

Linear elasticity obtained from finite elasticity by Γ -convergence under weak coerciveness conditions

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Abstract

The energy functional of linear elasticity is obtained as Γ -limit of suitable rescalings of the energies of finite elasticity. The quadratic control from below of the energy density $W(\nabla v)$ for large values of the deformation gradient ∇v is replaced here by the weaker condition $W(\nabla v) \geq |\nabla v|^p$, for some $p > 1$. Energies of this type are commonly used in the study of a large class of compressible rubber-like materials.

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1. Introduction

Consider an elastic body occupying a reference configuration $\Omega \subseteq \mathbb{R}^n$, with $n \geq 2$, subject to some deformation $v : \Omega \rightarrow \mathbb{R}^n$. Assuming that the body is homogeneous and hyperelastic, the stored energy can be written as

$$\int_{\Omega} W(\nabla v) dx,$$

where ∇v is the deformation gradient and the energy density $W(F) \geq 0$ is defined for every $F \in \mathbb{R}^{n \times n}$ and is finite only for $\det F > 0$. We assume that the energy density W is minimized at the value 0 by the identity matrix I , which amounts to saying that the reference configuration is stress free. We assume also that W is frame indifferent, i.e., $W(F) = W(RF)$ for every $F \in \mathbb{R}^{n \times n}$ and every R in the space $SO(n)$ of rotations.

Since the deformation $v(x) = x$ is an equilibrium when no external loads are applied, we expect that small external loads $\varepsilon l(x)$ will produce deformations of the form $v(x) = x + \varepsilon u(x)$, so that the total energy is given by

$$\int_{\Omega} W(I + \varepsilon \nabla u) dx - \varepsilon^2 \int_{\Omega} l u dx. \quad (1.1)$$

In the case ∇u bounded, by Taylor-expanding $W(I + \varepsilon \nabla u)$ around I and rescaling (1.1) by ε^{-2} , we obtain in the limit $\varepsilon \rightarrow 0$ the formula

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$$\frac{1}{2} \int_{\Omega} D^2 W(I) [\nabla u]^2 dx - \int_{\Omega} l u dx, \quad (1.2)$$

where $D^2 W(I)[F]^2$ is the second differential of W at I applied to the pair $[F, F]$. By frame indifference, the first summand in (1.2) depends only on the symmetric part $e(u)$ of the displacement gradient ∇u , i.e.,

$$\frac{1}{2} \int_{\Omega} D^2 W(I) [\nabla u]^2 dx = \frac{1}{2} \int_{\Omega} D^2 W(I) [e(u)]^2 dx.$$

This functional is the linearized elastic energy associated with the displacement u .

This elementary derivation of linear elasticity requires only C^2 regularity of W near I , and hence in a neighbourhood of $SO(n)$, by frame indifference. However, it does not guarantee that the minimizers of the most natural boundary value problems for (1.1) converge to the minimizer of the corresponding problems for the limit functional (1.2).

Convergence of minimizers has been established in [5] in the framework of Γ -convergence, under the assumption

$$W(F) \geq d(F, SO(n))^2, \quad (1.3)$$

where $d(F, SO(n))$ is the distance of F from $SO(n)$. The main result of the present paper is that the same conclusion holds if (1.3) is satisfied only in a neighbourhood of $SO(n)$, while the weaker condition

$$W(F) \geq c d(F, SO(n))^p, \quad \text{for some } 1 < p \leq 2 \text{ and } c > 0, \quad (1.4)$$

is assumed far from $SO(n)$. Similar results have been obtained in [11] assuming also a bound of order p from above.

The reason for considering energies satisfying (1.4) without any bound from above is not purely academic. Indeed, for a large class of compressible rubber-like materials, (1.4) is the appropriate behaviour (see Remark 2.8 and the discussion in [1] for a multiwell case).

The main tool for the proof of the compactness of the minimizers considered in [5] is the *Geometric Rigidity Lemma* of [8]. To obtain the same result when (1.3) holds only near $SO(n)$, while (1.4) holds far from $SO(n)$, we need a version with two exponents of the Geometric Rigidity Lemma, similar to those used in [3,10,11].

The proof of the Γ -convergence in the present paper has been renewed with respect to [5], and also with respect to the further improvements introduced in [12]. The main simplification relies on some arguments developed in [8] for the rigorous proof of dimension reduction results.

Moreover, the strong convergence in $W^{1,p}$ of the minimizers is obtained by adapting to our techniques some ideas introduced in [12] for the case of multiwell energies satisfying the analog of (1.3). We hope that all our results can be extended to multiwell energies satisfying only (1.4) far from the wells.

2. Setting of the problem and main results

Throughout the paper, $d(\cdot, \cdot)$ denotes the Euclidean distance both between two points and between a point and a set. The space of $n \times n$ real matrices is identified with $\mathbb{R}^{n \times n}$; $SO(n)$ is the set of rotations, $Sym(n)$ and $Skw(n)$ the sets of symmetric and skew-symmetric matrices, respectively, $Psym(n)$ the set of positive definite symmetric matrices, $Lin^+(n)$ the set of invertible matrices with positive determinant. Given $M \in \mathbb{R}^{n \times n}$, $sym M$ and $skw M$ denote the symmetric and the skew-symmetric part of M , respectively.

The reference configuration Ω is a bounded connected open set of \mathbb{R}^n with Lipschitz boundary $\partial\Omega$. We will prescribe a Dirichlet condition on a part $\partial_D\Omega$ of $\partial\Omega$ with Lipschitz boundary in $\partial\Omega$, according to the following definition.

Definition 2.1. Let us define

$$\begin{aligned} Q &:= (-1, 1)^n, & Q^+ &:= (-1, 1)^{n-1} \times (0, 1), \\ Q_0 &:= (-1, 1)^{n-1} \times \{0\}, & Q_0^+ &:= (-1, 1)^{n-2} \times (0, 1) \times \{0\}. \end{aligned}$$

We say that $E \subseteq \partial\Omega$ has *Lipschitz boundary in $\partial\Omega$* if it is nonempty and for every x in the boundary of E for the relative topology of $\partial\Omega$ there exist an open neighbourhood U of x in \mathbb{R}^n and a bi-Lipschitz homeomorphism $\psi : U \rightarrow Q$ such that

$$\psi(U \cap \Omega) = Q^+, \quad \psi(U \cap \partial\Omega) = Q_0, \quad \psi(U \cap E) = Q_0^+.$$

The Sobolev space $W^{1,p}(\Omega, \mathbb{R}^n)$ will be denoted by $W^{1,p}$. To deal with the Dirichlet boundary condition, for every $h \in W^{1,p}$ we introduce the set

$$W_h^{1,p} := \{u \in W^{1,p} : u = h \text{ } \mathcal{H}^{n-1}\text{-a.e. on } \partial_D \Omega\}, \tag{2.1}$$

where the equality on $\partial_D \Omega$ refers to the traces of the functions on the boundary $\partial \Omega$, and \mathcal{H}^{n-1} denotes the $(n - 1)$ -dimensional Hausdorff measure.

We suppose the material to be hyperelastic and that the stored energy density $W : \Omega \times \mathbb{R}^{n \times n} \rightarrow [0, \infty]$ is $\mathcal{L} \times \mathcal{B}$ -measurable, where \mathcal{L} and \mathcal{B} are the σ -algebras of the Lebesgue measurable subsets of \mathbb{R}^n and Borel measurable subsets of $\mathbb{R}^{n \times n}$, respectively. We assume that W satisfies the following properties for a.e. $x \in \Omega$:

- (i) $W(x, \cdot)$ is frame indifferent;
- (ii) $W(x, \cdot)$ is of class C^2 in some neighbourhood of $SO(n)$, independent of x , where the second derivatives are bounded by a constant independent of x ;
- (iii) $W(x, F) = 0$ if $F \in SO(n)$;
- (iv) $W(x, F) \geq g_p(d(F, SO(n)))$, for some $1 < p \leq 2$, where $g_p : [0, \infty) \rightarrow \mathbb{R}$ is defined by

$$g_p(t) := \begin{cases} \frac{t^2}{2}, & \text{if } 0 \leq t \leq 1, \\ \frac{t^p}{p} + \frac{1}{2} - \frac{1}{p}, & \text{if } t > 1. \end{cases} \tag{2.2}$$

Observe that these assumptions are compatible with the condition $W(x, F) = \infty$, if $\det F \leq 0$, which is classical in the context of finite elasticity. Also, observe that g_p is a convex function. In what follows D^2 denotes the second differential with respect to the variable $F \in \mathbb{R}^{n \times n}$, so that $D^2W(x, F)[M]^2$ means the second differential of the function W with respect to F , evaluated at the point (x, F) and applied to the pair $[M, M]$. By frame indifference, for a.e. $x \in \Omega$ we have that

$$D^2W(x, I)[M]^2 = D^2W(x, I)[\text{sym } M]^2, \quad \text{for every } M \in \mathbb{R}^{n \times n}. \tag{2.3}$$

Together with assumption (iv), this implies that the quadratic form $D^2W(x, I)[\cdot]^2$ is null on $Skw(n)$ and satisfies the coerciveness condition

$$D^2W(x, I)[\text{sym } M]^2 \geq |\text{sym } M|^2, \quad \text{for a.e. } x \in \Omega \text{ and every } M \in \mathbb{R}^{n \times n}. \tag{2.4}$$

Energy densities for which estimate (iv) holds only with $1 < p < 2$ are commonly used in the study of compressible elastomers (see Remark 2.8).

The load is modelled by a continuous linear functional $\mathcal{L} : W^{1,p} \rightarrow \mathbb{R}$. If $v \in W^{1,p}$ represents the deformation of the elastic body, the stable equilibria of the elastic body are obtained by minimizing the functional

$$\int_{\Omega} W(x, \nabla v) dx - \mathcal{L}(v),$$

under the prescribed boundary conditions. We are interested in the case where the load has the form $\varepsilon \mathcal{L}$ and we want to study the behaviour of the solution as ε tends to zero. We write

$$v = x + \varepsilon u$$

and we assume Dirichlet boundary condition of the form

$$v = x + \varepsilon h \text{ } \mathcal{H}^{n-1}\text{-a.e. on } \partial_D \Omega,$$

with a prescribed $h \in W^{1,\infty}$. The corresponding minimum problem for u becomes

$$\min_{W^{1,p}} \left\{ \int_{\Omega} W(x, I + \varepsilon \nabla u) dx - \varepsilon \mathcal{L}(\varepsilon u) \right\}, \tag{2.5}$$

where the term $\varepsilon \mathcal{L}(x)$ has been neglected since it does not depend on u . The following theorem is the main result of the paper. It describes the behaviour of the minimizers of (2.5).

Theorem 2.2. Assume that $W : \Omega \times \mathbb{R}^{n \times n} \rightarrow [0, \infty]$ satisfies conditions (i)–(iv) for some $1 < p \leq 2$, and let $h \in W^{1,\infty}$. For every $\varepsilon > 0$ let

$$m_\varepsilon := \inf_{u \in W_h^{1,p}} \left\{ \frac{1}{\varepsilon^2} \int_{\Omega} W(x, I + \varepsilon \nabla u) dx - \mathcal{L}(u) \right\}, \quad (2.6)$$

and let $\{u_\varepsilon\}_{\varepsilon > 0}$ be a sequence such that

$$\frac{1}{\varepsilon^2} \int_{\Omega} W(x, I + \varepsilon \nabla u_\varepsilon) dx - \mathcal{L}(u_\varepsilon) = m_\varepsilon + o(1). \quad (2.7)$$

Then, $\{u_\varepsilon\}$ converges strongly in $W^{1,p}$ to the unique solution of the problem

$$m := \min_{u \in W_h^{1,2}} \left\{ \frac{1}{2} \int_{\Omega} D^2 W(x, I) [e(u)]^2 dx - \mathcal{L}(u) \right\}, \quad (2.8)$$

where $e(u) := \text{sym}(\nabla u)$. Moreover, $m_\varepsilon \rightarrow m$.

In the case $1 < p < 2$, Theorem 2.2 asserts that a sequence of “almost minimizers” in $W_h^{1,p}$ for the ε -problems converges to a minimizer for the limit problem in a different Sobolev space: indeed, the limit problem is formulated in $W_h^{1,2}$.

In the case $p = 2$, weak convergence of the “almost minimizers” has already been proved in [5]. Theorem 2.2 extends this result to the case $1 < p \leq 2$ and provides also strong convergence. The proof is based on the following three results which are proved in Sections 3, 4, and 5, respectively. These involve the functionals $\mathcal{F}_\varepsilon, \mathcal{F} : W^{1,p} \rightarrow [0, \infty]$ defined by

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \frac{1}{\varepsilon^2} \int_{\Omega} W(x, I + \varepsilon \nabla u) dx, & \text{if } u \in W_h^{1,p}, \\ \infty, & \text{otherwise,} \end{cases} \quad (2.9)$$

and

$$\mathcal{F}(u) := \begin{cases} \frac{1}{2} \int_{\Omega} D^2 W(x, I) [e(u)]^2 dx, & \text{if } u \in W_h^{1,2}, \\ \infty, & \text{otherwise,} \end{cases} \quad (2.10)$$

and the functionals $\mathcal{G}_\varepsilon, \mathcal{G} : W^{1,p} \rightarrow (-\infty, \infty]$ defined by

$$\mathcal{G}_\varepsilon := \mathcal{F}_\varepsilon - \mathcal{L}, \quad \mathcal{G} := \mathcal{F} - \mathcal{L}. \quad (2.11)$$

Observe that, due to the growth property (iv) of W , the functionals \mathcal{G}_ε and \mathcal{G} are bounded from below.

Theorem 2.3. Assume that $W : \Omega \times \mathbb{R}^{n \times n} \rightarrow [0, \infty]$ satisfies conditions (i)–(iv) for some $1 \leq p \leq 2$. There exists a constant $C > 0$ depending on $\Omega, \partial_D \Omega$, and p such that for every $h \in W^{1,p}$ and every sequence $\{u_\varepsilon\} \subseteq W_h^{1,p}$ we have

$$\int_{\Omega} |\nabla u_\varepsilon|^p dx \leq C \left[1 + \mathcal{F}_\varepsilon(u_\varepsilon) + \left(\int_{\partial_D \Omega} |h| d\mathcal{H}^{n-1} \right)^2 \right], \quad (2.12)$$

for every $\varepsilon > 0$ sufficiently small.

The previous theorem ensures that, if $\{u_\varepsilon\}$ is a sequence in $W_h^{1,p}$ such that $\{\mathcal{F}_\varepsilon(u_\varepsilon)\}$ is bounded, then $\{u_\varepsilon\}$ is bounded in $W^{1,p}$, hence a subsequence converges weakly in $W^{1,p}$.

Theorem 2.4. Under the hypotheses of Theorem 2.2, for every $\varepsilon_j \rightarrow 0$ we have that

$$\mathcal{F}_{\varepsilon_j} \xrightarrow{\Gamma} \mathcal{F}, \quad \text{as } j \rightarrow \infty,$$

in the weak topology of $W^{1,p}$.

Theorem 2.4, together with the compactness result provided by Theorem 2.3, implies the convergence of minima and the weak convergence of minimizers, using standard results on Γ -convergence. The next theorem and the previous remarks allow us to obtain the strong convergence of minimizers.

Theorem 2.5. *Under the hypotheses of Theorem 2.2, let $\varepsilon_j \rightarrow 0$ and let $\{u_j\}$ be a recovery sequence for $u \in W_h^{1,2}$, that is $u_j \rightarrow u$ weakly in $W^{1,p}$ and $\mathcal{F}_{\varepsilon_j}(u_j) \rightarrow \mathcal{F}(u)$. Then $\{u_j\}$ converges strongly in $W^{1,p}$.*

Remark 2.6 (On the condition $\partial_D \Omega \neq \emptyset$). Observe that in Theorem 2.3 the assumption $\partial_D \Omega \neq \emptyset$ is crucial. When $\partial_D \Omega = \emptyset$, inequality (2.12) is false, as the following example shows. Consider the simple case $W(F) := g_p(d(F, SO(n)))$ for every $F \in Lin^+(n)$. For every $\varepsilon > 0$ and some $R \in SO(n) \setminus \{I\}$, set

$$u_\varepsilon(x) := \frac{R - I}{\varepsilon} x, \quad x \in \Omega.$$

In this case, we have that

$$\int_{\Omega} |\nabla u_\varepsilon|^p dx = \frac{|\Omega| |R - I|^p}{\varepsilon^p} \rightarrow \infty, \quad \text{as } \varepsilon \rightarrow 0^+,$$

whereas

$$\mathcal{F}_\varepsilon(u_\varepsilon) = \frac{1}{\varepsilon^2} \int_{\Omega} g_p(d(I + \varepsilon \nabla u_\varepsilon, SO(n))) dx = 0, \quad \text{for every } \varepsilon > 0.$$

Remark 2.7 (On the condition $h \in W^{1,\infty}$). In Theorems 2.2, 2.4 and 2.5 the hypothesis $h \in W^{1,\infty}$ cannot be replaced by $h \in W^{1,2}$, unless W satisfies suitable bounds from above, which are not natural in the context of finite elasticity. Consider the simple case $\partial_D \Omega = \partial \Omega$, $\mathcal{L} = 0$, and assume that for some $r > 2$ we have

$$W(F) \geq |F|^r \quad \text{for } |F| \text{ large enough.}$$

By well-known properties of the images of Sobolev spaces under the trace operator, there exists $h \in W^{1,2}$ such that

$$\{u \in W^{1,r} : u = h \text{ } \mathcal{H}^{n-1}\text{-a.e. on } \partial \Omega\} = \emptyset. \tag{2.13}$$

Let us prove that $\mathcal{F}_\varepsilon(u) = \infty$ for every $u \in W^{1,p}$. Assume by contradiction that there exists $u \in W^{1,p}$ with $\mathcal{F}_\varepsilon(u) < \infty$. By (2.9) we have that $\nabla u \in L^r$, hence $u \in W^{1,r}$, because Ω has Lipschitz boundary. This contradicts (2.13). Therefore $\{\mathcal{F}_\varepsilon\}$ cannot Γ -converge to \mathcal{F} , because $\mathcal{F}(h) < \infty$.

Remark 2.8 (Model energy densities). A large class of models where the energy density grows quadratically near the wells and less than quadratically elsewhere is provided by rubber elasticity, when one wishes to take into account the compressibility of the material. We recall that we have formalized this growth behaviour by introducing, as bound from below of our energies, the function

$$g_p(d(\cdot, SO(3))), \quad \text{for some } 1 < p < 2,$$

where g_p is the function defined in (2.2). For simplicity, we focus on the homogeneous case.

A common practice to pass from an incompressible model, with associated energy density \tilde{W} defined on $\{F \in \mathbb{R}^{3 \times 3} : \det F = 1\}$, to a corresponding compressible model W (see, e.g., [2,7,9]) is to define

$$W(F) := \tilde{W}((\det F)^{-1/3} F) + W_{vol}(\det F), \quad \text{for every } F \in Lin^+(3),$$

where W_{vol} is such that

$$W_{vol} \geq 0 \quad \text{and} \quad W_{vol}(t) = 0 \quad \text{if and only if} \quad t = 1.$$

For example, we can take W_{vol} of the form

$$W_{vol}(t) = c[t^2 - 1 - 2 \log t], \quad \text{for every } t > 0,$$

for $c > 0$. Consider first the *Neo-Hookean* incompressible model for hyperelastic materials, where the energy density is of the form

$$\tilde{W}_{\mathcal{N}}(F) := a(|F|^2 - 3), \quad \text{for every } F \in \mathbb{R}^{3 \times 3} \text{ with } \det F = 1,$$

for a certain $a > 0$. Following the procedure described above, we consider the corresponding compressible energy density defined for every $F \in \text{Lin}^+(3)$ by

$$\begin{aligned} W_{\mathcal{N}}(F) &:= \tilde{W}_{\mathcal{N}}\left(\frac{F}{(\det F)^{1/3}}\right) + W_{\text{vol}}(\det F) \\ &= a\left(\frac{|F|^2}{(\det F)^{2/3}} - 3\right) + W_{\text{vol}}(\det F). \end{aligned}$$

Let us check that $W_{\mathcal{N}}$ has “ g_p -growth”. By using the well-known inequality between arithmetic and geometric mean, it is easy to see that

$$W_{\mathcal{N}} \geq 0 \quad \text{and} \quad W_{\mathcal{N}}(F) = 0 \quad \text{if and only if} \quad F \in SO(3). \quad (2.14)$$

Moreover, recalling the Green–St. Venant strain tensor $E = \frac{1}{2}(F^T F - I)$ and using simple rules of tensor calculus, it turns out that in the small strain regime (that is, the regime of the deformation gradients which vary near $SO(3)$), W has the expression

$$W_{\mathcal{N}}(F) = \mu|E|^2 + \frac{\lambda}{2} \text{tr}^2 E + o(|E|^2), \quad (2.15)$$

where

$$\mu = 2a, \quad \lambda = 4\left(-\frac{a}{3} + c\right).$$

The parameters μ and $\lambda + \frac{2}{3}\mu$ have the physical meaning of a shear modulus and a bulk modulus, respectively. Since $|E|^2 \geq \frac{1}{3} \text{tr}^2 E$ for every $E \in \text{Sym}(3)$, from (2.15) we obtain that

$$W_{\mathcal{N}}(F) \geq \min\{\mu, 6c\}|E|^2 + o(|E|^2),$$

and in turn,

$$W_{\mathcal{N}}(F) \geq \frac{1}{2} \min\{\mu, 6c\}|E|^2, \quad (2.16)$$

if $|E|$ is small enough, that is, if $d(F, SO(3))$ is small enough. Since $|\sqrt{C} - I| \leq |C - I|$ for every $C \in \text{Psym}(3)$, from (2.16) we obtain that

$$W_{\mathcal{N}}(F) \geq \frac{1}{8} \min\{\mu, 6c\} |\sqrt{F^T F} - I|^2 = \frac{1}{8} \min\{\mu, 6c\} d^2(F, SO(3)), \quad (2.17)$$

if $d(F, SO(3))$ is sufficiently small. Now, we want to study the growth of W in the regime $|F| \rightarrow \infty$. In this case, if $\det F$ is bounded, then

$$W_{\mathcal{N}}(F) \geq C|F|^2 - 3a \geq \tilde{C}d^2(F, SO(3)) \quad (\det F \text{ bounded}), \quad (2.18)$$

for some $C, \tilde{C} > 0$. In the case $\det F \rightarrow \infty$, we have that

$$W_{\mathcal{N}}(F) \geq K\left(\frac{|F|^2}{\det^{2/3} F} + \det^2 F\right),$$

for some $K > 0$. By using Young’s inequality

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q} \quad \left(\frac{1}{p} + \frac{1}{q} = 1\right)$$

with $x = \left(\frac{|F|^3}{\det F}\right)^{1/2}$ and $y = (\det F)^{1/2}$, it is easy to show that

$$W_{\mathcal{N}}(F) \geq K|F|^{3/2} \geq \tilde{K}d^{3/2}(F, SO(3)) \quad (\det F \rightarrow \infty), \quad (2.19)$$

for some $\tilde{K} > 0$. (2.14), (2.17), (2.18) and (2.19) show that $W_{\mathcal{N}}$ has g_p -growth from below with $p = \frac{3}{2}$. It is important to notice that $W_{\mathcal{N}}$ has not quadratic growth everywhere. In particular, $W_{\mathcal{N}}$ has not quadratic growth in the regime $\det F \rightarrow \infty$. This can be checked by taking into account deformation gradients of the type

$$F = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{with } \lambda \gg 0. \tag{2.20}$$

As a second example, we consider the *Mooney–Rivlin* compressible model given, for some $a, b > 0$, by

$$\begin{aligned} W_{\mathcal{M}}(F) &:= a \left(\frac{|F|^2}{(\det F)^{2/3}} - 3 \right) + b \left((\det F)^{2/3} |F^{-1}|^2 - 3 \right) + W_{vol}(\det F) \\ &= W_{\mathcal{N}}(F) + b \left((\det F)^{2/3} |F^{-1}|^2 - 3 \right), \end{aligned} \tag{2.21}$$

for every $F \in Lin^+(3)$, and derived from the corresponding incompressible version as explained before. The inequality between arithmetic and geometric mean implies that the second summand in (2.21) is nonnegative, so that, from (2.14), we have that

$$W_{\mathcal{M}} \geq 0 \quad \text{and} \quad W_{\mathcal{M}}(F) = 0 \quad \text{if and only if} \quad F \in SO(3).$$

The formula for the small strain regime is given by (2.15), with

$$\mu = 2(a + b), \quad \lambda = 4 \left(-\frac{a + b}{3} + c \right).$$

From the fact that $W_{\mathcal{N}}$ has g_p -growth and from the positiveness of the second summand of (2.21) the g_p -growth of $W_{\mathcal{M}}$ trivially follows. Also in this case, deformation gradients of the type (2.20) show that $W_{\mathcal{M}}$ does not grow quadratically everywhere.

Finally, we mention some *Ogden-type* compressible energy densities:

$$W_{\mathcal{O}}(F) := \sum_{i=1}^m a_i \left(\frac{\text{tr}((F^T F)^{\gamma_i/2})}{(\det F)^{\gamma_i/3}} - 3 \right) + W_{vol}(\det F),$$

defined for every $F \in Lin^+(3)$, for some $m \geq 1$ and $a_i, \gamma_i > 0, i = 1, \dots, m$. The formula for $W_{\mathcal{O}}$ in the small strain regime is again given by (2.15), with

$$\mu = 2 \sum_{i=1}^m a_i, \quad \lambda = 4 \left(-\frac{1}{3} \sum_{i=1}^m a_i + c \right).$$

Arguing similarly to the Neo-Hookean and the Mooney–Rivlin models, we obtain that $W_{\mathcal{O}}$ attains its minimum 0 at $SO(3)$. By using Young’s inequality and proper counterexamples, it is possible to show that $W_{\mathcal{O}}$ has g_p -growth for some $1 < p < 2$ (p depending on the exponents γ_i), but not a quadratic growth in general, if $0 < \gamma_i < 3$ for every $i = 1, \dots, m$ and $\gamma_i > \frac{6}{5}$ for at least one index $i \in \{1, \dots, m\}$.

3. Compactness

In this and in the next sections we give the proofs of the results stated in Section 2. To simplify the exposition, the proofs are given only when W does not depend explicitly on x . The proofs in the general case require only minor modifications.

The compactness result requires the following extension of the well-known geometric rigidity result of [8], where a power of $d(\nabla v, SO(n))$ is replaced by $g_p(d(\nabla v, SO(n)))$.

Lemma 3.1 (*Geometric rigidity*). *Let g_p be the function defined in (2.2). There exists a constant $C = C(\Omega, p) > 0$ with the following property: for every $v \in W^{1,p}$ there exists a constant rotation $R \in SO(n)$ satisfying*

$$\int_{\Omega} g_p(|\nabla v - R|) dx \leq C \int_{\Omega} g_p(d(\nabla v, SO(n))) dx. \tag{3.1}$$

Similar versions of Lemma 3.1 can be found in [3,10,11]. For sake of completeness, we give the proof in Appendix A.

We need two more lemmas in order to prove Theorem 2.3.

Lemma 3.2. *Let $S \subseteq \mathbb{R}^n$ be a bounded \mathcal{H}^m -measurable set with $0 < \mathcal{H}^m(S) < \infty$ for some $m > 0$. Then*

$$|F|_S := \min_{\zeta \in \mathbb{R}^n} \int_S |Fx - \zeta| d\mathcal{H}^m$$

is a seminorm on $\mathbb{R}^{n \times n}$. Define

$$S_0 := \{x \in S: \mathcal{H}^m(S \cap B_\rho(x)) > 0 \text{ for every } \rho > 0\},$$

and let $\text{aff}(S_0)$ be the smallest affine space containing S_0 . Let $K \subseteq \mathbb{R}^{n \times n}$ be a closed cone such that

$$\dim(\text{Ker}(F)) < \dim(\text{aff}(S_0)), \quad \text{for every } F \in K \setminus \{0\}. \tag{3.2}$$

Then, there exists a constant $C = C(S) > 0$ such that

$$C|F| \leq |F|_S, \quad \text{for every } F \in K.$$

Proof. It is enough to repeat the proof of [5, Lemma 3.3], replacing the L^2 norm with the L^1 norm. \square

We will use the next lemma also in the proof of the Γ -convergence result. In what follows and in the rest of the paper we denote by C a positive constant which may change from line to line.

Lemma 3.3. *Let $\varepsilon > 0$ and $u_\varepsilon \in W_h^{1,p}$. Under the hypotheses of Theorem 2.3, let $R_\varepsilon \in SO(n)$ be a constant rotation satisfying (3.1) with $v = x + \varepsilon u_\varepsilon$. Then,*

$$|I - R_\varepsilon|^2 \leq C\varepsilon^2 \left[\mathcal{F}_\varepsilon(u_\varepsilon) + \left(\int_{\partial_D \Omega} |h| d\mathcal{H}^{n-1} \right)^2 \right],$$

where C depends only on Ω , $\partial_D \Omega$, and p .

Proof. Consider the deformation $v_\varepsilon := x + \varepsilon u_\varepsilon$. Lemma 3.1 tells us that there exists a constant rotation $R_\varepsilon \in SO(n)$ such that

$$\int_\Omega g_p(|\nabla v_\varepsilon - R_\varepsilon|) dx \leq C \int_\Omega g_p(d(\nabla v_\varepsilon, SO(n))) dx,$$

where C depends only on Ω and p . Then, by assumption (iv) on W , we have that

$$\int_\Omega g_p(|\nabla v_\varepsilon - R_\varepsilon|) dx \leq C \int_\Omega W(\nabla v_\varepsilon) dx = C\varepsilon^2 \mathcal{F}_\varepsilon(u_\varepsilon).$$

Jensen inequality thus implies

$$g_p\left(\frac{1}{|\Omega|} \int_\Omega |\nabla v_\varepsilon - R_\varepsilon| dx\right) \leq C\varepsilon^2 \mathcal{F}_\varepsilon(u_\varepsilon). \tag{3.3}$$

Poincaré–Wirtinger inequality and the continuity of the trace operator give

$$\int_{\partial_D \Omega} |v_\varepsilon - R_\varepsilon x - \zeta_\varepsilon| d\mathcal{H}^{n-1} \leq C \int_\Omega |\nabla v_\varepsilon - R_\varepsilon| dx,$$

where $\zeta_\varepsilon := \frac{1}{|\Omega|} \int_\Omega (v_\varepsilon - R_\varepsilon x) dx$ and C depends on Ω , so that, since $v_\varepsilon = x + \varepsilon h$ \mathcal{H}^{n-1} -a.e. on $\partial_D \Omega$, we obtain

$$\int_{\partial_D \Omega} |(I - R_\varepsilon)x - \zeta_\varepsilon| d\mathcal{H}^{n-1} \leq C \left(\int_\Omega |\nabla v_\varepsilon - R_\varepsilon| dx + \varepsilon \int_{\partial_D \Omega} |h| d\mathcal{H}^{n-1} \right). \tag{3.4}$$

Now, let us use Lemma 3.2 with $S = \partial_D \Omega$ and with K equal to the closed cone generated by $I - SO(n)$. Showing first that every $F \in K$ belongs to the cone generated by $I - SO(n)$ or to $Skw(n)$, it is easy to prove that every $F \in K \setminus \{0\}$ is such that

$$\dim(\text{Ker}(F)) < n - 1.$$

On the other hand, $\partial \Omega$ Lipschitz implies that the right-hand side of (3.2) is equal to $n - 1$. Thus, we can apply Lemma 3.2 to $(I - R_\varepsilon) \in K$ and write that

$$C|I - R_\varepsilon| \leq \min_{\zeta \in \mathbb{R}^n} \int_{\partial_D \Omega} |(I - R_\varepsilon)x - \zeta| d\mathcal{H}^{n-1}, \tag{3.5}$$

where C depends on $\partial_D \Omega$ and not on ε . From (3.4) and (3.5) we obtain that

$$|I - R_\varepsilon|^2 \leq C \left[\left(\frac{1}{|\Omega|} \int_{\Omega} |\nabla v_\varepsilon - R_\varepsilon| dx \right)^2 + \varepsilon^2 \left(\int_{\partial_D \Omega} |h| d\mathcal{H}^{n-1} \right)^2 \right]. \tag{3.6}$$

We conclude the proof by distinguishing two cases. If $\int_{\Omega} |\nabla v_\varepsilon - R_\varepsilon| dx \leq |\Omega|$, then (3.3) and the definition of g_p tell us that

$$\frac{1}{2} \left(\frac{1}{|\Omega|} \int_{\Omega} |\nabla v_\varepsilon - R_\varepsilon| dx \right)^2 \leq C\varepsilon^2 \mathcal{F}_\varepsilon(u_\varepsilon).$$

Using this last inequality in (3.6), it turns out (2.12). If $\int_{\Omega} |\nabla v_\varepsilon - R_\varepsilon| dx > |\Omega|$, again (3.3) and the definition of g_p tell us that

$$C\varepsilon^2 \mathcal{F}_\varepsilon(u_\varepsilon) > \frac{1}{2}.$$

This bound from below of $\varepsilon^2 \mathcal{F}_\varepsilon(u_\varepsilon)$ gives trivially (2.12), in view of the fact that $|I - R_\varepsilon| \leq 2\sqrt{n}$. \square

For the proof of Theorem 2.3 we will need the following estimate

$$g_p(s + t) \leq C[g_p(s) + t^2], \quad \text{for every } s, t \geq 0, \tag{3.7}$$

for a certain C depending on p . This estimate can be easily deduced from the convexity of g_p and from the growth properties of g_p which give

$$g_p(t) \leq \frac{1}{p} \min\{t^p, t^2\} \quad \text{and} \quad g_p(2t) \leq Cg_p(t), \quad \text{for every } t \geq 0,$$

for some C depending on p .

Proof of Theorem 2.3. Let R_ε be given by Lemma 3.1 for $v_\varepsilon := x + \varepsilon u_\varepsilon$, for every $\varepsilon > 0$. By using (3.7), we have that

$$\begin{aligned} \int_{\Omega} g_p(|\varepsilon \nabla u_\varepsilon|) dx &\leq C \int_{\Omega} [g_p(|\nabla v_\varepsilon - R_\varepsilon|) + |I - R_\varepsilon|^2] dx \\ &\leq C \left[\int_{\Omega} g_p(d(\nabla v_\varepsilon, SO(n))) dx + |I - R_\varepsilon|^2 \right], \end{aligned}$$

where in the last inequality we have used Lemma 3.1. Assumption (iv) on W and Lemma 3.3 then imply that for some C , depending on Ω , $\partial_D \Omega$, and p ,

$$\int_{\Omega} g_p(|\varepsilon \nabla u_\varepsilon|) dx \leq C\varepsilon^2 \left[\mathcal{F}_\varepsilon(u_\varepsilon) + \left(\int_{\partial_D \Omega} |h| d\mathcal{H}^{n-1} \right)^2 \right]. \tag{3.8}$$

In particular, from (3.8) and from the definition of g_p we obtain

$$\begin{aligned} \int_{\{x \in \Omega: |\varepsilon \nabla u_\varepsilon(x)| \leq 1\}} |\varepsilon \nabla u_\varepsilon|^2 dx &\leq 2 \int_{\Omega} g_p(|\varepsilon \nabla u_\varepsilon|) dx \\ &\leq C\varepsilon^2 \left[\mathcal{F}_\varepsilon(u_\varepsilon) + \left(\int_{\partial_D \Omega} |h| d\mathcal{H}^{n-1} \right)^2 \right], \end{aligned}$$

so that, by Hölder inequality, it turns out

$$\begin{aligned} \int_{\{x \in \Omega: |\varepsilon \nabla u_\varepsilon(x)| \leq 1\}} |\varepsilon \nabla u_\varepsilon|^p dx &\leq \left(\int_{\{x \in \Omega: |\varepsilon \nabla u_\varepsilon(x)| \leq 1\}} |\varepsilon \nabla u_\varepsilon|^2 dx \right)^{p/2} |\Omega|^{1-(p/2)} \\ &\leq C\varepsilon^p \left[\mathcal{F}_\varepsilon(u_\varepsilon) + \left(\int_{\partial_D \Omega} |h| d\mathcal{H}^{n-1} \right)^2 \right]^{p/2} \\ &\leq C\varepsilon^p \left[1 + \mathcal{F}_\varepsilon(u_\varepsilon) + \left(\int_{\partial_D \Omega} |h| d\mathcal{H}^{n-1} \right)^2 \right]. \end{aligned} \quad (3.9)$$

Note that in (3.9) we have used the fact that

$$t^{p/2} \leq 1 + t, \quad \text{for every } t \geq 0.$$

On the other hand, from (A.2) and again from (3.8) we obtain that

$$\begin{aligned} \int_{\{x \in \Omega: |\varepsilon \nabla u_\varepsilon(x)| > 1\}} |\varepsilon \nabla u_\varepsilon|^p dx &\leq C \int_{\{x \in \Omega: |\varepsilon \nabla u_\varepsilon(x)| > 1\}} g_p(|\varepsilon \nabla u_\varepsilon|) dx \\ &\leq C\varepsilon^2 \left[\mathcal{F}_\varepsilon(u_\varepsilon) + \left(\int_{\partial_D \Omega} |h| d\mathcal{H}^{n-1} \right)^2 \right]. \end{aligned} \quad (3.10)$$

Inequalities (3.9) and (3.10) imply that (2.12) holds. \square

In the next remark we construct a counterexample which shows that Theorem 2.3 is not true in general for $p \in (0, 1)$.

Remark 3.4. Let $p \in (0, 1)$ and consider the simple case in which Ω is the open unitary ball $B(0, 1)$ in \mathbb{R}^2 , $W(F) := g_p(d(F, SO(2)))$ for every $F \in Lin^+(2)$, $h = 0$, and $\mathcal{L} = 0$. For any $\varepsilon > 0$ and some $\alpha > 0$ to be chosen, we introduce the set

$$S_\varepsilon := \left\{ x \in \mathbb{R}^2: \frac{1}{2} < |x| < \frac{1}{2} + \varepsilon^\alpha \right\}.$$

For every $\varepsilon > 0$ sufficiently small, S_ε is an open annulus strictly included in Ω . We want to define a sequence $\{u_\varepsilon\} \subseteq W_0^{1,p}$ such that the values $\mathcal{F}_\varepsilon(u_\varepsilon)$ are equibounded and $\int_{\Omega} |\nabla u_\varepsilon|^p dx \rightarrow \infty$ as $\varepsilon \rightarrow 0^+$. In order to do this, we consider for every $\varepsilon > 0$ arbitrarily small a function $\varphi_\varepsilon \in C_c^\infty(\Omega, \mathbb{R})$ such that $\text{supp}(\varphi_\varepsilon) \subseteq \overline{B(0, \frac{1}{2})} \cup S_\varepsilon$, $0 \leq \varphi_\varepsilon \leq 1$, $\varphi_\varepsilon \equiv 1$ on $B(0, \frac{1}{2})$ and

$$|\nabla \varphi_\varepsilon| \leq \frac{C}{\varepsilon^\alpha} \quad \text{for some } C \text{ independent of } \varepsilon. \quad (3.11)$$

Then, we choose $R \in SO(2) \setminus \{I\}$ and define the function

$$u_\varepsilon(x) := \varphi_\varepsilon(x) \frac{R - I}{\varepsilon} x, \quad x \in \Omega,$$

which belongs to C^∞ for every $\varepsilon > 0$ sufficiently small. Observe that

$$\int_{\Omega} |\nabla u_\varepsilon|^p dx \geq \int_{B(0, \frac{1}{2})} |\nabla u_\varepsilon|^p dx = \frac{\pi |R - I|^p}{4\varepsilon^p},$$

so that $\int_{\Omega} |\nabla u_\varepsilon|^p dx \rightarrow \infty$ as $\varepsilon \rightarrow 0^+$ (for every choice of $\alpha > 0$). Now, let us compute

$$\nabla u_\varepsilon(x) = \frac{1}{\varepsilon} \{ \varphi_\varepsilon(x)(R - I) + [(R - I)x] \otimes \nabla \varphi_\varepsilon(x) \}$$

and observe that $\nabla u_\varepsilon \equiv 0$ on $\Omega \setminus [\overline{B(0, \frac{1}{2})} \cup S_\varepsilon]$, so that $d(I + \varepsilon \nabla u_\varepsilon, SO(2)) \equiv 0$ on the same set. Thus, recalling that g_p is increasing, it turns out that

$$\begin{aligned} \varepsilon^2 \mathcal{F}_\varepsilon(u_\varepsilon) &\leq \int_{B(0, \frac{1}{2}) \cup S_\varepsilon} g_p(|I + \varepsilon \nabla u_\varepsilon - R|) dx \\ &\leq \int_{S_\varepsilon} g_p(|R - I|(1 + |x| |\nabla \varphi_\varepsilon|)) dx, \end{aligned} \tag{3.12}$$

where in the last inequality we have also used the fact that $\varphi_\varepsilon \equiv 1$ on $B(0, \frac{1}{2})$. Therefore, from (2.2) and (3.12) we obtain that

$$\mathcal{F}_\varepsilon(u_\varepsilon) \leq \frac{C}{\varepsilon^2} \int_{S_\varepsilon} (1 + |\nabla \varphi_\varepsilon|^p) dx, \tag{3.13}$$

for some C independent of ε . Using (3.11) and noticing that $|S_\varepsilon| = \pi \varepsilon^\alpha + o(\varepsilon^\alpha)$, (3.13) implies that

$$\mathcal{F}_\varepsilon(u_\varepsilon) \leq \frac{C}{\varepsilon^2} [\pi \varepsilon^\alpha + o(\varepsilon^\alpha)] \left(1 + \frac{1}{\varepsilon^{\alpha p}} \right),$$

so that $\{\mathcal{F}_\varepsilon(u_\varepsilon)\}$ turns out to be bounded whenever $\alpha > \frac{2}{1-p}$.

We end this section with the following corollary.

Corollary 3.5. *Under the hypotheses of Theorem 2.3, the functionals \mathcal{G}_ε are equicoercive in the weak topology of $W^{1,p}$.*

Proof. Let $t \in \mathbb{R}$ and $\{u_\varepsilon\}$ be a sequence with $\mathcal{G}_\varepsilon(u_\varepsilon) \leq t$, so that $\{u_\varepsilon\} \subseteq W_h^{1,p}$. Thus, by the definition of \mathcal{G}_ε (2.11), we have

$$\mathcal{F}_\varepsilon(u_\varepsilon) \leq t + \mathcal{L}(u_\varepsilon).$$

Theorem 2.3 implies that for every ε sufficiently small

$$\int_{\Omega} |\nabla u_\varepsilon|^p dx \leq C \left[1 + \mathcal{L}(u_\varepsilon) + \left(\int_{\partial_D \Omega} |h| d\mathcal{H}^{n-1} \right)^2 \right],$$

for some C independent of ε . By Poincaré inequality, this gives

$$\|u_\varepsilon\|_{W^{1,p}}^p \leq C (\|u_\varepsilon\|_{W^{1,p}} + 1), \tag{3.14}$$

where C now depends also on h and \mathcal{L} . Therefore, since $p > 1$, from (3.14) we obtain that $\|u_\varepsilon\|_{W^{1,p}}$ is bounded. \square

Observe that the proofs of Theorem 2.3, Lemma 3.3 and Corollary 3.5 do not use the fact that $\partial_D \Omega$ has Lipschitz boundary in $\partial \Omega$ (see Definition 2.1): actually, these results hold under the weaker hypothesis $\mathcal{H}^{n-1}(\partial_D \Omega) > 0$.

4. Γ -convergence

Consider a sequence $\varepsilon_j \rightarrow 0^+$ as $j \rightarrow \infty$. By Theorem 2.3, we can characterize the Γ -limit in the weak topology of $W^{1,p}$ in terms of weakly converging sequences (see [6, Proposition 8.10]). In particular, we have that

$$\begin{aligned}\mathcal{F}'(u) &:= \Gamma\text{-}\liminf_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(u) = \inf \left\{ \liminf_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(u_j) : u_j \rightharpoonup u \text{ weakly in } W^{1,p} \right\}; \\ \mathcal{F}''(u) &:= \Gamma\text{-}\limsup_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(u) = \inf \left\{ \limsup_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(u_j) : u_j \rightharpoonup u \text{ weakly in } W^{1,p} \right\}.\end{aligned}\quad (4.1)$$

Thus, in order to prove Theorem 2.4, we will show that $\mathcal{F}(u) \geq \mathcal{F}''(u)$ and $\mathcal{F}(u) \leq \liminf_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(u_j)$, for every $u \in W^{1,p}$ and every $u_j \rightharpoonup u$ weakly in $W^{1,p}$.

Proof of Theorem 2.4. (I) We want to show that $\mathcal{F}(u) \geq \mathcal{F}''(u)$. Consider the nontrivial case $\mathcal{F}(u) < \infty$, so that $u \in W_h^{1,2}$ and

$$\mathcal{F}(u) = \frac{1}{2} \int_{\Omega} D^2 W(I)[e(u)]^2 dx.$$

Suppose first $u \in W^{1,\infty}$. The boundedness of ∇u and assumption (ii) on W , together with the fact that $W(I) = 0$ and $DW(I) = 0$, imply that

$$\lim_{j \rightarrow \infty} \frac{1}{\varepsilon_j^2} W(I + \varepsilon_j \nabla u(x)) = \frac{1}{2} D^2 W(I)[\nabla u(x)]^2, \quad \text{for a.e. } x \in \Omega,$$

and that there exists $C > 0$ such that for every $\varepsilon_j > 0$ sufficiently small

$$W(I + \varepsilon_j \nabla u) \leq \varepsilon_j^2 C |\nabla u|^2, \quad \text{a.e. in } \Omega.$$

Then, by dominated convergence and by (2.3), we obtain

$$\lim_{j \rightarrow \infty} \frac{1}{\varepsilon_j^2} \int_{\Omega} W(I + \varepsilon_j \nabla u) dx = \frac{1}{2} \int_{\Omega} D^2 W(I)[e(u)]^2 dx.$$

Therefore, by (4.1),

$$\mathcal{F}(u) = \lim_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(u) \geq \mathcal{F}''(u).\quad (4.2)$$

Consider now the general case $u \in W_h^{1,2}$. Since $\partial_D \Omega$ has Lipschitz boundary in $\partial \Omega$, from Proposition A.2 we have that there exists a sequence $\{u_k\} \subseteq W_h^{1,\infty}$ such that $u_k \rightarrow u$ strongly in $W^{1,2}$, as $k \rightarrow \infty$. Observe that by (4.2) we have $\mathcal{F}''(u_k) \leq \mathcal{F}(u_k)$ for every k . Thus, by the weak lower semicontinuity of \mathcal{F}'' in $W^{1,p}$ and the strong continuity of \mathcal{F} in $W_h^{1,2}$, it turns out that

$$\mathcal{F}(u) = \lim_{k \rightarrow \infty} \mathcal{F}(u_k) \geq \liminf_{k \rightarrow \infty} \mathcal{F}''(u_k) \geq \mathcal{F}''(u).$$

(II) We want to prove that, if $u_j \rightharpoonup u$ weakly in $W^{1,p}$, then $\mathcal{F}(u) \leq \liminf_j \mathcal{F}_{\varepsilon_j}(u_j)$. Consider the nontrivial case $\liminf_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(u_j) < \infty$ so that, up to a subsequence, we can suppose $\{\mathcal{F}_{\varepsilon_j}(u_j)\}$ bounded and, in particular, $\{u_j\} \subseteq W_h^{1,p}$. Let 1_{B_j} be the characteristic function of B_j , where

$$B_j := \left\{ x \in \Omega : |\nabla u_j(x)| \leq \frac{1}{\sqrt{\varepsilon_j}} \right\}.\quad (4.3)$$

Claim 1. We have that $\{1_{B_j} \nabla u_j\}$ is bounded in L^2 .

Proof. By Lemma 3.1 and by the growth hypothesis on W we have that for every j there exists $R_j \in SO(n)$ such that

$$\int_{\Omega} g_p(|I + \varepsilon_j \nabla u_j(x) - R_j|) dx \leq \varepsilon_j^2 C \mathcal{F}_{\varepsilon_j}(u_j) \leq C \varepsilon_j^2, \tag{4.4}$$

where the last inequality follows from the boundedness of $\{\mathcal{F}_{\varepsilon_j}(u_j)\}$. Considering the set

$$A_j := \{x \in \Omega : |I + \varepsilon_j \nabla u_j(x) - R_j| \leq 3\sqrt{n}\},$$

it is easy to check that $B_j \subseteq A_j$ for every j large enough, so that

$$\int_{B_j} |\nabla u_j|^2 dx \leq \frac{2}{\varepsilon_j^2} \int_{A_j} (|\varepsilon_j \nabla u_j + I - R_j|^2 + |I - R_j|^2) dx. \tag{4.5}$$

Therefore, by using (A.1) and the definition of A_j , from (4.5) we obtain that

$$\begin{aligned} \int_{B_j} |\nabla u_j|^2 dx &\leq \frac{C}{\varepsilon_j^2} \int_{A_j} [g_p(|\varepsilon_j \nabla u_j + I - R_j|) + |I - R_j|^2] dx \\ &\leq C \left(1 + \frac{|I - R_j|^2}{\varepsilon_j^2} \right), \end{aligned} \tag{4.6}$$

where in the last inequality we have used (4.4) and C depends on Ω and p . Since $\{\mathcal{F}(u_j)\}$ is bounded, Lemma 3.3 tells us that $|I - R_j|^2/\varepsilon_j^2$ is bounded. This fact, together with (4.6), gives the claim. \square

Claim 2. $\nabla u \in L^2$ and, up to a subsequence, we have that

$$1_{B_j} \nabla u_j \rightharpoonup \nabla u \quad \text{weakly in } L^2.$$

Proof. By Claim 1, we have that, up to a subsequence,

$$1_{B_j} \nabla u_j \rightharpoonup v \quad \text{weakly in } L^2, \tag{4.7}$$

for some $v \in L^2$. Let us prove that

$$1_{B_j^c} \nabla u_j \rightarrow 0 \quad \text{strongly in } L^\alpha, \tag{4.8}$$

for every $\alpha \in [1, p)$. We first observe that $|B_j^c| \rightarrow 0$, by Chebyshev inequality. Taking into account the boundedness of $\{u_j\}$ in $W^{1,p}$, by Hölder inequality we obtain

$$\int_{\Omega} |1_{B_j^c} \nabla u_j|^\alpha dx \leq \left(\int_{\Omega} |\nabla u_j|^p dx \right)^{\alpha/p} |B_j^c|^{(p-\alpha)/p} \leq C |B_j^c|^{(p-\alpha)/p} \rightarrow 0,$$

which proves (4.8).

The weak convergence of u_j to u in $W^{1,p}$ implies also that $\nabla u_j \rightharpoonup \nabla u$ weakly in L^α , for every $\alpha \in [1, p)$. This fact, together with (4.8), gives that

$$1_{B_j} \nabla u_j = (\nabla u_j - 1_{B_j^c} \nabla u_j) \rightharpoonup \nabla u \quad \text{weakly in } L^\alpha, \tag{4.9}$$

for every $\alpha \in [1, p)$. By (4.7) and (4.9) we conclude that $\nabla u = v \in L^2$ and Claim 2 follows. \square

From assumptions (ii) and (iii) on W it is easy to show that

$$W(I + F) \geq \frac{1}{2} D^2 W(I)[F]^2 - \eta(|F|)|F|^2, \quad \text{for every } F \in \mathbb{R}^{n \times n},$$

where η is an increasing function on $[0, \infty)$ such that $\eta(t) \rightarrow 0$ as $t \rightarrow 0^+$. Therefore, we can write

$$\begin{aligned} \mathcal{F}_{\varepsilon_j}(u_j) &\geq \int_{B_j} \left\{ \frac{1}{2} D^2 W(I)[e(u_j)]^2 - \eta(\varepsilon_j |\nabla u_j|) |\nabla u_j|^2 \right\} dx \\ &\geq \int_{\Omega} \left\{ \frac{1}{2} D^2 W(I)[1_{B_j} e(u_j)]^2 - \eta(\sqrt{\varepsilon_j}) 1_{B_j} |\nabla u_j|^2 \right\} dx, \end{aligned} \tag{4.10}$$

where in the last inequality we have used the definition of B_j and the monotonicity of η . Thus, from (4.10) we obtain that

$$\begin{aligned} \liminf_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(u_j) &\geq \frac{1}{2} \liminf_{j \rightarrow \infty} \int_{\Omega} D^2 W(I)[1_{B_j} e(u_j)]^2 dx - \lim_{j \rightarrow \infty} \eta(\sqrt{\varepsilon_j}) \int_{\Omega} 1_{B_j} |\nabla u_j|^2 dx \\ &= \frac{1}{2} \liminf_{j \rightarrow \infty} \int_{\Omega} D^2 W(I)[1_{B_j} e(u_j)]^2 dx \end{aligned} \tag{4.11}$$

$$\geq \frac{1}{2} \int_{\Omega} D^2 W(I)[e(u)]^2 dx, \tag{4.12}$$

where (4.11) follows from Claim 1 and from the convergence of $\eta(\sqrt{\varepsilon_j})$ to 0, while (4.12) is deduced from Claim 2 and from the lower semicontinuity of

$$w \mapsto \frac{1}{2} \int_{\Omega} D^2 W(I)[w]^2$$

in the weak topology of L^2 , which is a consequence of (2.3) and (2.4). In order to conclude the proof, it remains to show that $u \in W_h^{1,2}$, so that from (4.12) we have $\liminf_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(u_j) \geq \mathcal{F}(u)$. We already know, from Claim 2, that $\nabla u \in L^2$. Since u is at least in L^1 , it is easy to show, by using Sobolev embeddings, that $u \in L^2$. Therefore, $u \in W^{1,2}$. Since $u_j \rightharpoonup u$ weakly in $W^{1,p}$ and $\{u_j\} \subseteq W_h^{1,p}$, we have $u \in W_h^{1,p}$. Thus, $u \in W_h^{1,p} \cap W_h^{1,2} = W_h^{1,2}$. \square

Remark 4.1. In the case $p = 2$, one can prove a slightly different version of Theorems 2.2 and 2.4, assuming only that $\partial_D \Omega$ is a subset of $\partial \Omega$ with $\mathcal{H}^{n-1}(\partial_D \Omega) > 0$, as in [5]. In this case, in the definitions of the functionals (2.9)–(2.11) the space $W_h^{1,2}$ has to be replaced by the closure of $W_h^{1,\infty}$ in $W^{1,2}$.

5. Convergence of minimizers

Recall that a family $\mathcal{F} := \{f\} \subseteq L^1(\Omega)$ is *equiintegrable* if for every $\eta > 0$ there exists $M_\eta > 0$ such that

$$\int_{\{x \in \Omega : |f(x)| > M_\eta\}} |f| dx < \eta, \quad \text{for every } f \in \mathcal{F}. \tag{5.1}$$

Equivalently, \mathcal{F} is equiintegrable if for every $\eta > 0$ there exists $\delta_\eta > 0$ such that, if $A \subseteq \Omega$ and $|A| < \delta_\eta$, then

$$\int_A |f| dx < \eta, \quad \text{for every } f \in \mathcal{F}. \tag{5.2}$$

The following criterion of equiintegrability will be useful.

Lemma 5.1. *The family $\mathcal{F} := \{f\} \subseteq L^1$ is equiintegrable if and only if for every $\eta > 0$ there exists $M_\eta > 0$ and $p \in (1, \infty]$ such that any $f \in \mathcal{F}$ can be written as*

$$f = g + h, \quad \text{with } \|g\|_{L^1} < \eta \text{ and } \|h\|_{L^p} < M_\eta. \tag{5.3}$$

Proof. Suppose \mathcal{F} equiintegrable, so that, for every $\eta > 0$, there exists $M_\eta > 0$ such that (5.1) holds. By setting

$$g := f 1_{\{|f| > M_\eta\}} \quad \text{and} \quad h := f 1_{\{|f| \leq M_\eta\}},$$

we have that $f = g + h$ and

$$\|g\|_{L^1} = \int_{\{|f|>M_\eta\}} |f| dx < \eta, \quad \|h\|_{L^p}^p \leq |\Omega| M_\eta^p.$$

Conversely, assume (5.3). We want to prove that, for every $\eta > 0$, there exists $\delta_\eta > 0$ such that (5.2) holds, whenever $|A| < \delta_\eta$. By hypothesis, for every $f \in \mathcal{F}$ there exist g, h , and $p \in (1, \infty]$ such that (5.3) holds with $\frac{\eta}{2}$ in place of η . Thus, by using Hölder inequality, we have that

$$\int_A |f| dx \leq \int_A |g| dx + \int_A |h| dx < \frac{\eta}{2} + M_{\eta/2} |A|^{(p-1)/p},$$

so that, by imposing $\delta_\eta := (\frac{\eta}{2M_{\eta/2}})^{p/(p-1)}$, we can conclude. \square

In the next proof, we will make use of *Vitali’s Convergence Theorem*: if $\{f_j\}$ is a sequence of equiintegrable functions on Ω which converges pointwise to a function f , then

$$f \in L^1 \quad \text{and} \quad f_j \rightarrow f \quad \text{in } L^1.$$

Moreover, we will use the following result of geometric rigidity, for which we refer to [4].

Theorem 5.2. *Let $1 < p_1 < p_2 < \infty$. There exists $C = C(\Omega, p_1, p_2) > 0$ with the following property: for every $v \in W^{1,1}$ with*

$$d(\nabla v, SO(n)) = f_1 + f_2 \quad \text{a.e. in } \Omega, \quad \text{and } f_i \in L^{p_i}, \quad i = 1, 2,$$

there exist $g_i \in L^{p_i}, i = 1, 2$, and a constant rotation $R \in SO(n)$ such that

$$\nabla v = R + g_1 + g_2, \quad \text{a.e. in } \Omega, \quad \text{with } \|g_i\|_{L^{p_i}} \leq C \|f_i\|_{L^{p_i}}, \quad i = 1, 2.$$

Proof of Theorem 2.5. Let $\{u_j\}$ be a recovery sequence for $u \in W_h^{1,2}$. In order to prove that $\{u_j\}$ converges to u strongly in $W^{1,p}$, we show that

- (i) $e(u_j)1_{B_j} \rightarrow e(u)$ strongly in L^2 ,
- (ii) $\left\{ \frac{d^p(I + \varepsilon_j \nabla u_j, SO(n))}{\varepsilon_j^p} \right\}$ is equiintegrable,
- (iii) $\{|\nabla u_j|^p\}$ is equiintegrable,

where B_j is the set defined in (4.3). Once (i) and (iii) are proved ((ii) is an intermediate step to prove (iii)), we can conclude as follows. From (i) we have that, up to a subsequence,

$$e(u_j)1_{B_j} \rightarrow e(u) \quad \text{a.e. in } \Omega. \tag{5.4}$$

Moreover, $e(u_j)1_{B_j^c} \rightarrow 0$ strongly in L^1 by Hölder inequality:

$$\int_{B_j^c} |e(u_j)| dx \leq \|e(u_j)\|_{L^p} |B_j^c|^{(p-1)/p} \rightarrow 0, \tag{5.5}$$

where we have used the boundedness of $\{u_j\}$, which implies $|B_j^c| \rightarrow 0$ by Chebyshev inequality. Thus, by (5.4) and (5.5), we have that, up to a further subsequence,

$$e(u_j) = e(u_j)1_{B_j} + e(u_j)1_{B_j^c} \rightarrow e(u) \quad \text{a.e. in } \Omega. \tag{5.6}$$

Let us apply Vitali’s Convergence Theorem to the functions $f_j := |e(u_j) - e(u)|^p$ and $f = 0$. Since $f_j \rightarrow f$ a.e. in Ω by (5.6) and $\{f_j\}$ is equiintegrable by (iii), we obtain that

$$e(u_j) \rightarrow e(u) \quad \text{in } L^p.$$

Observe that, by the hypothesis $\mathcal{F}_{\varepsilon_j}(u_j) \rightarrow \mathcal{F}(u) < \infty$, $u_j = h$ on $\partial_D \Omega$ for every j , thus it is sufficient to apply *Korn's inequality* to deduce that $u_j \rightarrow u$ strongly in $W^{1,p}$.

We now prove (i)–(iii). Let us set, for every j ,

$$v_j := x + \varepsilon_j u_j, \quad \text{for a.e. } x \in \Omega.$$

Proof of (i). As shown in the proof of Theorem 2.4, the boundedness of $\{\mathcal{F}(u_j)\}$ for every j sufficiently large implies that, up to a subsequence, the sequence $\{1_{B_j} \nabla u_j\}$ converges to ∇u weakly in L^2 , and

$$\begin{aligned} \lim_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(u_j) &\geq \limsup_{j \rightarrow \infty} \frac{1}{\varepsilon_j^2} \int_{B_j} W(\nabla v_j) dx \geq \limsup_{j \rightarrow \infty} \int_{\Omega} \frac{1}{2} D^2 W(I) [e(u_j) 1_{B_j}]^2 dx, \\ \liminf_{j \rightarrow \infty} \frac{1}{\varepsilon_j^2} \int_{B_j} W(\nabla v_j) dx &\geq \liminf_{j \rightarrow \infty} \int_{\Omega} \frac{1}{2} D^2 W(I) [e(u_j) 1_{B_j}]^2 dx \geq \mathcal{F}(u). \end{aligned}$$

Since $\mathcal{F}_{\varepsilon_j}(u_j) \rightarrow \mathcal{F}(u)$, it turns out that

$$\begin{aligned} \frac{1}{\varepsilon_j^2} \int_{B_j} W(\nabla v_j) dx &\rightarrow \frac{1}{2} \int_{\Omega} D^2 W(I) [e(u)]^2 dx, \\ \int_{\Omega} D^2 W(I) [e(u_j) 1_{B_j}]^2 dx &\rightarrow \int_{\Omega} D^2 W(I) [e(u)]^2 dx. \end{aligned} \tag{5.7}$$

The latter, together with the positive definiteness of $D^2 W(I)$ on symmetric matrices and the weak convergence of $\{1_{B_j} e(u_j)\}$ to $e(u)$ in L^2 , proves (i).

Proof of (ii). Let us write

$$\frac{1}{\varepsilon_j^p} d^p(\nabla v_j, SO(n)) = \frac{1}{\varepsilon_j^p} d^p(\nabla v_j, SO(n)) (1_{B_j} + 1_{B_j^c}), \tag{5.8}$$

and prove that both terms of the sum in (5.8) are equiintegrable. Observe that

$$\begin{aligned} d(\nabla v_j, SO(n)) &\leq d(\nabla v_j, I + \varepsilon_j \operatorname{skw}(\nabla u_j)) + d(I + \varepsilon_j \operatorname{skw}(\nabla u_j), SO(n)) \\ &= \varepsilon_j |e(u_j)| + d(I + \varepsilon_j \operatorname{skw}(\nabla u_j), SO(n)). \end{aligned} \tag{5.9}$$

Since $\varepsilon_j \operatorname{skw}(\nabla u_j)$ is an element of the tangent space to the C^∞ manifold $SO(n)$ at I , we have that

$$d(I + \varepsilon_j \operatorname{skw}(\nabla u_j), SO(n)) \leq C \varepsilon_j^2 |\operatorname{skw}(\nabla u_j)|^2 \leq C \varepsilon_j^2 |\nabla u_j|^2, \tag{5.10}$$

for every ε_j small enough. Inequalities (5.9) and (5.10) imply that

$$\frac{1}{\varepsilon_j^p} d^p(\nabla v_j, SO(n)) \leq 2^p \{ |e(u_j)|^p + C \varepsilon_j^p |\nabla u_j|^{2p} \}. \tag{5.11}$$

Now, by using the definition of B_j and writing

$$|\nabla u_j|^{2p} 1_{B_j} = |\nabla u_j|^p |\nabla u_j|^p 1_{B_j} \leq \frac{1}{\varepsilon_j^{p/2}} |\nabla u_j|^p 1_{B_j},$$

from (5.11) we obtain that

$$\frac{1}{\varepsilon_j^p} d^p(\nabla v_j, SO(n)) 1_{B_j} \leq 2^p \{ |e(u_j) 1_{B_j}|^p + C \varepsilon_j^{p/2} |\nabla u_j 1_{B_j}|^p \}.$$

This last inequality gives that

$$\frac{1}{\varepsilon_j^p} d^p(\nabla v_j, SO(n)) 1_{B_j} \text{ is equiintegrable,}$$

in view of (i) and of the fact that $\{\nabla u_j 1_{B_j}\}$ converges weakly in L^2 . It remains to prove that $\{\frac{1}{\varepsilon_j^p} d^p(\nabla v_j, SO(n)) 1_{B_j^c}\}$ is equiintegrable. Indeed, it turns out that

$$\frac{1}{\varepsilon_j^p} \int_{B_j^c} d^p(\nabla v_j, SO(n)) dx \rightarrow 0. \tag{5.12}$$

In order to see this, we use the fact that

$$\frac{1}{\varepsilon_j^2} \int_{B_j^c} W(\nabla v_j) dx \rightarrow 0, \tag{5.13}$$

which descends from (5.7) and from the convergence of $\{\mathcal{F}_{\varepsilon_j}(u_j)\}$ to $\mathcal{F}(u)$. By the growth hypothesis on W and by the inequality $t^p \leq t^2 + 1$, for $t \geq 0$, it is easy to show that

$$\frac{1}{\varepsilon^p} d^p(I + \varepsilon F, SO(n)) \leq \frac{2}{\varepsilon^2} W(I + \varepsilon F) + 1, \quad \text{for every } F \in \mathbb{R}^{n \times n} \text{ and } \varepsilon \in (0, 1),$$

so that

$$\frac{1}{\varepsilon_j^p} \int_{B_j^c} d^p(\nabla v_j, SO(n)) dx \leq \frac{2}{\varepsilon_j^2} \int_{B_j^c} W(\nabla v_j) dx + |B_j^c|.$$

This last inequality, together with (5.13) and the fact that $|B_j^c| \rightarrow 0$, implies (5.12).

Proof of (iii). For every $M > 0$ and every j , we set

$$E_M^j := \{x \in \Omega : d^p(\nabla v_j(x), SO(n)) \geq \varepsilon_j^p M\}.$$

Let us fix $q > p$. By using (ii), it is easy to show that for every $\eta > 0$ there exists $M_\eta > 0$ with the following property. If

$$f_1^j := d(\nabla v_j, SO(n)) 1_{E_{M_\eta}^j} \quad \text{and} \quad f_2^j := d(\nabla v_j, SO(n)) 1_{(E_{M_\eta}^j)^c},$$

then $f_1^j \in L^p$, $f_2^j \in L^q$, $d(\nabla v_j, SO(n)) = f_1^j + f_2^j$, and

$$\|f_1^j\|_{L^p}^p < \eta \varepsilon_j^p, \quad \|f_2^j\|_{L^q}^q \leq |\Omega| M_\eta^{q/p} \varepsilon_j^q. \tag{5.14}$$

Applying Theorem 5.2, it turns out that for every j there exists $R_j \in SO(n)$ such that $\nabla v_j = R_j + g_1^j + g_2^j$ a.e. in Ω , with

$$\|g_1^j\|_{L^p} \leq C \|f_1^j\|_{L^p}, \quad \|g_2^j\|_{L^q} \leq C \|f_2^j\|_{L^q}. \tag{5.15}$$

In particular,

$$\frac{1}{\varepsilon_j^p} |\nabla v_j - R_j|^p \leq \left(\frac{2}{\varepsilon_j}\right)^p (|g_1^j|^p + |g_2^j|^p) \tag{5.16}$$

and, due to (5.14) and (5.15),

$$\frac{1}{\varepsilon_j^p} \int_{\Omega} |g_1^j|^p dx < C\eta, \quad \frac{1}{\varepsilon_j^p} \left(\int_{\Omega} |g_2^j|^{p\alpha} dx \right)^{1/\alpha} < CM_\eta, \tag{5.17}$$

for $\alpha = \frac{q}{p} > 1$. Therefore, by considering (5.16) and (5.17), and using Lemma 5.1, we have that

$$\left\{ \frac{|\nabla v_j - R_j|^p}{\varepsilon_j^p} \right\} \text{ is equiintegrable.} \tag{5.18}$$

Recalling that $v_j = x + \varepsilon_j h$ \mathcal{H}^{n-1} -a.e. on $\partial_D \Omega$, it turns out that

$$|I - R_j| \leq C \left(\int_{\Omega} |\nabla v_j - R_j| dx + \varepsilon_j \int_{\partial_D \Omega} |h| d\mathcal{H}^{n-1} \right), \quad (5.19)$$

where C depends on Ω and $\partial_D \Omega$. This can be shown as done in the proof of Lemma 3.3 by using Poincaré–Wirtinger inequality and Lemma 3.2. From (5.18) follows in particular that $\left\{ \frac{|\nabla v_j - R_j|^p}{\varepsilon_j^p} \right\}$ is bounded in L^1 so that, by (5.19), we obtain that

$$\left\{ \frac{|I - R_j|}{\varepsilon_j} \right\} \text{ is bounded.} \quad (5.20)$$

Finally, observe that for every measurable subset A of Ω

$$\int_A |\nabla u_j|^p dx \leq \frac{2^p}{\varepsilon_j^p} \left\{ \int_A |\nabla v_j - R_j|^p dx + |A| |I - R_j|^p \right\},$$

for every j . This inequality, together with (5.18) and (5.20), gives (iii). \square

Proof of Theorem 2.2. Consider a sequence $\varepsilon_j \rightarrow 0$. By using the notation introduced in (2.9)–(2.11), the infima m_{ε_j} and m (see (2.6) and (2.8)) can be rewritten as

$$m_{\varepsilon_j} = \inf_{W^{1,p}} \mathcal{G}_{\varepsilon_j}, \quad m = \min_{W^{1,p}} \mathcal{G}.$$

It is easy to show that \mathcal{G} has a unique minimizer $u \in W_h^{1,2}$ on $W^{1,p}$. By standard properties of Γ -convergence (see [6, Theorem 7.8]), Theorem 2.4 and Corollary 3.5 imply that

$$m_{\varepsilon_j} \rightarrow m = \mathcal{G}(u)$$

and in turn, by (2.7), that

$$\mathcal{G}_{\varepsilon_j}(u_{\varepsilon_j}) \rightarrow \mathcal{G}(u) < \infty, \quad (5.21)$$

when $\{u_{\varepsilon_j}\}$ is a sequence of “almost minimizers”. Again by standard arguments, (5.21) and Corollary 3.5 imply that

$$u_{\varepsilon_j} \rightharpoonup u \text{ weakly in } W^{1,p} \quad \text{and} \quad \mathcal{F}_{\varepsilon_j}(u_{\varepsilon_j}) \rightarrow \mathcal{F}(u).$$

This last result and Theorem 2.5 give that $\{u_{\varepsilon_j}\}$ converges to u strongly in $W^{1,p}$. Since this is true for every $\varepsilon_j \rightarrow 0$, the whole sequence $\{u_{\varepsilon}\}$ converges to u strongly in $W^{1,p}$ (and $m_{\varepsilon} \rightarrow m$). \square

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Appendix A

We collect here some estimates involving the function g_p , which describes the growth from below of our energy density. We use them mainly in the proof of Lemma 3.1.

For every $K > 0$, there exists C depending on p and K such that

$$t^2 \leq C g_p(t), \quad \text{for every } 0 \leq t \leq K, \quad (A.1)$$

$$t^p \leq C g_p(t), \quad \text{for every } t \geq K. \quad (A.2)$$

Moreover, since $g_p(t) \leq \frac{1}{2} \min\{t^p, t^2\}$ for every $t \geq 0$ and g_p is convex, there exists C depending on p such that

$$g_p(s+t) \leq C(s^p + t^2), \quad \text{for every } s, t \geq 0. \quad (A.3)$$

In order to prove Lemma 3.1 we need the following truncation result proved in [8].

Proposition A.1 (Truncation). *There exists a constant C depending on Ω and p with the following property: for every $v \in W^{1,p}$ and every $\lambda > 0$ there exists $V \in W^{1,\infty}$ such that*

- (i) $\|\nabla V\|_{L^\infty} \leq C\lambda,$
- (ii) $\|\nabla v - \nabla V\|_{L^p(\Omega)}^p \leq C \int_{\{x \in \Omega: |\nabla v(x)| > \lambda\}} |\nabla v|^p dx.$

Proof of Lemma 3.1. For $v \in W^{1,p}$, let $V \in W^{1,\infty}$ be given by Proposition A.1 (with $\lambda > 0$ to be chosen), and $R \in SO(n)$ arbitrary. Since g_p is nondecreasing, by using (A.3) we have

$$\int_{\Omega} g_p(|\nabla v - R|) dx \leq C \int_{\Omega} (|\nabla v - \nabla V|^p + |\nabla V - R|^2) dx, \tag{A.4}$$

where C depends on p . Let $S(x) \in SO(n)$ be such that $|\nabla v - S| = d(\nabla v, SO(n))$ a.e. in Ω . Observe that, in the set where

$$|\nabla v - S| \geq \sqrt{n}, \tag{A.5}$$

we have

$$|\nabla v|^p \leq 2^p (|\nabla v - S|^p + n^{p/2}) \leq 2^{p+1} d^p(\nabla v, SO(n)). \tag{A.6}$$

It is clear that (A.5) is satisfied if $|\nabla v| \geq 2\sqrt{n}$. Thus, by using (A.6) and Proposition A.1 (ii) with $\lambda = 2\sqrt{n}$, we have that

$$\begin{aligned} \int_{\Omega} |\nabla v - \nabla V|^p dx &\leq C \int_{\{x \in \Omega: |\nabla v(x)| > 2\sqrt{n}\}} |\nabla v|^p dx \\ &\leq C \int_{\{x \in \Omega: |\nabla v(x)| > 2\sqrt{n}\}} d^p(\nabla v(x), SO(n)) dx \end{aligned}$$

and in turn, by using (A.2), that

$$\int_{\Omega} |\nabla v - \nabla V|^p dx \leq C \int_{\Omega} g_p(d(\nabla v(x), SO(n))) dx. \tag{A.7}$$

In the case $p = 2$, the lemma we are proving is already well known (see [8]) and we apply it to V : there exist C independent of V and a constant rotation $R \in SO(n)$ such that

$$\int_{\Omega} |\nabla V - R|^2 dx \leq C \int_{\Omega} d^2(\nabla V, SO(n)) dx. \tag{A.8}$$

By rewriting (A.4) for such an $R \in SO(n)$, from (A.7) and (A.8) we obtain

$$\int_{\Omega} g_p(|\nabla v - R|) dx \leq C \int_{\Omega} \{g_p(d(\nabla v, SO(n))) + d^2(\nabla V, SO(n))\} dx, \tag{A.9}$$

where C depends on Ω and p . Next, we prove that

$$d^2(\nabla V, SO(n)) \leq C \{|\nabla v - \nabla V|^p + g_p(d(\nabla v, SO(n)))\} \quad \text{a.e. in } \Omega, \tag{A.10}$$

for some C depending on Ω and p . We use again the matrix $S(x) \in SO(n)$ such that $|\nabla v - S| = d(\nabla v, SO(n))$ a.e. in Ω .

(i) In the set where $|\nabla v - S| \leq 1$, the function $|\nabla V - \nabla v|$ is bounded by a constant independent of V :

$$|\nabla V - \nabla v| \leq |\nabla V| + |S| + 1 \leq C,$$

where in the last inequality we have used Proposition A.1 (i). Thus, since

$$t^2 \leq K^{2-p} t^p, \quad \text{for every } t \in [0, K] \text{ and } K \geq 1, \quad (\text{A.11})$$

we have

$$|\nabla V - \nabla v|^2 \leq C |\nabla V - \nabla v|^p$$

and in turn, using the definition of g_p ,

$$\begin{aligned} d^2(\nabla V, SO(n)) &\leq |\nabla V - S|^2 \leq 2|\nabla V - \nabla v|^2 + 2|\nabla v - S|^2 \\ &\leq C \{ |\nabla V - \nabla v|^p + g_p(d(\nabla v, SO(n))) \}, \end{aligned}$$

which gives (A.10).

(ii) In the set where $|\nabla v - S| > 1$, Proposition A.1 (i) and (A.11) give that

$$\begin{aligned} d^2(\nabla V, SO(n)) &\leq |\nabla V - S|^2 \leq C |\nabla V - S|^p \\ &\leq C \{ |\nabla V - \nabla v|^p + d^p(\nabla v, SO(n)) \}. \end{aligned}$$

From this inequality and from (A.2) we obtain (A.10).

Inequalities (A.9) and (A.10) imply that

$$\int_{\Omega} g_p(|\nabla v - R|) dx \leq C \int_{\Omega} \{ g_p(d(\nabla v, SO(n))) + |\nabla V - \nabla v|^p \} dx,$$

and in turn, by considering (A.7), give the thesis. \square

We finish by proving an approximation result for functions in $W_h^{1,p}$, which has been useful in the proof of the Γ -convergence results. We write $x \in \mathbb{R}^n$ in the form $x = (x'', x_{n-1}, x_n)$ and refer the reader to Definition 2.1 and to (2.1) for the notation.

Proposition A.2. *Suppose that $\partial_D \Omega$ has Lipschitz boundary in $\partial \Omega$, according to Definition 2.1, and let $W_h^{1,p}$ be defined in (2.1).*

If $h \in W^{1,\infty}$ and $1 \leq p < \infty$, then $W_h^{1,p}$ is the closure of $W_h^{1,\infty}$ in $W^{1,p}$.

In order to prove Proposition A.2, we need the following lemma.

Lemma A.3. *For $p \in [1, \infty)$, let $u \in W^{1,p}(Q^+)$ be such that $\text{supp}(u) \Subset Q$ and $u = 0$ \mathcal{L}^{n-1} -a.e. on Q_0^+ . Then, for every $\varepsilon > 0$ there exists $u_\varepsilon \in C^\infty(Q)$ such that $u_\varepsilon = 0$ on Q_0^+ and*

$$\|u_\varepsilon - u\|_{W^{1,p}(Q^+)} < \varepsilon. \quad (\text{A.12})$$

Proof. Let $u \in W^{1,p}(Q^+)$ satisfy the hypotheses of the lemma. Consider the subset $M := (-1, 1)^{n-2} \times (0, 1) \times (-1, 0]$ of Q and define

$$v := \begin{cases} u, & \text{on } Q^+, \\ 0, & \text{on } M. \end{cases}$$

It turns out that $v \in W^{1,p}(Q^+ \cup M)$. Up to extend v to a function in $W^{1,p}(Q)$ and to multiply it by a function $\zeta \in C_c^\infty(Q)$ such that $\zeta \equiv 1$ on $\text{supp}(u)$, we can suppose that $v \in W^{1,p}(Q)$ and that $\text{supp}(v) \Subset Q$. Starting from v , we want to construct a sequence $\{v_k\}$ which approximates u in $W^{1,p}(Q^+)$ and is such that $\text{supp}(v_k) \Subset Q \setminus M$. To this end, we define for every k

$$v_k(x) := v\left(x'', x_{n-1} + \frac{1}{k}, x_n - \frac{1}{k}\right), \quad \text{for every } x \in Q_k,$$

where

$$Q_k := (-1, 1)^{n-2} \times \left(-1 - \frac{1}{k}, 1 - \frac{1}{k}\right) \times \left(-1 + \frac{1}{k}, 1 + \frac{1}{k}\right).$$

Observe that

$$\text{supp}(v_k) \Subset Q \setminus M, \quad \text{for every } k \text{ sufficiently large.} \quad (\text{A.13})$$

Moreover, v and v_k are functions in $W^{1,p}(\mathbb{R}^n)$, up to extend them at 0 out of Q and Q_k , respectively. In this case, it is well known that $v_k \rightarrow v$ in $W^{1,p}(\mathbb{R}^n)$. In particular, we have obtained that

$$v_k \rightarrow u \quad \text{in } W^{1,p}(Q^+).$$

The last step of the proof consists in choosing k_ε such that

$$\|v_{k_\varepsilon} - u\|_{W^{1,p}(Q^+)} < \frac{\varepsilon}{2} \quad (\text{A.14})$$

and considering a standard family $\{\rho_m\}_m$ of mollifiers. By (A.13), there exists m_ε such that $u_\varepsilon := v_{k_\varepsilon} * \rho_{m_\varepsilon} \in C_c^\infty(Q \setminus M)$ (thus, $u_\varepsilon \equiv 0$ on Q_0^+) and

$$\|u_\varepsilon - v_{k_\varepsilon}\|_{W^{1,p}(Q)} < \frac{\varepsilon}{2}. \quad (\text{A.15})$$

Inequalities (A.14) and (A.15) give (A.12). \square

Proof of Proposition A.2. By a standard argument based on a partition of unity subordinate to a finite covering of $\overline{\Omega}$ and on local bi-Lipschitz charts, we can use Lemma A.3 to prove that $\{u \in W^{1,p}: u = 0 \text{ } \mathcal{H}^{n-1}\text{-a.e. on } \partial_D \Omega\}$ is contained in the closure of $\{u \in C^\infty(\overline{\Omega}): u = 0 \text{ on } \partial_D \Omega\}$ in $W^{1,p}$. The opposite inclusion is trivial. The result for a general boundary value $h \in W^{1,\infty}$ is obtained by adding h to both sets. \square

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