The space of 4-ended solutions to the Allen–Cahn equation in the plane

Michał Kowalczyk [a,1], Yong Liu [a,2], Frank Pacard [b,*]

[a] *Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático (UMI 2807 CNRS), Universidad de Chile, Casilla 170, Correo 3, Santiago, Chile*

[b] *Institut Universitaire de France et Centre de Mathématiques Laurent Schwartz, École Polytechnique, 91128 Palaiseau, France*

Abstract

We are interested in entire solutions of the Allen–Cahn equation $\Delta u - F'(u) = 0$ which have some special structure at infinity. In this equation, the function F is an even, double well potential. The solutions we are interested in have their zero set asymptotic to 4 half oriented affine lines at infinity and, along each of these half affine lines, the solutions are asymptotic to the one dimensional heteroclinic solution: such solutions are called 4-*ended solutions*. The main result of our paper states that, for any $\theta \in (0, \pi/2)$, there exists a 4-ended solution of the Allen–Cahn equation whose zero set is at infinity asymptotic to the half oriented affine lines making the angles $\theta, \pi - \theta, \pi + \theta$ and $2\pi - \theta$ with the x-axis. This paper is part of a program whose aim is to classify all $2k$-ended solutions of the Allen–Cahn equation in dimension 2, for $k \geqslant 2$.
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Keywords: Allen–Cahn equation; Classification of solutions; Entire solutions of semilinear elliptic equations

1. Introduction

In this paper, we are interested in entire solutions of the Allen–Cahn equation

$$\Delta u - F'(u) = 0, \tag{1.1}$$

in $\mathbb{R}^2$, where the function F is a smooth, double well potential. This means that F is even, nonnegative and has only two zeros which will be chosen to be at ± 1. Moreover, we assume that

$$F''(0) \neq 0, \qquad F''(\pm 1) \neq 0,$$

and also that

$$F'(t) \neq 0, \quad \text{for all } t \in (0, 1).$$

It is known that (1.1) has a solution whose nodal set is *any* given straight line. These special solutions, which will be referred to as the *heteroclinic solutions*, are constructed using the heteroclinic, one dimensional solution of (1.1), namely the function H defined on $\mathbb{R}$, solution of

$$H'' - F'(H) = 0, \tag{1.2}$$

which is odd and tends to -1 (respectively to $+1$) at $-\infty$ (respectively at $+\infty$).

More precisely, we have the:

Definition 1.1. Given $r \in \mathbb{R}$ and $e \in \mathbb{R}^2$ such that $|e| = 1$, the *heteroclinic solutions with end* $\lambda := r e^{\perp} + \mathbb{R} e$ is defined by

$$u(x) := H\big(x \cdot e^{\perp} - r\big),$$

where $\perp$ denotes the rotation of angle $\pi/2$ in the plane.

Observe that this construction extends in any dimension to produce solutions whose level sets are hyperplanes.

The famous *De Giorgi's conjecture* asserts that (in space dimension less than or equal to 8), if u is a bounded solution of (1.1) which is monotone in one direction, then u has to be one of the above defined heteroclinic solutions. This conjecture is known to hold when the space dimension is equal to 2 [10], in dimension 3 [1] and in dimensions 4 to 8 [17] under some mild additional assumption. Counterexamples have been constructed in all dimensions $N \geqslant 9$ in [7], showing that the conjecture is indeed sharp.

In this paper, we are interested in entire solutions of (1.1) which are defined in $\mathbb{R}^2$ and which have some special structure at infinity, namely their zero set is, at infinity, asymptotic to 4 oriented affine lines: such solutions are called 4-*ended solutions* and will be precisely defined in the next section. The main result of our paper states that, for any $\theta \in (0, \pi/2)$, there exists a 4-ended solution of (1.1) whose zero set is asymptotic at infinity to the half oriented affine lines making angles $\theta, \pi - \theta, \pi + \theta$ and $2\pi - \theta$ with the x-axis.

2. The space of 4-ended solutions

In order to proceed, we need to define precisely the class of entire solutions of (1.1) we are interested in. As already mentioned, these solutions have the property that their nodal sets are, away from a compact, asymptotic to a finite (even) number of half oriented affine lines, which are called the *ends* of the solutions. The concept of solutions with a finite number of *ends* was first introduced in [5] and, for the sake of completeness, we recall the precise definitions in the case of 4-ended solutions.

An *oriented affine line* $\lambda \subset \mathbb{R}^2$ can be uniquely written as

$$\lambda := r e^{\perp} + \mathbb{R} e,$$

for some $r \in \mathbb{R}$ and some unit vector $e \in S^1$, which defines the orientation of λ. We recall that $\perp$ denotes the rotation by $\pi/2$ in $\mathbb{R}^2$. Writing $e = (\cos\theta, \sin\theta)$, we get the usual coordinates (r, θ) which allow to identify the set of oriented affine lines with $\mathbb{R} \times S^1$.

Assume that we are given 4 oriented affine lines $\lambda_1, \ldots, \lambda_4 \subset \mathbb{R}^2$ which are defined by

$$\lambda_j := r_j e_j^{\perp} + \mathbb{R} e_j,$$

and assume that these oriented affine lines have corresponding angles $\theta_1, \ldots, \theta_4$ satisfying

$$\theta_1 < \theta_2 < \theta_3 < \theta_4 < 2\pi + \theta_1.$$

In this case, we will say that the 4 oriented affine lines are *ordered* and we will denote by Λ_{ord}^4 the set of 4 oriented affine lines. It is easy to check that for all $R > 0$ large enough and for all $j = 1, \ldots, 4$, there exists $s_j \in \mathbb{R}$ such that:

(i) The point $r_j e_j^\perp + s_j e_j$ belongs to the circle ∂B_R.

(ii) The half affine lines

$$\lambda_j^+ := r_j e_j^\perp + s_j e_j + \mathbb{R}^+ e_j \tag{2.3}$$

are disjoint and included in $\mathbb{R}^2 - B_R$.

(iii) The minimum of the distance between two distinct half affine lines λ_i^+ and λ_j^+ is larger than 4.

The set of half affine lines $\lambda_1^+, \ldots, \lambda_4^+$ together with the circle ∂B_R induce a decomposition of $\mathbb{R}^2$ into 5 slightly overlapping connected components

$$\mathbb{R}^2 = \Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_4,$$

where $\Omega_0 := B_{R+1}$ and

$$\Omega_j := \left(\mathbb{R}^2 - B_{R-1}\right) \cap \left\{x \in \mathbb{R}^2 \colon \operatorname{dist}\left(x, \lambda_j^+\right) < \operatorname{dist}\left(x, \lambda_i^+\right) + 2, \ \forall i \neq j\right\}, \tag{2.4}$$

for $j = 1, \ldots, 4$. Here, $\operatorname{dist}(\cdot, \lambda_j^+)$ denotes the distance to λ_j^+. Observe that, for all $j = 1, \ldots, 4$, the set Ω_j contains the half affine line λ_j^+.

Let $\mathbb{I}_0, \mathbb{I}_1, \ldots, \mathbb{I}_4$ be a smooth partition of unity of $\mathbb{R}^2$ which is subordinate to the above decomposition. Hence

$$\sum_{j=0}^{4} \mathbb{I}_j \equiv 1,$$

and the support of $\mathbb{I}_j$ is included in Ω_j. Without loss of generality, we can also assume that $\mathbb{I}_0 \equiv 1$ in

$$\Omega_0' := B_{R-1},$$

and $\mathbb{I}_j \equiv 1$ in

$$\Omega_j' := \left(\mathbb{R}^2 - B_{R-1}\right) \cap \left\{x \in \mathbb{R}^2 \colon \operatorname{dist}\left(x, \lambda_j^+\right) < \operatorname{dist}\left(x, \lambda_i^+\right) - 2, \ \forall i \neq j\right\},$$

for $j = 1, \ldots, 4$. Finally, without loss of generality, we can assume that

$$\|\mathbb{I}_j\|_{\mathcal{C}^2(\mathbb{R}^2)} \leqslant C.$$

With these notations at hand, we define

$$u_\lambda := \sum_{j=1}^{4} (-1)^j \mathbb{I}_j H\left(\operatorname{dist}^s(\cdot, \lambda_j)\right), \tag{2.5}$$

where $\lambda := (\lambda_1, \ldots, \lambda_4)$ and

$$\operatorname{dist}^s(x, \lambda_j) := x \cdot e_j^\perp - r_j \tag{2.6}$$

denotes the *signed distance* from a point $x \in \mathbb{R}^2$ to λ_j.

Observe that, by construction, the function u_λ is, away from a compact and up to a sign, asymptotic to copies of the heteroclinic solution with ends $\lambda_1, \ldots, \lambda_4$.

Let $\mathcal{S}_4$ denote the set of functions u which are defined in $\mathbb{R}^2$ and which satisfy

$$u - u_\lambda \in W^{2,2}\left(\mathbb{R}^2\right), \tag{2.7}$$

for some ordered set of oriented affine lines $\lambda_1, \ldots, \lambda_4 \subset \mathbb{R}^2$. We also define the decomposition operator $\mathcal{J}$ by

$$\mathcal{J} : \mathcal{S}_4 \to W^{2,2}\left(\mathbb{R}^2\right) \times \Lambda_{ord}^4$$

$$u \mapsto (u - u_\lambda, \lambda).$$

The topology on $\mathcal{S}_4$ is the one for which the operator $\mathcal{J}$ is continuous (the target space being endowed with the product topology).

We now have the:

Definition 2.1. The set $\mathcal{M}_4$ is defined to be the set of solutions u of (1.1) which belong to $\mathcal{S}_4$.

It is known that $\mathcal{M}_4$ is not empty. For example, the *saddle solution* constructed in [4] belongs to $\mathcal{M}_4$, the nodal set of this solution is the union of the two lines $y = \pm x$. Another important fact, also proven in [4] or in [11], is that up to a sign and a rigid motion, the *saddle solution* is the unique bounded solution whose nodal set coincides with the union of the two lines $y = \pm x$. The solutions constructed in [6] are also elements of $\mathcal{M}_4$ and we shall return to this point later on.

Recall from [5], that a solution u of (1.1) is said to be *nondegenerate* if there is no $w \in W^{2,2}(\mathbb{R}^2) - \{0\}$ which is in the kernel of

$$L := -\Delta + F''(u),$$

and which decays exponentially at infinity.

As far as the structure of the set of 4-ended solutions is concerned, the main result of [5] asserts that:

Theorem 2.1. *(See [5].) Assume that $u \in \mathcal{M}_4$ is* nondegenerate*, then, close to u, $\mathcal{M}_4$ is a 4-dimensional smooth manifold.*

Observe that, given $u \in \mathcal{M}_4$, translations and rotations of u are also elements of $\mathcal{M}_4$ and this accounts for 3 of the 4 formal dimensions of $\mathcal{M}_4$, moreover, if $u \in \mathcal{M}_4$ then $-u \in \mathcal{M}_4$.

All the 4-ended solutions constructed so far have two axes of symmetry and in fact, it follows from a result of C. Gui [12] that:

Theorem 2.2. *(See [12].) Assume that $u \in \mathcal{M}_4$. Then, there exists a rigid motion g such that $\bar{u} := u \circ g$ is even with respect to the x-axis and the y-axis, namely*

$$\bar{u}(x, y) = \bar{u}(-x, y) = \bar{u}(x, -y). \tag{2.8}$$

In addition, $\bar{u}$ is a monotone function of both the x and y variables in the upper right quadrant $Q^{\llcorner}$ defined by

$$Q^{\llcorner} := \left\{ (x, y) \in \mathbb{R}^2 \colon x > 0, \ y > 0 \right\},$$

and, changing the sign of $\bar{u}$ if this is necessary, we can assume that

$$\partial_x \bar{u} < 0 \quad and \quad \partial_y \bar{u} > 0,$$

in $Q^{\llcorner}$.

Thanks to this result, we can define the moduli space of 4-ended solutions by:

Definition 2.2. The set $\mathcal{M}_4^{even}$ is defined to be the set of $u \in \mathcal{S}_4$ which are solutions of (1.1), are even with respect to the x-axis and the y-axis and which tend to $+1$ at infinity along the y-axis (and tend to -1 at infinity along the x-axis). In particular,

$$\partial_x u < 0 \quad and \quad \partial_y u > 0,$$

in the upper right quadrant $Q^{\llcorner}$.

When studying $\mathcal{M}_4^{even}$, we restrict our attention to functions which are even with respect to the x-axis and the y-axis and, in this case, a solution $u \in \mathcal{M}_4^{even}$ is said to be *even-nondegenerate* if there is no $w \in W^{2,2}(\mathbb{R}^2) - \{0\}$, which is symmetric with respect to the x-axis and the y-axis, belongs to the kernel of

$$L := -\Delta + F''(u),$$

and which decays exponentially at infinity.

In the equivariant case (namely solutions which are invariant under both the symmetry with respect to the x-axis and the y-axis), Theorem 2.1 reduces to:

Theorem 2.3. *(See [5].) Assume that $u \in \mathcal{M}_4^{even}$ is even-nondegenerate, then, close to u, $\mathcal{M}_4^{even}$ is a 1-dimensional smooth manifold.*

Any solution $u \in \mathcal{M}_4^{even}$ has a nodal set which is asymptotic to 4 half oriented affine lines and, given the symmetries of u, these half oriented affine lines are images of each other by the symmetries with respect to the x-axis and the y-axis. In particular, there is at most one of these half oriented affine line

$$\lambda := r e^{\perp} + \mathbb{R} e,$$

which is included in the upper right quadrant $Q^{\llcorner}$. Writing $e = (\cos\theta, \sin\theta)$ where $\theta \in (0, \pi/2)$, we define

$$\mathcal{F} : \mathcal{M}_4^{even} \to (-\pi/4, \pi/4) \times \mathbb{R},$$
$$u \mapsto (\theta - \pi/4, r).$$

For example, the image by $\mathcal{F}$ of the *saddle solution* defined in [4] is precisely $(0, 0)$, while the images by $\mathcal{F}$ of the solutions constructed in [6] correspond to parameters (θ, r) where θ is close to $\pm\pi/4$ and r is close to $\mp\infty$.

Remark 2.1. Let us observe that, if $u \in \mathcal{M}_4^{even}$, then $\bar{u}$ defined by

$$\bar{u}(x, y) = -u(y, x),$$

also belongs to $\mathcal{M}_4^{even}$ and

$$\mathcal{F}(\bar{u}) = -\mathcal{F}(u).$$

In this paper, we are interested in the understanding of $\mathcal{M}_4^{even}$. To begin with, we prove that:

Theorem 2.4 *(Nondegeneracy). Any $u \in \mathcal{M}_4$ is nondegenerate and hence any $u \in \mathcal{M}_4^{even}$ is even-nondegenerate.*

As a consequence of this result, we find that all connected components of $\mathcal{M}_4^{even}$ are one-dimensional smooth manifolds. Moreover, as a byproduct of the proof of this result, we also obtain that the image by $\mathcal{F}$ of any connected component of $\mathcal{M}_4^{even}$ is a smooth immersed curve in $(-\pi/4, \pi/4) \times \mathbb{R}$. Thanks to Remark 2.1, we find that the image of $\mathcal{M}_4^{even}$ by $\mathcal{F}$ is invariant under the action of the symmetry with respect to $(0, 0)$.

To proceed, we define the *classifying map* to be the projection of $\mathcal{F}$ onto the first variable

$$\mathcal{P} : \mathcal{M}_4^{even} \to (-\pi/4, \pi/4),$$
$$u \mapsto \theta - \pi/4.$$

Our second result reads:

Theorem 2.5 *(Properness). The mapping $\mathcal{P}$ is proper, i.e. the pre-image of a compact in $(-\pi/4, \pi/4)$ is compact in $\mathcal{M}_4^{even}$ (endowed with the topology induced by the one of $\mathcal{S}_4$).*

The solutions with almost parallel ends constructed in [6] belong to one of the connected component of $\mathcal{M}_4^{even}$ and we also know that the saddle solution belongs to a connected component of $\mathcal{M}_4^{even}$. In principle, it could be possible that $\mathcal{M}_4^{even}$ contained many different connected components and it could also be possible that $\mathcal{M}_4^{even}$ contained connected components which are diffeomorphic to S^1. Nevertheless, we prove that:

Theorem 2.6. *All connected components of $\mathcal{M}_4^{even}$ are diffeomorphic to $\mathbb{R}$, i.e. there is no closed loop in $\mathcal{M}_4^{even}$.*

Looking at the image by $\mathcal{P}$ of the connected component of $\mathcal{M}_4^{even}$ which contains the saddle solution, we conclude from the above results that:

Theorem 2.7 *(Surjectivity of $\mathcal{P}$). The mapping $\mathcal{P}$ is onto.*

As a consequence, for any $\theta \in (0, \pi/2)$, there exists a solution $u \in \mathcal{M}_4^{even}$ whose nodal set at infinity is asymptotic to the half oriented affine lines whose angles with the x-axis are given by $\theta, \pi - \theta, \pi + \theta$ and $2\pi - \theta$.

Given all the evidence we have, it is tempting to conjecture that $\mathcal{M}_4^{even}$ has only one connected component and that the image of $\mathcal{M}_4^{even}$ by $\mathcal{F}$ is a smooth embedded curve. Moreover, it is very likely that $\mathcal{P}$ is a diffeomorphism from $\mathcal{M}_4^{even}$ onto $(-\pi/4, \pi/4)$. Observe that Theorem 2.7 already proves that $\mathcal{P}$ is onto.

To give credit to the above conjecture, in [15], we will show that $\mathcal{M}_4^{even}$ has only one connected component which contains both the saddle solution and the solutions constructed in [6]. The proof of this last result is rather technical and uses tools which are different from the one needed to prove the results in the present paper and this is the reason why, we have chosen to present it in a separate paper [15].

To complete this list of results, we mention an interesting by-product of the proof of Theorem 2.4. Assume that u is a solution of (1.1) and denote by

$$L := -\Delta + F''(u),$$

the linearized operator about u. Recall that, if Ω is a bounded domain in $\mathbb{R}^2$, then the index of L in Ω is given by the number of negative eigenvalues of the operator L which belong to $W_0^{1,2}(\Omega)$. Following [9], we have the:

Definition 2.3. The Morse index of a function u, solution of (1.1) is the supremum of the index of $-\Delta + F''(u)$ over bounded domains of $\mathbb{R}^2$.

For example the Morse index of the saddle solution is 1 [18]. In this paper, we prove the:

Theorem 2.8 *(Morse index). Any $2k$-ended solution of* (1.1) *has finite Morse index.*

The proof follows from the result of Proposition 3.1 together with a result of [8].

Since the Morse index of a $2k$-ended solution u is finite (equal to m), we know from [9], that there exists a finite dimensional subspace $E \subset L^2(\mathbb{R}^2)$, with $\dim E = m$, which is spanned by the eigenfunctions $\phi_1, \ldots, \phi_m$ of the operator L, corresponding to the negative eigenvalues $\mu_1, \ldots, \mu_m$ of L.

We now sketch the plan of our paper.

In Section 3, we prove that *any* element of $\mathcal{M}_4^{even}$ is even-nondegenerate (we also prove that it is nondegenerate, even though we do not need this result). The proof follows the line of the proof in [14] where it is proven that the saddle solution is nondegenerate. This will prove Theorem 2.4 and, thanks to this result, it will then follow from the Implicit Function Theorem (see Section 8 and Theorem 2.2 in [5]) that any connected component of $\mathcal{M}_4^{even}$ is 1-dimensional.

In Section 4, we recall two key tools which will be needed in the analysis of the properness of the classifying map $\mathcal{P}$. The first is a well known *a priori* estimate for solutions of (1.1) which states that, away from its zero set, the solutions of (1.1) tend to ± 1 exponentially fast. The second tool is a *balancing formula* which holds for any solution of (1.1). This balancing formula reflects the invariance of our problem under translations and rotations and can be understood as a consequence of Noether's Theorem.

In Section 5, we prove the properness of the classifying map $\mathcal{P}$. Assume that $(u_n)_{n \geq 0}$ is a sequence of solutions of $\mathcal{M}_4^{even}$ such that $\mathcal{P}(u_n)$ remains bounded away from $-\pi/4$ and from $\pi/4$, further assume that $(u_n)_{n \geq 0}$ converges on compacts to u (thanks to elliptic estimates, this can always be achieved up to the extraction of a subsequence). We will show that $u \in \mathcal{M}_4^{even}$ and also that

$$\mathcal{P}(u) = \lim_{n \to \infty} \mathcal{P}(u_n).$$

The key tool in the proof is the use of the balancing formula introduced in the previous section which allows to control the nodal sets of u_n as n tends to infinity. As we will see, this compactness result implies that the image by $\mathcal{P}$ of the connected component of $\mathcal{M}_4^{even}$ which contains the saddle solution, is the entire interval $(-\pi/4, \pi/4)$.

Section 6 is devoted to the proof of the nonexistence of compact components in $\mathcal{M}_4^{even}$. We will show that a connected component in $\mathcal{M}_4^{even}$ cannot be compact (i.e. cannot be diffeomorphic to S^1).

Our results are very much inspired from a similar classification result which was obtained in a very different framework: the theory of minimal surfaces. Let us briefly explain the analogy between our result and the corresponding result in the theory of minimal surfaces in $\mathbb{R}^3$.

In 1834, H.F. Scherk discovered an example of a singly-periodic, embedded, minimal surface in $\mathbb{R}^3$ which, in a complement of a vertical cylinder, is asymptotic to 4 half planes with angle $\pi/2$ between them (these planes are called ends). This surface, after an appropriate rigid motion and scaling, has two planes of symmetry, say the $x_2 = 0$ plane and the $x_1 = 0$ plane, and it is periodic, with period 2π in the x_3 direction. If $\theta \in (0, \pi/2)$ denotes the angle between the asymptotic end of the Scherk's surface contained in $\{(x_1, x_2, x_3) \in \mathbb{R}^3\colon x_1 > 0,\ x_2 > 0\}$ and the $x_2 = 0$ plane, then for the original Scherk surface corresponds to $\theta = \pi/4$. This surface is the so-called *Scherk's second surface* and it will be denoted here by $S_{\pi/4}$.

In 1988, H. Karcher [13] found a one parameter family of Scherk's type surfaces with 4-ends including the original example. These minimal surfaces are parameterized by the angle $\theta \in (0, \pi/2)$ between one of their asymptotic planes and the $x_2 = 0$ plane. The one parameter family $(S_\theta)_{\theta \in (0,\pi/2)}$ of these surfaces, normalized in such a way that the period in the x_3 direction is 2π, is the family of Scherk singly periodic minimal surfaces.

We note that the 4-ended elements of Scherk family are given explicitly in terms of the Weierstrass representation, or alternatively they can be represented implicitly as the solutions of

$$\cos^2\theta \cosh\left(\frac{x_1}{\cos\theta}\right) - \sin^2\theta \cosh\left(\frac{x_2}{\cos\theta}\right) = \cos x_3.$$

More generally, Scherk's surfaces with $2k$-ends have also been constructed by H. Karcher [13]. They have been classified by J. Perez and M. Traizet in [16]. In some sense our result can be understood as an analog of the classification result of J. Perez and M. Traizet for 4-ended Scherk's surfaces.

3. The nondegeneracy of 4-ended solutions

In this section, we prove that any $u \in \mathcal{M}_4^{even}$ is even-nondegenerate. The proof follows essentially the proof of the nondegeneracy of the saddle solution in [14]. The main result is the:

Theorem 3.1. *Assume that $u \in \mathcal{M}_4^{even}$ and $\delta > 0$. Further assume that $\varphi \in e^{-\delta(1+|x|^2)} W^{2,2}(\mathbb{R}^2)$ is a solution of*

$$\left(\Delta - F''(u)\right)\varphi = 0,$$

in $\mathbb{R}^2$ which is symmetric with respect to both the x-axis and the y-axis, then $\varphi \equiv 0$.

As in [14], the proof of this proposition relies on the construction of a supersolution for the operator L, away from a compact (similar idea was also used by X. Cabré in [3]). To explain the main idea of the proof, let us digress slightly and consider the heteroclinic solution $(x, y) \mapsto H(x)$ and define

$$L_0 := -\Delta + F''\left(H'\right),$$

the linearized operator about the heteroclinic solution. Clearly, the function

$$\Psi_0(x, y) := H'(x)$$

is positive and is a solution of $L_0\Psi_0 = 0$. Since any $u \in \mathcal{M}_4^{even}$ is asymptotic to a heteroclinic solution, we can transplant H' along the ends of u to build a positive supersolution for $L := -\Delta + F'(u)$. More precisely, we have the:

Proposition 3.1. *Under the above assumptions, there exist $R_0 > 0$ and a function $\Psi > 0$ defined in $\mathbb{R}^2$ such that*

$$\left(\Delta - F''(u)\right)\Psi \leqslant 0,$$

in $\mathbb{R}^2 - B(0, R_0)$.

Proof. The proof follows from a direct construction of the function Ψ. In the upper right quadrant $Q^{\llcorner}$, the zero set of u is asymptotic to the half of an oriented affine line

$$\lambda = r e^{\perp} + \mathbb{R} e,$$

with $e := (\cos\theta, \sin\theta)$. Without loss of generality, we can assume that $\theta \in [\pi/4, \pi/2)$ since, if this is not the case, we just compose $-u$ with a rotation by $\pi/2$. The intersection of λ with the closure of the upper right quadrant, $Q^{\llcorner}$ will be denoted by

$$\bar{\lambda}_1^+ := \overline{Q^{\llcorner}} \cap \lambda,$$

and its image by the symmetry with respect to the y-axis will be denoted by $\bar{\lambda}_2^+$, while its image by the symmetry with respect to the x-axis will be denoted by $\bar{\lambda}_4^+$. Finally, the image of $\bar{\lambda}_2^+$ by the symmetry with respect to the x-axis is equal to the image of $\bar{\lambda}_4^+$ by the symmetry with respect to the y-axis and will be denoted by $\bar{\lambda}_3^+$. So, at infinity, the zero set of u is asymptotic to $\bar{\lambda}_1^+, \ldots, \bar{\lambda}_4^+$. We denote by q_j^+ the end points of $\bar{\lambda}_j^+$, namely

$$\{q_j^+\} := \partial\bar{\lambda}_j^+.$$

Observe that $\{q_j^+ : j = 1, \ldots, 4\}$ contains at most 2 points and we denote by Γ the line segment joining these two points. Also observe that $\mathbb{R}^2 - (\bar{\lambda}_2^+ \cup \Gamma \cup \bar{\lambda}_4^+)$ has two connected components, one of which contains $\bar{\lambda}_1^+$ and will be denoted by U_1 while the other, which contains $\bar{\lambda}_3^+$, will be denoted by U_3.

The crucial observation is the following: If

$$f(x, y) := \left(1 - e^{-\mu y}\right) H'(x),$$

then, using the fact that $(\Delta - F''(H))H' = 0$, we get

$$\left(\Delta - F''(H)\right)f = -\mu^2 e^{-\mu y} H' < 0.$$

We define the function h by

$$h\left((r + s)e^{\perp} + te\right) = \left(1 - e^{-\mu t}\right) H'(s).$$

Observe that the function h is defined in all $\mathbb{R}^2$. Nevertheless, since we are only interested in this function in U_1, we define a smooth cutoff function χ which is identically equal to 1 in U_1 at distance 1 from ∂U_1 and which is identically equal to 0 in $U_3 := \mathbb{R}^2 - \overline{U_1}$. As usual, we assume that $|\nabla\chi| \leqslant C$, for some $C > 0$, as we are entitled to do.

Using χ and h, we build our supersolution in such a way that it is invariant under the symmetry with respect to the x-axis and under the symmetry with respect to the y-axis. We define

$$\Psi(x, y) := \chi(x, y)h(x, y) + \chi(-x, -y)h(-x, -y) + \chi(x, -y)h(x, -y) + \chi(-x, y)h(-x, y).$$

We know from the Refined Asymptotics Theorem (Theorem 2.1 in [5]) that, as t tends to infinity, $(s, t) \mapsto u((r + s)e^{\perp} + te)$ converges exponentially fast to $(s, t) \mapsto H(s)$ uniformly in $s \in [-\rho, \rho]$. Using this property, we see that we can chose $\mu > 0$ close enough to 0 such that

$$L\Psi < -\frac{\mu^2}{2}e^{-\mu t}H' \tag{3.9}$$

in a tubular neighborhood of width ρ around ∂U_1 and away from a ball of radius R_0 large enough, centered at the origin. Observe that the choice of μ only depends on the decay of $(s, t) \mapsto u((r + s)e^{\perp} + te)$ towards $(s, t) \mapsto H(s)$ and does not depend on the choice of ρ. However, increasing ρ affects the minimal value of R_0 for which (3.9) holds.

Now, we choose $\rho > 0$ and $R_0 > 0$ large enough, so that

$$L\Psi < -\frac{\mu^2}{2}e^{-\mu t}H',$$

away from B_{R_0} and away from a tubular neighborhood of width ρ around $\bar{\lambda}_1 \cup \cdots \cup \bar{\lambda}_4^+$. Here, we simply use the fact that $F''(u)$ converges uniformly to $F''(\pm 1)$ away from the nodal set of u. This completes the proof of result. $\quad\square$

Observe that this construction is not specific to the case of 4-ended solutions of (1.1) and in fact a similar construction would hold for any $2k$-ended solution. With this lemma at hand, we can adapt the argument in [14] where the nondegeneracy of the saddle solution is proven and we simply adapt it to the general case where u is any 4-ended solution.

Proof of Theorem 3.1. Recall that, by definition, since $u \in \mathcal{M}_4^{even}$, we have

$$\partial_y u > 0 \quad \text{when } y > 0,$$

and

$$\partial_x u < 0 \quad \text{when } x > 0.$$

Let φ be the function as in the statement of Theorem 3.1. It follows from the Linear Decomposition Lemma (Lemma 4.2 in [5]) that φ decays exponentially at infinity. More precisely, there exist constants $\alpha, C > 0$ such that

$$\left| \varphi(\mathsf{x}) \right| \leqslant C e^{-\alpha |\mathsf{x}|},$$

for all $\mathsf{x} \in \mathbb{R}^2 - B(0, 1)$.

Step 1. We assume that φ is not identically equal to 0 and define $\mathcal{Z}$ to be the zero set of φ. Since φ is assumed to be symmetric with respect to both the x-axis and the y-axis, so is the set $\mathcal{Z}$. Clearly Proposition 3.1 together with the maximum principle, implies that $\mathbb{R}^2 - \mathcal{Z}$ has no bounded connected component included in $\mathbb{R}^2 - B(0, R_0)$, since Ψ can be used as a supersolution to get a contradiction.

Step 2. We claim that any unbounded connected component of $\mathbb{R}^2 - \mathcal{Z}$ necessarily contains $\mathbb{R}^2 - B(0, R)$ for some R large enough. Indeed, if this were not the case then, using the symmetries of φ, one could find $\Omega \subset \mathbb{R}^2$, an unbounded connected component of $\mathbb{R}^2 - \mathcal{Z}$, which is included in one of the four half spaces $\{(x, y) \in \mathbb{R}^2 : \pm x > 0\}$ or $\{(x, y) \in \mathbb{R}^2 : \pm y > 0\}$. For example, let us assume that

$$\Omega \subset \left\{ (x, y) \in \mathbb{R}^2 : y > 0 \right\}.$$

Following [14], we adapt the proof of De Giorgi's conjecture in dimension 2 by N. Ghoussoub and C. Gui to derive a contradiction.

We define $\psi := \partial_y u$ which is a solution of $(\Delta - F''(u))\psi = 0$ in $\mathbb{R}^2$. Moreover, $\psi > 0$ in $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ and we check from direct computation that

$$\text{div}(\psi^2 \nabla h) = 0, \tag{3.10}$$

where $h := \frac{\varphi}{\psi}$. For all $R \geqslant 1$, we consider a cutoff function ζ_R which is identically equal to 1 in $B(0, R)$, identically equal to 0 outside $\mathbb{R}^2 - B(0, 2R)$ and which satisfies $|\nabla \zeta_R| \leqslant C/R$ for some constant $C > 0$ independent of $R \geqslant 1$.

We multiply (3.10) by $\zeta_R^2 h$ and integrate the result over Ω. We find after an integration by parts

$$\int_{\Omega} |\nabla h|^2 \psi^2 \zeta_R^2 \, d\mathsf{x} + 2 \int_{\Omega} \psi^2 h \zeta_R \nabla h \nabla \zeta_R \, d\mathsf{x} = 0.$$

Observe that, in the integration by parts, some care is needed when the boundary of Ω touches the x-axis but it is not hard to see that the integration by parts is also legitimate in this case (we refer to [14] for details). Then, Cauchy–Schwarz inequality yields

$$\int_{\Omega} |\nabla h|^2 \psi^2 \zeta_R^2 \, d\mathsf{x} \leqslant 2 \left(\int_{\Omega \cap A_R} |\nabla h|^2 \psi^2 \zeta_R^2 \, d\mathsf{x} \right)^{1/2} \left(\int_{\Omega \cap A_R} \varphi^2 |\nabla \zeta_R|^2 \, d\mathsf{x} \right)^{1/2}, \tag{3.11}$$

where $A_R := B(0, 2R) - \overline{B}(0, R)$ contains the support of $\nabla \zeta_R$. Hence,

$$\int_{\Omega} |\nabla h|^2 \psi^2 \zeta_R^2 \, d\mathsf{x} \leqslant 4 \left(\sup_{A_R} |\varphi|^2 \right) \int_{\Omega \cap A_R} |\nabla \zeta_R|^2 \, d\mathsf{x}.$$

By construction of ζ_R, the integral on the right hand side is bounded independently of R and since $\varphi \in W^{2,2}(\mathbb{R}^2)$ we know that φ tends to 0 at infinity. Letting R tend to infinity, we conclude that

$$\int_{\Omega} |\nabla h|^2 \psi^2 \, d\mathsf{x} = 0,$$

which then implies that $h \equiv 0$ in Ω and hence we also have $\varphi \equiv 0$ in this set. Finally, $\varphi \equiv 0$ in $\mathbb{R}^2$ by the unique continuation theorem. This is certainly a contradiction and the proof of the claim is complete.

Step 3. By the above, we know that φ does not change sign away from a compact and, without loss of generality, we can assume that $\varphi > 0$ in $\mathbb{R}^2 - B(0, R)$ for some $R > 0$ large enough. As in [14], we proceed by analyzing the projection of φ onto H' (composed with a suitable rotation).

The nodal set of u in the upper right quadrant $Q^{\llcorner}$ is asymptotic to an oriented half line which is denoted by λ and is given by

$$\lambda = r\mathrm{e}^{\perp} + \mathbb{R}\mathrm{e},$$

where $\mathrm{e} = (\cos\theta, \sin\theta)$. Up to a rotation by $\pi/2$ and a possible change of sign, we can assume that $\theta \in [\pi/4, \pi/2)$. The image of λ through the symmetry with respect to the y-axis will be denoted by $\bar{\lambda}$. Notice that, since u is symmetric with respect to the y-axis, $\bar{\lambda}$ is also asymptotic to the zero set of u.

Given the expression of λ, we define

$$\tilde{u}(s, t) := u\big((r + s)\mathrm{e}^{\perp} + t\mathrm{e}\big) \quad \text{and} \quad \tilde{\varphi}(s, t) := \varphi\big((r + s)\mathrm{e}^{\perp} + t\mathrm{e}\big).$$

We consider the function g defined by

$$g(t) := \int_{\mathbb{R}} \tilde{\varphi}(s, t) H'(s) \, ds.$$

Since $\varphi > 0$ away from a compact, we conclude that $g \geqslant 0$ for $t > 0$ large enough. We have, using the equation satisfied by φ

$$g''(t) := \int_{\mathbb{R}} \partial_t^2 \tilde{\varphi}(s, t) H'(s) \, ds = -\int_{\mathbb{R}} \big(\partial_s^2 - F''\big(\tilde{u}(s, t)\big)\big)\tilde{\varphi}(s, t) H'(s) \, ds,$$

and an integration by parts yields

$$g''(t) := -\int_{\mathbb{R}} \tilde{\varphi}(s, t)\partial_s^2 H'(s) \, ds + \int_{\mathbb{R}} F''\big(\tilde{u}(s, t)\big)\tilde{\varphi}(s, t) H'(s) \, ds.$$

Finally, using the equation satisfied by H', we conclude that

$$g''(t) = \int_{\mathbb{R}} \big(F''\big(\tilde{u}(s, t)\big) - F''\big(H(s)\big)\big)\tilde{\varphi}(s, t) H'(s) \, ds.$$

Observe that φ tends exponentially to 0 at infinity and hence so does g. Integrating the above equation from t to ∞, we conclude that

$$g'(t) = -\int_t^{\infty} \int_{\mathbb{R}} \big(F''\big(\tilde{u}(s, z)\big) - F''\big(H(s)\big)\big)\tilde{\varphi}(s, z) H'(s) \, ds \, dz.$$

Recall that $\bar{\lambda}$ is the image of λ through the symmetry with respect to the y-axis and that we can parameterize $\bar{\lambda}$ by

$$\bar{\lambda} = \bar{r}\bar{\mathrm{e}}^{\perp} + \mathbb{R}\bar{\mathrm{e}},$$

with obvious relations between e and $\bar{\mathrm{e}}$. We define

$$\bar{g}(t) := \int_{\mathbb{R}} \varphi\big((\bar{r} + s)\bar{\mathrm{e}}^{\perp} + t\bar{\mathrm{e}}\big) H'(s) \, ds.$$

Observe that, by symmetry of both u and φ, we have

$$\bar{g}(t) = g(t),$$

for all $t \geqslant 0$.

We claim that there exist constants $C > 0$ and $\beta > 0$ such that

$$\left|g'(t)\right| \leqslant Ce^{-\beta t} \int\limits_{t}^{+\infty} g(z)\,dz, \tag{3.12}$$

for all $t > 0$ large enough. Assuming that we have already proven this inequality, it is then a simple exercise to check that the only solution to this differential inequality is identically equal to 0. Hence

$$\int\limits_{\mathbb{R}} \tilde{\varphi}(s, t) H'(s)\,ds = 0,$$

for all $t > 0$ large enough. Since the integrand is nonnegative this implies that $\tilde{\varphi}(s, t) \equiv 0$ for all $s \in \mathbb{R}$ and all $t > 0$ large enough. Therefore, $\varphi \equiv 0$ by the unique continuation theorem. This is again a contradiction and hence this completes the proof of the theorem.

It remains to prove (3.12). Observe that, in the definition of g, the domain of integration contains both a half of λ and $\bar{\lambda}$ (this is where we use the fact that $\theta \in [\pi/4, \pi/2)$). We use the fact that, thanks to the Refined Asymptotics Theorem (Theorem 2.1 in [5]), u is exponentially close to the sum of the heteroclinic solution H' along λ and also along $\bar{\lambda}$. Close to λ, we can therefore estimate

$$\left|F''\big(\tilde{u}(s, t)\big) - F''\big(H(s)\big)\right| \leqslant Ce^{-\beta t},$$

for some $\beta > 0$. In fact this estimate holds at any point of $\{(x, y) \in \mathbb{R}^2\colon y > 0\}$ which is closer to λ than to $\bar{\lambda}$.

We can write any point $(r + s)\mathrm{e}^{\perp} + t\mathrm{e}$ close to $\bar{\lambda}$ as $(\bar{r} + \bar{s})\bar{\mathrm{e}}^{\perp} + \bar{t}\bar{\mathrm{e}}$. Therefore, at any such point which is closer to $\bar{\lambda}$ than to λ, we simply use the fact that

$$\left|\big(F''\big(u\big((r + s)\mathrm{e}^{\perp} + t\mathrm{e}\big)\big) - F''\big(H(s)\big)\big)H'(s)\right| \leqslant Ce^{-\beta \bar{t}} H'(\bar{s}),$$

and we conclude that

$$\left|\int\limits_{t}^{\infty} \int\limits_{\mathbb{R}} \big(F''\big(\tilde{u}(s, z)\big) - F''\big(H(s)\big)\big)\tilde{\varphi}(s, z) H'(s)\,ds\,dz\right| \leqslant Ce^{-\beta t} \int\limits_{t}^{\infty} \big(g(z) + \bar{g}(z)\big)\,dz.$$

Finally, the estimate (3.12) follows from the fact that $\bar{g} = g$. $\quad\square$

We now explain how to prove that any $u \in \mathcal{M}_4^{even}$ is nondegenerate. If $\phi \in e^{-\delta(1 + |x|^2)} W^{2,2}(\mathbb{R}^2)$ is a solution of $(\Delta - F'(u))\phi = 0$, we can decompose $\phi = \phi_e + \phi_o$ into the sum of two functions, one of which ϕ_e being even under the action of the symmetry with respect to the x-axis and the other one ϕ_o being odd under the action of the same symmetry. Since ϕ_o vanishes on the x-axis, we can use the argument already used in Step 2 to prove that $\phi_o \equiv 0$ and hence ϕ is even under the action of the symmetry with respect to the x-axis. Using similar arguments one also prove that ϕ is even under the action of the symmetry with respect to the y-axis and the nondegeneracy follows from Theorem 3.1.

Thanks to Theorem 3.1, we can apply the Implicit Function Theorem (see Section 8 and Theorem 2.2 in [5]) to show that any connected component of $\mathcal{M}_4$ is 4-dimensional and, equivalently, we conclude that any connected component of $\mathcal{M}_4^{even}$ is 1-dimensional (the rational being that, the formal dimension of the moduli space of solutions of (1.1) is equal to the number of ends but, because of the symmetries, elements of $\mathcal{M}_4^{even}$ have only one end in the quotient space). Moreover, a consequence of this Implicit Function Theorem is that close to $u \in \mathcal{M}_4^{even}$, the space $\mathcal{M}_4^{even}$ can either be parameterized by the angle θ or by the distance r and this implies that the image of *any* connected component of $\mathcal{M}_4^{even}$ by the mapping $\mathcal{F}$ is an immersed curve.

4. Two useful tools

4.1. An a priori estimate

It is well known that any solution u of (1.1) which satisfies $|u| < 1$ tends to ± 1 exponentially fast away from its nodal set. In particular, we have the:

Lemma 4.1. *Given $\delta \in (0, 1)$, there exists $\rho_\delta > 0$ such that, for any solution of* (1.1) *which satisfies $|u| < 1$, we have*

$$B(\mathbf{x}, 2\rho_\delta) \subset \mathbb{R}^2 - \mathcal{Z}(u) \quad \Rightarrow \quad |u^2 - 1| \leqslant \delta \quad in \ B(\mathbf{x}, \rho_\delta), \tag{4.13}$$

where

$$\mathcal{Z}(u) := \left\{ \mathbf{x} \in \mathbb{R}^2 \colon u(\mathbf{x}) = 0 \right\}$$

denotes the nodal set of the function u.

This result is a simple corollary of the result below, whose proof can already be found in [2] (see Lemma 3.1–Lemma 3.3 therein) and also in [10]:

Lemma 4.2. *There exist constants $C > 0$ and $\alpha > 0$ such that, for any solution of* (1.1) *which satisfies $|u| < 1$, we have*

$$\left| u(\mathbf{x})^2 - 1 \right| + \left| \nabla u(\mathbf{x}) \right| + \left| \nabla^2 u(\mathbf{x}) \right| \leqslant C e^{-\alpha \mathrm{dist}(\mathbf{x}, \mathcal{Z}(u))}, \tag{4.14}$$

for all $\mathbf{x} \in \mathbb{R}^2$.

Proof. Since this lemma plays a central role in our result, we give here a complete proof for the sake of completeness.

We denote by ϕ_R the eigenfunction which is associated to the first eigenvalue of $-\Delta$ on the ball of radius R, under 0 Dirichlet boundary conditions. We assume that ϕ_R is normalized so that $\phi_R(0) = \sup_{B(0,R)} \phi_R = 1$. Recall that the associated eigenvalue μ_R satisfies $\mu_R = \mu_1 / R^2$.

Given $\delta \in (0, 1)$, we choose $R_0 > 0$ such that

$$-F'(s) R_0^2 > \mu_1 s,$$

for all $s \in [0, 1 - \delta]$. Assume that $R > R_0$ and that $B(\mathbf{x}, 2R) \subset \mathbb{R}^2 - \mathcal{Z}(u)$. To simplify the discussion, let us also assume that $u > 0$ in $B(\mathbf{x}, 2R)$.

We claim that $u \geqslant 1 - \delta$ in $B(\mathbf{x}, R)$. Indeed, if this is not the case then there exists $\bar{\mathbf{x}} \in B(\mathbf{x}, R)$ such that $u(\bar{\mathbf{x}}) < 1 - \delta$. In this case, we define $\epsilon > 0$ to be the largest positive real such that

$$u \geqslant \epsilon \phi_R(\cdot - \bar{\mathbf{x}}),$$

in $B(\bar{\mathbf{x}}, R)$. Certainly $\epsilon \leqslant 1 - \delta$ and there exists $\mathbf{z} \in B(\bar{\mathbf{x}}, R)$ such that

$$u(\mathbf{z}) = \epsilon \phi_R(\mathbf{z} - \bar{\mathbf{x}}) \leqslant 1 - \delta.$$

By construction of R, we can write

$$-\epsilon \Delta \phi_R = \frac{\mu_1}{R^2} \epsilon \phi_R < F'(\epsilon \phi_R).$$

Since $-\Delta u = -F'(u)$ we conclude that

$$-\Delta(\epsilon \phi_R - u) < 0,$$

at the point $\mathbf{z}$ and this contradicts the fact that $u - \epsilon \phi_R$ has a local minimum at $\mathbf{z}$. The proof of the claim is complete.

We now fix $\alpha > 0$ such that $\alpha^2 < F''(1)$ and we choose $\delta \in (0, 1)$ close to 1 so that $F''(t) \geqslant \alpha^2$ for all $t \in [1 - \delta, 1]$. According to the above claim, we know that for all $R > R_0$, $u \geqslant 1 - \delta$ (or $u < \delta - 1$) in $B(\bar{\mathbf{x}}, R)$ provided $B(\bar{\mathbf{x}}, 2R) \subset \mathbb{R}^2 - \mathcal{Z}(u)$. Therefore, we get

$$-\Delta(1 - u) = -\frac{F'(u) - F'(1)}{u - 1}(1 - u) \leqslant -\alpha^2(1 - u),$$

in $B(\bar{\mathbf{x}}, R)$. A direct computation shows that

$$\left(-\Delta + \alpha^2 \right) e^{-\alpha \sqrt{1 + r^2}} \geqslant 0,$$

where $r := |\mathbf{x} - \bar{\mathbf{x}}|$. This, together with the maximum principle, implies the exponential decay of $1 - u^2$ away from $\mathcal{Z}(u)$, the zero set of u. $\quad \square$

4.2. The balancing formulae

We describe the balancing formulae for solutions of (1.1) in the form they were introduced in [5]. Assume that u is a smooth function defined in $\mathbb{R}^2$ and X a vector field also defined in $\mathbb{R}^2$. We define the vector field

$$\varXi(X, u) := \left(\frac{1}{2}|\nabla u|^2 + F(u)\right) X - X(u)\nabla u.$$

Recall that Killing vector fields are vector fields which generate the group of isometries of $\mathbb{R}^2$. They are linear combinations (with constant coefficients) of the constant vector fields ∂_x and ∂_y generating the group of translations and the vector field $x\partial_y - y\partial_x$ which generates the group of rotations in $\mathbb{R}^2$.

We have the:

Lemma 4.3 *(Balancing formulae). (See [5].) Assume that u is a solution of* (1.1) *and that X is a Killing vector field. Then* div $\varXi(X, u) = 0$.

Proof. To prove this formula, just multiply Eq. (1.1) by $X(u)$ and use simple manipulations on partial derivatives. $\quad\square$

This result is nothing but an expression of the invariance of (1.1) under the action of rigid motions. To see how useful this result will be for us, let us assume that $u \in \mathcal{M}_4^{even}$. By definition, the nodal set of u is, in the upper right quadrant $Q^{\llcorner}$ of $\mathbb{R}^2$, asymptotic to an oriented half line

$$\lambda := r e^{\perp} + \mathbb{R}e,$$

where $r \in \mathbb{R}$ and $e \in S^1$. We write $e := (\cos\theta, \sin\theta)$ and, since we assume that the oriented line λ lies in $Q^{\llcorner}$, we have $\theta \in (0, \pi/2)$.

Given $R > 0$, we define the plain triangle

$$T_R := \left\{(x, y) \in \mathbb{R}^2 : x > 0, \ y > 0 \text{ and } (x, y) \cdot e < R\right\}.$$

The divergence theorem implies that

$$\int_{\partial T_R} \varXi(X, u) \cdot \nu \, ds = 0, \tag{4.15}$$

where ν is the (outward pointing) unit normal vector field to ∂T_R.

We set

$$c_0 := \int_{-\infty}^{+\infty} \left(\frac{1}{2}(H')^2 + F(H)\right) ds. \tag{4.16}$$

Taking $X := \partial_x$ and letting R tend to infinity, we conclude, using the fact that u is asymptotic to $\pm H$ along the half line λ, that

$$c_0 \cos\theta = \int_{x=0, y>0} \left(\frac{1}{2}|\partial_y u|^2 + F(u)\right) dy. \tag{4.17}$$

Observe that we have implicitly used the fact that $\partial_x u = 0$ along the y-axis. Similarly, taking $X := \partial_y$ and letting R tend to infinity, we conclude that

$$c_0 \sin\theta = \int_{y=0, x>0} \left(\frac{1}{2}|\partial_x u|^2 + F(u)\right) dx. \tag{4.18}$$

Finally, taking $X = x\partial_y - y\partial_x$ and letting R tend to infinity, we get

$$c_0 r = \int_{x=0, y>0} \left(\frac{1}{2}|\partial_y u|^2 + F(u)\right) y \, dy - \int_{y=0, x>0} \left(\frac{1}{2}|\partial_x u|^2 + F(u)\right) x \, dx. \tag{4.19}$$

The key observation is that it is possible to detect both the angle θ and the parameter r which characterize the half line λ just by performing some integration over the x-axis and the y-axis. As one can guess this property will be very useful in the compactness analysis we are going to perform now. In some sense it will be enough to pass to the limit in the above integrals to guarantee the convergence of the parameters characterizing the asymptotics of the zero set of the solutions of (1.1). Note that similar type of identities has been used by Gui [11,12] to analyze the entire solutions of the Allen–Cahn equation.

5. Properness

In this section, we prove a compactness result for the set of 4-ended solutions of (1.1). More precisely, we prove that, given $(u_n)_{n\geqslant 0}$, with $u_n \in \mathcal{M}_4^{even}$, a sequence of solutions of (1.1) whose angles $\theta_n := \pi/4 + \mathcal{P}(u_n)$ converge to some limit angle $\theta_* \in (0, \pi/2)$, one can extract a subsequence which converges to a 4-ended solution u_* with angle θ_*. Naturally, the fact that one can extract a subsequence which converges uniformly (at least on compacts of $\mathbb{R}^2$) to a solution u_* of (1.1) is not surprising since the functions u_n are uniformly bounded and, by elliptic regularity, have gradient which is uniformly bounded, hence this compactness result simply follows from the application of Ascoli–Arzela's Theorem. In general, it is hard to say anything about the limit solution u_*. It turns out that it is possible to control the zero set of the limit solution u_* and prove that u_* is also a 4-ended solution. As we will see, a key ingredient in this analysis is provided by the balancing formulae defined in the previous section.

Theorem 5.1. *Assume that we are given a sequence $(u_n)_{n\geqslant 0}$, with $u_n \in \mathcal{M}_4^{even}$, which converge uniformly on compacts to a solution u_*. If $(\mathcal{P}(u_n))_{n\geqslant 0}$ converges in $(-\pi/4, \pi/4)$, then, $u_* \in \mathcal{M}_4^{even}$,*

$$\lim_{n\to\infty} \mathcal{P}(u_n) = \mathcal{P}(u_*),$$

and

$$\lim_{n\to\infty} u_n = u_*,$$

in the topology of $\mathcal{S}_4$.

The proof of this theorem is decomposed into many small lemmas. Assume that we are given a sequence $(u_n)_{n\geqslant 0}$, with $u_n \in \mathcal{M}_4^{even}$. Recall that

$$\partial_x u_n < 0 \quad \text{and} \quad \partial_y u_n > 0, \tag{5.20}$$

in the right upper quadrant

$$Q^\llcorner := \{(x, y) \in \mathbb{R}^2 : x > 0, \text{ and } y > 0\}.$$

We denote by

$$\mathcal{Z}_n := \{(x, y) \in \mathbb{R}^2 : u_n(x, y) = 0\},$$

the nodal set of u_n. Monotonicity of u_n in $Q^\llcorner$ implies that the zero set of u_n is either a graph over the x-axis or a graph over the y-axis. In particular, $\mathcal{Z}_n \cap \partial Q^\llcorner$ contains exactly one point which we denote by p_n

$$\mathcal{Z}_n \cap \partial Q^\llcorner = \{p_n\}.$$

We define $\theta_n := \pi/4 + \mathcal{P}(u_n)$ and r_n to be the parameters describing the asymptotics of $\mathcal{Z}_n$ in the right upper quadrant $Q^\llcorner$. In other words, $\mathcal{Z}_n \cap Q^\llcorner$ is asymptotic to the oriented half line

$$\lambda_n := r_n e_n^\perp + \mathbb{R} e_n,$$

where $e_n := (\cos\theta_n, \sin\theta_n)$. Finally, we assume that

$$\lim_{n\to\infty} \theta_n = \theta_* \in (0, \pi/2).$$

First, we prove that the point p_n where the zero set of u_n meets the boundary of the upper right quadrant $Q^\llcorner$ remains bounded as n tends to infinity.

Lemma 5.1. *Under the above assumptions, the sequence $(p_n)_{n \geqslant 0}$ remains bounded, and, up to a subsequence, can be assumed to converge.*

Proof. We argue by contradiction and for example assume that, up to a subsequence, $p_n = (0, y_n)$ with $\lim_{n \to \infty} y_n = +\infty$.

We define
$$w_n(x, y) := u_n(x, y + y_n).$$

Standard arguments involving elliptic estimates and Ascoli–Arzela's Theorem imply that, up to a subsequence, the sequence $(w_n)_{n \geqslant 0}$ converges uniformly on compacts to some function w which is defined on $\mathbb{R}^2$ and which is again a solution of (1.1). Since $\partial_y u_n > 0$ for $y > 0$, we conclude that
$$\partial_y w \geqslant 0$$
in $\mathbb{R}^2$. Moreover, $w(0, 0) = 0$ and $\partial_x w(0, y) = 0$ for all $y \in \mathbb{R}$ since u_n is even with respect to the y-axis.

Observe that, since w_n is not identically equal to 0, we may apply Lemma 4.1 to conclude that w is not identically equal to 0. Indeed, thanks to (5.20) we know that the zero set of u_n is a graph over the x-axis for some function which is increasing and $u_n < 0$ in the set $B_n := \{(x, y) \in \mathbb{R}^2 : |y| < y_n\}$. In particular, provided n is chosen large enough, discs of arbitrary large radii can be inserted in B_n and Lemma 5.20 implies that $|u_n^2 - 1| \leqslant 1/2$ in $\{(x, y) \in \mathbb{R}^2 : |y| < y_n - \rho_{1/2}\}$.

Therefore, w is bounded, monotone increasing in y, even with respect to the y-axis and according to De Giorgi's conjecture in dimension 2 which is proven in [10], we conclude that
$$w(x, y) = H(y).$$

Now, we use (4.17) which tells us that
$$c_0 \cos \theta_n = \int\limits_{x=0, y>0} \left(\frac{1}{2} |\partial_y u_n|^2 + F(u_n) \right) dy.$$

Passing to the limit as n tends to infinity, we conclude that
$$c_0 \cos \theta_* = \lim_{n \to \infty} \int\limits_{x=0, y>0} \left(\frac{1}{2} |\partial_y u_n|^2 + F(u_n) \right) dy.$$

However, given $y_* > 0$,
$$\int\limits_{x=0, y>0} \left(\frac{1}{2} |\partial_y u_n|^2 + F(u_n) \right) dy \geqslant \int\limits_{x=0, y-y_n \in [-y_*, y_*]} \left(\frac{1}{2} |\partial_y u_n|^2 + F(u_n) \right) dy$$
$$= \int\limits_{x=0, y \in [-y_*, y_*]} \left(\frac{1}{2} |\partial_y w_n|^2 + F(w_n) \right) dy,$$

for all n large enough so that $y_n - y_* > 0$. Passing to the limit as n tends to infinity, we conclude that
$$\lim_{n \to \infty} \int\limits_{x=0, y>0} \left(\frac{1}{2} |\partial_y u_n|^2 + F(u_n) \right) dy \geqslant \int\limits_{[-y_*, y_*]} \left(\frac{1}{2} |H'(s)|^2 + F(H(s)) \right) ds.$$

Hence
$$c_0 \cos \theta_* \geqslant \int\limits_{[-y_*, y_*]} \left(\frac{1}{2} |H'(s)|^2 + F(H(s)) \right) ds.$$

Since y_* can be chosen arbitrarily large, we get

$$c_0 \cos \theta_* \geqslant \int_{\mathbb{R}} \left(\frac{1}{2} |H'(s)|^2 + F(H(s)) \right) ds = c_0,$$

which is clearly in contradiction with the fact that $\theta_* > 0$. If $p_n = (x_n, 0)$ with x_n tending to infinity, similar arguments using (4.18) and the fact that $\theta_* < \pi/2$. This completes the proof of the result. □

Thanks to the above lemma, we know that u_* is not identically constant equal to 0 or ± 1. Therefore, according to Theorem 4.4 in [11] we know that the nodal set of u_* in the upper right quadrant $Q^\llcorner$ must be an asymptotically straight line which is not parallel to the x-axis nor to the y-axis. The Refined Asymptotic Theorem (Theorem 2.1 in [5]) then implies that u_* is a 4-ended solution. Some comment is due in the way the Refined Asymptotic Theorem (Theorem 2.1 in [5]) is used. Indeed, in the statement of this result, one start with a solution of (1.1) which differs from the model heteroclinic solution sharing the same end by some $W^{2,2}$ function. Nevertheless, close inspection of the proof of Theorem 2.1 in [5] shows that the result remains valid provided we start from a solution which is asymptotic to the model heteroclinic solution in the L^∞ sense, and this is precisely the situation in which we need the result.

To proceed, we prove that we have a uniform control on the nodal set of u_n away from a compact set. Again, the balancing formulae will play an important role in the control of the nodal set of u_n.

To fix the ideas, we assume that the nodal set of u_n in the upper right quadrant $Q^\llcorner$ is the graph of a function $y = f_n(x)$. Since u_n is a 4-ended solution and given the notations introduced at the beginning of this section, we know that f_n is asymptotic to the affine function given by

$$\tilde{f}_n(x) := \tan \theta_n x + \frac{r_n}{\cos \theta_n}.$$

In particular, given $\delta > 0$ and $n \geqslant 0$, there exists $x_{n,\delta} > 0$ such that

$$\left| f'_n(x) - \tan \theta_n \right| < \delta,$$

for all $x \geqslant x_{n,\delta}$. In the next lemma, we prove that $x_{n,\delta}$ can be chosen to be independent of $n \geqslant 0$. In other words, this provides a uniform control on the derivative of f_n away from a compact set.

Lemma 5.2. *For all $\delta > 0$, there exists $x_\delta > 0$ such that*

$$\left| f'_n(x) - \tan \theta_n \right| < \delta,$$

for all $n \geqslant 0$ and for all $x \geqslant x_\delta$.

Proof. We now argue by contradiction. Observe that the result is true if we restrict our attention to a finite number of the u_n. Hence, if the result were not true, there would exist $\delta_* > 0$ and sequences $(x_k)_{k \geqslant 0}$ and $(n_k)_{k \geqslant 0}$ both tending to infinity such that

$$\sup_{x \geqslant x_k} \left| f'_{n_k}(x) - \tan \theta_{n_k} \right| \geqslant \delta_*.$$

We define $\bar{x}_k \geqslant x_k$ to be the supremum of the $x > 0$ such that $|f'_{n_k}(x) - \tan \theta_{n_k}| \geqslant \delta_*$. Observe that $\bar{x}_k$ is well defined since

$$\lim_{x \to \infty} f'_{n_k}(x) = \tan \theta_{n_k}.$$

By definition, we have

$$\left| f'_{n_k}(\bar{x}_k) - \tan \theta_{n_k} \right| = \delta_*, \tag{5.21}$$

and

$$\sup_{x \geqslant \bar{x}_k} \left| f'_{n_k}(x) - \tan \theta_{n_k} \right| \leqslant \delta_*. \tag{5.22}$$

Moreover, the sequence $(\bar{x}_k)_{k \geqslant 0}$ tends to infinity as k tends to infinity and, if we define

$$\bar{y}_k := f_{n_k}(\bar{x}_k),$$

we find that the sequence $(\bar{y}_k)_{k\geqslant 0}$ also tends to infinity as k does. This latter fact is just a consequence of the fact that $(u_n)_{n\geqslant 0}$ converges on compacts to u_* which is a 4-ended solution and hence the zero set of u_* is the graph of a function which tends to infinity at infinity.

We now consider the domain D_k of $Q^{\llcorner}$ which contains the graph of f_{n_k} for x large enough and which is bounded by the half line $t \mapsto (0, \bar{y}_k - \bar{x}_k + t)$ for $t > 0$, the segment joining $(0, \bar{y}_k - \bar{x}_k)$ to $(\bar{x}_k + \bar{y}_k, 0)$ and the half line $t \mapsto (\bar{x}_k + \bar{y}_k + t, 0)$ for $t > 0$. Observe that ∂D_k contains the point $(\bar{x}_k, \bar{y}_k)$. Applying the analysis of Section 4, we conclude that

$$\int_{\partial Q^{\llcorner}} \varXi(\partial_x, u_{n_k}) \cdot v \, ds = \int_{\partial D_k} \varXi(\partial_x, u_{n_k}) \cdot v \, ds,$$

where v denotes the outward pointing normal vector to the sets $Q^{\llcorner}$ and D_k. Using (4.17), we conclude that

$$c_0 \cos \theta_{n_k} = \int_{\partial D_k} \varXi(\partial_x, u_{n_k}) \cdot v \, ds.$$

Observe that, up to a subsequence, the sequence of functions

$$(x, y) \mapsto u_{n_k}(x + \bar{x}_k, y + \bar{y}_k)$$

converges, uniformly on compacts, to the heteroclinic solution whose zero set is the line passing through the origin, of slope $\lim_{k\to\infty} f'_{n_k}(\bar{x}_k)$. Moreover, thanks to (5.22) we see that there exist constants $C > 0$ and $\beta > 0$, independent of $k \geqslant 0$, such that

$$\left| u_{n_k}^2(\mathrm{x}) - 1 \right| + \left| \nabla u_{n_k}(\mathrm{x}) \right| \leqslant C e^{-\beta|\mathrm{x} - \bar{\mathrm{x}}_k|},$$

for all $\mathrm{x} \in \partial D_k$, where $\bar{\mathrm{x}}_k := (\bar{x}_k, \bar{y}_k)$. This property, together with the result of Lemma 4.2 allows one to conclude that

$$\lim_{k\to\infty} \int_{\partial D_k} \varXi(\partial_x, u_{n_k}) \cdot v \, ds = c_0 \cos \tilde{\theta}_*,$$

where $\tilde{\theta}_*$ is defined by

$$\tan \tilde{\theta}_* = \lim_{k\to\infty} f'_{n_k}(\bar{x}_k).$$

This is clearly in contradiction with (5.21) which implies that $|\tan \theta_* - \tan \tilde{\theta}_*| = \delta_*$. This completes the proof of the result. $\quad\Box$

As a consequence, we have the:

Lemma 5.3. *Under the above assumptions, we have* $\lim_{n\to\infty} \mathcal{P}(u_n) = \mathcal{P}(u_*)$.

Since u_n converges on compacts to u_* which is a 4-ended solution, we conclude, with the help of the previous lemma that the distance from a point $\mathrm{x} \in \mathcal{Z}_n$ to the x-axis and the y-axis, tends to infinity as $|\mathrm{x}|$ tends to infinity. We now prove a more quantitative version of this assertion in the:

Lemma 5.4. *There exist constants $C > 0$ and $\alpha > 1$, such that*

$$\mathcal{Z}_n \cap Q^{\llcorner} \subset \left\{ (x, y) \in \mathbb{R}^2 \colon x > 0, \ y > 0, \ and \ \frac{x}{\alpha} - C \leqslant y \leqslant \alpha x + C \right\}.$$

Proof. According to Lemma 5.2, we have a uniform control on the slopes of the nodal sets of u_n away from a tubular neighborhood of the x-axis and y-axis. This means that these slopes are bounded away from 0 and ∞ independently of n. Next, in a ball of fixed radius, u_n converges uniformly to u_* and the result then follows at once. $\quad\Box$

Recall that

$$\mathcal{F}(u_n) = (\theta_n - \pi/4, r_n).$$

We set

$$(\theta_* - \pi/4, r_*) := \mathcal{F}(u_*).$$

Now that we have understood the behavior of the sequence $(\theta_n)_{n \geqslant 0}$, we turn to the behavior of the sequence $(r_n)_{n \geqslant 0}$, the other parameter which characterizes the asymptotic of the nodal set of u_n. We have the:

Lemma 5.5. *Under the above assumptions,* $\lim_{n \to \infty} r_n = r_*$.

Proof. Again, the proof uses the balancing formula (4.15) but this time, we will use the vector field

$$X = x \partial_y - y \partial_x.$$

Recall that (4.19) yields

$$c_0 r_n = \int\limits_{x=0, y>0} \left(\frac{1}{2} |\partial_y u_n|^2 + F(u_n) \right) y \, dy - \int\limits_{y=0, x>0} \left(\frac{1}{2} |\partial_x u_n|^2 + F(u_n) \right) x \, dx.$$

The key ingredients are Lemma 5.4 and Lemma 4.2, from which we get an exponential decay of the solution u_n along the coordinate axis as $|x|$ tends to infinity, the decay being uniform in $n \geqslant 0$. Once this decay is proven one uses the fact that $(u_n)_{n \geqslant 0}$ converges in $\mathcal{C}^1$ topology to u_* uniformly on any given ball.

Using these remarks, one can pass to the limit as n tends to infinity in the above equality to get

$$c_0 \lim_{n \to \infty} r_n = \int\limits_{x=0, y>0} \left(\frac{1}{2} |\partial_y u_*|^2 + F(u_*) \right) y \, dy - \int\limits_{y=0, x>0} \left(\frac{1}{2} |\partial_x u_*|^2 + F(u_*) \right) x \, dx.$$

Since the right hand side is equal to $c_0 r_*$, the proof is complete. □

At this point, we have shown that the sequence $(u_n)_{n \geqslant 0}$ converges uniformly on compacts to u_* and the ends of u_n also converges to the end of u_*. However, this is not quite enough since our aim is to show the convergence of $(u_n)_{n \geqslant 0}$ to u_* in $\mathcal{S}_4$.

Recall that $\mathcal{Z}_n$, the zero set of u_n, is asymptotic to

$$\lambda_n := r_n e_n^\perp + \mathbb{R} e_n,$$

in $Q^\llcorner$, where $e_n = (\cos \theta_n, \sin \theta_n)$. We define v_n in $Q^\llcorner$ by

$$v_n(x) := u_n(x) - H(x \cdot e_n^\perp - r_n).$$

Then $|v_n| \to 0$ as $|x|$ tends to infinity in $Q^\llcorner$.

In the next lemma, we prove that this convergence is in fact uniform in $n \geqslant 0$.

Lemma 5.6. *As $|x|$ tends to infinity, $|v_n(x)|$ converges to 0 uniformly with respect to $n \geqslant 0$.*

Proof. The proof is by contradiction. If the result were not true, there would exist $\epsilon > 0$, a sequence $(R_j)_{j \geqslant 0}$ tending to infinity, a sequence $(x_j)_{j \geqslant 0}$ such that $|x_j| \geqslant R_j$ and a sequence $(n_j)_{j \geqslant 0}$ such that

$$\left| v_{n_j}(x_j) \right| \geqslant \epsilon. \tag{5.23}$$

Up to a subsequence we can assume that $(\theta_{n_j}, r_{n_j})_{j \geqslant 0}$ converges to (θ_*, r_*).

Observe that the distance from x_j to λ_{n_j} is necessarily bounded since, according to Lemma 4.2, v_{n_j} tends to 0 away from λ_{n_j}. Let $\bar{x}_j$ be the orthogonal projection of x_j onto λ_{n_j}.

Making use of elliptic estimates and the Arzela–Ascoli Theorem, we can assume, up to a subsequence, that $(u_{n_j}(\cdot - \bar{x}_j))_{j \geqslant 0}$ converges uniformly on compacts to a solution of (1.1) which is nontrivial and which, thanks to

De Giorgi's conjecture, is a heteroclinic solution $\bar{u}$ of (1.1). The end of this heteroclinic solution is the affine line of angle $\bar{\theta}$. As in the proof of Lemma 5.4, we use the vector field $X = \partial_x$ in the balancing formula to conclude that $\theta_* = \bar{\theta}$.

Therefore, the parameters of the end of $\bar{u}$ are given by $(\theta_*, \bar{r})$. As in the proof of Lemma 5.4, we use the vector field $X = x\partial_y - y\partial_x$ in the balancing formula to conclude that $r_* = \bar{r}$. This is clearly a contradiction with (5.23). $\quad\square$

Thanks to the Refined Asymptotics Theorem (Theorem 2.1 in [5]) we can decompose

$$u_n = v_n + u_{\lambda_n},$$

and

$$u_* = v_* + u_{\lambda_*},$$

where $\lambda_n, \lambda_* \in \Lambda_{ord}^4$ and where $v_n, v_* \in e^{-\delta(1+|x|^2)} W^{2,2}(\mathbb{R}^2)$ for some $\delta > 0$. Observe that, *a priori*, the parameter δ can vary with n but close inspection of the proof of Theorem 2.1 in [5] shows that $\delta > 0$ can indeed be chosen independently of $n \geqslant 0$ since the ends λ_n converge to a fixed end λ_*.

This, together with the fact that $(u_n)_{n \geqslant 0}$ converges uniformly on compacts to u_* implies that $(u_n)_{n \geqslant 0}$ converges to u_* in the topology of $\mathcal{S}_4$. This completes the proof of the properness of the classifying map $\mathcal{P}$.

Let M be the connected component of $\mathcal{M}_4^{even}$ which contains the saddle solution. We claim that the properness of $\mathcal{P}$ implies that the image by $\mathcal{P}$ of M is the entire interval $(-\pi/4, \pi/4)$. The proof of this claim goes as follows: we argue by contradiction and assume that $\mathcal{P} : M \to (-\pi/4, \pi/4)$ is not onto. Recall that if $u \in \mathcal{M}_4^{even}$, then $\bar{u}$ defined by

$$\bar{u}(x, y) := -u(y, x)$$

also belongs to $\mathcal{M}_4^{even}$ and M is also invariant under this transformation. We will write $\bar{u} = Ju$. The properness of $\mathcal{P}$ implies that M is compact and one dimensional. Hence, it must be diffeomorphic to S^1. Obviously $J : M \to M$ is a diffeomorphism and the saddle solution is a fixed point of J. Since M is diffeomorphic to S^1, there must be at least another fixed element $v \in M$ which is a fixed point of J. Then, the zero set of v is union of the two lines $y = \pm x$. But, according to [4] or [11], a solution of (1.1) having as zero set the two lines $y = \pm x$ is the saddle solution. This is a contradiction and the proof of the claim is complete. Note that this argument does not guarantee that there are no other compact connected components in $\mathcal{M}_4^{even}$. To prove this fact, we will need one more result which will be described in the next section. In any case, instead of using the argument outlined above to show that $\mathcal{P}$ is onto, one can use the next section of the paper.

6. Connected component of $\mathcal{M}_4^{even}$ are not compact

We have shown in Section 3 that elements in $\mathcal{M}_4^{even}$ are even-nondegenerate. According to the moduli space theory for solutions of (1.1) (see Section 8 and Theorem 2.2 in [5]), any connected component of $\mathcal{M}_4^{even}$ is a one dimensional manifold and its image by $\mathcal{F}$ is a smooth (possibly immersed) curve in $(-\pi/4, \pi/4) \times \mathbb{R}$. In particular, any compact connected component $M \subset \mathcal{M}_4^{even}$ would have to be diffeomorphic to S^1. In this section, we show that this cannot happen.

Theorem 6.1. *All connected components of $\mathcal{M}_4^{even}$ are not compact, namely, there is no closed loop in $\mathcal{M}_4^{even}$.*

Proof. We argue by contradiction and assume that $\mathcal{M}_4^{even}$ contains a connected component M which is diffeomorphic to S^1. We choose a smooth regular parameterization of M by

$$\sigma \in S^1 \mapsto u(\cdot, \sigma) \in M,$$

so that

$$\Delta u(\cdot, \sigma) - F'\big(u(\cdot, \sigma)\big) = 0,$$

for all $\sigma \in S^1$ and $\partial_\sigma u \neq 0$ for all $\sigma \in S^1$. Differentiation with respect to σ implies that $\partial_\sigma u \in T_u \mathcal{S}_4$ satisfies

$$\left(\Delta - F''(u)\right)\partial_\sigma u = 0.$$

Observe that, for all $\mathrm{x} \in \mathbb{R}^2$,

$$0 = u(\mathrm{x}, 2\pi) - u(\mathrm{x}, 0) = \int\limits_0^{2\pi} \partial_\sigma u(\mathrm{x}, \sigma)\, d\sigma.$$

Choosing x to be the origin, this implies that there exists $\sigma_* \in S^1$, such that $\partial_\sigma u((0,0), \sigma_*) = 0$. We define

$$\phi := \partial_\sigma u(\cdot, \sigma_*).$$

Observe that $\phi \neq 0$ and that ϕ is even with respect to the symmetry about both the x-axis and the y-axis.

By definition, any element u of $\mathcal{S}_4$ can be decomposed into the sum of a function in $W^{2,2}(\mathbb{R}^2)$ and an element of the form u_λ as defined in (2.5). Moreover, because of the symmetries, u_λ only depends on the two parameters r and θ which characterize λ. In particular, the tangent space of $\mathcal{S}_4$ at u can be decomposed as

$$T_u \mathcal{S}_4 = W^{2,2}(\mathbb{R}^2) \oplus \mathfrak{D},$$

where

$$\mathfrak{D} := \mathrm{Span}\{\partial_r u_\lambda, \partial_\theta u_\lambda\}.$$

It is easy to check that $\partial_\theta u_\lambda$ is linearly growing along the zero set of u while $\partial_r u_\lambda$ is bounded.

Since $\phi(0,0) = 0$ and since ϕ is symmetric with respect to the x-axis and the y-axis, there exists Ω, a nodal domain of ϕ, which is included in one of the four half spaces $\{(x, y) \in \mathbb{R}^2 \colon \pm x > 0\}$ or $\{(x, y) \in \mathbb{R}^2 \colon \pm y > 0\}$. We claim that this nodal domain can be chosen so that ϕ is bounded on it. Indeed, if $\phi \in W^{2,2}(\mathbb{R}^2) \oplus \mathrm{Span}\{\partial_r u_\lambda\}$, then ϕ is bounded and one can select any nodal domain contained in a half space.

The other case to consider is the case where $\phi = a\partial_\theta u_\lambda + \tilde{\phi}$ where $\tilde{\phi}$ is bounded. Inspection of $\partial_\theta u_\lambda$ near the end of u shows that, away from a large ball $B(0, R)$, the function ϕ does not vanish along the zero set of u. In this case, it is enough to select a nodal domain of ϕ which is unbounded and which, away from $B(0, R)$, does not contain the zero set of u. It is easy to check that ϕ is bounded in such a nodal domain.

For example, let us assume that the nodal domain $\Omega \subset \{(x, y) \in \mathbb{R}^2 \colon x > 0\}$. Then, one can repeat the argument of Step 2 in the proof of Theorem 3.1, with $\psi = \partial_x u$, to prove that

$$\int\limits_\Omega |\nabla h|^2 \psi^2 \zeta_R^2\, d\mathrm{x} \leqslant 2\left(\int\limits_{\Omega \cap A_R} |\nabla h|^2 \psi^2 \zeta_R^2 d\mathrm{x}\right)^{1/2} \left(\int\limits_{\Omega \cap A_R} \phi^2 |\nabla \zeta_R|^2\, d\mathrm{x}\right)^{1/2}, \tag{6.24}$$

where $h := \frac{\phi}{\psi}$. Using the fact that ϕ is bounded, and letting R tend to infinity, we conclude that

$$\int\limits_\Omega |\nabla h|^2 \psi^2\, d\mathrm{x} < +\infty.$$

Using this information back into (6.24), and letting R tend to infinity, we conclude that

$$\int\limits_\Omega |\nabla h|^2 \psi^2\, d\mathrm{x} = 0,$$

and this implies that $\phi \equiv 0$ in Ω. The unique continuation theorem then implies that $\phi \equiv 0$, which is a contradiction. $\quad\Box$

We observe that from the above considerations, we can give a different proof of Theorem 2.7. Indeed, we choose M to be the connected component of $\mathcal{M}_4^{even}$ which contains the saddle solution. Using the Implicit Function Theorem (Theorem 2.2 in [5]), which applies since we have proven that any element of $\mathcal{M}_4^{even}$ is even-nondegenerate, we conclude that M is a smooth, one dimensional manifold. By Theorem 6.1, M is necessarily noncompact and the

image of M by $\mathcal{P}$ cannot be compact either. Hence the image of M by $\mathcal{P}$ contains either an interval of the form $(-\pi/4, \delta)$ or $(\delta, \pi/4)$. Since the image by $\mathcal{P}$ of the saddle solution is 0, we conclude that $(-\pi/4, \delta)$ or $(\delta, \pi/4)$ contains 0. Moreover, the image of M by $\mathcal{P}$ is symmetric with respect to 0 and hence it has to be the whole interval $(-\pi/4, \pi/4)$.

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Further reading

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