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Determination of the insolation function in the nonlinear Sellers climate model

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Abstract

We are interested in the climate model introduced by Sellers in 1969 which takes the form of some nonlinear parabolic equation with a degenerate diffusion coefficient. We investigate here some inverse problem issue that consists in recovering the so-called insolation function. We not only solve the uniqueness question but also provide some strong stability result, more precisely unconditional Lipschitz stability in the spirit of the well-known result by Imanuvilov and Yamamoto (1998) [22]. The main novelties rely in the fact that the considered model is degenerate and above all nonlinear. Indeed we provide here one of the first result of Lipschitz stability in a nonlinear case.

Keywords: Nonlinear parabolic equation; Degenerate diffusion; Climate models; Inverse problems; Carleman estimates; Hardy inequalities

1. Introduction

We are interested in a problem arising in climatology, coming more specifically from the classical energy balance model introduced by Sellers in 1969. This climate model aims at understanding the effects of many parameters (such as for instance greenhouse gazes, albedo or advection fluxes) on the ice covering of the Earth surface. It takes the form of some 1-dimensional nonlinear parabolic problem with degenerate diffusion.

The mathematical analysis of energy balance models like the Sellers one is the subject of many recent works. Questions such as well-posedness, uniqueness, asymptotic behavior, existence of periodic solutions, bifurcations have been investigated. Many interesting results on the subject have been proved by various authors, among them Diaz, Hetzer, Tello. For an overview of these studies, we may refer the reader to [16–18,21,5] and the references therein.

In this paper, we investigate some inverse problem issue that consists in recovering the insolation function in the Sellers model. To our knowledge, this is the first inverse problem result for such model. And with respect to the results that can be found in the literature concerning inverse problems for parabolic equations, the question we address here presents several novelties.

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Let us recall that one can find many references dealing with uniqueness results for parabolic inverse problems, for instance the pioneering work of Bukhgeim and Klibanov [6], the books of Isakov [23], Klibanov [24] and Klibanov and Timonov [26] and the references therein. Let us focus here on a specific result by Imanuvilov and Yamamoto [22] obtained for some standard inverse source problem for the linear heat equation. Their purpose is to retrieve the source term of the equation using some partial measurements of the solution. Their method is based on the global Carleman estimates for the heat equation that were developed by Fursikov and Imanuvilov [20] and used to solve null controllability issues. The novelty of their work is that they not only solve the uniqueness question but, they also provide some strong stability result concerning the reconstruction of the source (more precisely they prove unconditional Lipschitz stability). The idea of using global Carleman estimates to solve inverse problems was first introduced by Puel and Yamamoto [32] in the context of the wave equation and has proved its efficiency. Indeed it has been adapted to solve various situations, see for example [1–3,13,33–36,38]. In particular, the recent works [13,33] are devoted to the study of some inverse source problem for a class of linear degenerate parabolic equations. In this purpose, the authors used some recent global Carleman estimates specifically derived in [8] to treat controllability issues for such degenerate equations (see also the papers [9,10,12]).

In the present paper, we aim at obtaining some uniqueness and Lipschitz stability result for the insolation coefficient in the Sellers model in the spirit of the result by Imanuvilov and Yamamoto [22]. In this purpose, the works [13,33] may be seen as a very first step in view of solving the present question. However the problem we study here presents several additional difficulties:

– First of all, the degeneracy occurring in the diffusion coefficient in the Sellers model is more complex than the one considered in [13,33]. Instead of some power function vanishing at one extremity of the domain, one now has some more general function vanishing at both extremities of the domain. Therefore we will adapt other Carleman estimates proved in [31] in order to deal with such degeneracies.
– Next, as mentioned above, the Sellers equation is not linear which obviously constitutes a new and major difficulty. To our knowledge, the only paper dealing with Lipschitz stability for some inverse coefficient problem in a semilinear parabolic equation is [14]. Here again, we face additional difficulties. First of all, we consider solutions in Sobolev spaces, which prevents us from using strong maximum principles and requires a careful study of the regularity of weak solutions. Next, the coefficient that we assume to be unknown is located in front of a nonlinear term, which triggers some additional technical difficulties. Eventually, this unknown coefficient is assumed to be a bit more general than in [14] and has some known time dependence.

As a conclusion, let us mention that, although our paper aims at solving some inverse coefficient problem for the specific nonlinear Sellers model, our method may be successfully adapted in other situations and therefore constitutes a possible approach to obtain Lipschitz stability results for nonlinear equations.

The paper is organized as follows. In Section 2, we first describe the Sellers model and make precise our assumptions. In Section 3, we begin our study of the Sellers model with some preliminary results of well-posedness and regularity. Next we state our main result of Lipschitz stability for the considered inverse problem in Section 4. We make some more comments and describe open questions in Section 5. Finally, Sections 6–8 are devoted to the proofs.

2. The Sellers model and assumptions

2.1. Climate modelling

Let us introduce here Budyko–Sellers climate models. Our goal here is to present briefly the model. For more details, we refer the reader to [17,18] and the references therein. The purpose of climate models is to allow a better understanding of past and future climates and their evolution and sensitivity to some relevant solar and terrestrial parameters. Unlike weather prediction models, they involve a long time scale. The first one-dimensional energy balance climate models were introduced independently in 1969 by Budyko and Sellers. They aim at describing the ice covering on the Earth and have been used in the study of the Milankovitch theory of ice-ages (see for instance [30]). In both models, $u(x,t)$ represents the mean annual or seasonal temperature average on the latitude circles around the Earth (here $x = \sin\varphi$ where φ denotes the latitude). Then the two models are stated in the domain $I = (-1, 1)$ and take the form of the reaction–diffusion equation:

$$\begin{cases} u_t - \big(\rho(x)u_x\big)_x = \mathcal{R}_a(t,x,u) - \mathcal{R}_e(u), & x \in I,\ t > 0, \\ \rho(x)u_x(t,x) = 0, & x \in \partial I,\ t > 0, \\ u(0,x) = u_0(x), & x \in I. \end{cases} \tag{2.1}$$

Due to the peculiar expression of the diffusion operator on a meridian circle, the diffusion coefficient $\rho(x)$ vanishes at both extremities $x = \pm 1$ in the following way:

$$\rho(x) = \rho_0\big(1 - x^2\big) \quad \text{for some } \rho_0 > 0. \tag{2.2}$$

The right-hand side of the equation corresponds to the mean radiation flux depending on the solar radiation $\mathcal{R}_a$ and the radiation $\mathcal{R}_e$ emitted by the Earth. The two models differ here since different choices for $\mathcal{R}_a$ and $\mathcal{R}_e$, which are relevant in climatology, have been made by Budyko and Sellers.

The solar radiation $\mathcal{R}_a$ corresponds to the fraction of the solar energy absorbed by the Earth. It depends on the incoming solar flux $Q(x,t)$ and on the planetary coalbedo $\beta(u)$:

$$\mathcal{R}_a(t,x,u) = Q(t,x)\beta(u). \tag{2.3}$$

Note that the albedo (which is more used than coalbedo in the climatological setting) is actually

$$1 - \beta(u).$$

When the time scale is long enough, as for instance in annual models, one may assume that the insolation function $Q = Q(x)$ is a nonnegative function that does not depend on time t. But, when the time scale is smaller, as in seasonal models, one uses a more realistic description of the incoming solar flux by assuming that $Q = Q(t,x)$ is a time-periodic function.

The coalbedo is a function $\beta(u)$ of the temperature that represents the fraction of the incoming radiation flux which is absorbed by the surface of the Earth. Over ice-covered zones, reflection is greater than over ice-free regions like oceans therefore the coalbedo is smaller. One usually considers that β is roughly constant for temperatures far enough from the critical value for which ice becomes white (the snow-line) and that is usually taken as $u = u_s$ where $u_s = -10\,^\circ\mathrm{C}$. Different kind of assumptions can be made on β in a neighborhood of u_s to represent the sharp transition that occurs between zones of low and high coalbedo.

Budyko assumed that β is a discontinuous function taking the value a_i for $u < u_s$ and the value a_f for $u > u_s$ where $0 < a_i < a_f$. Here, for the mathematical analysis of the equation, the nonlinearity $\mathcal{R}_a$ has to be treated as a maximal graph in $\mathbb{R}^2$. This obviously generates difficulties in the mathematical treatment of the problem as uniqueness of solutions is not guaranteed (see for instance [17,18,21]).

On the contrary, Sellers assumed that β is a more regular function of u, which is at least Lipschitz continuous. For example, he considered a continuous linear function β that takes the value a_i for $u < u_s - \varepsilon$ and a_f for $u > u_s + \varepsilon$ for some small $\varepsilon > 0$. In the same spirit, it is still relevant to consider a more regular function providing it realizes a sharp transition near $u = u_s$ between the two values a_i and a_f. For example, one can take (see [16, Section 1]):

$$\beta(u) = a_i + \frac{1}{2}(a_f - a_i)\big(1 + \tanh\big(\gamma(u - u_s)\big)\big) \quad \text{for some } 0 < \gamma < 1. \tag{2.4}$$

The Earth radiation $\mathcal{R}_e$ corresponds to the energy emitted by the Earth. It may depend on the amount of greenhouse gases, clouds and water vapor in the atmosphere and may be affected by anthropo-generated changes. Several empiric relations are proposed in the literature. Budyko simply assumed that it is linear:

$$\mathcal{R}_e(u) = A + Bu \quad \text{with } A \in \mathbb{R},\ B > 0. \tag{2.5}$$

On the other hand, Sellers uses a Stefan–Boltzmann type law to obtain the nonlinear relation

$$\mathcal{R}_e(u) = \varepsilon(u)|u|^3 u. \tag{2.6}$$

Here u is measured in Kelvin degrees. The function $\varepsilon(u)$ represents the emissivity and is assumed to be regular positive and bounded. More precisely, one may take (see [17, Section 1]):

$$\varepsilon(u) = \sigma\left(1 - m \tanh\left(\frac{19u^6}{10^6}\right)\right), \tag{2.7}$$

where $\sigma > 0$ is the emissivity constant and $m > 0$ the atmospheric opacity. Let us mention that (2.5) corresponds to a linear approximation of (2.6) near the actual mean temperature of $15\,^\circ\mathrm{C}$.

2.2. *Assumptions for the Sellers model*

Let us now turn to the case that we consider in the present work. We study here the Sellers model. Let us make precise the assumptions under which we consider problem (2.1):

Assumption 1.

(i) The diffusion coefficient $\rho(x)$ is defined by (2.2).
(ii) The solar radiation $\mathcal{R}_a$ is given by (2.3) with $\beta:\mathbb{R}\to\mathbb{R}$ satisfying

$$\beta\in\mathcal{C}^1(\mathbb{R})\cap L^\infty(\mathbb{R}),\qquad \beta'\in L^\infty(\mathbb{R}),\quad \beta' \text{ is } k\text{-Lipschitz } (k>0), \tag{2.8}$$

$$\exists\beta_1>0,\ \forall u\in\mathbb{R},\quad \beta(u)\geqslant\beta_1. \tag{2.9}$$

Besides we assume that Q takes the form

$$Q(t,x)=r(t)q(x) \tag{2.10}$$

where

$$q\in L^\infty(I),\quad q\geqslant 0, \tag{2.11}$$

$$r\in\mathcal{C}^1(\mathbb{R}) \text{ is } \tau\text{-periodic function}\quad (\tau>0), \tag{2.12}$$

$$\exists r_1>0,\ \forall t\in\mathbb{R},\quad r(t)\geqslant r_1. \tag{2.13}$$

(iii) Finally, the Earth radiation is given by (2.6) with $\varepsilon:\mathbb{R}\to\mathbb{R}$ satisfying

$$\varepsilon\in\mathcal{C}^1(\mathbb{R})\cap L^\infty(\mathbb{R}),\qquad \varepsilon'\in L^\infty(\mathbb{R}),\quad \varepsilon' \text{ is } K\text{-Lipschitz } (K>0), \tag{2.14}$$

$$\exists\varepsilon_1>0,\ \forall u\in\mathbb{R},\quad \varepsilon(u)>\varepsilon_1. \tag{2.15}$$

Let us mention that our assumptions on β and ε cover a large class of functions that are relevant when considering the Sellers model, including in particular the choices suggested in (2.4) and in (2.7). On the other hand, as we assume that Q depends periodically on time t, our assumption allows to consider both annual and seasonal models.

In the following sections, we assume Assumption 1 is satisfied and we focus on the following model

$$\begin{cases} u_t-\big(\rho(x)u_x\big)_x=r(t)q(x)\beta(u)-\epsilon(u)|u|^3u, & x\in I,\ t>0,\\ \rho(x)u_x(t,x)=0, & x\in\partial I,\ t>0,\\ u(0,x)=u_0(x), & x\in I. \end{cases} \tag{2.16}$$

3. Well-posedness

Existence and uniqueness of a weak solution to problem (2.16) has been proved by Diaz in [16]. However, for inverse issues, it is well known that solutions must be quite regular. So in the present section, we show existence and uniqueness of a global regular solution to (2.16). In this purpose, we will write problem (2.16) in terms of a semilinear evolution equation governed by an analytic semigroup.

3.1. *Functional framework*

As the diffusion operator is degenerate, the natural energy space is a suitable weighted Sobolev space,

$$V=\big\{w\in L^2(I):\ \sqrt{\rho}w_x\in L^2(I)\big\},$$

endowed with the following inner product

$$(u,v)_V:=(u,v)_{L^2(I)}+(\sqrt{\rho}u_x,\sqrt{\rho}v_x)_{L^2(I)}$$

and the associated norm

$$\forall u \in V, \quad \|u\|_V := \sqrt{(u,u)_V}.$$

Notice that $V \subset H^1_{loc}(I)$. However let us mention that, due to the power of the degeneracy occurring at both extremities of the domain I, the trace at $x = \pm 1$ of an element of V does not exist. (We refer the reader to [11] where some similar situation is studied.)

Let us define the following symmetric continuous bilinear form a on V by

$$a : \begin{cases} V \times V \to \mathbb{R}, \\ (v,w) \mapsto \int_{-1}^{1} \sqrt{\rho} v_x \sqrt{\rho} w_x \, dx. \end{cases}$$

We immediately see that a is $V - L^2(I)$ coercive, i.e.

$$\exists \alpha > 0,\ \exists \beta \in \mathbb{R},\ \forall v \in V, \quad a(v,v) + \beta \|v\|^2_{L^2(I)} \geqslant \alpha \|v\|^2_V.$$

Following [4, p. 45], we associate with a the unbounded operator $A : D(A) \subset L^2(I) \to L^2(I)$ defined by

$$D(A) := \{ v \in V \mid w \mapsto a(v,w) \text{ is } L^2(I) \text{ continuous} \},$$
$$\forall v \in D(A),\ \forall w \in V, \quad (Av, w)_{L^2(I)} = a(v,w).$$

Theorem 3.1. *$(A, D(A))$ is the infinitesimal generator of an analytic semigroup.*

Proof. Use [4, Theorem 2.12, p. 46]. □

Then we prove that the operator $(A, D(A))$ may equivalently be characterized in another way:

Lemma 3.1. *The couple $(A, D(A))$ satisfies*

$$D(A) = \{ u \in V \mid \rho u_x \in H^1(I) \},$$
$$\forall u \in D(A), \quad Au = -(\rho u_x)_x.$$

We prove this lemma later in Section 6.1.

Let us mention that the reader may also refer to [7] for another way of defining $(A, D(A))$ and its study. See in particular in [7], Lemma 2.5 and Theorems 2.3 and 2.8.

Remark. The boundary condition appearing in (2.16) is not useful from the mathematical approach. Indeed, the fact that $(\rho u_x)_x$ belongs to $L^2(I)$ automatically implies that u satisfies $(\rho u_x)(x) = 0$, $x \in \partial I$ (see the proof of Lemma 3.1). Hence the boundary condition is contained in the definition of the operator.

As in [28, Proposition 2.1, p. 22], we denote by $[D(A), L^2(I)]_{\frac{1}{2}}$ the real interpolation space $(D(A), L^2(I))_{1/2,2}$ constructed by the trace method and we state some interpolation result.

Lemma 3.2. *The intermediate space $[D(A), L^2(I)]_{\frac{1}{2}}$ is the space V.*

Proof. Due to the variational context $V \subset L^2(I) \subset V'$, the proof is immediate using [28, Proposition 2.1, p. 22]. □

3.2. Local existence of regular solutions

Let us now give some results of local existence. We fix here $T > \tau$ and consider the problem for $t \in [0, T]$ (in practical situations, $T \gg \tau$). In order to apply the theory in [29], we first transform (2.16) into an evolution equation in $L^2(I)$. The first difficulty encountered is that $\mathcal{R}_e(u)$ is not a priori defined if u is living in $L^2(I)$. The following result will be used to overcome it.

Lemma 3.3. *For all $q \in [1, +\infty)$, V is continuously embedded in $L^q(I)$.*

We refer the reader to [17, Lemma 1(ii)] for the proof of Lemma 3.3. Next we define the $L^2(I)$-valued function G as

$$G : \begin{cases} [0,T] \times V \to L^2(I), \\ (t,u) \mapsto r(t)q\beta(u) - \epsilon(u)|u|^3 u. \end{cases}$$

Provided that G is well-defined, problem (2.16) on $[0,T]$ may be recast into the evolution equation:

$$\begin{cases} u_t(t) + Au(t) = G\big(t, u(t)\big), \quad t \in [0,T], \\ u(0) = u_0. \end{cases} \tag{3.1}$$

First we prove the following properties of G.

Lemma 3.4. *G is well-defined on $[0,T] \times V$ with values into $L^2(I)$. Moreover G satisfies*:

- $\forall t \in [0,T]$, $\forall R > 0$, $\exists C > 0$, $\forall u_1, u_2 \in B_V(0,R)$,

$$\big\|G(t,u_1) - G(t,u_2)\big\|_{L^2(I)} \leqslant C\|u_1 - u_2\|_V. \tag{3.2}$$

- $\forall R > 0$, $\exists \theta \in (0,1)$, $\exists C > 0$, $\forall u \in B_V(0,R)$, $\forall s,t \in [0,T]$,

$$\big\|G(t,u) - G(s,u)\big\|_{L^2(I)} \leqslant C|t-s|^\theta. \tag{3.3}$$

The proof of Lemma 3.4 is postponed to Section 6.2. We are now able to deduce our first result of local existence:

Theorem 3.2. *For all $u_0 \in D(A)$, there exists $T^\star(u_0) \in (0,+\infty]$ such that, for all $0 < T < T^\star(u_0)$, problem (3.1) has a unique solution $u \in \mathcal{C}([0,T], D(A)) \cap \mathcal{C}^1([0,T], L^2(I))$. Moreover, if $T^\star(u_0) < +\infty$, then $\|u(t)\|_V \to +\infty$ as $t \to T^\star(u_0)$.*

Proof. The proof relies on the book [29]. Indeed, using Theorem 3.1, Lemmas 3.4, 3.2 and applying [29, Theorem 7.1.2] to the evolution equation (3.1), we prove that there exists a unique mild solution in the sense of [29, Definition (7.1.1)] up to a maximal time, denoted by $T^\star(u_0)$. Then, [29, Proposition 7.1.8] implies that, if $T^\star(u_0) < +\infty$, then $\|u(t)\|_V \to +\infty$ as $t \to T^\star(u_0)$. Moreover, since $Au_0 + G(0,u_0) \in L^2(I)$, [29, Proposition 7.1.10] ensures that, for all compact interval $[0,T]$ with $T < T^\star(u_0)$, $u \in \mathcal{C}([0,T], D(A)) \cap \mathcal{C}^1([0,T], L^2(I))$. □

3.3. A weak maximum principle

Next we turn to some useful boundedness properties.

Theorem 3.3. *Consider $u_0 \in D(A) \cap L^\infty(I)$ and $0 < T < T^\star(u_0)$ where $T^\star(u_0)$ is defined as in Theorem 3.2. Then denote*

$$M := \max\left\{ \|u_0\|_{L^\infty(I)}, \left(\frac{\|q\|_{L^\infty(I)} \|r\|_{L^\infty(\mathbb{R})} \|\beta\|_{L^\infty(\mathbb{R})}}{\varepsilon_1} \right)^{1/4} \right\}. \tag{3.4}$$

Then the solution u of (2.16) satisfies

$$\|u\|_{L^\infty((0,T)\times I)} \leqslant M.$$

The proof of Theorem 3.3 is given later in Section 6.3. Observe that one may deduce from Theorem 3.3 that, for any u_0 in $D(A) \cap L^\infty(I)$, the L^2-norm of the solution of (2.16) does not blow up at time $T^\star(u_0)$. Yet, this is not sufficient to ensure global existence of the regular solution for it may happen that the V-norm of $u(t)$ blows up at $t \to T^\star(u_0)$. In order to get global existence results, it remains to show that this cannot happen.

3.4. Regularity of the time derivative of the solution of (2.16)

Let us now state some further regularity properties of the solution or more precisely of its time derivative. In this purpose, we restrict the initial conditions to the following space:

$$\mathcal{U} := \{u_0 \in D(A) \cap L^\infty(I):\ Au_0 \in L^\infty(I)\}.$$

We also recall the following standard notation:

$$W(0,T;V,V') := \{v \in L^2(0,T;V):\ v_t \in L^2(0,T;V')\}.$$

This allows us to state the following result proved later in Section 6.4.

Theorem 3.4. *Let $u_0 \in \mathcal{U}$ be given and consider u the corresponding solution of (2.16). Let T be such that $0 < T < T^\star(u_0)$ where $T^\star(u_0)$ is defined as in Theorem 3.2. Then the function $z := u_t$ belongs to $L^2(0,T;V)$ and is the solution of the variational problem*

$$\begin{cases} z \in W(0,T;V,V'), \\ \forall w \in V, \quad \langle z_t(t), w\rangle + b(t, z(t), w) = (r'(t)q\beta(u(t)), w)_{L^2(I)}, \\ z(0) = -Au_0 + G(0,u_0), \end{cases} \tag{3.5}$$

where $b : [0,T] \times V \times V \to \mathbb{R}$ is a weakly coercive time-dependent bilinear form defined by

$$b(t,v,w) = \int_I \sqrt{\rho}v_x\sqrt{\rho}w_x\,dx + \int_I \pi(t,x)vw\,dx$$

with $\pi(t,x) := \mathcal{R}_e'(u(t,x)) - r(t)q(x)\beta'(u(t,x))$.

Observe that b is well-defined. Indeed, $\mathcal{R}_e \in \mathcal{C}^1(\mathbb{R})$ and for all $s \in \mathbb{R}$, $\mathcal{R}_e'(s) = \varepsilon'(s)s|s|^3 + 4\varepsilon(s)s\,\mathrm{sign}(s)|s|^2$. Hence, $\pi \in L^\infty((0,T)\times I)$, thanks to Theorem 3.3. As a consequence of Theorem 3.4, we may state the following corollary:

Corollary 3.1. *Let $u_0 \in \mathcal{U}$ and $0 < T < T^\star(u_0)$ where $T^\star(u_0)$ is defined as in Theorem 3.2. Then the solution z of (3.5) satisfies*

$$\|z\|_{L^\infty((0,T)\times I)} \leqslant e^{(\|\pi\|_{L^\infty((0,T)\times I)}+1)T}N,$$

where

$$N := \max\{\|-Au_0 + G(0,u_0)\|_{L^\infty(I)}, \|r'\|_{L^\infty(\mathbb{R})}\|q\|_{L^\infty(I)}\|\beta\|_{L^\infty(\mathbb{R})}\}. \tag{3.6}$$

The proofs of Theorem 3.4 and Corollary 3.1 are given later in Sections 6.4 and 6.5.

3.5. Global existence of regular solutions

The last result of this section devoted to well-posedness is a global existence property, proved later in Section 6.6.

Theorem 3.5. *Let $u_0 \in \mathcal{U}$. Then the solution u of (2.16) is defined on $[0,+\infty)$, that is to say $T^\star(u_0) = +\infty$.*

Let us mention that, as a consequence, Theorem 3.3 and Corollary 3.1 hold true for $T^\star(u_0) = +\infty$.

4. A Lipschitz stability result

4.1. Statement of the result

In this part, our goal is to determine the coefficient $q(x)$ in problem (2.16) assuming that it satisfies some boundedness condition. In this purpose, we introduce the set of admissible coefficients: for all $D > 0$, we define the set $\mathcal{Q}_D$ as

$$\mathcal{Q}_D := \{q \in L^\infty(I):\ \|q\|_{L^\infty(I)} \leqslant D\}.$$

The main result of this section is the following theorem supplying with not only a uniqueness result but also some Lipschitz stability estimates.

Theorem 4.1. *Let $t_0 \in (0, T)$ be given and consider*

$$T' := \frac{T + t_0}{2}.$$

Let $\omega := (L_1, L_2)$ where $-1 < L_1 < L_2 < 1$ and take $u_0 \in \mathcal{U}$. Then, for all $D > 0$, there exists a constant $C > 0$ such that, for all q_1 and q_2 in $\mathcal{Q}_D$, the associated solutions u_1 and u_2 of problem (2.16) satisfy

$$\|q_1 - q_2\|^2_{L^2(I)} \leqslant C\big(\big\|(u_1 - u_2)\big(T', .\big)\big\|^2_{D(A)} + \|u_{1,t} - u_{2,t}\|^2_{L^2((t_0,T)\times\omega)}\big). \tag{4.1}$$

The proof of Theorem 4.1 is given later in Section 7.1. We follow the method introduced by Imanuvilov and Yamamoto in [22] to get Lipschitz stability results for inverse problems. This method is based on the use of global Carleman estimates for parabolic problems (see Fursikov and Imanuvilov [20]). Here we use specific Carleman estimates for degenerate parabolic equations (inspired by [9,31]). Thus, we first recall this fundamental tool in Section 4.2 before proving Theorem 4.1.

4.2. Main tool: a global Carleman estimate for degenerate parabolic equations with locally distributed observation

We recall here a fundamental result from [31]. Let us mention that, in [31], the space domain of the considered functions is $(0, 1)$ whereas it is $(-1, 1)$ in the present case. So in this section, we slightly modify the definitions and the statement of the result of [31] to adapt them to our situation.

As usual, the derivation of global Carleman estimates relies on the introduction of some suitable weight function of the form

$$\forall (t, x) \in (t_0, T) \times I, \quad \sigma(t, x) = \theta(t)p(x)$$

where the functions θ and p have to be specified.

As in [8,31], we introduce the following time weight function $\theta(t)$:

$$\forall t \in (t_0, T), \quad \theta(t) := \left[\frac{1}{(t - t_0)(T - t)}\right]^4.$$

Then we introduce a space weight function $p(x)$ specifically adapted to locally distributed observations in the case of a degenerate problem like (2.16), see [31]:

$$\forall x \in I = (-1, 1), \quad p(x) = G_0 - \int_{-1}^{x} \frac{\phi_-(s)}{\rho(s)} e^{S(\phi_+(s))^2}\, ds, \tag{4.2}$$

where G_0, S are positive constants (to be fixed later) and ϕ_- and ϕ_+ are the two functions defined below. Let $\widetilde{\omega} := (\tilde{L}_1, \tilde{L}_2)$ be such that $-1 < L_1 < \tilde{L}_1 < \tilde{L}_2 < L_2 < 1$ and let ϕ_1 and ϕ_2 be two smooth cut-off functions such that

$$\forall x \in [-1,1], \quad 0 \leqslant \phi_1(x) \leqslant 1, \qquad 0 \leqslant \phi_2(x) \leqslant 1,$$
$$\forall x \in [-1,\tilde{L}_1], \quad \phi_1(x) = 1, \qquad \phi_2(x) = 0,$$
$$\forall x \in [\tilde{L}_2, 1], \quad \phi_1(x) = 0, \qquad \phi_2(x) = 1,$$
$$\forall x \in [-1,1], \quad \phi_1(x) + \phi_2(x) > 0. \tag{4.3}$$

Next ϕ_+ and ϕ_- are defined by

$$\begin{cases} \forall x \in [-1,1], \quad \phi_+(x) := \dfrac{(1+x)}{2}\phi_1(x) + \dfrac{(1-x)}{2}\phi_2(x), \\ \forall x \in [-1,1], \quad \phi_-(x) := \dfrac{(1+x)}{2}\phi_1(x) - \dfrac{(1-x)}{2}\phi_2(x). \end{cases} \tag{4.4}$$

Observe that there exists some constant $C > 0$ such that

$$\forall x \in I, \quad \left|\phi_-(x)\right| \leqslant C(1+x)(1-x) = \frac{C}{\rho_0}\rho(x). \tag{4.5}$$

Then we state the following result from [31, Lemma 5.4]:

Lemma 4.1. *If G_0 and S are large enough, then p is a well-defined and strictly positive function on $I = [-1, 1]$.*

For the proof, we refer the reader to [31, Lemma 5.4] which is similar (except the fact that the space domain in [31] is $(0, 1)$ whereas it is $(-1, 1)$ in the present case). In the following, we choose G_0 and S large enough so that the statement of Lemma 4.1 holds true.

Eventually, we define as in [13] the second time weight function:

$$\forall t \in (t_0, T), \quad \gamma(t) := T + t_0 - 2t. \tag{4.6}$$

Let us now turn to the following linear initial–boundary value problem:

$$\begin{cases} z_t - (\rho z_x)_x = h, & (t,x) \in (t_0,T) \times I, \\ \rho(x) z_x(t,x) = 0, & (t,x) \in (t_0,T) \times \partial I, \end{cases} \tag{4.7}$$

where $h \in L^2(t_0, T; L^2(I))$. In the following, we denote

$$Q_T^{t_0} := (t_0,T) \times I \quad \text{and} \quad \omega_T^{t_0} := (t_0,T) \times \omega.$$

Now we are ready to state global Carleman estimates for locally distributed observation for system (4.7):

Theorem 4.2. *Let $\omega := (L_1, L_2)$ where $0 < L_1 < L_2 < 1$. There exist two constants $R_0 = R_0(T, t_0, \omega) > 0$ and $C_1 = C_1(T, t_0, \omega) > 0$ such that: $\forall R \geqslant R_0$,*

$$\iint\limits_{Q_T^{t_0}} \left(R^3\theta^3\left(1-x^2\right)z^2 + R\theta^{3/2}|\gamma| p z^2 + R\theta\left(1-x^2\right)z_x^2 + \frac{1}{R\theta} z_t^2 \right) e^{-2R\sigma}\, dx\, dt$$
$$\leqslant C_1 \left(\iint\limits_{Q_T^{t_0}} h^2 e^{-2R\sigma}\, dx\, dt + \iint\limits_{\omega_T^{t_0}} R^3\theta^3 z^2 e^{-2R\sigma}\, dx\, dt \right), \tag{4.8}$$

for all solutions $z \in L^2(t_0, T; D(A)) \cap H^1(t_0, T; L^2(I))$ of (4.7).

Part of these estimates were obtained in [31, Theorem 3.3] (in the case of a space domain $(0, 1)$ instead of $(-1, 1)$): the first estimate of z and the estimate of z_x (that were sufficient for control purposes). In view of obtaining inverse problem results, one also needs some other estimate of z and some estimate of z_t that we added here in the statement of Theorem 4.2. The proof, based on the methods developed in [20,31,13], is given later in Section 8.

5. Open questions

In this paper, we use and extend the approach by [22] in order to get an unconditional global Lipschitz stability of an unknown coefficient in a nonlinear term in the 1D Sellers climate model. Our method is not specific to such a model and may be successfully used for similar inverse coefficient problems in other kinds of nonlinear parabolic equations (even non-degenerate ones). A first additional question in the field of inverse problems for climate models is to study the two-dimensional Sellers model on the Earth surface. It comes to solve an inverse coefficient problem for a nonlinear heat equation posed on a Riemannian manifold. This is the subject of a forthcoming paper.

As we mentioned in the introduction of this paper, the albedo is a quite badly known function. Therefore it would be very interesting to solve the inverse problem of determining the albedo from measurements of the temperature. Yet this question leads to two main difficulties. Even if one assumes the coalbedo is smooth, the question of unconditional global stability results for a nonlinear smooth term in a parabolic equation is not well-understood (see [19] for a partial answer). If one considers the Budyko model (the coalbedo is seen as a maximal monotone graph), there are well-posedness problems such as non-uniqueness of solutions [16–18].

Moreover, the Sellers model described in Section 2 is a simplified version of some more complicated Budyko–Sellers climate models. For instance, one can consider a p-Laplace operator instead of a linear operator [18,17,16]. Very few results are known in the fields of controllability and inverse problems for equations involving the p-Laplace operator. The question of Lipschitz stability is completely open in this case.

6. Proofs related to well-posedness

6.1. Proof of Lemma 3.1

Let us denote $D := \{u \in V \mid \rho u_x \in H^1(I)\}$. First we prove that $D(A) \subset D$. Let $v \in D(A)$ and $w \in \mathcal{D}(I)$. Then $Av \in L^2(I)$ and

$$\begin{aligned}\langle Av, w\rangle_{\mathcal{D}',\mathcal{D}} &= (Av, w)_{L^2(I)} = a(v, w)\\ &= \int_I \sqrt{\rho} v_x \sqrt{\rho} w_x\, dx = \int_I \rho v_x w_x\, dx\\ &= \langle \rho v_x, w_x\rangle_{\mathcal{D}',\mathcal{D}} = -\big\langle (\rho v_x)_x, w\big\rangle_{\mathcal{D}',\mathcal{D}}.\end{aligned}$$

Then $Av = -(\rho v_x)_x$ in $\mathcal{D}'$. Since $Av \in L^2(I)$, $(\rho v_x)_x \in L^2(I)$ and $v \in D$. This proves that $D(A) \subset D$. Moreover we have shown that $Av = -(\rho v_x)_x$ for all $v \in D(A)$.

It now remains to show that $D \subset D(A)$. Let $v \in D$ and $w \in V$. Then

$$a(v, w) = \int_I \rho v_x w_x\, dx.$$

We wish to integrate the above expression by parts. Therefore we need to know the boundary values of $\rho v_x w$ at $x = -1$ and $x = 1$.

Let us first prove that $(\rho v_x)(1) = (\rho v_x)(-1) = 0$. Since $\rho v_x \in H^1(I)$, we have $\rho v_x \in \mathcal{C}^0([-1, 1])$. Therefore the quantity $|\rho v_x|$ has a limit as $x \to 1$ denoted by $L \geqslant 0$. We argue by contraction assuming that $L > 0$. Then, for x close to 1, $|\rho v_x| \geqslant L/2$. Hence $|\sqrt{\rho} v_x| \geqslant \frac{L}{2\sqrt{\rho}}$ which contradicts the fact that $\sqrt{\rho} v_x \in L^2(I)$ since $1/\sqrt{\rho} \notin L^2(I)$. Finally $L = 0$. We prove the same way that $(\rho v_x)(-1) = 0$.

Next we prove that

$$\forall x \in I, \quad \big|(\rho v_x)(x)\big| \leqslant \big\|(\rho v_x)_x\big\|_{L^2(I)} \sqrt{1 - x}. \tag{6.1}$$

Indeed, using $(\rho v_x)(1) = 0$, it suffices to write for all $x \in (0, 1)$,

$$\big|(\rho v_x)(x)\big| = \left|\int_x^1 (\rho v_x)_x(s)\, ds\right| \leqslant \left(\int_x^1 \big|(\rho v_x)_x(s)\big|^2 ds\right)^{1/2} \left(\int_x^1 ds\right)^{1/2}.$$

Let us finally deduce that

$$(\rho v_x w)(x) \to 0 \quad \text{as } x \to 1 \text{ and } x \to -1.$$

Let $x_1, x_2 \in (0,1)$. Since $v \in D$, we have

$$\int_{-x_1}^{x_2} \sqrt{\rho} v_x \sqrt{\rho} w_x \, dx = -\int_{-x_1}^{x_2} (\rho v_x)_x w \, dx + [\rho v_x w]_{-x_1}^{x_2}.$$

Moreover the quantities

$$\lim_{x_2 \to 1} \int_{-x_1}^{x_2} \sqrt{\rho} v_x \sqrt{\rho} w_x \, dx \quad \text{and} \quad \lim_{x_2 \to 1} -\int_{-x_1}^{x_2} (\rho v_x)_x w \, dx$$

are finite. Therefore the limit

$$\lim_{x_2 \to 1} (\rho v_x w)(x_2)$$

exists and we denote it by L. We assume that $L \neq 0$. Then for x_2 close to 1, we get $|(\rho v_x w)(x_2)| > |L|/2$. Moreover, using (6.1), we have

$$\left|(\rho v_x w)(x_2)\right| \leqslant \left\|(\rho v_x)_x\right\|_{L^2(I)} \sqrt{1 - x_2} \left|w(x_2)\right|.$$

Hence $|w(x_2)| \geqslant C/\sqrt{1-x_2}$ for some $C > 0$. This contradicts the fact that $w \in L^2(I)$. Therefore $L = 0$. We prove the same way that $(\rho v_x w)(x) \to 0$ also as $x \to -1$.

We conclude that

$$a(v, w) = -\int_I (\rho v_x)_x w \, dx.$$

Then $w \mapsto a(v, w)$ is $L^2(I)$-continuous so that $v \in D(A)$. This ends the proof of Lemma 3.1. □

6.2. Proof of Lemma 3.4

Let us first prove that G is well-defined on $[0,T] \times V$ with values in $L^2(I)$. We denote $Q = rq$ and $Q_1 = \|Q\|_{L^\infty(\mathbb{R}\times I)}$. Next we consider $t \in [0,T]$ and $u \in V$ and we write

$$\begin{aligned}\left\|G(t,u)\right\|_{L^2(I)}^2 &= \int_I \left|\mathcal{R}_a(t,u) - \mathcal{R}_e(u)\right|^2 dx \leqslant 2\int_I Q(t,x)^2 \beta(u)^2 \, dx + 2\int_I \varepsilon(u)^2 u^8 \, dx \\ &\leqslant 4Q_1^2 \|\beta\|_{L^\infty(\mathbb{R})}^2 + 2\|\varepsilon\|_{L^\infty(\mathbb{R})}^2 \int_I u^8 \, dx \leqslant 4Q_1^2 \|\beta\|_{L^\infty(\mathbb{R})}^2 + C\|u\|_V^8,\end{aligned} \tag{6.2}$$

according to Lemma 3.3.

Now, let us prove that (3.2) holds. Fix $t \in [0,T]$ and $R > 0$ and consider u_1, u_2 in $B_V(0,R)$. Then

$$\begin{aligned}\left\|G(t,u_1) - G(t,u_2)\right\|_{L^2(I)}^2 &= \int_I \left|Q(t,x)\big(\beta(u_1) - \beta(u_2)\big) + \mathcal{R}_e(u_1) - \mathcal{R}_e(u_2)\right|^2 dx \\ &\leqslant 2Q_1^2 \left\|\beta'\right\|_{L^\infty(\mathbb{R})}^2 \int_I |u_1 - u_2|^2 \, dx + 2\int_I \left|\mathcal{R}_e(u_1) - \mathcal{R}_e(u_2)\right|^2 dx \\ &\leqslant C\|u_1 - u_2\|_V^2 + 2\int_I \left|\mathcal{R}_e(u_1) - \mathcal{R}_e(u_2)\right|^2 dx.\end{aligned}$$

To conclude the proof of (3.2), it remains to show that

$$\int_I \left|\mathcal{R}_e(u_1)-\mathcal{R}_e(u_2)\right|^2 dx \leqslant C\|u_1-u_2\|_V^2, \tag{6.3}$$

for some $C>0$. In this purpose, some standard computations lead to

$$\int_I \left|\mathcal{R}_e(u_1)-\mathcal{R}_e(u_2)\right|^2 dx \leqslant 3\int_I \left|\varepsilon(u_1)-\varepsilon(u_2)\right|^2 |u_1|^8\,dx + 3\int_I \varepsilon(u_2)^2|u_1-u_2|^2|u_1|^6\,dx + 3\int_I \varepsilon(u_2)^2|u_2|^2\left(|u_1|^3-|u_2|^3\right)^2 dx. \tag{6.4}$$

So it remains to estimate the three above right-hand side terms. Using the properties of ε set in Assumption 1, we have:

$$\int_I \left|\varepsilon(u_1)-\varepsilon(u_2)\right|^2|u_1|^8\,dx \leqslant \left\|\varepsilon'\right\|^2_{L^\infty(\mathbb{R})}\|u_1-u_2\|^2_{L^4(I)}\|u_1\|^8_{L^{16}(I)},$$

$$\int_I \varepsilon(u_2)^2|u_1-u_2|^2|u_1|^6\,dx \leqslant \|\varepsilon\|^2_{L^\infty(\mathbb{R})}\|u_1-u_2\|^2_{L^4(I)}\|u_1\|^6_{L^{12}(I)},$$

and

$$\begin{aligned}\int_I \varepsilon(u_2)^2|u_2|^2\left(|u_1|^3-|u_2|^3\right)^2 dx &\leqslant \|\varepsilon\|^2_{L^\infty(\mathbb{R})}\int_I |u_2|^2\left(|u_1|-|u_2|\right)^2\left(|u_1|^2+|u_1||u_2|+|u_2|^2\right)^2 dx\\ &\leqslant \|\varepsilon\|^2_{L^\infty(\mathbb{R})}\|u_1-u_2\|^2_{L^4(I)}\left(\int_I |u_2|^4\left(|u_1|^2+|u_1||u_2|+|u_2|^2\right)^4 dx\right)^{1/2}.\end{aligned}$$

Using Lemma 3.3 and that u_1 and u_2 belong to $B_V(0,R)$, we achieve the proof of (3.2).

To end the proof of Lemma 3.4, we finally show that (3.3) holds. For all $t,s\in[0,T]$, we write

$$\begin{aligned}\left\|G(t,u)-G(s,u)\right\|^2_{L^2(I)} &= \int_I \left|r(t)-r(s)\right|^2 q(x)^2\beta\big(u(x)\big)^2 dx\\ &\leqslant 2\left\|r'\right\|^2_{L^\infty(\mathbb{R})}\|q\|^2_{L^\infty(I)}\|\beta\|^2_{L^\infty(\mathbb{R})}|t-s|^2.\end{aligned}$$

This obviously implies the required inequality (3.3). □

6.3. *Proof of Theorem 3.3*

In order to prove Theorem 3.3, we first state a preliminary result.

Lemma 6.1. *Let $u\in V$. Then, for all $M\geqslant 0$, $(u-M)^+ := \sup(u-M,0)\in V$ and $(u+M)^- := \sup(-(u+M),0)\in V$. Moreover*

$$\text{for a.e. } x\in I,\quad \big((u-M)^+\big)_x(x) = \begin{cases} u_x(x) & \text{if } (u-M)(x)>0,\\ 0 & \text{if } (u-M)(x)\leqslant 0,\end{cases} \tag{6.5}$$

and

$$\text{for a.e. } x\in I,\quad \big((u+M)^-\big)_x(x) = \begin{cases} 0 & \text{if } (u+M)(x)>0,\\ -u_x(x) & \text{if } (u+M)(x)\leqslant 0.\end{cases} \tag{6.6}$$

Proof. Let us consider $u\in V$. For all $\eta>0$, $u\in H^1(-1+\eta,1-\eta)$. By [15, Proposition 6, p. 934], it follows that $(u-M)^+\in H^1(-1+\eta,1-\eta)$ and

$$\text{for a.e. } x\in(-1+\eta,1-\eta),\quad \big((u-M)^+\big)_x(x) = \begin{cases} u_x(x) & \text{if } (u-M)(x)>0,\\ 0 & \text{if } (u-M)(x)\leqslant 0.\end{cases} \tag{6.7}$$

Then

$$\int_{-1+\eta}^{1-\eta} \rho\left|\left((u-M)^+\right)_x\right|^2 dx = \int_{A_\eta} \rho u_x^2\, dx,$$

where $A_\eta := \{x \in (-1+\eta, 1-\eta):\ (u-M)(x) > 0\}$. Since the quantity $\int_{A_\eta} \rho u_x^2\, dx$ is bounded from above by $\int_I \rho u_x^2\, dx$ which does not depend on η, we get (passing to the limit as $\eta \to 0$):

$$\int_I \rho\left|\left((u-M)^+\right)_x\right|^2 dx \leqslant \int_I \rho u_x^2\, dx < +\infty.$$

Hence, $(u-M)^+ \in V$. Moreover, (6.5) follows from (6.7). Similar arguments apply to treat the case of $(u+M)^-$. □

Now we are ready to proceed with the proof of Theorem 3.3. We consider $u_0 \in D(A) \cap L^\infty(I)$ and M defined by (3.4). Let $t \in [0,T]$ be fixed. Multiplying the equation satisfied by u by $(u-M)^+$, we get thanks to Lemma 6.1 and to the boundary condition satisfied by u,

$$\int_I u_t(u-M)^+\, dx + \int_I \rho\left|\left((u-M)^+\right)_x\right|^2 dx = \int_I \left[Q\beta(u) - \varepsilon(u)u|u|^3\right](u-M)^+\, dx.$$

Moreover denoting $\mathcal{A} := \{x \in I:\ u(t,x) > M\}$, one has

$$\int_I \left[Q\beta(u) - \varepsilon(u)u|u|^3\right](u-M)^+\, dx = \int_{\mathcal{A}} \left[Q\beta(u) - \varepsilon(u)u|u|^3\right](u-M)\, dx.$$

For $x \in \mathcal{A}$,

$$Q\beta(u) - \varepsilon(u)u|u|^3 \leqslant \|Q\|_{L^\infty(\mathbb{R}\times I)}\|\beta\|_{L^\infty(\mathbb{R})} - \varepsilon_1 M^4 \leqslant 0,$$

thanks to our choice of M. As a consequence, for all $t \in [0,T]$, we get

$$\frac{1}{2}\frac{d}{dt}\int_I \left|(u-M)^+\right|^2 dx = \int_I u_t(u-M)^+\, dx \leqslant 0.$$

Thus $t \mapsto \|(u-M)^+(t)\|^2_{L^2(I)}$ is decreasing on $[0,T]$. Since $(u_0 - M)^+ \equiv 0$, we deduce that, for all $t \in [0,T]$ and for a.e. $x \in I$, $u(t,x) \leqslant M$.

In the same way, we multiply the equation satisfied by u by $(u+M)^-$ and we obtain

$$\frac{d}{dt}\int_I \left|(u+M)^-\right|^2 dx \leqslant 0.$$

Since $(u_0+M)^- \equiv 0$, we conclude that for all $t \in [0,T]$ and a.e. $x \in I$, $u(t,x) \geqslant -M$. Thus, the proof of Theorem 3.3 is achieved. □

6.4. Proof of Theorem 3.4

Let us consider $u_0 \in \mathcal{U}$. Introducing $z := u_t$, we observe that the following equation holds in the sense of distributions:

$$z_t - (\rho z_x)_x = z\left[r(t)q\beta'(u) - \mathcal{R}_e'(u)\right] + r'(t)q\beta(u).$$

Thus, in order to check that z satisfies (3.5), it remains to prove that z belongs to $L^2(0,T;V)$. In this purpose, we use the method of differential quotients (see [27] for instance).

Let us consider $0 < \delta < \frac{T}{2}$, $t \in (\delta, T - \delta)$ and $-\delta < s < \delta$. Observe that

$$\begin{cases} u_t(t+s) - (\rho u_x)_x(t+s) = Q(t+s)\beta\big(u(t+s)\big) - \mathcal{R}_e\big(u(t+s)\big), \\ u_t(t) - (\rho u_x)_x(t) = Q(t)\beta\big(u(t)\big) - \mathcal{R}_e\big(u(t)\big). \end{cases} \tag{6.8}$$

Let us define for all $t \in (\delta, T - \delta)$,

$$u^s(t) := \frac{u(t+s) - u(t)}{s}.$$

Then, for all $t \in (\delta, T - \delta)$, $u^s(t) \in V$ and we deduce from (6.8) that

$$\frac{\partial u^s}{\partial t}(t) - \frac{\partial}{\partial x}\left(\rho \frac{\partial u^s}{\partial x}\right)(t) = \frac{Q(t+s)\beta(u(t+s)) - Q(t)\beta(u(t))}{s} + \frac{\mathcal{R}_e(u(t)) - \mathcal{R}_e(u(t+s))}{s}. \tag{6.9}$$

Multiplying (6.9) by $u^s(t)$ and integrating by parts with respect to x, one gets, for all $t \in (\delta, T - \delta)$,

$$\left(\frac{\partial u^s}{\partial t}(t), u^s(t)\right)_{L^2(I)} + \int_I \rho \left(\frac{\partial u^s}{\partial x}(t)\right)^2 dx$$
$$= \int_I \left[\frac{Q(t+s)\beta(u(t+s)) - Q(t)\beta(u(t))}{s} + \frac{\mathcal{R}_e(u(t)) - \mathcal{R}_e(u(t+s))}{s}\right] u^s(t)\, dx.$$

Integrating over $(\delta, T - \delta)$, this leads to

$$\begin{aligned} &\frac{1}{2}\left\|u^s(T-\delta)\right\|^2_{L^2(I)} + \int_\delta^{T-\delta}\int_I \rho\left(\frac{\partial u^s}{\partial x}(t)\right)^2 dx\, dt \\ &\quad = \frac{1}{2}\left\|u^s(\delta)\right\|^2_{L^2(I)} \\ &\qquad + \int_\delta^{T-\delta}\int_I \left[\frac{Q(t+s)\beta(u(t+s)) - Q(t)\beta(u(t))}{s} + \frac{\mathcal{R}_e(u(t)) - \mathcal{R}_e(u(t+s))}{s}\right] u^s(t)\, dx\, dt. \end{aligned} \tag{6.10}$$

Yet,

$$\begin{aligned} &\int_\delta^{T-\delta}\int_I \frac{Q(t+s)\beta(u(t+s)) - Q(t)\beta(u(t))}{s} u^s(t)\, dx\, dt \\ &\quad = \int_\delta^{T-\delta}\int_I \left[\frac{(Q(t+s) - Q(t))\beta(u(t+s)) + Q(t)(\beta(u(t+s)) - \beta(u(t)))}{s}\right] u^s(t)\, dx\, dt \\ &\quad \leqslant \|\beta\|_{L^\infty(\mathbb{R})} \int_\delta^{T-\delta}\int_I q(x)\frac{|r(t+s) - r(t)|}{s}\left|u^s(t)\right| dx\, dt \\ &\qquad + \|Q\|_{L^\infty((0,T)\times I)}\left\|\beta'\right\|_{L^\infty(\mathbb{R})} \int_\delta^{T-\delta}\int_I \frac{|u(t+s) - u(t)|}{s}\left|u^s(t)\right| dx\, dt \\ &\quad \leqslant \|\beta\|_{L^\infty(\mathbb{R})} \int_\delta^{T-\delta}\int_I \left(\frac{1}{2}\|q\|^2_{L^\infty(I)}\left\|r'\right\|^2_{L^\infty(\mathbb{R})} + \frac{1}{2}\left|u^s(t)\right|^2\right) dx\, dt \\ &\qquad + \|Q\|_{L^\infty(\mathbb{R}\times I)}\left\|\beta'\right\|_{L^\infty(\mathbb{R})} \int_\delta^{T-\delta}\int_I \left|u^s(t)\right|^2 dx\, dt \end{aligned}$$

$$
\begin{aligned}
&\leqslant T\|\beta\|_{L^\infty(\mathbb{R})}\|q\|^2_{L^\infty(I)}\|r'\|^2_{L^\infty(\mathbb{R})}\\
&\quad+\left(\frac{1}{2}\|\beta\|_{L^\infty(\mathbb{R})}+\|Q\|_{L^\infty(\mathbb{R}\times I)}\|\beta'\|_{L^\infty(\mathbb{R})}\right)\int_\delta^{T-\delta}\int_I |u^s(t)|^2\,dx\,dt.
\end{aligned}
\tag{6.11}
$$

On the other hand, we have

$$
\begin{aligned}
&\mathcal{R}_e\big(u(t,x)\big)-\mathcal{R}_e\big(u(t+s,x)\big)\\
&\quad=\big[\varepsilon(u)u|u|^3\big](t,x)-\big[\varepsilon(u)u|u|^3\big](t+s,x)\\
&\quad=\big[\varepsilon(u)|u|^3\big](t,x)\big(u(t,x)-u(t+s,x)\big)+\big(\varepsilon(u)(t,x)-\varepsilon(u)(t+s,x)\big)|u|^3(t,x)u(t+s,x)\\
&\qquad+\varepsilon(u)(t+s,x)\big(\big[|u|^3\big](t,x)-\big[|u|^3\big](t+s,x)\big)u(t+s,x)\\
&\quad=-\big[\varepsilon(u)|u|^3\big](t,x)\big(u(t+s,x)-u(t,x)\big)-\big(\varepsilon(u)(t+s,x)-\varepsilon(u)(t,x)\big)|u|^3(t,x)u(t+s,x)\\
&\qquad-\varepsilon(u)(t+s,x)\big(|u|(t+s,x)-|u|(t,x)\big)\big(|u|^2(t+s,x)+|u|(t+s,x)|u|(t,x)+|u|^2(t,x)\big)u(t+s,x).
\end{aligned}
$$

Since $u_0\in\mathcal{U}$, Theorem 3.3 implies that for a.e. $(t,x)\in(0,T)\times(0,1)$, $|u(t,x)|\leqslant M$, where M is defined by (3.4). Using also the fact that ε is $\|\varepsilon'\|_{L^\infty(\mathbb{R})}$-Lipschitz, we deduce

$$
\big|\mathcal{R}_e\big(u(t,x)\big)-\mathcal{R}_e\big(u(t+s,x)\big)\big|\leqslant C\big|u(t+s,x)-u(t,x)\big|,
$$

for some $C>0$. It follows that

$$
\int_\delta^{T-\delta}\int_I \frac{\mathcal{R}_e(u(t,x))-\mathcal{R}_e(u(t+s,x))}{s}u^s(t)\,dx\,dt\leqslant C\int_\delta^{T-\delta}\int_I |u^s(t,x)|^2\,dx\,dt.
\tag{6.12}
$$

Hence, thanks to (6.11) and (6.12), (6.10) turns into

$$
\int_I \rho\left(\frac{\partial u^s}{\partial x}(t)\right)^2 dx\leqslant\frac{1}{2}\|u^s(\delta)\|^2_{L^2(I)}+C+C\int_\delta^{T-\delta}\int_I|u^s(t,x)|^2\,dx\,dt.
$$

As $u\in\mathcal{C}^1([0,T];L^2(I))$, we deduce

$$
\int_I \rho\left(\frac{\partial u^s}{\partial x}(t)\right)^2 dx\leqslant\frac{1}{2}\sup_{t\in[0,T]}\|u_t\|^2_{L^2(I)}+C+CT\sup_{t\in[0,T]}\|u_t\|^2_{L^2(I)}.
$$

Therefore the quantity $\int_\delta^{T-\delta}\int_I\rho(\frac{\partial u^s}{\partial x}(t))^2\,dx\,dt$ is bounded by a positive constant which does not depend on s. So there exists a subsequence u^s that weakly converges to some $v\in L^2(\delta,T-\delta;V)$ as $s\to 0$. Yet, u^s converges to z in the distribution sense. Therefore $z=v\in L^2(\delta,T-\delta;V)$, and

$$
\|z\|^2_{L^2(\delta,T-\delta;V)}\leqslant\liminf_{s\to+\infty}\|u^s\|^2_{L^2(\delta,T-\delta;V)}\leqslant C+\left(\frac{1}{2}+CT\right)\sup_{t\in[0,T]}\|u_t\|^2_{L^2(I)}.
$$

Since the above right-hand side does not depend on δ, $z\in L^2(0,T;V)$ and z is the solution of (3.5). □

6.5. Proof of Corollary 3.1

The main difficulty of the proof relies in the lack of coercivity of the bilinear form b. Therefore we first introduce some perturbed variational problem that is coercive:

$$
\begin{cases}
y\in W\big(0,T;V,V'\big),\\
\forall w\in V,\quad \big\langle y_t(t),w\big\rangle+b_1\big(t,y(t),w\big)=\big(r'(t)e^{-(\|\pi\|_{L^\infty((0,T)\times I)}+1)t}q\beta\big(u(t)\big),w\big)_{L^2(I)},\\
y(0)=-Au_0+G(0,u_0),
\end{cases}
\tag{6.13}
$$

where $b_1 : [0,T] \times V \times V \to \mathbb{R}$ is the coercive time-dependent bilinear form defined by: $\forall (t,v,w) \in [0,T] \times V \times V$,

$$b_1(t,v,w) = b(t,v,w) + \int_I \big(\|\pi\|_{L^\infty((0,T)\times I)} + 1\big) vw\,dx.$$

Note that for a.a. $t \in (0,T)$, $y(t) \in V$. Consider N defined by (3.6). According to Lemma 6.1, for a.a. $t \in (0,T)$, $(y(t)-N)^+ \in V$, so that we can write

$$\begin{aligned}&\frac{d}{dt}\big(y(t), \big(y(t)-N\big)^+\big)_{L^2(I)} + b_1\big(t, y(t), \big(y(t)-N\big)^+\big)\\ &\quad = \big(r'(t)e^{-(\|\pi\|_{L^\infty((0,T)\times I)}+1)t} q\beta\big(u(t)\big), \big(y(t)-N\big)^+\big)_{L^2(I)}.\end{aligned}$$

One can easily check that

$$\begin{aligned}&b_1\big(t, y(t), \big(y(t)-N\big)^+\big) - \big(r'(t)e^{-(\|\pi\|_{L^\infty((0,T)\times I)}+1)t} q\beta\big(u(t)\big), \big(y(t)-N\big)^+\big)_{L^2(I)}\\ &\quad = \int_I \rho\big|\big(\big(y(t)-N\big)^+\big)_x\big|^2\,dx + \int_I \big(\|\pi\|_{L^\infty((0,T)\times I)} + \pi(t,x)\big)\big|\big(y(t)-N\big)^+\big|^2\,dx\\ &\qquad + \int_I \big(y(t) - r'(t)e^{-(\|\pi\|_{L^\infty((0,T)\times I)}+1)t} q\beta\big(u(t)\big)\big)\big(y(t)-N\big)^+\,dx \geqslant 0\end{aligned}$$

since $y(t) - r'(t)e^{-(\|\pi\|_{L^\infty((0,T)\times I)}+1)t} q\beta(u) \geqslant 0$ when $y(t) \geqslant N$. Therefore

$$\frac{d}{dt}\big(y(t), \big(y(t)-N\big)^+\big)_{L^2(I)} \leqslant 0.$$

As a consequence, the function $t \mapsto \|(y(t)-N)^+\|_{L^2(I)}$ is decreasing on $[0,T]$. Since $\|(y(0)-N)^+\|_{L^2(I)} = 0$, we get $\|(y(t)-N)^+\|_{L^2(I)} = 0$ for a.a. $t \in (0,T)$. Then for a.a. $(t,x) \in (0,T) \times I$, $y(t,x) \leqslant N$. For the same reasons, for a.a. $(t,x) \in (0,T) \times I$, $-y(t,x) \leqslant N$. Finally $\|y\|_{L^\infty((0,T)\times I)} \leqslant N$. As the solution z of problem (3.5) satisfies, for all $t \in (0,T)$, $z(t) = e^{(\|\pi\|_{L^\infty((0,T)\times I)}+1)t} y(t)$, we immediately have

$$\|z\|_{L^\infty((0,T)\times I)} \leqslant e^{(\|\pi\|_{L^\infty((0,T)\times I)}+1)T} N. \qquad \square$$

6.6. *Proof of Theorem 3.5*

Let us assume that $T^\star(u_0) < +\infty$. According to the remark after Theorem 3.3, it only remains to prove that the V-norm of the local solution $u(t)$ does not blow up at time $T^\star(u_0)$. For a.e. $0 < t < T^\star(u_0)$, let us multiply the equation satisfied by u at time t by $-(\rho u_x)_x(t)$. We get:

$$-\int_I u_t(t)(\rho u_x)_x(t)\,dx + \int_I \big|(\rho u_x)_x(t)\big|^2\,dx = -\int_I \big[Q\beta(u) - \varepsilon(u)u|u|^3\big](t)(\rho u_x)_x(t)\,dx.$$

Since $u_0 \in \mathcal{U}$, Theorem 3.3 implies $\|u\|_{L^\infty((0,t)\times I)} \leqslant M$ where M is defined by (3.4). Therefore,

$$\begin{aligned}-\int_I \big[Q\beta(u) - \varepsilon(u)u|u|^3\big](t)(\rho u_x)_x(t)\,dx &\leqslant \frac{1}{2}\int_I \big[Q\beta(u) - \varepsilon(u)u|u|^3\big]^2(t)\,dx + \frac{1}{2}\int_I \big|(\rho u_x)_x(t)\big|^2\,dx\\ &\leqslant \|Q\|^2_{L^\infty(\mathbb{R}\times I)}\|\beta\|^2_{L^\infty(\mathbb{R})} + \|\varepsilon\|^2_{L^\infty(\mathbb{R})}M^8 + \frac{1}{2}\int_I \big|(\rho u_x)_x(t)\big|^2\,dx.\end{aligned}$$

Thus, we get

$$-\int_I u_t(t)(\rho u_x)_x(t)\,dx + \frac{1}{2}\int_I \big|(\rho u_x)_x(t)\big|^2\,dx \leqslant \|Q\|^2_{L^\infty(\mathbb{R}\times I)}\|\beta\|^2_{L^\infty(\mathbb{R})} + \|\varepsilon\|^2_{L^\infty(\mathbb{R})}M^8 = C. \tag{6.14}$$

Moreover, according to Theorem 3.4, $u_t \in L^2(0,T;V)$ for all $0 < T < T^\star(u_0)$, so that for a.e. s

$$-\int_I u_t(s)(\rho u_x)_x(s)\,dx = \int_I \sqrt{\rho}u_{tx}(s)\sqrt{\rho}u_x(s)\,dx = \frac{1}{2}\frac{d}{dt}\int_I \rho\left|u_x(s)\right|^2 dx. \tag{6.15}$$

Therefore, integrating over $(0,t)$,

$$\frac{1}{2}\int_I \rho\left|u_x(t)\right|^2 dx \leqslant Ct + \frac{1}{2}\int_I \rho|u_{0,x}|^2\,dx \leqslant CT^\star(u_0) + \frac{1}{2}\|u_0\|_V^2. \tag{6.16}$$

According to Theorem 3.3, we also have $\|u(t)\|_{L^2(I)} \leqslant \sqrt{2}M$ where M is given by (3.4). Therefore, (6.16) leads to

$$\left\|u(t)\right\|_V^2 \leqslant \sqrt{2}M + CT^\star(u_0) + \frac{1}{2}\|u_0\|_V^2,$$

which ensures that $\|u(t)\|_V$ does not blow up at time $T^\star(u_0)$. □

7. Proofs related to Lipschitz stability

7.1. Proof of Theorem 4.1

For reader's convenience, the proof is divided in several steps and the technical points are postponed to the end of the proof (see Sections 7.2–7.5).

Step 1: reduction to some linear inverse problem. Let $T > 0$ and u_1 and u_2 belonging to $\mathcal{C}([0,T]; D(A)) \cap \mathcal{C}^1([0,T]; L^2(I))$ be the solutions of (2.16) respectively associated to q_1 and q_2. Let us define $w := u_1 - u_2$. Then $w \in \mathcal{C}([0,T]; D(A)) \cap \mathcal{C}^1([0,T]; L^2(I))$ and the calculations below are justified:

$$\begin{aligned}
w_t &= u_{1,t} - u_{2,t}\\
&= (\rho u_{1,x})_x - (\rho u_{2,x})_x + rq_1\beta(u_1) - rq_2\beta(u_2) + \varepsilon(u_2)u_2|u_2|^3 - \varepsilon(u_1)u_1|u_1|^3\\
&= (\rho w_x)_x + r(q_1-q_2)\beta(u_1) + rq_2\big(\beta(u_1)-\beta(u_2)\big) + \varepsilon(u_2)u_2|u_2|^3 - \varepsilon(u_1)u_1|u_1|^3\\
&= (\rho w_x)_x + h_1 + h_2 + h_3,
\end{aligned}$$

where

$$\begin{aligned}
h_1 &:= r(q_1-q_2)\beta(u_1),\\
h_2 &:= rq_2\big(\beta(u_1)-\beta(u_2)\big),\\
h_3 &:= \varepsilon(u_2)u_2|u_2|^3 - \varepsilon(u_1)u_1|u_1|^3.
\end{aligned}$$

It follows that $w \in \mathcal{C}([0,T]; D(A)) \cap \mathcal{C}^1([0,T]; L^2(I))$ is the solution of

$$\begin{cases}
w_t - (\rho w_x)_x = h_1 + h_2 + h_3, & (t,x) \in (0,T)\times I,\\
\rho(x)w_x(t,x) = 0, & (t,x) \in (0,T)\times \partial I,\\
w(0,x) = 0, & x \in I.
\end{cases} \tag{7.1}$$

Since the functions r and β are bounded from below (see Assumption 1), estimating h_1 will be sufficient to estimate the quantity $q_1 - q_2$. So we have reduced our inverse problem into the determination of h_1 in the *linear* initial–boundary value problem (7.1). In [13], the authors treated the problem of the determination of a source term in degenerate equations similar to (7.1). However we cannot directly apply the result of [13] here for several reasons:

- First, the degenerate diffusion coefficient $\rho(x)$ is more complicated than the one studied in [13]. The diffusion coefficient in [13] takes the form x^α for $x \in (0,1)$ and $\alpha \in [0,2)$. In particular it vanishes only at one extremity of the domain $(0,1)$ and has the form of a power function. Here the diffusion coefficient ρ is more general and vanishes at both extremities of the domain. However this difficulty can be overcome using the Carleman estimates stated in Theorem 4.2.

– Next the source term $h_1 + h_2 + h_3$ does not necessarily satisfy the standard assumption that is generally required on source terms, see for example in [22,13]. However we will see in Step 2 that the part h_1 that we need to recover satisfies such assumption and this point will be essential to adapt the proofs in [22,13] to the present case.
– Finally and above all, we need here to recover the quantity h_1 which is only a part of the source term $h_1+h_2+h_3$. In Step 3, we will show that in some way we can get rid of $h_2 + h_3$ in (7.1) so that our inverse problem is transformed into a problem that is more similar to the one studied in [22,13]. To do that, we will use the Carleman estimates of Theorem 4.2.

Step 2: condition satisfied by h_1. Let us recall that in inverse source problems, the source term has to satisfy some condition otherwise uniqueness may be false. Let $C_0 > 0$ be given. In [22,13], the authors make the assumption that source terms h satisfy the condition

$$\left|\frac{\partial h}{\partial t}(t,x)\right| \leqslant C_0\left|h\left(T',x\right)\right| \quad \text{for almost all (a.a.) } (t,x)\in(0,T)\times I. \tag{7.2}$$

Therefore they define the set $\mathcal{G}(C_0)$ of admissible source terms as

$$\mathcal{G}(C_0) := \left\{h\in H^1\left(0,T;L^2(I)\right) \;\middle|\; h \text{ satisfies (7.2)}\right\}.$$

Coming back to (7.1), we prove that the part h_1 of the source term $h_1 + h_2 + h_3$ (which is the part that we wish to identify) satisfies the above essential condition:

Lemma 7.1. *The quantity* $h_1 = r(q_1 - q_2)\beta(u_1)$ *belongs to* $\mathcal{G}(C_0)$ *for* $C_0 > 0$ *defined by*

$$C_0 := \frac{\|r'\|_{L^\infty(\mathbb{R})}\|\beta\|_{L^\infty(\mathbb{R})} + \|r\|_{L^\infty(\mathbb{R})}\|\beta'\|_{L^\infty(\mathbb{R})}e^{(\|\pi\|_{L^\infty((0,T)\times I)}+1)T}N}{\beta_1 r(T')},$$

where N *is defined by* (3.6).

The proof of Lemma 7.1 is given later in Section 7.2.

Step 3: application of global Carleman estimates and link with some more standard inverse source problem. In the following computations, C stands for a generic constant depending on T, t_0, D, ω and the parameters set in Assumption 1. Let us now introduce $z := w_t = u_{1,t} - u_{2,t}$ where w is the solution of (7.1). Then, using standard regularization results for linear parabolic equations (see [13, Lemma 2.2]), $z\in L^2(t_0,T;D(A))\cap H^1(t_0,T;L^2(I))$ and satisfies

$$\begin{cases} z_t - (\rho z_x)_x = h_{1,t} + h_{2,t} + h_{3,t}, & (t,x)\in(t_0,T)\times I,\\ \rho(x)z_x(t,x) = 0, & (t,x)\in(t_0,T)\times\partial I. \end{cases} \tag{7.3}$$

Applying the Carleman estimate of Theorem 4.2 to problem (7.3), we get:

$$\begin{aligned} I_0 &:= \iint\limits_{Q_T^{t_0}} \left(R^3\theta^3\left(1-x^2\right)z^2 + R\theta^{3/2}|\gamma|pz^2 + R\theta\left(1-x^2\right)z_x^2 + \frac{1}{R\theta}z_t^2\right)e^{-2R\sigma}\,dx\,dt \\ &\leqslant C\left(\iint\limits_{Q_T^{t_0}} h_{1,t}^2 e^{-2R\sigma}\,dx\,dt + \iint\limits_{Q_T^{t_0}}\left(h_{2,t}^2 + h_{3,t}^2\right)e^{-2R\sigma}\,dx\,dt + \iint\limits_{\omega_T^{t_0}} R^3\theta^3 z^2 e^{-2R\sigma}\,dx\,dt\right). \end{aligned} \tag{7.4}$$

Inequality (7.4) is the first step when dealing with standard inverse source problems, see [22,13]. Here our problem consists in retrieving only the part h_1 of the source term $h_1 + h_2 + h_3$. Hence our goal is now to absorb the term $\iint_{Q_T^{t_0}}(h_{2,t}^2 + h_{3,t}^2)e^{-2R\sigma}\,dx\,dt$ into the left-hand side of (7.4). In that purpose, we state the following fundamental lemma:

Lemma 7.2. *There exists a positive constant $C > 0$ such that*

$$\iint_{Q_T^{t_0}} \left(h_{2,t}^2 + h_{3,t}^2\right) e^{-2R\sigma}\,dx\,dt \leqslant C\left(\iint_{Q_T^{t_0}} z^2 e^{-2R\sigma}\,dx\,dt + \int_I w\left(T',x\right)^2 dx\right). \tag{7.5}$$

The proof of Lemma 7.2 is given later in Section 7.3.

We also need to estimate the term $\iint_{Q_T^{t_0}} z^2 e^{-2R\sigma}\,dx\,dt$. In this purpose, we apply some Hardy-type inequalities to prove:

Lemma 7.3.

$$\iint_{Q_T^{t_0}} z^2 e^{-2R\sigma}\,dx\,dt \leqslant C\left(\iint_{Q_T^{t_0}} \left(1-x^2\right)\left(z_x^2 + R^2\theta^2 z^2\right) e^{-2R\sigma}\,dx\,dt + \iint_{\omega_T^{t_0}} z^2 e^{-2R\sigma}\,dx\,dt\right).$$

The proof of Lemma 7.3 is given later in Section 7.4. Let us mention that Hardy inequalities have been largely used when dealing with degenerate parabolic equations not only for controllability matters [9,31] but also for inverse problems [13]. Here again, Hardy-type inequalities appear to be an essential tool to solve our problem.

From Lemmas 7.2 and 7.3, we deduce

$$\begin{aligned}\iint_{Q_T^{t_0}} \left(h_{2,t}^2 + h_{3,t}^2\right) e^{-2R\sigma}\,dx\,dt \leqslant C\Bigg(&\iint_{Q_T^{t_0}} \left[\left(1-x^2\right)z_x^2 + R^2\theta^2\left(1-x^2\right)z^2\right] e^{-2R\sigma}\,dx\,dt \\ &+ \int_I w^2\left(T',x\right) dx + \iint_{\omega_T^{t_0}} z^2 e^{-2R\sigma}\,dx\,dt\Bigg).\end{aligned}$$

Coming back to (7.4), we get:

$$\begin{aligned}I_0 \leqslant C\Bigg(&\iint_{Q_T^{t_0}} h_{1,t}^2 e^{-2R\sigma}\,dx\,dt + \iint_{Q_T^{t_0}} \left(1-x^2\right) z_x^2 e^{-2R\sigma}\,dx\,dt + \iint_{Q_T^{t_0}} R^2\theta^2\left(1-x^2\right) z^2 e^{-2R\sigma}\,dx\,dt \\ &+ \int_I w\left(T',x\right)^2 dx + \iint_{\omega_T^{t_0}} R^3\theta^3 z^2 e^{-2R\sigma}\,dx\,dt + \iint_{\omega_T^{t_0}} z^2 e^{-2R\sigma}\,dx\,dt\Bigg).\end{aligned} \tag{7.6}$$

For all $t \in (t_0, T)$, $1 \leqslant C\theta(t)$ and $\theta^2(t) \leqslant C\theta^3(t)$, so that, for R large,

$$C \iint_{Q_T^{t_0}} \left(1-x^2\right) z_x^2 e^{-2R\sigma}\,dx\,dt \leqslant \frac{1}{2} \iint_{Q_T^{t_0}} R\theta\left(1-x^2\right) z_x^2 e^{-2R\sigma}\,dx\,dt,$$

and

$$C \iint_{Q_T^{t_0}} R^2\theta^2\left(1-x^2\right) z^2 e^{-2R\sigma}\,dx\,dt \leqslant \frac{1}{2} \iint_{Q_T^{t_0}} R^3\theta^3\left(1-x^2\right) z^2 e^{-2R\sigma}\,dx\,dt.$$

As a conclusion, one can absorb these two right-hand side terms into I_0 and there exist $R_1 > 0$ and $C > 0$, such that $\forall R \geqslant R_1$, the following estimate holds:

$$I_0 \leqslant C\underbrace{\left(\iint_{Q_T^{t_0}} h_{1,t}^2 e^{-2R\sigma}\,dx\,dt + \int_I w\left(T',x\right)^2 dx + \iint_{\omega_T^{t_0}} R^3\theta^3 z^2 e^{-2R\sigma}\,dx\,dt\right)}_{:=I_1}. \tag{7.7}$$

Let us note that, without the term $\int_I w(T',x)^2\,dx$, the inequality (7.7) would be the kind of estimate that one would obtain when dealing with the more standard inverse problem that consists in retrieving the source term h_1 in the equation $w_t - (\rho w_x)_x = h_1$. Let us also observe that this extra term satisfies

$$\int_I w\big(T',x\big)^2\,dx = \big\|(u_1-u_2)\big(T',\cdot\big)\big\|^2_{L^2(I)} \leqslant \big\|(u_1-u_2)\big(T',\cdot\big)\big\|^2_{D(A)}.$$

Therefore it can easily be estimated by the right-hand side of (4.1). Hence the next step mainly consists in adapting the reasoning of [13] to the present case, taking into account this extra term and the fact that the considered degeneracy is not the same.

Step 4: estimate from above of I_1. In this step, our purpose is to show that there exists some constant $C>0$ such that

$$I_1 \leqslant C\left[\frac{1}{\sqrt{R}}\int_I h_1\big(T',x\big)^2 e^{-2R\sigma(T',x)}\,dx + \|w_t\|^2_{L^2(\omega_T^{t_0})} + \big\|w\big(T',.\big)\big\|^2_{L^2(I)}\right]. \tag{7.8}$$

Let us recall that, by Lemma 4.1, there exists some constant $p_1>0$ such that $p(x)\geqslant p_1$ for all $x\in I$. Therefore we have

$$R^3\theta(t)^3e^{-2R\sigma(t,x)} \leqslant R^3\theta(t)^3e^{-2Rp_1\theta(t)} \to 0 \quad \text{as } t\to t_0 \text{ or } T.$$

As a consequence, setting $C=\sup_{\mathbb{R}}\{y\mapsto y^3e^{-2Rp_1y}\}>0$, we have

$$\iint_{\omega_T^{t_0}} R^3\theta^3z^2e^{-2R\sigma}\,dx\,dt \leqslant C\|z\|^2_{L^2(\omega_T^{t_0})} = C\|w_t\|^2_{L^2(\omega_T^{t_0})}. \tag{7.9}$$

In order to complete the proof of (7.8), it remains to prove the following lemma:

Lemma 7.4. *There exists some constant $C>0$ such that*

$$\iint_{Q_T^{t_0}} h_{1,t}^2e^{-2R\sigma}\,dx\,dt \leqslant C\frac{1}{\sqrt{R}}\int_I h_1\big(T',x\big)^2e^{-2R\sigma(T',x)}\,dx. \tag{7.10}$$

We omit the proof of Lemma 7.4 which is classical and we refer the reader to [22]. Using (7.9) and (7.10), we obtain (7.8).

Step 5: estimate from below of I_0. The purpose of the step is to provide the following estimate: there exists a constant $C=C(T,t_0)>0$ such that

$$\int_I z\big(T',x\big)^2e^{-2R\sigma(T',x)}\,dx \leqslant CI_0 + C\iint_{\omega_T^{t_0}} z^2e^{-2R\sigma}\,dx\,dt. \tag{7.11}$$

Since $z(t,x)^2e^{-2R\sigma(t,x)}\to 0$ as $t\to t_0$ for a.a. $x\in I$, we can write

$$\begin{aligned}\int_I z\big(T',x\big)^2e^{-2R\sigma(T',x)}\,dx &= \int_{t_0}^{T'}\frac{\partial}{\partial t}\left(\int_I z(t,x)^2e^{-2R\sigma(t,x)}\,dx\right)dt\\ &= \int_{t_0}^{T'}\int_I \big[2zz_t - 2R\sigma_tz^2\big]e^{-2R\sigma}\,dx\,dt. \end{aligned}\tag{7.12}$$

First we estimate

$$\int_{t_0}^{T'}\int_I 2zz_t e^{-2R\sigma}\,dx\,dt = \int_{t_0}^{T'}\int_I 2\sqrt{R\theta}\,ze^{-R\sigma}\frac{1}{\sqrt{R\theta}}z_t e^{-R\sigma}\,dx\,dt$$
$$\leqslant \int_{t_0}^{T'}\int_I \left(R\theta z^2 e^{-2R\sigma} + \frac{z_t^2}{R\theta}e^{-2R\sigma}\right)dx\,dt$$
$$\leqslant \int_{t_0}^{T'} R\theta\left(\int_I \left(ze^{-R\sigma}\right)^2 dx\right)dt + CI_0.$$

With a proof similar to the proof of Lemma 7.3, we get

$$\int_{t_0}^{T'}\int_I 2zz_t e^{-2R\sigma}\,dx\,dt \leqslant C\left(\int_{t_0}^{T'}\int_I R\theta\left(1-x^2\right)\left(z_x^2 + R^2\theta^2 z^2\right)e^{-2R\sigma}\,dx\,dt + \iint_{\omega_T^{t_0}} z^2 e^{-2R\sigma}\,dx\,dt + I_0\right),$$

so that,

$$\int_{t_0}^{T'}\int_I 2zz_t e^{-2R\sigma}\,dx\,dt \leqslant C\left(I_0 + \iint_{\omega_T^{t_0}} z^2 e^{-2R\sigma}\,dx\,dt\right). \tag{7.13}$$

Next we estimate

$$\int_{t_0}^{T'}\int_I 2R|\sigma_t|z^2 e^{-2R\sigma}\,dx\,dt = \int_{t_0}^{T'}\int_I 2R|\theta_t|pz^2 e^{-2R\sigma}\,dx\,dt$$
$$\leqslant C\int_{t_0}^{T'}\int_I R\theta^{3/2}|\gamma|z^2 e^{-2R\sigma}\,dx\,dt \leqslant CI_0, \tag{7.14}$$

where we used the fact that $|\theta_t(t)| = |-4\theta(t)^{4/5}\gamma(t)| \leqslant C\theta(t)^{3/2}$.

Eventually, (7.12) associated to (7.13) and (7.14) gives (7.11).

Step 6: conclusion. Using (7.11), (7.7) and next (7.8), there exists some constant $C > 0$ such that

$$\int_I z\left(T',x\right)^2 e^{-2R\sigma(T',x)}\,dx \leqslant C\frac{1}{\sqrt{R}}\int_I h_1\left(T',x\right)^2 e^{-2R\sigma(T',x)}\,dx + C\|w_t\|^2_{L^2(\omega_T^{t_0})} + C\left\|w\left(T',.\right)\right\|^2_{L^2(I)}. \tag{7.15}$$

On the other hand, let us recall that

$$z\left(T',x\right) = w_t\left(T',x\right) = (\rho w_x)_x\left(T',x\right) + h_1\left(T',x\right) + h_2\left(T',x\right) + h_3\left(T',x\right).$$

Therefore,

$$\int_I h_1\left(T',x\right)^2 e^{-2R\sigma(T',x)}\,dx \leqslant C\left(\int_I z\left(T',x\right)^2 e^{-2R\sigma(T',x)}\,dx + \int_I \left|(\rho w_x)_x\left(T',x\right)\right|^2 e^{-2R\sigma(T',x)}\,dx\right.$$
$$\left. + \int_I h_2\left(T',x\right)^2 e^{-2R\sigma(T',x)}\,dx + \int_I h_3\left(T',x\right)^2 e^{-2R\sigma(T',x)}\,dx\right).$$

Applying (7.15) to estimate the term $\int_I z(T',x)^2 e^{-2R\sigma(T',x)}\,dx$, we get

$$\int_I h_1(T',x)^2 e^{-2R\sigma(T',x)}\,dx \leqslant C\biggl(\frac{1}{\sqrt{R}}\int_I h_1(T',x)^2 e^{-2R\sigma(T',x)}\,dx + \|w_t\|^2_{L^2(\omega_T^{t_0})} + \bigl\|w(T',.)\bigr\|^2_{D(A)} + \int_I h_2(T',x)^2 e^{-2R\sigma(T',x)}\,dx + \int_I h_3(T',x)^2 e^{-2R\sigma(T',x)}\,dx\biggr).$$

Choosing R large enough such that $C/\sqrt{R} = 1/2$, we get

$$\frac{1}{2}\int_I h_1(T',x)^2 e^{-2R\sigma(T',x)}\,dx \leqslant C\biggl(\|w_t\|^2_{L^2(\omega_T^{t_0})} + \bigl\|w(T',.)\bigr\|^2_{D(A)} + \int_I h_2(T',x)^2 e^{-2R\sigma(T',x)}\,dx + \int_I h_3(T',x)^2 e^{-2R\sigma(T',x)}\,dx\biggr). \tag{7.16}$$

Let us now estimate the two last terms of the right-hand side of (7.16). First we recall that $|h_2| = |rq_2(\beta(u_1) - \beta(u_2))| \leqslant \|r\|_{L^\infty(\mathbb{R})} D\|\beta'\|_{L^\infty(\mathbb{R})}|u_1 - u_2|$. Therefore

$$\int_I h_2(T',x)^2 e^{-2R\sigma(T',x)}\,dx \leqslant C\int_I w(T',x)^2 e^{-2R\sigma(T',x)}\,dx \leqslant C\bigl\|w(T',\cdot)\bigr\|^2_{L^2(I)}. \tag{7.17}$$

Next we write

$$\begin{aligned}
|h_3| &= \bigl|\varepsilon(u_2)u_2|u_2|^3 - \varepsilon(u_1)u_1|u_1|^3\bigr| \\
&= \bigl|\bigl(\varepsilon(u_2) - \varepsilon(u_1)\bigr)u_2|u_2|^3 + \varepsilon(u_1)(u_2 - u_1)|u_2|^3 + \varepsilon(u_1)u_1\bigl(|u_2|^3 - |u_1|^3\bigr)\bigr| \\
&\leqslant \bigl\|\varepsilon'\bigr\|_{L^\infty(\mathbb{R})}|u_2 - u_1||u_2|^4 + \|\varepsilon\|_{L^\infty(\mathbb{R})}|u_2 - u_1||u_2|^3 \\
&\quad + \|\varepsilon\|_{L^\infty(\mathbb{R})}|u_1|\bigl||u_2| - |u_1|\bigr|\bigl(|u_2|^2 + |u_2u_1| + |u_1|^2\bigr).
\end{aligned}$$

Recall that, thanks to Theorem 3.3, for $i = 1, 2$, $\|u_i\|_{L^\infty((0,T)\times I)} \leqslant C$. Hence,

$$|h_3| \leqslant C|u_2 - u_1| + C\bigl||u_2| - |u_1|\bigr| \leqslant C|u_2 - u_1|.$$

We deduce

$$\int_I h_3(T',x)^2 e^{-2R\sigma(T',x)}\,dx \leqslant C\bigl\|w(T',.)\bigr\|^2_{L^2(I)}. \tag{7.18}$$

Finally, putting (7.17) and (7.18) into (7.16), we get

$$\int_I h_1(T',x)^2 e^{-2R\sigma(T',x)}\,dx \leqslant C\bigl[\|w_t\|^2_{L^2(\omega_T^{t_0})} + \bigl\|w(T',.)\bigr\|^2_{D(A)}\bigr].$$

On the other hand, R being now fixed, there exists some constant $C_1 > 0$ such that $e^{-2R\sigma(T',x)} \geqslant C_1 > 0$. So we can write

$$\begin{aligned}
\int_I h_1(T',x)^2 e^{-2R\sigma(T',x)}\,dx &= \int_I r(t)^2\bigl|q_1(x) - q_2(x)\bigr|^2 \beta\bigl(u_1(T',x)\bigr)^2 e^{-2R\sigma(T',x)}\,dx \\
&\geqslant C_1 r_1^2 \beta_1^2 \|q_1 - q_2\|^2_{L^2(I)}.
\end{aligned}$$

Hence

$$\|q_1 - q_2\|^2_{L^2(I)} \leqslant C\bigl[\|w_t\|^2_{L^2(\omega_T^{t_0})} + \bigl\|w(T',.)\bigr\|^2_{D(A)}\bigr],$$

which concludes the proof. $\square$

7.2. Proof of Lemma 7.1

For a.a. $(t,x) \in (0,T) \times I$, we can write

$$\left|\frac{\partial h_1}{\partial t}(t,x)\right| \leqslant \left|r'(t)\big(q_1(x)-q_2(x)\big)\beta\big(u_1(t,x)\big)\right| + \left|r(t)\big(q_1(x)-q_2(x)\big)\beta'\big(u_1(t,x)\big)u_{1,t}(t,x)\right|$$
$$\leqslant \|r'\|_{L^\infty(\mathbb{R})}\left|q_1(x)-q_2(x)\right|\|\beta\|_{L^\infty(\mathbb{R})} + \|r\|_{L^\infty(\mathbb{R})}\left|q_1(x)-q_2(x)\right|\left\|\beta'\right\|_{L^\infty(\mathbb{R})}\left|u_{1,t}(t,x)\right|.$$

Since $u_0 \in \mathcal{U}$, Corollary 3.1 implies that for a.a. $(t,x) \in (0,T) \times I$,

$$\left|u_{1,t}(t,x)\right| \leqslant e^{(\|\pi\|_{L^\infty((0,T)\times I)}+1)T} N,$$

where N is defined by (3.6). Therefore

$$\left|\frac{\partial h_1}{\partial t}(t,x)\right| \leqslant C_0'\left|q_1(x)-q_2(x)\right|,$$

for $C_0' := \|r'\|_{L^\infty(\mathbb{R})}\|\beta\|_{L^\infty(\mathbb{R})} + \|r\|_{L^\infty(\mathbb{R})}\|\beta'\|_{L^\infty(\mathbb{R})}e^{(\|\pi\|_{L^\infty((0,T)\times I)}+1)T}N$. Since β satisfies $\beta(\cdot) \geqslant \beta_1$, we get

$$\left|\frac{\partial h_1}{\partial t}(t,x)\right| \leqslant \frac{C_0'}{\beta_1}\left|q_1(x)-q_2(x)\right|\beta\big(u_1\big(T',x\big)\big).$$

Finally, as for a.a. $x \in I$, $h_1(T',x) = r(T')(q_1(x)-q_2(x))\beta(u_1(T',x))$, Lemma 7.1 is proved. □

7.3. Proof of Lemma 7.2

Let us first mention that the proof will require the following technical lemma, which is classical in the theory of inverse problems (see [37]):

Lemma 7.5. *There exists a positive constant $C > 0$ such that, for all $z \in L^2(Q_T^{t_0})$,*

$$\iint_{Q_T^{t_0}} \left|\int_t^{T'} z(\tau,x)\,d\tau\right|^2 e^{-2R\sigma(t,x)}\,dx\,dt \leqslant C \iint_{Q_T^{t_0}} z(t,x)^2 e^{-2R\sigma(t,x)}\,dx\,dt.$$

For the reader convenience, we give a short proof of this lemma later in Section 7.5. Now we are ready to proceed with the proof of (7.5). Let us first estimate the term $\iint_{Q_T^{t_0}} h_{2,t}^2 e^{-2R\sigma}\,dx\,dt$. We recall that $h_2 = rq_2(\beta(u_1)-\beta(u_2))$. Therefore

$$\begin{aligned} h_{2,t} &= r'q_2\big(\beta(u_1)-\beta(u_2)\big) + rq_2\beta'(u_1)u_{1,t} - rq_2\beta'(u_2)u_{2,t} \\ &= r'q_2\big(\beta(u_1)-\beta(u_2)\big) + rq_2\beta'(u_1)[u_{1,t}-u_{2,t}] + rq_2\big(\beta'(u_1)-\beta'(u_2)\big)u_{2,t} \\ &= r'q_2\big(\beta(u_1)-\beta(u_2)\big) + rq_2\beta'(u_1)z + rq_2\big(\beta'(u_1)-\beta'(u_2)\big)u_{2,t}. \end{aligned}$$

Then,

$$\begin{aligned} \iint_{Q_T^{t_0}} h_{2,t}^2 e^{-2R\sigma}\,dx\,dt \leqslant{}& 3\iint_{Q_T^{t_0}} r'^2 q_2^2\big(\beta(u_1)-\beta(u_2)\big)^2 e^{-2R\sigma}\,dx\,dt \\ &+ 3\iint_{Q_T^{t_0}} r^2 q_2^2 \beta'(u_1)^2 z^2 e^{-2R\sigma}\,dx\,dt \\ &+ 3\iint_{Q_T^{t_0}} r^2 q_2^2\big(\beta'(u_1)-\beta'(u_2)\big)^2 u_{2,t}^2 e^{-2R\sigma}\,dx\,dt. \end{aligned}$$

Hence, using the fact that q_2 belongs to $\mathcal{Q}_D$ and the assumptions on r and β (Assumption 1), we get

$$\iint_{Q_T^{t_0}} h_{2,t}^2 e^{-2R\sigma}\,dx\,dt \leqslant 3\|r'\|_{L^\infty(\mathbb{R})}^2 D^2 \|\beta'\|_{L^\infty(\mathbb{R})}^2 \iint_{Q_T^{t_0}} w^2 e^{-2R\sigma}\,dx\,dt$$
$$+ 3\|r\|_{L^\infty(\mathbb{R})}^2 D^2 \|\beta'\|_{L^\infty(\mathbb{R})}^2 \iint_{Q_T^{t_0}} z^2 e^{-2R\sigma}\,dx\,dt$$
$$+ 3\|r\|_{L^\infty(\mathbb{R})}^2 D^2 k^2 \iint_{Q_T^{t_0}} w^2 u_{2,t}^2 e^{-2R\sigma}\,dx\,dt.$$

Now, applying Corollary 3.1 to $u_{2,t}$, we have $|u_{2,t}| \leqslant C$ for some $C > 0$. Therefore, we obtain:

$$\iint_{Q_T^{t_0}} h_{2,t}^2 e^{-2R\sigma}\,dx\,dt \leqslant C\left(\iint_{Q_T^{t_0}} z^2 e^{-2R\sigma}\,dx\,dt + \iint_{Q_T^{t_0}} w^2 e^{-2R\sigma}\,dx\,dt\right), \tag{7.19}$$

for some other constant $C > 0$.

Let us now estimate the second term $\iint_{Q_T^{t_0}} h_{3,t}^2 e^{-2R\sigma}\,dx\,dt$. We recall that $h_3 = \varepsilon(u_2)u_2|u_2|^3 - \varepsilon(u_1)u_1|u_1|^3$. Therefore one has

$$\begin{aligned} h_{3,t} &= u_{2,t}\varepsilon'(u_2)u_2|u_2|^3 - u_{1,t}\varepsilon'(u_1)u_1|u_1|^3 + 4\varepsilon(u_2)u_{2,t}|u_2|^3 - 4\varepsilon(u_1)u_{1,t}|u_1|^3 \\ &= (u_{2,t} - u_{1,t})\varepsilon'(u_2)u_2|u_2|^3 + u_{1,t}\big(\varepsilon'(u_2) - \varepsilon'(u_1)\big)u_2|u_2|^3 \\ &\quad + u_{1,t}\varepsilon'(u_1)(u_2 - u_1)|u_2|^3 + u_{1,t}\varepsilon'(u_1)u_1\big(|u_2|^3 - |u_1|^3\big) \\ &\quad + 4\varepsilon(u_2)(u_{2,t} - u_{1,t})|u_2|^3 + 4\big(\varepsilon(u_2) - \varepsilon(u_1)\big)u_{1,t}|u_2|^3 + 4\varepsilon(u_1)u_{1,t}\big(|u_2|^3 - |u_1|^3\big). \end{aligned}$$

Using the assumptions on ε and β (Assumption 1), we deduce

$$\begin{aligned} |h_{3,t}| &\leqslant |u_{2,t} - u_{1,t}|\|\varepsilon'\|_{L^\infty(\mathbb{R})}|u_2|^4 + |u_{1,t}|K|u_2 - u_1||u_2|^4 \\ &\quad + |u_{1,t}|\|\varepsilon'\|_{L^\infty(\mathbb{R})}|u_2 - u_1||u_2|^3 \\ &\quad + |u_{1,t}|\|\varepsilon'\|_{L^\infty(\mathbb{R})}|u_1|\big||u_2| - |u_1|\big|\big(|u_2|^2 + |u_2||u_1| + |u_1|^2\big) \\ &\quad + 4\|\varepsilon\|_{L^\infty(\mathbb{R})}|u_{2,t} - u_{1,t}||u_2|^3 + 4\|\varepsilon'\|_{L^\infty(\mathbb{R})}|u_2 - u_1||u_{1,t}||u_2|^3 \\ &\quad + 4\|\varepsilon\|_{L^\infty(\mathbb{R})}|u_{1,t}|\big||u_2| - |u_1|\big|\big(|u_2|^2 + |u_2||u_1| + |u_2|^2\big). \end{aligned}$$

From Theorem 3.3 and Corollary 3.1, we know that the quantities $|u_1|$, $|u_2|$, $|u_{1,t}|$ and $|u_{2,t}|$ are bounded. Therefore, recalling that $w = u_1 - u_2$ and $z = u_{1,t} - u_{2,t}$,

$$|h_{3,t}| \leqslant C|z| + C|w| + C\big||u_2| - |u_1|\big| \leqslant C|z| + C|w|.$$

We finally get

$$\iint_{Q_T^{t_0}} h_{3,t}^2 e^{-2R\sigma}\,dx\,dt \leqslant C\left(\iint_{Q_T^{t_0}} z^2 e^{-2R\sigma}\,dx\,dt + \iint_{Q_T^{t_0}} w^2 e^{-2R\sigma}\,dx\,dt\right). \tag{7.20}$$

Both in (7.19) and (7.20), we need to estimate the last term $\iint_{Q_T^{t_0}} w^2 e^{-2R\sigma}\,dx\,dt$ in order to conclude. In this purpose, we write for a.a. $(t,x) \in (t_0, T) \times I$,

$$w(t,x) = w(T',x) + \int_{T'}^{t} w_t(\tau,x)\,d\tau = w(T',x) + \int_{T'}^{t} z(\tau,x)\,d\tau.$$

Then, applying Lemma 7.5 and coming back to (7.19) and (7.20), one achieves the proof of the expected estimate (7.5). □

7.4. Proof of Lemma 7.3

Let us recall the Hardy inequalities stated in [31, Theorem 3.10] that will be used here:

Theorem 7.1. *Let $a : \bar{I} = [-1,1] \to \mathbb{R}$ be such that $a \in \mathcal{C}^1(\bar{I})$, $a(-1) = 0$, $a(1) = 0$ and assume there exist $\alpha_1, \alpha_2 \in (1,2)$ such that*

$$\frac{(1+x)a'(x)}{a(x)} \underset{x\to -1, x>-1}{\longrightarrow} \alpha_1, \qquad \frac{(1-x)a'(x)}{a(x)} \underset{x\to 1, x<1}{\longrightarrow} \alpha_2. \tag{7.21}$$

Let L_1, L_2 be given such that $-1 < L_1 < L_2 < 1$. Then there exists some constant $C = C(\alpha_1, \alpha_2, L_1, L_2) > 0$ such that, for all $Z \in L^2(I)$ satisfying $\sqrt{a} Z_x \in L^2(I)$,

$$\int_I \frac{a(x)}{(1+x)^2(1-x)^2} Z^2\,dx \leqslant C \int_I a(x) Z_x^2\,dx + C \int_{L_1}^{L_2} Z^2\,dx. \tag{7.22}$$

Let us mention that the statement of Theorem 7.1 slightly differs from [31, Theorem 3.10] since it has been adapted to the case of functions $a(x)$ defined on $[-1,1]$ instead of $[0,1]$. Typically Theorem 7.1 applies to functions $a(x)$ taking the form $a(x) = (1-x)^{\alpha_1}(1+x)^{\alpha_2}$ with $\alpha_1, \alpha_2 \in (1,2)$. It cannot directly be applied to the present weight function $1 - x^2 = (1-x)(1-x)$ since the critical exponents $\alpha_1 = 1$ and $\alpha_2 = 1$ are not covered by Theorem 7.1. Therefore we introduce some $\eta \in (1,2)$ and we define $a(x) = (1-x)^\eta (1+x)^\eta = (1-x^2)^\eta$ for all $x \in [-1,1]$. This function satisfies the assumptions required by Theorem 7.1. Besides we observe that any function $z \in V$ satisfies the assumption $\sqrt{a} z_x \in L^2(I)$ since $a(x) \leqslant (1-x^2)$ for all $x \in I$ (because $\eta \in (1,2)$). Therefore (7.22) leads to

$$\forall Z \in V, \quad \int_I \frac{a(x)}{(1+x)^2(1-x)^2} Z^2\,dx \leqslant C \int_I a(x) Z_x^2\,dx + C \int_{L_1}^{L_2} Z^2\,dx.$$

Since $a(x) \leqslant 1 - x^2$ and $a(x)/(1-x^2)^2 = 1/(1-x^2)^{2-\eta} \geqslant 1$ for all $x \in I$, we finally obtain

$$\forall Z \in V, \quad \int_I Z^2\,dx \leqslant C \int_I \left(1-x^2\right) Z_x^2\,dx + C \int_\omega Z^2\,dx. \tag{7.23}$$

Now we will apply (7.23) to $Z(t,\cdot) = z(t,\cdot)^{-R\sigma(t,\cdot)}$ for a.a. $t \in (0,T)$. First we check that, for a.a. $t \in (0,T)$, $Z(t,\cdot)$ belongs to V. Clearly we have $Z(t,\cdot) \in L^2(I)$. Moreover one computes

$$Z_x = z_x e^{-R\sigma} - R\sigma_x z e^{-R\sigma} = z_x e^{-R\sigma} - R\theta p_x z e^{-R\sigma}, \tag{7.24}$$

with

$$p_x(x) = -\frac{\phi_-(x)}{1-x^2} e^{S(\phi_+(x))^2}.$$

We recall that $\rho(x) = \rho_0(1-x^2)$ and $|\phi_-(x)| \leqslant C\rho(x)$. Thus there exists some constant $C > 0$ such that $|p_x(x)| \leqslant Ce^{S(\phi_+(x))^2}$. Since ϕ_+ is bounded and S is fixed, it follows that $|p_x(x)| \leqslant C$ for some other constant $C > 0$. Associated to (7.24), we deduce that $Z(t,\cdot) \in V$ for a.a. $t \in (0,T)$. As a consequence, (7.23) can be applied to $Z(t,\cdot) = z(t,\cdot)^{-R\sigma(t,\cdot)}$ and leads to

$$\iint_{Q_T^{t_0}} z^2 e^{-2R\sigma}\,dx\,dt \leqslant C \iint_{Q_T^{t_0}} \left(1-x^2\right)\left(z_x^2 + R^2\theta^2 p_x^2 z^2\right) e^{-2R\sigma}\,dx\,dt + C \iint_{\omega_T^{t_0}} z^2 e^{-2R\sigma}\,dx\,dt. \tag{7.25}$$

Finally, since p_x^2 is bounded, we obtain the result claimed in Lemma 7.3. □

7.5. *Proof of Lemma 7.5*

The idea is to split the domain of integration $Q_T^{t_0}$ into $(t_0, T') \times I$ and $(T', T) \times I$ so that the quantity $T' - t$ keeps being positive or negative on each sub-domain.

First, we carry out the integration over $(t_0, T') \times I$. By a standard Cauchy–Schwarz argument, we write

$$\int_I \int_{t_0}^{T'} \left| \int_t^{T'} z(\tau, x)\, d\tau \right|^2 e^{-2R\sigma(t,x)}\, dt\, dx \leqslant \int_I \int_{t_0}^{T'} \left(\int_t^{T'} d\tau \right) \left(\int_t^{T'} z(\tau, x)^2\, d\tau \right) e^{-2R\sigma(t,x)}\, dt\, dx$$
$$\leqslant (T - t_0) \int_I \int_{t_0}^{T'} \int_t^{T'} z(\tau, x)^2 e^{-2R\sigma(t,x)}\, d\tau dt\, dx.$$

Using the Fubini–Tonelli Theorem, we also can write:

$$(T - t_0) \int_I \int_{t_0}^{T'} \int_t^{T'} z(\tau, x)^2 e^{-2R\sigma(t,x)}\, d\tau\, dt\, dx = (T - t_0) \int_I \int_{t_0}^{T'} z(\tau, x)^2 \left(\int_{t_0}^{\tau} e^{-2R\sigma(t,x)}\, dt \right) d\tau\, dx.$$

Since the function θ is decreasing on the interval (t_0, T'), we get

$$(T - t_0) \int_I \int_{t_0}^{T'} z(\tau, x)^2 \left(\int_{t_0}^{\tau} e^{-2R\sigma(t,x)}\, dt \right) d\tau\, dx \leqslant (T - t_0)^2 \int_I \int_{t_0}^{T'} z(\tau, x)^2 e^{-2R\sigma(\tau,x)}\, d\tau\, dx.$$

Eventually, we have shown that

$$\int_I \int_{t_0}^{T'} \left| \int_t^{T'} z(\tau, x)\, d\tau \right|^2 e^{-2R\sigma(t,x)}\, dt\, dx \leqslant (T - t_0)^2 \int_I \int_{t_0}^{T'} z(\tau, x)^2 e^{-2R\sigma(\tau,x)}\, d\tau\, dx. \tag{7.26}$$

Next, we carry out the integration over $(T', T) \times I$ in a similar way and we get

$$\int_I \int_{T'}^{T} \left| \int_t^{T'} z(\tau, x)\, d\tau \right|^2 e^{-2R\sigma(t,x)}\, dt\, dx \leqslant (T - t_0)^2 \int_I \int_{T'}^{T} z(\tau, x)^2 e^{-2R\sigma(\tau,x)}\, d\tau\, dx. \tag{7.27}$$

Finally, using (7.26) and (7.27), the proof of Lemma 7.5 is complete. □

Let us mention that Lemma 7.5 is a variant of standard lemmas used in inverse issues such as for example in [25] with the difference that the present specific weight functions do not satisfy the same monotony hypotheses.

8. Proof related to Carleman estimates

In this section, our goal is to prove Theorem 4.2. Some part of the estimate (4.8) is already proved in [31] and, even if we refer to [31] a few times, our proof is quite self-contained. In [31], the authors prove a Carleman inequality that estimates the integrals of $R^3\theta^3(1-x^2)z^2$ and $R\theta(1-x^2)z_x^2$. Let $z \in L^2(t_0, T; D(A)) \cap H^1(t_0, T; L^2(I))$ be a solution of problem (4.7) and $R > 0$ be a positive number. As in [31], define $w(t, x) := e^{-R\sigma(t,x)}z(t, x)$ for a.a. $(t, x) \in Q_T^{t_0}$.

First of all, observe that w satisfies $P_R^+ w + P_R^- w = he^{-R\sigma}$ where

$$P_R^+ w = R\sigma_t w + R^2\rho\sigma_x^2 w + (\rho w_x)_x,$$
$$P_R^- w = w_t + R(\rho\sigma_x)_x w + 2R\rho\sigma_x w_x.$$

In [31], the authors prove that there exist two positive constants $C_1 = C_1(T, t_0, \omega)$ and $R_1 = R_1(T, t_0, \omega)$ such that, for all $R \geqslant R_1$,

$$\|P_R^+ w\|^2_{L^2(Q_T^{t_0})} + \|P_R^- w\|^2_{L^2(Q_T^{t_0})} + \iint_{Q_T^{t_0}} R^3\theta^3(1-x^2)w^2\,dx\,dt + \iint_{Q_T^{t_0}} R\theta(1-x^2)w_x^2\,dx\,dt$$
$$\leqslant C_1\left(\iint_{Q_T^{t_0}} h^2 e^{-2R\sigma}\,dx\,dt + \iint_{\omega_T^{t_0}} R^3\theta^3 z^2 e^{-2R\sigma}\,dx\,dt\right). \tag{8.1}$$

From this estimate, we aim at proving estimate (4.8) that concerns the variable z.

Step 1: Estimate of $\iint_{Q_T^{t_0}} R^3\theta^3(1-x^2)z^2e^{-2R\sigma}\,dx\,dt$ and $\iint_{Q_T^{t_0}} R\theta(1-x^2)z_x^2e^{-2R\sigma}\,dx\,dt$. Replacing w by $ze^{-R\sigma}$, we immediately get from (8.1)

$$\iint_{Q_T^{t_0}} R^3\theta^3(1-x^2)z^2e^{-2R\sigma}\,dx\,dt \leqslant C_1\left(\iint_{Q_T^{t_0}} h^2e^{-2R\sigma}\,dx\,dt + \iint_{\omega_T^{t_0}} R^3\theta^3z^2e^{-2R\sigma}\,dx\,dt\right). \tag{8.2}$$

Moreover $w_x = -R\sigma_x ze^{-R\sigma} + z_xe^{-R\sigma}$. Therefore,

$$\iint_{Q_T^{t_0}} R\theta(1-x^2)z_x^2e^{-2R\sigma}\,dx\,dt \leqslant 2\iint_{Q_T^{t_0}} R^2\theta^2(1-x^2)p_x^2z^2e^{-2R\sigma}\,dx\,dt + 2\iint_{Q_T^{t_0}} R\theta(1-x^2)w_x^2\,dx\,dt. \tag{8.3}$$

Yet, for all $x\in\bar I$, $p_x(x) = -\frac{\phi_-(x)}{\rho(x)}e^{S(\phi_+(x))^2}$. Using (4.5) and the fact that $e^{S(\phi_+(.))}$ is a bounded function on $\bar I$, there exists $C = C(T,t_0,\omega) > 0$ such that

$$\iint_{Q_T^{t_0}} R^2\theta^2(1-x^2)p_x^2z^2e^{-2R\sigma}\,dx\,dt \leqslant C\iint_{Q_T^{t_0}} R^2\theta^2(1-x^2)z^2e^{-2R\sigma}\,dx\,dt.$$

As there exists $C = C(T,t_0) > 0$ such that $\theta^2 \leqslant C\theta^3$, we get

$$\iint_{Q_T^{t_0}} R^2\theta^2(1-x^2)p_x^2z^2e^{-2R\sigma}\,dx\,dt \leqslant C\iint_{Q_T^{t_0}} R^3\theta^3(1-x^2)z^2e^{-2R\sigma}\,dx\,dt, \tag{8.4}$$

choosing R large enough. As a conclusion, thanks to (8.1), (8.2), (8.3), (8.4) we have:

$$\iint_{Q_T^{t_0}} R\theta(1-x^2)z_x^2e^{-2R\sigma}\,dx\,dt + \iint_{Q_T^{t_0}} R^3\theta^3(1-x^2)z^2e^{-2R\sigma}\,dx\,dt$$
$$\leqslant C\left(\iint_{Q_T^{t_0}} h^2e^{-2R\sigma}\,dx\,dt + \iint_{\omega_T^{t_0}} R^3\theta^3z^2e^{-2R\sigma}\,dx\,dt\right). \tag{8.5}$$

Step 2: Estimate of $\iint_{Q_T^{t_0}} R\theta^{\frac{3}{2}}|\gamma|pz^2e^{-2R\sigma}\,dx\,dt$. First, observe that:

$$P_R^+w = -4R\gamma\theta^{\frac{5}{4}}pw + R^2\rho\sigma_x^2w + (\rho w_x)_x. \tag{8.6}$$

We recall that

$$\gamma(t)\geqslant 0 \quad \text{if } t\in[t_0,T'] \quad\text{and}\quad \gamma(t)\leqslant 0 \quad \text{if } t\in[T',T].$$

As a consequence, we define:

$$\zeta : [t_0,T]\to\mathbb{R}$$
$$t\mapsto \begin{cases} 1 & \text{if } t\in[t_0,T'),\\ -1 & \text{if } t\in[T',T].\end{cases}$$

Then, for all $t \in [t_0, T]$, $\zeta(t)\gamma(t) = |\gamma(t)|$ and $|\zeta(t)| \leqslant 1$. Multiplying (8.6) by $\zeta\theta^{\frac{1}{4}}w$, we obtain

$$\zeta\theta^{\frac{1}{4}}P_R^+ww = -4R|\gamma|\theta^{\frac{3}{2}}pw^2 + R^2\theta^{\frac{1}{4}}\zeta\rho\sigma_x^2w^2 + \zeta\theta^{\frac{1}{4}}(\rho w_x)_x w.$$

Then, taking the integral over $Q_T^{t_0}$ and integrating by parts with respect to x in the last right-hand term,

$$\iint\limits_{Q_T^{t_0}} R\theta^{\frac{3}{2}}|\gamma|pw^2\,dx\,dt = -\frac{1}{4}\iint\limits_{Q_T^{t_0}} \zeta P_R^+w\theta^{\frac{1}{4}}w\,dx\,dt + \frac{1}{4}\iint\limits_{Q_T^{t_0}} R^2\theta^{\frac{1}{4}}\zeta\rho\sigma_x^2w^2\,dx\,dt - \frac{1}{4}\iint\limits_{Q_T^{t_0}} \zeta\theta^{\frac{1}{4}}\rho w_x^2\,dx\,dt.$$

Since $|\zeta| \leqslant 1$, it follows that

$$\iint\limits_{Q_T^{t_0}} R\theta^{\frac{3}{2}}|\gamma|pw^2\,dx\,dt \leqslant \frac{1}{4}\iint\limits_{Q_T^{t_0}} \left|P_R^+w\theta^{\frac{1}{4}}w\right|dx\,dt + \frac{1}{4}\iint\limits_{Q_T^{t_0}} R^2\theta^{\frac{1}{4}}\rho\sigma_x^2w^2\,dx\,dt + \frac{1}{4}\iint\limits_{Q_T^{t_0}} \theta^{\frac{1}{4}}\rho w_x^2\,dx\,dt.$$

First, we have:

$$\frac{1}{4}\iint\limits_{Q_T^{t_0}} \left|P_R^+w\theta^{\frac{1}{4}}w\right|dx\,dt \leqslant \underbrace{\frac{1}{8}\left\|P_R^+w\right\|^2_{L^2(Q_T^{t_0})}}_{J_1} + \underbrace{\frac{1}{8}\iint\limits_{Q_T^{t_0}} \theta^{\frac{1}{2}}w^2\,dx\,dt}_{J_2}.$$

Moreover,

$$\begin{aligned}
\frac{1}{4}\iint\limits_{Q_T^{t_0}} R^2\theta^{\frac{1}{4}}\rho\sigma_x^2w^2\,dx\,dt &= \frac{1}{4}\iint\limits_{Q_T^{t_0}} R^2\theta^{\frac{9}{4}}\rho_0\left(1-x^2\right)p_x^2w^2\,dx\,dt \\
&= \frac{1}{4}\iint\limits_{Q_T^{t_0}} R^2\theta^{\frac{9}{4}}\rho_0\left(1-x^2\right)\left(\frac{\phi_-(x)}{\rho_0(1-x^2)}e^{S(\phi_+(x))^2}\right)^2 w^2\,dx\,dt \\
&\leqslant \frac{C}{4}\iint\limits_{Q_T^{t_0}} R^2\theta^{\frac{9}{4}}\left(1-x^2\right)w^2\,dx\,dt
\end{aligned}$$

using (4.5) and the fact that $e^{S(\phi_+(.))^2}$ is bounded on $\bar{I}$. Therefore,

$$\iint\limits_{Q_T^{t_0}} R\theta^{\frac{3}{2}}|\gamma|pw^2\,dx\,dt \leqslant J_1 + J_2 + \frac{C}{4}\iint\limits_{Q_T^{t_0}} R^2\theta^{\frac{9}{4}}(1-x)^2w^2\,dx\,dt + \frac{1}{4}\iint\limits_{Q_T^{t_0}} \theta^{\frac{1}{4}}\rho w_x^2\,dx\,dt. \tag{8.7}$$

The term J_1 is estimated thanks to (8.1). As for J_2, we need in a way, to make the weight $1-x^2$ appear. In this purpose, we use the Hardy inequality (7.22) choosing $a(x) = (1-x^2)^{\alpha^\star}$ with $1 < \alpha^\star < 2$. In this case, (7.22) becomes

$$\int\limits_I \frac{1}{(1-x^2)^{2-\alpha^\star}}Z^2\,dx \leqslant C\int\limits_I \left(1-x^2\right)^{\alpha^\star}Z_x^2\,dx + C\int\limits_{L_1}^{L_2} Z^2\,dx.$$

Since $(1-x^2)^{\alpha^\star} \leqslant (1-x^2)$ and $1/(1-x^2)^{2-\alpha^\star} \geqslant 1$, it implies

$$\int\limits_I Z^2\,dx \leqslant C\int\limits_I \left(1-x^2\right)Z_x^2\,dx + C\int\limits_{L_1}^{L_2} Z^2\,dx. \tag{8.8}$$

Therefore

$$J_2 = \frac{1}{4}\iint\limits_{Q_T^{t_0}} \theta^{\frac{1}{2}}w^2\,dx\,dt \leqslant C\iint\limits_{Q_T^{t_0}} \theta^{\frac{1}{2}}\left(1-x^2\right)w_x^2\,dx\,dt + C\iint\limits_{\omega_T^{t_0}} \theta^{\frac{1}{2}}w^2\,dx\,dt. \tag{8.9}$$

Then, using the definition of θ and large parameter R, one can estimate the terms $\iint_{Q_T^{t_0}} \theta^{\frac{1}{2}}(1-x^2)w_x^2\,dx\,dt$ and $\iint_{Q_T^{t_0}} R^2\theta^{\frac{9}{4}}(1-x)^2w^2\,dx\,dt + \frac{1}{4}\iint_{Q_T^{t_0}} \theta^{\frac{1}{4}}\rho w_x^2\,dx\,dt$ of the right-hand side of (8.7) by the left-hand side of (8.1).

To sum up, we have shown that

$$\iint\limits_{Q_T^{t_0}} R\theta^{\frac{3}{2}}|\gamma|pw^2\,dx\,dt \leqslant \Bigg(\iint\limits_{Q_T^{t_0}} h^2e^{-2R\sigma}\,dx\,dt + \iint\limits_{\omega_T^{t_0}} R^3\theta^3z^2e^{-2R\sigma}\,dx\,dt\Bigg).$$

Finally, coming back to z, we get:

$$\iint\limits_{Q_T^{t_0}} R\theta^{\frac{3}{2}}|\gamma|pz^2e^{-2R\sigma}\,dx\,dt \leqslant C\Bigg(\iint\limits_{Q_T^{t_0}} h^2e^{-2R\sigma}\,dx\,dt + \iint\limits_{\omega_T^{t_0}} R^3\theta^3z^2e^{-2R\sigma}\,dx\,dt\Bigg). \tag{8.10}$$

Step 3: Estimate of $\iint_{Q_T^{t_0}} \frac{1}{R\theta}w_t^2\,dx\,dt$. First of all, coming back to the definition of P_R^-w, we have:

$$\begin{aligned} P_R^-w &= w_t + R(\rho\sigma_x)_xw + 2R\rho\sigma_xw_x \\ &= w_t + R\theta\big(\rho_0\big(1-x^2\big)p_x\big)_xw + 2R\theta\rho_0\big(1-x^2\big)p_xw_x. \end{aligned}$$

Let us set, for all $x \in \bar{I}$, $\kappa(x) := \rho_0(1-x^2)p_x(x)$. Then

$$P_R^-w = w_t + R\theta\kappa_xw + 2R\theta\rho_0\big(1-x^2\big)p_xw_x.$$

Therefore

$$\frac{1}{\sqrt{R\theta}}w_t = -\sqrt{R\theta}\kappa_xw - 2\sqrt{R\theta}\rho_0\big(1-x^2\big)p_xw_x + \frac{P_R^-w}{\sqrt{R\theta}}.$$

Note that κ_x and p_x are bounded functions on $\bar{I}$ and $\frac{1}{\sqrt{\theta}}$ is bounded on $[t_0,T]$. Hence, there exists $C = C(T,t_0,\omega)$ such that, for R large enough

$$\begin{aligned} \iint\limits_{Q_T^{t_0}} \frac{1}{R\theta}w_t^2\,dx\,dt &\leqslant C\Bigg(\iint\limits_{Q_T^{t_0}} R\theta w^2\,dx\,dt + \iint\limits_{Q_T^{t_0}} R\theta\big(1-x^2\big)^2w_x^2\,dx\,dt + \big\|P_R^-w\big\|^2_{L^2(Q_T^{t_0})}\Bigg) \\ &\leqslant C\Bigg(\iint\limits_{Q_T^{t_0}} R\theta w^2\,dx\,dt + \iint\limits_{Q_T^{t_0}} R\theta\big(1-x^2\big)w_x^2\,dx\,dt + \big\|P_R^-w\big\|^2_{L^2(Q_T^{t_0})}\Bigg) \end{aligned} \tag{8.11}$$

since $0 \leqslant (1-x^2) \leqslant 1$.

We then estimate $\iint_{Q_T^{t_0}} R\theta w^2\,dx\,dt$ thanks to Hardy inequality (7.22), as we have done in the previous step in (8.8). We have

$$\iint\limits_{Q_T^{t_0}} R\theta w^2\,dx\,dt \leqslant C\Bigg(\iint\limits_{Q_T^{t_0}} R\theta\big(1-x^2\big)w_x^2\,dx\,dt + \iint\limits_{\omega_T^{t_0}} R\theta w^2\,dx\,dt\Bigg).$$

Finally, using (8.11) and (8.1), and taking R large, one has

$$\iint\limits_{Q_T^{t_0}} \frac{1}{R\theta}w_t^2\,dx\,dt \leqslant C(T,t_0,\alpha)\Bigg(\iint\limits_{Q_T^{t_0}} h^2e^{-2R\sigma}\,dx\,dt + \iint\limits_{\omega_T^{t_0}} R^3\theta^3z^2e^{-2R\sigma}\,dx\,dt\Bigg). \tag{8.12}$$

At this step of the proof, we do not get the estimate of z_t directly from (8.12) and we have to use the estimate of Step 2.

Step 4: Estimate of $\iint_{Q_T^{t_0}} \frac{1}{R\theta} z_t^2 e^{-2R\sigma}\,dx\,dt$. We have: $w_t = z_t e^{-R\sigma} - R\sigma_t w$. Then:

$$\iint_{Q_T^{t_0}} \frac{1}{R\theta} z_t^2 e^{-2R\sigma}\,dx\,dt \leqslant 2\left(\iint_{Q_T^{t_0}} \frac{1}{R\theta} w_t^2\,dx\,dt + \iint_{Q_T^{t_0}} \frac{R^2\sigma_t^2}{R\theta} w^2\,dx\,dt\right). \tag{8.13}$$

Since we have already estimated the term $\iint_{Q_T^{t_0}} \frac{1}{R\theta} w_t^2\,dx\,dt$ in (8.12), we just need to bound the second term in the above right-hand side. For a.a. $(t,x) \in Q_T^{t_0}$, we have:

$$R\sigma_t(t,x) = R\theta_t(t)p(x) = -4R\gamma(t)\theta^{\frac{5}{4}}(t)p(x).$$

Then

$$\iint_{Q_T^{t_0}} \frac{R^2\sigma_t^2}{R\theta} w^2\,dx\,dt = 16\iint_{Q_T^{t_0}} R\theta^{\frac{3}{2}}\gamma^2 p^2 w^2\,dx\,dt \leqslant C\iint_{Q_T^{t_0}} R\theta^{\frac{3}{2}}|\gamma| p w^2\,dx\,dt, \tag{8.14}$$

where $C = 16\sup_{(t,x)\in[t_0,T]\times\bar{I}} |\gamma(t)|p(x)$. Hence, using (8.13), (8.12), (8.14) and (8.10), we get

$$\iint_{Q_T^{t_0}} \frac{1}{R\theta} z_t^2 e^{-2R\sigma}\,dx\,dt \leqslant C\left(\iint_{Q_T^{t_0}} h^2 e^{-2R\sigma}\,dx\,dt + \iint_{\omega_T^{t_0}} R^3\theta^3 z^2 e^{-2R\sigma}\,dx\,dt\right). \tag{8.15}$$

Conclusion: We immediately deduce the expected Carleman estimate (4.8) from (8.2), (8.5), (8.10) and (8.15). □

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