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Liouville-type theorems and decay estimates for solutions to higher order elliptic equations ☆

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Abstract

Liouville-type theorems are powerful tools in partial differential equations. Boundedness assumptions of solutions are often imposed in deriving such Liouville-type theorems. In this paper, we establish some Liouville-type theorems without the boundedness assumption of nonnegative solutions to certain classes of elliptic equations and systems. Using a rescaling technique and doubling lemma developed recently in Poláčik et al. (2007) [20], we improve several Liouville-type theorems in higher order elliptic equations, some semilinear equations and elliptic systems. More specifically, we remove the boundedness assumption of the solutions which is required in the proofs of the corresponding Liouville-type theorems in the recent literature. Moreover, we also investigate the singularity and decay estimates of higher order elliptic equations.

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1. Introduction

The aim of this paper is to establish some Liouville-type theorems without the boundedness assumption on nonnegative solutions to certain classes of elliptic equations. Liouville-type theorems are powerful tools to prove a priori bounds for nonnegative solutions in a bounded domain. Using the rescaling method (also called the "blow-up" method) in the elegant paper [16], an equation in a bounded domain will blow up to become another equation in the whole Euclidean space or a half space. With the aid of the corresponding Liouville-type theorem in the Euclidean space $\mathbb{R}^N$ and half space $\mathbb{R}^N_+$ and a contradiction argument, the a priori bounds could be deduced. Moreover, the existence of nonnegative solutions to elliptic equations is established by the topological degree method using a priori estimates (see e.g. [11]).

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In [20], the authors develop a general method for the derivation of universal, pointwise a priori estimates of local solutions from the Liouville-type theorem. Their results show that the universal bounds theorems for local solutions and the Liouville-type theorem are essentially equivalent. By further exploring their rescaling method, we also obtain that the boundedness assumption of solutions in proving the Liouville-type theorem is not essential in many cases.

We denote the Sobolev critical exponent by

$$p_S := \begin{cases} \frac{N+2m}{N-2m} & \text{if } N > 2m, \\ \infty & \text{if } N \leqslant 2m, \end{cases}$$

where $m \in \mathbb{N}$ and N is the dimension of Euclidean space $\mathbb{R}^N$ or half space $\mathbb{R}^N_+$. The Liouville-type theorem for the subcritical higher order elliptic equations in Euclidean space was established in [25] as follows:

Theorem A. *Let $1 < q < p_S$. If u is a classical nonnegative solution of*

$$(-\Delta)^m u = u^q \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

then $u \equiv 0$.

For the higher order elliptic equation in half space, generally speaking, two types of boundary conditions, i.e. the Dirichlet boundary problem and the Navier boundary problem, are considered. The Liouville theorem for the Dirichlet boundary type of higher order equation in half space has been obtained in [21].

Theorem B. *Let $1 < q < p_S$. If u is a classical solution of*

$$\begin{cases} (-\Delta)^m u = u^q & \text{in } \mathbb{R}^N_+, \\ u \geqslant 0 & \text{in } \mathbb{R}^N_+, \\ u = \dfrac{\partial u}{\partial x_N} = \cdots = \dfrac{\partial^{m-1} u}{\partial x_N^{m-1}} = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{1.2}$$

then $u \equiv 0$.

The Liouville-type theorem for the Navier boundary problem of higher order elliptic equation in half space has been studied in [3,13], and [23]. The authors consider higher order elliptic equation

$$\begin{cases} (-\Delta)^m u = u^p & \text{in } \mathbb{R}^N_+, \\ u \geqslant 0 & \text{in } \mathbb{R}^N_+, \\ u = \Delta u = \cdots = \Delta^{m-1} u = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{1.3}$$

under the following boundedness assumption on the solution:

$$\sup u < \infty, \qquad \sup\left|\Delta^i u\right| < \infty, \quad i = 1, 2, \ldots, m-1. \tag{1.4}$$

They established that the solutions in (1.3) are trivial if $2m + 1 < N$ and $1 < p < \frac{N-1+2m}{N-1-2m}$ or if $2m + 1 \geqslant N$ and $1 < p < \infty$. The proof of their results is based on an idea of [9]. The idea is the following: if there exists a solution of (1.3) and one is able to show that any solution is increasing in the x_N direction, then passing the limit as $x_N \to \infty$, one could get a solution of the same equation in $\mathbb{R}^{N-1}$, which in turn allows the use of the Liouville-type theorem in the whole space. (Note that the critical exponent in (1.3) is $\frac{N-1+2m}{N-1-2m}$.) We would like to mention that the Liouville-type theorem for (1.3) in [9] is for the case of $m = 1$ with the assumption that the solution is bounded. Later on this result was improved to hold for unbounded solution in [2], among many other results.

We also mention that Liouville-type theorems for differential inequalities and systems involving polyharmonic operators and related to the Hardy–Littlewood–Sobolev inequalities have also been extensively studied. We refer the reader to the paper by Caristi, D'Ambrosio and Mitidieri [5] and references therein (see also [19]).

A natural question is then whether the boundedness assumption for the Liouville-type theorem is necessary for higher order elliptic equation or not. In particular, we are interested in the Liouville-type theorems for solutions to polyharmonic equations. We show that it is indeed unnecessary for such equations and establish that

Theorem 1. *If u is a classical solution of (1.3) with $2m+1<N$ and $1<p<\frac{N+2m}{N-2m}$ or with $2m+1\geqslant N$ and $1<p<\infty$, then $u\equiv 0$.*

In addition, we consider the a priori estimates of possible singularities for local solutions of higher order elliptic equation

$$(-\Delta)^m u = f(u) \tag{1.5}$$

when f satisfies certain conditions. More precisely, we will obtain

Theorem 2. *Let $1<p<p_S$ and $\Omega\neq\mathbb{R}^N$ be a domain in $\mathbb{R}^N$. Assume that the function $f:[0,\infty)\to\mathbb{R}$ is continuous,*

$$\lim_{u\to\infty} u^{-p} f(u) = \rho \in (0,\infty). \tag{1.6}$$

Then there exists $C(N,f,m)>0$ independent of Ω and u such that for any positive solution u of (1.5) in Ω, there holds

$$\sum_{|\nu|=0}^{2m-1} \left|D^\nu u(x)\right| \leqslant C(N,f,m)\bigl(1+dist^{-\frac{2m}{p-1}}(x,\partial\Omega)\bigr), \quad x\in\Omega, \tag{1.7}$$

where $\nu=\{\nu_1,\ldots,\nu_N\}$, $D^\nu u = D_{x_1}^{\nu_1} u \cdots D_{x_N}^{\nu_N} u$ and $dist(x,\partial\Omega)$ is the distance of x from $\partial\Omega$.

In particular, if $\Omega = B_R\setminus\{0\}$ for some $R>0$, then

$$\sum_{|\nu|=0}^{2m-1} \left|D^\nu u(x)\right| \leqslant C(N,f,m)\bigl(1+|x|^{-\frac{2m}{p-1}}\bigr), \quad 0<|x|\leqslant\frac{R}{2},$$

where B_R is a ball centered at 0 with radius R.

In the special case of $f(u)=u^p$, more precise results can be given in bounded or exterior domains.

Theorem 3. *Let $1<p<p_S$ and $\Omega\neq\mathbb{R}^N$ be a domain in $\mathbb{R}^N$. Then there exists $C(N,p,m)>0$ independent of Ω and u such that any nonnegative solution u in (1.5) satisfies*

$$\sum_{|\nu|=0}^{2m-1} \left|D^\nu u(x)\right| \leqslant C(N,p,m)\, dist^{-\frac{2m}{p-1}}(x,\partial\Omega), \quad x\in\Omega. \tag{1.8}$$

In particular if Ω is an exterior domain, i.e. the set $\{x\in\mathbb{R}^N; |x|>R\}\subset\Omega$ for some $R>0$, then

$$\sum_{|\nu|=0}^{2m-1} \left|D^\nu u(x)\right| \leqslant C(N,p,m)|x|^{-\frac{2m}{p-1}}, \quad |x|\geqslant 2R.$$

The Liouville-type theorems for semilinear elliptic equations in Euclidean space $\mathbb{R}^N$ and half space $\mathbb{R}^N_+$ are well known in the literature. Among many results in [15], the Liouville-type theorem for the subcritical elliptic equation is shown. Namely, there is no nontrivial C^2 solution of the problem

$$\begin{cases} -\Delta u = u^p & \text{in } \mathbb{R}^N, \\ u\geqslant 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.9}$$

if $N\geqslant 3$ and $1<p<\frac{N+2}{N-2}$. See also [6] for a simpler proof with the Kelvin transform and moving plane method of Gidas, Ni and Nirenberg [14]. In [10], the authors consider the following mixed (Dirichlet–Neumann) boundary conditions in a half space, i.e. the problem

$$\begin{cases} -\Delta u = u^p & \text{in } \mathbb{R}^N_+, \\ u \geqslant 0 & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{on } \Gamma_0 := \{x \in \partial\mathbb{R}^N_+ \mid x_N = 0,\ x_1 > 0\}, \\ \dfrac{\partial u}{\partial x_N} = 0 & \text{on } \Gamma_1 := \{x \in \partial\mathbb{R}^N_+ \mid x_N = 0,\ x_1 < 0\}, \end{cases} \tag{1.10}$$

where $N \geqslant 3$ and $1 < p < \frac{N+2}{N-2}$. Using the idea of the Kelvin transform combined with the moving plane method, they prove the nonexistence of nontrivial solutions in (1.10) under the assumption that the solution is bounded. In the next theorem, we are able to remove this boundedness assumption of the solution. In fact, our theorem is stated as:

Theorem 4. *If $u \in C^2(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$ is a solution of (1.10) with $N \geqslant 3$ and $1 < p < \frac{N+2}{N-2}$, then $u \equiv 0$.*

For the semilinear elliptic system, an important model is the Lane–Emden system:

$$\begin{cases} -\Delta u_1 = u_2^p & \text{in } \mathbb{R}^N, \\ -\Delta u_2 = u_1^q & \text{in } \mathbb{R}^N. \end{cases} \tag{1.11}$$

It is conjectured that if

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n},$$

then there are no nontrivial classical solutions of (1.11) in $\mathbb{R}^N$. The conjecture has been proved to be true for radial solutions in all dimensions in [18]. The cases of $N = 3, 4$ for the conjecture in general have been also solved recently in [20] and [24] respectively. The interested reader can refer to the above papers and references therein for detailed descriptions (see also the works [4,22], etc.). In [12], the authors study the elliptic system

$$\begin{cases} \Delta u_1 + u_1^{\alpha_1} + u_2^{\alpha_1 \frac{\alpha_2 - 1}{\alpha_1 - 1}} = 0 & \text{in } \mathbb{R}^N, \\ \Delta u_2 + u_2^{\alpha_2} + u_1^{\alpha_2 \frac{\alpha_1 - 1}{\alpha_2 - 1}} = 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.12}$$

and prove that (1.12) does not have bounded positive classical solutions in $\mathbb{R}^N$ if

$$1 < \alpha_1, \alpha_2 < \frac{N+2}{N-2}. \tag{1.13}$$

We also show that the boundedness assumption of the solution for the elliptic system (1.12) is not necessary. Indeed, we can establish that

Theorem 5. *The classical positive solutions u_1 and u_2 in (1.12) are trivial under the assumption of (1.13).*

We end our introduction by mentioning that nonexistence results of nonnegative solutions have been recently established by the first and third authors [17] for certain classes of integral equations which are closely related to polyharmonic equations in half spaces by a completely different method from that used in the current paper, namely, the method of moving planes in integral form developed in [7].

The outline of the paper is as follows. Section 2 is devoted to obtaining the Liouville-type theorem, then singularity and decay estimates for higher order elliptic equations. In Section 3, the Liouville-type theorem with mixed boundary condition for semilinear elliptic equation is shown. The Liouville-type theorem for the elliptic systems is considered in Section 4. Throughout the paper, C and c denote generic positive constants, which are independent of u and may vary from line to line.

2. Higher order elliptic equations

We state the following technical lemma used frequently in this paper.

Lemma 1 *(Doubling lemma). Let (X,d) be a complete metric space and $\emptyset \neq D \subset \Sigma \subset X$, with Σ closed. Define $M: D \to (0,\infty)$ to be bounded on compact subsets of D and fix a positive number k. If $y \in D$ is such that*

$$M(y)\,dist(y,\Gamma) > 2k,$$

where $\Gamma = \Sigma \setminus D$, then there exists $x \in D$ such that

$$M(x)\,dist(x,\Gamma) > 2k, \qquad M(x) \geqslant M(y)$$

and

$$M(z) \leqslant 2M(x), \quad \forall z \in D \cap \bar{B}\big(x, kM^{-1}(x)\big).$$

Remark 1. If $\Gamma = \emptyset$, then $dist(x,\Gamma) := \infty$.

The proof of the above lemma is given in [20]. Based on the doubling property, we can start the rescaling process to prove local estimates of solutions of superlinear problems. The idea is by a contradiction argument. One can argue that the local estimates of solutions are violated. By an appropriate rescaling, the blow-up sequence of solutions converges to a bounded solution of a limiting equation in $\mathbb{R}^N$. Then from the nonexistence of solutions in the limiting equation, we could infer that the local estimates exist. In this spirit, we conclude the proof of Theorem 1. We first state the lemma about the local a priori estimates for higher order elliptic equation (see [1] or [21]). Let u satisfy the following equation

$$(-\Delta)^m u = g(x) \quad \text{in } \Omega. \tag{2.1}$$

Then we have

Lemma 2. *Let Ω be a ball $\{x \in \mathbb{R}^N\colon |x| < R\}$. Suppose $u \in W^{2m,q}$ satisfies (2.1). Then there exits a constant $C > 0$ depending only on Ω, N, m and R such that for any $\sigma \in (0,1)$*

$$\|u\|_{W^{2m,q}(\Omega \cap B_{\sigma R})} \leqslant \frac{C}{(1-\sigma)^{2m}}\big(\|g\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega)}\big). \tag{2.2}$$

The following lemma is the Liouville-type theorem for higher order elliptic equations with boundedness assumption in [23].

Lemma 3. *The higher order elliptic equation (1.3) does not have positive solutions under assumption of (1.4) provided $p < \frac{N-1+2m}{N-1-2m}$ and $2m+1 < N$ (or $N \geqslant 2m+1$ and $p < \infty$).*

Proof of Theorem 1. Suppose that a solution u to Eq. (1.3) is unbounded in the sense that (1.4) is violated. Namely, the following boundedness is not true

$$\sup u < \infty, \qquad \sup\big|\Delta^i u\big| < \infty, \quad i = 1, 2, \ldots, m-1.$$

Then, there exists a sequence of $(y_k) \in \mathbb{R}^N$ such that

$$\sum_{i=0}^{m-1}\big|\Delta^i u(y_k)\big| \to \infty$$

as $k \to \infty$. Set

$$M(y) := \left(\sum_{i=0}^{m-1}\big|\Delta^i u(y)\big|\right)^{\frac{p-1}{2m}} : \mathbb{R}^N_+ \to \mathbb{R}.$$

Then $M(y_k) \to \infty$ as $k \to \infty$. By taking $D = \Sigma = X = \overline{\mathbb{R}^N_+}$ in the doubling lemma and Remark 1 (see also, e.g. [21]), there exists another sequence of (x_k) such that

$$M(x_k) \geqslant M(y_k)$$

and

$$M(z) \leqslant 2M(x_k), \quad \forall z \in B_{k/M(x_k)}(x_k) \cap \overline{\mathbb{R}^N_+}.$$

Define

$$d_k := x_{k,N} M(x_k)$$

and

$$H_k := \left\{\xi \in \mathbb{R}^N \;\middle|\; \xi_N > -d_k\right\}.$$

We introduce a new function

$$v_k(\xi) := \frac{u(x_k + \frac{\xi}{M(x_k)})}{M^{\frac{2m}{p-1}}(x_k)}.$$

Then, $v_k(\xi)$ is the nonnegative solution of

$$\begin{cases} (-\Delta)^m v_k = v_k^p & \text{in } H_k, \\ v_k = \Delta v_k = \cdots = \Delta^{m-1} v_k = 0 & \text{on } \partial H_k \end{cases} \tag{2.3}$$

satisfying

$$\sum_{i=0}^{m-1} \left|\Delta^i v_k(0)\right|^{\frac{p-1}{2m+2(p-1)i}} = 1 \tag{2.4}$$

and

$$\sum_{i=0}^{m-1} \left|\Delta^i v_k(\xi)\right|^{\frac{p-1}{2m+2(p-1)i}} \leqslant 2, \quad \forall \xi \in H_k \cap B_k(0). \tag{2.5}$$

Two cases may occur as $k \to \infty$, either case (1)

$$x_{k,N} M(x_k) \to \infty$$

for a subsequence still denoted as before, or case (2)

$$x_{k,N} M(x_k) \to d$$

for a subsequence still denoted as before, here $d \geqslant 0$. If case (1) occurs, i.e. $H_k \cap B_k(0) \to \mathbb{R}^N$ as $k \to \infty$, then for any smooth compact D in $\mathbb{R}^N$, there exists k_0 large enough such that $D \subset (H_k \cap B_k(0))$ as $k \geqslant k_0$. By the classical $W^{2m,q}$ estimates for higher order elliptic equation in Lemma 2.2 and (2.5), we have $\|v_k\|_{W^{2m,q}(\bar{D})} \leqslant C$ for any $1 < q < \infty$. Therefore, we can extract a convergent subsequence $v_k \to v$ in D, where $v \in C^{2m-1,\tau}$ for some $\tau > 0$. Furthermore, using a diagonal line argument, $v_k \to v$ in $C^{2m-1,\tau}_{loc}(\mathbb{R}^N)$ and v solves

$$(-\Delta)^m v = v^p \quad \text{in } \mathbb{R}^N.$$

From Theorem A, u is trivial. However, (2.4) implies that

$$\sum_{i=0}^{m-1} \left|\Delta^i v(0)\right|^{\frac{p-1}{2m+2(p-1)i}} = 1,$$

which indicates that u is nontrivial. Obviously a contradiction is arrived.

If case (2) occurs, we make a further translation. Set

$$\tilde{v}_k(\xi) := v_k(\xi - d_k e_N) \quad \text{for } \xi \in \overline{\mathbb{R}^N_+}.$$

Then $\tilde{v}_k$ satisfies

$$\begin{cases} (-\Delta)^m \tilde{v}_k = \tilde{v}_k^p & \text{in } \mathbb{R}_+^N, \\ \tilde{v}_k = \Delta \tilde{v}_k = \cdots = \Delta^{m-1} \tilde{v}_k = 0 & \text{on } \partial \mathbb{R}_+^N. \end{cases} \tag{2.6}$$

While

$$\sum_{i=0}^{m-1} \left| \Delta^i \tilde{v}_k (d_k e_N) \right|^{\frac{p-1}{2m+2(p-1)i}} = 1 \tag{2.7}$$

and

$$\sum_{i=0}^{m-1} \left| \Delta^i \tilde{v}_k (\xi) \right|^{\frac{p-1}{2m+2(p-1)i}} \leqslant 2, \quad \forall \xi \in \mathbb{R}_+^N \cap B_k(d_k e_N). \tag{2.8}$$

Observe that (2.6) could also be reduced to a system of elliptic equations. Let

$$\big(\tilde{v}_k, \ldots, (-\Delta)^{m-1} \tilde{v}_k\big) := \big(w_k^{(0)}, \ldots, w_k^{(m-1)}\big).$$

Then, $(w_k^{(0)}, \ldots, w_k^{(m-1)})$ solves

$$\begin{cases} -\Delta w_k^{(m-1)} = \big(w_k^{(0)}\big)^p & \text{in } \mathbb{R}_+^N, \\ -\Delta w_k^{(i)} = w_k^{(i+1)} & \text{in } \mathbb{R}_+^N,\ i = 0, \ldots, m-2, \\ w_k^{(0)} = \cdots = w_k^{(m-1)} = 0 & \text{on } \partial \mathbb{R}_+^N. \end{cases} \tag{2.9}$$

Furthermore,

$$\sum_{i=0}^{m-1} \left| w_k^{(i)} (d_k e_N) \right|^{\frac{p-1}{2m+2(p-1)i}} = 1$$

and

$$\sum_{i=0}^{m-1} \left| w_k^{(i)} (\xi) \right|^{\frac{p-1}{2m+2(p-1)i}} \leqslant 2, \quad \forall \xi \in \mathbb{R}_+^N \cap B_k(d_k e_N).$$

For any smooth compact D in $\overline{\mathbb{R}_+^N}$, there exists k_0 large enough such that $D \subset (\overline{\mathbb{R}_+^N} \cap B_k(d_k e_N))$ for any $k > k_0$. By the classical elliptic estimates,

$$\sum_{i=0}^{m-1} \left\| w_k^{(i)} (\xi) \right\|_{C^{2,\tau}(D)} \leqslant C$$

for some $\tau > 0$. Thanks to the Arzelá–Ascoli Theorem, there exists a function $(v^{(0)}, \ldots, v^{(m-1)})$ such that $(w_k^{(0)}, \ldots, w_k^{(m-1)})$ converges to $(v^{(0)}, \ldots, v^{(m-1)})$ in $C^2(\bar{D})$. Through a diagonal line argument, $(w_k^{(0)}, \ldots, w_k^{(m-1)})$ converges in $C^2_{loc}(\overline{\mathbb{R}_+^N})$ to $(v^{(0)}, \ldots, v^{(m-1)})$, which solves

$$\begin{cases} -\Delta v^{(m-1)} = \big(v^{(0)}\big)^p & \text{in } \mathbb{R}_+^N, \\ -\Delta v^{(i)} = v^{(i+1)} & \text{in } \mathbb{R}_+^N,\ i = 0, \ldots, m-2, \\ v^{(0)} = \cdots = v^{(m-1)} = 0 & \text{on } \partial \mathbb{R}_+^N, \end{cases} \tag{2.10}$$

and satisfies

$$\sum_{i=0}^{m-1} \left| v^{(i)} (d e_N) \right|^{\frac{p-1}{2m+2(p-1)i}} = 1 \tag{2.11}$$

and

$$\sum_{i=0}^{m-1} \left| v^{(i)} (\xi) \right|^{\frac{p-1}{2m+2(p-1)i}} \leqslant 2, \quad \forall \xi \in \mathbb{R}_+^N. \tag{2.12}$$

From (2.10), $v^{(0)}$ actually solves

$$\begin{cases} (-\Delta)^m v^{(0)} = \big(v^{(0)}\big)^p & \text{in } \mathbb{R}^N_+, \\ v^{(0)} = \Delta v^{(0)} = \cdots = \Delta^{m-1} v^{(0)} = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{2.13}$$

Hence $v^{(0)} \equiv 0$ by Lemma 3. Note that $\frac{N-1+2m}{N-1-2m} > \frac{N+2m}{N-2m}$. Then $v^{(1)} \equiv \cdots \equiv v^{(m-1)} \equiv 0$, which contradicts (2.11). Therefore, the proof is completed. □

Next, we consider the singularity and decay estimates of the higher order elliptic equation. The main idea is the same as the case of the semilinear elliptic equation in [20].

Proof of Theorem 2. Suppose (1.7) is not true. Then there exist sequences of Ω_k, u_k, $y_k \in \Omega_k$ such that u_k solves (1.5) on Ω_k and the function

$$M_k := \sum_{|\nu|=0}^{2m-1} \big|D^\nu u_k\big|^{\frac{p-1}{2m+(p-1)|\nu|}}$$

satisfies

$$M_k(y_k) \geqslant 2k\big(1 + dist^{-1}(y_k, \partial\Omega)\big) \geqslant 2k\, dist^{-1}(y_k, \partial\Omega).$$

Adapting from the doubling lemma, there exists $x_k \in \Omega_k$ such that

$$M_k(x_k) \geqslant M_k(y_k),$$

$$M_k(x_k) > 2k\, dist^{-1}(x_k, \partial\Omega_k)$$

and

$$M_k(z) \leqslant 2M_k(x_k), \quad \text{if } |z - x_k| \leqslant k M_k^{-1}(x_k).$$

We introduce a new function

$$v_k(\xi) := \frac{u_k(x_k + \frac{\xi}{M_k(x_k)})}{M_k^{\frac{2m}{p-1}}(x_k)}, \quad \forall |\xi| \leqslant k.$$

Then, the function v_k solves

$$(-\Delta)^m v_k(\xi) = f_k\big(v_k(\xi)\big) := M_k^{\frac{-2mp}{p-1}}(x_k) f\big(M_k^{\frac{2m}{p-1}}(x_k) v_k\big). \tag{2.14}$$

Also,

$$\sum_{|\nu|=0}^{2m-1} \big|D^\nu v_k\big|^{\frac{p-1}{2m+(p-1)|\nu|}}(0) = 1 \tag{2.15}$$

and

$$\sum_{|\nu|=0}^{2m-1} \big|D^\nu v_k\big|^{\frac{p-1}{2m+(p-1)|\nu|}}(\xi) \leqslant 2, \quad \forall |\xi| \leqslant k. \tag{2.16}$$

By (1.6) and the continuity of f, we have

$$-c \leqslant f(s) \leqslant C\big(s^p + 1\big), \quad \text{for } s \geqslant 0.$$

Furthermore, it follows that

$$-c M_k^{\frac{-2mp}{p-1}}(x_k) \leqslant f_k\big(v_k(\xi)\big) \leqslant C, \quad \forall |\xi| \leqslant k.$$

For any smooth compact D in $\mathbb{R}^N$, there exists k_0 large enough such that $D \subset B_k(0)$ if $k \geqslant k_0$. Using the classical estimate of the higher order elliptic equation in (2.2) and (2.16), $v_k(\xi) \in W^{2m,q}_{loc}(\mathbb{R}^N)$ for any $1 < q < \infty$ and

$$\|v_k\|_{W^{2m,q}(\bar{D})} \leqslant C.$$

By the Sobolev imbedding and a diagonal line argument, $v_k(\xi)$ converges in $C^{2m-1,\tau}_{loc}(\mathbb{R}^N)$ to a function $v \in C^{2m-1,\tau}_{loc}(\mathbb{R}^N)$ for some $\tau > 0$. Since $v_k(\xi)$ is positive for $\xi \in \mathbb{R}^N$, then

$$f_k(v_k) \to \rho v^p \quad \text{as } k \to \infty.$$

Consequently, v is a nonnegative solution of

$$(-\Delta)^m v(\xi) = \rho v^p(\xi), \quad \xi \in \mathbb{R}^N.$$

By an appropriate rescaling, v is trivial from Theorem A, which contradicts (2.15). Hence, the conclusion holds. □

Next, we turn to the special case of $f(u) = u^p$.

Proof of Theorem 3. Suppose (1.8) fails, then, there exist sequences of Ω_k, u_k, $y_k \in \Omega_k$ such that u_k solves (1.5) in Ω_k and the function

$$M_k := \sum_{|\nu|=0}^{2m-1} \left|D^\nu u_k\right|^{\frac{p-1}{2m+(p-1)|\nu|}}$$

satisfies

$$M_k(y_k) > 2k\, dist^{-1}(y_k, \partial\Omega_k).$$

By the doubling lemma, it follows that there exists $x_k \in \Omega_k$ such that

$$M_k(x_k) \geqslant M_k(y_k),$$

$$M_k(x_k) > 2k\, dist^{-1}(x_k, \partial\Omega_k)$$

and

$$M_k(z) \leqslant 2M_k(x_k), \quad \text{if } |z - x_k| \leqslant k M_k^{-1}(x_k).$$

As before, we introduce a new function

$$v_k(\xi) := \frac{u_k(x_k + \frac{\xi}{M_k(x_k)})}{M_k^{\frac{2m}{p-1}}(x_k)}, \quad \forall |\xi| \leqslant k.$$

Then, the function v_k satisfies

$$(-\Delta)^m v_k(\xi) = v_k^p(\xi), \quad \forall |\xi| \leqslant k \tag{2.17}$$

with

$$\sum_{|\nu|=0}^{2m-1} \left|D^\nu v_k\right|^{\frac{p-1}{2m+(p-1)|\nu|}}(0) = 1 \tag{2.18}$$

and

$$\sum_{|\nu|=0}^{2m-1} \left|D^\nu v_k\right|^{\frac{p-1}{2m+(p-1)|\nu|}}(\xi) \leqslant 2, \quad \forall |\xi| \leqslant k.$$

For any smooth compact D in $\mathbb{R}^N$, there exists k_0 large enough such that $D \subset B_k(0)$ if $k \geqslant k_0$. By the classical $W^{2m,q}$ estimates for higher order elliptic equation in Lemma 2.2, we have $\|v_k\|_{W^{2m,q}(\bar{D})} \leqslant C$ for any $1 < q < \infty$.

Furthermore, using the Sobolev imbedding theorem and a diagonal line argument, $v_k \to v$ in $C^{2m-1,\tau}_{loc}(\mathbb{R}^N)$ and v solves

$$(-\Delta)^m v = v^p.$$

With the aid of Theorem A, u is trivial. Nevertheless, from (2.18), it is impossible to be trivial. Hence, a contradiction is arrived. Then the theorem is verified. □

3. Elliptic equation with mixed boundary condition

In this section, we consider the Liouville-type theorem with mixed boundary condition.

Lemma 4. *Let $u \in W^{1,2}_{loc}(\mathbb{R}^N_+) \cap C^0(\overline{\mathbb{R}^N_+})$ be a weak solution of (1.10). Then the only bounded solution is $u \equiv 0$.*

The lemma above is the Liouville-type theorem with boundedness assumptions of solutions in [10].

Proof of Theorem 4. Suppose that there exists an unbounded solution u. Then, there exits a sequence $(y_k) \in \mathbb{R}^N_+$ such that $u(y_k) \to \infty$ as $k \to \infty$. Let $M := u^{\frac{p-1}{2}} : \mathbb{R}^N_+ \to \mathbb{R}$, then $M(y_k) \to \infty$ as $k \to \infty$ since $p > 1$. Following from the doubling lemma by taking $D = \Sigma = X = \overline{\mathbb{R}^N_+}$ and Remark 1, there exists another sequence of (x_k) such that

$$M(x_k) \geqslant M(y_k)$$

and

$$M(z) \leqslant 2M(x_k), \quad \forall z \in B_{k/M(x_k)}(x_k) \cap \overline{\mathbb{R}^N_+}.$$

Define

$$d_k := x_{k,N} M(x_k)$$

and

$$H_k := \{\xi \in \mathbb{R}^N \mid \xi_N > -d_k\}.$$

As in Theorem 1, we introduce a new function

$$v_k(\xi) = \frac{u(x_k + \frac{\xi}{M(x_k)})}{M^{\frac{2}{p-1}}(x_k)}.$$

Then, $v_k(\xi)$ is the nonnegative solution of

$$\begin{cases} -\Delta v_k = v_k^p & \text{in } H_k, \\ v_k \geqslant 0 & \text{in } H_k, \\ v_k = 0 & \text{on } \Gamma_0^k := \{\xi \in H_k \mid \xi_N = -d_k,\ \xi_1 > 0\}, \\ \dfrac{\partial v_k}{\partial \xi_N} = 0 & \text{on } \Gamma_1^k := \{\xi \in H_k \mid \xi_N = -d_k,\ \xi_1 < 0\} \end{cases} \tag{3.1}$$

satisfying

$$v_k(0)^{\frac{p-1}{2}} = 1 \tag{3.2}$$

and

$$v_k^{\frac{p-1}{2}}(\xi) \leqslant 2, \quad \forall \xi \in H_k \cap B_k(0). \tag{3.3}$$

As before, two cases may occur as $k \to \infty$, either case (1)

$$x_{k,N} M(x_k) \to \infty$$

for a subsequence still denoted as before, or case (2)

$$x_{k,N} M(x_k) \to d$$

for a subsequence still denoted as before, here $d \geqslant 0$. If case (1) occurs, i.e. $H_k \cap B_k(0) \to \mathbb{R}^N$ as $k \to \infty$, then for any smooth compact D in $\mathbb{R}^N$, there exists k_0 large enough such that $D \subset (H_k \cap B_k(0))$ as $k \geqslant k_0$. By the classical elliptic equation estimates and (3.3), we have $\|v_k\|_{W^{2,q}(\bar{D})} \leqslant C$ for any $1 < q < \infty$. Therefore, from the Sobolev imbedding theorem, we can extract a convergent subsequence v_k that converges to v in D, where $v \in C^{1,\tau}(\bar{D})$. Furthermore, using a diagonalization argument, $v_k \to v$ in $C^{1,\tau}_{loc}(\mathbb{R}^N)$ and v solves

$$-\Delta v = v^p \quad \text{in } \mathbb{R}^N.$$

By the classical Liouville theorem of semilinear elliptic equation in Euclidean space, u is trivial, which contradicts (3.2).

If case (2) occurs, we make a further translation. Define

$$\tilde{v}_k(\xi) := v_k(\xi - d_k e_N) \quad \text{for } \xi \in \overline{\mathbb{R}^N_+}.$$

Then $\tilde{v}_k$ satisfies

$$\begin{cases} -\Delta \tilde{v}_k = \tilde{v}_k^p & \text{in } \mathbb{R}^N_+, \\ \tilde{v}_k \geqslant 0 & \text{in } \mathbb{R}^N_+, \\ \tilde{v}_k = 0 & \text{on } \tilde{\Gamma}^k_0 := \{\xi \in \mathbb{R}^N_+ \mid \xi_N = 0,\ \xi_1 > 0\}, \\ \dfrac{\partial \tilde{v}_k}{\partial \xi_N} = 0 & \text{on } \tilde{\Gamma}^k_1 := \{\xi \in \mathbb{R}^N_+ \mid \xi_N = 0,\ \xi_1 < 0\}. \end{cases} \tag{3.4}$$

While,

$$\tilde{v}_k^{\frac{p-1}{2}}(d_k e_N) = 1 \tag{3.5}$$

and

$$\tilde{v}_k^{\frac{p-1}{2}}(\xi) \leqslant 2, \quad \forall \xi \in \mathbb{R}^N_+ \cap B_k(d_k e_N). \tag{3.6}$$

For any smooth compact neighborhood D of the origin in $\bar{\mathbb{R}}^N_+$, we have $\{\tilde{v}_k\}$ is uniformly bounded in $C^{2,\tau}(K) \cap C^\tau(\bar{D})$ for some $0 < \tau < \frac{1}{2}$, where K is a compact set of $\bar{\mathbb{R}}^N_+$ with $dist(K, \tilde{\Gamma}) > 0$ (the a priori estimates could be done in Section 6 in [8]) and where $\tilde{\Gamma} := \{\xi \in \mathbb{R}^N_+ \mid \xi_N = 0,\ \xi_1 = 0\}$. By the Arzelá–Ascoli Theorem, $\tilde{v}_k$ converges to $\tilde{v}$ in $C^2(K) \cap C(\bar{D})$. In addition, by a diagonalization argument, $\tilde{v}_k$ converges locally in $\overline{\mathbb{R}^N_+}$ to $\tilde{v} \in C^2(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$ such that v satisfies

$$\begin{cases} -\Delta \tilde{v} = \tilde{v}^p & \text{in } \mathbb{R}^N_+, \\ \tilde{v}(\xi) \leqslant 2^{\frac{2}{p-1}} & \text{in } \mathbb{R}^N_+, \\ \tilde{v} = 0 & \text{on } \Gamma_0 = \{\xi \in \mathbb{R}^N_+ \mid \xi_N = 0,\ \xi_1 > 0\}, \\ \dfrac{\partial \tilde{v}}{\partial \xi_N} = 0 & \text{on } \Gamma_1 = \{\xi \in \mathbb{R}^N_+ \mid \xi_N = 0,\ \xi_1 < 0\}. \end{cases} \tag{3.7}$$

Deduced from Lemma 4, we have $\tilde{v} \equiv 0$. However, from (3.5), $\tilde{v}(de_N) = 1$. Clearly, a contradiction is achieved. Therefore, the proof of Theorem 4 is complete. □

4. Elliptic systems

In the last section, we consider the elliptic systems. We could show that boundedness is removable in the Liouville theorem of (1.12). The following lemma is established in [12].

Lemma 5. *The system* (1.12) *does not have a bounded positive solution in* $\mathbb{R}^N$*, provided*

$$1 < \alpha_1, \alpha_2 < \frac{N+2}{N-2}.$$

Proof of Theorem 5. Suppose by contradiction that there exists a sequence of (y_k) such that $u_1(y_k) \to \infty$ or $u_2(y_k) \to \infty$ as $k \to \infty$. Let

$$M(y) := u_1^{\frac{\alpha_1-1}{2}}(y) + u_2^{\frac{\alpha_2-1}{2}}(y).$$

Then, $M(y_k) \to \infty$ as $k \to \infty$. From the doubling lemma by letting $D = \Sigma = X = \overline{\mathbb{R}^N}$ and Remark 1, there exists a sequence of (x_k) such that

$$M(x_k) \geqslant M(y_k)$$

and

$$M(z) \leqslant 2M(x_k), \quad \text{if } |z - x_k| \leqslant kM^{-1}(x_k).$$

Let $\alpha = \frac{2}{\alpha_1-1}$ and $\beta = \frac{2}{\alpha_2-1}$. We rescale (u_1, u_2) by

$$\tilde{v}_{1,k}(\xi) := \frac{u_1(x_k + \frac{\xi}{M(x_k)})}{M^\alpha(x_k)}$$

and

$$\tilde{v}_{2,k}(\xi) := \frac{u_2(x_k + \frac{\xi}{M(x_k)})}{M^\beta(x_k)}.$$

Then, $(\tilde{v}_{1,k}, \tilde{v}_{2,k})$ satisfies

$$\begin{cases} \Delta \tilde{v}_{1,k} + \tilde{v}_{1,k}^{\alpha_1} + \tilde{v}_{2,k}^{\alpha_1 \frac{\alpha_2-1}{\alpha_1-1}} = 0, \\ \Delta \tilde{v}_{2,k} + \tilde{v}_{2,k}^{\alpha_2} + \tilde{v}_{1,k}^{\alpha_2 \frac{\alpha_1-1}{\alpha_2-1}} = 0. \end{cases} \tag{4.1}$$

Moreover

$$\tilde{v}_{1,k}^{\frac{1}{\alpha}}(0) + \tilde{v}_{2,k}^{\frac{1}{\beta}}(0) = 1$$

and

$$\tilde{v}_{1,k}^{\frac{1}{\alpha}}(\xi) + \tilde{v}_{2,k}^{\frac{1}{\beta}}(\xi) \leqslant 2, \quad \forall |\xi| \leqslant k.$$

For any smooth compact D in $\mathbb{R}^N$, there exists k_0 large enough such that $D \subset B_k(0)$ if $k \geqslant k_0$. Using the classical $W^{2,q}$ estimates for elliptic equation, we have

$$\sum_{i=1}^{2} \|\tilde{v}_{i,k}\|_{W^{2,q}(D)} \leqslant C$$

for $1 < q < \infty$. By the standard Sobolev imbedding theorem, there exists a sequence of $(\tilde{v}_{1,k}, \tilde{v}_{2,k})$ that converges to $(\tilde{v}_1, \tilde{v}_2)$ in $C^{1,\tau}(D)$ for some $\tau > 0$. Employing a diagonal line argument, we readily deduce that $(\tilde{v}_{1,k}, \tilde{v}_{2,k})$ converges in $C^{1,\tau}_{loc}(\mathbb{R}^N)$ to a solution $(\tilde{v}_1, \tilde{v}_2)$ in $\mathbb{R}^N$ which satisfies

$$\begin{cases} \Delta \tilde{v}_1 + \tilde{v}_1^{\alpha_1} + \tilde{v}_2^{\alpha_1 \frac{\alpha_2-1}{\alpha_1-1}} = 0, \\ \Delta \tilde{v}_2 + \tilde{v}_2^{\alpha_2} + \tilde{v}_1^{\alpha_2 \frac{\alpha_1-1}{\alpha_2-1}} = 0. \end{cases} \tag{4.2}$$

Furthermore,

$$\tilde{v}_1^{\frac{1}{\alpha}}(0) + \tilde{v}_2^{\frac{1}{\beta}}(0) = 1.$$

So $(\tilde{v}_1, \tilde{v}_2)$ is nontrivial. This contradicts Lemma 5. Therefore, we then complete the proof of the theorem. □

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