\|j(t)\|_{L^{3/(2-\alpha)}}^2+C\|j(t)\|_{L^2}\|\omega_1(t)\|_{L^{6/(2\alpha-1)}}^2 \\
&\leqslant C\|j(t)\|_{L^2}\|j(t)\|_{L^2}^{1/\alpha}\|\Lambda^\alpha j(t)\|_{L^2}^{(2\alpha-1)/\alpha}+C\|j(t)\|_{L^2}\|\omega_1(t)\|_{L^{(6\alpha-2)/(2\alpha-1)}}^{(6\alpha-2)/3} \\
&\leqslant \frac{1}{24}\|\Lambda^\alpha j(t)\|_{L^2}^2+C\|j(t)\|_{L^2}^{2\alpha}\|j(t)\|_{L^2}^2+C\|j(t)\|_{L^2}\|\omega_1(t)\|_{L^2}^{(6\alpha-5)/3}\|\Lambda^\alpha\omega_1(t)\|_{L^2} \\
&\leqslant \frac{1}{24}\left(\|\Lambda^\alpha j(t)\|_{L^2}^2+\|\Lambda^\alpha\omega(t)\|_{L^2}^2\right)+C\|j(t)\|_{L^2}^{2\alpha}\|j(t)\|_{L^2}^2+C\|\omega(t)\|_{L^2}^{(2(6\alpha-5))/3}\|j(t)\|_{L^2}^2,
\end{aligned} \tag{3.19}$$

where we used the $L^\infty$-bounds of $\omega_1(t)$.

Combining (3.4), (3.6), (3.7), (3.10) and (3.12)–(3.19), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\omega(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2\right)+\|\Lambda^\alpha\omega(t)\|_{L^2}^2+\|\Lambda^\alpha j(t)\|_{L^2}^2 \\
&\quad\leqslant \left(\rho^{-\delta}+K\right)^{2\alpha}\left(\|\omega(t)\|_{L^2}^{2\alpha}+\|j(t)\|_{L^2}^{2\alpha}\right)\left(\|j(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2\right) \\
&\qquad+C\|\omega(t)\|_{L^2}^{4\alpha/(8\alpha-3)}\|j(t)\|_{L^2}^2+C\|\omega(t)\|_{L^2}^{(2(6\alpha-5))/3}\|j(t)\|_{L^2}^2.
\end{aligned} \tag{3.20}$$

Thanks to the Gagliardo–Nirenberg inequality, we have

$$\|\omega(t)\|_{L^2} \leqslant C\|\nabla u(t)\|_{L^2} \leqslant C\|u(t)\|_{L^2}^{(\alpha-1)/\alpha}\|\Lambda^\alpha u(t)\|_{L^2}^{1/(\alpha)},$$

where $C$ is an absolute constant independent of $u$ and $t$. Similar inequality holds for $j(x,t)$.

By using the interpolation inequality

$$\|\Lambda u\|_{L^2} \leqslant C\|u\|_{L^2}^{1-1/\alpha}\|\Lambda^\alpha u\|_{L^2}^{1/\alpha},$$

then energy inequality tells us that

$$\int_0^T \left(\|\omega(t)\|_{L^2}^{2\alpha} + \|j(t)\|_{L^2}^{2\alpha}\right) \mathrm{d}t \leqslant C \int_0^T \left(\|\Lambda u(t)\|_{L^2}^{2\alpha} + \|\Lambda b(t)\|_{L^2}^{2\alpha}\right) \mathrm{d}t$$
$$\leqslant C\left(\|u\|_{L^{\infty,2}}^{2\alpha-2}\|\Lambda^\alpha u\|_{L^{2,2}}^2 + \|b\|_{L^{\infty,2}}^{2\alpha-2}\|\Lambda^\alpha b\|_{L^{2,2}}^2\right)$$
$$\leqslant C\left(\|u_0\|_{L^2}^2 + \|b_0(x)\|_{L^2}^2\right)^\alpha. \tag{3.21}$$

Finally, by (3.21) and noting that

$$\frac{4\alpha}{8\alpha-3}, \qquad \frac{2(6\alpha-5)}{3} < 2\alpha,$$

(2.3) is a straight consequence by applying the Gronwall inequality on (3.20).

This finishes the proof. □


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